INVESTIGATING PRE-SERVICE CANDIDATES' IMAGES OF MATHEMATICAL REASONING: AN IN–DEPTH ONLINE ANALYSIS OF COMMON CORE MATHEMATICS STANDARDS

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ABSTRACT

This paper details the outcomes of a qualitative in–depth investigation into teacher education mathematics preparation. This research is grounded in the notion that mathematics teacher education students (as "degree seeking candidates") need to develop strong foundations of mathematical practice as defined by the Common Core State Standards for Mathematics' (CCSSM). In this investigation mathematics Pre-Service Candidates ("PSCs") participated in an online 15-week methods course that infused writing prompts. This research activity probed the PSCs images of mathematical reasoning. It is based on the idea that in mathematical teacher education, teacher preparation requires teaching mathematical standards. In teaching, the standards activities are required that infuse mathematical reasoning. This will aid PSCs in further infusing mathematical reasoning in their teaching both now and in the future.

Keywords: Common Core, Common Core State Standards for Mathematics'(CCSSM), Mathematical Practice, Mathematical Knowledge, Mathematics, National Research Council (WRC), National Council of Teachers of Mathematics' (NCTM), Pre-Service Candidate (PSC), Standards for Mathematical Practice (SMP), State Standards, and Tri–Squared

INTRODUCTION

Recent adoptions of the Common Core State Standards across numerous states have called into question teacher education preparation. The Standards for Mathematical Practice as defined on pages 6 – 8 of the Common Core State Standards for Mathematics reflect a need for teachers to strengthen and build "processes and proficiencies" for their students (NGACBP & CCSSO, 2010, p.6). In order to assist Pre-Service Candidates (PSC) to develop strong foundations of mathematical practice, as defined by the Common Core, for their future students, the following research activity probed their images of mathematical reasoning. This idea solicits the notion that in teaching the standards one should seek out activities to infuse mathematical reasoning throughout their teaching. To analyze PSCs' knowledge of mathematical reasoning, writing prompts were distributed to students online during a 15-week methods course.

Standards for Mathematical Practice (SMP)

The Common Core State Standards for Mathematics' (CCSSM) Standards for Mathematical Practice (SMP) were heavily influenced by the National Research Council's (NRC). Adding It Up (NRC, 2001) stands of mathematical proficiency; as well as the National Council of Teachers of Mathematics' (NCTM) Principles and Standards for School Mathematics (NCTM, 2000) process standards. The strands of mathematical proficiency are defined as: Conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (NRC, 2001). In the NCTMs' Principles and Standards for School Mathematics the process standards are listed as: Problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2000). The Table 1 gives a brief description of the strands for mathematical proficiency, from NRC's Adding it Up, that were used to develop the SMP.

These five strands are interdependent and interlaced together in the creation of mathematical proficiency. The NRC states that "helping children acquire mathematical proficiency calls for instructional programs that address all its strands" (NRC, 2001, p.116). As students' progress through elementary and middle school the framework provided by the five strands should empower them to contend with the mathematical challenges they face, as well as provide opportunities for their mathematical success in future studies. It should be noted that reasoning and reflection are essential components to developing mathematical proficiency.

The NCTM's book, Principles and Standards for School Mathematics (PSSM), has served as a guide for mathematics instruction for over twelve years. The PSSM provides a recommended framework for instructional programs in mathematics through a set of six principles (Equity, Curriculum, Teaching, Learning, Assessment, and Technology) and ten general standards that construct school mathematics curriculum across several grade-bands (Preschool to 2, 3 to 5, 6 to 8, and 9 to 12). The ten standards are divided into two different types: Content and process. There are five content standards, they are: Number and Operation, Measurement, Geometry, Data Analysis and Probability and Algebra. The process standards are listed in Table 2 below.

The process standards are seen as the mathematical processes in which students gain and use mathematical knowledge. Similar to the strands for proficiency, the NCTM's principles and standards are seen as interrelated. They should not be seen as separate content, principles, and standards, but necessary components in a mathematics curriculum. It can clearly be shown that one's ability to reason mathematically is an idea that threads through each of the process standards. The principles and standards are to be used to guide the methods and processes for teaching and learning mathematics. To incorporate the process standards within instruction is seen as integral to creating proficient learners of mathematics

The National Governors Association Center for Best Practices and Council of Chief State School Officers' Common Core State Standards for Mathematics (CCSSM) has been adopted by most of the United States. The CCSSM, in addressing best practices for teaching mathematics, lists eight Standards for Mathematical Practice (SMP). The SMP infuse both the processes, as mentioned in the NCTM's PSSM (NCTM, 2000), as well as the proficiencies from the NRC's Adding it Up (NRC, 2001). The CCSSM's SMP are listed in Table 3 below.

The SMP describe actions for teachers to incorporate within their classrooms to support the development of the NCTM's mathematical processes and NRC's proficiencies. These types of actions, in many instances, engage students' mathematical reasoning to become mathematically prepared and confident during their studies of

NRC's Strands for Mathematical Proficiency	
Conceptual Understanding	Integrated comprehension of concepts, operations and procedures.
Procedural Fluency	Ability to perform procedures appropriately, with flexibility, accuracy, and efficiency.
Strategic Competence	Abilities to formulate, represent, and solve problems.
Adaptive Reasoning	Aptitudes to think logically, reflect, explain, and justify among concepts and ideas.
Productive Disposition	Tendency to see content as sensible, valuable, useful, and worthwhile, combined with a belief that, with steady effort, one can effectively produce results.

Table 1. NRC's Strands for Mathematical Proficiency (NRC, 2001)

NCTM's Process Standards	
	Instructional Mathematics programs should enable all students to:
Problem Solving	1. Build new knowledge through problem solving; 2. Solve problems that arise in various contexts; 3. Incorporate a variety of strategies to solve problems; and 4. Reflect on the process of mathematical problem solving.
	Instructional Mathematics programs should enable all students to:
Reasoning and Proof	1. Recognize and create conjectures based on observed patterns; 2. Investigate conjectures and prove that all cases are true or that a counter example shows that it is not always true; and 3. Explain and justify solutions.
	Instructional Mathematics programs should enable all students to:
Communication	1. Organize and consolidate thinking in both written and verbal communication; 2. Communicate thinking clearly to peers, teachers, and others; and 3. Use appropriate vocabulary to express ideas precisely.
	Instructional Mathematics programs should enable all students to:
Connections	1. Understand that ideas are interconnected and that they build and support each other; 2. Recognize and apply connections to other contents; and 3. Solve problems that arise in various contexts with mathematical connections.
	Instructional Mathematics programs should enable all students to:
Representations	1. Emphasize a variety of representations to communicate ideas; 2. Select, apply, and translate among representations to solve problems; and 3. Use representations to model and interpret real life situations.

Table 2. NCTM's Process Standards (NCTM, 2000)

CCSSM Standards for Mathematical Practices	
Make sense of problems and persevere in solving them.	Mathematically proficient students: 1. Explain to themselves the meaning of a problem and look for entry points to its solution; 2. Analyze givens, constraints, relationships, and goals; 3. Make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt; 4. Consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution; 5. Monitor and evaluate their progress and change course if necessary; 6. Explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends; 7. Check their answers to problems using "different methods"; and 8. Understand the approaches of others to solving complex problems and identify correspondences between different approaches.
Reason abstractly and quantitatively.	Mathematically proficient students: 1. Make sense of quantities and their relationships in problem situations; 2. Bring two complementary abilities to bear on problems involving quantitative relationships: the abilities to decontextualize and contextualize ; and 3. Create a coherent representation of the problem at hand, considers the units involved, attends to the meaning of quantities, and with knowledge and flexibility uses different properties of operations and objects.
Construct viable arguments and critique the reasoning of others.	Mathematically proficient students: 1. Understand and use stated assumptions, definitions, and previously established results in constructing arguments; 2. Make conjectures and build a logical progression of statements to explore the truth of their conjectures; 3. Analyze situations by breaking them into cases, and can recognize and use counterexamples; 4. Justify their conclusions, communicate them to others, and respond to the arguments of others; 5. Reason inductively about data, making plausible arguments that take into account the context from which the data arose; 6. Compare the effectiveness of two plausible arguments; and 7. Listen or read the arguments of others, and decide whether they make sense, and ask useful questions to clarify or improve the arguments.
Model with mathematics.	Mathematically proficient students: 1. Apply the mathematics they know to solve problems arising in everyday life, society, and the workplace; 2. Apply what they know to make assumptions and approximations to simplify a complicated situation, and realize that these may need revisions later; 3. Identify important quantities in a practical situation and map their relationships using such tools as diagrams, two - way tables, graphs, flowcharts and formulas; 4. Analyze relationships mathematically to draw conclusions; and 5. Routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense.
Use appropriate tools strategically.	Mathematically proficient students: 1. Consider the available tools when solving a mathematical problem; 2. Are sufficiently familiar with appropriate tools to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and the limitations. 3. Detect possible errors by strategically using estimation and other mathematical knowledge; 4. Know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data; 5. Identify relevant external mathematical resources and use them to pose or solve problems; and 6. Use technological tools to explore and deepen their understanding of concepts.
Attend to precision.	Mathematically proficient students: 1. Try to communicate precisely to others; 2. Try to use clear definitions in discussion with others and in their own reasoning; 3. State the meaning of the symbols they choose, including using the equal sign consistently and appropriately; 4. Are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem; and 5. Calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context.
Look for and make use of structure.	Mathematically proficient students: 1. Look closely to discern a pattern or structure. 2. Recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. 3. Can step back for an overview and shift perspective; and 4. Can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects.
Look for and express regularity in repeated reasoning.	Mathematically proficient students: 1. Notice if calculations are repeated, and look both for general methods and for shortcuts; 2. As they work to solve a problem, they maintain oversight of the process, while attending to the details; and 3. Continually evaluate the reasonableness of their intermediate results.

Table 3. The CCSSM Standards for Mathematical Practices (NGACBP & CCSSO, 2010).

mathematics. Teachers, as well as other educational professionals, should seek ways to bridge mathematical practices with the content during instruction. The overall goal is for students to develop both procedural and conceptual understandings of mathematics. In order for this to occur, mathematics classrooms need to incorporate ideas of discourse, problem-based learning, as well as seek out other opportunities to understand students' mathematical reasoning.

The Rationale for Mathematical Reasoning

Over the course of the last thirty years, teacher education programs took on the responsibility for much of the research, training, and support of pre-service candidates' (PSC) Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). Given the possible repercussion that teachers' knowledge of subject matter has on their pedagogy, many teacher educators considered ways to incorporate discussions of subject matter into teacher education programs (Grossman et al., 1989). Ma (1999) stated that it is during teacher education programs that PSC have one of three opportunities to cultivate their knowledge of school mathematics and that "their mathematical competence starts to be connected to a primary concern about teaching and learning school mathematics" (p.145). Several researchers worked with PSC to advance teacher education programs and develop courses that promote teachers' SMK and PCK (Ball, 2002, 1988; Ball & McDiarmid, 1990).

Grossman, Wilson, and Shulman (1989) pointed out that having knowledge of only the content is not sufficient. For example, a PSC with a strong content knowledge may not be able to make connections and illustrate relationships between mathematical topics and may teach them as if they are fragmented pieces of information. Thus, disciplinary knowledge must also include knowledge of the underlying structures (substantive knowledge) and knowledge of how to conduct inquiry (syntactic knowledge). Pre-service candidates' of mathematical reasoning is based on their understandings of how students develop syntactic knowledge and substantive knowledge. In support of this idea the National Council of Teachers of Mathematics' (NCTM) Professional Standards for Teaching Mathematics (1991) stated:

"Knowing mathematics includes understanding specific concepts and procedures as well as the process of doing mathematics. Mathematics involves the study of concepts and properties of numbers, geometric objects, functions, and their uses - identifying, counting, measuring, comparing, locating, describing, constructing, transforming, and modeling. At any level of mathematical study, there are appropriate concepts and procedures to be studied (p.132)."

A general definition for substantive knowledge of teaching refers to understandings of particular topics within a discipline, procedures, concepts, and their relationships to each other. Mathematical substantive knowledge can be seen as the knowledge of mathematics and includes an understanding of particular mathematical topics, procedures, concepts, and the connections and organizing structures within mathematics.

Teachers' substantive knowledge of discipline structures has strong implications for what and how teachers choose to teach. To learn mathematics with understanding, students must be exposed to relevant mathematical relationships and connections in their mathematics courses (Ball, 1988; NCTM, 2000). One of the implications is the influence teachers' substantive knowledge can have on curricular decisions. Teachers with a strong knowledge of mathematics make connections among relationships within topics that promote students' conceptual understandings (Ball, 2002, 1991; Borko & Putnam, 1996; Ma, 1999; Stigler & Hiebert, 1999). "Given the potential impact teachers' knowledge of substantive structures may have on their pedagogy, teacher educators need to consider ways to incorporate discussions of substantive structures into programs of teacher education" (Grossman et al., 1989, p.29).

The syntactic knowledge of teaching is the evidence and proof that guide inquiry within the discipline. It focuses on where the discipline comes from, how it changes, and how truth is established within the discipline. Ball (1991) emphasized that syntactic knowledge of mathematics is knowledge about mathematics. Syntactic knowledge is seen as the nature of the knowledge in a field of study.

"Knowledge about mathematics also includes what it means to 'know' and 'do' in mathematics, the relative centrality of different ideas, as well as what is necessary or logical, and a sense of philosophical debate within the discipline" (Ball, 1991, p. 7).

PSCs' knowledge of syntactic structures has many different components. These components are concerned with establishing truth within a discipline. Truth within a discipline comes from its preserved foundations and new evidence or inquiry that gives rise to debate (Lakatos, 1976). For example, Schoenfeld (1999) asked researchers and professionals within the discipline of education to "characterize fundamentally important educational are as for investigation, in which theoretical and practical progress can be made over the century to come" (p. 4). Since this challenge for debate and inquiry, research has focused and made some improvement in many of the areas Schoenfeld called "sites for progress." Grossman, Wilson and Shulman (1989) stressed that teachers' lack of syntactic knowledge can limit their abilities to learn new information in their fields. Teachers with limited syntactic knowledge may not be able to distinguish between more or less legitimate claims within a discipline. Furthermore, a lack of syntactic knowledge may also cause teachers to misrepresent the mathematics they are teaching. Teachers with a limited syntactic knowledge may be unable to sufficiently explain relationships or engage in discourse to allow their students to explore of mathematics, in this case, impeding their abilities to teach as reflected in the SMP and therefore impacting their students' capabilities to reason mathematically.

PSCs' perspectives of their discipline influence their views of the roles of factual knowledge, evidence and inquiry. The syntactic knowledge of teachers is instrumental in determining the classroom environment that they nurture. Teachers with a strong sense of mathematical syntactic structures are more likely to have classrooms that incorporate mathematical reasoning by including discussions and activities aimed at developing their students' awareness and understanding.

Research Focus

The focus of this research is to provide information about PSCs' knowledge of mathematical reasoning. Borko and Putnam (1996) suggested that learning opportunities for teachers be grounded in the teaching of subject matter and "provide opportunities for teachers to enhance their own subject matter knowledge and beliefs" (p. 702). In response to this assertion, and Schoenfeld's (1999) call for theoretical research that is focused on practical and relevant applications throughout education, the focus of this research is to explore PSCs' images of mathematical reasoning. It is this researcher's belief that in order for teachers to get a better understanding of students' conceptions about mathematical reasoning, they themselves should evaluate their own ideas and notions about mathematical reasoning.

Research Conceptual Frameworks

The conceptual frameworks used in this research help explain the PSCs' images of mathematical reasoning when examining their responses to several related questions about mathematical reasoning and proof. The conceptual framework uses components of Shulman's (1986) knowledge base in teaching and incorporates Pirie-Kieren's (1994) notions of primitive knowledge and images from their model for growth in mathematical understanding that was adapted to teacher preparation by Berenson, Cavey, Clark and Staley (2001).

Shulman (1986) described three categories of content knowledge: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. Content knowledge refers to "the amount of knowledge per sec in the mind of the teacher" (p. 9). Subject matter content knowledge is seen as more than the knowledge of facts and concepts of a subject, but also includes understanding of the substantive and syntactic structures (Grossman et al., 1989; Shulman, 1986). PCK is a type of SMK that is improved and embellished by knowledge of the learners, knowledge of the curriculum, and knowledge of context pedagogy. Curriculum knowledge is seen as the knowledge of the range in programs, the materials available, and characteristics of a curriculum, at any grade level. Lastly, Shulman described three forms of knowledge that teachers use in their practice: propositional knowledge, case knowledge, and strategic knowledge.

Propositional knowledge is knowledge from examples of literature that contain useful principles about teaching. Propositional knowledge is also the wisdom of practice, empirical principles, norms, values, and the ideological and philosophical principles of teaching. Case knowledge is specific, well-documented, descriptive events of propositional knowledge. Cases are specific instances in practice that are detailed and complete descriptions that exemplify theoretical claims and communicate principles of practice and norms. Strategic knowledge is used when a teacher cannot rely on propositional or case knowledge, and must formulate answers when no simple solution seems possible. These strategies about teaching go beyond principles or specific experiences and are formulated using alternate approaches. The PSCs' examples, definitions, activities, and explanations are examined using Shulman's three forms of knowledge to see if they are using propositional, case, or strategic knowledge.

The Pirie and Kieren (1994) model for growth in mathematical understanding was adapted by Berenson, Cavey, Clark and Staley (2001) creating a model for studying prospective teachers' understanding of what and how to teach high school mathematics. The teacher preparation is a conceptual framework that is designed to capture the process of learning as prospective teachers come to an understanding of both what and how to teach in high school mathematics. The "what" to teach refers to the school mathematics that prospective teachers will teach after their teacher education training. The "how" to teach is the teaching strategies that are used by the prospective teachers. Both models define primitive knowledge as what is known. Making/having images is when PSCs use their primitive knowledge in new ways, and these images can lend insight into a PSCs' primitive knowledge.

Synchronous and Asynchronous Methods of Inquiry

The investigation of PSC's knowledge of mathematical reasoning was part of a study that investigated prospective teachers' knowledge of what and how to teach concepts in an elementary mathematics methods course. The larger study was concerned with developing prospective teacher knowledge in the concepts of mathematical content and processes. The study was done over a 15-week semester, involving several PSCs in two classes. One class was taught on-campus and the other was taught strictly online.

Over twenty-five pre-service candidates, who were scheduled to take the course on-campus and online, responded to the questions through Blackboard. The data source included a series of responses from writing prompts regarding mathematical reasoning. Weekly, the PSCs were given one reflection question, through Blackboard, in which they had forty-five minutes to answer. The use of technology allowed for the flexibility in the time for which PSCs could take to reflect upon the questions. By giving this opportunity for online reflections to both the on-campus section of the course, as well as the online section, it allowed all PSCs to answer the reflections when they had time to reflect and did not restrict anyone to a classroom setting.

The questions were developed from a call for proposals for the NCTM's publication Mathematics Teaching in the Middle School. The PSCs were not privy to the questions beforehand, unless they accessed the questions through Mathematics Teaching in the Middle School. The only criteria expected from the PSC were that their responses be at least a half page in length, with other formatting guidelines, which included font and font size, and single-spacing. Written statements for each of the eight questions, listed below, were analyzed for the PSCs' images of mathematical reasoning.

The questions used during the study were taken from a call for manuscripts from the NCTM publication Mathematics Teaching in the Middle School. The questions that the PSC were asked to respond to were:

1. What habits of mind does mathematical reasoning entail? How can those habits be cultivated?
2. What strategies, processes, or resources can students use to reason mathematically?
3. How can teachers help students progress from concrete reasoning to symbolic or abstract reasoning?
4. How can teachers elicit students' inductive or

deductive reasoning?

5. What are teachers' beliefs and attitudes toward mathematical reasoning in learning mathematics? How can these attitudes be supported, enhanced, or changed?

6. How can teachers help students develop mathematical understanding through mathematical reasoning, and vice versa? What challenges will teachers encounter?

7. How can mathematical reasoning be fostered by representations, communication, curriculum, technology, or the learning environment?

8. How can teachers help their students develop an appreciation for the value of proofs and enhance their students' capacity to construct proofs? (NCTM, 2010).

The conceptual frameworks used were instrumental in the analysis of the data. Analysis of the data for the research involved coding, sorting into categories of significance, and establishing patterns among the categories. Data from question responses were coded and evidence of mathematical reasoning was extracted and examined. To analyze the images of mathematical reasoning that PSCs had were coded using Shulman's (1986) three forms of teacher knowledge. They were looked at for consistency throughout the semester.

Data Analysis Methodology

The Total Transformative Trichotomous–Squared Test (Tri–Squared) provides a methodology for the transformation of the outcomes from qualitative research into measurable quantitative values that are used to test the validity of hypotheses. The advantage of this research procedure is that it is a comprehensive holistic testing methodology that is designed to be static way of holistically measuring categorical variables directly applicable to educational and social behavioral environments where the established methods of pure experimental designs are easily violated. The unchanging base of the Tri–Squared Test is the 3×3 Table based on Trichotomous Categorical Variables and Trichotomous Outcome Variables. The emphasis the three distinctive variables provide a thorough rigorous robustness to the test that yields enough outcomes to determine if differences truly exist in the environment in which the research takes place (Osler, 2012).

Trichotomous Categorical Variables—Research Questions

1. Is Mathematical Reasoning effective as a teaching method?

2. Does Mathematical Reasoning have an effect on student outcomes?

3. What were teacher perceptions of Mathematical Reasoning?

Trichotomous Outcome Variables—Responses

1. Yes
2. No
3. No Opinion

Research Hypotheses

The below hypotheses were used to assess the research questions one and two. Each research question addresses a null hypothesis with anticipation of a non–significant association, and an alternative hypothesis that suggests that a significant association does occur between the variables.

H_0: There is no significant difference in the Pre-Service Candidate (PSC) perceptions on Mathematical Reasoning in regards to effectiveness as a teaching method, effectiveness in terms of student outcomes, and effectiveness as a method of classroom instructional delivery.

H_1: There is a significant difference in the Pre-Service Candidate (PSC) perceptions on Mathematical Reasoning in regards to effectiveness as a teaching method, effectiveness in terms of student outcomes, and effectiveness as a method of classroom instructional delivery.

Mathematical Hypotheses

The Mathematical Hypotheses used in the study in terms of the Tri–Squared Test to determine PSC candidate perspectives regarding Mathematical Reasoning are as follows:

H_0: $\boldsymbol{Tri^2 = 0}$

H_1: $\boldsymbol{Tri^2 \neq 0}$

Quantitative Research Results

Table 1 shows the Assessment of Mathematical Reasoning Tri–Squared Test Qualitative Outcomes.

Data Analyzed Using the Trichotomous–Squared 3x3 Table designed to analyze the research questions from an Inventive Investigative Instrument with the following Trichotomous Categorical Variables: a_1 = [Mathematical Reasoning effectiveness as a teaching method?] = Qualitative Instrument Items: 1, 2, 4, 6, and 7; a_2 = [Mathematical Reasoning and its effect on student outcomes?] = Qualitative Instrument Items: 3 and 8; and a_3 = [Teacher perceptions of Mathematical Reasoning as it relates to classroom instruction?] = Qualitative Instrument Item: 5. The 3 × 3 Table has the following Trichotomous Outcome Variables: b_1 = Yes; b_2 = No; and b_3 = No Opinion. The Inputted Qualitative Outcomes are reported as follows:

$$\boldsymbol{Tri}^{2}\ d.f. = [C-1][R-1] = [3-1][3-1] = 4 = \boldsymbol{Tri}^{2}_{[]}$$

The Tri–Square Test Formula for the Transformation of Trichotomous Qualitative Outcomes into Trichotomous Quantitative Outcomes to Determine the Validity of the Research Hypothesis:

$$\boldsymbol{Tri}^{2} = \boldsymbol{T}_{Sum}\left[\left(\boldsymbol{Tri}_{x} - \boldsymbol{Tri}_{y}\right)^{2} : \boldsymbol{Tri}_{y}\right]$$

Tri ^{2} Critical Value Table = 0.207 (with d.f. = 4 at α = 0.975). For d.f. = 4, the Critical Value for p > 0.975 is 0.207. The calculated Tri–Square value is 17.47, thus, the null hypothesis (H_0) is rejected by virtue of the hypothesis test which yields the following: Tri–Squared Critical Value of 0.484 < 17.47 the Calculated Tri–Squared Value. Thus the null hypothesis is rejected and there is strong evidence that supports that there is a significant difference in the Pre-Service Candidate (PSC) perceptions on Mathematical Reasoning in regards to effectiveness as a teaching method, effectiveness in terms of student outcomes, and effectiveness as a method of classroom instructional delivery.

nTri = 19 □ = 0.975		TRICHOTOMOUS CATEGORICAL VARIABLES		
		a_1	a_2	a_3
TRICHOTOMOUS OUTCOME VARIABLES	b_1	100	24	22
	b_2	37	30	7
	b_3	6	4	0

Qualitative Research Outcomes

In examining the PSCs' responses to the questions there is a need to define terminology to keep consistent when explaining their images. In framing their responses, the PSCs define a student and a teacher as we would expect them to be defined in a K-12 setting. As the PSCs draw upon their images of mathematical reasoning, the data has strong implications for their propositional knowledge while examples of case knowledge and strategic knowledge are combined throughout to exemplify either observations in a K-12 mathematical classroom or personal experiences they have had as a student in such a classroom.

In qualitatively reviewing the data, it becomes evident that the PSCs have strong images with regards to propositional knowledge. The data reflects that the PSCs problem-solving, reasoning and proof, communication, connections, and representations as processes needed to foster mathematical reasoning. The PSCs view mathematical reasoning as an idea that can be achieved in the classroom through a multitude of strategies that can be modeled by the teacher. As one PSC commented the following:

"Students must have the habit of providing a rationale as a major part of every answer. They need to learn to justify their ideas through logical argument. Students should be in the habit of logical thinking in order to decide if our answers make sense and why they do so. It isn't enough to show the right answer, but a student needs to know why it is right. These habits of mind must be cultivated by teacher example."

Some PSCs view mathematical reasoning as student metacognition and self-reflection. As represented in this comment:

"When performing math tasks it is important to remember to be reflective in your thinking. Always asking yourself, "Does this make sense?" This will allow you to see that if your answer does not make sense, or you realize you came across the incorrect answer, that you can use this as an opportunity for learning."

The example below exemplifies that mathematical reasoning is a process that is student-centered, however, it should go beyond the individual to the nurturing of the process by the classroom teacher. As stated in this reflective comment:

"I think that I will start will the learning environment because I think that can really foster or take away from a child's ability to use mathematical reasoning. If a child feels safe in their classroom; not just physically safe, but safe to explore their thinking; then I believe they will be more likely to try new things and grow in their reasoning skills."

In looking at ways that a classroom teachers can nurture students' mathematical reasoning, many of the PSCs focused on Polya's Problem-Solving Process or a similar strategy as a framework to elicit discussion. Several images explored incorporating technology and real-life applications to make mathematics more relevant to the students' lives as stated by one PSC in the following reflection:

"Mathematical reasoning can be fostered by technology because it allows students to focus on the process of problem solving instead of the process of calculating numbers or amounts. Mathematical reasoning is also fostered by technology because it gives the opportunity for students to get acquainted with interesting problems."

The PSCs believe that mathematical reasoning must occur on an ongoing basis. They believe that in order for teachers to expect mathematical reasoning from their students that the classroom environment must be positive and allow for discourse, so that the students do not feel intimidated and are encouraged to explore. This is demonstrated in this PSC reflection:

"Students should also see that math is not simply memorization; they need to become comfortable with the topics so that they are able to see that many ideas in math are interrelated and they can use one concept to help them solve another. Overall, I would say that the best habit of mind to have when learning math concepts is persistence, because seeing math as something that you are determined to figure out is the best way to want to keep learning."

Interestingly, the PSCs propositional images of mathematical reasoning were vastly different then their personal experiences as students. They felt as if the opportunities that they were given to reason mathematically were limited. As honestly demonstrated in following two comments:

"This was one thing about learning Mathematics that hindered me, as I wasn't taught in a variety of ways. I never really learned mathematical reasoning; I was taught to perform mathematical algorithms without an explanation of what they represented."

"When I was in elementary school, we didn't a lot of the reasoning behind the concepts, just the algorithms. I think this has really hurt my mathematical reasoning skills."

While the PSCs may not have had K-12 experiences that allowed them to reasoning mathematically consistently, they do recognize its importance. This supports that notion that they would rely heavily on their propositional knowledge with regards to how mathematics will be taught in their classrooms.

Conclusion

Current reform efforts such as the Core Common Standards guiding curriculum decisions, it is imperative that teacher education programs look forward to the Standards for Mathematical Practice, so that instruction can become better aligned with these efforts. These practices are built on established processes and proficiencies for mathematics education that rely heavily on PSCs' knowledge and beliefs of mathematical reasoning. Two important themes emerged during the qualitative image investigation of the Pre-Service Candidates:

1. Opportunities for students to communicate, reflect, and explore a variety of mathematical representations are seen as important for fostering mathematical reasoning.

The PSCs saw a lack of exploration of mathematics content as a detriment to their learning. They felt as if they were taught in ways that required understanding through rote memorization without opportunities for exploration of the content. In looking back on their own limited experiences with reflection and communication, some PSCs felt as if they could have been more familiar and self-assured in their learning of mathematics content if they had additional occasions in which to discuss and internally process their learning.

2. Mathematical learning environments must nurture a climate of mathematical reasoning.

One of the most promising results of the study was the PSC's image of the importance of mathematical reasoning and the role in which the classroom teacher plays in facilitating this process. They knew that teachers needed to afford students multiple chances to reflect and communicate their mathematical understanding. They saw a teacher's role as going beyond "just telling" to establishing a classroom environment where explanations are processed and explored in ways that incorporate technology or other tools for learning.

Investigations into PSCs' beliefs on mathematical practices need to be continually explored and best practices further defined for all mathematics teachers. It is hoped that through opportunities, such as this study, PSCs will reflect on their successes and their possible shortcomings of their own K-12 educational experiences and will find ways to incorporate the Standards for Mathematical Practice in their own classrooms.

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