Multiple solutions of a problem: Find the best point of the shot

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In a recent issue of *Australian Senior Mathematics Journal* there has been published an interesting article by Galbraith and Lockwood (2010). In that article the problem of finding the most favorable points for a shot at goal in Australian football is considered from different points of view. A similar problem was considered by Galbraith and Stillman (2006) in the context of soccer.

Some time ago, at the Olympiad 'Lomonosov' held in Moscow for high school students, a problem with a the similar plot was proposed by the author of this article:

> The football player moves to the goal in parallel with the touchline of the rectangular field at a distance of 20 yards from it (Figure 1). He wants to strike at the goal at a time when the goal will be seen under the largest possible angle. At what distance from the goal-line (the side of the rectangle in the centre of which the goal is located) must he strike if it is known that the width of a football field is 72 yards and the distance between goalposts is 8 yards?

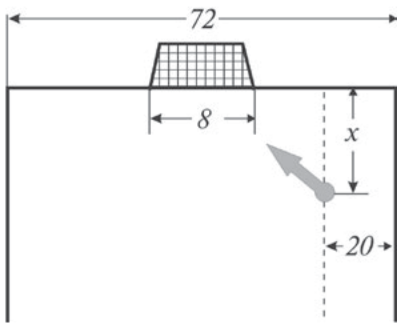

Figure 1. Statement of the problem.

We are talking of course about European football, however the differences between the two sports are not important here. This problem allows a few different solutions. It is important that among them there are both solutions by means of calculus and geometric solutions. We can recommend a teacher to offer a similar problem for students to solve in high school and after some time carefully to analyse with students all their results and all the solutions described below.

The solution with the help of the cosine rule and the calculus

The first algebraic solution is quite natural but it involves cumbersome calculations.

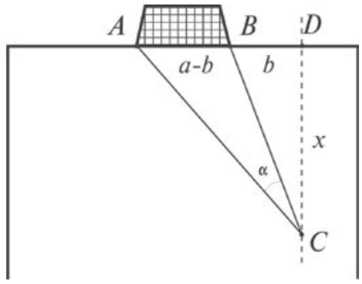

Figure 2. Angle of opportunity.

Similar to Galbraith and Lockwood (2010) we introduce the term the 'angle of opportunity' (α in Figure 2) which is the angle subtended by the goalmouth at the point of the shot. This angle can be expressed by the cosine rule in the triangle ABC. Denoting $AD = a$, $BD = b$, we get $AB = a - b$ and

$$\cos\alpha = \frac{AC^2 + BC^2 - AB^2}{2 \cdot AC \cdot BC}$$

$$= \frac{a^2 + x^2 + b^2 + x^2 - (a-b)^2}{2 \cdot \sqrt{(a^2 + x^2) \cdot (b^2 + x^2)}}$$

$$= \frac{x^2 + ab}{\sqrt{x^4 + (a^2 + b^2) \cdot x^2 + a^2 b^2}}$$

We need to find the value of x at which the value of the angle α will be the largest, that is, the value cos α will be the smallest. This can be done with the help of the derivative but further solution is quite time-consuming. Only the most advanced students could successfully carry out these calculations. This is shown in Appendix 1.

The solution with the help of the tangents and the calculus

The second algebraic solution is harder to find however, it is much easier to solve.

Since $\alpha = \angle ACD - \angle BCD, \tan\angle ACD = \frac{a}{x}, \tan\angle BCD = \frac{b}{x}$

then
$$\begin{aligned}\tan\alpha &= \tan(\angle ACD - \angle BCD)\\ &= \frac{\tan\angle ACD - \tan\angle BCD}{1+\tan\angle ACD\cdot\tan\angle BCD}\\ &= \frac{\frac{a}{x}-\frac{b}{x}}{1+\frac{a\cdot b}{x^2}}\\ &= \frac{(a-b)\cdot x}{x^2+a\cdot b}\end{aligned}$$

Denoting $f(x) = \frac{(a-b)\cdot x}{x^2+a\cdot b}$

we find the greatest value of this function with the derivative:

$$f'(x) = \frac{(a-b)\cdot\left(a\cdot b - x^2\right)}{\left(x^2+a\cdot b\right)^2}$$

Hence the function $f(x)$ has a maximum value at the point

$$x = \sqrt{ab} = \sqrt{20\cdot 12} = 4\sqrt{15} \approx 15.5 \text{ yards.}$$

Algebraic solution without the use of the derivative

The greatest value of the function

$$f(x) = \frac{(a-b)\cdot x}{x^2+a\cdot b}$$

can be found without the use of the derivative.

Since $f(x) = \frac{(a-b)\cdot x}{x^2+a\cdot b} = \frac{a-b}{x+\frac{ab}{x}}$

then $f(x) \le \frac{a-b}{2\sqrt{x\cdot\frac{ab}{x}}} = \frac{a-b}{2\sqrt{ab}}$

It follows from the inequality between the arithmetic mean and geometric mean:

$$\frac{z+t}{2} \ge \sqrt{z\cdot t}$$

The greatest value $f(x) = \frac{a-b}{2\sqrt{ab}}$ is achieved when $x = \frac{ab}{x}$, that is, when $x = \sqrt{ab}$.

Geometric solution

Similar problems were offered to students of senior mathematically-oriented classes in Moscow. As a rule, most students find the solutions that are close enough to those given above. However the use of geometric methods is much more effective and more efficient.

If we draw the circle, as shown in Figure 3, through the points A and B (the two goalposts) which will be tangent with the line on which the player moves (C is the tangent point), then the angle ACB is the desired maximum possible angle!

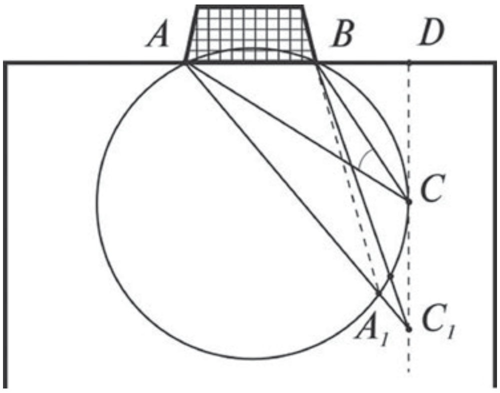

Figure 3. Point of contact.

Indeed this angle is equal to half of the arc AB of the inscribed angle. The angle with any other vertex which lies on this line (for example, angle AC_1B) will be less than angle ACB because $\angle AC_1B < \angle AA_1B$ (the exterior angle AA_1B of the triangle BA_1C_1 is bigger than the remote interior angle $\angle AC_1B$) and $\angle AA_1B = \angle ACB$ (inscribed angles).

As the secant DA and the tangent DC are drawn (see Figure 4) to the circle from the external point D then the square of the length of the tangent equals the product of the lengths to the circle on the secant: $DC^2 = AD{\cdot}BD$. This means that $x = \sqrt{a \cdot b} = \sqrt{240} = 4\sqrt{15} \approx 15.5$ yards.

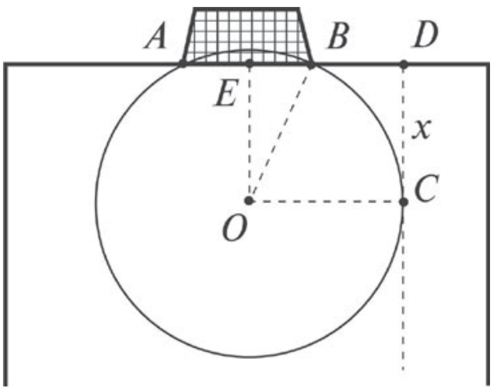

Figure 4. Find the distance.

The level lines

Usually when after a tedious algebraic solution the students are shown this geometric solution. Students are surprised by the elegance and simplicity of this solution. However simultaneously they have a natural question: how to 'invent' such a solution?

Here reasoning in the spirit of George Polya (Polya, 1957) will help. The ideas of geometric solution can appear if the students have knowledge about 'level lines' (otherwise called 'level curves', 'isolines' or 'contour lines'). The students here should have been informed that they are viewing level lines when they see on a map the lines along which the height of the relief (or depth of the ocean) has a constant value. Similar lines to represent constant values of different phenomena are isobars (lines of equal pressure) and isotherms (lines of equal temperature) in physics.

It is known that the geometric locus of points from which the segment AB is seen at any given angle is a pair of symmetric arcs with endpoints A and B. To substantiate this fact we can use the fact that the inscribed angle is equal to half of the arc on which it rests or the sine rule

$$\frac{AB}{\sin\alpha} = 2R$$

(see Appendix 2).

In our case, one of the arcs which is located 'outside the gate' (i.e., the goal-mouth) should not be considered. Therefore in Figure 5 it shows an example for the arc corresponding to the angle of 30°. This is the level line for the angle 30°: from all points lying on this line the segment AB is seen at an angle 30°.

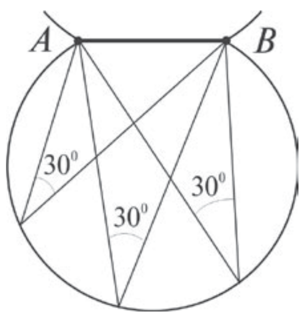

Figure 5. Level line for the angle 30 degrees.

For another angle the level line will be different. In this case the greater the radius of the circle, the less the sine of the angle, since

$$\sin\alpha = \frac{AB}{2R}$$

(see Appendix 2). Thus each point of a football field will be on one of the level lines (see Figure 6).

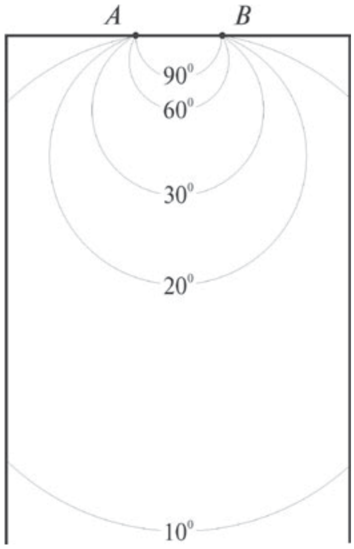

Figure 6. Level lines.

Therefore a player must strike at the moment when they will be on the 'smallest' of these circles and this circle will have as tangent the straight line on which the player moves. Any other point of this line lies on a circle of a larger radius. Note that these arguments are suitable if the player is moving not only on a given straight line but also for any trajectory: a kick must be applied at the moment when this trajectory contacts with the circle corresponding to the maximum angle.

Problems for independent decision

Finally you can invite the students to solve two similar problems.

1. The statue height of a metres stands on a pedestal height of b metres (see Figure 7). At what distance from the base of the statue should the observer stand in order to see the statue at the largest angle? The height of the observer to eye level is c metres ($c < b$). The width of the pedestal is ignored.

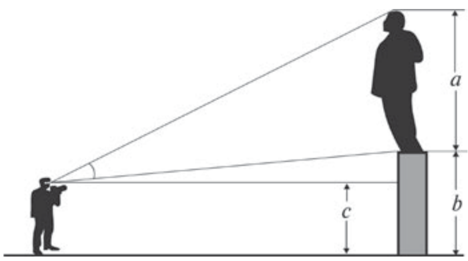

Figure 7. Statement of the problem about the statue.

This problem is placed in the textbook (Nikol'skii, Potapov, Reshetnikov & Shevkin, 2009) in the section 'The application of the derivative'. We hope that after the approach described above, students will find a much more elegant geometric solution. To check it, we give as answer to this problem: $\sqrt{(a+b-c)(b-c)}$ metres.

2. A tour bus is travelling along the straight road. The palace is located at the side of the road at an angle to it. At what point of the road shall the bus stop in order for the tourists to have the best visibility of the facade of the palace from the bus?

This problem is from Vasil'ev and Gutenmaher (1978). It is easy to verify that there is no data in the condition. The student's aim is to construct an adequate mathematical model of this situation. The following mathematical formulation of the problem is possible (but not unique):

Consider a segment AB that does not intersect a given line. Find on this line a point C for which the angle ACB has the greatest possible value.

Concluding thoughts

The above approaches to solving the problems are somewhat different from the approaches described in Galbraith and Lockwood (2010) and Galbraith and Stillman (2006). These solutions have their advantages and disadvantages. The main feature is the absence of the use of numerical methods. This is connected with the tradition of Russian mathematics education, in which unfortunately the full implementation of mathematical modelling is carried out with great difficulty.

However the use of the problems with multiple approaches is very useful for students. Due to multiple methods, students have flexible knowledge and are able to solve unfamiliar problems. So the material presented here as well as some results of Galbraith and Lockwood (2010) and Galbraith and Stillman (2006) may serve as a basis for a cycle of lessons in high school and for extracurricular activities.

They will be relevant to senior mathematics courses, especially for such subjects as Mathematical Methods and Specialist Mathematics (ACARA, 2012) because these subjects are designed for students with a strong interest in mathematics and whose future paths may involve mathematical studies at university.

In the case where consideration of the problem is not feasible as a whole, then different elements of this problem and its solutions can be used for studying and repetition of the following concepts and skills: the sine rule, the cosine rule, applications of the derivative, the arithmetic mean and geometric mean, the tangent and the secant, the exterior angle of a triangle, the inscribed angle, and others (ACARA, 2012).

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