Geometric series:
A new solution to the Dog Problem

Peter Dion
Niagara Health System, Canada
<sleepdoc.awning@sympatico.ca>

Anthony Ho
The Chinese University of Hong Kong
<hoamh@yahoo.com>

An interesting mathematical problem, 'The Dog Problem', is easily stated.

> Four dogs each at the corners A, B, C and D of a large square of side length L (Figure 1) start walking directly towards the dog on his right (Dog A to B, B to C, C to D, and D to A) all at the same speed. They do not to anticipate the target dog's movement and attempt a short cut. The dogs continually change direction, and we know by symmetry that they will spiral around each other and to meet in the middle of the square. What is the path length walked by each dog?

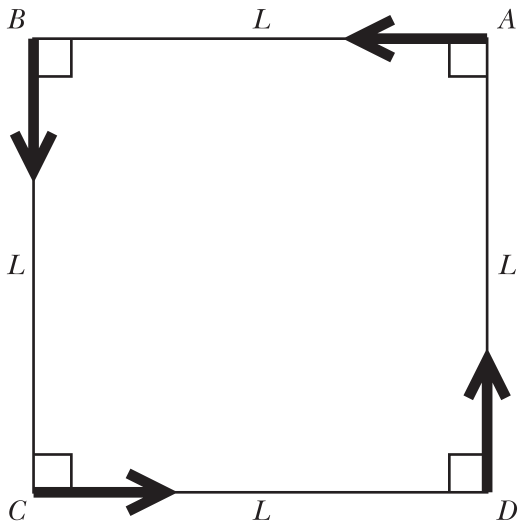

Figure 1. The square dog problem.

This problem is also known as the beetle, mice, ant or turtle problem, and perhaps others. Numerous internet sites (see for example, http://mathworld.wolfram.com/MiceProblem.html, http://mathproblems.info/prob29s.htm, or http://www.cut-the-knot.org/Curriculum/Geometry/FourTurtles.shtml) provide animations, solutions and generalisations to the n-gon dog problem. One need only Google 'dog/ beetle/ mice problem mathematics' to get started, but include the word 'mathematics' or you will be referred to the services of domestic exterminators. It is often used in computer programming simulation exercises, and variations involve asymmetries where the dogs do *not* all walk at the same speed or where the initial n-gon is not regular.

Solutions to this problem exist, some being variations of each other, which involve mathematics of a wide range of complexity. One can use intuition alone or calculus and polar co-ordinates, but herein we offer a solution we have not seen presented elsewhere using the sum of a geometric series applied in an unexpected way.

First the intuitive solution. The dogs start distance L apart, end up together, so distance L has to be covered somehow. At every instant the dogs form a square, which gets smaller with time. Since each *target* dog always walks at right angles to its pursuing dog, and all dogs walk at the same speed, nothing in the *target* dog's motion can increase or decrease the total distance the *pursuing* dog has to cover. Hence, each pursuing dog covers distance L. QED. Now this intuitive solution is not at all mathematically rigorous, and is better termed a plausibility argument, which some people might find convincing. Perhaps it becomes more persuasive when one considers an n-gon generalisation of the problem, where each target dog does, to some extent, walk away from (or toward for $n < 4$) the pursuing dog.

We can learn a lot more about this problem and its solution by thinking about it in more detail. For example, how many times do the dogs spiral around before they meet? Considering that the dogs are at every instant at the corners of an ever shrinking square, let us assume they have spiralled down until our original square is now just a tiny speck. Let us mentally examine that tiny square 'with a microscope'. Except for a matter of scale, we have our original problem again! Now if we had originally speculated that the dogs have X revolutions to walk before they meet, in our new tiny square, the dogs still have X revolutions to walk, albeit much smaller revolutions. So we see that no matter how many revolutions the dogs walk, they always have X to go. Hence the shape of the future spiral never changes, and the dogs must circle around themselves an infinite number of times before they meet.

The calculus solution in polar coordinates is beyond the scope of this article. However the result is that the distance of any dog from the centre of the square is

$$r = \frac{Le^{-\theta}}{\sqrt{2}}$$

and the total path length is $L(1 - e^{-\theta})$, where θ is the angle that each dog has walked counter clockwise around the centre of the square. After one revolution $\theta = 2\pi$, and $e^{-\theta} = 0.0019$, so the spiral converges extremely quickly, the entire remaining square reduced by this factor. This confirms also that r becomes zero and the path length becomes L only as θ becomes infinite, so the dogs circle an infinite number of times before covering a finite length L. Thus we have a most interesting curve.

We now provide a completely new solution. Let us assume (see Figure 2) that dog A has taken a small step αL towards vertex B, where α is a small fraction, say $\frac{1}{100}$, and each dog moves similarly. For clarity in the figure we exaggerate this step. The dogs now form a new smaller square $EFGH$ with corners situated distance αL away from those of square $ABCD$. Side length EF is the hypotenuse of right angled triangle EBF with sides EB of length $L - \alpha L$, and BF equal to αL. Since side αL is so tiny, and extends at right angles to side EB, side EB and the hypotenuse are almost the same length. For example, if $L = 100$ metres, and $\alpha L = 1$ metre, side EB has length 99 metres and EF is about 99.005 metres, about 5 millimetres longer, or about 5 parts in 100 000. Physically this is easy to understand. If you were to have a 12 foot ladder standing on the ground but held flatly against a wall, and then pull the ladder out one foot from the wall, the top of the ladder barely moves down at all.

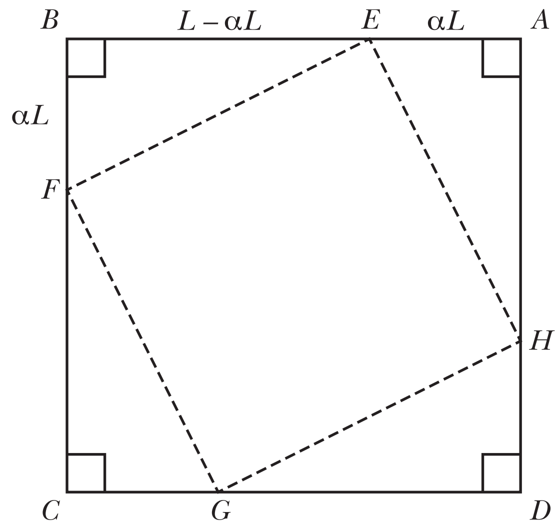

Figure 2. The square dog problem after a small step αL.

So for the moment we will say our new inner square has sides of length $L - \alpha L$ or $L(1 - \alpha)$, even though it is not strictly true, and that our smaller square has sides slightly decreased by a factor of $(1 - \alpha)$. Later we will redo the problem exactly and show that this approximation, perhaps plausible to some readers, is good enough to give the correct answer. But now let us get on to the solution.

Again we have the dogs take a step α times the length of the new side, or $\alpha L(1 - \alpha)$. This creates another still smaller square, and by the above argument it will have sides decreased by a factor $(1 - \alpha)$ again, to become length $L(1 - \alpha)^2$.

The pattern is clear. The dogs keep taking steps of α times the side of the current square, which become progressively smaller. The total path length is the sum of an infinite number of these steps which are easy to enumerate:

$$\text{Distance} = \alpha L + \alpha L(1-\alpha) + \alpha L(1-\alpha)^2 + \alpha L(1-\alpha)^3 + \ldots \qquad (1)$$

This is an infinite geometric series of the form:

$$A + Ar + Ar^2 + Ar^3 + \ldots \tag{2}$$

with sum $\frac{A}{1-r}$ if the absolute value of r is less than one. Here $A = \alpha L$ and $r = (1 - \alpha)$ is less than 1, and the sum is

$$\frac{\alpha L}{1-(1-\alpha)} = \frac{\alpha L}{\alpha} = L$$

So each dog walks distance L, as we found by intuition alone or calculus. QED!

How to use this in the classroom? An overriding theme would be that there are often many valid solutions to a given problem. A teacher might use the following sequence, after the topic of geometric series has been covered. First, pose the problem and have the students brainstorm solutions either individually or in groups. (They might come up with something original!) Second, it would be instructive to geometrically try to step through it, that is, draw carefully a large square (say 20 centimetres side length) on a piece of paper, then using a ruler going from dog A to dog B, measure off one centimetre and make this the new position of dog A, and do similarly for dogs B, C, and D. One could simply repeat this again and again, perhaps taking smaller steps near the centre. We find this works well. Even without great care a spiral will emerge, and a string laid out to measure the length of the arc can then be compared to the side of the square, and be found to fit almost exactly. This suggests a path length equal to L. Then, in discussing this apparent result, the teacher could introduce the idea behind the 'intuitive' solution. Finally the teacher might ask, "Could the recent topic of geometric series be of use?" Since this solution we suggest is in fact, new, one should not expect the students to immediately reproduce the solution we present here. However, the teacher might suggest that the students have the dogs step a distance αL, for some small factor α, and see where this gets them. This could be repeated a few times, and perhaps some students might catch on. At some point the teacher would present the solution. Finally, for the stronger students, the calculus solution could be hinted at. The rest of the article deals with some rather more subtle points which the teacher might wish to discuss with the mathematically stronger students.

Now let us deal with the annoying approximation we made above. We will redo the problem more exactly. Since EF is the hypotenuse of triangle EBF, it has an exact length of

$$\left(L^2(1-\alpha)^2 + \alpha^2 L^2\right)^{\frac{1}{2}}$$

Taking out a factor $L(1 - \alpha)$ from the square root we have:

$$EF = L(1-\alpha)\left(1 + \frac{\alpha^2}{(1-\alpha)^2}\right)^{\frac{1}{2}}$$

We have three factors, L, $(1-\alpha)$, and a square root which is difficult to work with so we will eliminate it with an approximation.

Inside the square root we have a 1 and the term $\frac{\alpha^2}{(1-\alpha)^2}$ which is tiny.

If say, α were $\frac{1}{1000}$, $\frac{\alpha^2}{(1-\alpha)^2}$ would be about $\frac{1}{1000000}$.

Again we will make a small approximation, but do not worry because in the end we really will do it exactly. Given the square root of 1 plus something very small, which we might call ε, there is a trick to estimate it very closely. We set $\sqrt{1+\varepsilon} = 1 + \beta$, and find β. Since ε is almost zero, $1 + \beta$ is almost equal to 1, so β is also very small. Squaring both sides we get $(1 + \varepsilon) = 1 + 2\beta + \beta^2$, or $\varepsilon = 2\beta + \beta^2$. If we solve this quadratic equation for β, surprisingly we just get our original equation again! So we try something else. Since we know that β is very small, β^2 is very small compared to 2β, so we can disregard it and get $\beta = \frac{\varepsilon}{2}$. So finally we have $\sqrt{1+\varepsilon} = 1 + \frac{\varepsilon}{2}$ to a good approximation. For example, the square root of 1.001 differs from $1 + \frac{0.001}{2}$ by 1.25 parts in 10 000 000. So, accepting this approximation again, we will proceed for the moment. So we can now say:

$$EF = L(1-\alpha)\left(1+\frac{\alpha^2}{2(1-\alpha)^2}\right)$$

Now the sides of our new smaller square will be multiplied not by $r = (1-\alpha)$, but instead by factor

$$r = (1-\alpha)\left(1+\frac{\alpha^2}{2(1-\alpha)^2}\right)$$

and for our sum of the geometric series $\frac{A}{1-r}$ we get:

$$\text{Path length} = \frac{\alpha L}{1-(1-\alpha)\left(1+\frac{\alpha^2}{2(1-\alpha)^2}\right)} \tag{3}$$

Expanding this gives:

$$\text{Path length} = \frac{\alpha L}{1-1+\alpha-\frac{\alpha^2}{2(1-\alpha)^2}+\frac{\alpha^3}{2(1-\alpha)^2}}$$

The ones subtract out, and if we divide top and bottom by α we finally have:

$$\text{Path length} = \frac{L}{1-\frac{\alpha}{2(1-\alpha)^2}+\frac{\alpha^2}{2(1-\alpha)^2}} \tag{4}$$

We recognise that α is small, as small as desired. In the limit of α approaching zero, all the terms in the denominator become negligible except the 1, and the path length approaches L. QED again!

But wait a minute! We still made an approximation by using the 'trick' that our square root $\sqrt{1+\varepsilon}$ was equal to $1+\frac{\varepsilon}{2}$. That is true, but our trick is actually a shortcut of a more formal expansion of a square root using the binomial theorem for real numbers.

We are familiar with the expansion $(a + b)^2 = a^2 + 2ab + b^2$, but if we had $(a + b)^{17}$ we would have 18 terms with large and unfamiliar coefficients. Fortunately the binomial theorem gives all these coefficients, and it has various forms. One form is:

$$(a+b)^N = a^N + Na^{(N-1)}b + N(N-1)a^{(N-2)}\frac{b^2}{2!} + N(N-1)(N-2)a^{(N-3)}\frac{b^3}{3!} + \ldots$$

where N is a whole number. There are $N + 1$ terms and the series continues until we reach a term with a $(N - N = 0)$ factor and the series ends.

Isaac Newton discovered an amazing extension of this formula by assuming that N can be any real number! In this case, the various $N - k$ factors before successive terms *never* equal zero if N is not a positive integer, and the series goes on forever. If a is larger than b, it converges to a finite sum. Typically this formula is used in physics, and almost always things are contrived to have $a = 1$, and b (<1) as small as possible. For $N = \frac{1}{2}$, this formula gives the square root, which is just what we want. For consistency of notation, let us replace b with ε and our binomial formula becomes simply:

$$(1+\varepsilon)^{\frac{1}{2}} = 1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \frac{\varepsilon^3}{16} - \frac{5\varepsilon^4}{128} + \ldots \tag{5}$$

exactly true for any ε with absolute value < 1.

So if instead of using $1+\frac{\varepsilon}{2}$ for the square root of $1+\varepsilon$, we could have used the infinite series (5), and for the path length formula (4) in the denominator we would have had a '1' and an infinite series of terms, all containing α, and all of which would disappear in the limit of α approaching zero. So the path length is L, exactly. QED finally!

The whole basis for originally accepting that $EF = L(1 - \alpha)$, even though it strictly is not, is that it differs from it by an amount involving the square of α. When α is very small, α^2 is negligible.

An alert reader might have noticed something peculiar in our original solution. We had the dogs step αL but did not specify α. Our smaller square had sides reduced by $r = (1 - \alpha)$. We created a geometric series with sum $\frac{A}{1-r}$, in which α does not appear! So it seems to work for any α! We have the right answer, but never specified that α goes to zero or is even small! How did we do that?

The answer is that we did make α small, without noticing it, when we made our annoying approximation that EF, the hypotenuse of triangle EBF, was close to the same length as side EB. This can only be true if the other side of the triangle, BF, is *very small compared to EB*. So we stipulated, actually forced BF to become much smaller than EB. In symbols this means that $\alpha L << L(1 - \alpha)$, hence $2\alpha << 1$.

This suggests *another* solution to the dog problem! Originally we said dog A took a step αL. We could have set $\alpha = \frac{1}{2}$, creating a new problem. Here the dogs would walk half the length of the square, then stop. They would now sit on the vertices of a new smaller square, of side length $\frac{L}{\sqrt{2}}$, rotated ninety degrees, inside the original square. The dogs would now turn and walk to the midpoints of the sides of this new smaller square. They would keep doing this, making a jerky, jagged 'spiral' to the centre. Once again we can now do a geometric series with $A = \frac{L}{2}$ and $r = \frac{1}{\sqrt{2}}$ to get the path length of this jagged spiral to be

$$\frac{\left(\frac{L}{2}\right)}{\left(1-\frac{1}{\sqrt{2}}\right)} = 1.707L \text{ approximately}$$

Now we could also have the dogs take step αL equal to $\frac{L}{3}$, or $\frac{L}{4}$, or in general, $\frac{L}{k}$ for some integer k. As k approaches infinity, the more it is like our original problem. If k is infinite, we get infinitely small steps and our original problem again. The algebra is easy. In Figure 2 our step BF would now be of length $\frac{L}{k}$, and EB would be $L - \frac{L}{k}$. The new smaller square would have sides of length

$$\left(\left(L-\frac{L}{k}\right)^2 + \left(\frac{L}{k}\right)^2\right)^{\frac{1}{2}}$$

and the reduction factor for this would be

$$r = \left(\left(1-\frac{1}{k}\right)^2 + \frac{1}{k^2}\right)^{\frac{1}{2}} = \left(1-\frac{2}{k}+\frac{2}{k^2}\right)^{\frac{1}{2}}$$

We now let k become large, and we have the square root of '1 plus something small'. Using our trick for that, we make it '1 + something small/2'.

Then $$r = 1-\frac{1}{k}+\frac{1}{k^2}$$

With initial step $A = \frac{L}{k}$, the sum is $\frac{A}{1-r}$ which is

$$\frac{\left(\frac{L}{k}\right)}{\left(1-1+\frac{1}{k}-\frac{1}{k^2}\right)}$$

On simplifying and multiplying top and bottom by k we get

$$\text{path length} = \frac{L}{1-\frac{1}{k}}$$

Letting k go to infinity, we have path length L again. QED for the last time!

For fun we will solve the dog problem for n dogs on a regular polygon with n vertices. With the square, the target dog always walks at right angles to the pursuing dog, but now the target dog walks somewhat away from the pursuing dog for $n \geq 5$, and somewhat towards him for $n \leq 3$. We use the same approach, concentrating on side AB of an n-gon as in Figure 3. Dog A

takes step αL towards vertex B while dog B takes step $BF = \alpha L$ toward vertex C. BF projects a component KB on the extension of line AB to the left. KB is $\alpha L \cos(\phi)$. Hence side $EK = L - \alpha L + \alpha L \cos(\phi)$. The angle KBF is ϕ, which is the angle the polygon 'turns' at each vertex. Since the sides turn a total of one revolution with n vertices, $\phi = \frac{2\pi}{n}$. The distance the dogs are separated apart after one step is EF. Since EKF is a right-angled triangle and KF is tiny compared to the other two sides, we once again postulate that hypotenuse EF can be taken to be equal to side EK, using the same logic and mathematics as above for the square. Thus the new smaller polygon's sides will be decreased a factor of not $1 - \alpha$, but $1 - \alpha + \alpha \cos(\phi)$. We create a geometric series as we did for the square, calculate its sum, and this time obtain the value for the total path length for a regular n-gon:

$$\frac{L}{1-\cos\left(\frac{2\pi}{n}\right)}$$

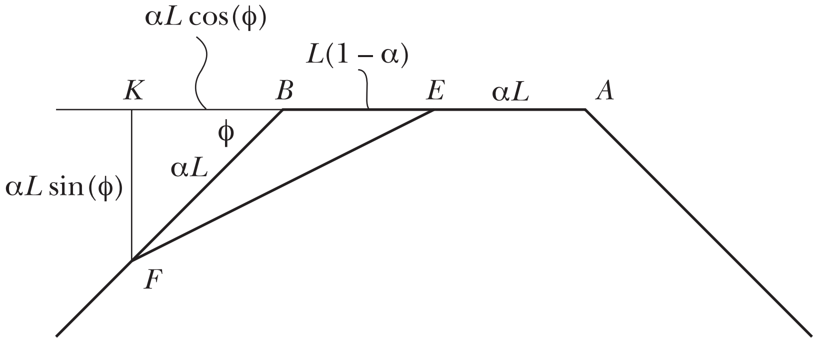

Figure 3. The n-polygon dog problem.

It is amusing to examine some special cases. For $n = 4$, we have a square, the cosine term is zero, and get distance L as before. For very large n we approach the cosine of zero which is 1 and the distance will be $\frac{L}{x}$ as x approaches 0, thus giving a path length approaching infinity. Here the polygon approaches a circle, where a ring of an infinite number of dogs chase each other endlessly. For $n = 3$ we have a triangle and the dogs meet after distance $\frac{2L}{3}$. For $n = 2$ the cosine term is -1 and there are only two dogs walking directly towards each other, a distance $\frac{L}{2}$ as expected. And finally, for $n = 1$ there is only one vertex, one dog, and no sides. Here $\cos(2\pi)=1$, the distance is L divided by zero, which is undefined. Our poor exhausted dog, lost by himself on a mathematical plane, wanders 'indefinitely' in search of a mate (pun intended).