Factorising a quadratic expression with geometric insights

Anwar H. Joarder
University Brunei Darussalam
anwarul.joarder@ubd.edu.bn, ajstat@gmail.com

An algorithm is presented for factorising a quadratic expression to facilitate instruction and learning. It appeals to elementary geometry which may provide better insights to some students or teachers.

Introduction

Factorisation of quadratic expressions appears at Year 10 in the *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], n.d.) in the Patterns and algebra subsection of Number and algebra. This also appears in Year 10 or lower levels in the British curriculum (Bostock, Chandler, Shepherd & Smith, 2014). There have been many methods for factorising a quadratic expression described in school text books. However, students often seem to struggle with grasping this skill. Some geometric explanation is presented with the hope that it will help visualise the steps of the solution. In particular, an algorithm is presented with an example to factorise any quadratic expression $q(x) = ax^2 + bx + c$. We start with a simple multiplication to provide better insight.

Observe that $(y + b)(y + b) = y(y + b) + b(y + b) = y^2 + 2yb + b^2$. The left hand side is the area of a square with each side of $y + b$ units. Thus we have following formula:

$$y^2 + 2yb + b^2 = (y + b)^2 \qquad (1)$$

It is interesting to note that the first term y^2 in the left hand side of (1) is the area of a square with each side of y units, the second term $2by$ is linear in y and is the area of twin rectangles (rectangles having the same area) with sides of y and b units, and the third term is the area of a square with each side of b units, as shown in Figure 1.

Draw a square with each side of $y + b$ units as in the right side of the above figure. Mark y units and then b units of $y + b$ units in any side of the square.
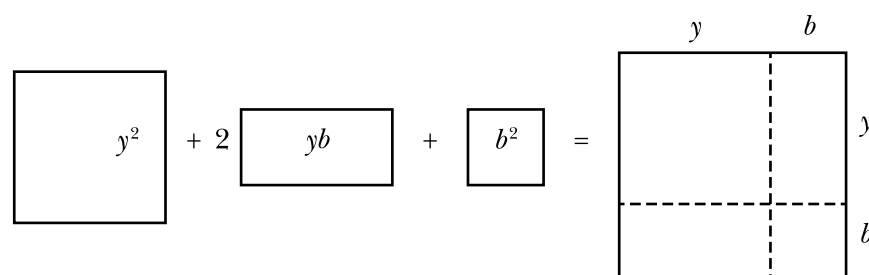
Figure 1. The area of interest, decomposed and composed.

Decompose the square into four regions: a square with each side of y units, another square with each side of b units, and twin rectangles, each of sides y and b units. The following is an example of a quadratic expression that can be simplified to a perfect square.

Example 1
Express $q = 64x^2 + 16x + 1$ as a perfect square.

Solution
The first term $64x^2 = (8x)^2$ is the area of a square with each side of $8x$ units, the second term $2(8x)1 = (8x)(1) + (1)(8x)$ is the area of twin rectangles, and third term is the area of a unit square. Since $q = 64x^2 + 16x + 1 = (8x)2 + 2(8x)1 + 1$ by comparing with (1), or Figure 1, we obtain $y = 8x$ and $b = 1$ so that the quadratic expression can be expressed as the area of a square with each side of $8x + 1$, i.e., $q = (8x + 1)^2$. From now on, by a square or a rectangle, we will often mean its area. The square $(8x)^2$, based on the leading term, will hereinafter be called *first square* and the augmented square $(8x + 1)^2$ with each side of $8x + 1$ units will be called a *new square*.

The two sides of (1) are the same though the right hand side of (1) is easily understood to be the area of a perfect square with each side of $y + b$ units. From (1), we have

$$y^2 + 2yb = (y + b)^2 - b^2 \qquad (2)$$

While both sides of (1) is a perfect square, neither side of (2) is a perfect square unless $b = 0$. The left hand side $y^2 + 2yb$ of (2) requires an additional b^2 to make it a perfect square $(y + b)^2$. That is why b^2 is subtracted from $(y + b)^2$ in the right hand side of (2). Note that y^2 in the left hand side is changed to a new square $(y + b)^2$ in the right hand side with an excess of b^2 units. Henceforth, the right hand side of (2) will be called *difference of two squares.* The following two examples are given below to help students to achieve the new square $(y + b)^2$ given in (2).

Example 2

Express $256x^2 + 64x$ as a new square and an excess.

Solution

It is easy to check that $256x^2 + 64x = (16x)^2 + 2(16x)2$. Then by (2), we obtain $256x^2 + 64x = (16x + 2)^2 - 2^2$ where the first square $(16x)^2$ is augmented to a new square $(16x + 2)^2$ and an excess of 2^2.

Note that while converting the quadratic expression in the above example into a new square $(16x + 2)^2$ and an excess of 2^2, we have actually got the *difference of two squares.* In the following example, we will see that any quadratic expression $4aq$ can be converted into a *first square, new square* and then the *difference of two squares.* Interestingly, this fact is generally true.

Example 3

Express $q = 8x^2 + x$ as the difference of two squares.

Solution

Since the first term is not a perfect square as that in (1) or (2), let us consider $4aq = 4 \times 8(8x^2 + x) = 4 \times 8 \times 8x^2 + 4 \times 8 \times x$. By comparing with (2), we have $y = 16x$ and $2yb = 4 \times 8 \times x$ or, $b = 1$ so that $4aq = (16x)^2 + 2(16x)(1) = (16x + 1)^2 - 1^2$ (difference of two squares). Since this can be converted to the area of a rectangle as we will see in (3), the factorisation of q is obvious.

The right hand side of $(z + d)(z - d) = z(z - d) + d(z - d) = z^2 - d^2$ is the difference of two squares but the left hand side is the area of a rectangle whose sides are $z + d$ and $z - d$. We thus have the formula:

$$z^2 - d^2 = (z + d)(z - d) \tag{3}$$

While, the left hand side will be called *difference of two squares,* the right hand side will be called a *rectangular form.* In a general notation, we may write $a^2 - b^2 = (a + b)(a - b)$.

A geometric explanation is presented below. Suppose that we want to find the difference of the two squares, say, $ABCD - EFGD$ as shown in the Figure 2.
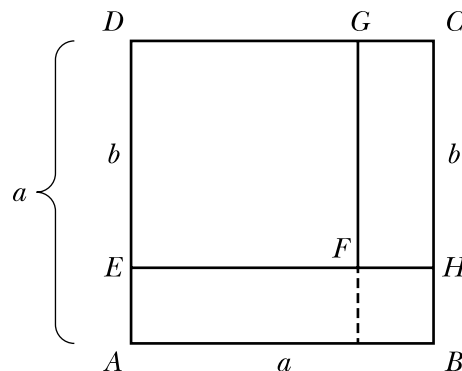

Figure 2. A geometric representation of the difference of two squares.

Obviously, the area of the rectangle $ABCD$ is $\Box ABCD = a^2$ and that of the rectangle $EFGD$ is $\Box EFGD = b^2$. The difference of the area of these two squares is then given by

$$a^2 - b^2 = \Box ABHE + \Box FHCG$$

The rectangle $FHCG$ can be placed beside the rectangle $ABHE$ so that the above can be written as

$$a^2 - b^2 = \Box ABHE + \Box FHCG = (AB + HC)AE = (a + b)(a - b)$$

Hence $a^2 - b^2 = (a + b)(a - b)$.

Example 4
Express $(8x + 1)^2 - 5^2$ in a rectangular form.

Solution
By using (3), we have $(8x+1)^2 - 5^2 = [(8x+1) + 5][(8x+1) - 5] = 8(4x+3)(2x-1)$.

An algorithm

An algorithm to factorise is presented below:

$$q = ax^2 + bx + c \tag{4}$$

We will first factorise $4aq = 4a(ax^2 + bx + c) = 4a^2x^2 + 4abx + 4ac$ and then simplify for $q = ax^2 + bx + c$.

Step 1: First square
Express the first term of $4aq$ as a perfect square, say, $4a^2x^2 = (2ax)^2$ which is the area of a square with each side of $2ax$ units. This will be called *first square.* Then, we have

$$4aq = (2ax)^2 + 4axb + 4ac \tag{5}$$

Step 2: Twin rectangles
Express the second term $4abx$ of $4aq$ in (5) as $4abx = 2(2ax)b$ to match it with twin rectangles (rectangles with the same area), each with sides of $2ax$ and b units. Then we have

$$4aq = (2ax)^2 + 2(2ax)b + 4ac \tag{6}$$

Note that $2ax$ in the expression $4abx = 2(2ax)b$ is 2 times the square root of the first term of (5) times a constant b.

Step 3: New square

By comparing the first two terms of the right hand side of (6) to (2), we obtain $4aq = (2ax + b)^2 - b^2 + 4ac$ which can also be written as

$$4aq = (2ax + b)^2 - (b^2 - 4ac) \tag{7}$$

The term $(2ax + b)^2$ is the new square with each side of $2ax + b$ units.

Step 4: Difference of two squares

The quantity $4aq$ in (6) has been converted into a new square $(2ax + b)^2$ and an excess of $b^2 - 4ac$ in (7). The latter may be expressed as a perfect square, say, $b^2 - 4ac = d^2$. This requires that $b^2 - 4ac$ is 0 or positive, i.e., b^2 is equal to $4ac$ or larger than it. In most cases when factorising at school level, d is an integer. Using this notation in (7), we have

$$4aq = (2ax + b)^2 - d^2 \tag{8}$$

The right hand side of (8) matches with the left hand side of the rectangular form given by (3). If $d^2 < 0$ there will be no real factorisation.

Step 5: New rectangle

By using the rectangular form in (3) to (8), we have

$$4aq = (2ax + b + d)(2ax + b - d) \tag{9}$$

The algorithm to factor $4aq = 4a(ax^2 + bx + c) = 4a^2x^2 + 4abx + 4ac$ is presented in Table 1.

Table 1. The steps in the solution process.

Steps	Names of steps	Algebraic expressions
Step 1	First square: $(2ax)^2$	$4aq = (2ax)^2 + 4abx + 4ac$
Step 2	Twin rectangles: $2(2ax)b$	$4aq = (2ax)^2 + 2(2ax)b + 4ac$
Step 3	New square: $(2ax + b)^2$	$4aq = (2ax + b)^2 - b^2 + 4ac$
Step 4	Difference of two squares	$4aq = (2ax + b)^2 - d^2$ where $d^2 = b^2 - 4ac$
Step 5	New rectangle	$4aq = (2ax + b + d)\ (2ax + b - d)$

A tabular method

The process above can also be represented by the following three tables (Tables 2, 3 and 4)

Table 2. Two-way multiplication table.

	$2ax$	b	$-d$
$2ax$	$4a^2x^2$: First square	$2abx$: Rectangle	$-2adx$
b	$2abx$: Rectangle	b^2	$-bd$
d	$2adx$	bd	$-d^2$

Multiplying $2ax + b + d$ (see the first column) by $2ax + b - d$ (see the first row), we got 9 elements in the bottom right 3×3 table. The sum of the two terms $2adx$ and bd in the last row and the two terms $-2adx$ and $-bd$ in the last column in the bottom right 3×3 table above vanishes. By (9), or Step 5 of the algorithm, the sum of the elements in the bottom right 3×3 table is $4aq$. Hence q is obvious. A simplified form is presented in Table 3. By using (1), or Figure 1 in Table 2, we have the following:

Table 3. New square and difference of two squares.

	$2ax$	b	$-d$
$2ax$			
b	$(2ax + b)^2$: New square		
d	$2adx$	bd	$-d^2$

By using (3) in Table 3, we have the following:

Table 4. New rectangle.

	$2ax$	b	$-d$
$2ax$			
b	4aq = $(2ax + b + d)$ $(2ax + b - d)$: New Rectangle		
d			

An example is provided below to illustrate how the algorithm can be applied.

Example 5
Factorise $q = 8x^2 + 2x - 3$.

Solution
We will factorise $4aq$, i.e., $4(8)q = 4(8)8x^2 + 4(8)2x - 4(8)3$. Calculate $d^2 = (2)^2 - 4(8)(-3) = 10^2$ so that $d = \pm 10$. In the following table, elements in the first column are $2ax$, b and d whereas those in the first row are $2ax$, b and $-d$. By cross multiplication, fill in the 9 elements in the bottom right 3×3 table of Table 5 given below.

Table 5. Two-way multiplication table.

	$2(8)x = 16x$	2	-10
$2(8)x = 16x$	$(16x)^2$: First square	$(16x)2$: Rectangle	$-160x$
2	$2(16x)$: Rectangle	2^2	-20
10	$160x$	20	-10^2

By (9) or Step 5 of the algorithm, the total of the 9 elements in the bottom right 3×3 table is $4aq$. In the following table, we show that $4aq$ is the difference of two squares. By using (1), or, Figure 1 in Table 1, we have Table 6:

Table 6. New square and difference of two squares.

	$2(8)x = 16x$	2	−10
$2(8)x = 16x$	$(16x + 2)^2$: New square		−160x
2			−20
10			-10^2

By using (3) in Table 6, we have the following, Table 7:

Table 7. New Rectangle.

	$2(8)x = 16x$	2	−10
$2(8)x = 16x$	$(16x + 2 + 10)(16x + 2 - 10)^2$: Rectangular Form		
2			
10			

Thus we have $(16x + 2 + 10)(16x + 2 - 10) = 4(8)q(x)$. Finally, we have $q(x) = (4x + 3)(2x - 1)$.

Conclusion

Though the method in the paper is long but it helps factorise any quadratic expression as long as d^2 is 0 or positive. Otherwise, the factors will involve imaginary numbers. For step (8), students are suggested to memorise the square of whole numbers up to 20, say. If a is a perfect square, one can easily factorize $4q$ instead of factorising $4aq$. We believe, the algorithm or the tabular method will be a relief for students and teachers.

Acknowledgements

References

Australian Curriculum, Assessment and Reporting Authority. (n.d.). *Australian Curriculum. F–10 Curriculum: Mathematics.* Retrieved from http://www.australiancurriculum.edu.au.

Bostock, L, Chandler, S., Shepherd, A. & Smith, E. (2014). *ST(P) Mathematics 3A.* Oxford, UK: Oxford University Press.