On solving systems of equations by successive reduction using 2×2 matrices

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Usually a student learns to solve a system of linear equations in two ways: 'substitution' and 'elimination'. While the two methods will of course lead to the same answer they are considered different because the thinking process is different. In this paper we solve a system in these two ways to demonstrate the similarity in the computation. We then see that changing the point of view leads us to a 'simpler' way to solve a system of equations. This leads naturally to two other consequences, viz., what is known as Chio's pivotal condensation process for computing determinants and Cramer's Rule. While the condensation process for computing determinants is known, it is not widely known, and the manner of solving equations developed here has not been seen elsewhere. This should be of interest to anyone teaching solving systems of linear equations (especially by hand) and can be the basis for teaching the basics of solving systems of equations, or for use as a guided project. This material is particularly relevant for the topic of matrices in unit 2 of the Specialist Mathematics, the topic of Algebra and matrices in unit 1 of the General Mathematics curriculum, as well as anywhere where multivariate applications appear such as finding regression lines in the data collection topic in unit 3 of essential mathematics (perhaps as a special project) and the bivariate data analysis topic of unit 3 of the general mathematics curriculum.

Solving systems of linear equations

While we can deal with any size system of equations, we will begin by considering the system of equations given by:

$$\begin{cases} 2x+3y+z=4 \\ x+2y-z=-1 \\ -x+2y+2z=3 \end{cases}$$

We can solve this in two ways, viz., 'substitution' and 'elimination'. Let us consider one at a time.

Substitution

Let us solve each of the equations for *x*. We get the system

$$\begin{cases} x = \dfrac{-3y - z + 4}{2} \\ x = -2y + z - 1 \\ x = \dfrac{-2y - 2z + 3}{-1} \end{cases}$$

Substituting the right-hand side (RHS) of the first equation in for in the second and third equation and cross multiplying we find:

$$\begin{cases} 2x + 3y + z = 4 \\ 2\left(-2y + z - 1\right) = 1(-3y - z + 4) \\ 2\left(-2y - 2z + 3\right) = -1(-3y - z + 4) \end{cases}$$

or, by multiplying both sides of the second and third equation by (–1) (distributing through the parentheses) and collecting variables on the left-hand side, we have

$$\begin{cases} 2x + 3y + z = 4 \\ (2\cdot 2 - 1\cdot 3)y + \left((2\cdot(-1)) - 1\cdot 1\right)z = 2\cdot(-1) - 1\cdot 4 \\ \left(2\cdot 2 - (-1)\cdot 3\right)y + \left(2\cdot 2 - (-1)\cdot 1\right)z = 2\cdot 3 - (-1)\cdot 4 \end{cases}$$

or, by recognizing the coefficients as determinants of 2×2 matrices, we have

$$\begin{cases} 2x + 3y + z = 4 \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} y + \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} z = \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} y + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} z = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} \end{cases}$$

or,

$$\begin{cases} 2x + 3y + z = 4 \\ y - 3z = -6 \\ 7y + 5z = 10 \end{cases}$$

So now each of the second and third equation can be solved for and cross multiplying yields:

$$\begin{cases} 2x + 3y + z = 4 \\ y = 3z - 6 \\ y = \dfrac{-5z + 10}{7} \end{cases}$$

or

$$\begin{cases} 2x+3y+z=4 \\ y=3z-6 \\ 7(3z-6)=1\cdot(-5z+10) \end{cases}$$

So we have

$$\begin{cases} 2x+3y+z=4 \\ y-3z=-6 \\ (1\cdot 5-7\cdot(-3))z=1\cdot 10-7(-6) \end{cases}$$

or

$$\begin{cases} 2x+3y+z=4 \\ y-3z=-6 \\ \left|\begin{vmatrix} 1 & -3 \\ 7 & 5 \end{vmatrix} z = \begin{vmatrix} 1 & -6 \\ 7 & 10 \end{vmatrix}\right| \end{cases}$$

or,

$$\begin{cases} 2x+3y+z=4 \\ y-3z=-6 \\ 26z=52 \end{cases}$$

This can then be solved (via back substitution) to yield the solution $z = 2$, $y = 0$, $x = 1$.

There is another option for this problem which we now describe. Going back to

$$\begin{cases} 2x+3y+z=4 \\ y=3z-6 \\ y=\dfrac{-5z+10}{7} \end{cases}$$

we see that we can extend this substitution step to include the first equation (substituting the value of y found from the second equation in both the first and the third):

$$\begin{cases} y=\dfrac{-2x-z+4}{3} \\ y=3z-6 \\ y=\dfrac{-5z+10}{7} \end{cases}$$

to get

$$\begin{cases} 3(3z-6)=1\cdot(-2x-z+4) \\ y=3z-6 \\ 7(3z-6)=1\cdot(-5z+10) \end{cases}$$

or, by multiplying both sides of the first and third equation by −1 (distributing that −1 through the parentheses) and collecting variables, we have

$$\begin{cases} (2\cdot 1-0\cdot 3)x+\big(1\cdot 1-3\cdot(-3)\big)z=1\cdot 4-3\cdot(-6) \\ y=3z-6 \\ \big(1\cdot 5-7\cdot(-3)\big)z=1\cdot 10-7\cdot(-6) \end{cases}$$

or, by recognizing the coefficients as determinants of 2×2 matrices, we have

$$\begin{cases} \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} x - \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} z = -\begin{vmatrix} 3 & 4 \\ 1 & -6 \end{vmatrix} \\ y-3z=-6 \\ \begin{vmatrix} 1 & -3 \\ 7 & 5 \end{vmatrix} z = \begin{vmatrix} 1 & -6 \\ 7 & 10 \end{vmatrix} \end{cases}$$

or,

$$\begin{cases} 2x+10z=22 \\ y-3z=-6 \\ 26z=52 \end{cases}$$

Now substituting in for z in both the first and second equation:

$$\begin{cases} z=\dfrac{-2x+22}{10} \\ z=\dfrac{-y-6}{-3} \\ z=\dfrac{52}{26} \end{cases}$$

So,

$$\begin{cases} 52\cdot 10=26(-2x+22) \\ 52(-3)=26(-y-6) \\ z=\dfrac{52}{26} \end{cases}$$

or,

$$\begin{cases} \begin{vmatrix} 2 & 10 \\ 0 & 26 \end{vmatrix} x = -\begin{vmatrix} 10 & 22 \\ 26 & 52 \end{vmatrix} \\ \begin{vmatrix} 1 & -3 \\ 0 & 26 \end{vmatrix} y = -\begin{vmatrix} -3 & -6 \\ 26 & 52 \end{vmatrix} \\ 26z = 52 \end{cases}$$

or

$$\begin{cases} 52x = 52 \\ 26y = 0 \\ 26z = 52 \end{cases}$$

from which we can readily see that, as we found earlier, $x = 1$, $y = 0$ and $z = 2$.

Elimination

Now let us solve the same problem using elimination (boxing what is known as the pivot). We begin with the equation

$$\left(\begin{array}{ccc:c} \boxed{2} & 3 & 1 & 4 \\ 1 & 2 & -1 & -1 \\ -1 & 2 & 2 & 3 \end{array}\right)$$

and see that we should replace the second row (R2) by twice row 2 minus row 1 $(2 \times \mathrm{R2} - \mathrm{R1})$ and the third row (R3) by twice row three minus negative one times row one $(2 \times \mathrm{R3} - (-1)\mathrm{R1})$. Of course, $2 \times \mathrm{R3} - (-1)\mathrm{R1} = 2 \times \mathrm{R3} + \mathrm{R1}$, but the formulation we use is more algorithmic in that it allows for easy adaptation to a different problem by replacing 2 and −1 by appropriate numbers taken from the matrix for that problem. So the new row two is

$$\begin{aligned} &\begin{pmatrix} (2\cdot 1 - 1\cdot 2) & (2\cdot 2 - 1\cdot 3) & (2\cdot(-1) - 1\cdot 1) & (2\cdot(-1) - 1\cdot 4) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} \end{pmatrix} \end{aligned}$$

and the new row three is

$$\begin{aligned} &\begin{pmatrix} (2\cdot(-1) - (-1)\cdot 2) & (2\cdot 2 - (-1)\cdot 3) & (2\cdot 2 - (-1)\cdot 1) & (2\cdot 3 - (-1)\cdot 4) \end{pmatrix} \\ &= \begin{pmatrix} 0 & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} \end{pmatrix}. \end{aligned}$$

So we have

$$\left(\begin{array}{ccc|c} \boxed{2} & 3 & 1 & 4 \\ 1 & 2 & -1 & -1 \\ -1 & 2 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} \\ 0 & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & \boxed{1} & -3 & -6 \\ 0 & 7 & 5 & 10 \end{array}\right)$$

Now using the boxed 1 as the pivot we want to replace R3 by $(1 \times \text{R3} - 7 \times \text{R2})$ and (optionally if back-substitution will be used) R1 by $(1 \times \text{R1} - 3 \times \text{R2})$, i.e., the new R3 are R1 are

$$R1 = \begin{pmatrix} (1 \cdot 2 - 0 \cdot 3) & (1 \cdot 3 - 3 \cdot 1) & (1 \cdot 1 - (-3) \cdot 3) & (1 \cdot 4 - (-6) \cdot 3) \end{pmatrix}$$

$$= \begin{pmatrix} -\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & 0 & \begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} & \begin{vmatrix} 3 & 4 \\ 1 & -6 \end{vmatrix} \end{pmatrix}$$

and

$$R3 = \begin{pmatrix} (1 \cdot 0 - 0 \cdot 7) & (1 \cdot 7 - 7 \cdot 1) & (1 \cdot 5 - 7 \cdot (-3)) & (1 \cdot 10 - 7 \cdot (-6)) \end{pmatrix}$$

$$= \begin{pmatrix} -\begin{vmatrix} 0 & 1 \\ 0 & 7 \end{vmatrix} & 0 & \begin{vmatrix} 1 & -3 \\ 7 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -6 \\ 7 & 10 \end{vmatrix} \end{pmatrix}$$

giving

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & 1 & -3 & -6 \\ 0 & 7 & 5 & 10 \end{array}\right) = \left(\begin{array}{ccc|c} \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & 0 & -\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 1 & -6 \end{vmatrix} \\ 0 & 1 & -3 & -6 \\ 0 & 0 & \begin{vmatrix} 1 & -3 \\ 7 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -6 \\ 7 & 10 \end{vmatrix} \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 2 & 0 & 10 & 22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & \boxed{26} & 52 \end{array}\right)$$

While at this point we could solve using back substitution, we can also use the boxed value as the pivot we can replace R1 with $(26 \times \text{R1} - 10 \times \text{R3})$ and R2 with $(26 \times \text{R2} - (-3) \times \text{R3})$ giving the new rows:

$$R1 = \begin{pmatrix} -\ (26 \cdot 2 - 0 \cdot 10) & 0 & (26 \cdot 10 - 10 \cdot 26) & (26 \cdot 22 - 10 \cdot 52) \end{pmatrix}$$

$$= \begin{pmatrix} -\begin{vmatrix} 2 & 10 \\ 0 & 26 \end{vmatrix} & -\begin{vmatrix} 0 & 10 \\ 0 & 26 \end{vmatrix} & 0 & -\begin{vmatrix} 10 & 22 \\ 26 & 52 \end{vmatrix} \end{pmatrix}$$

$$R2 = \begin{pmatrix} -\ (26\cdot 0-(-3)\cdot 0) & -(26\cdot 1-(-3)\cdot 0) & 0 & -(26\cdot(-6)-(-3)\cdot 52) \end{pmatrix}$$
$$= \begin{pmatrix} -\begin{vmatrix} 0 & -3 \\ 0 & 26 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 0 & 26 \end{vmatrix} & 0 & -\begin{vmatrix} -3 & -6 \\ 26 & 52 \end{vmatrix} \end{pmatrix}$$

giving

$$\left(\begin{array}{ccc|c} 2 & 0 & 10 & 22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & \boxed{26} & 52 \end{array}\right) = \left(\begin{array}{ccc|c} \begin{vmatrix} 2 & 10 \\ 0 & 26 \end{vmatrix} & 0 & 0 & -\begin{vmatrix} 10 & 22 \\ 26 & 52 \end{vmatrix} \\ 0 & \begin{vmatrix} 1 & -3 \\ 0 & 26 \end{vmatrix} & 0 & -\begin{vmatrix} -3 & -6 \\ 26 & 52 \end{vmatrix} \\ 0 & 0 & 26 & 52 \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 52 & 0 & 0 & 52 \\ 0 & 26 & 0 & 0 \\ 0 & 0 & 26 & 52 \end{array}\right)$$

from which we can again see that $x = 1$, $y = 0$ and $z = 2$.

Comments on the two methods and a change of the point of view

Both of the above methods yield the following calculation in matrix notation:

$$\left(\begin{array}{ccc|c} \boxed{2} & 3 & 1 & 4 \\ 1 & 2 & -1 & -1 \\ -1 & 2 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 1 & -1 \end{vmatrix} \\ 0 & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} \end{array}\right)$$

which is equal to

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & \boxed{1} & -3 & -6 \\ 0 & 7 & 5 & 10 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & \begin{vmatrix} 1 & -3 \\ 7 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -6 \\ 7 & 10 \end{vmatrix} \end{array}\right)$$
$$= \left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 26 & 52 \end{array}\right)$$

or

$$\left(\begin{array}{ccc:c} 2 & 3 & 1 & 4 \\ 0 & \boxed{1} & -3 & -6 \\ 0 & 7 & 5 & 10 \end{array}\right) = \left(\begin{array}{ccc:c} \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & 0 & -\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 1 & -6 \end{vmatrix} \\ 0 & 1 & -3 & -6 \\ 0 & 0 & \begin{vmatrix} 1 & -3 \\ 7 & 5 \end{vmatrix} & \begin{vmatrix} 1 & -6 \\ 7 & 10 \end{vmatrix} \end{array}\right)$$

$$= \left(\begin{array}{ccc:c} 2 & 0 & 10 & 22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 26 & 52 \end{array}\right)$$

which can be written as

$$\left(\begin{array}{ccc:c} 2 & 0 & 10 & 22 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & \boxed{26} & 52 \end{array}\right) = \left(\begin{array}{ccc:c} \begin{vmatrix} 2 & 10 \\ 0 & 26 \end{vmatrix} & 0 & 0 & -\begin{vmatrix} 10 & 22 \\ 26 & 52 \end{vmatrix} \\ 0 & \begin{vmatrix} 1 & -3 \\ 0 & 26 \end{vmatrix} & 0 & -\begin{vmatrix} -3 & -6 \\ 26 & 52 \end{vmatrix} \\ 0 & 0 & 26 & 52 \end{array}\right)$$

$$= \left(\begin{array}{ccc:c} 52 & 0 & 0 & 52 \\ 0 & 26 & 0 & 0 \\ 0 & 0 & 26 & 52 \end{array}\right)$$

from which we can again see that $x = 1$, $y = 0$ and $z = 2$.

The points of view of the methods of substitution and elimination are different (that is to say that thought processes are different). The substitution point of view involves a fair amount of algebraic manipulations to proceed from one step to another (and the augmented matrix formulation is unnatural using this approach), the elimination point of view involves dealing with adding multiples of vectors to each other in order to form the next augmented matrix (which is often done on the side). But in both cases the outcome of each step can be written simply by replacing certain elements of the augmented matrix with 2×2 determinants which are formed by using 'submatrices' and minus signs according to where the pivot is in relation to the opposite corner of the submatrix (the 'counter pivot'). The sign situation is summarised in the following diagram where the pivot is in the centre and the plus or minus signs are the locations of the counter pivot:

$$\begin{pmatrix} + & * & - \\ * & \text{pivot} & * \\ - & * & + \end{pmatrix}$$

It should also be noted that there is some freedom in choosing pivots (which variables to substitute or which columns to 'wipe out') and they should be chosen in order that they not be zero (for then there is a division by zero or a non-invertible row replacement, in the substitution and elimination views, respectively).

Example 1

Here is another example where different pivots are used.

$$\left(\begin{array}{ccc:c} 0 & -1 & 1 & 1 \\ 2 & 1 & \boxed{-1} & 1 \\ 2 & 0 & -2 & -2 \end{array}\right) = \left(\begin{array}{ccc:c} -2 & 0 & 0 & -2 \\ 2 & 1 & -1 & 1 \\ 2 & \boxed{2} & 0 & 4 \end{array}\right)$$

$$= \left(\begin{array}{ccc:c} \boxed{-4} & 0 & 0 & -4 \\ 2 & 0 & -2 & -2 \\ 2 & 2 & 0 & 4 \end{array}\right)$$

$$= \left(\begin{array}{ccc:c} -4 & 0 & 0 & -4 \\ 0 & 0 & 8 & 16 \\ 0 & -8 & 0 & -8 \end{array}\right)$$

which leads to $x = 1$, $y = 1$, and $z = 2$.

Note: This procedure works for any size system of linear equations and can be used to solve equations of the form $Ax = b$ for various b simultaneously; therefore it can easily be used to compute the inverse of a matrix (by starting with the identity on the right side of the dotted line).

Determinants

Note that a determinant can be computed by using row operations, that is, by elimination. This procedure is based on the transformations described above and the following properties of determinants:

- $a \cdot |A| = |A'|$ (or $|A| = \frac{1}{a}|A'|$), where the ith row has been replaced by a times the ith row plus b times the jth row.
- $\begin{vmatrix} a & * \\ 0 & A' \end{vmatrix} = a \cdot |A'|$
- $|A| = -|A'|$ if A' is obtained from A by exchanging two adjacent rows or columns.

So the determinant of the matrix

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}$$

can be computed by using the exact same procedure as above with the adjustments coming from the three changes above of which, in this case, only the first two are used (the third point is used when other pivots are chosen as we will demonstrate with examples below). We perform the calculation here:

$$\det\begin{pmatrix} \boxed{2} & 3 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix} = \frac{1}{2^2}\det\begin{pmatrix} \boxed{2} & 3 & 1 \\ 0 & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} \\ 0 & \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \frac{1}{2^2}\det\begin{pmatrix} 2 & 3 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 7 & 5 \end{pmatrix}$$

Now we will use the 1 as a pivot, but since the determinant of a triangular matrix is as easy to compute as that of a diagonal matrix, we will only apply the operation with the pivot to row 3:

$$\frac{1}{2^2}\det\begin{pmatrix} 2 & 3 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 7 & 5 \end{pmatrix} = \frac{1}{2^2}\frac{1}{1^1}\det\begin{pmatrix} 2 & 3 & 1 \\ 0 & \boxed{1} & -3 \\ 0 & 0 & \begin{vmatrix} 1 & -3 \\ 7 & 5 \end{vmatrix} \end{pmatrix}$$

$$= \frac{1}{2^2}\frac{1}{1^1}\det\begin{pmatrix} 2 & 3 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 26 \end{pmatrix}$$

$$= \frac{2}{2^2}\frac{1}{1^1}\cdot 26 = 13$$

Generally, the determinant of an $n \times n$ matrix $A = (a_{ij})$ with $a_{11} \neq 0$ can be written as the reciprocal of ${a_{11}}^{n-2}$ times the determinant of an $(n-1) \times (n-1)$ matrix thusly:

$$\det(A) = \frac{1}{a_{11}^{n-2}}\det(A')$$

where A' is an $(n-1) \times (n-1)$ matrix whose ijth entry is

$$\det\begin{pmatrix} a_{11} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{pmatrix}$$

The pivot (which in the above case is a_{11}) may also be taken to be in the kl-position, provided that number is not zero, in which case the above formulation is that

$$\det(A) = \frac{(-1)^{\text{row+column of pivot}}}{a_{kl}^{n-2}}\det(A')$$

where A' is an $(n-1) \times (n-1)$ matrix whose entries are $|a_{kl}a_{ij} - a_{kj}a_{il}|$, $i \neq k$ and $j \neq l$, arranged in the same order as the subscripts ij appear in the matrix (since $i \neq k$ and $j \neq l$ there is one less row and one less column in A' than in A). Note that $a_{kl}a_{ij} - a_{kj}a_{il}$ is plus or minus (depending on the relationship between kl and ij) the determinant of a 'sub-matrix'. This method is known as Chio's pivotal condensation method, though according to Howard Eves, there are traces of the method in an earlier paper by Hermite (see Eves, 1966, p. 129). It is likely that it has been discovered a number of times.

We give a few more examples to illustrate the procedure (using different pivots which are boxed).

Example 2

This example demonstrates why a (–1) shows up when using some pivot values (note the column swap needed to put the matrix in diagonal form).

$$\begin{vmatrix} 2 & \boxed{3} & 1 \\ 3 & 2 & -1 \\ 1 & -2 & 1 \end{vmatrix} = \frac{1}{3^2}\begin{vmatrix} 2 & 3 & 1 \\ 5 & 0 & -5 \\ 7 & 0 & 5 \end{vmatrix} = (-1)\cdot\frac{1}{3^2}\begin{vmatrix} 3 & 2 & 1 \\ 0 & \boxed{5} & -5 \\ 0 & 7 & 5 \end{vmatrix}$$

where we have swapped the first two rows to avoid a non-zero pivot. Continuing,

$$= (-1)\cdot\frac{1}{3^2}\frac{1}{5^1}\cdot\begin{vmatrix} 3 & 2 & 1 \\ 0 & 5 & -5 \\ 0 & 0 & 60 \end{vmatrix} = (-1)\cdot\frac{3}{3^2}\frac{5}{5^1}\cdot 60 = (-1)\cdot\frac{1}{3^1}\frac{1}{5^0}\cdot 60 = -20$$

Example 3

This is an example using a larger matrix.

$$\det\begin{pmatrix} \boxed{2} & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 0 & 2 \\ 0 & 1 & 3 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} = \frac{1}{2^3}\det\begin{pmatrix} \boxed{1} & 2 & 1 & 1 \\ -2 & 4 & -2 & 2 \\ 2 & 6 & 6 & 0 \\ 0 & 2 & -2 & 2 \end{pmatrix}$$

$$= \frac{1}{2^3}\frac{1}{1^2}\det\begin{pmatrix} \boxed{8} & 0 & 4 \\ 2 & 4 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$

$$= \frac{1}{2^3}\frac{1}{1^2}\frac{1}{8^1}\det\begin{pmatrix} 32 & -24 \\ -16 & 8 \end{pmatrix}$$

$$= \frac{1}{2^3}\frac{1}{1^2}\frac{1}{8^1}8^2(-2)$$

$$= -2$$

Example 4

This example uses various choice of pivots.

$$
\begin{aligned}
\begin{vmatrix} 1 & 2 & 1 & 2 & 1 \\ -1 & 1 & \boxed{-2} & 1 & -2 \\ 2 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix}
&= \frac{-1}{(-2)^3} \begin{vmatrix} -1 & -5 & -5 & 0 \\ -4 & -2 & 0 & \boxed{2} \\ -1 & -1 & -1 & -4 \\ -1 & -3 & -3 & 0 \end{vmatrix} \\
&= \frac{-1}{(-2)^3 2^2} \begin{vmatrix} \boxed{-2} & -10 & -10 \\ -14 & -10 & -2 \\ -2 & -6 & -6 \end{vmatrix} \\
&= \frac{-1}{(-2)^3 2^2} \frac{(-1)^{1+1}}{(-2)^{3-2}} \begin{vmatrix} -120 & -136 \\ -8 & -8 \end{vmatrix} \\
&= \frac{-1}{(-2)^3 2^2 (-2)} 8(-16) \\
&= 2
\end{aligned}
$$

One variation of the method includes the application of the property of determinants: $|A| = a|A'|$ if A' is obtained by A by dividing every member of one row or column by a. For example, during the previous computation we could have proceeded from the second line as follows:

Example 4b

$$
\begin{aligned}
&= \frac{-1}{(-2)^3} \frac{1}{2^2} \begin{vmatrix} \boxed{-2} & -10 & -10 \\ -14 & -10 & -2 \\ -2 & -6 & -6 \end{vmatrix} = \frac{-1}{(-2)^3 2^2} (-2)^3 \begin{vmatrix} \boxed{1} & 5 & 5 \\ 7 & 5 & 1 \\ 1 & 3 & 3 \end{vmatrix} \\
&= \frac{-1}{2^2} \frac{(-1)^{1+1}}{1^{3-2}} \begin{vmatrix} -30 & -34 \\ -2 & -2 \end{vmatrix} \\
&= \frac{-1}{2^2} 2 \cdot (-4) \\
&= 2
\end{aligned}
$$

This method of computing determinants by hand has its advantages. As an illustration of one advantage, it should be noted however that in the 3×3 determinant above, there a multiplier that can not be 'gotten rid of' in the same way as the $(-2)^3$. Similarly, in Example 2, the multiplier of $1/3$ could not be cancelled by using the properties of determinants. So the fact that our final answer is an integer is encouraging. If we had made a mistake along the way, we may have arrived at a non-integer answer and been forced to admit our mistake!

Cramer's rule

Let us now discover Cramer's rule in the context of an example. Consider the solution to the same equation as before:

$$\left(\begin{array}{ccc|c} \boxed{2} & 3 & 1 & 4 \\ 1 & 2 & -1 & -1 \\ -1 & 2 & 2 & 3 \end{array}\right)$$

We can solve this for z by putting in triangular form:

$$\left(\begin{array}{ccc|c} \boxed{2} & 3 & 1 & 4 \\ 1 & 2 & -1 & -1 \\ -1 & 2 & 2 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & \boxed{1} & -3 & -6 \\ 0 & 7 & 5 & 10 \end{array}\right) = \left(\begin{array}{ccc|c} 2 & 3 & 1 & 4 \\ 0 & 1 & -3 & -6 \\ 0 & 0 & 26 & 52 \end{array}\right)$$

So we find that $z = 52/26$. But the operations on the third and fourth column were identical and the same as the operations used to find the determinant of the matrix as described in the previous section. That is to say that

$$\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 2 \end{vmatrix} = K \cdot 2 \cdot 1 \cdot 26$$

where K is a number depending on the pivots and their locations. But

$$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & -1 \\ -1 & 2 & 3 \end{vmatrix} = K \cdot 2 \cdot 1 \cdot 52$$

for the same K. Therefore

$$z = \frac{52}{26} = \frac{K \cdot 2 \cdot 1 \cdot 52}{K \cdot 2 \cdot 1 \cdot 26} = \frac{\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & -1 \\ -1 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 1 \\ 1 & 2 & -1 \\ -1 & 2 & 2 \end{vmatrix}}$$

which is what Cramer's rule says.

Rearranging columns we see that

$$\left(\begin{array}{ccc|c} \boxed{1} & 3 & 2 & 4 \\ -1 & 2 & 1 & -1 \\ 2 & 2 & -1 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & \boxed{5} & 3 & 3 \\ 0 & -4 & -5 & -5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & \boxed{5} & 3 & 3 \\ 0 & 0 & -13 & -13 \end{array}\right)$$

So, by the same argument

$$x=\frac{-13}{-13}=\frac{K'\cdot 1\cdot 5\cdot(-13)}{K'\cdot 1\cdot 5\cdot(-13)}=\frac{\begin{vmatrix}1&3&4\\-1&2&-1\\2&2&3\end{vmatrix}}{\begin{vmatrix}1&3&2\\-1&2&1\\2&2&-1\end{vmatrix}}=\frac{\begin{vmatrix}4&3&1\\-1&2&-1\\3&2&3\end{vmatrix}}{\begin{vmatrix}2&3&1\\1&2&-1\\-1&2&3\end{vmatrix}}$$

And finally:

$$\left(\begin{array}{ccc:c}\boxed{2}&1&3&4\\1&-1&2&-1\\-1&2&2&3\end{array}\right)=\left(\begin{array}{ccc:c}2&1&3&4\\0&\boxed{-3}&2&-6\\0&5&7&10\end{array}\right)=\left(\begin{array}{ccc:c}2&1&3&4\\0&-3&2&-6\\0&0&-31&0\end{array}\right)$$

So,

$$y=\frac{0}{-31}=\frac{K''\cdot 2\cdot(-3)\cdot 0}{K''\cdot 2\cdot(-3)\cdot(-31)}=\frac{\begin{vmatrix}2&1&4\\1&-1&-1\\-1&2&3\end{vmatrix}}{\begin{vmatrix}2&1&3\\1&-1&2\\-1&2&2\end{vmatrix}}=\frac{\begin{vmatrix}2&4&1\\1&-1&-1\\-1&3&2\end{vmatrix}}{\begin{vmatrix}2&3&1\\1&2&-1\\-1&2&2\end{vmatrix}}$$

Conclusion

We have shown in the context of an example that the methods of substitution and elimination for solving systems of linear equations lead to the same computation, though through different thought processes. If we adjust the point of view still again to a component-wise computation, the calculations become simpler (no side-work is needed). This directly leads to the discovery of what is known as Chio's pivotal condensation process for computing determinants and Cramer's rule.

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