

Are future teachers ready to embrace mathematical inquiry?

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Introduction

In Liping Ma's 1999 seminal work, *Knowing and Teaching Elementary Mathematics*, the author discusses a facet of teaching that relies on both content knowledge and attitudes toward mathematics, namely, exploring a student conjecture: as the perimeter of a closed figure increases, so too must the area. USA teacher responses to the conjecture were largely non-mathematical—blindly accepting the claim as true or deferring mathematical inquiry by expressing intentions to “look it up” or ask a more knowledgeable colleague. By contrast, Chinese teacher responses were largely mathematical, exemplifying what Ma termed four levels of understanding: (L1) disproving the claim, (L2) exploring possibilities, (L3) clarifying conditions, and (L4) explaining the conditions. These teachers had not been exposed to much content beyond elementary algebra and geometry, and yet provided arguments and justifications much like mathematicians would.

Ten years after the first publication of *Knowing and Teaching Elementary Mathematics*, the National Governors' Association in the US set about a collaborative effort among the states to create what is now known as the *Common Core State Standards in Mathematics*, including a set of eight standards of mathematical practice, one of which is to “construct viable arguments and critique the reasoning of others” (CCSSM, 2014). This standard places a high demand on teachers—to not only foster rigorous thinking, but to develop in their students the process of questioning, exploring and evaluating possibilities, forming conjectures, piecing together arguments to prove conjectures—a special type of investigation usually termed “mathematical inquiry.” Must not the teacher be in possession of this ability first? Liping Ma concludes her chapter “Exploring New Knowledge” with: “Yet my assumption is that only teachers who are acculturated to mathematics can foster their students' ability to conduct mathematical inquiry. To foster such an ability in their students, the teachers must have it first” (Ma, 1999).

In this short paper we describe a project inspired by Ma's "Exploring New Knowledge" in which future 7–12 grade mathematics teachers enrolled in a fall 2013 capstone course at a typical regional state university were assigned fictitious student conjectures. These future teachers (all current high school mathematics teachers) were classified as undergraduate mathematics majors with a concentration in mathematics education. The goal of the capstone course was for future teachers to connect the key ideas in middle school and high school mathematics with the higher-level mathematics studied in college courses through explorations, laboratory activities, technology and service-learning experiences, and ideally the course runs immediately prior to the final semester of student teaching. We mention that the Conference Board of the Mathematical Sciences, an umbrella organisation of sixteen professional societies including the American Mathematical Society, the Mathematical Association of America, and the National Council of Teachers of Mathematics, recommends "9 semester-hours explicitly focused on high school mathematics from an advanced standpoint" (AMS, 2012).

Here are the conjectures:

- C1. The classical proof (Euclid) that $\sqrt{2}$ is irrational can be used to prove that the square root of any whole number that is not a perfect square is irrational.
- C2. I have discovered a characterisation for kites: A quadrilateral is a kite if and only if the area of the quadrilateral is $\frac{1}{2}$ times the product of the lengths of the two diagonals.
- C3. We learned that SSS is a congruence property for triangles. I think that SSSS is a congruence property for quadrilaterals.

The candidates read Ma's chapter on exploring new knowledge, discussed the four levels of understanding outlined therein, and then were charged with writing and then orally presenting a response to an assigned conjecture to their classmates. What can be expected when mathematics majors concentrating in math education are given such tasks? We discuss the responses in the following sections and draw the conclusion that such activities are necessary to illuminate the mathematical process of inquiry, argument, and critique in a setting that is relevant to prospective teachers, and, moreover, prove useful in measuring the mathematical confidence and attitudes toward mathematical inquiry possessed by these future teachers. All names in what follows are pseudonyms.

Responses to C1

Malia first sketched the classical proof for the irrationality of $\sqrt{2}$, familiar from here previous coursework in discrete mathematics:

Suppose, by way of contradiction, that $\sqrt{2}$ is rational and thus can be written as a fraction in reduced form:

$$\sqrt{2} = \frac{p}{q} \text{ with } \gcd(p, q) = 1.$$

Then,

$$2 = \frac{p^2}{q^2} \Rightarrow p^2 = 2q^2 \Rightarrow 2 \mid p^2$$

and thus $2 \mid p$ by a lemma from number theory. This in turn implies $4 \mid 2q^2$, or $2 \mid q^2$, and thus $2 \mid q$. But $2 \mid p$ and $2 \mid q$ contradicts our assumption that $\gcd(p, q) = 1$.

Malia's notation " $2 \mid p$ " means that 2 divides p , that is, 2 is a factor of p . Also, the "lemma from number theory" references the fact that if p^2 is even, then p is even. Malia then tested the proof for $\sqrt{6}$, ultimately identifying the necessity of a lemma that $6 \mid p^2 \Rightarrow 6 \mid p$. Convinced of this lemma, Malia was satisfied that Euclid's argument demonstrated the irrationality of $\sqrt{6}$. However, the necessity of $n \mid p^2 \Rightarrow n \mid p$ for n other than 2 or 6 gave her pause, and indeed chose $n = 12$ to demonstrate that the lemma is false in general: " $12 \mid 6^2$ but $12 \nmid 6$."

Again, Malia:

Although it is certainly true that \sqrt{n} is irrational (for n not a perfect square), the technique of Euclid's proof only demonstrates this fact for those n for which we have the lemma $n \mid p^2 \Rightarrow n \mid p$, $p \in \mathbb{Z}$.

Up to this point, Malia had disproved the claim (L1), identified the possibilities (L2), and was beginning to clarify the conditions (L3). The remaining discourse rose to level four.

Define a natural number n to be thin if the standard prime decomposition for $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_3^{k_3} \dots p_l^{k_l}$ has $k_i = 1$ for $i = 1, 2, 3 \dots l$, that is, n is a product of distinct primes, each occurring only once. For example, $6 = 2 \cdot 3$ and $30 = 2 \cdot 3 \cdot 5$ are *thin*, but $12 = 2^2 \cdot 3$ is not.

We mention that Malia's definition for a "thin number" was non-standard—mathematical texts would instead use the phrase, "Let n be a product of distinct primes." Finally, Malia wrote a proof that $n \mid p^2 \Rightarrow n \mid p$ for thin n and concluded that for thin n Euclid's proof shows \sqrt{n} is irrational.

Carrie responded to the same conjecture, similarly demonstrating that the critical part of the proof, $n \mid p^2 \Rightarrow n \mid p$, does not always hold. This future teacher quoted a theorem from number theory that maintained the truth of $n \mid p^2 \Rightarrow n \mid p$ for n a prime number and concluded that Euclid's proof can be used to show \sqrt{n} is irrational for n a prime. At this point Carrie abandoned Euclid's proof and instead described a different proof of the irrationality of $\sqrt{2}$ she had encountered in a previous number theory course which, when extended to \sqrt{n} , examined the prime factorisation of n and, in particular, a prime divisor a of n so that the maximum power k for which $a^k \mid n$ was odd. In this case $p^2 = nq^2$ produces a contradiction of an even maximal power of a on the left hand side of the equation, yet an odd maximal power on the right hand side.

Assume $\sqrt{6}$ is rational. Then

$$\sqrt{6} = \frac{c}{d}$$

for some c and d which are relatively prime. Then we can square both sides and obtain

$$6 = \frac{c \times c}{d \times d}$$

If we multiply both sides by $d \times d$ we get $6 \cdot d \cdot d = c \cdot c$. So $6 \cdot d \cdot d$ is a different form of $c \cdot c$. Since every composite number can be written uniquely as a product of primes, we write 6 as $2 \cdot 3$ and now count how many times 2 appears in the equation $6 \cdot d \cdot d = c \cdot c$. On the left, 2 appears once in the prime decomposition of 6, k times (let us say) in the decomposition of d , and another k times in the second d factor, for a total of $2k + 1$ times—an odd number of times. On the right hand side, 2 appears m times (let us say) in the prime decomposition of c , and another m times in the second c factor, for a total of $2m$ times—an even number. By the uniqueness of prime decompositions, this is a contradiction since the power of 2 cannot be both an odd and an even number.

Carrie thus concluded that this proof for the irrationality of $\sqrt{2}$ could indeed be generalised to show the irrationality of \sqrt{n} for all n with some odd powered prime p in the prime decomposition for n , which she then rigorously demonstrated was all non-perfect squares.

Both Malia's and Carrie's responses to C1 were mathematically mature and went well beyond merely disproving the conjecture (L1). Malia discovered exactly which natural numbers were amenable to Euclid's proof for irrationality; Carrie proved that Euclid's proof carried through for primes, but then reinterpreted the conjecture as a challenge to find a proof for irrationality that *would* work for all non-square integers.

Responses to C2

Future teacher Haylee was assigned C2—the equivalence of a quadrilateral Q being a kite and the area for Q being half the product of the lengths for the diagonals. Haylee maintained that the conjecture was true, constructing an incorrect proof of necessity and offering no discussion of the converse. (Note that the converse is indeed false.) First, Haylee assumed, rather than proved, that the diagonals of a kite are perpendicular—a necessary step in order to prove that a kite does indeed have the desired area formula. It was unclear to Haylee as to what properties of a kite she could use without proof, and this confusion could be traced to the fact that Haylee never articulated a definition for a kite—a quadrilateral with two pairs of congruent adjacent sides. Furthermore, Haylee merely verified the formula for a particular example, as shown in Figure 1, failing to extend her reasoning to the general case:

$$\text{Area of kite is } \frac{1}{2} \cdot 4 \cdot 3 + \frac{1}{2} \cdot 4 \cdot 7 = 20.$$

$$\text{And, } \frac{1}{2} \cdot d_1 \cdot d_2 = \frac{1}{2} \cdot 10 \cdot 4 = 20.$$

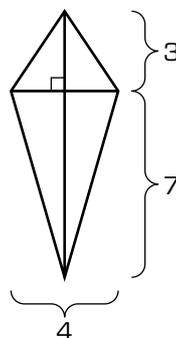


Figure 1. Haylee verifies the formula for a special case.

Finally, Haylee never considered the possibility of kites that are not convex and the question as to whether the area formula still holds true. (It does.)

By contrast, Kelsey constructed a careful argument that the diagonals for a kite are indeed perpendicular, but also assumed convexity (see Figure 2). She demonstrated the congruence of triangles ABD and ACD by SSS, and then triangles ABE and ACE by SAS in order to conclude the supplementary angles $\angle AEB$ and $\angle AEC$ are congruent and therefore right. The area formula then followed from the area of the triangles ABD and ACD as half the base times the height.

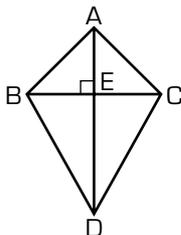


Figure 2. Kelsey's diagram.

Kelsey focused on this key property of perpendicular diagonals. She was able to construct counterexamples to the converse implication by starting with two perpendicular segments (the diagonals) and then completing to a (convex) quadrilateral Q for which $A = \frac{1}{2} \cdot d_1 \cdot d_2$, but clearly was not a kite, as shown in Figure 3.

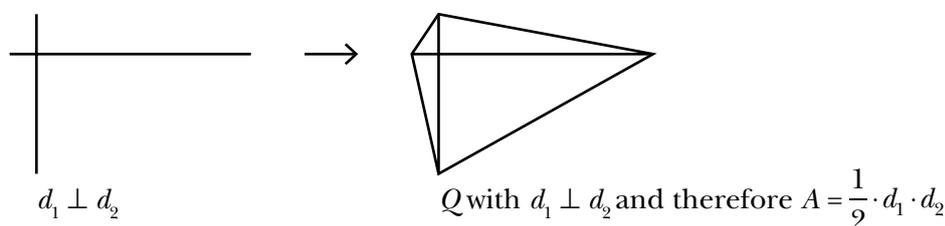


Figure 3. Kelsey's counterexample for the converse implication.

Kelsey had thus disproved the claim (L1) and was beginning to examine conditions (L2). This is where she stopped. Kelsey did not independently pursue a true characterisation for kites (for example, d_1 a perpendicular bisector for d_2), nor addressed the concave kites.

Malia and Carrie had recently completed a number theory course. Haylee and Kelsey had not recently completed a geometry course, and perhaps a lack of confidence in relevant content knowledge contributed to their limited responses. More importantly however, Candidate Haylee's written response and oral presentation clearly revealed a limited appreciation for the role of mathematical inquiry in the classroom—much like the US teachers in Ma's study. For Haylee, mathematical inquiry simply was not an integral part of school mathematics, especially when compared to using formulas and performing computations.

Response to C3

Darren was assigned the third conjecture, which was also geometric in nature. From the outset Darren was confounded by the conjecture and could not take any type of initial investigative step. The instructor introduced *Geostrips*—plastic strips of various lengths that can be joined together at the ends by using paper fasteners to produce various polygons. Darren discovered the rigidity of triangles and the flexing of polygons with more than three sides, and ultimately disproved the conjecture that SSSS is a congruence property for quadrilaterals. Darren used four *Geostrips* to construct a quadrilateral Q_1 which then flexed to produce a second quadrilateral Q_2 with the same four sides as Q_1 , but with different internal angles—two non-congruent quadrilaterals.

If we call my original quadrilateral $ABCD$, and then compare it to my new quadrilateral $A'B'C'D'$, which is just $ABCD$ flexed to produce different angles,

we have SSSS: $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{CD} \cong \overline{C'D'}$, $\overline{DA} \cong \overline{D'A'}$. But the two quadrilaterals are not congruent because the angles do not match.

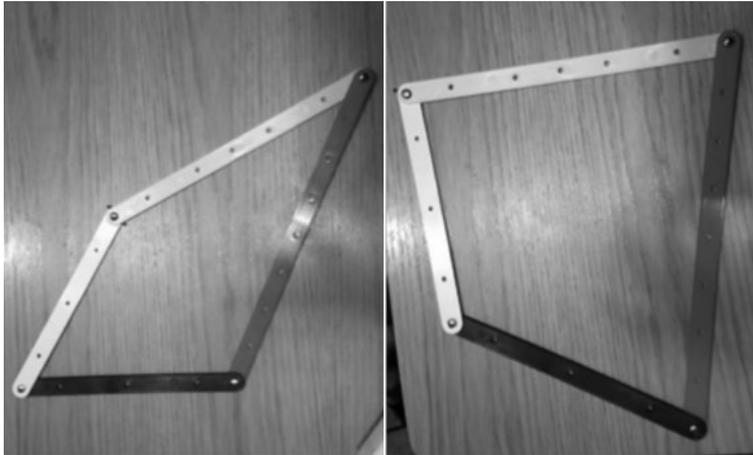


Figure 4. Darren's use of Geostrips to explore SSSS.

Darren's presentation was well-organised, providing the Common Core definition for congruence, illustrating examples where SSS is used in high school mathematics, and ultimately focused on the key “flexing” property which led to an interesting discussion on possible flexing for polyhedral surfaces. Yet, Darren did not attempt to refine C3 to a conjecture that would indeed lead to a congruence property for quadrilaterals (SASSS?). Convexity was mentioned in passing, but not used to restate the conjecture or produce cases in a proof. In short, Darren provided a L2 response.

Conclusion

The responses to the student conjectures were varied with respect to Ma's L1—L4. All candidates had successfully completed courses in calculus, linear algebra, abstract algebra, and other upper level mathematics courses, and yet only some candidates seemed to exhibit the instincts and qualities that characterise mathematicians when confronted with new knowledge. Indeed, the desire and ability to resolve new student conjectures to complete and intellectually satisfying solutions do not evidently develop from content knowledge alone. Likewise, acculturation to mathematics, in particular, recognising that mathematical inquiry is an integral part of mathematics, seemingly does not occur automatically from successful completion of upper level mathematics courses. Subsequently, inquiry and critique should be addressed directly, by reading studies such as Ma's and active participation in exploring mathematical conjectures whose content can be found in school mathematics.

The activity discussed in this paper will be extended in several ways.

- Future teachers responding to C2 and those responding to C3 will collaborate on discovering relationships between their findings. For example, since SSSS is not a congruence property for quadrilaterals, no area formula for a kite given solely in terms of the lengths of the four sides is possible.
- A second round of student conjectures will follow the first to gauge possible changes in attitudes toward mathematical inquiry.
- Future teachers enrolled in the course will collaboratively construct a scoring rubric for this activity.
- The critical responses from one future teacher to another will be the subject of analysis.
- Follow-up interviews and visits with current teachers who took part in the course will be conducted in order to measure the degree to which mathematical inquiry is a part of their classrooms.

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