Visualising and generalising with SQUARE ARRAYS

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Introduction

Almost 20 years ago, Cuoco, Goldenberg and Mark wrote a seminal paper for the *Journal of Mathematical Behavior* entitled "Habits of Mind: An Organizing Principle for Mathematics Curricula" (Cuoco et al., 1996). The article remains as relevant today as when it was originally published. The premise of their paper is that mathematical habits of mind such as searching for patterns, creating, experimenting, describing, visualising and conjecturing are far more important considerations than specific mathematical content, and that providing students with meaningful opportunities to engage in such habits of mind is a powerful means of exposing them to genuine experiences of mathematical activity.

One explorative context that has the potential to provide just such an opportunity for engagement is that of figurate or polygonal numbers such as triangular, square, pentagonal and hexagonal numbers. Although figurate numbers are a well known means of exploring aspects of generalisation and elementary number theory in a visually engaging manner, this article restricts itself to square arrays and draws on Cuoco et al.'s (1996) mathematical "habits of mind" to foster an authentic classroom experience of mathematical exploration while at the same time providing students with an opportunity to investigate and articulate expressions of generality in a pictorial context. As Lannin (2005) reminds us, statements of generality, along with the discovery and investigation of generality, "are at the very core of mathematical activity" (p. 233).

The problem statement

An $n \times n$ square array contains n^2 individual units (Figure 1). Given such an array, subdivide it into imagined 'component parts'. Try to do this in as many ways as possible. For each of these subdivisions explore the number of units in each of the component parts in relation to the side length of the original square array (i.e., n). Finally, determine generalised expressions (in terms of n) for the number of units in each of the component

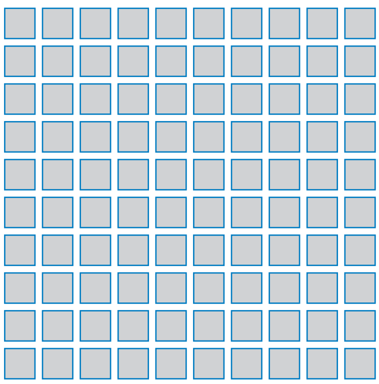

Figure 1. An $n \times n$ square array.

parts and show that the sum of the number of units in the different component parts simplifies to n^2 for each different subdivision.

Example 1

One way to subdivide the square array is in terms of consecutive odd numbers (Figure 2). This visual representation is a well known 'proof without words' that the sum of the first n consecutive odd numbers equals n^2. Expressed algebraically:

$$\text{sum} = \sum_{i=1}^{n}(2i-1) = 1+3+5+7+\ldots+(2n-1)$$

Using the standard formula for the sum of an arithmetic sequence we have:

$$\text{sum} = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}\left[2(1)+(n-1)(2)\right] = \frac{n}{2}\left[2+2n-2\right] = n^2$$

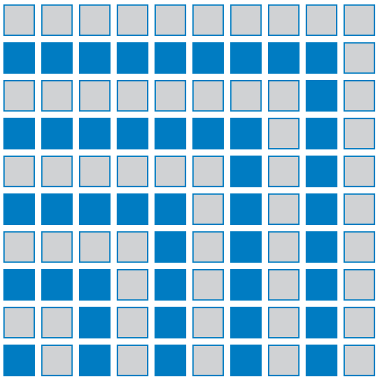

Figure 2. Splitting the n × n array into consecutive odd numbers.

Example 2

Another way to subdivide the square array is in terms of two abutting triangles (Figure 3).

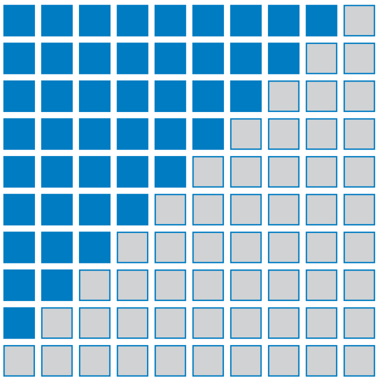

Figure 3. Splitting the n × n array into two triangles.

Careful inspection of the two triangles should reveal that for an $n \times n$ square array, the bigger of the two triangles comprises the sum of the first n natural numbers while the smaller triangle comprises the sum of the first $(n-1)$ natural numbers. Expressing this algebraically, and once again using the standard formula for the sum of an arithmetic sequence, we have:

$$\text{sum} = (1+2+3+\ldots+n)+(1+2+3+4+\ldots+(n-1)) = \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i$$

$$\begin{aligned}\therefore \text{sum} &= \frac{n}{2}\left[2(1)+(n-1)(1)\right]+\frac{(n-1)}{2}\left[2(1)+\left((n-1)-1\right)(1)\right]\\ &= \frac{n(n+1)}{2}+\frac{(n-1)(n)}{2}\\ &= n^2\end{aligned}$$

From the above we can make note of a useful formula for the sum of the first k natural numbers, i.e. the so-called triangular numbers:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

Example 3

Instead of splitting the square array into two triangles, rotating the array through 45° reveals an interesting pattern. Once rotated, the square array can be thought of as being composed of either a sequence of columns or rows. Counting the number of units in each of the rows or columns reveals a palindromic structure. For a 10 by 10 array as shown in Figure 4 the palindromic sequence is:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1$$

Once the generality of this observation is understood, the palindromic sequence for an $n \times n$ square array can be expressed as:

$$1 + 2 + 3 + 4 + \ldots + (n-1) + n + (n-1) + \ldots + 4 + 3 + 2 + 1$$

Splitting this series into two parts gives:

$$\text{sum} = [1 + 2 + 3 + 4 + \ldots + (n-1) + n] + [(n-1) + \ldots + 4 + 3 + 2 + 1]$$

Algebraically this is identical to the scenario in Example 2:

$$\therefore \text{sum} = \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i = \frac{n(n+1)}{2} + \frac{(n-1)(n)}{2} = n^2$$

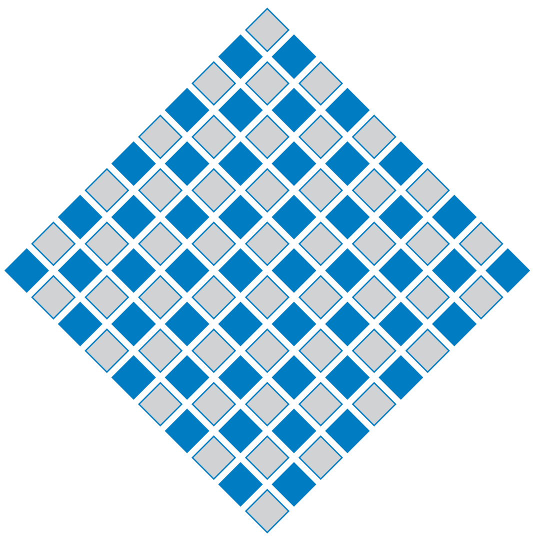

Figure 4. Splitting the n × n array into a palindromic sequence of columns or rows.

Example 4

Having previously established a formula for the nth triangular number it seems reasonable to use this to our advantage by looking at possible triangular sub-divisions. One possibility is shown in Figure 5.

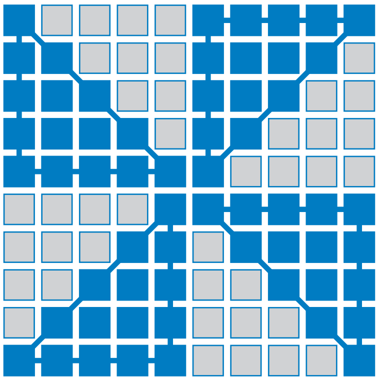

Figure 5. Splitting the n × n array into triangles.

Figure 5 shows a 10 by 10 square array subdivided into eight triangular component parts. Four of these triangular components can be represented by the fifth triangular number while the other four can be represented by the fourth triangular number. Careful exploration of the generality of this particular subdivision reveals that for an $n \times n$ square array the larger four triangular components can be represented by the $\left(\frac{n}{2}\right)$th triangular number while the smaller four triangular components can be represented by the $\left(\frac{n}{2}-1\right)$th triangular number. Using our formula for triangular numbers with k equal to $\left(\frac{n}{2}\right)$ and $\left(\frac{n}{2}-1\right)$ gives:

$$\text{sum} = 4 \times \left(\frac{n}{2}\right)\text{th triangular number} + 4 \times \left(\frac{n}{2}-1\right)\text{th triangular number}$$

$$\begin{aligned} \therefore \text{sum} &= 4 \times \left[\frac{\frac{n}{2}\left(\frac{n}{2}+1\right)}{2}\right] + 4 \times \left[\frac{\left(\frac{n}{2}-1\right)\left(\left(\frac{n}{2}-1\right)+1\right)}{2}\right] \\ &= 2\left(\frac{n^2}{4}+\frac{n}{2}\right) + 2\left(\frac{n^2}{4}-\frac{n}{2}\right) \\ &= n^2 \end{aligned}$$

One slight catch is that this particular subdivision only works for square arrays with even side lengths. In the case of odd side lengths a slight modification is necessary (Figure 6).

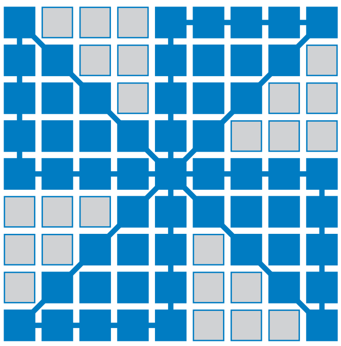

Figure 6. Splitting an odd n × n array into triangles.

In this case the four larger triangles are each representative of the $\left(\frac{n+1}{2}\right)$th triangular number while the four smaller triangles are representative of the $\left(\left(\frac{n+1}{2}\right)-2\right)$th triangular number. Since the four larger triangles overlap in the centre of the square array, a correction needs to be made to the tally which would otherwise give an overcount of three units. We thus have:

$$\text{sum} = 4 \times \left(\frac{n+1}{2}\right)\text{th } \Delta \text{ number} + 4 \times \left(\left(\frac{n+1}{2}\right) - 2\right)\text{th } \Delta \text{number} - 3$$

$$\begin{aligned}\therefore \text{sum} &= 4 \times \left[\frac{\left(\frac{n+1}{2}\right)\left(\frac{n+1}{2}+1\right)}{2}\right] + 4 \times \left[\frac{\left(\frac{n+1}{2}-2\right)\left(\left(\frac{n+1}{2}-2\right)+1\right)}{2}\right] - 3 \\ &= \frac{n^2+4n+3}{2} + \frac{n^2-4n+3}{2} - 3 \\ &= n^2\end{aligned}$$

An alternative modification in the case of odd side lengths is shown in Figure 7. It is left for the reader to determine and justify a generalised expression for this particular modified subdivision.

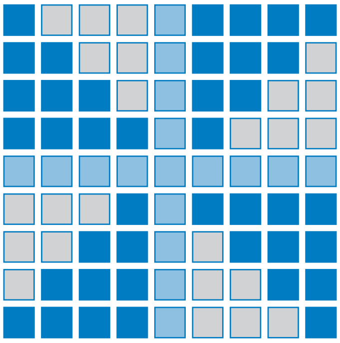

Figure 7. A modified splitting of an odd n × n array.

Example 5

One can also subdivide square arrays into concentric 'borders' as shown in Figure 8 and Figure 9. The scenario for arrays with even side lengths is slightly different for arrays with odd side lengths.

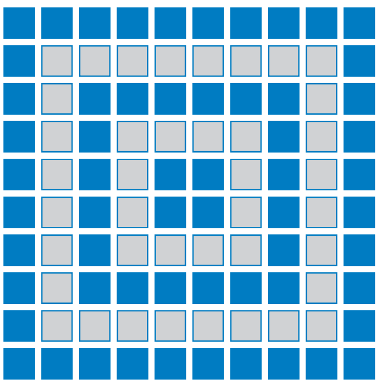

Figure 8. Concentric border subdivision for an even n × n array.

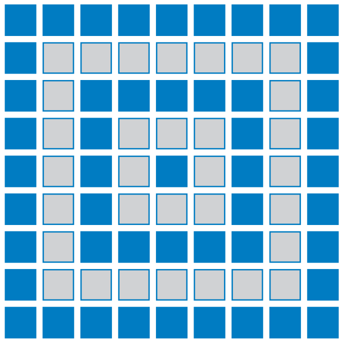

Figure 9. Concentric border subdivision for an odd n × n array.

In the case of the even array (Figure 8) the concentric borders generate the number sequence 4; 12; 20; 28; 36… for which we can express the kth term as $T_k = 8k - 4$ with $k \geq 1$. For an n-sided square array (where n is even) there will be $\left(\frac{n}{2}\right)$ concentric borders. The total number of units in an even-sided $n \times n$ square array can thus be expressed as:

$$\text{sum} = \sum_{k=1}^{\frac{n}{2}} (8k - 4)$$

Using the standard formula for the sum of an arithmetic sequence we have:

$$\begin{aligned} \text{sum} &= \frac{\left(\frac{n}{2}\right)}{2}\left[2(4) + \left(\frac{n}{2} - 1\right)(8)\right] \\ &= \frac{n}{4}\left[8 + 4n - 8\right] \\ &= n^2 \end{aligned}$$

In the case of the odd array (Figure 9) the concentric borders generate the number sequence 1; 8; 16; 24; 32… Let us ignore the 1, since it does not seem to conform to the pattern, and simply focus on the sequence 8; 16; 24; 32… for which the kth term is simply $8k$ with $k \geq 1$. For an n-sided square array (where n is odd) there will be $\frac{n-1}{2}$ concentric borders around a single central unit. The total number of units can thus be expressed as:

$$\text{sum} = 1 + \sum_{k=1}^{\frac{n-1}{2}} 8k$$

Using the standard formula for the sum of an arithmetic sequence we have:

$$\begin{aligned} \text{sum} &= 1 + \frac{\left(\frac{n-1}{2}\right)}{2}\left[2(8) + \left(\frac{n-1}{2} - 1\right)(8)\right] \\ &= 1 + \left(\frac{n-1}{4}\right)\left[16 + 4n - 12\right] \\ &= n^2 \end{aligned}$$

Example 6

In this example, let us split the square array into four triangular regions. Once again the scenario for arrays with even side lengths is slightly different for arrays with odd side lengths. In the case of even side lengths, each of the four triangular regions is what we can refer to as a 'double triangular number'. Since the kth triangular number is given by $\frac{k(k+1)}{2}$, the kth double triangular number, i.e. the kth number of the sequence 2; 6; 12; 20; 30… is given by $k(k + 1)$.

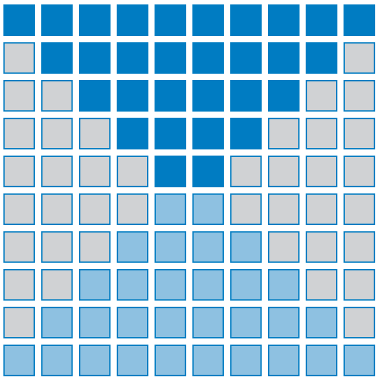

Figure 10. Four triangular subdivisions for an even n × n array.

Careful exploration of the generality of this particular subdivision reveals that for an even sided $n \times n$ square array the larger two triangular components can be represented by the $\left(\frac{n}{2}\right)$th double triangular number while the smaller two triangular components can be represented by the $\left(\frac{n}{2}-1\right)$th double triangular number. Using our formula for 'double triangular numbers' with k equal to $\left(\frac{n}{2}\right)$ and $\left(\frac{n}{2}-1\right)$ gives:

$$\text{sum} = 2\times\left(\frac{n}{2}\right)\text{th double } \Delta \text{ number} + 2\times\left(\frac{n}{2}-1\right)\text{th double } \Delta \text{ number}$$

$$\begin{aligned}\therefore \text{sum} &= 2\times\left[\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)\right]+2\times\left[\left(\frac{n}{2}-1\right)\left(\left(\frac{n}{2}-1\right)+1\right)\right]\\ &= 2\left(\frac{n^2}{4}+\frac{n}{2}\right)+2\left(\frac{n^2}{4}-\frac{n}{2}\right)\\ &= n^2\end{aligned}$$

In the case of odd side lengths, each of the triangular regions is composed of a sum of consecutive odd numbers and can thus be represented by a square number. In Figure 11, for example, the triangular regions on the left and right each contain 16 units while the triangular regions at the top and bottom (which overlap at the centre) each contain 25 units.

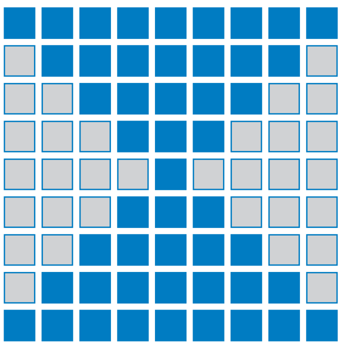

Figure 11. Four triangular subdivisions for an odd n × n array.

In general, for the case of odd side lengths, the two larger triangles are each representative of the $\left(\frac{n+1}{2}\right)$th square number while the two smaller triangles are representative of the $\left(\left(\frac{n+1}{2}\right)-1\right)$th square number. Since the two larger triangles overlap in the centre of the square array, a correction needs to be made to the tally which would otherwise give an overcount of one unit. We thus have:

$$\text{sum} = 2\times\left(\frac{n+1}{2}\right)\text{th square number} + 2\times\left(\left(\frac{n+1}{2}\right)-1\right)\text{th square number} - 1$$

$$\begin{aligned}\therefore \text{sum} &= 2\times\left(\frac{n+1}{2}\right)^2 + 2\times\left(\left(\frac{n+1}{2}\right)-1\right)^2 - 1\\ &= \frac{n^2+2n+1}{2}+\frac{n^2-2n+1}{2}-1\\ &= n^2\end{aligned}$$

Example 7

As a final example let us consider the square array as a spiral composed of a sequence of straight sections (Figure 12). Starting in the bottom left-hand corner and moving in a counter-clockwise direction we generate, for a 10 by 10 square array, the following sequence of section lengths: 10; 9; 9; 8; 8; 7; 7; 6; 6; 5; 5; 4; 4; 3; 3; 2; 2; 1; 1. This sequence can be interpreted as two interlinked arithmetic sequences:

10; **9**; 9; **8**; 8; **7**; 7; **6**; 6; **5**; 5; **4**; 4; **3**; 3; **2**; 2; **1**; 1

In general, for an $n \times n$ square array, the total number of units can be determined by summing the first n natural numbers and the first $n - 1$ natural numbers:

$$\text{sum} = \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i = \frac{n(n+1)}{2} + \frac{(n-1)(n)}{2} = n^2$$

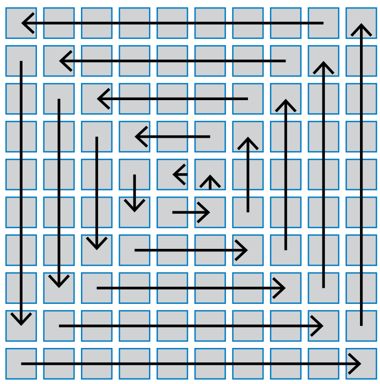

Figure 12. An n × n array viewed as a spiral.

Concluding comments

It was the purpose of this article to provide an example of a classroom context in which students could experience mathematical exploration through investigating and articulating expressions of generality in a pictorial context. While some examples are shown of how a square array could be subdivided into different component parts, there are no doubt many other potential subdivisions.

There are three important aspects that one needs to consider with such figural pattern generalisation activities. The first relates to each student's ability to discern structural 'features' in the visual stimulus. The second relates to developing an awareness of the regularity or generality of these structural features, while the third relates to the symbolic articulation of this generality. Each of these three aspects will have different challenges for different students, and the amount of scaffolding that a teacher would need to provide with such an activity would of course be dependent on each individual classroom situation. Using Cuoco et al.'s (1996) mathematical 'habits of mind' (such as searching for patterns, experimenting, describing, visualizing and conjecturing) as a framework for this scaffolding process has the potential to foster an authentic classroom experience of mathematical exploration and activity.

In addition to aspects of mathematical exploration, this activity also results in the generation of multiple algebraic expressions whose equivalence can be assessed by a process of expansion and simplification. Not only does this create an opportunity to discuss the notion of algebraic equivalence, but students will be able to engage in algebraic expansion and simplification in a context that arises naturally from the exploration process rather than having to resort to a series of decontextualised exercises designed for 'drill and practice'. Furthermore, by justifying the validity of their different algebraic expressions to other students in the class, students will take ownership of the different expressions of generality that they produce.

Quite apart from any pedagogic or epistemological concerns, I would also argue that there is a strong moral or ethical dimension to acknowledging the importance of multiple representations. This ethical dimension relates to the idea that different pupils have different learning styles, different ways of engaging with or making sense of mathematical situations, and different ways of 'seeing' the world. I believe that teachers have a moral obligation not only to value and embrace these different 'ways of knowing', but to provide appropriate classroom environments that support and actively encourage a multi-representational view of mathematics.

References

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Lannin, J. K. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning, 7*(3), 231–258.