Quantitative reasoning in problem solving

 

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In this article, Ajay Ramful and Siew Yin Ho explain the meaning of quantitative reasoning, describing how it is used in the to solve mathematical problems. They also describe a diagrammatic approach to represent relationships among quantities and provide examples of problems and their solutions.

We open this article with the following snapshot problem and an excerpt from a Grade 6 student's interview.

> **Paper Dolls Problem**
> Lili spent 4 days making paper dolls for her friends. Each day she managed to make 2 paper dolls more than the day before. She made a total of 24 paper dolls. How many paper dolls did she make on the last day? (modified from SEAB, ©2013)

Initially, the Grade 6 student subtracted 8 (2 paper dolls x 4 days) from 24. Then she crossed out the calculation and subtracted 6 from 24.

I: And the 6 you obtained by?
S: Three days of adding two more each day.
I: Three days, adding two. And then you got 18.
S: Yes.
I: So, does the 18 represent anything?
S: How many dollars she would make if she did not make 2 more (paper dolls) than the day before.
I: You were saying you want to divide. Right?
S: Umm (nodding).
I: Are you still thinking about what you want to divide?
S: Yes (hesitating); That does not work.
I: What does not work?
S: 18 divided by 4.
(I: Interviewer; S: student)

The lack of divisibility led her to abort this path and started to use a guess-and-check procedure. Ten minutes had already elapsed and we decided to end the interview as she mentioned "I am quite confused now."

What was the missing element in the student's solution? What elements of the problem made it challenging for the student? What form of reasoning does such a task require?

In a previous study (Ramful & Ho, 2014), we explained how the same Grade 6 student was constrained in reasoning with quantitative relations and consequently had to resort to numerical reasoning. In this complementary article, we focus on helping students represent and reason with quantitative relationships.

Background

The solution to a mathematical problem requires not only understanding the statement of the problem, making a plan or applying particular operations but it also necessitates establishing relationship(s) among the quantities. Identifying the quantities in a problem and setting the relationship(s) among them is a crucial aspect of problem solving. This is often facilitated when one can reason quantitatively (Thompson, 1993), i.e., make sense of the relationship(s) among quantities rather than working with particular values of the quantities. In this article, we explain the meaning of quantitative reasoning and explicate how it is used in the solution of mathematical problems. In particular, we focus on problems involving additive differences with unknown starting quantities. In problems involving additive differences, the difference between two quantities is specified rather than the individual values of the two quantities. For example, in the following Baker's problem, the additive difference between the number of rolls made on Saturday and Sunday is given, that is "15 more rolls", rather than the number of rolls made on Saturday and on Sunday.

Baker's Problem

A baker made a total of 175 rolls on the weekend. She made 15 more rolls on Saturday than on Sunday. How many rolls were made on Sunday?
(ACARA, ©2009)

Thompson (1995) uses the concept of "quantitative reasoning" to explain how working with quantities is operationally different from working with numbers: "quantitative reasoning" is not about numbers, it is reasoning about objects and their measurements (i.e., quantities) and relationships among quantities" (p.8). Quantitative reasoning involves analysing the quantities and relationships among quantities in a situation, creating new quantities, and making inferences with quantities.

How can we help students develop quantitative reasoning skills?

One way to help students develop quantitative reasoning skills is to represent quantities diagrammatically so that relationships among them can be made more explicit. In Singapore, such a diagrammatic approach (model method) has been formalised and is a commonly taught problem-solving procedure in classrooms where relationships among quantities are represented in terms of boxes (Kho, Yeo & Lim, 2009). We use the Baker's problem, stated above, as a starting example to illustrate the model method. This problem involves two quantitative relationships:

1. The total number of rolls made on Saturday and Sunday is 175 .
2. 15 more rolls were made on Saturday than on Sunday.

As mentioned above, the model method uses rectangular bars to represent quantities and relationships in the problem. Specifically, the rectangular bar represents an unknown quantity whose values will be determined by the end of the problem-solving process. Figure 1 shows the model for the second relationship, that is, 15 more rolls were made on Saturday than on Sunday.

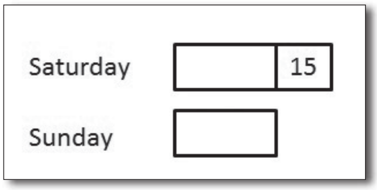

Figure 1. Model showing second relationship.

By incorporating the first relationship (i.e., the total number of rolls made on Saturday and Sunday is 175 rolls.) into Figure 1, we obtain the following representation (see Figure 2).

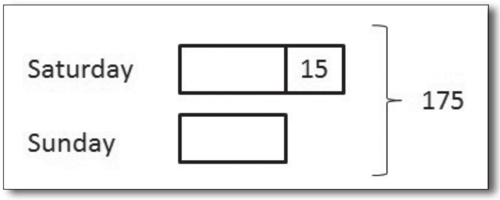

Figure 2. Model incorporating the first relationship.

Thus, the model attempts to make the relationship between the two quantities explicit by using a concrete embodiment in the form of bars.
Now, we can reason as follows:
2 $\square$ = 175 – 15 =160 (removing the additive difference, 15, so that the number of rolls made on Saturday and on Sunday are the same). Therefore, $\square$ = 160 ÷ 2 = 80. Thus, 80 rolls were made on Sunday.

Let us consider a second example (Machine problem) to illustrate how the model method makes the quantitative relationships inherent in a problem explicit by using external representations.

Machine Problem

Every minute Machine A prints 12 pages more than Machine B. Machine A and Machine B together print a total of 528 pages in 3 minutes. At this rate, how many pages does Machine B print in 1 minute? (SEAB, ©2010)

In this problem, the exact number of pages that each machine prints is not known. It involves two quantitative relationships:

1. Machine A prints 12 pages more than Machine B (every minute).
2. Machine A and Machine B together print a total of 528 pages (in 3 minutes).

The first relationship is represented as follows:

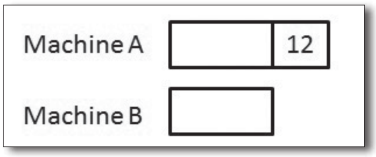

Figure 3. First relationship in Machine problem.

By incorporating the second relationship into Figure 3, we obtain the following model (see Figure 4) which serves as a trigger for making the next set of inferences. In 1 minute machine A and B together print 528 ÷ 3 = 176 pages.

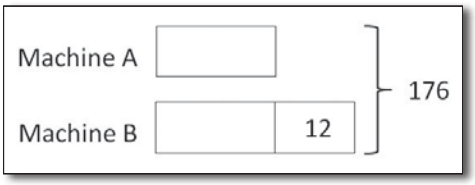

Figure 4. Model incorporating second relationship

2 $\square$ = 176 – 12 = 164 pages (situation where each machine prints an equal number of pages)
Therefore, 1 $\square$ = 164 ÷ 2 = 82 pages. That is, Machine B prints 82 pages in 1 minute.

In summary, the representation of the quantities in terms of bars aim at diminishing the cognitive load associated with the multiple pieces of information.

We now consider a task parallel to the Paper dolls problem presented at the beginning of the article. In this task (Siti's problem), the value of the starting quantity is unknown and there is a constant difference between successive amounts.

Siti's Problem

Siti started saving some money on Monday. On each day from Tuesday to Friday, she saved 20 cents more than the amount saved the day before. She saved a total of $6 from Monday to Friday. How much money did she save on Monday? (SEAB, ©2010

Here also, there are two quantitative relationships:

1. On each day from Tuesday to Friday, she saved 20 cents more than the amount saved the day before.
2. A total of $6 was saved on the 5 days (Monday to Friday).

Let $\square$ be the amount of money Siti saved on Monday. Then the model (see Figure 5) representing the quantitative relationship between the money saved on Monday and Tuesday is:

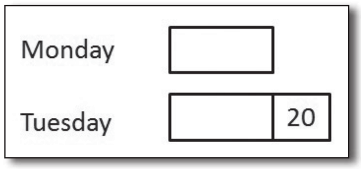

Figure 5. Model showing quantitative relationship between Day 1 and Day 2.

The same reasoning can be extended to form a picture of the situation from Monday to Friday (Figure 6).

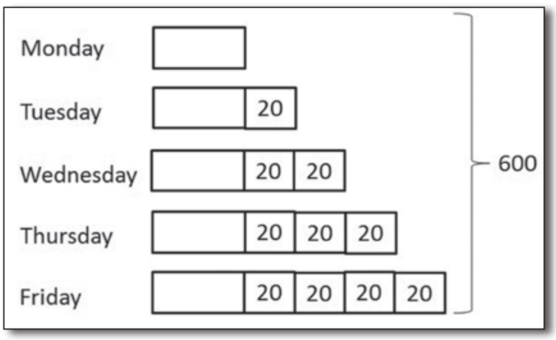

Figure 6. Model for the complete problem.

The model attempts to make the problem situation easier to interpret and can be used to make the following chain of inferences

10 × 20 cents = 200 cents
600 – 200 = 400
400 ÷ 5 = 80 cents
Thus, Siti saved 80 cents on Monday.

How do students cope with the above problems in the absence of explicit quantitative reasoning skills?

We provide another illustration of the approach used by the Grade 6 student to solve Siti's problem. In this task, she resorted to numerical reasoning by substituting particular values through systematic guess-and-check until the relationships in the problem were satisfied.

She started with zero cents on Monday (see Figure 7(a)) and successively increased the amount by 20 cents from Tuesday to Friday. She then added the amounts (in the first row of the table) to see if it summed to $6. As this was not the case, she increased the amount on Monday to 20 cents and performed a similar calculation to obtain 100 cents on Friday. Again, she observed that the amounts did not sum to $6. She then increased the 100 cents on Monday to 120 cents and worked backwards, i.e., by assigning 100, 80, 60, 40 to Thursday back to Monday. Once again she observed that the quantities for the 5 days did not sum to $6. She finally started with 160 cents for Friday and rewrote the solution in terms of dollars and cents (see Figure 7(b)).

In the foregoing scenario, numerical reasoning took over as a fallback strategy in the absence of a known method to approach the problem quantitatively.

Conclusion

We encourage teachers to experiment with this procedure (model method) and measure its potential as a pedagogical tool to articulate quantitative relations in problem solving situations involving unknown quantities. In one way, the model method can be regarded as a pre-algebraic tool where students are not yet introduced to the representation of unknown quantities in terms of abstract algebraic symbols. Moreover, the model method also provides students with opportunities to use heuristics such as "Draw a diagram/model". The model method can be used across the primary mathematics curriculum. For example, at the lower primary level, we may ask students to represent the following problem using rectangular bars so that they see the relationship among the quantities.

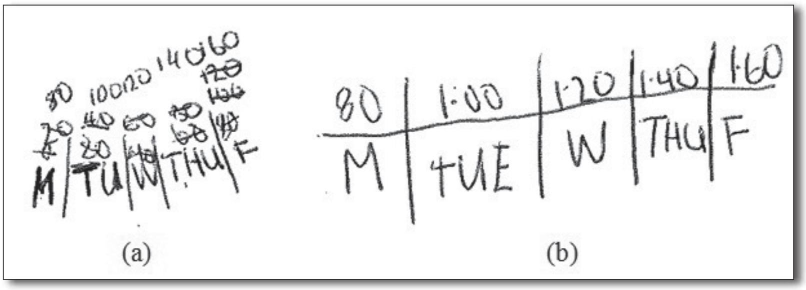

Figure 7. Solution to Siti's problem.

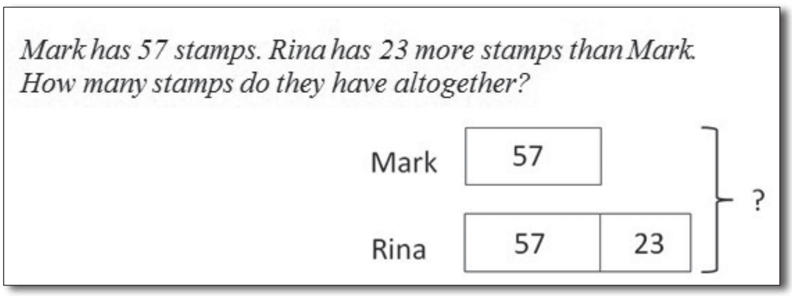

Figure 8. Use of model method at lower primary level.

Instruction is necessary for students to develop quantitative reasoning skills. Students need to be taught how to coordinate the quantities and quantitative relations in terms of diagrammatic representations. Students should be encouraged to explain their reasoning and pose questions about quantities and relationships about quantities.

Such instructional conversations provide opportunities for students to develop a disposition to think about "what one does with quantities values in specific situations and with patterns of known information" (Thompson, 2011, pp. 42–43). Although difficulties to use the model method have been reported by some researchers (e.g., Yan, 2002), we find if it a laudable approach if judiciously used.

We asked the reader to reflect on the Paper dolls problem at the beginning of the article. Throughout the paper, we illustrated the model method. We now invite the reader to use the model method to solve the following Singapore Grade 6 problem:

Postcard Problem

Gilbert and Hazel have some postcards. After Gilbert gives 18 postcards to Hazel, he has 20 postcards more than her. How many more postcards than Hazel does Gilbert have at first? (SEAB, ©2013)

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