AUSTRALIA'S NEXT top fraction model

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Peter Gould suggests Australia's next top fraction model should be a linear model rather than an area model. He provides a convincing argument and gives examples of ways to introduce a linear model in primary classrooms.

Although children's first knowledge of fractions is often associated with sharing food, their formal introduction to fractions commonly involves shading in parts of shapes. Of the different ways of representing fractions, the region or area model is clearly the most popular of the models currently used in Australian schools. Yet the use of the area model brings with it a number of limitations in the fraction concepts students form. In this article I argue for focussing on the linear aspects of fraction models as the primary representation of fractions. With the introduction of the Australian Curriculum: Mathematics, the time has come to recognise the need for a new top fraction model.

What do we mean by a fraction?

Understanding the meaning of common fractions and how to operate with them is for many people a very difficult aspect of learning mathematics (Davis, Hunting & Pearn, 1993; Pearn & Stephens, 2004). A substantial component of the difficulty students encounter when studying fractions is due to the symbol system employed to represent fractions (Ellerton & Clements, 1994; Mack, 1995). The symbols used to represent fractions, one whole number written above another whole number, do not transparently communicate the meaning of fractions; that is, students cannot 'see through' the symbols to the underlying meaning of a fraction.

Over the decades, far more teaching time has been dedicated to mastering the procedures necessary to manipulating the fraction symbol system than to seeking to understand what fractions are (Thompson & Saldanha, 2003). This intensive focus on the symbol system has had some unexpected outcomes. When a Year 3 student was asked, "Can you see how part of this shape (Figure 1) is shaded in? Do you know what we call that part that is shaded in?" he responded in terms of the symbol system.

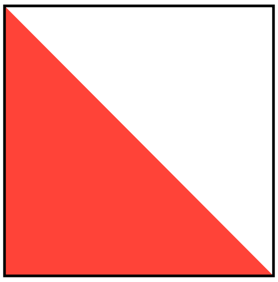

Figure 1. A square with one-half of the area shaded.

Student: Two out of one.
Teacher: It's two out of one?
Student: I mean two… two but, but one's out, so they write like two then, then they write a symbol, then a one.
Teacher: OK. What would we call this if this was a sandwich and I cut this down here (indicates the diagonal) and I gave you that part (indicates the shaded part)? How much would you have?
Student: One.

This student's response underscores the two different meanings we give to fractions in teaching: fractions in context (partitioned fractions) and fractions as abstract numbers (quantity fractions) (Gould, 2008; Isoda, Stephens, Ohara, & Miyakawa, 2007; Yoshida, 2004). A half of a sandwich and one-eighth of a pizza are examples of partitioned fractions, fractions in context. In contrast, when we ask, "Which is larger, one half or five-eighths?" we are referring to fractions as abstract numbers without specified units. Although $\frac{1}{2}$ of a family size pizza can be more than $\frac{5}{8}$ of a smaller pizza, the $\frac{1}{2}$ number is always smaller than the number $\frac{5}{8}$.

Introducing the models

Common fractions are frequently introduced to students in Australia through contexts such as sharing food; that is, partitioned fractions or fractions in context are introduced first. Shading parts of common shapes then follows discussions of what constitutes 'half an apple or a quarter of a sandwich'. Shapes such as circles or squares are used to model the relationship between the parts and the whole.

The move from introducing fraction language in contexts using apples or sandwiches to shading two-dimensional shapes often takes place rapidly. However, when we name half an apple or a quarter of a sandwich we often do so without explicit attention to the feature of the object we use as the basis of our judgement. How do we know that half an apple is indeed half an apple? It is not the number of pieces but rather the mass or volume of the pieces that informs our decision that we do have one-half of an apple. This can make it quite difficult for students to follow what we mean when we introduce fraction models. For many students, the perceived feature initially identifying half an apple or half a strawberry (Figure 2) is the number of pieces. Consequently, it is not unusual to hear young students refer to the 'bigger half'.

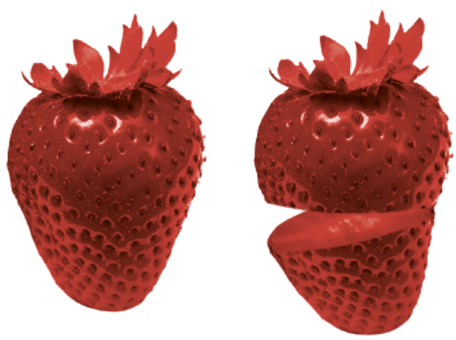

Figure 2. A strawberry cut at halfway producing two pieces with different volumes.

In teaching, *models* are often used to represent mathematical ideas. A fraction model refers to the instructional materials used to represent the mathematical idea of fractions. There are three common fraction models typical of school textbooks: the linear model, the area model and the discrete or set model (Watanabe, 2002).

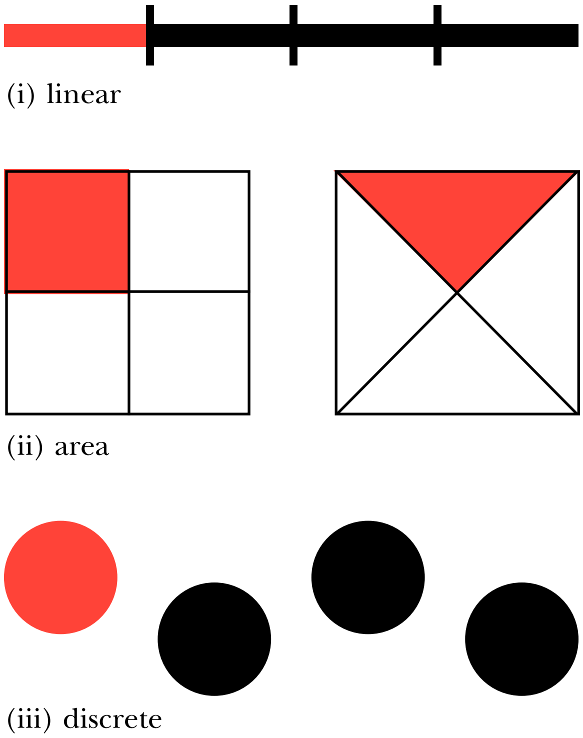

Figure 3. Three common fraction models.

Of these three models (Figure 3), Australia's current most popular fraction model is the area model. Textbooks abound in examples of the area model. However, we should not confuse the way students interpret a model with the fraction models themselves. A student may interpret an area model as if the components were discrete. The way that a student chooses to interpret a given fraction model is not determined by the model.

Interpreting models

Is the popularity of the area model justified? The area model readily encourages the use of fractional language. In Figure 4 we can describe the area model as showing three-eighths shaded, corresponding to three parts out of eight equal parts.

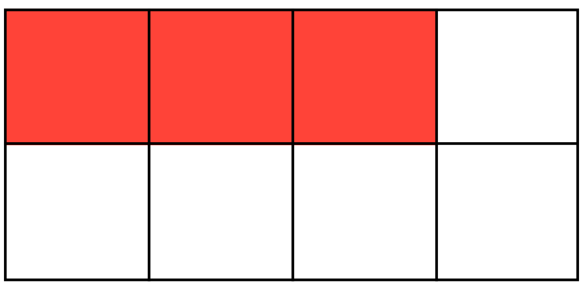

Figure 4. An area model for three-eighths.

However, a student may correctly describe the shaded area in Figure 4 as representing three-eighths without referring to area. For example, a student may reason that three parts (or squares) are shaded out of a total of eight parts. This is a comparison of two counts that does not explicitly make any use of area.

Using a fraction model needs to do more than elicit the language of fractions; it must also strengthen understanding of the *relationship* between the parts and the whole. Over 30 years ago, Kieren expressed concern over the ineffective use of fraction models in teaching.

> Because part-whole models of fractions conveniently help produce fractional language, the school mathematics fraction language of teacher and texts alike tend to orient a student to a static double count image and knowledge of fractions. The child, while being able to produce "correct" answers to questions, develops a mental model which is inappropriately inclusive (parts of a whole), rather than a powerful measure of inclusion (comparison to a unit)…
>
> (Kieren, 1988, p. 177)

The defining feature of a fraction area model is comparison of areas. However, using pre-partitioned shapes in models of fractions removes the necessity for students to engage with area, and makes it difficult to know if the student is relying on area or simple object counts as the defining feature of the model.

Indeed, when the fraction notation is linked to pre-partitioned shapes, student responses are more likely to reflect a double count of discrete parts than a comparison of areas. This is because students are usually taught to count the total number of parts, the number of parts shaded and then to place one count over the other, $\frac{a}{b}$. This has led some students to believe that fractions only require knowledge of counting (Figure 5).

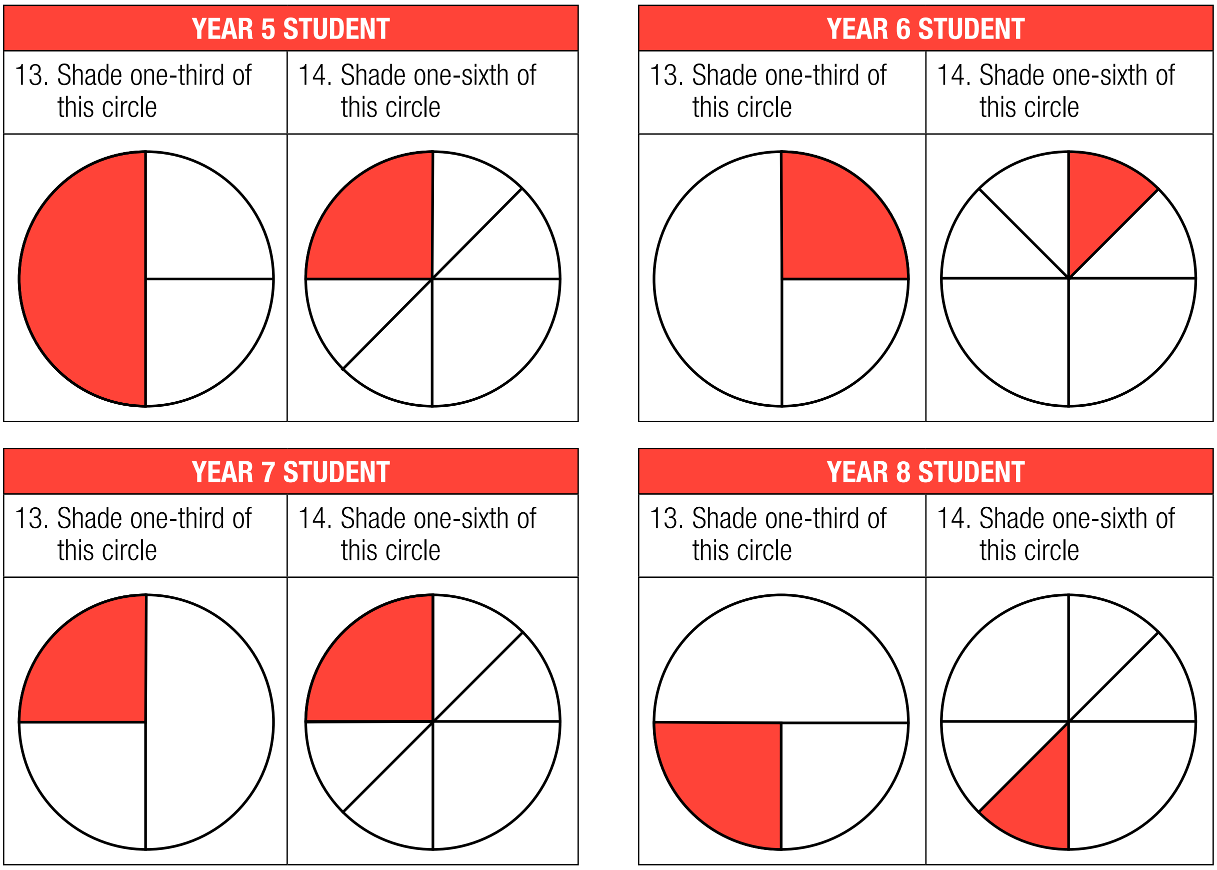

Figure 5. Discrete interpretations of area models.

A 'number of pieces' interpretation elicits three common variations as responses to the questions shown in Figure 5 (Gould, 2008). The first 'number of pieces' interpretation creates thirds and sixths as segments of a circle, using *parallel partitioning*. Parallel partitioning is the description I apply to the use of equidistant partitioning of the length of a diameter as in Figure 6.

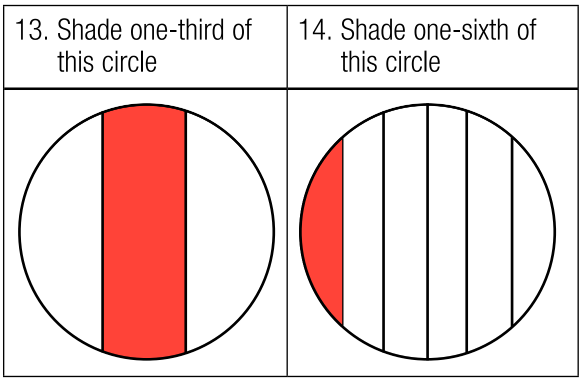

Figure 6. Equidistant partitioning of length to produce parallel partitioning (Year 6 student).

The second interpretation is to construct a number of pieces corresponding to the denominator and then, for unit fractions, to shade in one of the pieces as in Figure 5. A third interpretation is to shade in a number of pieces corresponding to the denominator (Figure 7).

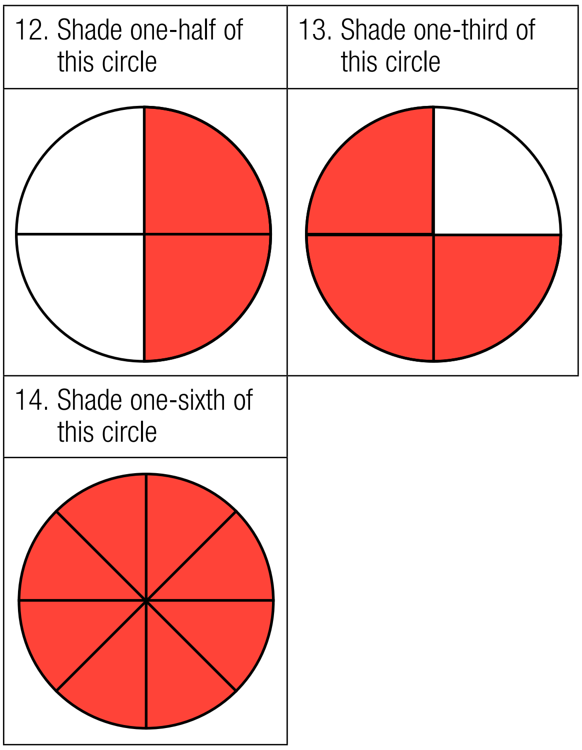

Figure 7. Shading the number of pieces indicated by the denominator (Year 5 student)

Although students are introduced to fractions using area models, many students do not use the feature of area when interpreting the model. In Figure 8, a Year 6 student's response explains why one-quarter is bigger than one-third using an area model interpreted as a number of parts.

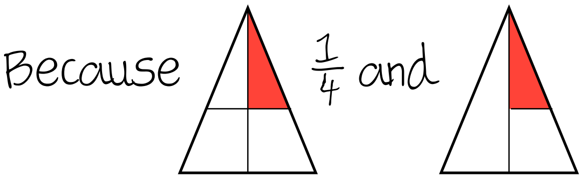

Figure 8. Explaining why one-quarter is bigger than one-third with shaded parts.

As this student is the same Year 6 student whose representation of one-third of a circle appears in Figure 5, it is clear that for this student shading parts of shapes need not be indicative of a comparison of areas.

Students cannot use area to compare two quantities (the area of a part to the whole area) before they have developed an understanding of how to compare areas. More specifically, using the area model of fractions requires students to:

- know what area is;
- identify the area of the part;
- identify the area of the whole; and
- compare the two areas by direct or indirect measurement.

Two areas are compared directly by placing one figure over another so that all like parts coincide. This method of directly comparing areas is sometimes called using superposition (Schwartzman, 1994, p. 213). Two areas can be compared indirectly when the shapes cannot be moved and an informal unit of covering is used to compare the two areas. For example, cardboard tiles could be used to indirectly compare the areas of two shapes.

Fractions in the Australian Curriculum: Mathematics

Students cannot use a fraction model without an understanding of the essential property used by the model. Using a linear model relies upon comparing units of length, using an area model relies upon comparing units of area and using a discrete model of fractions relies upon comparing abstract units composed of other units (abstract composite units). Students typically develop a capacity to work with units of length, area and abstract composite units at different times. For example, the *Australian Curriculum: Mathematics* locates direct and indirect comparison of length two years before a comparable understanding of area. Consequently, when following the *Australian Curriculum: Mathematics* to teach fractions, linear models of fractions are introduced before area models of fractions.

Students learn to compare and order several shapes based on area using appropriate uniform, informal units in Year 2 in the *Australian Curriculum: Mathematics*. At the same time, students are expected to recognise and interpret common uses of halves, quarters and eighths of shapes and collections. How is this possible if area models of fractions cannot be introduced before students have a robust understanding of how to compare areas?

The answer is that students have learnt to measure and compare the lengths of pairs of objects using uniform, informal units in Year 1. Added to this, the Year 2 fraction content is based on using repeated halving to form halves, quarters and eighths. One common method students use to form halves is to locate halfway. We made use of this in a recent lesson study on introducing eighths as fractions in Year 2. For our model we chose a liquorice strap. Repeated halving in one direction to compare lengths relies only on the linear aspect of the model and a long liquorice strap (Figure 9) provided an opportunity to find halfway.

Figure 9. Halving a liquorice strap.

In the lesson, the students were asked to share a liquorice strap fairly between Chris and Elaine and to explain orally why what is formed is a fair share. Students used thin paper strips to model the liquorice strap. The problem then progressed to one of sharing the liquorice strap fairly among four people and ultimately halving to form eighths (Figure 10).

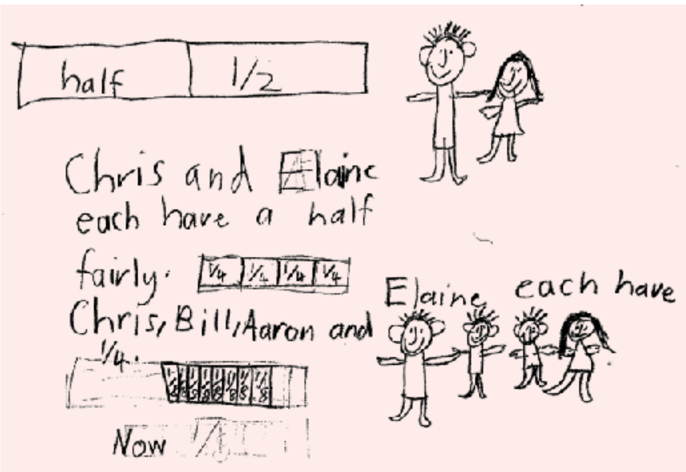

Figure 10. Repeated halving to create and record halves, quarters and eighths.

Folding has the advantage of forming equal halves by demonstrating the process of aligning and matching to create equal parts. In this lesson, we were particularly interested in how students would record the fraction pieces they had formed, as this was the first time they had encountered eighths. Understanding the meaning of the fraction notation is difficult and it has been argued that students should use words in the beginning rather than the symbol, to emphasise the fraction unit (Gunderson & Gunderson, 1957). Students' notations often provide insights into what they understand fractions to be (Brizuela, 2005) and we learnt by observing and questioning that some students believe that you can write 'eighths' as $\frac{8}{8}$.

Both the process of repeated halving and students' recordings of what was formed required careful monitoring. Some students initially used a process of partitioning that was akin to rolling up the paper strip and the teacher provided these students with opportunities to fold a line of cut out paper people. This activity supported students in attending to aligning the shapes and focusing on equal covering.

Figure 11. Folding a line of paper people.

Using linear models with discrete objects

Linear models of fractions are important for a number of reasons. Perhaps foremost among these reasons is the link between the linear model and the number line (Larson, 1980). Young students can begin to make this link by working with linear arrangements of quantity. In the next lesson, the students had an opportunity to link the linear aspect of repeatedly halving a liquorice strap to using a similar process with a line of penguins, played by the students.

Figure 12. Repeated halving of a line of students.

The students then used the process of repeated halving to find one-eighth of a strip of 16 penguins (Figure 13). Having the pictures on the strip of paper appeared to help the students to make the link between fraction units and quantities.

Figure 13. Repeated halving of a line of 16 penguins.

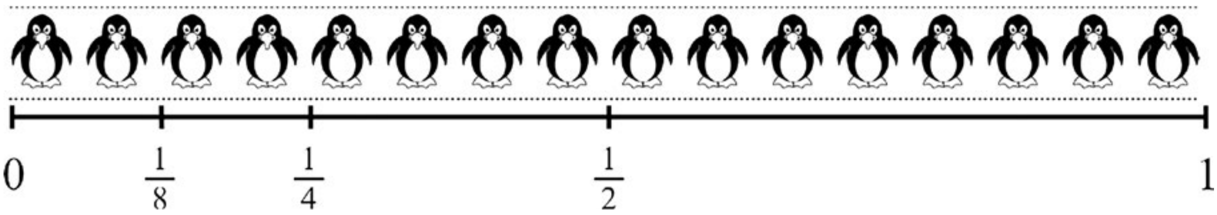

Figure 14. Linking repeated halving of 16 close identical penguins to the number line.

Just as students may interpret the components of an area model discretely, students can initially use paper strips as fraction models by attending only to a single feature such as length. Introducing side-by-side identical images to the paper strips helps them to link units of quantity to the process of repeated halving.

Any fraction model used in teaching needs to do more than elicit the language of fractions; it must also strengthen understanding of the *relationship* between the parts and the whole. Moving from a focus on the length of a liquorice strap to a 'line of penguins' helped the Year 2 students to see how repeated halving could be used with both continuous and discrete quantities. However, not all students came to this understanding at the same time and some needed extra opportunities to practise the process of linear halving.

The transition from a linear fraction model in context (represented by a liquorice strap) to partitioning collections of discrete elements can be achieved in a Year 2 class. That is, the *Australian Curriculum: Mathematics'* expectation of students recognising and interpreting common uses of halves, quarters and eighths of shapes and collections is achievable.

The transition from partitioned fractions (i.e., fractions as parts of things) to quantity fractions (i.e., fractions as numbers) is a necessary progression in developing an appreciation of fractions as mathematical objects. However, to make this transition, using fractions in context (partitioned fractions) also needs to link the process of partitioning continuous quantities to discrete objects arranged in lines. This process assists students in working with fractions portrayed as composite units. With care, partitioning close linear arrangements of identical discrete objects can be linked to subdividing measurement units on the number line (Figure 14).

With the introduction of the Australian Curriculum, the linear model appears set to replace the area model as Australia's next top fraction model. This may bring Australia into line with some top-performing mathematics countries. Unlike most Australian and US textbooks, in which area models dominate, linear models are the primary graphical representation of fractions in Japanese textbooks (Watanabe, 2007).

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References

Brizuela, B. M. (2006). Young children's notations for fractions. *Educational Studies in Mathematics, 62(3),* 281–305.

Davis, G., Hunting, R. P. & Pearn, C. (1993). What might a fraction mean to a child and how would a teacher know? *Journal of Mathematical Behavior, 12,* 63–76.

Ellerton, N. F. & Clements, M. A. (1994). *Fractions: A weeping sore in mathematics education, Set 2.* Camberwell, Vic: Australian Council for Educational Research.

Gould, P. (2008). *Children's quantitative sense of fractions.* Unpublished PhD Thesis, Macquarie University, Sydney.

Gunderson, A. G. & Gunderson, E. (1957). Fraction concepts held by young children. *Arithmetic Teacher, 4,* 168–174.

Isoda, M., Stephens, M., Ohara, Y. & Miyakawa, T. (Eds). (2007). *Japanese lesson study in mathematics: Its impact, diversity and potential for educational improvement.* Singapore: World Scientific Publishing.

Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development.

In J. Hiebert & M. J. Behr (Eds), *Number concepts and operations in the middle grades* (Vol. 2, pp. 162–181). Reston, VA: National Council of Teachers of Mathematics.

Larson, C. N. (1980). Locating proper fractions on number lines: Effect of length and equivalence. *School Science and Mathematics, 53*(5), 423–428.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*, 422–441.

Pearn, C. & Stephens, M. (2004). Why do you have to probe to discover what Year 8 students really think about fractions? In I. Putt, R. Faragher & M. McLean (Eds), *Mathematics education for the third millennium: Towards 2010. Proceedings of the 27th Annual Conference of the Mathematics Education Research Group of Australasia* (Vol. 2, pp. 430–437). Sydney: MERGA.

Schwartzman, S. (1994). *The words of mathematics: An etymological dictionary of mathematical terms used in English*. Washington, DC: The Mathematical Association of America.

Thompson, P. W. & Saldanha, L. A. (2003). Fractions and Multiplicative Reasoning. In J. Kilpatrick, G. Martin & D. Schifter (Eds), *A research companion to Principals and Standards for School Mathematics* (pp. 95–114). Reston, VA: National Council of Teachers of Mathematics.

Watanabe, T. (2002). Representations in teaching and learning fractions. *Teaching Children Mathematics, 8*, 457–463.

Watanabe, T. (2007). Initial treatment of fractions in Japanese textbooks. *Focus on Learning Problems in Mathematics, 29*(2), 41–60.

Yoshida, K. (2004). *Understanding how the concept of fractions develops: A Vygotskian perspective*. Paper presented at the 28th Conference of the International Group for the Psychology of Mathematics Education, Bergen, Norway.