Teaching 3-D Geometry

– the multi representational way

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Sonja Kalbitzer and Esther Loong provide an excellent range of activities that promote geometric thinking through the exploration of three-dimensional objects. They also provide some discussion on assessing the tasks and providing student feedback.

Many students have difficulties in geometric and spatial thinking (see Pittalis & Christou, 2010). These include the following:

- creating three-dimensional structures of unit-cubes;
- making and working with two-dimensional representations of three-dimensional objects, including plans and isometric diagrams;
- using and making two-dimensional nets of three-dimensional objects; and
- recognising and comparing mathematical properties of three-dimensional shapes.

Students who are asked to construct models of geometric thought not previously learnt may be forced into rote learning and only gain temporary or superficial success (Van de Walle & Folk, 2008, p. 431). Therefore it is imperative for instruction that promotes geometric thinking and spatial ability to provide a variety of activities that promote visual imagery, as well as use language that is appropriate to the level of the students. Research has shown that in the primary mathematics classroom, both computer and concrete manipulatives can be used interchangeably to learn two-dimensional geometry (Olkun, 2003). Manipulatives, whether in the form of physical concrete objects or computer manipulatives, are able to enhance learning only if they are used in a meaningful way (Clements & McMillen, 1996; Swan & Marshall, 2010). Open-ended geometry tasks have been shown to foster engagement and independent mathematical thinking with children as young as six years old (McKnight & Mulligan, 2010). Dienes (1960) emphasises the need for 'multiple embodiments' in mathematical concept development as being necessary to produce abstractive learning rather than associative learning.

In this article we describe a number of open-ended tasks that draw upon the use of multiple representations to develop the spatial ability and geometric thinking of students. Simple tools like multi-linked blocks, isometric dot paper and the use of the 'Insert Shapes' tool in *Microsoft Word* are used. These tasks have been adapted and revised from lessons prepared and implemented by the first author with her Year 5/6 mixed ability class. They are not meant to be prescriptive but merely represent several activities that we feel could engage students in the visualisation of three-dimensional objects, help develop geometric thinking, and present opportunities for students to carry out the non-trivial task of drawing three-dimensional objects in two-dimensional representations, both by hand and using the computer. The content description for the Space sub-strand in the Measurement and Geometry strand of the *Australian Curriculum: Mathematics* states that in Year 5 students "Connect three-dimensional objects with their nets and other two-dimensional representations" (Australian Curriculum Assessment and Reporting Authority, n.d.). These tasks aim to provide opportunities for students to:

- rearrange and fit shapes together in three dimensions;
- test spatial conjectures;
- analyse three-dimensional shapes and draw two-dimensional representations of the three-dimensional objects;
- develop visual imagery;
- develop problem-solving strategies;
- develop appropriate geometric language.

The history of the Soma Cube

The Soma Cube was invented by a Dane named Piet Hein (1905–1996) who was a gifted poet, writer, designer, inventor and mathematician. In 1936, he conceived the idea of the Soma Cube when he was sitting at a lecture on quantum physics as a university student. History has it that he pictured in his head the six three-dimensional shapes that could be made out of four adjoining cube pieces. By arranging the six different 'bent' four-cubed pieces and a three-cubed piece he was able to create a larger 3 x 3 x 3 cube (see *The official history of SOMA*, 1998).

To set the scene for the tasks that follow, show students a Soma Cube made from the seven configurations (see Figure 1) and tell students the story about how the Soma Cube came about, giving details of Piet Hein, the inventor.

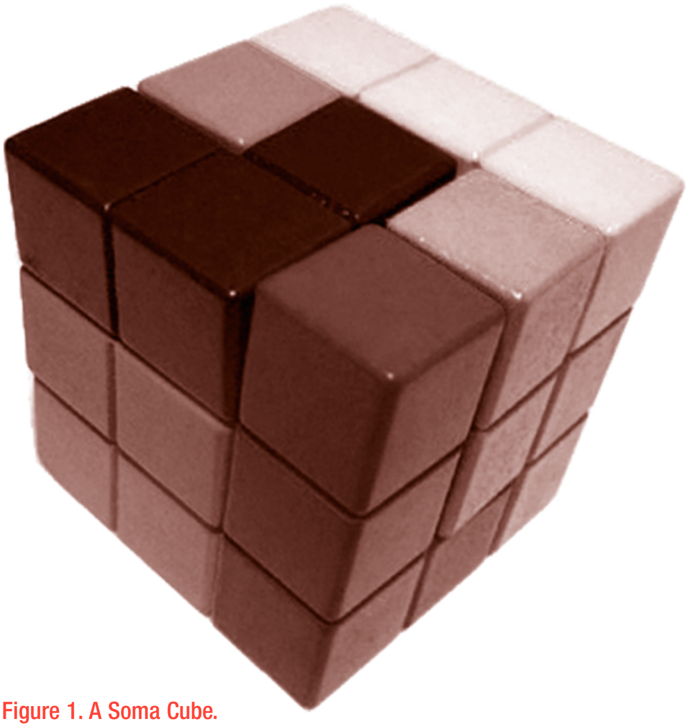

Figure 1. A Soma Cube.

Activity A: Constructing the Soma pieces

Give each child 30 multi-linked cube pieces and ask them to create shapes from three or four cubes which are not a straight line or square (for a similar activity, see Williams, n.d.). Ensure that there are four cubes of each colour and encourage students to make each shape in one colour to enhance visual imagery. Encourage the students to flip or rotate the different four-cube pieces to ensure that they are different. Model for them how to do this as some students may have heard the terms 'flip' and 'rotate' but may not have any idea how this is carried out. The seven three-cube or four-cube pieces that they can create would look like those in Figure 2.

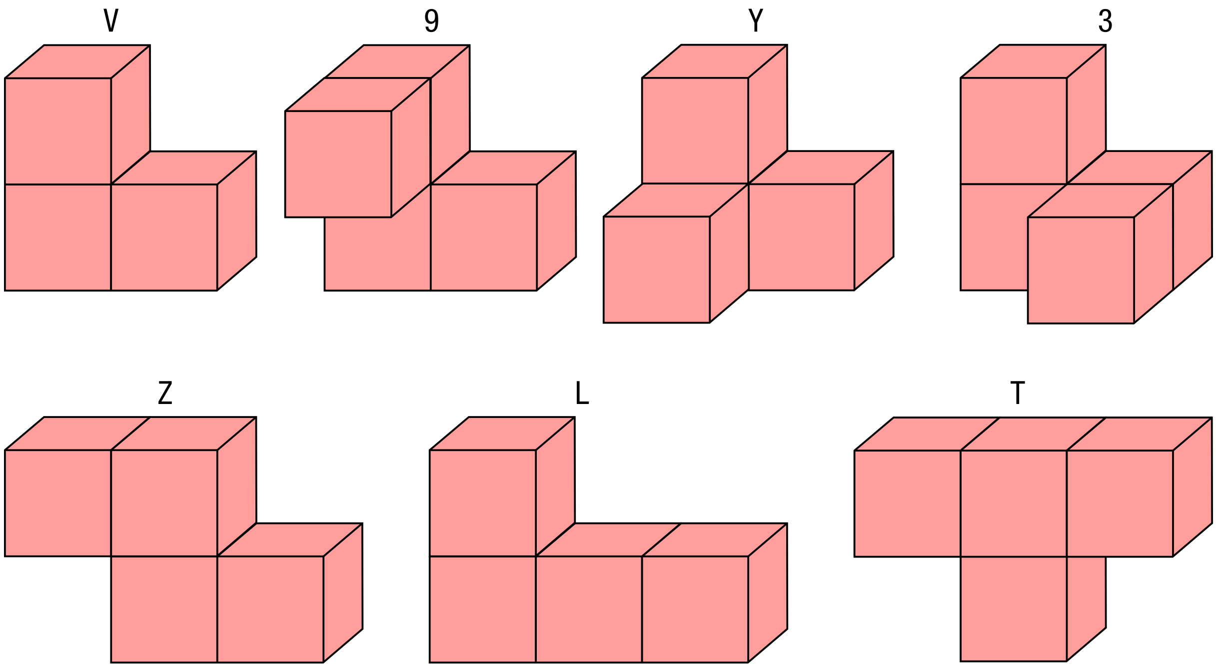

Figure 2. The three-cube piece and six four-cube pieces.

To distinguish the pieces, use a simple naming convention linking the shapes with alphabets (Balmoral Software, 2000). The piece named '**3**' is derived from the appearance of the hands at three o'clock: the minute hand is in front pointing upward, and the hour hand is at the back pointing to the right. Similarly, the piece named '9' corresponds to a front clock hand pointing upward and a back hand pointing to the left. Highlight to the students that there are two pieces (labelled 3 and 9) that are mirror images of each other and that they are different.

Activity B: The Soma pieces in two-dimensional representation

Drawing three-dimensional objects can be challenging and we suggest that teachers use isometric grid or dot paper to help students build the skill of drawing such three-dimensional diagrams. Students can practise drawing by recording each of the above shapes on isometric dot paper. Ensure that students join the correct dots to represent the line. Some students may still need to see the teacher actually manipulating unit-cubes, and then showing how the physical three-dimensional objects can be represented as two-dimensional isometric images. Three-dimensional objects are often drawn in isometric perspective because the drawing is in three dimensions, gives a correct visual impression (top, front and side views) and the distances and lengths are to scale. While talking about this, incorporate appropriate language to describe the cube such as face, edge, vertex, front, side, slant, above, top, below, right angle, each, every, adjacent and touching.

If an interactive whiteboard (a SmartBoard is used in this example) is available, the teacher can model how to make an isometric drawing of a four-cube piece using an isometric grid on the board. Using the line tool, draw a cube on the board. Group the lines in the cube and then clone the cube. When an object is 'cloned' on a SmartBoard, the 'infinite cloner' feature can be clicked to create infinite copies of the same object. The cloned object can then be clicked and dragged to create assembled images built from these copies—similar to physically building with unit-cubes. Ask students to re-create a diagrammatic representation of the four-cube pieces they have created by dragging the cube on the interactive whiteboard to form the four-cube piece they have made (see Figure 3).

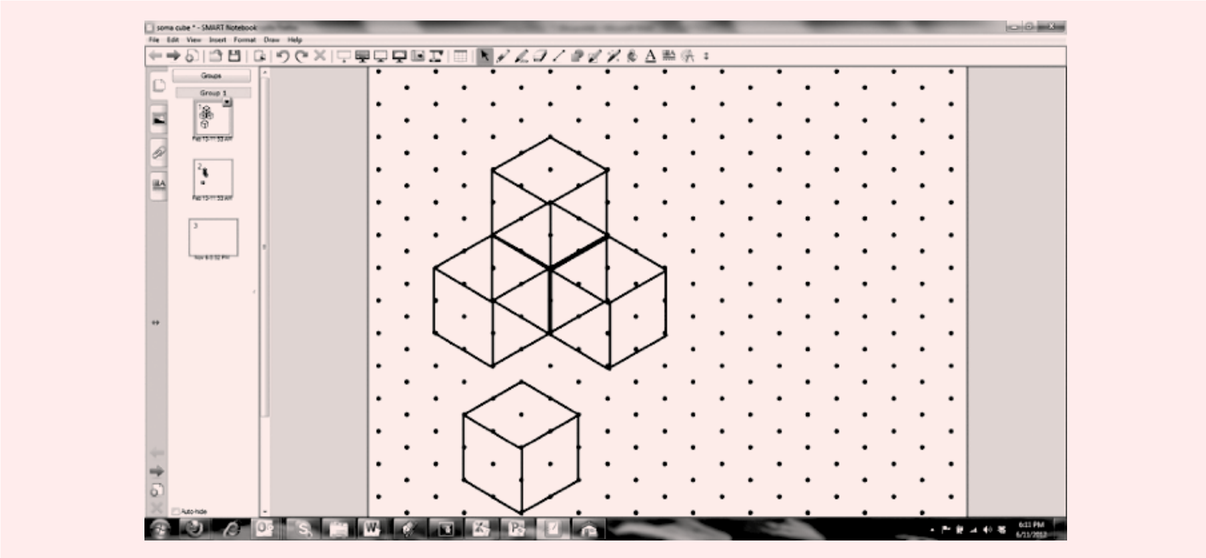

Figure 3. Drawing on isometric dot grid on an interactive whiteboard.

Activity C: The Soma Cube challenge

In small mixed-ability groups, ask students to construct the Soma Cube using the seven pieces they have created. The challenge is to put together the seven pieces to form a cube in 20 seconds or less. This activity may be quite challenging for some and may take longer, thus requiring the target time to be adjusted accordingly.

Activity D: The consolidation puzzle

As a consolidation activity, give each group six blocks: two red, two blue, one yellow and one green. In groups of five or six, each student draws out a task card from an envelope and reads the instructions within. Each card contains one of the following instructions:

- There are six blocks in all. One of the blocks is yellow.
- The green block shares one face with each of the other five blocks.
- The two red blocks do not touch each other.
- The two blue blocks do not touch each other.
- Each red block shares an edge with the yellow block.
- Each blue block shares one edge with each of the red blocks.

As a group they have to determine what the shape is by following the instructions. This activity has been adapted from Erickson (1989) and is one of five puzzles that can be used.

Activity E: My dream house (hand-drawn)

Once students have mastered how to draw on isometric dot paper, ask each child to construct a shape that look like a house using the supplied multi-link blocks and then design a dream house, based on that shape, on A4-size isometric dot paper. The teacher can model on the interactive whiteboard or isometric dot paper how to add extras such as roads, paths, windows and pitched roofs, and encourage the students to add these to their designs. Figure 4 shows an example of a student's work.

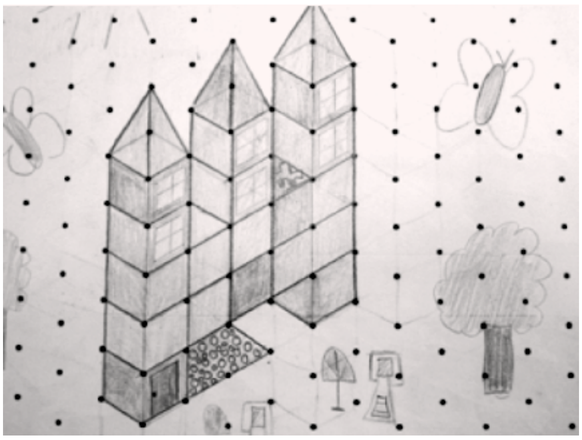

Figure 4. My dream house on isometric dot paper.

Activity F:
My dream house (computer-generated)

Ask students to replicate their hand-drawn picture on the computer using *Microsoft Word* (see Figure 5). On the top menu bar, go to <Insert> and select <Shapes> and under <Lines> select <Freeform> (Figure 5a). Left click on the mouse and then let go; move the mouse to draw a straight line of desired length, then left click again to start another line, continuing like this until the desired shape is obtained by ending at the starting point (Figure 5b). When the shape is closed, the resultant shape will be coloured (Figure 5c). The colour can be changed by clicking in the centre of the shape and selecting a colour from the colour palette. Figure 6 shows the drawing in Figure 5 reworked on the computer.

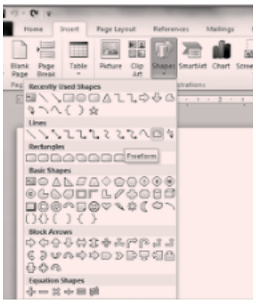

Figure 5a

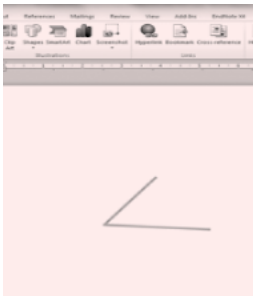

Figure 5b

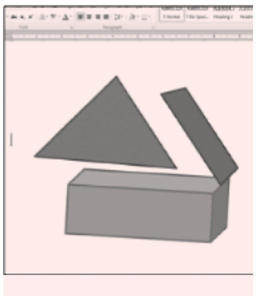

Figure 5c

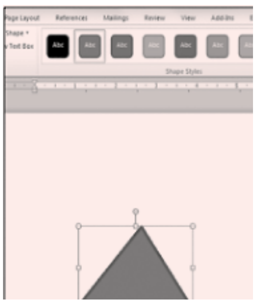

Figure 5d

Figure 6. My dream house drawn on the computer and printed.

Assessment of learning and student feedback

The tasks described above are sequenced in a way that build on or scaffold students' knowledge of geometry. Each task in itself provides information about students' knowledge. In the first author's class, observations and anecdotal records of the students' conversations, discussions and actions taken during the Soma Cube task gave an indication of their initial understanding of the language and concepts of geometry. Their first drawings of the shapes (in Activity B) were kept as a record of their initial ability to make two-dimensional representations of three-dimensional objects. The hand-drawn dream house becomes the assessment piece of a student's ability to make a three-dimensional drawing. Some students required and requested assistance during the process, while others required no assistance or persisted without assistance and subsequently encountered difficulties. The computer-generated dream house gave those students who continued to have difficulties an opportunity to revisit their hand-drawn version and to make alterations if they saw errors. The use of computer generated drawings of the dream house was intentionally delayed until students had had the opportunity to draw by hand the house they built with the multi-linked blocks. Swan & Marshall (2010) advocated from their observations of students that it is best "to delay the use of virtual manipulatives until students have had experience of the 'real thing'" (p. 14).

In Activity D, the first author looked for consistent and accurate use of terminology. Groups were asked to prove that the shape they had built was correct; through that discussion, students displayed their ability or inability to use appropriate geometrical language. Overall, the tasks required a considerable amount of communication amongst students and with the teacher. Students' use of appropriate language and their understanding of the concepts were assessed through conversations with them and observations of their interaction with one another.

Swan and Marshall (2010) defined a mathematics manipulative material as "an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered" (p. 14). The use of the multi-linked blocks in creating the Soma pieces and the Soma Cube were intended to have that effect. There was a high level of engagement with all of these tasks in the class. The students persevered to solve the Soma Cube puzzle, with some groups spending a considerable amount of time on it. Where one would normally see a few taking a 'back seat' in some class activities, every child was engaged and participated in all the above tasks. The students were eager to continue with each activity, with some asking, "Are we going to be doing some more of those card puzzles?" The pride students took in drawing and adding details to their dream home, both hand-drawn and computer-generated, suggested that they were highly engaged. It was heartening to see students using the cubes to make and draw three-dimensional shapes during 'finish off time' and some even requested to stay in during recess and lunch times to explore building and drawing three-dimensional shapes. It is interesting to note that many students, like the student in the example given above, presented different perspectives of their house on dot paper and on the computer. This displays a developed visual imagery and an understanding about two-dimensional and three-dimensional representations as well as plan and elevation perspectives.

Concluding remarks

The aim of this set of tasks is for students to develop skills in visual imaging, perspective drawing, and problem solving, while using appropriate geometric language. It builds on students' prior knowledge of two-dimensional shapes and the associated language and extends their knowledge of three-dimensional shapes by getting them to use concrete and computer manipulatives. The problem solving scenarios are useful in extending geometric knowledge and reasoning. The manipulation of concrete materials is complemented through the use of technology so that students have virtual manipulatives to explore—to drag, resize, transform, move, copy, paste, colour in or delete at will. Undoubtedly, a different kind of hand–eye coordination is at play. The combination of teaching strategies such as storytelling, the open-ended use of manipulatives, problem solving, group work, drawing, together with the appropriate use of technology, result in a set of rich-task-based lessons which can engage students in the learning of geometry. O'Shea (2009) describes the teacher's role in an ideal mathematics classroom as being to "encourage students to experience mathematics classes as fun, as social interaction and a liberation from the routine, a challenge and an assertion of individual creative intellectual work" (p. 23). From the feedback given by students, we believe that this multi-representational way of teaching geometry can provide all the above conditions as well as achieve the learning intentions for which it was designed.

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