Student understandings of numeracy problems:

Semantic alignment and analogical reasoning

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Introduction

Despite compulsory mathematics throughout primary and junior secondary schooling, many schools across Australia continue in their struggle to achieve satisfactory numeracy levels. Numeracy is not a distinct subject in school curriculum, and in fact appears as a general capability in the Australian Curriculum, wherein all teachers across all curriculum areas are responsible for numeracy. This general capability approach confuses what numeracy should look like, especially when compared to the structure of numeracy as defined on standardised national tests. In seeking to define numeracy, schools tend to look at past NAPLAN papers, and in doing so, we do not find examples drawn from the various aspects of school curriculum. What we find are more traditional forms of mathematical worded problems.

In mathematics, worded problems tend to be associated with teaching contextualised mathematics, aiming towards relational understanding in mathematics, rather than superficial instrumental understanding (Skemp, 1989). Given the heavy content of the Australian curriculum, the use of rich, context based mathematics is not a common approach to teaching mathematics. Teaching the contexts, as well as the mathematics, is often too time consuming for most schools. In cases where context, relational understanding and the use of analogies have been studied, the involvement students have in the analogical reasoning processes have been limited (Richland, Holyoak & Stigler, 2004). This limited student control over the reasoning processes is a possible reason why students misunderstand worded problems (Richland, Holyoak & Stigler, 2004). As a consequence, the teaching of mathematical worded problems is not something that happens in great depth in mathematics classrooms, and it certainly does not happen in the form of numeracy, in other curriculum areas.

Identifying the difficulty students experience with worded problems, one school has sought to address this by explicitly teaching students to interpret and understand numeracy worded problems. The aim of this paper is to present a theoretical rationale for this school's approach, and to describe the methodology and teaching experiences of an innovative extra numeracy program for Year 9 students.

Study method and context

This study is an analysis of one schools approach to addressing low levels of numeracy. The school is a state high school in regional southern Queensland with approximately 1000 students. The primary barrier to students achieving better numeracy outcomes was identified in terms of the specific literacy requirements for solving worded problems. These worded problems appear in NAPLAN tests, and as such they define the nature of required numeracy skills in our schools.

Numeracy: A cognitive sciences perspective

A cognitive sciences perspective may view numeracy worded problems as simplistic, contrived situations to represent context, involving relationships between concrete objects or illustrations that require resolution using quantitative methods. Understanding the language of the problem and relating the language to mathematics requires a unique set of cognitive skills. These cognitive skills may be defined conceptually, in terms of semantic alignment and analogical reasoning.

Semantic alignment refers to the alignment of the text of worded problems with an intended mathematical structure that will lead to a solution. This results in a unique textual structure that requires students' to possess an ability to follow a specialised set of rules to comprehend the language of worded problems. These rules are often not self evident to the student, nor to the teacher. These semantic rules tend to be embedded in worded problems and appear to be taken for granted by educators (Bassok, 2001). For this reason, worded problems adopt unique text characteristics, constructed by educators to omit irrelevant text. This keeps the problems brief, contrived and structured around standardised terminology and phrasing (Bassok, 2001). For students to understand worded problems there is clearly a need to explicitly teach this standardised terminology and phrasing, to enable understanding of semantic alignment between worded language and mathematical structure.

Concurrent with comprehending the language to understand semantic alignment, students need to apply their semantic skills in the process of analogical reasoning. Analogical reasoning was first studied in relation to children and learning by Jean Piaget, and includes the derivation of solutions by understanding the relationships between concrete illustrations and abstract mathematical concepts (Bassok, 2001). The skill of analogical reasoning has been identified as a core process in learning and thinking, throughout a child's cognitive development. Analogies, and analogical reasoning, work by starting with an analogue with which the student is familiar, and then bridging student knowledge through alignment or mapping, to explain an abstract concept. The analogue is referred to as the base, and the abstract concept, to be understood, is referred to as the target (Fluellen, 2007; Genter, 1983).

The base-target relationship between context and mathematical structure varies from learning to assessment situations (Richland & McDonough, 2010). Worded problems in learning situations require concrete, context based illustrations to be generated as a familiar start for students (Richland & McDonough, 2010). Learning then focuses on the steps required to generate an abstract mathematical representation of the context, with the learning goal being the relational understanding between context and abstract mathematics. Context is the base analogue, mathematics is the target concept.

In contrast, during assessment situations students are assumed to have developed their prior mathematical knowledge, and are required to apply this knowledge to new contextualised, concrete situations. In situations such as NAPLAN assessment, the mathematical structure is assumed to be known and this forms the familiar analogue base. The context becomes the target concept, as students need to show their understanding of the context by applying their mathematical knowledge (Bassok, 2001). From a cognitive sciences perspective this process is fundamental to solving worded problems and operates in tandem with the student's skills of textual comprehension.

The pedagogical model: RULES & T

To foster the student's skills of textual comprehension and their capacity to engage in analogical reasoning when confronted with numeracy worded problems, the school in this study developed a specific pedagogical model. The model was deployed within a Year 9 Extra Numeracy Program involving eight mathematics and English teachers. Teachers were briefed on the structure and application of the model and provided standardised presentations and teaching resources.

The stated objectives of the model were to help students to:

- understand what the question was asking;
- explore the steps required to solve numeracy worded problems;
- calculate the answer to worded problems.

The model was structured around a standardised problem solving process that students would learn and be able to apply to worded problems in the future.

Table 1. The pedagogical model, RULES & T.

Acronym	Definition	Significant links to cognitive sciences
R	Read the problem in detail.	Textual comprehension of semantic alignment
U	Underline key words and variables	Textual comprehension of semantic alignment
L	List the variables	Textual comprehension of semantic alignment
E	Estimate a likely answer via a mental calculation	Analogical reasoning
S	Steps to show working and derive an answer	Analogical reasoning
&		
T	Test the likely correctness of the answer by comparing to your estimate, and understanding of the problem	Back testing comprehension of semantic alignment and analogical reasoning

This was defined as RULES & T (See Table 1), and used by teachers as follows:

1. Read
   RULES & T required students to slow down their approach to worded problems by firstly reading the question in full.
2. Underline
   Underlining key operational terms was an important step in making their reading of the problem an active reading process. Key operational terms were explicitly discussed, modelled, and their identification by students was practiced (See Table 2). These words were modelled in problems such as the example at Figure 1.
3. List
   Each variable and its unit was identified and listed along with the variable that needed to be found.
4. Estimate
   An estimate of the answer was made using a rough mental calculation based on the known and unknown variables, and the key operational words. Students were encouraged to write statements such as "the answer is between 10 and 20" or "the answer should be less than 25".
5. Steps
   Students were taught that most questions would have multiple steps, and so planning calculations was important. At this stage, students were encouraged to write equations that could help at each step of the solution. The final part of the Steps stage was to solve the equations.
6. Test
   The final solution was tested against the estimate made earlier, and solutions revised if they were not reasonable.

Table 2. Key operational terms. Adapted from Stapel (2013).

Words that might mean	
ADDITION	increased by more than combined, together total of sum added to
SUBTRACTION	decreased by minus, less difference between/of less than, fewer than
MULTIPLICATION	of times, multiplied by product of increased/decreased by a factor of
DIVISION	per, a, each out of ratio of, quotient of rate average percent (divide by 100)
EQUALS	is, are, was, were, will be gives, yields sold for

Table 3. Sample problems to identify key operational terms.

Identify the key operational word in this question.	The key word is…
An airline bought 6 new aircraft for a total cost of \$720 million. Each aircraft cost the same amount. How much did each aircraft cost? A. \$120 million B. \$360 million C. \$714 million D. \$4320 million	Each, indicating DIVISION
How many hours and minutes are between 7:25 am and 5:05 pm on the same day?	Between, indicating SUBTRACTION

Classroom experiences: RULES & T

In the classroom, the initial approach was to model RULES & T using standardised worded problems and solutions provided across all classes as a PowerPoint presentation. Modelling was followed by student practice using NAPLAN styled and formatted problems. This approach met with resistance from students who, seeing the multiple choice options simply shaded the answer without showing any working, or at best scribbled part of the RULES & T steps on to the side of the question sheet. Subsequent worksheets were scaffolded with the RULES & T acronym positioned vertically in a box at the bottom of each page. This improved compliance to some degree, but the thoroughness and correctness of student work was still lacking.

When we moved onto the topic of ratio, an alternative approach was adopted. This comprised of modelling solutions to ratio questions using a simple worded analogy of the parts of water and flour to be used in baking a small cake, compared to a larger cake. After modelling two examples using an instrumental approach, students practiced about a dozen mathematically structured questions from their mathematics textbook. Once instrumental understanding was confirmed, a worded problem was presented and the solution modelled using the RULES & T model. In this case, the Read, Underline, List and Step stages were highlighted on the whiteboard, as being the most important. Students then copied the worded problems into their workbooks from the projection screen, prior to applying RULES & T. This approach yielded the best results, probably due to their need to read the question carefully as they copied it into their notebooks. There was a higher degree of compliance with the model, and overall better success in achieving correct answers.

Table 4. Sample problem, student processes and outcomes.

Key action	Student process	Outcomes
Read	Students read or write out the question.	Mental outcome: Student is familiarised with the problem.
Underline	Keywords and variables are underlined.	Written outcome: An airline bought 6 new aircraft for a total cost of $720 million. Each aircraft cost the same amount. How much did each aircraft cost? A. $120 million B. $360 million C. $714 million D. $4320 million
List	Keywords and variables are listed beside the question.	Written outcome: *Keyword:* each suggesting DIVISION *Variables:* number of aircraft = 6 total cost = $720 million cost of each aircraft = ?
Estimate	Mental maths calculations are done to reach an approximation or estimate.	Written outcome: 10 aircraft = $72 million, 5 = $144 million 6 aircraft will be a little less than $144 million
Steps	Steps are set out to show working.	Written outcome: Cost per aircraft = total cost ÷ number of aircraft = 720 ÷ 6 = 120 Therefore the cost per aircraft is $120 million or Option A, above.
Test	The answer is compared to the estimate to check for reasonableness	Written outcome: $120 million is a little less than the estimated $144 million, which is reasonable.

Table 5. Sample problem, student processes and outcomes.

Key Action	Student Process	Outcomes
Read	Students read or write out the question	Mental outcome: Student is familiarised with the problem.
Underline	Keywords and variables are underlined	Written outcome: How many <u>hours and minutes</u> are <u>between 7:25 am</u> and <u>5:05 pm</u> on the same day?
List	Keywords and variables are listed beside the question	Written outcome: *Keyword:* <u>between</u> suggesting SUBTRACTION *Variables:* Time one = 05:05 pm Time two = 07:25 am Time difference = ?
Estimate	Mental maths calculations are done to reach an approximation or estimate	Written outcome: Time difference between 7 am and 5 pm is 10 hours. Answer will be a little less than 10 hours.
Steps	Steps are set out to show working	Written outcome: Time difference one = 5:05 pm – 12:00 am = 5 hours 5 minutes Time difference two = 12:00 am – 7:25 am = 4 hours 35 minutes Total time difference = 5 + 4 h & 5 + 35 min = 9 hours 40 minutes Therefore the time difference is 9 hours 40 minutes.
Test	The answer is compared to the estimate to check for reasonableness	Written outcome: Answer is a little less than 10 hours which is reasonable.

When introducing this model to future students, it would be worth experimenting with an intermediate version of the model whereby students were not required to estimate, solve or test. In effect, rather than aiming to provide an answer to the equations, an initial phase of teaching could focus on the development of equations. The aim of this modification would be to focus learning around textual comprehension skills and analogical reasoning processes as these are clearly the core cognitive skills that lead to success in solving worded problems.

Conclusion

This study presents one school's experience of developing and implementing a model to address student misunderstandings about numeracy worded problems. The theoretical and empirical research from the cognitive sciences supports a need to explicitly teach the unique text structures of numeracy worded problems. These problems appear technically difficult for students, due to their contrived contexts, terminologies and phrases that are aligned with intended mathematical structures. The most important components of solving numeracy worded problems are comprehending the text and applying analogical reasoning to construct equations and other mathematical structures out of the text. The experience in this study suggests that students tend to focus only on the final solution and proceed directly to it without fully understanding the text, mathematical structures or semantic alignments. From a theoretical perspective, by slowing student thinking to a step by step process that emphasises the relationships between text and mathematical structure, student success in numeracy may be improved. Further evaluation of this model is needed to confirm its potential role in improving learning outcomes.

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