Linking models

reasoning from patterns to tables and equations

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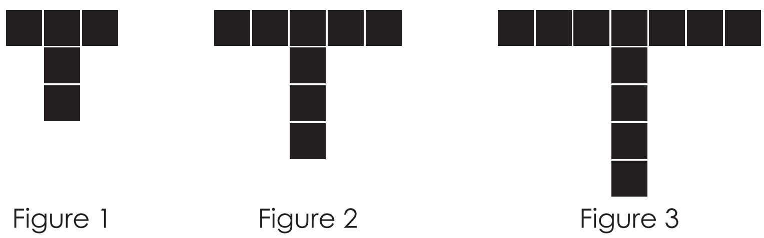
Patterns are commonly used in middle years mathematics classrooms to teach students about functions and modelling with tables, graphs, and equations. Grade 6 students are expected to, "continue and create sequences involving whole numbers, fractions and decimals," and "describe the rule used to create the sequence." (Australian Curriculum, Assessment and Reporting Authority (ACARA), 2012). Modelling functions, such as pattern problems, and making links between models in analysing patterns of change is an essential part of middle years mathematics (Lloyd, Herbel-Eisenmann & Star, 2011).

My students' approach to pattern problems often entailed extending the pattern and collecting data, which they organised in a table. They used the table to graph their data and used common differences, guess and check, or other strategies to find an equation to model the problem. While this procedure 'worked' for many students, others struggled with generating an equation from the data in their table. Many students who were able to generate an equation struggled to explain what the individual parts of the equation meant, how they were related to the original problem, what the variables represented, or the relationships between the table and equation. These students exhibited Kieran's (2007) finding that students use of tables in generalisation activities often lead to a disconnect between numerical and geometric relationships "shortcircuit[ing] all the richness of the process of generalization" (p. 725).

To address these issues, I had my students generate tables in such a way that the variant and invariant quantities were evident and related to the posed task. In doing so, the process of generating an equation to model the situation became clearer and more meaningful for the students as they made connections between the numerical and geometric relationships. In



(a) How many squares are in the 200th figure?
(b) How many squares are in the *n*th figure?

Figure 1. Typical geometric growth problem.

this paper, I provide a typical geometric growth pattern encountered in middle years mathematics to illustrate this method, see Figure 1.

My students normally approached such problems by extending the pattern, counting the number of squares in each figure, and recording the data in a table. They then looked for patterns in the data to find the number of squares in the 200th and *n*th figures. However, this approach may introduce the unintended and potentially problematic issue of focusing students attention on data for individual figures and recursive patterns of change with little attention on explicit patterns of change, including variant and invariant quantities, or how the data was related to the original figures. For instance, my students tended to examine the data in the table, such as examining common differences, without considering how these changes were related to the original pattern. I also found that once my students generated their table they tended to ignore the original figures, relying solely on the numerical data in the table.

In order to help students make connections between the figures, table of data, and subsequent equation, I used a strategy that forced them to consider the change in the pattern from one figure to the next and incorporate it in their table entries in order to make connections between the figures, table, equation, and the recursive and explicit patterns. As the students began exploring the problem, they entered 5 in their table for the number of squares in the first figure, as they normally did. However, when they entered the number of squares in the second figure, they had to consider and include how the number of squares changed from the first to the second figure. In this case, the second figure had three more squares than the first figure, shown as the blue squares in Figure 2.

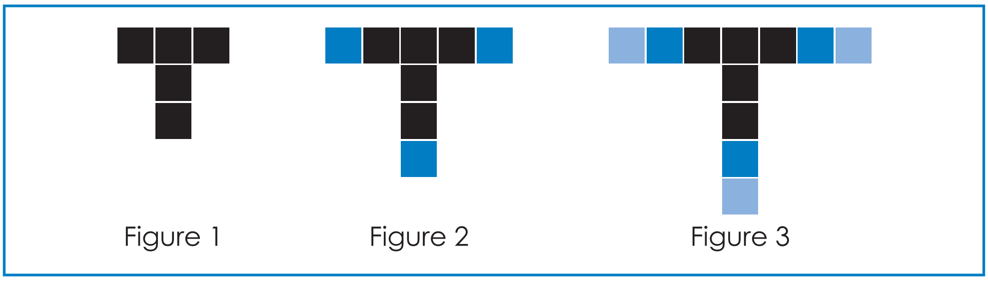


Figure 2. Making the geometric growth problem more explicit.

Instead of recording eight squares in the table for the second figure, we incorporated the fact that the second figure had three more squares than the first figure. Therefore, the entry in the table for the number of squares for the second figure was 5 + 3, the five squares in the first figure plus the three additional squares in the second figure. From the second to the third figure, we added another three squares, the light blue squares in the third figure. Likewise, for the third figure, we added three squares, +3, to the number of squares in second figure, 5 + 3. Therefore, the number of squares for the third figure was entered in the table as 5 + 3 + 3. Students extended this recursive pattern to generate the number of squares in the next two figures, see Table 1.

Table 1. Alternative table of values.

Figure	Number of squares
1	5
2	5 + 3
3	5 + 3 + 3
4	5 + 3 + 3 +3
5	5 + 3 + 3 + 3 + 3

Using this strategy focused students' attention on what the values in the table were referencing in the context of the problem, including the change from figure to figure. However, students recognised that extending the table in this way would quickly become an inefficient way of recording the number of squares. I asked them if there would be a more concise way to write the expressions for the number of squares in each figure. Students recognised that they could use multiplication to rewrite the number of squares in a more concise form. For example, they rewrote the number of squares in the fifth figure as 5 + 4(3); the number of squares for the fourth figure as 5 + 3(3), and the number of squares for figure 3 as 5 + 2(3). We extended this pattern to rewrite the number of squares for the second figure as 5 + 1(3), one group of three squares added to the original figure, and the number of squares for the first figure as 5 + 0(3), no groups of three yet added, see Table 2.

Table 2. More concise alternative table of values.

Figure	Number of squares
1	5 + 0(3)
2	5 + 1(3)
3	5 + 2(3)
4	5 + 3(3)
5	5 + 4(3)

This table revealed two invariant quantities: the number of squares in the original figure (5), and the number of squares added to any figure to get the next figure (3), and the variant quantity of the number of groups of three added to the original figure. However, not all of my students had recognised this pattern or how this pattern related to the original figures. To assist students in making this connection, I asked what each of these variant and invariant quantities meant in the context of the original problem. Students recognised that the 5 was the number of squares in the first figure and the 3 was the number of squares added to any figure to get the number of squares in the next figure. I then asked what the 4(3) meant for the number of squares in the fifth figure. They explained that the 4(3) showed that we had added four groups of 3 squares to the number of squares in the first figure (5) to get the number of squares in the fifth figure.

Before determining the number of squares in the 200th and *n*th figure, I wanted them to make one more connection from the table. While they recognised that the number of groups of three squares that had been added was the only invariant quantity in the expressions for the number of squares for each figure, not all students connected this quantity with the figure number. Many of these students were still focusing on the recursive pattern of adding one set of three squares to a figure to get the next figure and needed help connecting this recursive pattern to an explicit pattern. What I wanted them to notice was how these variant quantities were related to the figure numbers, the independent variables. In other words, I wanted them to move from a recursive view of adding one more group of three squares for each subsequent figure to an explicit relationship between the expression for the number of squares and the figure number.

Using what students had discovered about how the pattern grew, I asked how they could determine how many groups of three squares had been added to the original figure for any other figure number. After discussing this, the students concluded that the number of groups of three squares added to the original figure was one less then the figure number because they did not begin adding these groups of three squares until the second figure number because we had not yet added any groups of three for the first figure. The first time we added a group of three was in the second figure (i.e., we had added one group of three squares to find the number of squares in the second figure, two groups of three squares for the third figure, and so on). As one student stated, "For the first figure we have not added any groups of three squares yet, or we could say that we have added zero groups of three squares." Another student extended on this idea when they stated, "We always start with the five squares in the first figure and then just keep adding three squares starting with the second figure." This student had begun to attend to the invariant quantities, squares in Figure 1 and the addition of three squares, and the variant quantity, groups of three squares added. From this discussion and their understanding of the growth of the pattern, students recognised that the number of groups of three squares added was always one less than the figure number. Therefore we rewrote the number of groups of three squares added as the figure number minus one, to reflect their finding, see Table 3.

Table 3. More concise alternative table of values.

Figure	Number of squares
1	$5 + (1 - 1)(3)$
2	$5 + (2 - 1)(3)$
3	$5 + (3 - 1)(3)$
4	$5 + (4 - 1)(3)$
5	$5 + (5 - 1)(3)$

Students were then ready to determine how many squares were in the 200th figure, $5 + (200 - 1)3$ and explain what this meant in the context of the original problem. They could also generalise the results for the nth figure, $5 + (n - 1)3$ and explain how the parts of the equation related to the original problem. Besides providing students with a strategy for exploring patterns that made connections between the context, table, and equation, this strategy also made the variant and invariant quantities in the data explicit, all crucial concepts for later success in mathematics. Students continued to employ this strategy often discussing "the numbers that change" and "the numbers that stay the same," providing a foundation for their understanding of variable.

Conclusion

Providing students with opportunities to make connections across tables, graphs, equations, and contexts is a critical aspect of the teaching and learning of mathematics (Lloyd, et al., 2011). The National Council of Teachers of Mathematics (NCTM) suggests that middle years students study of patterns and relationships focus on linear functions (NCTM, 2000, p. 223). However, students should also have similar experiences with nonlinear functions (NCTM, 2000). Providing students with opportunities to make connections among these models and using strategies such as those described here, enables them to explore and understand the range of functional relationships they will encounter throughout the remainder of their mathematical experiences.

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