WORKING ON EXTREMUM PROBLEMS WITH THE HELP OF DYNAMIC GEOMETRY SYSTEMS

Iordanka Gortcheva

Abstract: Two problems from high school mathematics on finding minimum or maximum are discussed. The focus is on students' approaches and difficulties in identifying a correct solution and how dynamic geometry systems can help.

Key words: Dynamic geometry systems, Extremum problems, High school mathematics

1. Introduction

Many mathematical ideas included in modern high school curriculum, as well as the methods to teach them, are more than two millennia old. Legend has it that in those distant times a great teacher of mathematics, while drawing with a stick on the sand, told a great future ruler of the world [1], yet a student, that *there was no royal road to geometry* [1]. Since then students and teachers have changed and so have the teaching tools. Children of today are called *digital generation* [2, 3] and dynamic geometry systems (DGS) rapidly replace blackboard and chalk. Teaching styles are also being transformed, trying hard to meet modern children's individual needs and imagination.

Being a substantial part of high school mathematics curriculum, extremum problems can also be regarded as an opportunity to creatively apply students' mathematical knowledge. Such problems can be introduced even before derivatives have been taught. Implementing DGS in this process brings modern technology into mathematics classroom which, in turn, results in a number of important activities like exploration, communication, tracking mistakes, and filling knowledge gaps [4].

2. Modifying the traditional formulation of extremum problems

A traditional extremum problem in high school mathematics curriculum requires students to find local extrema of a function over a given interval. Such a problem is considered routine and poses little difficulty to the students. However, sometimes a minor modification in its formulation can turn it into an obstacle for the students. For example, if an end of the interval is represented by a parameter and not a number, a non-standard mathematical situation is created, in which even well-prepared students may feel uncertain. Here is a problem of the type:

Problem 1. Find the maximum value of the function $f(x) = x^3 - 12x$ on the interval $x \in [-1, b]$, assuming that $b > -1$.

To put in practice students' theoretical knowledge on derivatives, as an initial step the teacher asks the students to explore function's behavior on the whole real line. Thus the three x-intercepts: $x = 0$, $x = 2\sqrt{3}$, and $x = -2\sqrt{3}$, as well as the two critical points $x = -2$ and $x = 2$ of function f are found. The students show that $f(x)$ attains its local maximum equal to 16 at $x = -2$, and its local minimum equal to -16 at $x = 2$. They easily identify that for $x \in (-\infty, -2] \cup [2, +\infty)$ the function increases, and for $x \in [-2, 2]$ decreases. This information allows them to roughly sketch the graph of $f(x)$ for $x \in (-\infty, +\infty)$ (*Figure 1*).
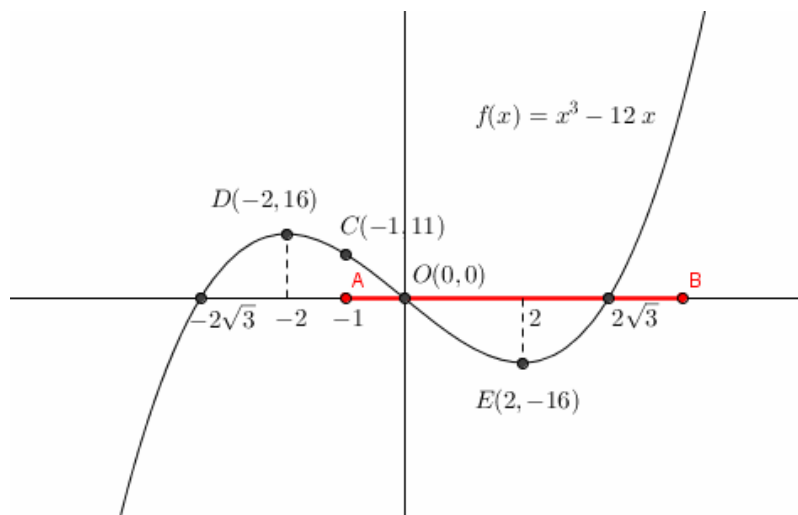
Figure 1. *Graph of function f(x) over the interval* $(-\infty, +\infty)$

Having completed the task, some of the students come up with the idea to use their graphs to model the situation described in *Problem 1*. For the purpose, they place interval $[-1, b]$ on the real line and try to figure out how the position of the right end b influences the maximum value of f. Thus, quite naturally, DGS learning environment with its visualization and computing means comes to use. Its role is to help students experiment, explore, and figure out how to apply in specific practical situations the theoretical concepts studied. It is appropriate, at that stage, to recall an important practical issue: to find the maximum value of a continuous function over a compact interval, the students compare its local maxima (if such exist) inside the interval with the functional values at the ends of the interval.

The most advanced students willingly take the challenges of problem solving, not using supplementary tools like DGS. They notice that point $x = -1$, the left end of the interval $[-1, b]$, is internal for the interval $[-2, 2]$, where function f decreases (*Figure 1*). Thus they consider the following two cases for the position of point $x = b$:

Case 1: $b \in (-1, 2]$. Since interval $[-1, b] \subset [-1, 2]$ and $f(x)$ decreases on it, the function attains its maximum at the left end $x = -1$, i.e. $\max_{-1 \le x \le b} f(x) = f(-1) = 11$;

Case 2: $b \in (2, +\infty)$. Since function $f(x)$ increases over the interval $[2, b]$, $\max_{2 \le x \le b} f(x) = f(b) = b^3 - 12b$. To find the maximum of $f(x)$ over the interval $[-1, b]$, however, one more step is needed: to compare the values $f(b)$ and $f(-1)$. Most students leave their results written in the form $\max_{-1 \le x \le b} f(x) = \max(11, b^3 - 12b)$.

However, there are students, for whom it is important to figure out when $b^3 - 12b$ is greater than 11. Exploring this situation, they find that equation $b^3 - 12b - 11 = 0$ has an integer root $b = -1$. They write the cubic equation as product $(b + 1)(b^2 - b - 11) = 0$. Solving quadratic equation $b^2 - b - 11 = 0$ allows them to re-write inequality $b^3 - 12b - 11 > 0$ as $(b + 1)(b - \dfrac{1 + 3\sqrt{5}}{2})(b - \dfrac{1 - 3\sqrt{5}}{2}) > 0$ and solve it by the method of intervals. The students conclude that the maximum value of function $f(x)$ over the interval $[-1, b]$ equals 11, if $-1 < b \le \dfrac{1 + 3\sqrt{5}}{2}$ and $b^3 - 12b$, if $b > \dfrac{1 + 3\sqrt{5}}{2}$.

The students who solved the problem analytically based their reasoning on scarce graphical representations. As a quick-to-complete task that follows, the teacher can ask them to draw the graph of a new function, defined as $h(t) = \max_{-1 \le x \le t} f(x)$. This additional task gives them a new perspective to the results already obtained (*Figure 2*).
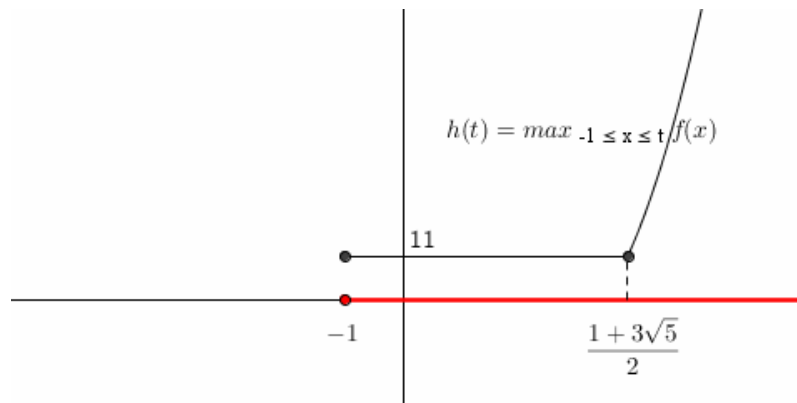
Figure 2. *Graph of function h(t) over the interval* $[-1, +\infty)$

For the students, who do not feel motivated enough to work on the problem, experimenting with DGS helps them understand that due to the parameter the solution holds not only for one fixed interval, but for a whole "family" of intervals. If a GeoGebra applet[2] is used, parameter *b* can be represented by a slider named *b*. For further pedagogical purposes, the fixed left end $x = -1$ of the interval can be chosen as the largest value of another slider named *a*; later it can be used by the students to make changes in problem formulation and explore their effect. To help students work effectively on the problem, the teacher may provide everybody with an applet (*Figure 3*).

If the students are to find the maximum value of function *f*(x) on a particular interval $[-1, 4.5]$, through GeoGebra command **Max[f, -1, 4.5]** they may obtain point $(-1, 11)$. According to this result, function *f*(x) should attain its maximum value equal to 11 at the left end $a = -1$ of the interval. However, this is not the case: the maximum value 37.13 is attained at the right end $b = 4.5$ of the interval. Students guess that **Max[f, -1, 4.5]** command has not taken into account the values of function *f*(x) at the two ends of the interval. Therefore, for finding an adequate solution in a DGS environment, the students need to use their attentiveness, theoretical knowledge, and common "geometrical" sense. Auxiliary tools like horizontal lines, sliding parallelly to the *x*-axis and tied to the values of $f(x)$ at the ends of the interval are also helpful (*Figure 3*).
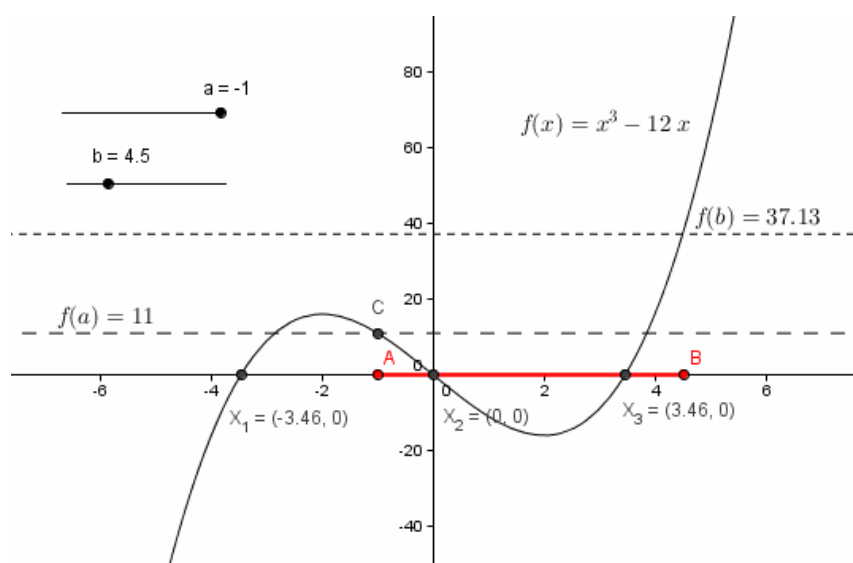

Figure 3. *Exploring the maximum of function f(x) over the interval* $[-1, b]$ *through a GeoGebra applet*

The excitement to find the maximum value only through their applets stimulates the students to try another approach. They use GeoGebra command **Sequence[f(i), i, -1, 4.5, d]**, which "makes" x run over the interval $[-1, 4.5]$ with a small increment d and creates a list of respective functional values. However, the **Max[List]** command returns the students number 37.12 and not 37.13. Although the value 37.12 is much closer to it than 11, they do not like the result. Somebody's question: "How may computers be wrong?" intrigues even the students who have solved the problem analytically. They suggest explicit comparison of the value obtained by **Max[List]** command with $f(-1)$ and $f(4.5)$ and choosing the largest one.

The answer to the problem comes quite unexpectedly: to figure out what is going on, some students take their cell phones to calculate the expression $f(4.5)$. The value of 37.125 displayed is marked by cheerful exclamations. The students speculate that the differing functional and maximum values are probably due to different rounding of the decimal 37.125 to the nearest hundredth appearing either as 37.12 or as 37.13. Thus a specific situation teaches the students that in DGS not only the accuracy of calculations matters, but also the accuracy of displaying the results. Going to button "Options" of GeoGebra toolbar menu and changing the rounding from two to three decimal places eliminates the discrepancy.

One student reflected on the situation she experienced by saying: "To solve a hard mathematical problem, except knowledge and skills sometimes we also need insight; the same we need even if a computer works for us". Her comment was followed by a classmate's remark: "Computers do what they are prescribed to do."

In such unintentional way, exploring the problem through DGS becomes a part of the process of developing students' mathematical thinking [5]. Their experience with continuous functions broadens to the ideas of modern computing: to obtain supremum of $f(x)$ approximately by using **Sequence[f(i), i, -1, 4.5, d]** and **Max[List]** commands, function f should be "smooth" enough and d − small enough. Searching to explain why in this particular case GeoGebra commands do not return a proper functional value helps the students develop numerical literacy considered crucial for young generation [6], [7]. Working on the problem in DGS environment engages the whole class to observe, systematize the results, share opinions, and formulate conjectures. The task to draw the graph of the function $h(t) = \max_{-1 \le x \le t} f(x)$ (*Figure 2*) can appeal to everybody, especially if the teacher chooses to introduce it as a test for mathematical intuition.

3. Facing up the challenges of dynamic constructions

To save time in his geometry class, my math teacher used to draw even complex figures on the blackboard by hand. Since the result was rarely perfect, he liked to joke that "*Geometry is the art of correct reasoning on incorrect figures.*" Advanced students with abstract mathematical thinking tend to neglect the resources of DGS; solving a hard geometric problem, they often need a rough sketch only. Those who do not feel confident in geometry look at DGS as a tool that not only draws pictures, but also solves the homework assignments for them. Let us consider a problem discussed in class:

Problem 2. Triangle ABC is given with side AB = 9 and radius $r = 3$ of its inscribed circle. Find the minimum value of area of Δ ABC.

In contrast to the previous problem, this one provides the students with an opportunity to combine their geometric and algebraic knowledge. From a pedagogical perspective, it can be solved before derivatives have been taught, applying the inequality between arithmetic and geometric means.

Following the students' work on the problem, we denote AM = AP = x, BM = BN = y, CN = CP = z, $\angle$ A = α, $\angle$ B = β, $\angle$ C = γ (*Figure 4*). Since $x + y = 9$, area of Δ ABC can be expressed as $S = pr = 3(x + y + z) = 27 + 3z$. At first glance the students like this equation of one variable because it looks simple. Later they figure out that although z is non-negative, its minimum value cannot be 0: otherwise the triangle inequality AC + BC > AB does not hold. The most advanced students try exploring the impact of angle values. Since AO, BO, and CO are angular bisectors, they express segments x, y, and z

as $x = 3 \cot \dfrac{\alpha}{2}$, $y = 3 \cot \dfrac{\beta}{2}$, and $z = 3 \cot \dfrac{\gamma}{2} = 3 \tan\left(\dfrac{\alpha}{2} + \dfrac{\beta}{2}\right)$ respectively.
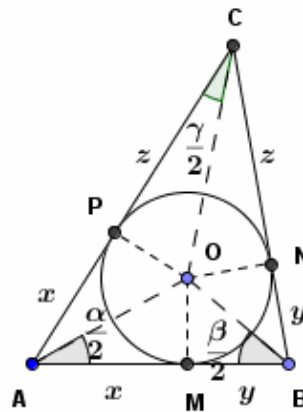
Figure 4. *Illustration of the variables used in the solution of **Problem 2***

Using trigonometric identities leads the students to the following expression:

$$z = 3\frac{\tan\dfrac{\alpha}{2} + \tan\dfrac{\beta}{2}}{1 - \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}} = 3\frac{\dfrac{3}{x} + \dfrac{3}{y}}{1 - \dfrac{3}{x}\dfrac{3}{y}} = 9\frac{x + y}{xy - 9} = \frac{81}{xy - 9}.$$

Thus triangle area is represented by a function of two variables: $S(x, y) = 27 + \dfrac{243}{xy - 9}$. This form

does not allow the students to apply the theory for finding extrema of one-variable functions, even if they have already studied it. To give them a clue, the teacher poses the question if there is a relationship between the values of the product xy and the sum $x + y$ they can suggest. Some students guess that the inequality between arithmetic and geometric means can help: $x + y \geq 2\sqrt{xy}$ re-written

as $xy \leq \dfrac{(x + y)^2}{4} = \dfrac{81}{4}$ leads to the conclusion that the minimum of triangle area is 48.6. The students

also figure out what the triangle of minimum area looks like: according to the same inequality this is the case when $x = y$, i.e. $\triangle$ ABC is isosceles.

A group of students do not use the teacher's hint. Using $y = 9 - x$, they express triangle area in the

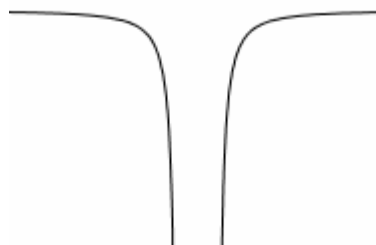form $S(x) = 27 + \dfrac{243}{-x^2 + 9x - 9}$. Their first attempt to graph function $S(x)$ is shown in *Figure 5*:

Figure 5. *Such student representation of function S (x) omits a lot of details and an entire branch of the graph*

It contains only a few details about the shape, but nothing about the functional values: neither coordinate axes, nor the local minimum are present. The students manage to adjust the scale properly. However, identifying the asymptotes needs more elaborate exploration until the students find the zeros of quadratic function $-x^2 + 9x - 9$ in the denominator. The graph in *Figure 6* confirms the minimum value obtained through the approach using the arithmetic-geometric means inequality.
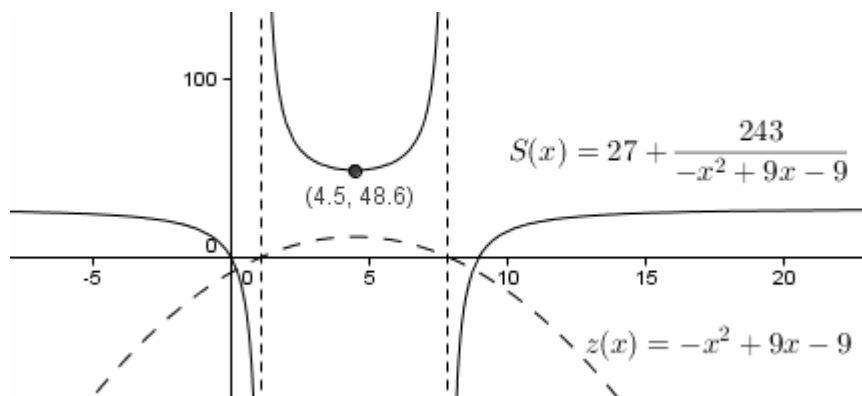

$$S(x) = 27 + \frac{243}{-x^2 + 9x - 9}$$

$$z(x) = -x^2 + 9x - 9$$

Figure 6. *The graph of function S (x)*

The variety of mathematical situations is appreciated also by those students who do not have strong mathematical background or do not like making theoretical effort. Their idea is to approach the problem, making a GeoGebra applet of their own. The students suppose that two segments of fixed lengths are given: $c = 9$ and $r = 3$. Since the circle inscribed in Δ ABC is tangent to its sides, after having constructed segment AB = c, they draw line $l \equiv A_1B_1$ which is parallel to AB and lies at distance $r = 3$ from it. Center O of the circle can be moved inside segment A_1B_1, which is a side of rectangle AA_1B_1B (*Figure 7-a, b*). It causes vertex C of dynamic Δ ABC to move and area of Δ ABC to change.
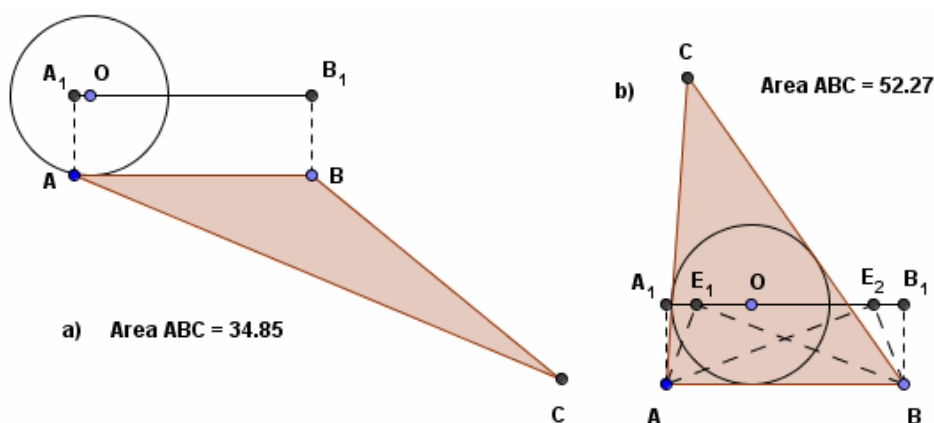

Figure 7-a, b. *Dealing with the position of center O*

The students notice that if point O belongs to some parts of segment A_1B_1, the circle with center O and radius r is not inscribed in Δ ABC (*Figure 7-a*). This raises the question how they can keep the circle "staying inscribed". They reason that O is intersection point of the angular bisectors of Δ ABC, therefore $\angle$ AOB = $180° - (\angle$ A + $\angle$ B)/2 should be obtuse. Due to that speculation they construct the two points E_1 and E_2 on the line l, for which $\angle$ A E_1B = $\angle$ A E_2B = 90°, restricting point O as internal for segment E_1E_2 (*Figure 7-b*).

The students use the **Area[Polygon]** command to calculate area of dynamic ΔABC, then record it in a spreadsheet, and point at its minimum value (*Figure 8*). Their observations, however, are not enough without theoretical support.
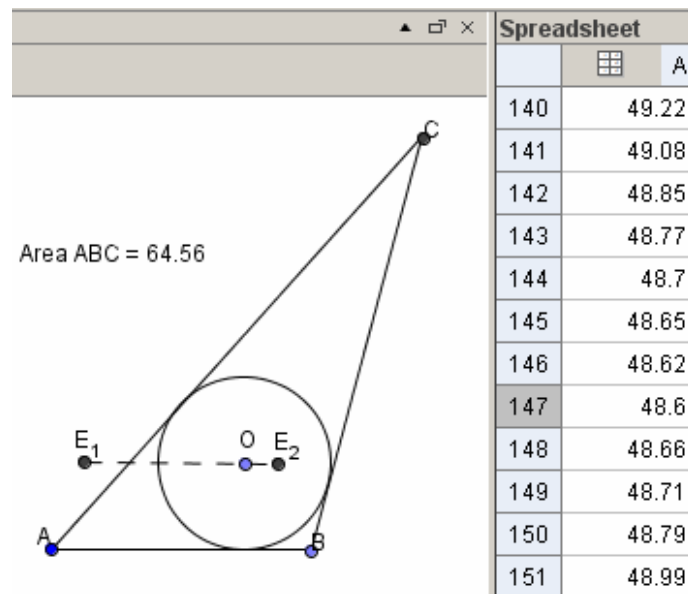

Figure 8. *Students' experiments with area of* Δ ABC

Although they have not been entirely successful, the students can only benefit from their experience with DGS. In the future they may face similar problems for which proper numerical exploration is essential.

The situation can be further developed if the teacher provokes the students to *calculate the area of a right triangle with hypotenuse c = 9 and radius r = 3 of the inscribed circle*. They will need to figure out that such a triangle does not exist.

4. Concluding remarks

The problems discussed so far have been selected not to be trivial and to promote inquiry-based learning. They allow the students to fill the gaps in their mathematical knowledge, using DGS to gather mathematical experience and learn from it. My professional observations have led me to the conclusion that before starting work on a mathematical assignment, many students need to "digitally" check it. These activities often result in unpredicted situations, which the students use to test their peers' and teachers' mathematical knowledge.

Even though mathematics is one of the most abstract fields of human knowledge, DGS have the potential to build self-confidence in mathematics and make the subject enjoyable for every student regardless of their age and knowledge levels. DGS permit the students to create and apply their own approach, as well as work at their individual pace. In my opinion, while teaching the students use DGS, we as teachers also learn and benefit. It allows me to say that in a modern math class the teacher is also a student.

References

[1] Boyer, C. B. (1991). *A history of mathematics* (2nd Edition). New York: John Wiley & Sons.

[2] Clancy, R. E. (Ed.). (2002). *A nation online: How Americans are expanding their use of the Internet.* Hauppauge, NY: Nova Science Publishers.

[3] Jukes, I., McCain, T., & Crockett, L. (2010). *Understanding the digital generation: Teaching and learning in the new digital landscape.* Kelowna, BC: 21st Century Fluency Project.

[4] *Principles and standards for school mathematics* (2000). Reston, VA: NCTM.

[5] Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.

[6] Usiskin, Z. (2001). Quantitative literacy for the next generation. In Steen, L A. (Ed.), *Mathematics and democracy: A case for quantitative literacy*. Princeton, NJ: National Council on Education and the Disciplines.

[7] Hughes-Hallet, D. (2001). Achieving numeracy: The challenge of implementation. In: Steen, L A. (Ed.), *Mathematics and democracy: A case for quantitative literacy*. Princeton, NJ: National Council on Education and the Disciplines.

Notes

[1] Alexander the Great (356 – 323 BC), creator and ruler of one of the largest empires of antiquity;

[2] *GeoGebra*: *Free mathematics software for learning and teaching*. http://www.geogebra.org, June 10, 2012.

Author

Iordanka Gortcheva, Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences; "Acad. Georgi Bonchev" Str, Block 8; 1113 Sofia, Bulgaria; **e-mail:** gorcheva@math.bas.bg