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Calculus Students' Understanding of Volume

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Abstract

Researchers have documented difficulties that elementary school students have in understanding volume. Despite its importance in higher mathematics, we know little about college students' understanding of volume. This study investigated calculus students' understanding of volume. Clinical interview transcripts and written responses to volume problems were analyzed. One finding is that some calculus students, when asked to find volume, find surface area instead and others blend volume and surface area elements. We found that some of these students believe adding the areas of an object's faces measures three-dimensional space. Findings from interviews also revealed that understanding volume as an array of cubes is connected to successfully solving volume problems. This finding and others are compared to what has been documented for elementary school students. Implications for calculus teaching and learning are discussed.

Introduction

We know from research that volume presents challenges to elementary school students (Battista & Clements, 1998; De Corte, Verschaffel, & Van Collie, 1998; Fuys, Geddes, & Tischler, 1998; Hirstein, Lamb & Osborne, 1978; Iszák, 2005; Lehrer. 2003; Lehrer, Jenkins, & Osana, 1998; Mack, 2011; Nesher, 1992; Peled & Nesher,1998; Simon & Blume, 1994). These difficulties are also reflected in student performance on standardized test items. For example, on an eighth grade National Assessment of Educational Progress (NAEP) multiple-choice question (U.S. Department of Education, 2007), students were given the dimensions of five rectangular prisms and asked which had the greatest volume. Only 75% of students answered correctly, which indicates that students may enter high school (where instruction builds on presumed competence with volume concepts) without proficiency in volume calculations. It would be useful to know if the difficulties elementary school students face persist through high school and into their study of college-level mathematics.

We conducted this study within a cognitivist framework (Byrnes, 2000), giving students mathematical tasks and analyzing the reasoning underlying their answers. This is consistent with the cognitivist orientation toward focusing on "the cognitive events that subtend or cause behaviors (e.g., [a student's] conceptual understanding of the question)" (Byrnes, 2001, p. 3). We collected written survey data and conducted clinical interviews to investigate the following research questions:

1. How successful are calculus students at volume computational problems?
2. Do calculus students find surface area when directed to find volume?

Our major finding is that nearly all students correctly calculate the volume of a rectangular prism, but many students perform surface area calculations or calculations that combine volume and surface area elements when asked to find the volume of other shapes.

Student Thinking about Volume

Volume is first learned in elementary school (NCTM, 2000; NGA & CCSSO, 2010) and, as noted above, little literature exists about calculus students' understanding of volume. Key issues that have been the focus of research include elementary school students' understanding of cross-sections.

Elementary School Students' Volume Understanding

Volume computations rely on the idea of array of cubes. A three-dimensional array is formed by the iteration of a cube into rows, columns, and layers such that there are no gaps or overlaps. Two difficulties students have are understanding an array's unit structure (Battista & Clements, 1996) and using an array to compute volume (Curry & Outhred, 2005).

These are related difficulties. One source of difficulty with using an array for computation is not seeing the relationships between rows, columns, and layers. Some students, given an array of cubes and asked to find volume, counted individualized cubes with "no global organizational schema" and seemed to view the answer as representing "a large number of randomly arranged objects" rather than a count that represented the array's volume (Battista & Clements, 1998, p. 228-229). Other researchers have concluded that elementary school students seem to see units as individual pieces to count rather than fractional parts of an initial whole (Hirstein, Lamb, & Osborne, 1978; Mack, 2011). Students who counted individual cubes neglected the innermost cubes and sometimes double-count edge and corner cubes (Battista & Clements, 1996).

Battista and Clements (1996) studied students' enumeration of three-dimensional cube arrays using written and manipulative tasks and found that only 23% of third graders and 63% of fifth graders could determine the number of cubes in a 3x4x5 cube building made from interlocking centimeter cubes. The researchers concluded that "students might see the three-dimensional array strictly in terms of its faces" (Battista & Clements, 1998, p. 229). In other words, these students may have been thinking about surface area when asked about volume.

Curry and Outhred (2005) found that students who are successful at enumerating arrays of cubes seem to have a mental picture of arrays and use a computational strategy of counting the units in the base layer and multiplying by the number of layers (Curry & Outhred, 2006) while the unsuccessful students typically covered only the base of the box. The researchers concluded that although "most students seem to have achieved a sound understanding of length and area measurement by Grade 4, the same cannot be said for volume [arrays]" (p. 272).

Some elementary school students use area and volume formulae without understanding them (De Corte, Verschaffel, & van Collie, 1998; Fuys, Geddes, & Tischler, 1988; Nesher, 1992; Peled & Nesher, 1988). In a study about students' multiplication strategies, De Corte el al. (1998) included area computation problems and found that students may multiply length times width to find area (not [as] a result of a 'deep' understanding of the problem structure and a mindful matching of that understanding with a formal arithmetical operation, but...based on the direct and rather mindless application of a well-known formula" (p. 19). Echoing this, Battista and Clements (1998) found some students use V=LWH "with no indication that they understand it in terms of layers" (Battista & Clements, 1998, p. 222). Even some prospective elementary school teachers use the A=lw formula without being able to explain why it finds area (Simon & Blume, 1994).

Secondary School Students' Understanding of Cross-Sections

Identifying the shape of a solid's cross section is difficult for middle school and high school students (Davis, 1973). This finding is important because some volumes can be thought of as V=Bh where B is the area of the base of the solid and the base is, in fact, a cross-section. This finding carries particular importance if it is also true for calculus students, as volumes of solids of revolution problems require identifying the shape of a cross-section.

The present study was designed to investigate calculus students' computations of volume, their understanding of volume formulae, as well as the issues noted above that other researchers have documented in younger students.

Research Design

Data Sources and Instrument

The data analyzed here are from written surveys completed by 198 differential calculus students and 20 clinical interviews with a subset of those students. Subjects were enrolled in differential calculus at a large public northeastern university and the researchers recruited volunteers to complete written surveys and clinical interviews. Data were collected for three semesters: spring 2011, summer 2011, and fall 2012. The university offers a single track of calculus for all majors in the physical sciences, engineering, biological sciences and education, as well as other disciplines.

Data collection and analysis had two phases. First, students completed written tasks. Since our focus was on the reasoning behind students' answers, we interviewed a subset of the students so that we could hear how students reasoned through the problems and ask questions about their reasoning. This methodology allowed for a quantitative analysis of a large number of written responses and a qualitative analysis of student thinking about those responses.

The written survey tasks consisted of diagrams of solids with dimensions labeled. Students were directed to compute the volume of the solid and explain their work. The rectangular prism task is shown in Figure 1. The other tasks were:

- A right triangular prism; triangular base 1=3 ft, h=4 ft; h_{prism}=8 ft
- A cylinder, r=3 in., h=8 in.

The complete statements of these tasks can be found in Appendix A.

Interviewees completed that written instrument but were asked to "think aloud" as they worked on the tasks. Clarifying questions were asked to probe understanding. Commonly-asked questions of this sort were "Can you tell me about that formula? Why is the 2 there?" and "Can you tell me why that formula finds volume?" Interviews were audio recorded and transcribed.

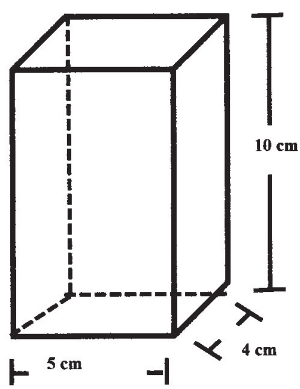

Figure 1.
Volume of a Rectantular Prism
What is the volume of the box? Explain how you found it.

Method of Data Analysis

Data were analyzed using a Grounded Theory-inspired approach (Glaser & Strauss, 1967). This entailed looking for patterns in a portion of the data and forming categories, then creating category descriptions and criteria. Those criteria were then used to code all the data, refining categories until new categories ceased to emerge. One departure from classic Grounded Theory was accessing literature prior to coding. A second was the use of anecdotal evidence that calculus students sometimes find surface area when asked to find volume. These departures informed coding in that prior to looking at data, we had ideas about what categories might emerge.

Analyzing written responses required deciding which parts of a response were relevant to the research question. We used the magnitude of the answer to judge correctness and we looked at written work (arithmetic) as it gave clues to student thinking. Units were not taken into account here, though the units students use for spatial computations is an additional issue that was part of a larger study (Dorko, 2012). Analysis was done by shape, not by student. That is, the data presented are the percent of students whose responses fell into each category for that task. The initial analysis resulted in three categories for students' work; volume, surface area [instead of volume], and other. The categories and their criteria are presented in Table 1. We used these criteria to develop coding algorithms for the three volume problems. An example of a task and its algorithm are shown in Figure 2. All algorithms were created in a way that sorted responses based on identifying parts of a student's work that might represent finding surface area, parts of a student's work that might represent finding volume, and an "other" category for responses that had neither of the aforementioned ideas. That is, algorithms for the other shapes are similar to the algorithm presented below (see Appendix B).

Table 1.
Categories for written responses.

Category	**Found Volume**	**Found Surface Area Instead of Volume**	**Other**
Criteria	Magnitude is the correct magnitude of the object's volume, or magnitude is incorrect for the object's volume but the work/explanation is consistent with volume-finding (i.e., multiplication or appropriate addition)	Magnitude is the magnitude of the object's surface area or the student work/explanation contains evidence of surface-area like computations, such as addition. To allow for computational errors, magnitude may or may not be the actual magnitude of surface area.	Student found neither volume nor surface area

Coding algorithm for the cylinder (correct volume 72π units^3; correct surface area: 66π units^2)

1. If the work says $\pi r^2 h$, $2\pi r^2 h$, 72π, or 144π, categorize as "found volume." If not, go to step 2.
2. Did the student write $\pi r^2 +$ ______ or $2\pi r^2 +$ ______ where ______ is something that looks like it might be πdh or some other computation that looks like an area of a lateral face? Did the student write 66π? In either case, categorize as "found surface area instead of volume." If not, proceed to step 3.
3. Categorize as "other."

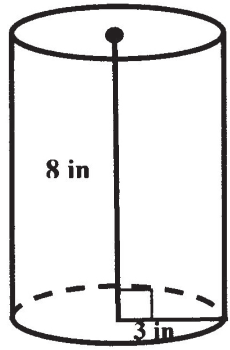

Figure 2.
Volume of Cylinder task.

The method of analysis for interview data mirrored the method of analysis for survey data. As interview data included both transcripts and students' written work, there were two parts to the analysis. First, written data were categorized according to the aforementioned algorithms. Then, transcripts were used to investigate the thinking that led to answers for each category. For instance, we looked for students who used the formulae $2\pi r^2 h$ and asked the student to "unpack" the formulae. Specifically, we looked for explanation of why the 2 was there. Did the student think the area of a circle was $2\pi r^2$? Did the two refer to the two bases of a cylinder? A 'yes' to the first would be consistent with thinking of volume as *area of base times height,* albeit with an incorrect formula for the area of the base. A 'yes' to the second would be consistent with the surface area idea of including the area of all faces. This analysis was used in two ways: to sort student formulae and to investigate why students find surface area instead of volume.

In the next section, we present the findings from our analyses. We use interview data to further explain some of these results. In particular, we use interview quotes to clarify how we classified students' volume formulae. The second section of the results contain students' success rates on the tasks, relying on coding from the written data. We return to the interview data to discuss why some students find surface area and the relationship between students' understanding of arrays and their computational success.

Results and Discussion

We present our findings in four parts. One finding is about students' reasoning and computational formulae, and since this is a good overview of the issues students have with volume, we begin with that and follow it with success rates on the problems. We then discuss why some students find surface area instead of volume. The final section is about the relationship between success on problems and understanding arrays.

Students' Volume Formulae

We believe there is an important link between students' formulae and their reasoning: that is, our data leads us to believe that students' formula are not (as is commonly assumed) remembered or misremembered, but are instead representative of ideas students have about volume. This finding, based on the synthesis of interview data with written work, let us to categorize students' formulae according to their surface area and volume elements. What we mean by "surface area and volume elements" is what we alluded to in discussing how the 2 in $2\pi r^2 h$ might from the ill-remembered area formula and might be from accounting for two bases. Categorizing students' formula in this way gave us the categories and component formulae shown in Table 3. The example given is for the cylinder; similar tables exist for each shape and are included in Appendix C. (Note the appearance of $2\pi r2h$ in both the "incorrect volume, no surface area element" and "surface area and volume elements" categories, per the reasoning stated above).

Table 2.
Categories for student responses to the cylinder task.

Correct volume	Incorrect volume, no surface area element	Surface area and volume elements	Surface area	Perimeter
$\pi r^2 h$	$2\pi r^2 h$	$2\pi r^2 h$	$2\pi r^2 h + 2\pi rh$	$d+h$
	$(1/3)\pi r^2 h$	$2\pi rh$	$2\pi r^2 h + \pi dh$	
	$(1/2)\pi r^2 h$	$2\pi r + \pi rh$		
	$(4/3)\pi r^2 h$	$\pi r^2 + 2\pi d$		
	πrh	$2\pi r^2 + 2rh$		
	$(1/2)\pi rh$			
	hdr			

This table includes all formulae that appeared in students' written work and interviews. Interview data provided help in placing the formula, and interview data are the basis of our claim that students' formulae are a reflection of their reasoning. For instance, consider Nell's reasoning about the volume of the cylinder:

Nell: I don't know the formula for this one. Two pi r squared... times the height. Sure. We'll go with that one. So you have two circles at the ends, which is two pi r squared... you have two pi r squared because that's the area on the top and the bottom so you can just double it, then you have to times it by the height.

Interviewer: Why do I have two areas?

Nell: You have two circles/

Interviewer: What about this multiplying by the height? Why do we do that?

Nell: It gives you the space between the two areas. Volume is all about the space something takes up so you need to know how tall it is.

Table 3.
Counts and percentages of students who did and did not find volume.

	Rectangular prism (n=198)	Cylinder (n=198)	Triangular Prism (n=122)
Found Vol.	194 (98.0%)	172 (86.9%)	95 (77.9%)
Found SA	3 (1.52 %)	10 (5.1%)	17 (13.9%)
Other	1 (0.5%)	16 (8.0%)	10 (8.2%)

Nell's reference to the space between two areas is indicative that she was thinking about volume. However, her formula ($2\pi r^2h$) included a surface area idea: she explained it as "the area on the top and the area on the bottom, so you can just double [πr^2]". We thus put the formula $2\pi r^2h$ in the "surface area and volume elements" category (see Table 3). It is also included in the "incorrect volume, no surface area elements" because other students talked about this formula as *area of base times height* where the area of the base was $2\pi r^2$. In this case, the two is not a nod to two bases, it is an incorrect formula for area but correct reasoning for volume.

Nell was not the only student who thought about including both circles when finding volume: Jo went back and forth about whether she could use the formula $2\pi r^2h$ or πr^2h. The interviewer asked her to make the case for both one and two circles as a way to investigate her reasoning:

Jo: The area of the circle is pi r squared times the height, but I can't decide if I need one or two circles.

Interviewer: Convince me that you need two circles.

Jo: You need two because you have the top and the bottom of the cylinder. But you don't actually need two... you just need the one. Because you get the area of the circle and you multiply it by the height... the circle is the same throughout the whole layer so you just multiply it by the height.

Jo's final reasoning was correct, but it's noteworthy that her initial response to the problem involved a surface area idea. Thus, despite her correct final response, we believe this is evidence that some students have mixed and combined surface area and volume ideas.

Most of the elements of the categories shown in Table 3 come from students' written work. No interviewee used a formula like $(4/3)\pi r^2h$ (or any of the others with a fractional coefficient for an otherwise correct volume formula), but we suspect students mixed and combined the formula for a cylinder with that of sphere, pyramid, or cone − all shapes whose volume formulas have fractional coefficients. Further, we suspect that students who use these formulae do not have an understanding of volume as *area of base times height.* Our evidence for this claim is that for a student who understands volume as area of base times height, a formula like $(1/3)\pi r^2h$ makes little sense.

The other formulae in the table provide additional evidence that some calculus students have difficulties with volume and surface area. We think answers like 'hdr' and πrh (both from "incorrect volume, no surface area elements") in which it seems the student has multiplied whatever dimensions were given (and in the latter, probably remembered that circle calculations often involve π) may result from translating a V=lwh form to a different shape. That is, we speculate that the students have a schema for volume to the effect of "volume is the product of measured attributes" and roughly equated V=lwh to V=hdr. Other instances of this included multiplying all of the dimensions given in the triangular prism; for instance, students' volume calculations included formulae like and 3 x 4x 5, (½) x 3 x 4 x 5 x 8. These lend further evidence that some students may hold "multiply whatever numbers you're given" as an accurate way to find volume, and moreover, don't understand volume as *area of base times height.*

In conclusion, we found that calculus students who are unsuccessful at finding volume often find surface area or a number that represents a combination of surface area and volume ideas. We speculate that many students do not understand volume as area of base times height, and construct formula based on ideas about area and volume. Some of those ideas include volume formula often having fractional coefficients (as in the (½) x 3 x 4 x 5 x 8 and (4/3)πr2h cases), or the more troublesome cases in which students have combined surface area and volume ideas. In the next section, we discuss the prevalence of these sorts of difficulties.

Students' Performance on Volume Tasks

The counts and percentages of students who found volume, surface area, or other for the four solids are shown in Table 4.

This table shows that 98% of students found the volume of the rectangular prism; 87% of students found the volume of the cylinder, and 78% of students found the volume of the triangular prism. We speculate that a rectangular prism is more common shape and thus students have its volume formula memorized, but struggle when they encounter other shapes. Certainly, Nell and Jo struggled with how they might find the volume of the cylinder. We also note that students were more likely to find surface area for the cylinder and the triangular prism (5.1% and 13.9%, respectively). Further research is needed to explain exactly why different shapes are harder in terms of finding volume, but we suspect the reason may be students not understanding volume as *area of base times height* or the combination some students hold of surface area and volume ideas.

Why Some Students Find Surface Area

We used interview data to examine why some students find surface area when directed to find volume. One reason is that some students seem to have combined elements from formulae, as Nell and Jo did. Despite the fact that both of these students understood volume as three-dimensional space, as evidenced by Nell's comment about volume being the "space between the two areas" and Jo's comment that the circle was "the same all the way through so we can just multiply by the height." A different reason that some students find surface area is the belief that *adding the areas of the faces measures three-dimensional space.* For instance, Geddy's description of filling an object with sugar cubes is indicative of understanding the concept of volume:

> **Geddy:** Volume is the amount of units it takes to occupy a space, like a three dimensional space. If you think of a box of sugar cubes, like a Domino box, I think when they come packaged they are usually just full of the little sugar cubes and there's no space between those cubes. So that's what volume is. It's when you have a bunch of little smaller pieces combining to fill a space without any gaps.

Geddy's "volume" computation, however, was actually a surface area computation (see Figure 3).

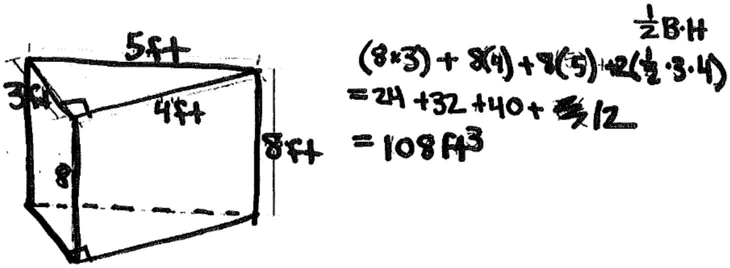

Figure 3. Surface area of the triangular prism

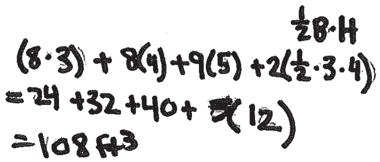

Figure 3. Next Best Version
Surface area of the triangular prism

Alex also understood volume but found surface area. She described volume as "when I think of volume I think of, like, this water bottle − what's the volume of water it can hold." Talking about holding water is evidence of understanding volume as three-dimensional space. However, Alex found the surface area of the triangular prism. Her work is shown in Figure 4.

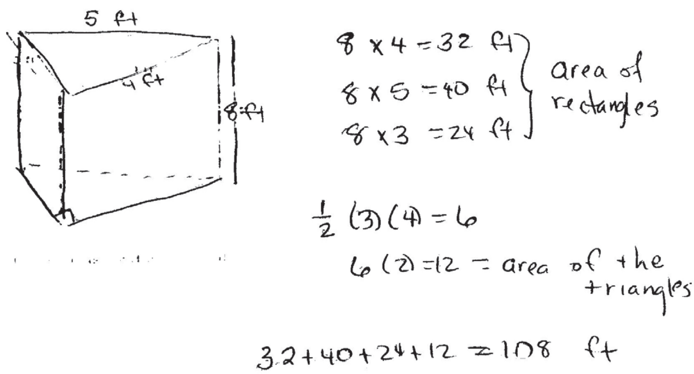

Figure 4.
Alex's triangular prism.

We asked Alex to explain her work.

Alex: I took the area of each rectangle and added that up, then I took the area of the triangles and added that to the rectangles to get the overall area. And I couldn't remember the area for the triangle. I thought it was 1/2 times base times height, which is 6. And there are 2, so 6 times 2 equals 12, so 12 is the area of the triangles... and then the area of the rectangles... and I just added them all together.

There is a discrepancy in Alex, Geddy, and other students' understanding of volume as a concept and their calculations such that these students understanding volume, but think adding the areas of the faces accounts for the measure of a three-dimensional space. This reasoning, and the combination of surface-area-and-volume formula discussed above, are the two reasons that students in this study found surface area when directed to find volume.

An understanding that seems to be connected to students' success on area and volume tasks is their understanding of arrays. We discuss this finding in the next section.

Students' Understanding of Arrays

A number of students used the formula V=1wh to find the volume of the rectangular prism. We asked interviewees to "unpack" this formula to see if they were simply reciting a formula or if they understood why it finds volume. Students' responses led us to several findings about their understanding of arrays. In this section, we discuss these findings and compare them to findings about elementary school students' understanding of arrays.

One finding about calculus students is that some students can describe the formula V=1wh in terms of relationships between rows, columns, and layers for a rectangular prism but not for other shapes. For instance, Amelia used the lwh formula for the volume of the rectangular prism and talked about an array when the researcher asked her to unpack that formula.

> **Amelia:** If we think about this in terms of area − you still have like this box [points to the 5 cm x 10 cm face], as long as you can figure out that there's like [draws a 5 x 10 array of squares on the face] so this represents 50 boxes, then you know that you have four of these... so you can think of it as having four sheets of 50 squares.

Amelia's drawing is shown in Figure 5. She has drawn 'boxes' (unit squares) on the front face and indicated the four 'sheets' (layers) along the 4 cm orthogonal face.

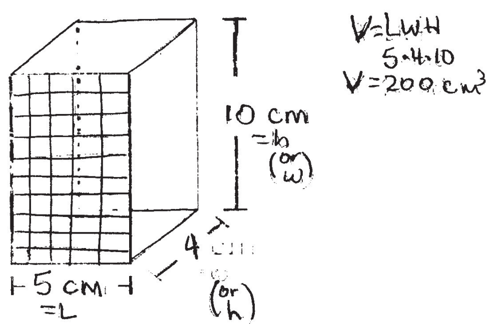

Figure 5.
Amelia's rectangular prism.

While this work is indicative that Amelia understands arrays, her attempt to apply an array to the triangular prism problem indicated that her understanding of arrays as specific to the rectangular prism. We had asked her if the "sheets" idea applied to the triangular prism, and she had trouble imagining cutting a unit cube to fit an acute angle:

> **Amelia:** Where its a triangle, you obviously can't squeeze a square into an acute angle. I guess that's why we have formulas, because we can't physically put a cubed object into that space there.

A thorough understanding of arrays would include the idea of fractions of unit cubes, an idea that eluded Amelia. She had found the correct volume of the rectangular prism, but found surface area in the triangular prism task. Geddy's work was similar: she explained volume as an array for the rectangular prism but found surface area for the triangular prism (see Figure 3). In contrast to Amelia, however, Geddy did seem to apply the array model to the rectangular prism. She said

> **Geddy:** Well, since it's 108, it's an even number of cubes. You'd be able to use squares equal to volume 1 ft cubed and you should be able to fit them all in without having any gaps.

We take Geddy's phrase about "squares equal to volume 1 ft cubed" to mean unit cubes and the statement about "fitting them in without gaps" to be indicative of an array of cubes.

A second finding about calculus students is that some do not have an array model for volume at all. Carly, one of the students who found the surface area of the rectangular prism, did not seem to have an understanding of arrays. She discussed her reasoning about the rectangular prism:

> **Carly:** I know there's an equation for volume. I don't remember what the equation is, but you know this is 5, this is 4, this is 10 [labels diagram]. I know you can find each side but I don't that gives you the volume... like find 10 times 4 so you know this side is 40 cm and this side is 10 times 5 so this side is 50. And I would just assume that this side is the same so I'd say the back side is 50 and the bottom would be 5 times 4 so that would be 20 and the top would be 20. But if you add all those together I don't think that would give you the volume because volume includes all the space in between − like in the middle of the box.

Carly had the correct idea that volume should account for "the middle" of the shape, but was not able to extend the idea of accounting for "the middle" to appropriate mathematics. Rather, she reverted to two-dimensional ideas. This leads us to the following: we hypothesize that there may be a relationship between students' array understanding and their success on these tasks.

This finding is strengthened by a number of students who described volume as an array or as layers and answered all of the problems correctly. For instance, Luke found the correct volumes for all shapes. He talked about depth and planes in the rectangular prism, which we consider analogous to layers in an array:

> **Luke:** The volume of the box is 10 cm x 4 cm x 5 cm, and that is 200 cm^3. I think of it as having an area, which is one plane, and you're multiplying it over 4 cm so you multiply your one plane by the depth of the object and that gives you the volume.

Wendell, who also found volume on all the tasks, discussed the volume of the rectangular prism similarly:

> **Wendell:** I'm a hands-on kind of person so I think it would be easiest to explain by giving them 5 one-centimeter cubes and show them that's one stack, then do it by 4, then tell them there are 10 stacks high. Then you tell them if there are 20 in the bottom and 10 stacks high, 20 x 10 is how you find the volume.

We did not ask Luke and Wendell to sketch arrays for the other shapes, but based on their descriptions for the rectangular prism array and their success on the other task, we suspect they would have sketched and described accurate array models for these shapes. Furthermore, we suspect that having these models is related to their computational success across all of the tasks. In contrast, the less-robust understanding of arrays in the other students (Geddy, Amelia, and Carly) may be related to their surface area finding in other tasks. Carly found surface area on all the tasks and did not understanding arrays; Geddy found surface area on two of the tasks but did understand arrays; Amelia understood an array only in terms of the rectangular prism and found volume for that task but surface area on the others. Luke and Wendell understood arrays and solved all of the problems correctly. We believe these data suggest that having an array model of volume for a shape has some connection to successfully finding volume of that shape (but not necessarily others), while not having an array model of volume for a shape may be connected to surface area computations (or computations involving both surface area and volume elements).

These calculus students' difficulties with arrays are similar to the difficulties faced by elementary students about the same topic. Elementary school students have trouble understanding the unit structure of an array (Battista & Clements, 1996) and using an array to compute volume (Curry & Outhred, 2005). We found that some calculus students, like elementary school students, have trouble with the structure of an array (e.g., Carly's work.) However, while elementary school students often do not see the relationship among the rows, columns, and layers in an array, many calculus students do (e.g., Wendell and Luke) and can use them for computation. Between the extremes of 'no array model' and 'array model', there are calculus students who have a model for a rectangular prism but not other shapes. We suspect a student's array model, robust or otherwise, is related to computational ability. In conclusion, while some calculus students' have overcome the difficulties elementary school students face with arrays, others continue to struggle with array models and their use in volume computation.

In the next section, we state some final conclusions as well as implications for instruction and suggestions for further research.

Conclusions, Implications for Instruction, and Suggestions for Further Research

One of the questions this study sought to answer was "How successful are calculus students at solving computational volume problems?" Success depends on shape. For the rectangular prism, 98.5% of calculus students found volume; 94.5% of students found the volume of the cylinder; and 84.2% of students found the volume of the triangular prism. It's important to note that students were more successful with the rectangular prism than the assumedly less familiar cylinder and triangular prism.

This may have implications for volume-finding in calculus; for instance, volumes of solids of revolution are rarely elementary shapes. A related finding with relevance to volumes of solids of revolution is that students who successfully found volume often thought of it in terms of *area of base times height* or as an array with layers. The base or a layer is a cross-section of the solid. Solving a volume of revolution problem often requires identifying the shape of a cross section and an expression to represent its area. The area expression is then integrated to find the volume of the solid. We think this is similar to *area of base times height* and to a layer model of volume because the integration sums the volume of infinitesimally thin layers (cross-sections). We suggest that instructors include these models of volume as part of instruction about volumes of solids of revolution.

The other research questions concerned whether or not calculus students find surface area when directed to find volume. As with volume-finding, the percentage differs by shape (1.5% for the rectangular prism, 5.5% for the cylinder, and 15.2% for the triangular prism). Further research is needed to know exactly what causes the differences in surface area finding for different shapes, but findings from the present study provides some insights as to why students find surface area at all. One reason is that some students think that adding the areas of faces measures three-dimensional space. A second is that some students have a clear conceptual understanding of volume, but blend of surface area and volume elements in computational formula. This has important implications for calculus learning, particularly in optimization problems. Standard optimization problems require students to minimize the surface area for a given volume (or vise versa). Students must construct formulae for both, solve one for a variable (often height) that can be substituted into the other equation, and only *then* can a student begin the calculus involved. It's possible that difficulties with optimization may be linked to these first few non-calculus steps. While further research is needed to confirm if this is the case, we suggest that instructors provide opportunities for students to revisit surface area and volume concepts and formulae, and perhaps give students these formulae on exams to ensure that calculus knowledge, rather then geometry knowledge is tested.

Issues Shared By Calculus Students and Elementary School Students

Finding surface area when directed to find volume is an issue that has been documented with elementary school students. In a Battista and Clements (1998) study, three tasks were given in which third- and fifth- grades students were directed to find the volume of a three-dimensional array of cubes. About 18% found surface area using pictures of a 4x2x2 array, a 4x3x3 array and a manipulative 3x4x5 array. In this study, 1.5% of students found surface area of a picture of a rectangular prism. We conclude that, at least for this shape, calculus students are more successful than elementary school students at finding volume, but it exists as an issue in both populations.

An additional issue shared by calculus students and elementary school students is that some students from both populations struggle with representing volumes with arrays and using the arrays for volume computations.

Implications for Instruction

In addition to the instructional implications mentioned above, we think instructors can use students' computational formulae to diagnose their ideas. Our findings indicate that students' formulae are often indicative of ideas they hold about surface area and volume, such as Nell and Jo's thoughts about whether or not to include the two bases in finding the area of the cylinder. We think that sorting students' formulae, as we did in Table 3, can be useful to identify ideas they bring to a computation. Instruction can then target those particular conceptions.

Additionally, we believe that many of students' errors result from not understanding volume as an array. We thus suggest that instructors provide students with educational opportunities to model volumes with arrays and connect the models to volume formulae. In a similar vein, we think that the conception of volume as *area of base times height* should be emphasized and, in calculus, connected to the idea of cross-sections. This could improve student success on volumes of solids of revolution, a notably difficult calculus topic (Orton, 1983).

Finally, we found that students' success in finding volume was somewhat shape-dependent. We suggest that volume learners proactive finding the volumes of a variety of shapes, rectangular solids and otherwise.

Suggestions for Further Research

We are particularly interested in how students' understanding of volume, and the surface area-volume combinations found here, are brought to bear in calculus topics like optimization, related rates, and volumes of solids of revolution. Many optimization problems use both surface area and volume. We are interested in how students who have difficulty with volume work through these problems. We suspect that, as in other areas of research about calculus learning, the issues students have with calculus topics is rooted in issues with underlying concepts. Further research is needed to investigate if this is also the case with volume and the calculus topics that use it.

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APPENDIX A: RESEARCH INSTRUMENT

What is the volume of the box? Explain how you found it.

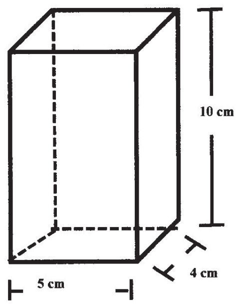

Figure 5. Volume of Rectangular Prism Task

What is the volume of the prism? Explain how you found it.

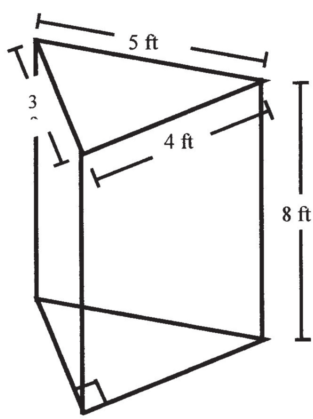

Figure 6. Right Triangular Prism Task

What is the volume of the cylinder? Explain how you found it.

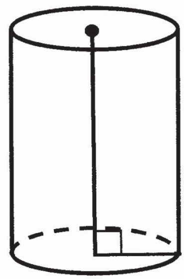

Figure 7. Volume of Cylinder Task

APPENDIX B: CODING ALGORITHMS

Rectangular prism:

Correct volume: 200 [units^3]

Correct surface area: 220 [units^2]

1. Did the student multiply the length, width, and height? This might be expressed as "lwh" or "5x4x10." Did the student write 200? In either case, categorize as "found volume." If not, proceed to #2.
2. Did the student find the areas of several faces? If so, proceed to #2a. If not, proceed to #3. Did the student write 220? If so, categorize as "found surface area instead of volume." Did the student find the areas of several faces but not add them? Categorize as "found surface area instead of volume."
   a. Did the student add those areas? If so, categorize as "found surface area instead of volume." If not, proceed to step 3.
3. Categorize as "other."

The researcher chose to put an answer in which the student found the areas of the faces but did not add them as "found surface area instead of volume" because finding the areas of many faces is *closer* to finding surface area than it is volume. Further, a student who finds the areas of many different faces clearly does not understand volume as lwh or V=Bh and thus that response certainly does not fit into a "found volume" category. The researcher thinks that finding the areas of several faces, even if the student didn't add those, is close enough to finding surface area that it should be categorized as such.

Triangular prism:

Correct volume: 48 [units^3]

Correct surface area: 108 [units^2]

1. Did the student write the formulae V=Bh? Did the student write the formula ½lwh? Did the student write 48? Did the student write 96 If so, categorize as "found volume." If not, proceed to #2.
2. Did the student write 108? Did the student write "area of two triangles plus area of faces?" Did the student do arithmetic that was finding the areas of the three lateral faces? If so, categorize as "found surface area instead of volume." If not, proceed to #3.
3. Categorize as "other."

In this task, many students wrote something like 'volume is the area of the base times the height,' and had mathwork to that effect. This is indicative of volume-finding and was categorized as "found volume," as were answers of magnitude 48. Magnitudes of 96 were also classified as "found volume" because it appeared that some students used the idea of *½lwh*, but forgot to multiply *lwh* by ½. If the student wrote 108, the correct surface area, the response was categorized as "found surface area instead of volume." If the student showed arithmetic that was finding the areas of lateral faces, the response was immediately categorized as "found surface area instead of volume" because the volume computation for a triangular prism simply does not involve finding the areas of the lateral faces.

APPENDIX C: FORMULAE BY VOLUME AND SURFACE AREA ELEMENTS

Correct volume	Incorrect volume, no surface area element	Surface area and volume elements	Surface area	Perimeter
Bh or lwh	[Cateogry DNE for this shape]	[Category DNE for this shape]	2lh+2wh+2lw	l + w + h

Table 4. Categories for the rectangular prism

Correct volume	Incorrect volume, no surface area element	Surface area and volume elements	Surface area	Perimeter
(1/2)Bh (1/2) lwh	(1/2)*3*4*5*8 (1/3)(5*3*4*8)	2*(3*4*5*8) 2[5*4*3]8	2(.5*3*4) + 8*5 + 8*4 + 8*3	3+4+5+8
	3*4*5*8 lwh, 8*4*3 w*w*h = 7*5*8 = 280 ft (3+4)(5)(8)			