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From Number Lines to Graphs in the Coordinate Plane: Investigating Problem Solving Across Mathematical Representations

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This article reports on students' problem-solving approaches across three representations—number lines, coordinate planes, and function graphs—the axes of which conventional mathematics treats in terms of consistent geometric and numeric coordinations. I consider these representations to be a part of a *hierarchical representational narrative* (HRN), a discursive narrative around a set of representations that model conventional mathematics in structurally consistent ways. A paper-and-pencil assessment was administered to students in grades 5 and 8 along with videotaped interviews with a subset of students. Results revealed students' application of particular meta-rules, which reflect their attempts to find and make use of recurring patterns in mathematics discourse. One such meta-rule, consistent with the HRN, was characterized by students' coordination of geometric and numeric properties of an axis, whereas alternate meta-rules reflected coordinations inconsistent with conventional mathematics. Detailed analyses of problem-solving strategies are reported, and implications for theory, curriculum, and instruction are discussed.

Researchers and mathematicians have highlighted the promise of geometric representations of quantities—including the number line, the coordinate plane, and functions on the plane—to support students' understanding of the number system (Bass, 1998; Saxe et al., 2010; Wu, 2005, 2009) and algebraic functions (Carraher, Schliemann, & Schwartz, 2008; Kaput, 2008; Schliemann, Carraher, & Caddle, 2013). Such claims are consistent with widely adopted standards in the United States, in which geometric representations of quantities are introduced in grade 2 and appear in every grade through grade 12 (National Governors Association Center for Best Practices [NGA Center] & Council of Chief State School Officers [CCSSO], 2010). In this article, I explore a conjecture about how students solve problems across three such representations. Recent research argues that a rich and flexible understanding of a number line's mathematical properties are rooted in a conceptual coordination of geometric and numeric properties of the line (Saxe et al., 2010; Schliemann et al., 2013); I investigate this geometric and numeric coordination in the context of the number line as well as the coordinate plane and function graphs. Rather than consider each representation in isolation, the present study investigates the possibility that solution

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approaches across representations may reveal continuities in students' understandings due to structural similarities across representations, even if students encounter such representations in different grades. Findings from this cross-sectional assessment and interview study provide evidence of continuities in problem-solving approaches among students in grades 5 and 8 across the three geometric representations of quantities.

As indicated in the name, geometric representations of quantities have both geometric and numeric components to them: Linear units are coordinated with numeric units. In one common treatment in conventional mathematics, a linear unit is defined by tick marks with consistent numeric units to indicate intervals on one axis or two perpendicular axes. While mathematically speaking the number line, coordinate plane, and functions on the plane have no order or positionality with respect to one another, school mathematics introduces children to each of the three representations at about ages 7, 11, and 14, respectively. Given this sequence, a premise of this study is that the representations introduced to students in middle and high school mathematics often have important conceptual roots in elementary school instruction.

Standards documents introduce the three focal representations across seemingly disparate instructional grades, with number lines introduced in grade 2, the coordinate plane in grade 5, and function graphs in grade 8 (National Council of Teachers of Mathematics [NCTM], 2000; NGA Center & CCSSO, 2010). Yet standards also reflect how content may be sequenced in generative and hierarchical ways so that ideas build on and connect with each other and are developmentally appropriate for how children at particular ages reason about the world around them (Case, 1991). Citing Schmidt, Houang, and Cogan (2002), the Common Core State Standards in Mathematics emphasize that the sequence of topics within and across grades was designed to reflect "the key ideas that determine how knowledge is organized and generated within" mathematics (NGA Center & CCSSO, 2010, p. 3). Schmidt, Wang, and McKnight (2005) further made clear that practices of school mathematics cohere when they "reflect, where appropriate, the sequential or hierarchical nature of the disciplinary content from which the subject matter derives" (p. 528). In the case of the three geometric representations of quantities, linear and numeric units and their coordination are first introduced with the number line and then subsumed within representations introduced in later grades. The line's early structure ideally serves as a generative tool as students progress through elementary, middle, and high school. For example, consider a high school student interpreting rate of change on a linear function graph. To do so, that student may first determine two ordered pairs by projecting between points along each axis and the function line; in turn, the value of points along each axis is determined by the inner workings of a single number line, a representation students typically first encounter in grade 1 or 2.

While in conventional mathematics there is no underlying sequence to the three representations in question, school mathematics imposes an order for two reasons. First, school mathematics reflects decisions about sequence: which mathematical concepts to teach and when to teach them. Yet as one may observe, this is not simply a matter of sequence of content but also one of development. A second grader understands quantities in the world in ways that are qualitatively different than a high schooler. A second reason, therefore, is that children possess an increasingly more complex mental representation of quantities across different points of development (Case, 1985, 1991). We can interpret school mathematics as necessarily considering a hierarchy of content and representations for these two reasons.

The group of representations under consideration share key coordinations of geometric and numeric properties, and for this reason I consider them to constitute a particular narrative within mathematics discourse. By narrative, I refer to what Sfard (2007) defined as "a description of objects, of relations between objects, or activities with or by objects" that is part of a particular discourse (p. 574). The number line is the initial representation to support geometric interpretations of number. Discourse involving subsequent representations, such as the coordinate plane and then functions plotted on the plane, reflect quantities and quantitative relations in complex ways that subsume the geometric structure of the number line. This article considers such a group of representations as a part of a *hierarchical representational narrative* (HRN), which I define as a discursive narrative around a set of representations that model conventional mathematics in structurally consistent ways. In the case of geometric representations of quantities positioned across grades, discourse involving linear function graphs subsumes discourses on number lines and the coordinate plane. Prior research has pointed to the idea of a hierarchical narrative involving one representation used to support the meaningful introduction of another representation (Goldin & Shteingold, 2001; Saxe, Shaughnessy, Gearhart, & Haldar, 2013). Examples include teaching experiments that leverage number line understandings to introduce the plane (Schliemann et al., 2013) or that use similar rectangles plotted on a coordinate plane to introduce simple linear functions (Boester & Lehrer, 2008). Nevertheless, perhaps because geometric representations of quantities are typically associated with disparate grade levels, research rarely considers problem-solving patterns across them. As a result, little systematic research exists on students' reasoning that may reveal continuities in how mathematical objects mediate talk and action across representations, whether or not such continuities are consistent with convention.

Consideration of an HRN involving the number line, the coordinate plane, and function graphs may illuminate a seamless and developmentally appropriate discursive use of representations across the K–12 sequence. The number line illuminates a geometric structure for number and quantity. Building on this, the plane, comprised of two orthogonal number lines, allows a geometric representation of relations between two quantities. Given the premise of this study, an underlying assumption is that as young students eventually use graphs to represent and analyze mathematical functions on the plane, they may build on their understanding of the syntactic rules of the coordinate plane by itself. In turn, that understanding depends on their knowledge of the inner workings of a single number line. This research considers that an HRN in school mathematics has the potential to support rich and generative understandings by allowing students to bootstrap their prior understandings as one representation is subsumed within another; at the same time, this research also considers the possibility that students' understandings may reflect alternate trajectories of discourse development.

STUDENTS' INTERPRETATIONS OF GEOMETRIC REPRESENTATIONS OF QUANTITIES

I consider here students' coordination of *numeric units* and *linear units* as they reason about geometric representations of quantities. Numeric and linear units are two independent systems that must be conceptually coordinated to make meaning of intervals featured in geometric representations of quantities (e.g., Carraher, Schliemann, Brizuela, & Earnest, 2006; Gravemeijer & Stephan, 2002; Saxe et al., 2009, 2010, 2013; Schliemann et al., 2013; Treffers, 1991). I define numeric units as units of discrete quantity, and they are a focus of instruction beginning

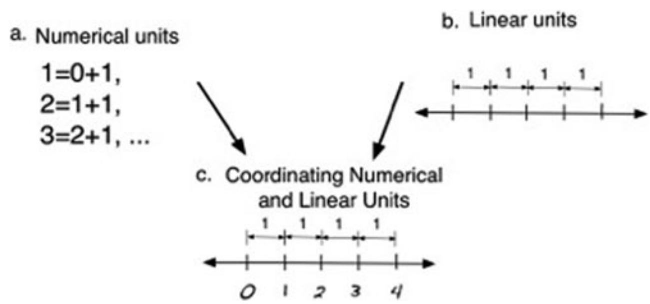

FIGURE 1 (a) Numeric units and (b) linear units that are (c) coordinated in the design of a number line (after Saxe et al., 2013).

in preschool and early primary grades (Figure 1a). Building on Saxe et al. (2013), I define linear units as the congruent segments that are concatenated to constitute a single line (Figure 1b). These two representational systems may function independently. Numeric units and counting principles may be learned without any linear context, and at the same time, linear units may be analyzed and utilized in contexts without reference to numeric units, whether such linear units are inscribed on paper or embodied in tools or manipulatives like Cuisenaire rods. Conventional mathematics[1] reflects the coordination of these two different types of units in the design of a number line (Figure 1c) and the axes of the plane. Rather than existing in either of these two unit types, discursive meaning involving the mathematical objects exists through the coordination of objects, or to which Nachlieli and Tabach (2012) refer as existing "'in between' symbols rather than in any one of them" (p. 11). In this section, I consider conventional mathematics as well as prior research on cognition involving each of the three geometric representations of quantities. As I present prior work, I reflect on how it may inform an understanding of each representation as a coordination of linear units and numeric units.

Number Lines

The number line is a line of infinite extent on which positions mark magnitudes from zero. The canonical number line features equal partitioning (linear units) with consecutive integers (numeric units) inscribed below successive tick marks and is a staple of elementary instruction (Bass, 1998; Carraher & Schliemann, 2007; Carraher et al., 2006; Corwin, Russell, & Tierney, 1990; Kaput, 2008; Saxe et al., 2007; Wu, 2009). Standards recommend the number line from grades 2 to 6 for varied content such as measurement, geometry, fractions and decimals, ratio and proportional relationships, the number system, and statistics and probability (NGA Center & CCSSO, 2010). While the number line is typically introduced in grade 1 or 2 (e.g., see standard 2.MD.6), mathematics content modeled by the number line advances across elementary grades;

[1]Note that the narrative of coordinating linear and numeric units in this way is one endorsed narrative of the discipline, yet other narratives are also endorsed in which intervals take on a different mathematical meaning, such as logarithmic number lines (in which equal intervals represent exponential increases in the underlying quantity for a given base) or bar graphs (in which intervals may represent discrete rather than continuous quantity).

students are expected to understand positive integers and rational number placement on the line by the end of elementary school. Although not reflected in recent standards, research has also indicated that upper elementary children can meaningfully position negative integers on the line as well as positive integers and fractions (Saxe et al., 2010).

In this article, I treat the number line as a tool on which, once any two numbers are positioned, the precise location of all numbers is determined through a coordination of linear and numeric units. To illustrate a coordination and a non-coordination, consider the number line in Figure 2a. Because 0, 1, and 2 are placed on the line, thereby establishing the length of a unit interval on this line, the location of all numbers is determined. A child that coordinates linear and numeric units may identify the unit interval and iterate this twice to the right to position 4 at the endpoint of the resulting linear distance (Figure 2b). Alternatively, a child's understanding of number line properties may reflect numeric order relations without coordination of the two unit types. For example, a child might position 4 to the right of 2 without regard for unit interval, resulting in incongruent intervals (Figure 2c). Despite the ubiquity of number lines in elementary school and the foundational idea of congruent unit intervals (e.g., Lehrer, 2003; Lehrer, Jaslow, & Curtis, 2003), children often do not conceptualize the number line in terms of geometric and numeric coordinations (Saxe et al., 2007, 2013).

Coordinate Plane

The coordinate plane is comprised of two orthogonal number lines with an intersection point at the origin. This geometric representation involves a coordination of linear units and numeric units on two independent axes. The plane has additional representational features not available with a single number line, including the property that a point in the plane represents two quantities' values. A point is named based on the intersection of two projections, one from a value on the horizontal axis and the other from a value on the vertical axis. As a part of the HRN, the plane subsumes properties of linear and numeric units with the number line, while at the same time the two orthogonal number lines enable a more complex way to highlight quantities in two dimensions.

U.S. state standards introduce the plane in grade 5 geometry as two perpendicular number lines on which to plot points (NGA Center & CCSSO, 2010). Reflecting the hierarchical narrative, the standard for an ordered pair (5.G.1) states that students should "[u]nderstand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis" (p. 38). Each of those distances traveled on either axis corresponds to the mathematics of a number line. Nonetheless, children tend to experience the canonical coordinate plane, which features equal intervals with identical

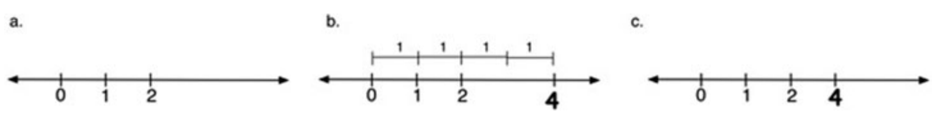

FIGURE 2 A number line (a) featuring 0, 1, and 2, thereby providing the length of the unit interval; (b) with 4 positioned at the appropriate linear distance; and (c) with 4 positioned without regard for linear distance.

scale on each axis (Leinhardt, Zaslavsky, & Stein, 1990). Such a treatment does not necessarily allow children to grapple with the two unit types.

In spite of the canonical treatment of the plane, the scale of each axis in conventional mathematics need not be the same, a property that may remain hidden in the typical unit scaling of axes in mathematics education. The canonical treatment may obscure the structural similarities between the plane's axes and an independent number line. A pitfall of this common instructional treatment is that instead of building on students' prior understandings of a number line, students may come to understand the coordinate plane in ways that do not consider geometric properties of the axes. We may imagine an alternate interpretation for the plane in which an ordered pair is determined by discrete numerals inscribed at some location along the axis instead of indicating distance traveled from the origin along either axis.

Function Graphs

Function graphs are a geometric representation of a relationship between quantities. In such, they display the joint variation of quantities—the change of one quantity in relation to a given change of the other quantity—such as the joint variation of time and distance. Geometric representations of mathematical functions are realized as a coordination of the function's ordered pairs with values along each axis, thereby providing a representation of both a quantitative relationship and the variation (Kaput, 1987). While standards do not suggest mastery of function graphs until grade 8 (NGA Center & CCSSO, 2010), research has suggested function graphs are a promising representation in upper elementary school to reason with functions and support the development of algebraic reasoning (Brizuela & Earnest, 2008; Kaput, 2008; Schliemann et al., 2013). The HRN suggests that, like the number line and the coordinate plane, conceptual activity with the function graph may involve coordinations of linear and numeric units in order to interpret ordered pairs or overall trends of a linear function.[2] At the same time, this coordination may not be salient for students. In such cases, students would solve new problems or nonroutine problems in ways that do not build on those underlying coordinations across the hierarchical narrative.

Upon the introduction of graphs of linear functions, the wording for the relevant standard (8.F.1) reflects the HRN, stating that, "[t]he graph of a function is the set of ordered pairs consisting of an input and the corresponding output" (NGA Center & CCSSO, 2010, p. 55). Given that ordered pairs were defined for grade 5 standards as the distance traveled on two axes, the provided language is consistent with a hierarchical narrative of these geometric representations across grades. At the same time, the sequence of content traditionally positions function graphs much later than number lines, thereby potentially obscuring structural similarities from a learner's perspective.

Like the coordinate plane, the scale of each axis for a function graph need not be the same. In the case of function graphs, the scale of the axes is consequential to the orientation of the line in the plane, particularly with respect to common descriptive properties (e.g., the steepness of the function line). While geometric representations of quantities are not typically considered in terms of linear and numeric units, I once again consider the coordination of these units as critical

[2]A common exception to this are graphs without any values at all that are used in instruction to highlight observable trends in the shape of a graph. I do not consider this type of graph in the present study.

for engaging in relevant mathematics discourse. For example, a salient feature of linear function graphs is slope, the ratio of change of the dependent variable to a unit change in the independent variable. In the geometric representation, slope refers to both a unit change on the horizontal axis—the linear unit corresponding to the numeric inscriptions 0 and 1 (or another interval of 1)—and the corresponding interval of change on the vertical axis. Nonetheless, many students come to interpret slope by drawing upon irrelevant or disconnected representational features, such as the change in a single variable (e.g., a change in *y*), the orientation of the line in the plane (e.g., steepness), or gridline units independent of values along the axes (Caddle & Earnest, 2009; Lobato, Ellis, & Muñoz, 2003; Zaslavsky, Sela, & Leron, 2002).

COGNITIVE FRAMEWORK

Conventional representations do not have meaning a priori; rather, the meaning of any representation—as in, what the representation represents—depends on a user's interpretation (Von Glassersfeld, 1987). Typically, students do not immediately understand a representation in the mathematically conventional sense. Rather, mathematical understandings emerge and are developed in the context of discourse and social activity[3] (Case, 1991; Cole, 1996; Piaget, 1965/1995; Saxe, 2012; Saxe, Guberman, & Gearhart, 1987; Schliemann, 2002; Sfard, 2007, 2008; Stevens & Hall, 1998; Vygotsky, 1978, 1986). The ways in which an individual interprets the mathematical objects of a representation—or, how particular signifiers are "realized" in discursive activity (Nachlieli & Tabach, 2012; Sfard, 2012)—is ideally consistent with conventional mathematics. A concern of the present study is in revealing discursive rules at play for students related to linear and numeric units as they problem solve.

Mathematics discourse refers to a particular type of communication that is identifiable as mathematical through four features (Sfard, 2007, 2012). These include: *narratives*, defined above as relations between or activities with objects and includes more broadly definitions, theorems, and algorithms; *rules* and *meta-rules* (to which Sfard refers as "routines") characterized by well-defined, repetitive patterns; unique *visual mediators*, such as the numbers and linear units of the focal geometric representations of quantities; and distinct *words* used in school or conventional mathematics, such as *function* or *half*. Sfard (2007, 2012) treats development as changes in discourses. In this section, I further elaborate on the discursive role of meta-rules and visual mediators.

As students engage in the discourse of school mathematics, they begin to interpret the content and problem solving through meta-rules (or meta-discursive rules) that reflect participants' experiences in finding and making use of discursive patterns (Kjeldsen & Blomhøj, 2012; Sfard, 2012). Meta-rules reflect repetition in the ways in which problems are typically posed in instruction and experienced by students. Meta-rules govern particular actions in discursive activity; in other words, they indicate both when to do what and how to do it (Sfard, 2008). Ideally students'

[3]The classroom is one important context of structured activity that supports the development of mathematical concepts, although mathematics may also be learned through out-of-school contexts, including (but not limited to) store exchanges (Brenner, 1998; Saxe & Esmonde, 2005; Taylor, 2009), street vending (Carraher, Carraher, & Schliemann, 1985; Saxe, 1988; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998; Sitabkhan, 2009), religious practices (Taylor, 2013), and sports (Nasir, 2000; Nasir & Hand, 2008).

take-up of meta-rules is consistent with conventional mathematics, although a premise of the present study is that alternate meta-rules that conflict with convention—yet that nonetheless indicate when to do what and how to do it—may be at play for students as they encounter particular problems.

The meta-rules being investigated for this study are inseparable from the symbolic artifacts of the written representation. Sfard (2007) refers to such artifacts as *visual mediators*, which "are means with which participants of discourses identify the object of their talk and coordinate their communication" (p. 573). As mediators, such objects of talk and action are more than mere auxiliary support; rather, mathematical objects are inseparable from an individual's communicative goals and thought processes. The various symbolic artifacts of geometric representations of quantities therefore serve as visual mediators that, for individuals making meaning in social contexts, speak to a particular meta-rule for solving that type of problem.

We may consider variations in meta-rules across students to suggest different mediating roles of the symbolic artifacts of the representation. Consider the number lines featured in Figure 2. We may imagine that the objects of the number line in Figure 2a serve particular mediating roles leading to approaches (e.g., Figure 2b and 2c) that reflect individuals' take-up of a particular meta-rule. One may conceptually coordinate linear and numeric units to iterate twice the unit interval bound by 1 and 2 to position 4 two unit intervals to the right of 2. Alternatively, a student may position 4 on the right side of 2 by adhering to order relations and following the established linear pattern. In Figure 2b, linear and numeric units are coordinated with one another. In Figure 2c, tick marks serve as positions for naming numbers in some increasing order yet where linear units are not coordinated with numeric units to determine the placement of a value.

The implication of this cognitive framing is twofold. First, this framing implies that individuals applying a meta-rule for function graphs in which linear units are coordinated with numeric units—a meta-rule consistent with convention—may apply that rule to the number line due to structural similarities across geometric representations. Second, individuals that apply an alternate meta-rule to function graphs in which linear and numeric units are not coordinated may apply that alternate rule to the number line. A goal of the present study is to reveal the meta-rules at play for students.

Empirical Design to Capture Student Understanding

A key challenge in identifying meta-rules at play is in capturing reliable data that may allow the various meta-rules to emerge. This is especially challenging considering that an individual's understandings are not readily visible to an outside observer. In order to address this challenge, I consider an empirical design framed around linear and numeric units with each as key visual mediators for geometric representations of quantities. I present here routine and nonroutine problems and the affordance of analyzing and comparing performances on each of them.

Routine problems feature a canonical representational treatment and, in such, are a part of well defined, repetitive patterns of problem-posing in instruction that characterize conventional mathematics. Children typically encounter routine or canonical representational treatments in day-to-day school mathematics. Though less common in instruction, nonroutine problems may also be a part of school mathematics. Two key features characterize nonroutine problems in this study. First, nonroutine problems are not routine; such problems are not a part of well defined or repetitive tasks that students typically encounter. Second, while such problems from the perspective of conventional mathematics may be considered ill-formed, school mathematics may legitimately utilize such problem design for assessment or instructional purposes, as nonroutine problems may highlight unconventional meta-rules at play (see also Kjeldsen & Blomhøj, 2012). Nonroutine problems and the unearthing of meta-rules have the capacity to reveal how objects of the representation mediate students' thinking and communication. Such discursive features may remain hidden or unnoticed in the context of routine problems alone. Research has shown that nonroutine problem designs have bearing on children's interaction with the task (e.g., Bouwmeester & Verkoeijen, 2012; Saxe et al., 2010, 2013; Schliemann, 2002) and, consequently, nonroutine tasks may serve as a window into the ways in which representational features visually mediate thinking and communication.

Both routine and nonroutine problems serve critical functions in school mathematics. Routine representations reflect canonical design and, therefore, conventional mathematics. For example, a foundational concept for second graders involves additive relations; a routine, equally partitioned number line with consecutive positive integers provides a geometric realization of number for children to use "hops" to record distance traveled, thereby enabling a geometric exploration of magnitudes. In fact, standards are explicit about a routine number line with "equally spaced points corresponding to the numbers 0, 1, 2 ... " (NGA Center & CCSSO, 2010, p. 20) in the grade 2 standard for measurement and data. Furthermore, a seventh grade student working with a canonical function graph with unit scaling may have important access to geometric properties of slope as embodied in the rise and run, a mathematical aspect that would be inconveniently hidden with anything but unit scaling. The routine design enables access for students.

Nevertheless, important properties of conventional mathematics are repeatedly hidden in the symbolic artifacts of routine problem design. A danger of routine problems then is that they hide in their encoding—the same encoding that distinguishes their affordances—the underlying mathematical properties and coordinations that characterize their utility in conventional mathematics. Schliemann (2002) advised caution with respect to student learning with routine problems due to "a risk of learning procedural rules without the proper understandings implicit in its procedural uses" (p. 302). An exclusive use of routine design assumes a mastery of the mathematical underpinnings reflected in the representation, thereby making that conceptual work invisible and removing it from the focus of children's activity. Problem solving in school mathematics may reflect observed patterns that have led to success on routine textbook problems yet are inconsistent with conventional mathematics.

In this study, empirical techniques involve the use of nonroutine problem design and a comparison with routine problems to investigate meta-rules at play. Consider the routine and nonroutine problems presented in Figure 3. Each features a multiunit interval from 1 to 3. The defined multiunit interval thereby determines the location of all values on the number line: the number

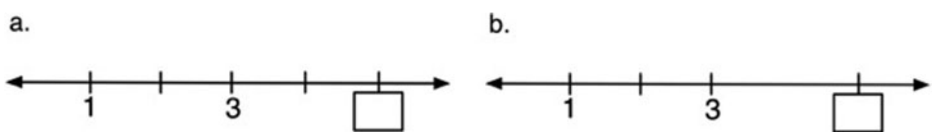

FIGURE 3 (a) Routine and (b) nonroutine versions of a problem.

belonging in each box is 5. In the case of the routine problem (Figure 3a), the equal partitioning of tick marks relieves a student from needing to consider that a coordination of the two unit types determines the unit interval. Students responding accurately to the task may have considered and coordinated the two unit types or, alternatively, may have counted-on by consecutive numbers from 3 without necessarily considering congruent intervals. In the case of the nonroutine problem with an omitted tick mark (Figure 3b), the unequal partitioning has the potential to reveal alternate approaches. One approach may lead to students' coordinating linear and numeric units to identify the missing point as 5. Alternatively, a student may have counted-on by tick marks to name the point 4, a meta-rule that would remain hidden with exclusive use of routine problems.

I argue that a comparison of performances across representations on routine and nonroutine tasks affords insights into the meta-rules students apply, whether or not such meta-rules are consistent with conventional mathematics. Alternate meta-rules that are inconsistent with conventional mathematics may lead to success with routine problems but do not support success with nonroutine problems. If performance on nonroutine tasks is lower than that for routine tasks, this suggests that further investigation of nonroutine problems would reveal the following: how symbolic artifacts of the representations mediate students' problem solving, and whether such approaches reflect meta-rules applied across the geometric representations.

Research Questions and Analytical Approach

The study employed quantitative and qualitative analytic techniques in order to address the following three questions: (a) Is there a difference in performance between routine and nonroutine problems in each grade? (b) What is the character of solution approaches to nonroutine problems for each of the three representations? (c) Do such approaches reveal meta-rules that individuals apply to solution approaches across the three representations?

To address these questions, I administered a paper-and-pencil assessment to students in grades 5 and 8. In my analysis, I first turn to quantitative evidence of differential performance (correct/incorrect) to consider student performance on routine and nonroutine problems. To address the second research question, I conducted clinical interviews (Ginsberg, 1997) with a subset of students in each grade. I analyzed the character of students' solution approaches, including incorrect responses, through a qualitative analysis that enabled the emergence of a coding scheme. Returning to the full data set, I apply the coding scheme from interviews back to the paper-and-pencil assessment responses. I then address the third research question in a final analysis regarding patterns in solution approaches to determine meta-rules across representations.

METHODS

Participants

Participants included 126 grade 5 students and 131 grade 8 students drawn from four elementary and middle schools in Northern California. Grade 5 students were selected for two reasons. First, they typically have not yet had targeted instruction on content involving graphs, and thereby illuminate how students without prior relevant instruction on graphs draw upon linear

TABLE 1
Breakdown of Four Schools Used in Sample

School	*Grade 5*	*Grade 8*	*Totals*
Beacon Community Charter School	50	64	114
The Penn School	47	45	92
Venkman Elementary School	29		29
Doyle Middle School		22	22
Totals	126	131	257

and numeric units. Second, recent research has emphasized the role of geometric representations of quantities, particularly function graphs, in elementary school (e.g., Schliemann et al., 2013), thereby warranting further research on elementary students' problem-solving approaches. Grade 8 students were selected because they were in the middle of an Algebra 1 course and thereby provide data on how students with instruction in relevant content respond to the same tasks.

School Information

All students were drawn from four schools in the San Francisco Bay Area: Beacon Community Charter School,[4] the Penn School, Venkman Elementary School, and Doyle Middle School. Schools were recruited in order to represent a cross section of socioeconomic status as well as race and ethnicity in the overall sample. I briefly provide some background information here about each of the schools, including details from teacher informational interviews. Table 1 displays the four schools and the number of students in the sample drawn from that school by grade, while Table 2 provides broader curricular and demographic information as available from the California Department of Education (2014). Classroom populations roughly reflected the overall school demographics reported to the state.

School 1, Beacon Community Charter School, is a tuition-free public charter school located in City A in the San Francisco Bay Area. Grades 5 and 8 both use reform-oriented curricula (see Table 2). In an informational interview, the grade 5 teacher stated that students had no instruction prior to data collection relevant to the plane or graphs. The grade 8 teacher had taught concepts related to graphing including linear function graphs, parabolas, and translations to equations.

School 2, The Penn School, is a private Quaker school located in City B in the San Francisco Bay Area with grades ranging from Kindergarten to grade 8. Many enrolled students do not live in the same neighborhood as the school. Fifth and eighth grade students from Penn participated in this study. Both grades used reform-oriented curricula. The two grade 5 teachers each stated that they explored the coordinate plane and graphs in the context of science class but not for math class. The grade 8 curriculum included linear functions in the first quadrant, as well as scatterplot graphs and the line of best fit.

School 3, Venkman Elementary School, is a public K–6 elementary school located in a third large city in the San Francisco Bay Area with students residing in the same neighborhood as the

[4]All school names are pseudonyms.

TABLE 2
School Demographics and Curricula

	Beacon Community Charter School	*The Penn School*	*Venkman Elementary School*	*Doyle Middle School*
School type	Public charter	Private	Public	Public
Grade span	K–8	K–8	K–6	6–8
Location	City A	City B	City C	City B
Curriculum				
Grade 5	Investigations in Number, Data and Space	Investigations in Number, Data and Space	Math Wise	N/A
Grade 8	College Preparatory Mathematics	College Preparatory Mathematics, other materials	N/A	Glencoe
Total Pupils (2010-2011)	474	424	541	1,144
Pupil to Teacher Ratio	23.5:1	11.2:1	17.2:1	21.9:1
Sex (F/M)	47.9%/52.1%	46.8%/53.2%	52.7%/47.3%	48.7%/51.3%
Race/ethnicity				
American Indian/ Alaskan		0.2%	0.02%	<0.01%
Asian/ Pacific Islander	7.2%	7.3%	15.7%	57.0%
Black (non-Hispanic)	11.8%	1.4%	6.8%	5.3%
Latin@[a]	78.1%	5.0%	69.3%	23.3%
White (non-Hispanic)	2.7%	62.7%	7.6%	8.2%
Two or more races		23.3	0.04%	<0.01%
Free and reduced lunch	85%	N/A	76.2%	55.8%
Adjusted tuition	N/A	26%	N/A	N/A

Note. Source: http://dq.cde.ca.gov/dataquest/dataquest.asp. Demographics reflect information reported to the California Department of Education (2014). Because research questions did not concern issues related to racial or gender identity, these data were not collected from individual students in this sample.

[a]Consistent with Gutierrez (2013), Latin@ is represented with the "@" symbol in order to represent a demographic group without proliferating artifacts of language that assert dominance of a particular gender identity.

school. The population breakdown by race and ethnicity resembles the neighborhood. Grade 5 participants in this project were using a textbook described as outdated and traditional by the classroom teacher. Students had worked with graphs prior to data collection for this project, including ordered pairs on the plane, linear function tables, and graphs in the four quadrants, though the teacher also indicated that such work was superficial and quick and in no way approached the complexity of some of the routine problems and all of the nonroutine problems in this study.

School 4, Doyle Middle School, is a public 6–8 middle school in City B, although in a different neighborhood than the Penn School. Most students reside in the same neighborhood as the school. The population breakdown by race resembles the neighborhood. Grade 8 participants in this project were enrolled in a tracked Algebra Readiness course with a textbook the teacher described as "traditional." In an informational interview, the teacher stated that she covered the slope of a line and multiple representations of functions (equations, tables, and graphs) prior to data collection for this project.

Assessment

The assessment featured a total of 32 items. Of these, 18 items were matched pairs. Each of these pairs featured one routine item and one nonroutine problem. Of the 18 problems, six were number line tasks, six were coordinate plane tasks, and six were function graphs tasks (see Appendix A). The remaining 14 problems were unmatched and included four number line tasks, five coordinate plane tasks, and five function graphs tasks and are not a focus of this analysis.

Number line tasks were adapted from assessments given as part of the Learning Mathematics through Representations (LMR) project at the University of California, Berkeley (Gearhart, Saxe, Halder, Earnest, Sitabkhan & Lewis, 2011; Saxe et al., 2009, 2010, 2013). All coordinate plane tasks were designed for this study. Assessment tasks involving function graphs were drawn from, adapted from, or inspired by examples in literature (Carraher et al., 2008; Schliemann et al., 2008; Zaslavsky et al., 2002) and curriculum development projects (Early Algebra, Early Mathematics, 2014), with other items designed for this study. Regardless of the source or inspiration, all items went through multiple rounds of piloting with students in grades 5, 6, and 8 over a period of 15 months from October 2009 to December 2010.

Assessment Administration and Scoring

Teachers administered the assessment in January or February 2011. The researcher was present to oversee each administration. Students had 30 minutes to complete the assessment; all students finished. Administration was staggered over four weeks across classrooms due to logistics of data collection at multiple research sites. All assessments were collected and scanned.

The assessment was scored immediately following administration. Each item received two codes in a database. First, I coded a response as correct or incorrect. Second, I entered each student's individual response choice.

To address the potential proximity of students to one another during assessment administration, three orders of the assessment were administered. All items were featured in each assessment order. Principles for the ordering included: (a) Matched routine and nonroutine items should not follow one another; (b) for matched items, the routine item always should appear before than the nonroutine; and (c) number line, coordinate plane, and function graphs tasks should be intermingled. Order distribution to students was determined randomly; the student closest to the teacher's desk was selected for the first order, the student closest to the first would receive the second order, the third the third order, the fourth the first order, and so on.

Interviews

A subset of grade 5 ($n = 33$) and grade 8 ($n = 26$) students participated in follow-up clinical interviews (see Table 3). All interviews were conducted in one-on-one (interviewer–interviewee) sessions. The location varied depending on the school. In all cases, the interview was conducted

TABLE 3
Subset of Sample Involved in Both Assessments and Interviews by School

School Name	*Grade 5*	*Grade 8*	*Totals*
1. Beacon Community Charter School	8	15	23
2. The Penn School	20	7	27
3. Venkman Elementary School	4	—	4
4. Doyle Middle School	—	5	5
Totals	33	26	59

Note. School representation in grade 5 interviews was in part affected by a coordinated tutorial study (not reported in this paper) that targeted lower-performing fifth graders in schools 1, 2, and 3 who were eligible to participate in either the interviews for the present study or the tutorial study but not both.

in a semiprivate location on school grounds. In the interview, the researcher provided a copy of the original assessment; students were permitted to make additional marks on the assessment copy as needed. The number of questions asked during the interview varied based on each interviewee's original assessment responses (see criteria below). In all cases, the interviewer asked the student about at least one nonroutine number line task, one nonroutine coordinate plane task, and one nonroutine function graphs task.

The interview protocol adhered to the following procedure for each item: the researcher (1) read the prompt for the target task, (2) asked the student to read her or his original answer, (3) asked the student to explain how she or he was thinking about the problem, and (4) presented a countersuggestion to the student. If the student provided a correct written solution, the researcher provided a predetermined, incorrect countersuggestion. In the case where a student provided an incorrect written solution, the researcher provided the correct response as a countersuggestion.

While all students with permission were eligible for interviews, two criteria determined the interview group: having all common responses represented and having a diversity of students (as determined by overall assessment score) represented. First, students were selected so that interviews captured common responses on the assessment; I considered a response to be common if it had a frequency of five students or greater across all participating classrooms. If possible, I interviewed at least one student from each classroom for each common response. Second, I used total score on the assessment to ensure the interview subset was representative of overall scores across the sample. This was an effort to represent in the interview subset an authentic range of students from low to high-achieving across each of the participating schools.

Interviews were recorded using a single video camera and attached flat microphone. The camera was positioned to focus on the paper and student's hands to record gestures and inscriptions as a student explained or changed a response.

RESULTS

The results are presented in three sections each linked to an organizing research question as depicted in Table 4. Across sections, my focus is on revealing the character of students' problem-solving approaches in terms of linear and numeric units in order to determine if such approaches reflect one or more meta-rules students apply across representations. First, I conduct quantitative

TABLE 4
Overview of Research Questions, Data Sources, and Analyses

Research Question	*Data Sources*	*Analyses*
Is there a difference in performance between routine and nonroutine problems in each grade?	Assessments: Correct/incorrect performance on 18 matched tasks	2 (problem type) × 2 (grade) repeated measures ANOVA
What is the character of solution approaches to nonroutine problems for each of the three representations?	Videotaped interviews of student solutions to target problems	Qualitative analysis of problem-solving approaches in terms of coordinations of linear and numeric units
Do such approaches reveal meta-rules that individuals apply to solution approaches across the three representations?	Written assessment responses	Statistical Permutation Tests

analyses to demonstrate a difference in performance on routine and nonroutine problems. In order to investigate the meta-rules at play and whether these are consistent with conventional mathematics, the overall analysis quickly moves beyond a simple correct–incorrect coding of answers. I next analyze interviews to understand how differential answers and approaches reflect the role of linear and numeric units as visual mediators, and then consider these results in light of the entire sample. Finally, I consider correlations in solution approaches across the three representations to reveal meta-rules at play.

Performance on Routine and Nonroutine Problems

A subscale of 18 problems on the assessment included nine routine items matched to nine analogous nonroutine items to create nine pairs. Recall that three pairs were number line tasks, three pairs were coordinate plane tasks, and three pairs were function graphs tasks. Each student was given one point for each correct response; incorrect responses were assigned 0 points. Means and standard deviations are presented in Table 5, and results are presented in boxplots in Figure 4. To determine whether there was a difference in performance between routine and nonroutine problems, I conducted a Two (Problem Type) × Two (Grade) repeated measures analysis of variance (ANOVA) for performance. A main effect emerged for problem type, $F(1, 255) = 375.948$, $p < .0001$, with better performance on routine problems than on nonroutine problems.

TABLE 5
Means and Standard Deviations for Each Grade for Performance on 18 Matched Assessment Problems

	Grade 5		*Grade 8*	
	Mean	*Std. Deviation*	*Mean*	*Std. Deviation*
Routine (9 problems)	6.198	1.811	7.627	1.237
Nonroutine (9 problems)	3.849	2.433	5.382	2.473

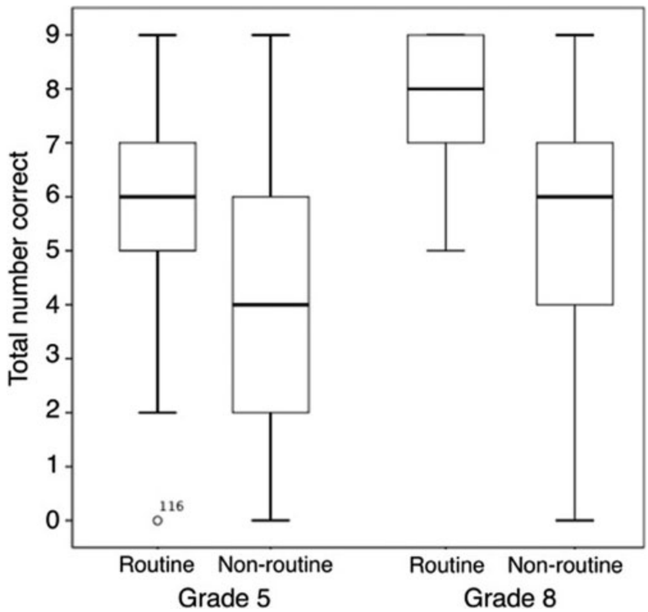

FIGURE 4 Boxplots showing assessment performance by grade and problem type.

A main effect also emerged for grade, $F(1, 255) = 44.113$, $p < .0001$, with grade 8 students outperforming grade 5 students.

There was no significant Problem Type × Grade interaction, suggesting that the discrepancy in performance on routine and nonroutine problems was roughly equivalent across grades ($p = .805$). I conducted post hoc analyses for problem type for each of the grades. Grade 5 students were significantly more successful at routine problems than nonroutine problems ($t(125) = 14.240$, $p < .0001$), and grade 8 students likewise were significantly more successful at routine problems than nonroutine problems ($t(130) = 13.247$, $p < .0001$); effect sizes for both groups from routine to nonroutine ($d = 1.095$ and $d = 1.170$) were found to exceed Cohen's (1988) convention for a large effect ($d = .80$). Regardless of grade, there was a significant decrease in score from routine to nonroutine tasks. Given the significant decrease in performance, problem-solving approaches to nonroutine tasks are a focus of further analysis.

The Character of Solution Approaches to Nonroutine Problems

In this section, I present interview data on student solution approaches on three nonroutine problems for each of the three representations. The role of qualitative data in this study is to inform further categorization of the larger quantitative data set. Recall that I conducted interviews with a subset of grade 5 ($n = 33$) and grade 8 ($n = 26$) students, with selection criteria determined by goals of representing common responses across all students as well as a range in overall student performance. For each problem, I first present interview data to establish patterns between

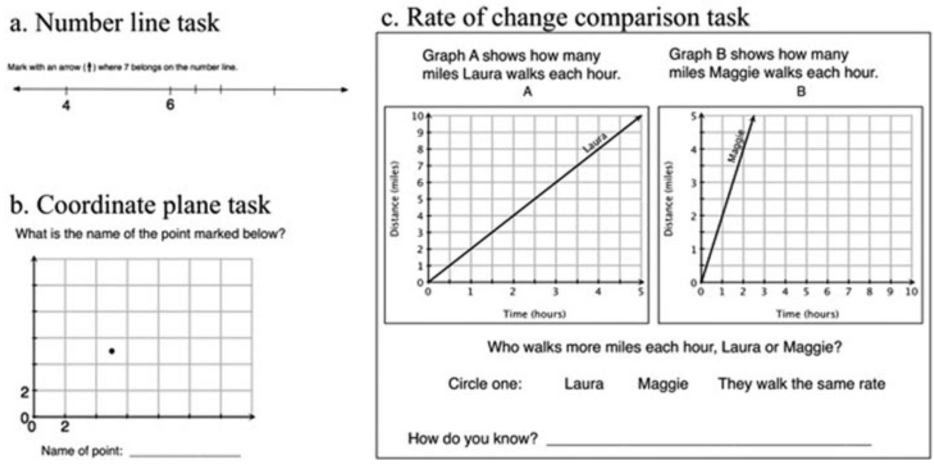

FIGURE 5 Target nonroutine problems for (a) the number line, (b) the coordinate plane, and (c) function graphs.

problem-solving approach and answer choice; the result of this analysis is a coding scheme. I then apply this coding scheme back to the paper-and-pencil assessments for the entire sample of students for each of the three target representations.

Figure 5 displays the three target problems for analysis, and Figure 6 presents the performance for students in each grade. I first present a nonroutine number line task in which students are asked to place 7 on a number line with the locations of only 4 and 6 provided. The pass rate on this problem was 34.1% among grade 5 students and 36.6% among grade 8 students. I then turn to a nonroutine coordinate plane task in which students are asked to identify the coordinates for an unnamed point that does not fall along provided gridlines. The pass rate on this problem was 41.3% among grade 5 students and 58% among grade 8 students. Lastly, I analyze performances

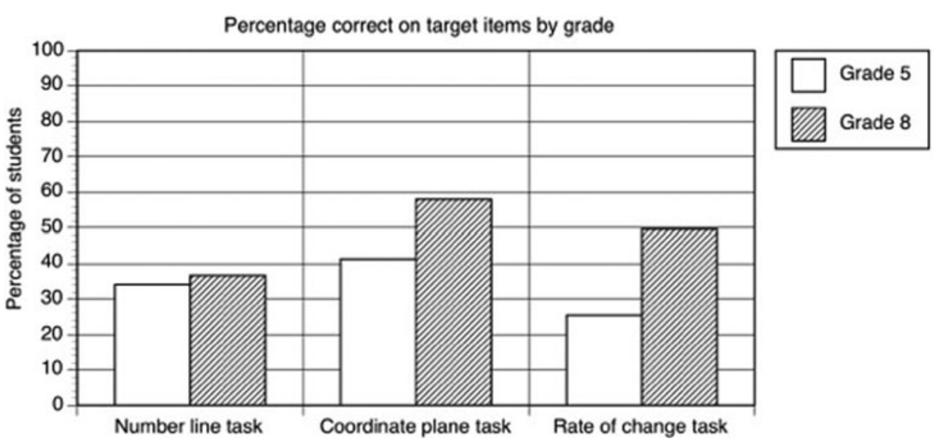

FIGURE 6 Percentage correct on target nonroutine problems for grades 5 and 8 students.

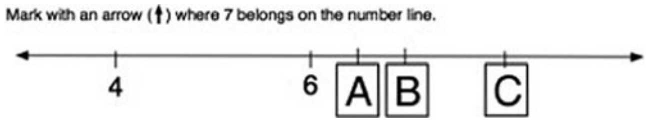

FIGURE 7 Focal number line task with tick marks labeled.

on a nonroutine rate of change comparison task in which students are presented with two graphs and asked to identify whether one displays a greater rate of change. The graphs feature the same function yet with different scales, resulting in different line orientations. The pass rate on this problem was 25.4% among Grade 5 students and 48.1% among grade 8 students. Recall that conventional mathematics may consider nonroutine tasks to be ill-formed; nonetheless, from the perspective of school mathematics, solution approaches to such problems may illuminate the mediating role the two unit types serve in children's solution approaches.

Number Line

The focal number line task assessed the coordination of linear and numeric units on a single number line. A multiunit interval of 2 was provided from 4 to 6, with no additional tick marks placed within the resulting linear unit. Because two numbers were placed on the line, the position of all numbers in the number system was determined (in the data, no students interpreted a number line as logarithmic). The prompt asked students to place 7 on the number line. Tick marks were provided at the points 6.5, 7, and 8. No tick mark was provided at the location of 5.

A subset of 23 students—10 grade 5 and 13 grade 8—were interviewed on this task in order to determine solution approaches across students for placing 7 at one of the unlabeled tick marks. Interviews confirmed three strategy codes, each of which I describe in turn: (a) Coordinated Linear and Numeric Units, (b) Linear Unit Treated as Unit Interval, and (c) Numeric Units Privileged. Interviewees that changed their response before providing a rationale were coded as Changed Response. I refer to the unlabeled tick marks as A, B, or C (see Figure 7).

Six students—two in grade 5 and four in grade 8—were coded Coordinated Linear and Numeric Units. Of these six students, all positioned 7 correctly at tick mark B. I provide two examples, each of which features a student identifying a unit interval. Fifth Grade Participant 1 indicated the 4 on the line and stated (see Figure 8a), "There's 4, then 5, 6. Maybe there could be a line right here <*draws a tick mark at 5*>. Then this <*tick mark A*> would be a half, and this <*tick mark B*> would be 7." This student coordinated the linear unit from 4 to 6 with additional numeric units, leading to the placement of 5 exactly midway between 4 and 6. Alternatively, Eighth Grade Participant 1 first positioned 8 on the line before accurately locating 7 (Figure 8b): "I kind of thought that it's going up by 2s, since it was 4, 6. So I thought the 8 would be here <*indicating tick mark C*>, so I put the 7 right here <*tick mark B*>. Each number has its own gap." This student's reference to both linear units ("its own gap") together with numeric units indicates a coordination of the two unit types.

Four students—one in grade 5 and three in grade 8—were coded as Linear Unit Treated as Unit Interval. All positioned 7 at tick mark C. Each of the four students identified the linear unit marked by 4 and 6; treating this as a unit interval, each student iterated this linear unit once to

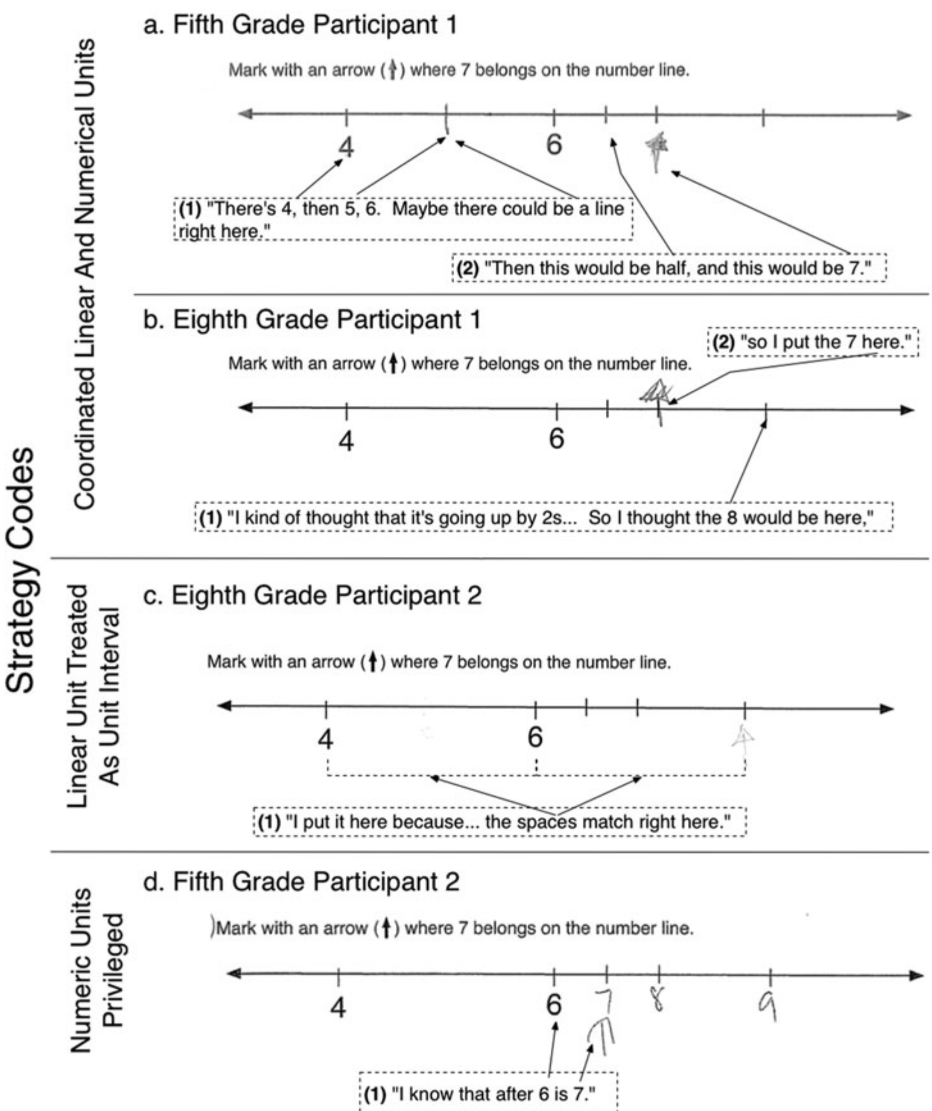

FIGURE 8 Annotated responses to the focal number line task illustrating three solution approaches.

the right to position 7. For example, Eighth Grade Participant 2 stated (Figure 8c), "I put it here because it was after, the spaces match right here." Four additional interviewees also located 7 at tick mark C; of these four students, all four changed their response at the beginning of the interview prompts.

Five students—three in grade 5 and two in grade 8—were coded as Numeric Units Privileged, indicating a consideration of numeric units absent of any overt attention to linear unit. All five

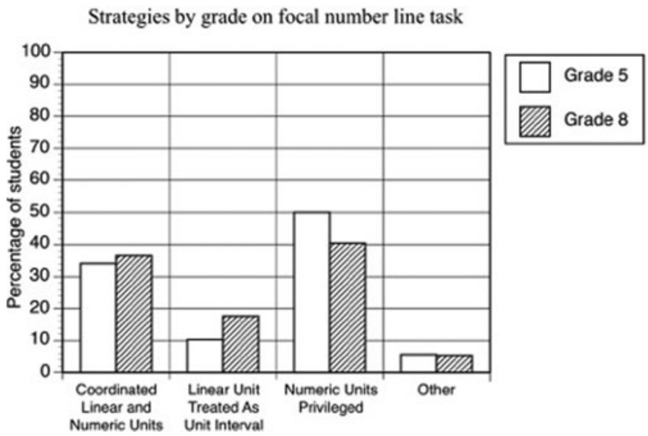

FIGURE 9 Percentage of students for each number line strategy code as a function of grade.

students positioned 7 at tick mark A. The three grade 5 students made no reference to the 4 on the line in their explanations; for example, Fifth Grade Participant 2 stated (Figure 8d), "I know after 6 is 7." The two grade 8 students each made reference both to the 4 and the 6 and—unlike the grade 5 students—they also referenced the location of 5 before locating 7 at tick mark A. For example, Eighth Grade Participant 3 stated, "because it's like 4, then 5 right here *<indicating midway between 4 and 6>*, then there's 6. And then there's this line *<tick mark A>*. So there's 7 *<tick mark A>*." Despite an accurate placement of 5, the two grade 8 students did not coordinate numeric and linear units to place 7 on the line.

Extrapolating to the Sample

Interviews confirmed that students' strategy codes on the number line task corresponded exactly to the location of one of the three unnamed tick marks, suggesting three common solution approaches involving linear and numeric units. Because of this correspondence—an anticipated result of the problem design—I take answer choice as an indicator for the entire sample of individuals' problem-solving approaches. I treat those who located 7 at the appropriate place, tick mark B, to have Coordinated Linear and Numeric Units (34.1% of grade 5 and 36.6% of grade 8 students; see Figure 9). I treat those who chose tick mark C as Linear Unit Treated as Unit Interval (10.3% of grade 5 and 17.6% of grade 8 students). I treat those who chose tick mark A as Numeric Units Privileged (50.0% of grade 5 and 40.5% of grade 8 students). In both grades 5 and 8, the most common approach to solve the nonroutine number line task was to privilege numeric units without any overt consideration to linear units, an approach that conflicts with convention.

Coordinate Plane

The nonroutine coordinate plane task asked students to identify an unnamed point in the plane, with each axis marked by a multiunit interval of 2. Gridlines are provided at intervals of 2. The point does not correspond to gridlines.

A subset of 29 students—18 in grade 5 and 11 in grade 8—were interviewed on this task in order to determine solution approaches across students. Interview data confirmed the same three categories as with the number line task: (a) Coordinated Linear and Numeric Units, (b) Linear Unit Treated as Unit Interval, and (c) Numeric Units Privileged. Once again, students that changed their response before providing a rationale in the interview were coded as Changed Response. Three responses had frequencies greater than five across students in the sample: point 5, 5; point 4.5, 4.5; and point 3.5, 3.5. While many responses with a frequency less than five were provided, most of these were related to grade 5 students' idiosyncratic efforts with fractional notation.

Ten students—seven in grade 5 and three in grade 8—were coded as Coordinated Linear and Numeric Units. All ten accurately named the point 5, 5.[5] These students all found a unit interval on each axis by partitioning either the interval from 0 to 2 to locate 1 or the interval from 4 to 6 to locate 5 along either axis; they then projected from axes to the point in the plane to name the point 5, 5. For example, Fifth Grade Participant 3 explained (Figure 10a), "Here it is 0 and it jumps right to 2 without a line in the middle *<indicating the location of 1>*. And the same here *<indicating the location of 3>*. So right in the middle should be 1, and then that would be 3, and then 4, and then right there would be 5. And the same over here *<on the vertical axis>*." In this explanation, linear units—the "jump" from 0 to 2—are coordinated with numeric units to find the unit interval.

Three students—one in grade 5 and two in grade 8—were coded Linear Unit Treated as Unit Interval. All three students identified the point as 4.5, 4.5 or $4\frac{1}{2}$, $4\frac{1}{2}$. For example, Eighth Grade Participant 3 explained, "Because it goes by 2s. It's 2, 4, and one half." In a similar way, Fifth Grade Participant 4 indicated the 2 along the horizontal axis and stated (Figure 10b), "So that's a 2. That *<midway between 2 and the tick mark to the right>* would be 2 and a half, and that *<tick mark to the right>* would be 4, because it's counting by 2s. So it would be 4 and a half here *<corresponding to the point in the plane>*. It goes up *<along the vertical axis>*, and 4 and a half over here. And they meet at the 4 and a half point on both of [the axes]." Students responding 4.5, 4.5 or $4\frac{1}{2}$, $4\frac{1}{2}$ on this task made some effort to coordinate numeric units with linear units yet at the same time treated each of the linear units as unit intervals rather than a multiunit interval of 2.

Seven students—four in grade 5 and three in grade 8—were coded as Numeric Units Privileged. Of these seven students, all named the point 3.5, 3.5 or $3\frac{1}{2}$, $3\frac{1}{2}$. Fifth Grade Participant 5 captured the spirit of other interviewees on this answer (Figure 10c): "I thought it was the next number, was 3. Then it was 4. And 5, and 6. That little dot was in the middle, and it was right between 4 and 3." Similarly, Eighth Grade Participant 4 counted on from the 2 on either axis, stating, "Because it's going 2, 3. You have to make it a decimal because it's in the middle." Interviewees who wrote 3.5, 3.5 did not, in any overt way, coordinate the 0 and 2 with linear units. Instead, they used the 2 to count on by ones, using each tick mark as the location for numeric units.

Extrapolating to the Sample

Interviews confirmed that students' strategy codes on the focal coordinate plane task corresponded to one of three common responses: point 5, 5; point 4.5, 4.5; and point 3.5, 3.5,

[5]One grade 5 interviewee that had accurately written point 5, 5 as a response did not provide a rationale during the interview and received a code of "Other."

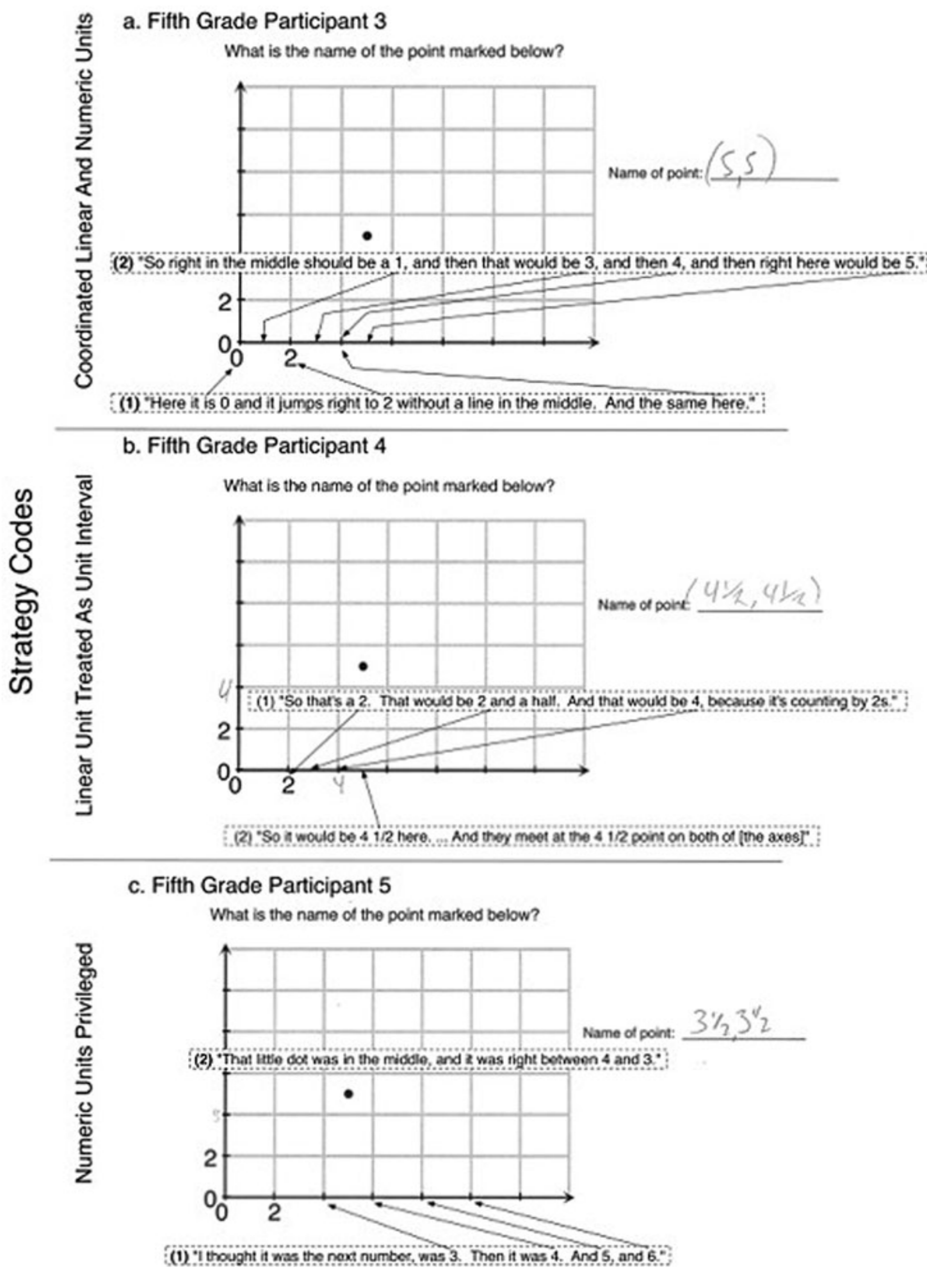

FIGURE 10 Annotated responses to the focal coordinate plane task illustrating three solution approaches.

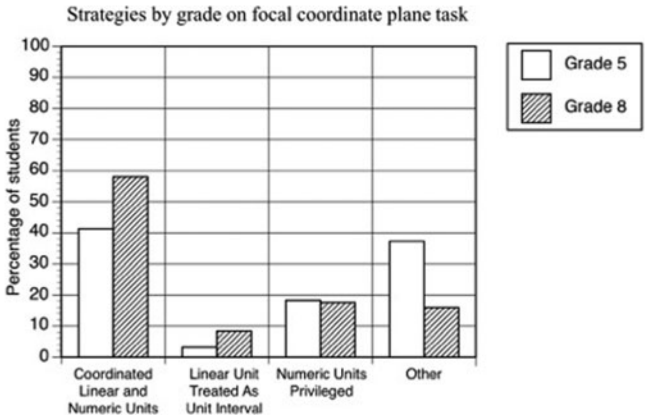

FIGURE 11 Percentage of students for each strategy code on the coordinate plane task as a function of grade.

suggesting three unique solution approaches. Because of this correspondence, I take response choice as an indicator of individuals' problem-solving approaches for all students in the sample (see Figure 11). Turning back to the paper-and-pencil assessment responses, I treat those who named the point 5, 5 as Coordinated Linear and Numeric Units (41.3% of grade 5 and 58% of grade 8 students). I treat those naming the point 4.5, 4.5 as Linear Unit Treated as Unit Interval (3.2% of grade 5 and 8.4% of grade 8 students). I treat those naming the point 3.5, 3.5 as Numeric Units Privileged (18.3% of grade 5 and 17.6% of grade 8 students). Unlike the number line item, the coordinate plane task was an open response item. This resulted in many students providing idiosyncratic responses. I treat those who provided such responses—none of which had a frequency greater than five—as Other (37.3% of grade 5 and 16.0% of grade 8 students).

Function Graphs

The graphing task Rate of Change assessed the coordination of linear units and numeric units across two graphs. The item featured two grids, each with a simple linear function, $f(x) = 2x$. The grids feature different scales, which rendered the orientation of the line in the plane as different between the two graphs. The prompt asked, "Who walks more miles each hour?" Students could choose among three options: "Graph A: Laura," "Graph B: Maggie," or "They walk the same rate." An additional prompt asked children to provide a written explanation for the answer choice; these written explanations are analyzed below. In order to identify the rate of change, students would have to determine ordered pairs through numeric and linear units without quantifying gridline units or interpreting the orientation of the line in the plane in isolation (see Figure 5).

A subset of 40 students—22 in grade 5 and 18 in grade 8—were interviewed on this task in order to determine solution approaches across students. Unlike the three strategy codes for number line and coordinate plane tasks, interviews confirmed four solution approaches: (a) Coordinated Linear and Numeric Units, (b) Linear Unit Treated as Unit Interval, (c) Discrete Numeric Units, and (d) Orientation of the Line. Students that changed their response before providing a rationale during the interview were coded as Changed Response.

Fourteen students—11 in grade 5 and three in grade 8—were coded as Coordinated Linear and Numeric Units. All students' justifications referred to two quantities (time and distance) on both graphs. Interview data indicate two ways in which students Coordinated Linear and Numeric Units; students either compared data points across graphs, or determined the rate of change. First, some students compared a single data point (e.g., 1, 2) on each graph. For example, Fifth Grade Participant 6 first explained Laura's graph, "Since you can see that it stopped here *<indicating the point (1, 2) on Laura's graph>*, and *<moving over to Maggie's graph>* I saw at 1 [hour] it stops here *<indicating the point (1, 2) on Maggie's graph>*, then goes straight across all the way to 2 *<on the vertical axis>*." Other students compared the point on the function line that corresponded to the greatest value along the vertical axis. For example, Fifth Grade Participant 7 explained, "Laura walks 10 miles every 5 minutes, and Maggie walks 5 miles every 2 and a half minutes. They *<referring to the two graphs>* are just different numbers, but they're equivalent. It's just that the graph is showing different numbers." Other students determined the rate of change for each graph. For example, Fifth Grade Participant 4 justified that each graph had the same rate of change by stating: "Because they walk about 2 times as many miles as the time it took them."

Three students—one in grade 5 and two in grade 8—were coded as Linear Unit Treated as Unit Interval. All three students, each of whom referred to the two quantities (time and distance) on both graphs, selected "Graph B: Laura." In each interview, students explicitly referred to linear units along Graph A's horizontal axis as unit intervals instead of intervals of 0.5. For example, Eighth Grade Participant 5 wrote, "Maggie's graph has a slope of 2, so she walks 2 miles each hour, and Laura's graph has a slope of 1, so she walks 1 mile each hour." In the interview, she further explained (Figure 12a), "To find the slope it's rise over run. So when it meets a point for Laura, it goes up one and across one. So it goes up 1 each time. And this one *<indicating Maggie's graph>* is in units of 2, so when you go up one you go over 2 (sic), so it's a slope of 2."

Six students—three in grade 5 and three in grade 8—were coded as Discrete Numeric Units. Of these students, three (one grade 5 and two grade 8 students) chose "Graph A: Laura," and three (two grade 5 and one grade 8 student) chose "Graph B: Maggie". In all six cases, students justified responses with reference to one discrete quantity on each graph, either time or distance. Three pairs of values were mentioned across interviewees: (a) the greatest value along each vertical axis (distance), (b) the greatest value along each horizontal axis (time), or (c) the horizontal axis value corresponding to the "end" of the function line (time). First, Fifth Grade Participant 6 compared values along the vertical axes to choose "Graph A: Laura": "She [Maggie] didn't do it the same as Laura. She [Maggie] did 5 miles. Laura does 10 miles," indicating values along the vertical axis only. Second, Fifth Grade Participant 8 compared values along the horizontal axes to choose "Graph B: Maggie": "I picked Maggie, because if this is 10 and this is 5, wouldn't that mean, like, she [Maggie] walked double?" Third, Fifth Grade Participant 9 made the third comparison to choose "Graph A: Laura" (Figure 12b): "I knew it *<Graph A: Laura>* was more because it *<gesturing to Graph B: Maggie>* stops at 2 *<gestures from the endpoint of the function line down to the horizontal axis and incorrectly identified this point as "2">*, and this one *<Graph A: Laura>* goes to 5." Across these six students, all compared two discrete values between graphs that were disconnected from other representational features.

Lastly, six students—all in grade 8—were coded as Orientation of the Line. Three of these students selected "Graph A: Laura," and three selected "Graph B: Maggie." Five of the six

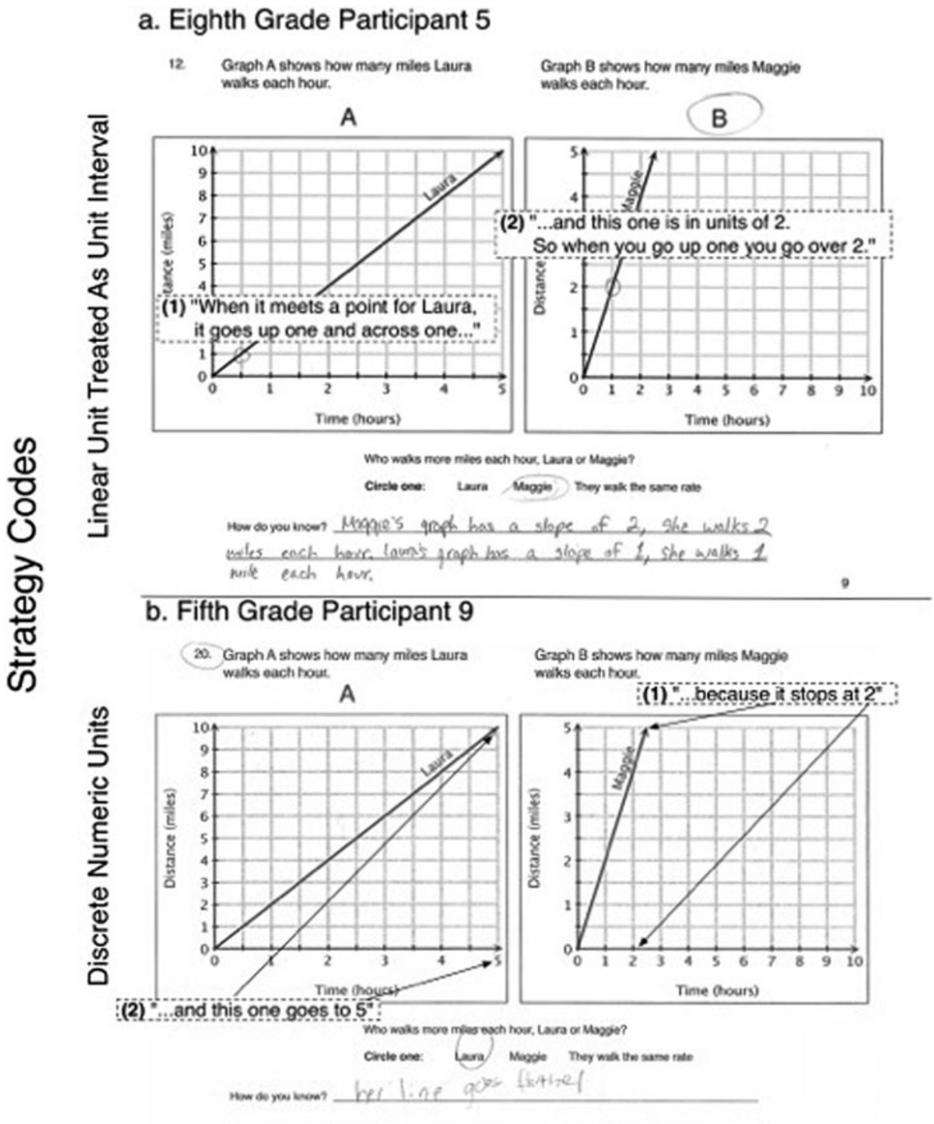

FIGURE 12 Annotated responses to the focal rate of change comparison task illustrating two solution approaches.

students made no reference to quantity on either graph, instead referring only to the function line; the remaining student made passing reference to a discrete quantity (five miles) on one graph. Students' explanations had one or more of the following three words in oral justifications: Students referenced (a) the steepness, (b) the length, and/or (c) the straightness of the line in the plane. These elements did not necessarily correspond with one answer choice. For example, Eighth Grade Participant 6 selected "Graph A: Laura," with an explanation that involved steepness

TABLE 6
Answer Choice and Strategy Codes for Interviewees on Item Rate of Change

	Grade 5			*Grade 8*			
	Graph A:Laura	*Graph B:Maggie*	*Same*	*Graph A:Laura*	*Graph B:Maggie*	*Same*	*Totals*
a. Coordinated Linear and Numeric Units	—	—	11	—	—	4	15
b. Linear Unit Treated as Unit Interval	—	1	—	—	2	—	3
c. Discrete Numeric units	1	2	—	2	1	—	6
d. Orientation of the Line	—	—	—	3	3	—	6
e. Changed Response	2	4	—	—	1	—	7
f. Other	1	—	—	2	—	—	3
Totals	4	7	11	7	7	4	40

and straightness of the line in the plane: "She [Laura] goes straight, and this [Graph B: Maggie] goes steeper." Meanwhile, Eighth Grade Participant 7 also made references to the steepness of the line in addition to the length of the line in the plane to select "Graph B: Maggie": "Because she was steeper, so it didn't take her that long. And Laura's was really long, so it took her longer." Eighth Grade Participant 8 referred just to the straightness of the line in the plane to select "Graph A: Laura," stating, "Because right here [on Graph A: Laura], there a line, it's going straight;" at which point she gestured along Laura's function line beginning at the origin moving up and to the right.

Extrapolating to the Sample

Interviews for the rate of change comparison task revealed four strategy codes: Coordinated Linear and Numeric Units, Linear Unit Treated as Unit Interval, Discrete Numeric Units, and Orientation of the Line. The four codes suggest four unique solution approaches. Yet unlike prior interview analyses for target number line and coordinate plane tasks, strategy codes did not necessarily lead to a unique answer choice (Table 6). Recall that the rate of change comparison task featured a prompt for students both to choose one of the three options ("Graph A: Laura," "Graph B: Maggie," or "They Walked The Same Rate") as well as to justify that choice. Due to the lack of correspondence in interviews between answer choice and strategy code, I turn to students' written justifications.

To consider the entire sample, I apply the four codes that emerged through interview analyses to students' written justifications. If justifications did not correspond to one of the four codes or if insufficient detail was provided, that justification received a code of Other. Note that one code, Orientation of the Line, emerged in interviews with grade 8 students only; nonetheless, this code was included as a possibility for grade 5 written responses. Table 7 provides exemplars from

TABLE 7
Exemplars of Strategy Codes to Students' Written Justifications on Rate of Change

Codes	*Written Justifications*
1. Coordinated Linear and Numeric Units	"They walk the same rate because in one hour they both walked 2 miles."
2. Linear Unit Treated as Unit Interval	"Because it shows that Maggie walks 2 miles in 1 hour and Laura walks 1 hour for 1 miles."
3. Discrete Numeric Units	"Laura walked more because she ran 10 miles and Maggie walked 5 miles."
4. Orientation of the Line	"Laura. Hers goes straight through and Maggie goes a lot to the left."
Other (insufficient detail)	"Maggie, because she ran faster."

coding categories for written justifications, and Table 8 presents results of the analysis of written justifications.

Patterns in Solution Approaches Across Representations

The analysis of interviews suggested a small set of solution approaches for each of the three representations, with each approach making differential uses of linear and numeric units as visual mediators. To address the final research question regarding meta-rules in students' solutions across the three geometric representations of quantities, I now consider the strategy codes for individual students across the three target problems and correlations of these codes.

Recall that three identical solution codes emerged from the analysis of both number line and coordinate plane tasks, while four codes emerged in the function graphs task. I first consider codes for number line and coordinate plane tasks, which included: (a) Coordinated Linear and Numeric Units, (b) Linear Unit Treated as Unit Interval, and (c) Numeric Units Privileged, along with a minority of students coded Other. Recall that in particular, the coordinate plane task resulted in many responses that were idiosyncratic (frequencies less than five) and therefore coded as Other, yielding a potential challenge in considering meta-rules across the three representations. In order to address this challenge, I reframe the number line and coordinate plane coding categories into two groups: those students that considered linear unit (Coordinated Linear and Numeric Units

TABLE 8
Strategy Codes on Rate of Change Comparison Task by Grade

	Grade 5 (n = 126)		*Grade 8 (n = 131)*	
	Frequency	*Percent*	*Frequency*	*Percent*
1. Coordinated Linear and Numeric Units	28	22.2	59	45.0
2. Linear Unit Treated as Unit Interval	4	3.2	14	10.7
3. Discrete Numeric Units	38	30.2	16	12.2
4. Orientation of the Line	32	25.4	27	20.6
Other	24	19	15	11.5

and Linear Unit Treated as Unit Interval) and those that did not (Numeric Units Privileged and Other). An affordance of this scheme is that the subsequent analysis may consider all students in the sample, including those that provided idiosyncratic answers. At the same time, this new scheme has two limitations to it.

First, those students coded as Linear Unit Treated as Unit Interval—an approach inconsistent with conventional mathematics—are now included with those whose solutions coordinated the two unit types. This particular limitation is mitigated by the fact that a low percentage of students in each grade were coded as Linear Unit Treated as Unit Interval. Furthermore, the majority of that small amount of students coded as Linear Unit Treated as Unit Interval on the number line task were subsequently coded as Coordinated Linear and Numeric Units on the coordinate plane task. This suggests that this group of students, while not fully coordinating linear and numeric units for the number line, are making concerted efforts to consider the two unit types.

A second limitation is that those students coded as Other, particularly for the coordinate plane task, are treated together with those coded as Numeric Units Privileged. Because interview criteria led to interviewing students only on responses with frequencies greater than five, those responses coded Other were not a focus of analysis. Therefore, data do not support any assertion that Other solution approaches were similar or dissimilar from those coded Numeric Units Privileged. A resulting issue is in potentially confounding Other students that may have in fact considered linear unit in their approach. This concern is mitigated in part by the fact that the majority of students coded Other on the coordinate plane task ($n = 68$) were in fact coded as Numeric Units Privileged on the number line task (70.6%), suggesting these students in fact were not considering linear unit in their approaches.

When considering the number line and coordinate plane tasks together, three categories result: those that attended to linear unit on both number line and coordinate plane task (Both, $n = 89$), those that attended to linear units on one of the two tasks but not both (Either, $n = 92$), and those that did not attend to linear units on either the number line or coordinate plane tasks (Neither, $n = 76$). To consider patterns across geometric representations of quantities, I compare students' use of linear units on number line and coordinate plane problems (Both, Either, Neither) with the four strategy codes on the function graphs task: Coordinated Linear and Numeric Units, Linear Unit Treated as Unit Interval, Discrete Numeric Values, and Orientation of the Line. Figure 13 features a mosaic plot with Both, Either, and Neither categories displayed as three columns with the overall width proportional to the *n* for each (note that because *n*s for each column are similar, the difference in column width is subtle). Strategy codes for the function graphs task are displayed as rows (see Figure 13), with each of the four graphing codes and Other featuring a unique shading. Function graph strategy codes are displayed as a proportion of each of the three columns.

The top row of the mosaic plot (in the darkest grey) presents data for students coded as Coordinated Linear and Numeric Units. The majority of students that attended to linear units on both number line and coordinate plane tasks also Coordinated Linear and Numeric Units on the function graphs task (61.8%); conversely, in the Neither column only 6.6% of students coordinated the two unit types on the graphing task. Students in the Neither column were much more likely to have used discrete numeric units (36.8% of Neither) or interpret the orientation of the line in isolation of other representational features (34.2% of Neither). Trends in data suggest a meta-rule involving the coordination of linear and numeric units across representations as well as two alternate meta-rules, one involving drawing on numeric units independent from linear

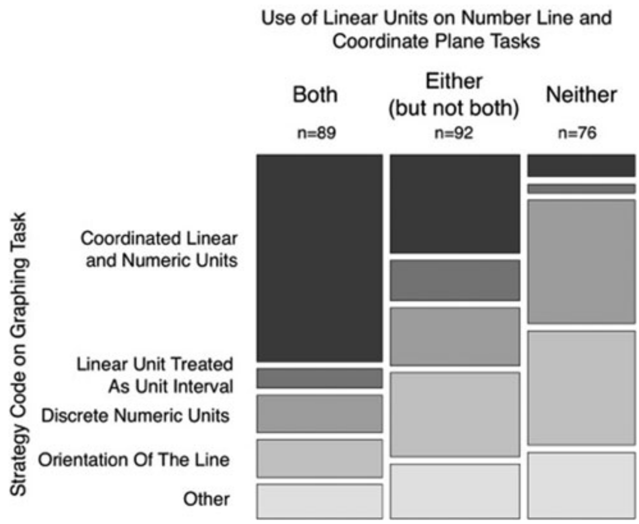

FIGURE 13 Mosaic plot displaying proportions of strategy codes on graphing task as a function of the use of linear units on number line and coordinate plane tasks.

units and another involving drawing on a single representational feature in isolation of others. Those students that did not consider linear units for early representations were likely to solve the graphing task using discrete numerals or the orientation of the line in the plane in isolation. The mosaic plot indicates that students' attention to linear units is consistent across representations.

To confirm these visual trends statistically, permutation tests (Davison & Hinkley, 2003; Yung, Freimund, & Chandler-Pepelnjak, 2008) were applied to the data to determine whether this finding—that those students that attended to linear units on both number line and coordinate plane tasks also coordinated linear and numeric units on the function graphs task—was due to chance. Or, in other words, is the top left box (Figure 13) featuring 68% of students coded Both indicative of a correlation in solution approaches across represenations, or could it be the result of chance? I first provide an explanation of this statistical test, and present the results below.

Determining correlation across representations first involved calculating two different statistics: the proportion of students using linear units on both number line and coordinate plane problems, and the proportion of students coded as Coordinated Linear and Numeric Units on the rate of change comparison task. These two proportions then allow for the calculation of the test statistic, which represents the proportion coded as Both that also coordinated linear and numeric units on the graphing task. Permutation tests provide a way to address whether a result other than Coordinated Linear and Numeric Units is more likely if the graphing category labels are disregarded and results are permuted. Addressing this question first involved calculating the value of the test statistic for the data labeled correctly (see Table 9) and then randomly permuting the graphing strategy codes 2,000 times, calculating the test stastitic with each new permutation. The measure of the extremity of the real test statistic as compared to the permuted test statistics is the *p*-value, which if significant would indicate that the real test statistic is not due to chance.

TABLE 9
Proportions of Students' Solution Approaches by Grade With Resulting Test-Statistics

	Used Linear Units on both Number Line and Coordinate Plane Tasks	*Coordinated Linear and Numeric Units on Rate of Change Task*	*Test-Statistic*	*P-Value*
Grade 5	27.7%	22.2%	.167	$p < .0001$
Grade 8	42.0%	45.0%	.267	$p < .0001$

Results of the permutation tests are presented in Table 9. Test statistics are .167 for grade 5 students and .267 for grade 8 students. Permutation tests revealed significant p-values for each grade ($p < .0001$), thereby confirming that the extremity for the real test statistic is not due to chance. In other words, permutation tests support the finding of continuities across solution approaches in how units were coordinated, and thereby suggest a meta-rule for each grade involving the coordination of linear and numeric units that students apply across the three representations.

DISCUSSION

A goal of the present study was to investigate whether students' problem-solving efforts across geometric representations of quantities reflect particular meta-rules. Such approaches may or may not be consistent with conventional mathematics. Results of this assessment and interview study revealed a small set of meta-rules students applied across representations. In this discussion, I consider these meta-rules as well as implications for the purposes of theory, curriculum, and instruction.

Patterns of Student Understanding Across Representations

In this section, I discuss problem-solving approaches across representations in terms of the meta-rules suggested by the data. Recall that meta-rules reflect participants' use of patterns in discourse and thereby indicate when to do what and how to do it. A premise of the study was that individuals that coordinated linear and numeric units on a function graph would apply the same approach to a number line due to structural similarities across representations. Conversely, students that did not coordinate the two unit types on a function graph may be applying an alternate meta-rule across the three representations.

A Meta-Rule Consistent with Conventional Mathematics

This study considered geometric representations of quantities to be a part of a hierarchical narrative based on consistent linear and numeric properties. In the data, a meta-rule emerged that was consistent with conventional mathematics in which students coordinated linear units with numeric units across the three representations. The character of problem solving was consistent across representations. Despite the fact that relevant content across the three representations is quite varied and introduced in disparate grades, students that coordinated linear and numeric units with the graph were most likely to do so with number lines and the coordinate plane. An aspect to this finding is that structurally similar features indicated to students how to solve this set of problems.

Recall that the sample involved students both with (grade 8) and without (grade 5) relevant instructional experience for some of the content presented in this study. Because of this, a reasonable expectation of this data is that grade 8 students would outperform grade 5 students; results confirmed this expectation. Nonetheless, many grade 5 students did respond accurately to the target function graphs task. These successful grade 5 students applied the meta-rule involving a coordination of the two unit types across representations, and permutation tests confirmed this. This trend indicates, first, that their understanding of the number line involves coordinating linear and numeric units. Yet perhaps more importantly, this finding also suggests that, despite little to no prior instruction, they successfully bootstrapped their understandings of the number line and coordinate plane to solve the function graphs task. The meta-rule consistent with conventional mathematics served as a resource for a novel and more complex problem. I further reflect on meta-rules by grade below as well as limitations to these interpretations.

Meta-Rules Inconsistent with Conventional Mathematics

Alternate meta-rules emerged that are inconsistent with conventional mathematics. These meta-rules reflect how symbolic artifacts of the representation differently mediated students' problem-solving approaches. I draw attention to three alternate meta-rules suggested by the data: (a) over-generalizing the function of numeric units, (b) isolating representational features from the overall representational context, and (c) treating linear units as unit intervals.

A first alternate meta-rule involves over-generalizing the role of numeric units across geometric representations of quantities. For both number line and coordinate plane tasks, students whose problem-solving strategies reflected this meta-rule counted-on by consecutive numeric units using successive tick marks without any overt attention to linear units. On the function graphs tasks, these students similarly drew on numeric units by over-generalizing the meaning of a discrete value along one of the axes, such as the maximum value along either the horizontal or the vertical axis.

A second alternate meta-rule involves isolating a symbolic artifact from the greater representational context. Consistent with the first alternate meta-rule described above, students in this meta-rule counted-on by consecutive numeric units on number line and coordinate plane tasks, treating each tick mark without coordination with linear units. On the function graphs task, these students drew on an isolated representational feature, the orientation of the line in the plane, by referring to the steepness, the length, or the straightness of the function lines. In these cases, students did not coordinate their interpretation of the function line with any additional information (i.e., numeric or linear units) along the axes.

A third alternate meta-rule involves treating linear units as unit intervals regardless of numeric units provided. Unlike the prior two alternate meta-rules, this meta-rule reflects some efforts on the part of children to coordinate linear and numeric units; however, with multiunit intervals treated as unit intervals, these efforts are inconsistent with conventional mathematics. While such findings echo prior research on children's performances involving graphs (Caddle & Earnest, 2009; Lobato et al., 2003; Zaslavsky et al., 2002), such an approach is not well reflected in studies involving the number line or coordinate plane. Nonetheless, this approach was represented in performances across each of the three target problems. Because these students demonstrated concern for both unit types, this particular meta-rule appears to more closely approximate conventional mathematics than the prior two mentioned above.

Meta-Rules by Grade

Grade 8 students outperformed grade 5 students on the assessment, an expected result given the difference in years of instruction. Nonetheless, the approaches employed by grades 5 and 8 students were similar, with no single problem-solving approach represented in one grade and not the other. I draw attention to elements from data that are particularly striking. Each of the target tasks allows us to glean the character of understanding between the two grades. I briefly refer back to findings on each of the three target problems.

First, despite three years of instructional difference between the two groups, students performed remarkably similarly on the target number line task. In both grades, the most common response (50% of grade 5 and 40.5% of grade 8 students) reflected the first and second alternate meta-rules above in which students either overgeneralized the role of numeric units by counting on using tick marks or drew on one numeral in isolation of other information provided. While the two grades in the sample have three years of instructional difference, suggesting that grade 8 students are better prepared for certain content, instruction targeted to the number line is most commonly associated with elementary school. Grade 8 students may have received little to no additional instruction on number lines since grade 5, thereby speaking to the similar performance between the two grades on this particular task. This meta-rule, inconsistent with conventional mathematics, may have little chance for remediation after grade 5. In these cases, geometric and numeric properties of a single number line are, in fact, not resources for problem solving for representations that come later in the hierarchical narrative. This also suggests that students in grade 5 who have not mastered geometric and numeric properties of the number line may lack future instructional opportunities to challenge that alternate meta-rule.

Second, students in grade 5 (37.3%) were much more likely to be coded Other on the coordinate plane problem as compared to grade 8 students (16%). Recall that students were coded as Other if a given response had a frequency less than five for all students across the sample. I conjecture that the difference in years of instruction has bearing on frequencies of idiosyncratic responses. Based on standards and typical curricular sequences, students in grade 8 have had much more additional experience plotting and interpreting points on the coordinate plane, whereas students in the middle of their grade 5 year have had limited instructional experience in this. While state standards (NGA Center & CCSSO, 2010) indicate students should be able to place rational numbers and integers on an axis, it is reasonable that grade 5 students would refine their understandings with subsequent work in middle school, thereby speaking to the difference in performance.

Third, grade 8 students did much better on the function graphs task as compared to grade 5 students. This is likely because of the direct instruction grade 8 students have received as a part of their algebra coursework. Nevertheless, a minority of grade 5 students was successful on the function graphs task despite a lack of prior instruction. These students not only Coordinated

Linear and Numeric Units on the graphing task, but also did so for number line and coordinate plane tasks (see Table 9). This implies that students without much or any prior instruction on function graphs tasks may bootstrap their understanding of the coordination of linear and numeric units due to structural similarities; conversely, those grade 5 students that did not consider linear unit in their solutions to number line and coordinate plane tasks may lack a conventional meta-rule to support meaningful interpretation of an unfamiliar representation that comes later in the sequence.

Implications

Patterns among students' approaches to problems featuring geometric representations of quantities revealed particular meta-rules across the hierarchical narrative. I discuss theoretical implications and curricular and instructional implications.

Theoretical Implications

The design and subsequent analysis of data draws on sociocultural theory, in particular that of Sfard (2007, 2008, 2012), to situate children's mathematical understandings in both the hierarchical structure of conventional mathematics as well as the discourse of school instruction across grades from elementary to high school. This lens afforded a theoretically grounded investigation into the character of problem solving across grades. With each meta-rule that data suggested, mathematical objects visually mediated thinking and communication in particular ways that were consistently applied across the three representations. The present study positions these meta-rules as a product of discourse and social activity in which students have been participants.

The HRN construct may offer a meaningful way to operationalize investigations of mathematics learning and thinking across grades. We can and already do consider mathematics concepts independent of particular representations and their visual mediators. At the same time, problem-solving approaches in school mathematics are inseparable from the symbolic artifacts of the representation, and as such we need to consider the role of representations—particularly routine ones—in supporting particular meta-rules and whether these reflect conventional mathematics. While outside the scope of the present study, we may imagine additional HRNs in which children apply one of a small set of meta-rules that may or may not be consistent with conventional mathematics. For example, research has shown that children's difficulties with fractional notation are related to an overgeneralization of whole number understandings (e.g., Mack, 1995; Saxe, Taylor, McIntosh, & Gearhart, 2005). Framing numeric symbols as a part of a hierarchical narrative in school mathematics—one that begins with the whole numbers of early primary grades that are later subsumed in numeric representations of fractions, negative numbers, and exponents—may make salient particular meta-rules that may or may not be consistent with conventional mathematics. Once again, considering meta-rules across representations with structural similarities may allow us to more deeply understand connections across grades for the range of learners and explore possible interventions where needed. Future research that seeks to investigate this or other potential HRNs may also utilize nonroutine problems in order to well reveal how symbolic artifacts of representations serve as visual mediators in children's problem solving.

Curricular and Instructional Implications

A goal of school mathematics is not that children merely have the capacity to solve problems they may encounter on a standardized test or in a workbook, but rather that they are successful problem solvers with deep mathematical understandings that enable flexibility across mathematical situations. In the present study, students performed better on routine problems as compared to nonroutine problems. While routine tasks are necessary fixtures in mathematics instruction, nonroutine problems can reveal whether routine problem performance reflects an application of procedural rules without the underlying mathematical meaning. I further reflect here on implications for curriculum and instruction.

In order to address coherence across grades, curricula ought to reflect the hierarchical narrative of geometric representations of quantities and allow the meta-rule consistent with convention to be an object of focus. This assertion suggests that the discourse involving one representation ought to build on or foreshadow the discourse of another, with that structure disturbed as little as possible across representations. Curriculum should support the conceptualization of geometric representations of quantities in terms of both the geometric properties and the numeric properties, ideally providing students opportunities to grapple with the two unit types. This is of particular concern given that elementary curricula have been found to provide limited opportunities to learn about the geometric and numeric features of measurement tools (Smith, Males, Dietiker, Lee, & Mosier, 2013).

I argue that nonroutine problems have a critical role in instruction to support an understanding of conventional mathematics. In the present study, children applied alternate meta-rules across the three representations. While inconsistent with conventional mathematics, such meta-rules were not spontaneously invented by students. Rather, they have been supported by discourse and activity in school mathematics. As Sfard (2012) herself stated, the development of meta-rules in mathematical discourses is "a product of unintended, often serendipitous variations by individual reproducers" (p. 4), which in this case include curriculum and instruction. While we may imagine that educators would never intentionally support a meta-rule that clearly conflicts with convention, research has confirmed that students typically see routine problems and representations in their schooling. As a result, the current reality may be that children do not have opportunities to grapple with the two unit types in instruction. School mathematics may simply fail to provide appropriate counterexamples to students that would disprove or challenge those alternate meta-rules that conflict with convention. As noted in prior research (Kjeldsen & Blomhøj, 2012), learners themselves are not likely to provoke a meta-level change without support through instructional experiences, indicating that instruction must devise improved methods to address this. The use of nonroutine problems has potential to make alternate meta-rules an object of focus in classroom discourse.

A challenge in considerations of curriculum and instruction is that the representations in question are typically associated with disparate grade levels. Often times the focus of curriculum developers or researchers does not require explicit focus across the K–12 sequence; instead, focus tends to be on a subset of years (e.g., K–5, 6–8, or 9–12). Instruction faces a similar challenge, with in- and preservice elementary teachers and middle and high school teachers often having few opportunities to consider the development of mathematical ideas across the K–12 sequence. Developing curriculum or providing instruction consistent with any hierarchical narrative must overcome this challenge in order to consider the development of mathematical ideas across grades.

LIMITATIONS

Although data revealed meta-rules students apply across the three representations, conclusions are limited by three important factors. First, despite the fact that this study framed the HRN and associated meta-rules as across-grades, the empirical techniques employed were not longitudinal. Instead, a cross-sectional approach was used to select two different grade levels, one grade before and one grade after those that typically work with linear function graphs. While this approach affords insights into how children at two different ages reason across representations, such an approach does not afford consideration of how children interpret these symbolic artifacts upon introduction. In light of this, the empirical techniques do not well reflect research that demonstrates the qualitatively different character of children's quantitative reasoning across childhood (e.g., Case, 1991). The data in this study cannot speak to whether the solution approaches on the number line task, for example, is similar to or different from how a second grade student may engage with such a task. Second, the assessment did not gather data on negative integers, nor did it explicitly target students' understanding of rational number. Research has highlighted that a key property of geometric representations of quantities is that they represent the real numbers (Bass, 1998; Saxe et al., 2007, 2010; Wu, 2005, 2009), not just whole numbers or integers. Subsequent research that seeks to document student understanding around this hierarchical narrative and associated meta-rules should consider the role of negative integers and fractions, moving to the left of 0 on the number line and beyond Quadrant I of the coordinate plane, in order to further illuminate the findings of the present study. Lastly, children in each grade solved problems in one thirty-minute session. Children are clever problem solvers, and not just in the sense of solving individual problems as posed and intended by the study. Rather, the "problem" children may have solved is, in fact, deducing what the test-maker might be asking through repeated nonroutine problems and, if determining this, solving such problems in similar ways. In such cases, the assessment would reflect how children might try to crack the code of the test rather than the meaning embodied by visual mediators of the representations. Future research ideally would employ techniques that enable investigation of solution approaches over time in order to corroborate the claims of this article.

CONCLUSION

Conceptualizing a set of related mathematical representations in terms of a hierarchical representational narrative across grades has the potential to support students' rich and generative understandings of relevant content. Geometric representations of quantities—from the number line to the coordinate plane to functions on the plane—share structural properties with similar mathematical underpinnings. As data in this article have shown, students' problem solving across these representations reflects continuity in how they draw upon representational features, whether consistent or not with conventional mathematics. In future studies, it would be useful to conduct teaching experiments (e.g., Steffe, 2001; Tzur, 1999) in which instruction is designed with attention to the hierarchical narrative. In addition, such research ideally would consider alternate meta-rules that children may have and ways to support opportunities to grapple with and understand the conventional treatment of the two unit types. In turn, school mathematics could become a site in which the range of all learners, from low to high, have the opportunity to bootstrap their prior understandings as they explore mathematical ideas across grades.

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APPENDIX A

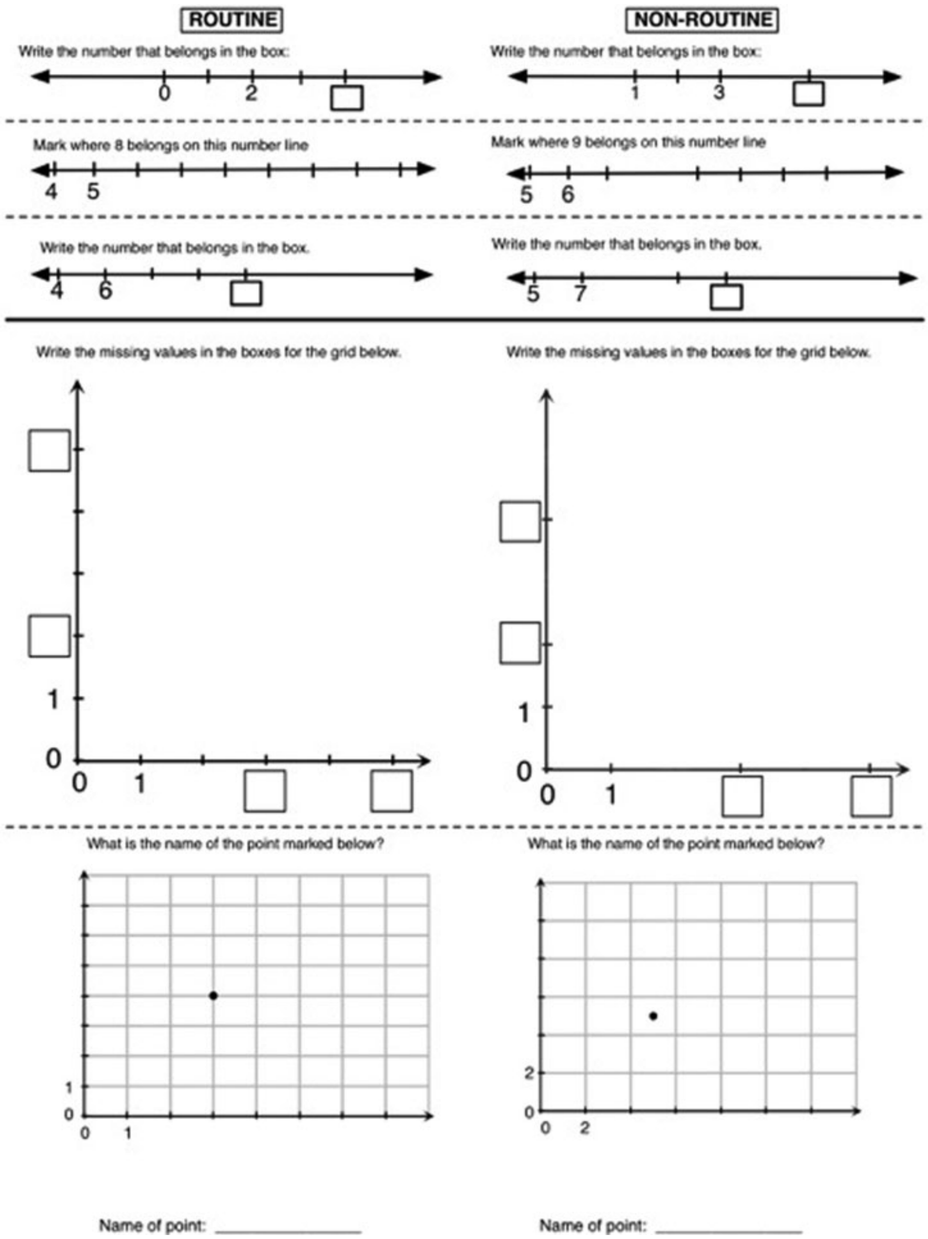

FIGURE A1 Matched routine and nonroutine problems on the assessment.

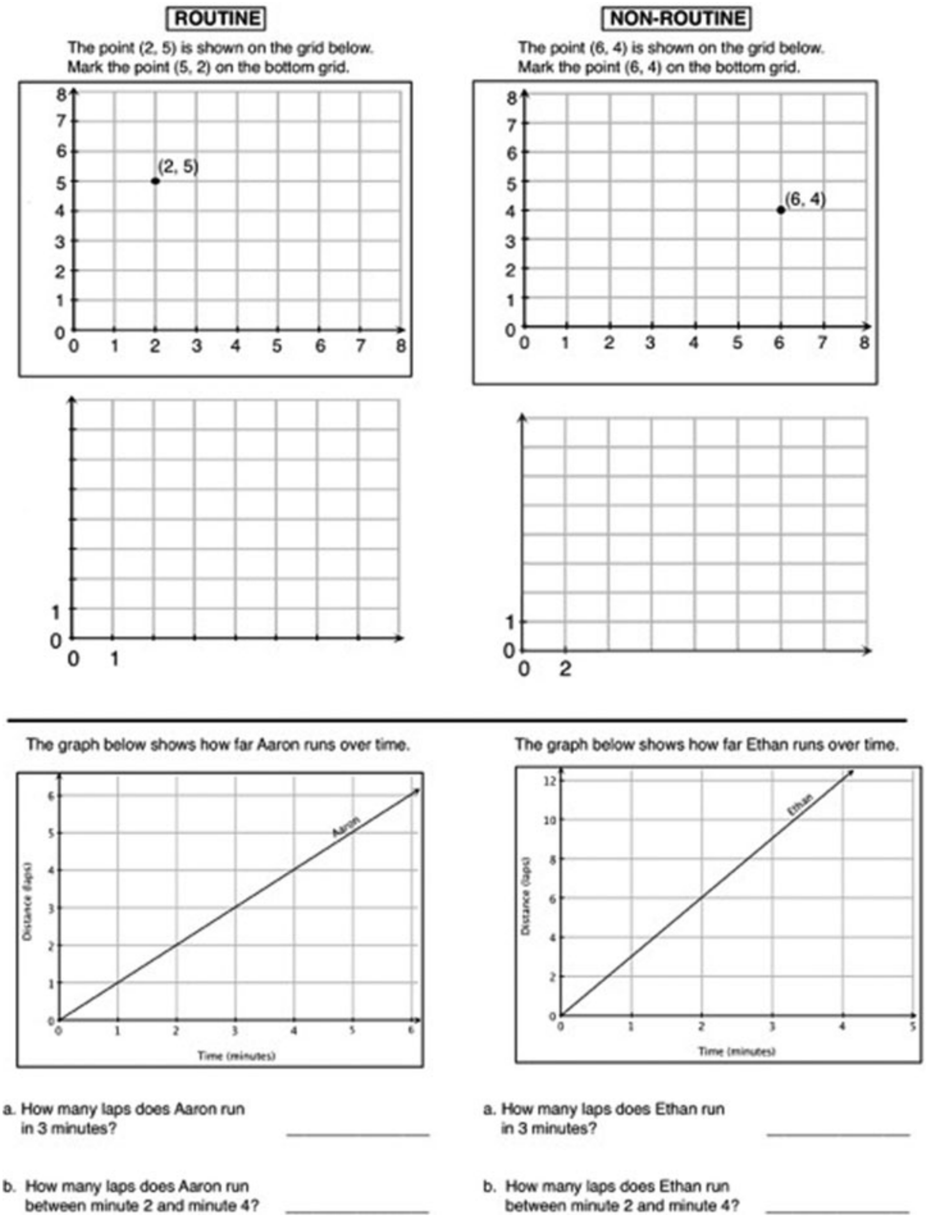

FIGURE A2 Matched routine and nonroutine problems on the assessment.

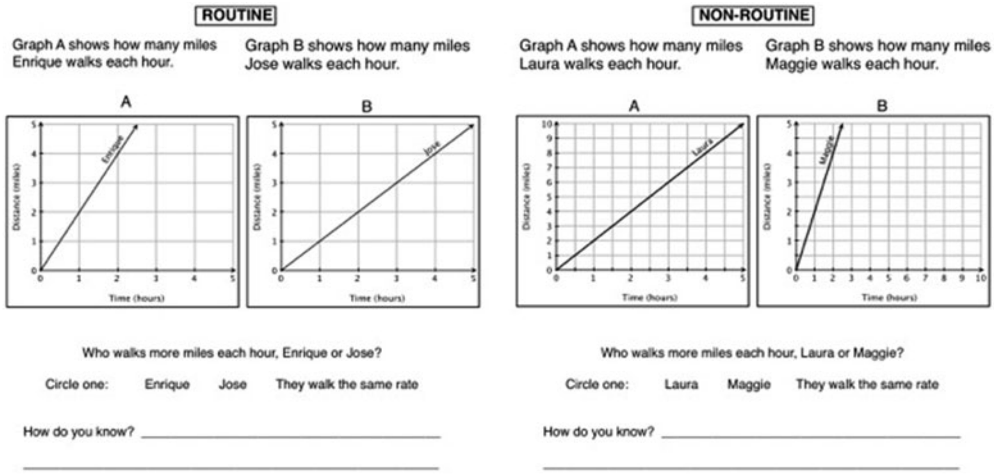

FIGURE A3 Matched routine and nonroutine problems on the assessment.