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EXPLORING THE INSIGHTS INTO GRADE 11 LEARNERS' UNDERSTANDING OF THE "ROOTS" OF QUADRATIC EQUATIONS

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ABSTRACT

This article provides insights into Grade 11 learner understanding of the roots of quadratic equations, how learners used solution strategies to find equations with given roots, how they determine the unknown values and the other root when one root is given and how they determine the unknown values when two roots of the equation are given. The study was underpinned by Skemp's relational and instrumental understanding, and Kilpatrick, Swafford and Findel's conceptual and procedural understanding to gain the insights into Grade 11 learners' understanding of the roots of quadratic equations. A case study design was espoused to answer the main research questions of this study. A written test was administered to 42 learners, and eight learners were purposively sampled to participate in a clinical interview based on their responses to the test. The findings of this study revealed that learners find it difficult to describe the roots, determine the other root when one root is given, to solve quadratic equations written in standard form, to represent the roots to determine the value of unknown variable(s), and to execute the reversal process using the given roots to find the equation.

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INTRODUCTION

A quadratic equation (QE) is a polynomial equation of the standard form $ax^2 + bx + c = 0$, where $a \neq 0$ with a, b and c are coefficients and x as unknown variable (Kabar, 2018; Hirawaty et al., 2021). It is a second-degree equation, which comprises three terms, the first being a with the second degree of x, the second b with the first degree of x and the last c, which is a constant. So, the zero-product property says if $a \times b = 0, \text{then } a = 0 \text{ or } b = 0$, and if $a = 0 \text{ or } b = 0, \text{then } a \times b = 0$ (Pike et al., 2011). The equations always have two solutions, roots, or zeros. QE can be solved in factored form by using the zero-product property which states that if the product of two quantities equals to zero, at least one of the quantities should be zero. The zero-product property in QE can be written in factored form $(a + b)(a - b) = 0, then\ a = b\ or\ a = -b$. If one side of the equation is not equal to zero, then it will not be possible to use zero-product property to make the same deduction. For example, if $(a + b)(a - b) = 3$, then you cannot conclude that $a = 3$. However, some of the equations may have solutions of the same value, for instance $x^2 - 2x + 1 = 0, (x - 1)(x - 1) = 0, x = 1$, and some of the equations may have solutions which cannot be real numbers, such as $x^2 + 1 = 0, x^2 = -1$.

Quadratic equations is part of the school mathematics curriculum that is compulsory in secondary school mathematics (Cahyani & Rahaju, 2019; Kim How et al., 2021). They form a bridge between mathematical topics such as linear equations, functions, and polynomials (Sağlam & Alacacı, 2012). Kim How et al. (2021) and Didis and Erbas (2015) point out that quadratic equations can be used in other disciplines such as physics, engineering, and designs. In addition, López, Robles, and Martínez-Planell* (2016) are affirm that the ability to solve QE with one unknown is fundamental to advanced studies in mathematics and sciences. However, solving QE is one of the challenging topics in the curriculum (Thomas & Mahmud, 2021; Vaiyavutjamai & Clements, 2006). Research has shown that many secondary-school learners and even undergraduate students at tertiary institutions find these equations and the rules to solve them difficult to understand (López et al., 2016). In their work, Didis and Erbas (2015) found that the understanding of QE is procedural (or instrumental), rather than conceptual. Learners are found to have memorised procedures to solve QE without knowing the reasons why they used those procedures to solve the equations.

Although research has been conducted in the teaching and learning of quadratic equations (for example, Gözde and Kabar, 2018; Herawaty et al., 2021; López et al., 2016; Tendere & Mutambara, 2020; Thomas & Mahmud, 2021), not much research has been conducted in understanding the topic of the roots of quadratic equations in South Africa. Thomas and Mahmud (2021) found one of the reasons for these difficulties in this topic is lack of prior knowledge from previous grades, which contributes towards the failure to master mathematics at Higher Education Institutions (Tendere & Mutambara, 2020). Gözde and Kabar's (2018) study found that learners lack understanding of the variable concept, and confusing quadratic equations and linear equations.

Sağlam and Alacacı (2012) conducted a study in which they compared the way in which quadratic equations are dealt with in mathematics textbooks in different countries and Clark (2012) in Austin (United States of America) conducted a study which focused on the application of the history of quadratic equations in teacher preparation programmes. Thomas and Mahmud (2021) in Malaysia investigated learners' errors in solving QE using Newman's Procedure and found learner committed errors in comprehension and transformation phases. Wilkie (2021) explored the coordinating and algebraic reasoning with quadratic functions in Australia and found learners likely to connect the generality of structural aspects of figures to symbolic QE. Kim How et al. (2021) identified the teaching styles and challenges led to higher-order teaching skills (HOTS) in quadratic equations. Their (ibid) study found that strategies to teach QE is mainly teacher-centered emphasising memorisation of procedures to solve problems. The teacher-centered approach affects learning as learners cannot understand the concept well with this approach which then impedes the learning of subsequent concepts.

Notwithstanding the above discussion, little attention, however, has been given to describing the roots of quadratic equations, solving quadratic equations to determine the roots, determining the unknown variable(s) when one or two roots is or are given, finding quadratic equations when two roots are given, and finding the other root of an equation when one root is given. This focus was therefore the focus of the study reported on in this article. The study also aimed to some extent on the solving of quadratic equations using the factorisation method, which is the starting point for learners' understanding of the roots of equations.

This article emanates from the Department of Mathematics Education community engagement research project, whose focus is to explore the improvement of pedagogical content knowledge and proficiency in mathematics teaching and learning. The research project has assisted the researcher to establish how Grade 11 learners understand the roots of quadratic equations and contains responses to the following research questions (RQ):

- RQ1: How do learners describe and determine the roots of QE?
- RQ2: How do learners determine the unknown variable(s) when the roots are given and the other roots?
- RQ3: How do learners determine QE by using two given roots?
- RQ4: What are learners' underlying misconceptions regarding the roots of QE, and their origins?

A lot might have been done in understanding mathematics concepts; however, few empirical investigations have been carried out on understanding the roots of quadratic equations in the South African context.

Quadratic equations, understanding, teaching styles, difficulties, and misconceptions learners face in solving quadratic equations.

International literature shows that many secondary-school learners find solving QE difficult from a conceptual point of view (Prima Sari & Jailani, 2019; Vaiyavutjamai et al, 2005). In their study in Turkey, Didis and Erbas (2015) showed that learners had difficulty in solving QE by factorisation, especially if the structure of the equation is different from the ones they are used to. Similarly, Tendere and Mutambara (2020) in Zimbabwe found that learners experience difficulties when solving QE by factoring to determine the roots. These researchers found that learners also attempted to factorise non-factorable quadratic equations and incorrectly guessed factors when using the cross-multiplication method. In other words, the learners incorrectly factorised QE into two linear factors and determined incorrect roots, as they made false guesses when using the cross-multiplication method.

The QE is not just a fundamental topic in secondary school mathematics, but also in the developmental of algebraic understanding (Slamet et al., 2020). This concept is viewed as simple to deal with and can be used in real-life situation such as architecture and sport (Yeow et al., 2019). Gözde and Kaber (2018) argue that learners need to learn QE conceptually, that is for understanding in secondary schools. To solve QE problems, learners can use various methods, factorisation, completing a square, using QE formula, graphical or geometric methods (Gözde & Kaber, 2018). However, solving QE to determine the roots using factoring is emphasized by most teachers over other methods (Hewaraty et al., 2021), which means learners to lose interest in learning this concept. In addition, Hu et al. (2021) found solving QE as challenging compared to other mathematical concepts. Most of the learners struggle to solve QE by the factoring method to determine the roots, which is the most preferred method by the teachers (Herawaty et al., 2021).

The key issue in solving QE is to determine the roots or x-intercept values of the equations (Kim How et al., 2022). The roots of QE are the answers found after solving the equation either by using factoring, completing a square or using QE formula. For learners to understand the roots of QE, they need to know and understand the formal definition of QE (Herawaty et al., 2021). The word "understanding" is designated by Skemp (1976) as critical in his work where he distinguishes between instrumental and relational understanding: these theoretical perspectives are elaborated in this study. For learners to understand how to determine the roots of QE, teachers need to design a teaching and learning environment that would be conducive to learners, and which can promote reasoning, considering both rote learning and algebraic reasoning (Lithner, 2008).

Various teaching styles can be used to teach QE for understanding, for example, representations including arithmetic or numerical, algebraic or symbolic, and visual or geometric representations (Katz & Barton, 2007), to determine the roots of QE. In addition, Hearawaty et al. (2021) showed that squares, rectangular, and unit square cards can be used to teach QE for understanding using factorisation methods. Teachers can use concept images to teach QE, which can assist them to understand their learners better to improve their teaching (Kabar, 2018). However, Sosa-Moguel and Apricio-Landa (2021) found that teachers find it difficult to choose an appropriate style to teach QE for understanding when determining the roots.

Some teachers are found to teach QE at too fast a pace QE which diminishes the interest of learners learning this concept (Prima Sari & Jailani, 2019), which then leads to lack an understanding of QE. Vaiyavutjamai and Clements (2006) revealed that teaching of QE was found to be teacher-centered reinforcing procedural understanding without translating the meaning of symbols, which can contribute learners' difficulties in determining the roots of QE.

Quadratic equations present learners with a number of difficulties, for instance with regard to the algebraic procedures in using the quadratic equation method (the quadratic formula), and consequently they struggle to understand this topic (Didis & Erbas, 2015). Learners find it difficult to solve even simple quadratic equations using the factorisation method, for example $x^2 + bx + c = 0$, where b and $c \in R$, and find non-simple quadratic equations such as $ax^2 + bx + c = 0$, where a, b and $c \in R$ and $a \neq 1$ very difficult to solve. Kotsopoulos (2007) suggests that the cause of the problem in solving quadratic equations by factorisation may lie in the recalling the main multiplication facts, as this lack directly influences learners' ability to solve equations.

Kotsopoulos (2007) revealed that learners could not solve this equation when it was given in the non-standard form of $x^2 + 3x + 1 = x + 4$. The study found, moreover, that most of the learners failed to write the equation in the standard form, and also failed to factorise it. Didis and Erbas (2015) add that learners experience problems when solving quadratic equations when the leading coefficient and/or constant terms have many pairs of possible factors.

Some studies have focused on learner misconceptions about solving quadratic equations. Misconceptions are regarded as the misapplication or misunderstanding of rules (Ojose, 2015). Teachers need to understand their learners' misconceptions, as these can be barriers to learning mathematical concepts (Yassin, 2017). They can be sources of errors which can be exacerbated by the faulty or deficient thinking systems (Mestre, 1987). Teachers' understanding of the prior knowledge of the learners can enable them to shape their strategies (Murphy & Alexander, 2004), as misconceptions can be permanent obstacles to learning new concepts.

Kim How et al. (2021) show that memorisation of facts and procedures can cause learners' misconceptions in quadratic equations. Tendere and Mutambara (2020) support that procedural knowledge can cause misconceptions when learners learn quadratic equations. Tall et al. (2014) found misconceptions about solving quadratic equations to originate from a lack of conceptual understanding of the procedures of linear equations. Vaiyavutjamai and Clements (2006) concur that the difficulty learners experience in solving quadratic equations may arise from a lack of both instrumental understanding and of the relational understanding of mathematics which is associated with quadratic equations. The study conducted by Lima (2008) showed that learners adopt a purely instrumental approach to solving quadratic equations, without paying attention to the unknown as a fundamental characteristic of the equation. Lima and Tall (2010) found that teaching learners to solve QE needs emphasis on using quadratic formula as a general method to determine the roots.

Vaiyavutjamai and Clements (2006) argue that difficulties in, and misconceptions relating to, solving quadratic equations emanate from insufficient instrumental and relational understanding. Tenedere and Mutambara's (2020) study revealed that misconceptions in quadratic equations are due to lack of pre-requisite knowledge of basic concepts and skills. For example, these researchers found that learners thought that the two xs in the equation $(x-3)(x-5)=0$ stood for different variables, even though most of them obtained the correct solutions. Learners simultaneously checked the roots by substituting $x = 3\ into\ (x-3) and\ x = 5\ into\ (x-5)$, finding $0 \times 0 = 0$, and concluded that the roots were correct. With similar misconceptions, Tall et al. (2014) cite instances where learners were unable to find correct solutions to the equation $m^2 = 9$, and attempted to transform quadratic equations into linear equations. In attempting to solve $m^2 = 9$, learners attempted to use exponents associated with the unknown as if it were the coefficient, that is m^2 is the same as $2m$

Didis and Erbas (2015) found that most of the learners used incorrect factors while using the cross-multiplication method, attempted to factorise non-factorable quadratic equations, and incorrectly used either the difference of two squares or the greatest common factor techniques. For example, in the case of $x^2 + 5x + 6 = 0$ learners supplied the answers $-6\ and\ 1$ instead of $-2\ and\ -3$ respectively. Secondly, learners also factorise non-factorable quadratic equations, $x^2 + 2x - 1 = 0\ as\ (x+1)^2\ or\ (x-1)(x+1)$ which has no factors. Lastly, learners have challenges in finding the difference of two squares $9x^2 - 25$ as $(9x-5)(9x+5)\ or\ (3x-5)^2$ and the greatest common factor technique of the equation $3x - 6x^2\ as\ 3x(x-3)\ or\ 3x(x-6)\ or\ 3x(3x-2)$. Vaiyavutjamai and Clements (2006) reported that only a few learners were able to solve the greatest common factor equations $t^2 - 2t = 0\ and\ k^2 - k = 0$.

Theoretical perspectives

In this article, I used Skemp's (1976) relational and instrumental understanding to gain the insights of the Grade 11 learners understanding of the roots of quadratic equations. The word understanding was brought to Skemp's attention by Stieg Mellin-Olsen, of Bergen University. It has two meanings and those meanings have been distinguished as instrumental and relational understanding. Van de Walle (2007) defines understanding as the quality and quantity of connections of pre-existing and new ideas. Understanding depends on how the pre-existing ideas of algebraic linear equations and quadratic expressions can be used to create new connection of ideas to determine the roots of quadratic equations. Skemp (1976) and Van de Walle (2014) describe understanding the rich and strongly interconnected end of the continuum, and is then referred to as 'relational understanding', while the other end of the continuum where ideas are completely isolated, it can be referred to as instrumental understanding.

According to Skemp, relational understanding is when an individual knows what to do and the reasons why learners use certain procedures to solve mathematical problems. Moosavi et al. (2022) refers to relational understanding as an understanding of all the parts, how they are related and why they are used in a particular way. Relational understanding promotes self-discovery which provides meaning to learning and facilitates building cognitive structures (Van de Walle et al., 2014).

Learners with relational mathematics arrive at concepts that are developed from cognitive schemas, which is a network of connected ideas (Machaba, 2016). Machaba (2016) refers to learners with a relational understanding of mathematics as the ones who can think independently, and as knowledge developers and users, rather than merely storage systems and performers. Relational understanding can be understood as making sense of the roots of quadratic equations, working back-and-forth to determine quadratic equations with given roots, determining the other root if one root of quadratic equation is given and determining the unknown coefficients with two given roots.

By contrast, instrumental understanding refers to the use of procedures and formulae without knowing where those procedures and formulae are emanating from. For example, a learner can often get the correct answer more quickly and reliably by instrumental thinking than relational thinking (Skemp, 1976). Instrumental understanding is knowledge learned by rote, which is completely isolated and disconnected, and which promotes memorisation of ideas. It encourages learners to use procedures to solve mathematical problems without making sense or meaning of those problems. Skemp (1976) has called instrumental and relational understanding the two components of conceptual understanding. Conceptual understanding has also been included as one of the strands of mathematical proficiency (Kilpatrick et al. 2001). Instrumental understanding depends on memorisation of procedures to find factors of quadratic equations to determine the roots without making sense of those roots.

Kilpatrick et al. (2001) describes conceptual knowledge, as learners' comprehension of concepts, operations, symbols, diagrams, procedures and relations between conceptions in mathematics. This type of understanding brings forth the integration of mathematics understanding to both content and context. Conceptual understanding is evident when Grade 11 learners can relate the roots of quadratic equations either given in the form $ax^2 + bx + c = 0$, or when one of the equations is given to determine the other root, or when two roots are given to determine the equation, or when solving simultaneous quadratic equations when the two roots of the equation are given.

Bransford et al. (2000) and Carpenter and Lehrer (1999) suggest that learners who are taught how to develop conceptual knowledge can organise knowledge into a coherent whole, and to connect previous knowledge with new ideas. Skemp (1976) refers to the interconnection of networks between ideas as cognitive schemas that are the product of constructing knowledge and tools to add to new knowledge that can be constructed. Learners would therefore represent their mathematical solutions in various ways and recognise how those representations can be useful for different purposes; this would allow them to connect mathematical ideas to various representation and identify the similarities and differences between them when solving quadratic equation problems, and hence mathematical problems.

The learners who acquire knowledge with conceptual understanding can generate new knowledge and solve new and unfamiliar problems (Bransford et al., 1999). Learners who possess conceptual understanding can solve more complex quadratic equations such as determining the root of the equations with one given root, can perform the reversal of quadratic equations when two roots are given, and determine the unknown values of the coefficients when given two roots.

Kilpatrick et al. (2001) describe procedural knowledge as a focus on procedures, when and how to use them appropriately, and having skills to carry them out flexibly, accurately and efficiently. Skemp (1976) agrees that it is insufficient for learners merely to understand *how* to solve mathematical problems (the application of instrumental understanding), and states that those learners need to justify why certain procedures are used for a particular task performed (in other words, they should apply relational understanding). Pegg (2010) postulates that initial processing of information in working memory, which of is of limited capacity, can result in learners following incorrect procedures to solve problems. Hiebert (1999) suggests that memorised procedures can lead learners to the concepts difficult to understand as they may have little knowledge to justify those procedures used to solve a mathematical problem. Learners would memorise procedures to determine the roots of quadratic equations without making sense of those roots. For the effective solution of mathematical problems, it is vital that procedural knowledge be coupled with conceptual knowledge.

The concepts of this framework for this study are the types understanding described by Skemp (1976) as relational and instrumental understanding, which Kilpatrick et al. (2001) describe as conceptual and procedural understanding. In this study, relational and conceptual understanding reveal similar features as the "what" and then "why" certain procedures are used to solve a particular problem and were used interchangeably. In the same vein, instrumental and procedural understanding reveal similar features as learners use memorised procedures to solve mathematical problems without making sense of those problems. These types of understanding were used as a lens to analyse the findings of this study when learners solve QE.

METHODOLOGY

The qualitative approach was adopted to collect data for the study. A case study design was espoused to establish how learners understand the roots of quadratic equations and the misconceptions that emerged when determining the roots of QE. Qualitative data was therefore used for the analysis and interpretation of learners' understanding of the roots of quadratic equations (QE), their determination of the equation with the two given roots, their determination of the unknown values when one root of the equation is given, and learners' misconceptions and the reasons for these. The study was organised around the diagnostic test administered to 42 learners, of which eight learners were purposively sampled to participate in the clinical interviews. The purpose of the interviews was to gain an insight into learners' understanding of the roots of QE, their misconceptions in this regard and the reasons for these. The interviews with the learners were conducted a day after the test was administered to avoid retention loss.

The sampling strategy for learners who participated in the interviews entailed the identification of those who performed poorly $(n = 8)$ in the test. The test instrument comprised 12 question items that were drawn from an examination question bank, some of which were representative of questions in a Grade 11 mathematics textbook. The test was administered during regular class time and learners were given 45 minutes to complete it. Since the intention of the study was to gain insight into and give an account of learners' understanding of and misconceptions regarding the roots of QE, purposive sampling was considered appropriate. The school had already given me permission to conduct a broader study, of which this formed a part.

At the time the data was collected, the learners had studied QE by means of three methods, namely: factorisation, completing a square and using the quadratic formula as stipulated in the Curriculum and Assessment Policy Statement (CAPS) and mathematics textbooks. Other topics that were covered during data collection were solving QE by substitution and finding an equation when the roots are given.

Table 1: Questions assessing learners' understanding of QE

	Question	Motivation for question
1	1.1 How can you define the 'roots' of quadratic equations? 1.2 Find the roots of the following quadratic equations: 1.2.1 $2x^2 - 10x = 0$ 1.2.2 $x^2 + x = 20$ 1.2.3 $8x^2 - 47x - 6 = 0$	1.1 Assess if learners understand the meaning of the roots of QE. 1.2 Assess if learners can use different methods to determine the roots of QE.
2	2.1 If 2 is one root of the equation $kx^2 + x - 3 = 0$. Determine k and the other root. 2.2 One of the roots of the equation $2x^2 - x + m = 0$ is $-\frac{3}{2}$. Determine m and the other root.	The question assesses learners' understanding of QE when one roots of the equation is given to determine the other root.
3	Find a quadratic equation, which has the given roots: 3.1.1 $5; -4$ 3.1.2 $1\frac{1}{2}; -\frac{3}{4}$	This question assesses learners if they can use the two given roots of QE to find the quadratic equation.
3	3.2 Find the values of $a\ and\ b$ if $\frac{1}{2}\ and\ 4$ are the roots of the equation $2x^2 + ax + b = 0$.	This question assesses learners' understanding when given two roots to simultaneously determine the two unknown variables in QE.

Ethical considerations

A letter granting permission to conduct the study was obtained from the Department of Education and the principal of the school at which the study was conducted by the Department of Mathematics Education community engagement project leader. Informed consent was requested from Grade 11 learners after prior permission to conduct the research had been granted. The purpose and the rationale of the study were explained to the learners. Moreover, I established a rapport with the learners by assuring them that my intention was not to evaluate their competencies for the purposes of promotion to the next grade. The roles of learners, and their right to choose whether to participate in this study, was explained to them (Johnson & Christensen, 2012).

Learners who agreed to participate in this study were assured of confidentiality and that their participation was voluntary: they were given permission to withdraw from the study at any time without being prejudiced, and they were assured that pseudonyms would be used to protect their identities. An undertaking was given that pseudonyms such as L_1, L_2 and L_3 would be used instead of learner's real names throughout the study. The confidentiality of information collected at the school was also guaranteed.

Validity and trustworthiness

I collected the data by means of the test instrument and interviews. Prior to the main study, I administered a test comprising 9 items to 25 learners as a means of piloting the test instrument. The test items were discussed with two senior academics in the department of mathematics education with Doctoral Degrees in Mathematics Education and two experienced Grade 11 mathematics teachers to validate the correctness of the content and pedagogical alignment. Learners were given 45 minutes to complete the test. After the piloting phase, two questions were added to the test instrument to elicit a response to the first research question focusing on learner understanding and description of the roots of QE. One question item was eliminated from each of questions 3 and 4, as they displayed similar structural properties, and these were replaced by roots with fractions. This allowed the inclusion of roots of different structural properties in the test instrument, thus providing structural diversity in terms of question items.

The selection of learners to participate in interviews was determined by their responses to the questionnaire. I selected four learners who had performed well and four learners who had performed poorly in the test to obtain data relating to learners' understanding and descriptions of the roots of QE, and learner misconceptions and their causes.

The research was presented at a conference and underwent peer review by two experienced professors in mathematics education, who advised me and made suggestions about the study before it was finalised. The recommendation for the instrument was to add a question that could assess learners to determine the unknown values using simultaneous equations when given two roots. I then added the question 3.2 as in the question paper to assess how learners could find the unknown values using simultaneous equations when given the two roots. The study cannot be generalised, as a small number of learners were interviewed, and they cannot represent the population.

For trustworthiness of this study, I welcomed scrutiny in the process by peers, colleagues and academics to deal with biases and assumptions relating to interpretation of findings. Furthermore, after marking of scripts, I shared the marked scripts with the mathematics teacher for moderation and minor errors of addition of marks were found and corrected. Member checking was also done with the learners who participated in the interviews for them to confirm if the interpretation of the finding reflected exactly what they said.

Data analysis procedures

Thematic analysis is used to analyse the collected data as the study espoused a qualitative approach. Firstly, data collected from the diagnostic test and the interviews were prepared and organised for analysis. The test used Didis and Erbas' (2015) correct, incorrect, incomplete, and blank responses by using absolute numbers to calculate the percentages. Secondly, the interview data were transcribed and captured using Microsoft Excel arranged according to the questions used to establish the reasons why they used a particular method to determine the roots of QE. The researcher repeatedly read through the data for analysis and interpretation purposes (Creswell & Creswell, 2018). Lastly, data were coded by bracketing chunks of text from which themes were generated. The generated themes were compared with the collected data to ensure accuracy of the data. Furthermore, the generated data were informed by the constructs emanating from the conceptual framework used to underpin this study. Four themes were developed which provided a thick description of the collected data. The meanings of the themes generated were then interpreted. Learners were coded as $L_{1,}$ $L_{2,}$ $L_{3,}$ L_4 and so on, L_1 represents learner number 1 who wrote the test and participated in the interviews.

FINDINGS AND DISCUSSION

The data generated from the test followed the categories of learner responses devised by Didis and Erbas (2014, p. 1141), namely: *Correct Responses (CR), Incorrect Responses (InR), Incomplete Responses (IR), and Blank Responses (BR).* To obtain an overall view of the performance of the learners, the percentages were calculated using the absolute numbers (Didis & Erbas, 2014). Any responses that were not entirely complete, which included those that were correct mathematically, were categorised as *IR* which show that learners used procedures to find the roots of QE with limited understanding. In addition, *InR* were computationally incorrect answers where learners could have memorised the procedures used to determine the roots of QE, and *BR* were those that the learners left unanswered where learners could have seen the roots of QE being isolated or disconnected.

The rationale for using the categories devised by Didis and Erbas was to obtain a descriptive picture of the learners' responses in the test using relational/ conceptual understanding and instrumental/ procedural understanding to support the four different responses of the learners. This part of the study presents a discussion of the findings from the data collected from Grade 11 learners. Reflection on the findings will lead to implications for learners' conceptual/ relational understanding and procedural/ instrumental understanding of the roots of QE. Table 2 below shows the responses to questions one to three which consists of nine sub-questions, which revealed that learners performed poorly in solving quadratic equation problems. The highest percentage for the test in terms of *CR* was found to be *69.1%* and the lowest *4.8%*. The test comprised nine sub-questions altogether.

Table 2*: Distribution of Correct, Incorrect, Incomplete, and Blank Responses to Test Questions* $(N = 42)$

Question Items	*Correct f (%)*	*Incorrect f (%)*	*Incomplete f (%)*	*Blank f (%)*
1.1	(0.%)	33 (78,6%)	0 (0.0%)	9 (21.45)
1.2.1	29 (69.1%)	13 *(31.0 %)*	0 *(0.0 %)*	0 *(0.0 %)*
1.2.2	19 (45.2%)	20 *(47.6 %)*	3 *(7.1 %)*	0 *(0.0 %)*
1.2.3	8 *(19.1%)*	13 *(31.0 %)*	15 *(35.7%)*	6 *(14.3 %)*
2.1	15 (35.7%)	9 *(21.4 %)*	4 *(9.5 %)*	14 *(33.3 %)*
2.2	11 (26.2%)	13 *(31.0 %)*	7 *(16.7%)*	11 *(26.2 %)*
3.1.1	17 (40.5%)	9 *(21.4 %)*	7 *(16.7%)*	10*(23.8 %)*
3.1.2	6 (14.3%)	9 *(21.4 %)*	10 *(23.8%)*	17 *(40.5%)*
3.2	3 (7.1%)	8 *(19.1 %)*	9 *(21.4%)*	22 *(52.4 %)*

Table 2 above depicts the responses to question one to nine items, based on QE. The results are discussed below. In this study I used the four themes developed to analyse and discuss the findings of this study.

How learners understand the roots of QE

The findings showed that when describing the roots of QE, most of the learners viewed the roots of QE as factors of $ax^2 + bx + c = 0$, understand it when presented in standard form. Learners failed to describe the root of QE as the solution or zeros of *x*, and were unable to understand the zero-product principle, which states that to obtain two distinct roots, we use the fact that $a \times b = 0, then\ a = 0\ or\ b = 0$. Tendere and Mutambara (2020) postulate that learners find it difficult to solve QE to determine the roots.

Figure 1: Sample from L_3 and L_6

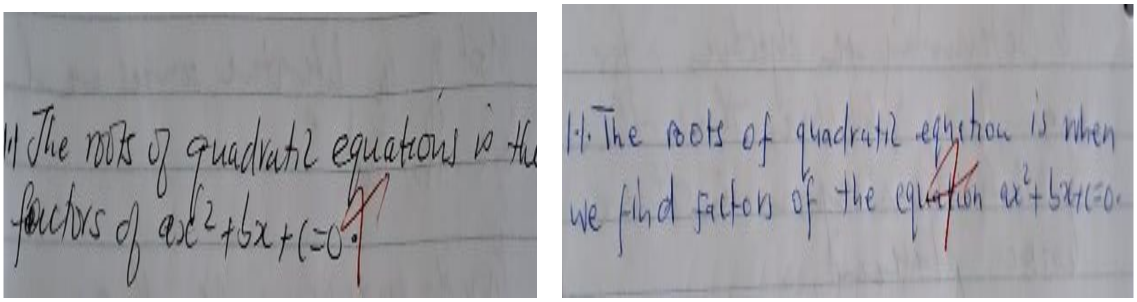


Figure 1 Samples from L3 and L6

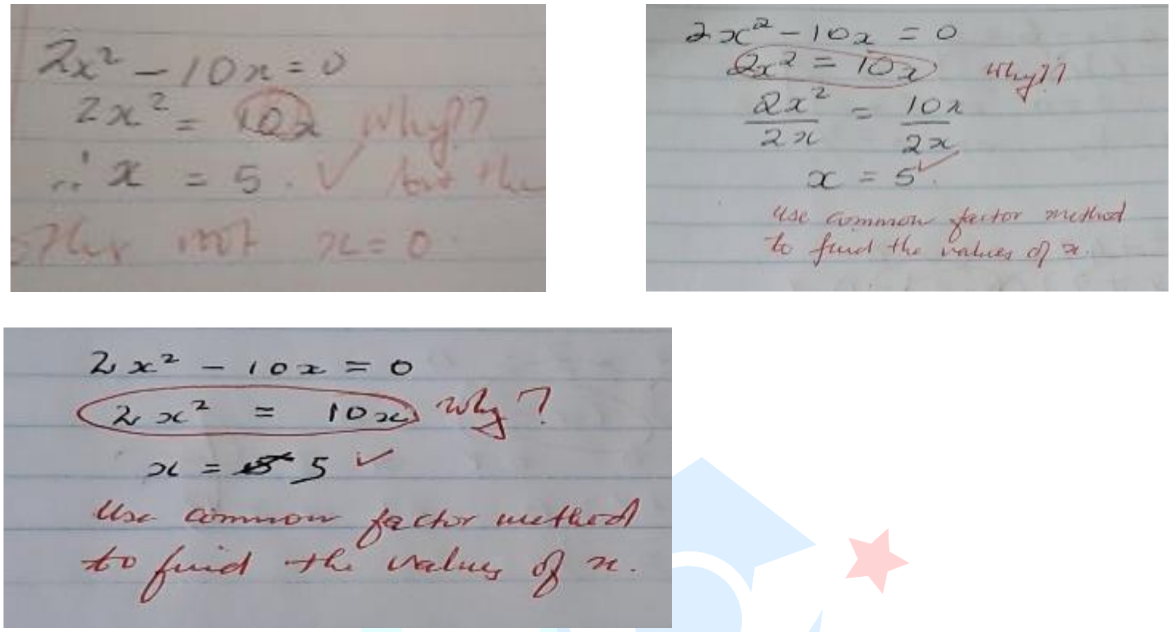
When asked why they define the roots of QE like this, L_3 (quoted verbatim) said *"When we find the roots of quadratic equations, we factorise the equation"*, and L_6 said *"We find the factors of the equation when we find the roots"*. This finding was also evident in the research of Kim et al., (2021) and Tendere and Mutambara (2020) that learners appeared to have memorised procedures when solving QE to determine the roots. Herawaty et al. (2021) claim that for learners to understand the roots of QE, they need to know the formal definition of the formal QE. For example, those learners could not find the roots of the equations presented in the form $2x^2 - 10x = 0$ and $x^2 + x = 20$, as these QEs are not presented in standard form. This supports Pegg's (2010) and Hiebert's (1999) studies that reported that memorised procedures can result in learners following incorrect procedures.

Lack of understanding of the zero-product property law was evident in instances where learners were asked to determine the roots in sub-questions 1.2.1 to 1.2.3 in table 2. Learners who gave correct responses were 69.1%, 45.2% and 19.1% respectively, this depicts that learners appeared to have learned procedures to determine the roots of the equations as they struggled to solve sub-question 1.2.3 and some learners used procedures to solve this problem. Skemp (1976) suggests that learners should possess both instrumental understanding (knowing the procedures for solving problems) and in addition relational understanding (knowing how to solve problems and why).

Most of the learners appeared to have procedural understanding of how to determine the roots of QE. They appeared to have memorised and practised procedures that they appeared to find difficult to understand (Kim et al., 2021); this may have led to reduced understanding of the significance of and reasoning behind those procedures, as suggested by Hiebert (1999) and Tendere and Mutambara (2020). Learners should have back-multiplied and checked whether they would obtain the original equation and should also have back-substituted the roots in the original equation to check whether they satisfied the equation by getting the answer zero.

In question 2.1.1, some of learners had transposed $-10x$ to have $2x^2 = 10x$ and had divided both sides by $2x$ to get the root as $x = 5$. The reason given by L_1 and L_2 was that the equation should have the third term and can only be solved by eliminating x to get the answer $x = 5$. The learners appeared not to have paid attention to the equation before solving it. Below are the samples for L_1, L_3, and L_4.

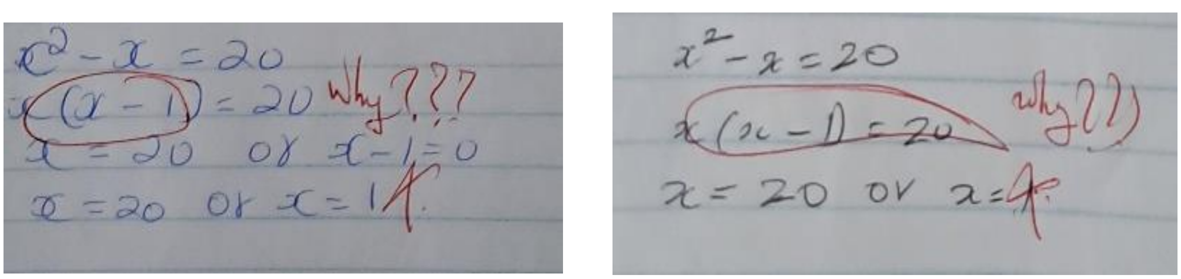
***Figure 2: Sample from L*₁*, L*₃ *and L*₄**

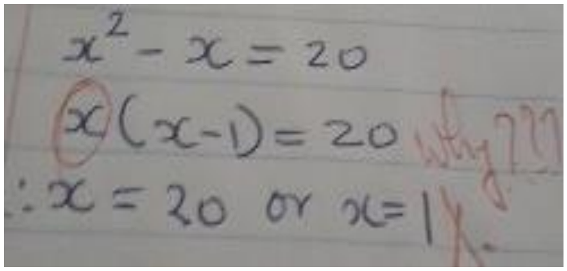


These samples from L₁, L₃, and L₄ have transposed $-10x$ and divided both sides by $2x$. Tendere and Mutambara (2020) and Vaiyavutjamai and Clements (2006) reported similar findings, stating that only a few learners are able to answer greatest common factor equations, giving the examples $t^2 - 2t = 0$ and $k^2 - k = 0$. Similarly in question 1.2.2 ($x^2 + x = 20$) where learners 45.2% obtained CR, and other shared amongst *InR (47.6%)* and *IR (7.1%)*, most of the learners gave factors without understanding them, as in sub-question 1.2.3; when determining the roots, those factors given are $(x-10)(x+2), (x-5)(x+4), (x-20)(x-1), (x-5)(x-4), (x-2)(x+10)$.

The first, the second and the last responses showed that learners had used incorrect procedures to determine the roots of the equation, which demonstrated an instrumental understanding where they appeared randomly found factors without considering the middle term and last term without considering their signs. Others used the common factor method $x(x+1) = 20$ to obtain the roots of the equation which was the inappropriate method.

***Figure 3: Sample from L*₁*, L*₂ *and L*₄**



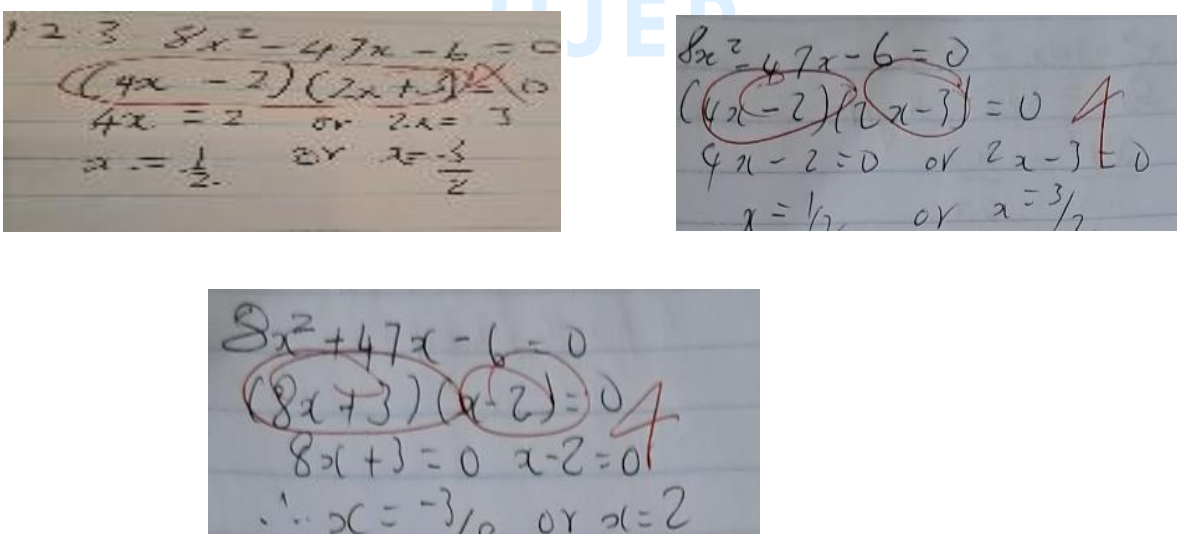


These responses of learners showed lack of understanding of quadratic equations that are given in non-standard form as suggested in Hu et al. (2021) and Kotsopoulos (2007). In learners' interviews, memorisation of procedures seems to have contributed in determining the roots by factorisation. For example, L_1, L_2 and L_4 said that when the equation is given in the form $x(x+1) = 20$, it means that $x = 20\ or\ (x+1) = 0$. Only sub-question 1.2.1 required them to use common factor method to determine the roots of quadratic equations. This sample is from L_2 who supported her answer *(quoted verbatim)* that $x(x+1) = 20$ is the same as $ab = 0$, however, the given QE is not equal to zero (*the equation* $x(x+1) = 20$ *is same as* $ab = 0$*, therefore* $x = 20\ or\ (x+1) = 0$).

The findings of Didis and Erbas (2015) showed that learners give incorrect factors when using cross-multiplication methods and attempting to factorise QE to determine the roots. L_1, L_2, and L_4 in the interview reasoned that the factors are found by using the first term $x^2\ and - 20$, without checking whether the factors that satisfy the equation or not. Furthermore, learners showed lack of understanding to the given equations as they did not mention the importance of considering the middle term and its sign when finding the factors of to determine the correct roots of the equation.

These findings concur with Kotsopoulos (2007) study learners' problems in factoring quadratic equations lie in multiplying facts which influences the ability to solve the equations. Similar results were found in sub-question 1.2.3 as earlier noted, learners had just randomly gave factors to determine the roots of the equation $8x^2 - 47x - 6 = 0$, where 19.1% of learners gave *CR* and 80.9% of learners shared *InR (31.%), IR (35.7%)* and *BR (14.3%)*. Some learners just gave factors of this equation without considering the middle term as $-47x$, which showed a lack of instrumental and relational understanding.

Figure 4: Sample from L_3, L_7 L_8



In answering this question $8x^2 - 47x - 6 = 0$, most of learners gave factors of the equation as $(4x-2)(2x+3), (8x-6)(x-1), (4x+2)(2x-3), (8x-1)(x+6),\ and\ (8x+6)(x-1)$ instead of giving factors as $(8x-1)(x-6) = 0$. For example L_5, L_8 and L_{11} gave factors of $8x^2 - 47x - 6 = 0$ as $(4x-3)(2x+2), (4x-2)(2x-3), and\ (8x+3)(x-2)$ respectively, without justifying the procedures they used to solve the equation; however, L_3 gave the correct factors of the equation $(8x+1)(x-6) = 0$ and gave the roots as $x = \frac{1}{8}\ or\ x = 6$, and was able to justify the procedures used to solve the equation.

Furthermore, he (L_3) indicated that when the middle term is negative, *re swanetše go be le positive and negative numbers ka gare ga di-brackets (we should have positive and negative numbers in the brackets) and the bigger number should be negative because of the middle term.* The learners did not consider the middle term when finding the roots of this equation which revealed memorisation of procedures as they seem to have forgotten some of the procedures when multiplying those factors, such as considering the middle term and its sign.

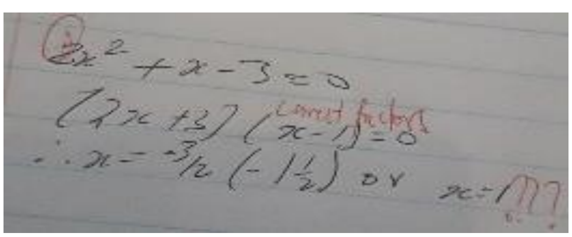
In the interview, L_1, L_2 and L_4 said that when finding the factors of QE, they concentrate on the multiplication of first factors and last factors of the equations and had not considered the middle term and their signs. These findings are similar to the findings of Prima Sari and Jailani (2019) that many secondary school learners find it difficult to solve QE from a conceptual point of view.

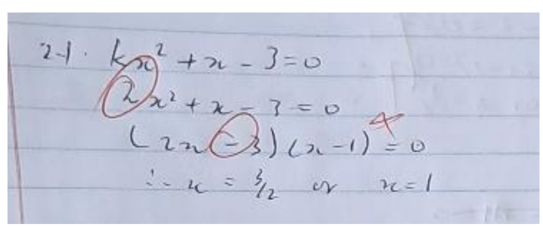
How learners determined the unknown variable(s) when the roots were given and the other roots

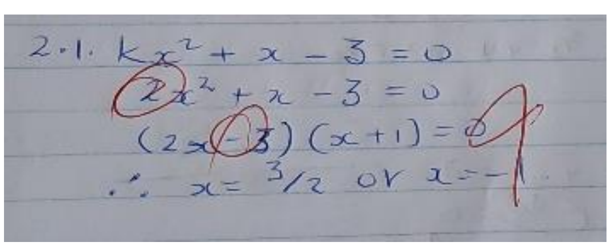
The findings revealed that that learners did not comprehend what to do if one root was given and the requirement was to find the other root. It is not clear why they followed incorrect procedures. Skemp (1976) and Kilpatrick et al. (2001) postulate that if learners lack comprehension of what procedures they follow and why they follow those procedures, they lack conceptual relational understanding of the concept. This lack of understanding is revealed in sub-question 2.1 and 2.2 where only 35.7% and 26.2% gave *CR*, others were shared amongst *InR (21.4%,31.0%), IR (9.5%, 16.7%)* and *BR (33.3%, 26.2%)*, where learners have used wrong procedures when one root is given to determine the other root of the two QEs.

For example, in solving sub-question 2.1 $kx^2 + x - 3 = 0$, most the learners substituted the value of *k* by 2 and found the new equation to be $2x^2 + x - 3 = 0$, which they ultimately factorised to determine the roots. It was evident that L_1, L_2, and L_4 had followed the same procedure, and gave the incorrect factors as $(2x-3)(x-1), (2x+1)(x-3), and\ (2x-3)(x+1)$, which was indicative of incorrect cross-multiplication, as postulated in the study of Vaiyavutjamai and Clements (2006).

Figure 5: Sample of L_4, L_7 and L_8



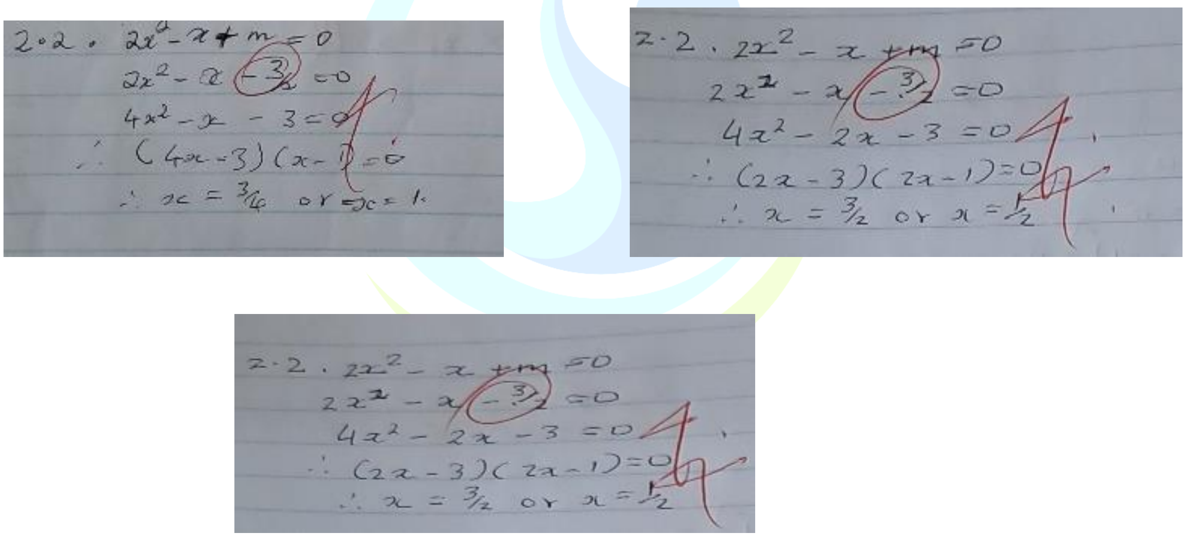




In the sample of L_4 who found the correct factors of $2x^2 + x - 3 = 0$ as $(2x + 3)(x - 1) = 0$ and found the roots as $x = -\frac{3}{2}\ or\ x = 1$, though this learner did not follow the correct procedures to determine the other root of the QE. L_4 said *"the question said, if 2 is one root of the equation $kx^2 + x - 3 = 0$, it means we have to substitute k by 2 and factorise the equation"*. This showed lack of understanding of the concept as learners followed wrong procedures without understanding why they use those procedures to solve equation to find the other root of the given QE. The learners' responses contracts what Moosavi et al. (2022) argue in their study that learners should have relational understanding as an understanding which procedure to follow and the logic behind those procedures used to solve mathematical problems.

Similarly, most of the learners had substituted $m\ by -\frac{3}{2}$ in sub-question 2.2 $2x^2 - x + m = 0$ and found the new equation as $2x^2 - x - \frac{3}{2} = 0$, which was given as after multiplying QE both sides by $2, 4x^2 - 2x - 3 = 0$ and had factorised the equation to determine the roots as follows:$(4x - 3)(x - 1), (2x - 3)(2x - 1), (2x + 3)(2x - 1), (2x - 3)(2x + 1) and\ (4x + 3)(x - 1)$.

Figure 6: Sample of L_1, L_2 and L_5



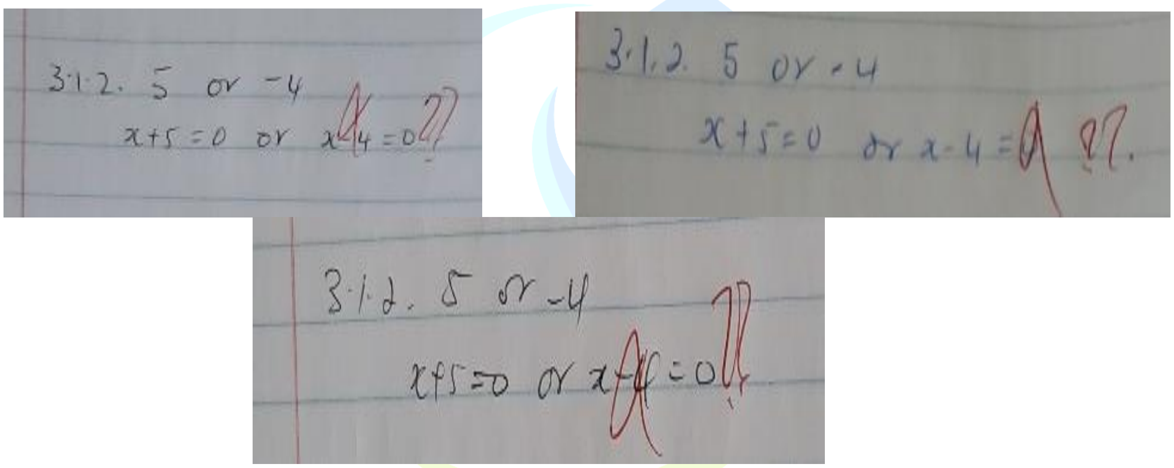
Learners also did not realise that the new equation $4x^2 - 2x - 3 = 0$ could not be factorised to determine the roots; either using the completing the square method or the quadratic formula could have been used to solve this equation. For example, L_1 (quoted verbatim) in the interviews said *"ye e ya re hlakahlantšha o ka se tsebe gore yona di-fekethara tša nnete ke tše dife (this one is confusing you will not know which factors are the correct ones)"*. This finding concurs with the study of Didis and Erbas (2015) who maintain that learners struggle to solve QA when the leading coefficient and/ or constant terms have more than one pair of factors.

Gözde and Kaber (2018) further claim that learners who have conceptual understanding are able to generate knowledge and to solve new and unfamiliar problems. These sub-question 2 items appeared unfamiliar to them, and the lack of conceptual knowledge appeared to have had a negative outcome when using a root that is given to determine the other root of QA. Furthermore, learners' memorisation and practising of procedures that they found difficult to understand, as suggested by Kim et al. (2021), could have caused this problem. This in turn caused learners to have less understanding of the meaning of and reasoning behind those procedures when finding the value of unknown variable(s) and the other root(s) when there are given roots.

How learners determined QE by using two given roots

The findings relating to this question revealed procedural knowledge rather than conceptual knowledge (Kilpatrick et al., 2001). In sub-question 3.1.1 and 3.1.2, learners obtained 40.5% and 14.3% of *CR* were given and the first sub-question appeared to have been easier for the learners to answer than the second one. A high percentage of learners' outcomes were distributed across *InR, IR* and *BR* respectively, which showed a lack of relational understanding; they struggled to observe the connectedness of the QE in the form $ax^2 + bx + c = 0$ and their roots. It was not noticed by the learners that the two given roots can be used to find QEs, 3.1.1 $5\ or - 4$ and 3.1.2 $1\frac{1}{2}\ or \frac{3}{4}$. Most of the learners simply gave answers by $x + 5 = 0\ or\ x - 4 = 0$ instead of starting with $x = 5\ or\ x = -4$.

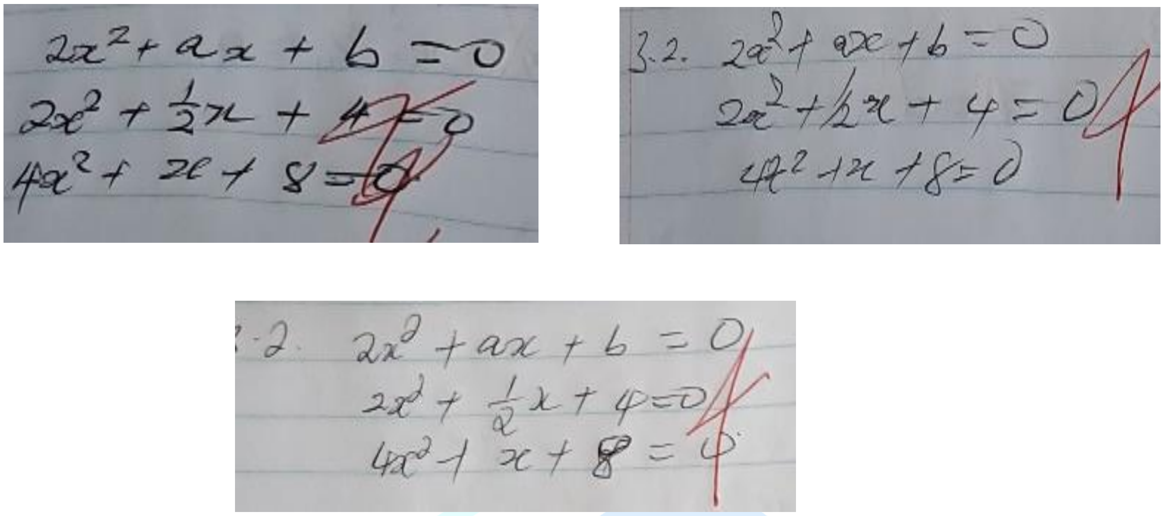
Figure 7: Sample from L_2, L_5 and L_6



They did not know that the roots 5 and -4 represent $x = 5\ or\ x = -4$. The same conclusion was reached in relation to sub-question 3.2: learners began with $x + 1\frac{1}{2} = 0\ or\ x - \frac{3}{4} = 0$ instead of analysing the roots as $x = 1\frac{1}{2}\left(\frac{3}{2}\right)\ or\ x = -\frac{3}{4}$. Some of the learners abandoned this problem at the first statement. Most of the learners did not attempt to answer this question, possibly as a result of their lack of conceptual knowledge of QE (Prima Sari & Jailani, 2019). Learners did not know that determining the equation with the given roots was the reverse process of finding the roots of QE by factorisation. Skemp (1976) maintains that it is not enough for learners to have instrumental understanding only; this needs to be coupled with relational understanding to solve unfamiliar problems.

The same conclusion was reached when learners solved sub-question 3.2, Here they regarded finding the roots, and the equation in quadratic form as disconnected and isolated. Without an integration of the concepts, the only recourse learners have is memorisation of procedures. This finding was supported by Skemp (1976). Most of the learners substituted the value of a by $\frac{1}{2}$ and b by 4 and obtained the new equation as $2x^2 + \frac{1}{2}x + 4 = 0$ and have multiplied all the terms both sides by 2 to have $4x^2 + x + 8 = 0$.

Figure 8: Sample from L_1, L_2 and L_5



Learners did not realise that since they were given two roots and two unknown values, they should have used simultaneous equations to determine the values of a and b, which revealed that sub-question 3.2 was unfamiliar to them. However, the findings of the current study do not confirm with Bransford et al's. (1999) study pointing out learners should have conceptual understanding to be able to generate new knowledge and solve new and unfamiliar problems.

Learners' misconceptions regarding the roots of QE, and their origins

Misconceptions regarding variables in solving QE constitute obstacles to learning other concepts. The first misconception identified in this study was learners' perception of QE as determining the unknown variable of any given equation, with no grasp of the processes for solving those equations. Skemp (1976) states that it is not enough for learners merely to know how to solve mathematical problems; they should also know why the procedures are appropriate for the problems.

For example, L_1, L_2, L_3, and L_4 and their responses recorded in the test scripts described the roots as the factors of quadratic equation $ax^2 + bx + c = 0$. This showed that learners may have perceived QE as a mere calculation without understanding underlying concepts such the roots of the equation. The heart of this problem may lie in misunderstanding of the meaning of the roots of QE. Furthermore, the problem may arise from symbols used to perform operations, since, as Lima (2008) has suggested, learners may not be aware of the concepts that are involved.

The second misconception was found to be a misunderstanding of the use of cross-multiplication when finding the factors of QE to determine the roots. For example, most of the learners were not able to find the correct roots of $2x^2 - 10x = 0$, identifying only one solution by transposing $-10x$ and dividing both sides by $2x$ to get their roots as $x = 5$. This finding confirms the idea in the study by Tall et al. (2014), which revealed that only a few learners are able to factorise the equations, $t^2 - 2t = 0\ and\ 3k^2 - k = 0$. L_1, L_2 and L_4 stated in the interviews that since they had calculated the value of *x*, they had to eliminate by transposing $-10x$ and divided by $2x$ to find the value of *x*.

The findings are also consistent with those of Didis and Erbas (2015) and suggest that learners were not aware of the missing zero when cancelling an *x* in the equation $2x^2 = 3x$, suggesting that they lacked understanding of the zero-product property. The other problem was that of determining the roots of non-standard equations and those in which the leading coefficient and constant have more than one pair of factors. This was found in the study of Didis and Erbas (2015) which showed that learners find it difficult to solve equations with the leading coefficient and constant with more than one pair of factors.

For example, learners solved the two equations $x^2 + x = 20$ and $8x^2 - 47x - 6 = 0$. In the first example, the equation was in non-standard form, and learners were required to write it in the standard form before factoring the equation. Although the equation is factorable, some learners used the common factor method to find the roots of the equation $x(x+1) = 20, x = 20\ or\ x = -1$ instead of $(x-4)(x+5) = 0$. Rote learning and lack of relational understanding can be the underlying misconceptions of this nature (Vaiyavutjamai & Clements, 2006). The second example had more than one pair of factors of the coefficient and a constant to determine the roots of the equation and had given the factors as $(4x-3)(2x+2), (4x-2)(2x-3), and\ (8x+3)(x-2)$ which appeared to have been challenging to learners to determine the roots of the equation. Learners appeared to use false guesses when using the cross-multiplication method to determine the roots.

The third misconception identified was the misrepresentation of the roots given to determine the value of unknown variables and the other roots in the equation. For example, most learners had misrepresented the value of $k = 2$ in the equation $kx^2 + x - 3 = 0$ and $m = \frac{3}{2}$ in the equation $2x^2 - x + m = 0$. Leaners need to have substituted the variable x by 2 in the first equation to determine the value of k and substituted the variable x by $\frac{3}{2}$ in the second equation to determine the value of m. This finding matched those observed in the study of Lima (2008) pointing out that most learners focus predominantly on the variables to determine factors without any understanding of the concepts.

Most of the learners appeared not to have read sub-questions 2.1 and 2.2 with understanding, as they simply substituted the value of *k* by 2 and the value of *m* by $\frac{3}{2}$. L_1, L_2 and L_4 stated that since *k* is unknown in the equation, it could be represented by 2. A similar response was given by these three learners (L_1, L_2 and L_4) to question 2.2, as *m* was substituted by $\frac{3}{2}$. The problem of solving non-factorable equations became evident when learners tried to determine the roots in 2.2, with most of the learners giving factors in $2x^2 - x + \frac{3}{2} = 0\ (4x^2 - 2x + 3 = 0)$ as $(4x-3)(x-1), (2x-3)(2x-1), (2x+3)(2x-1), (2x-3)(2x+1) and\ (4x+3)(x-1)$. The learners used false guesses in finding the factors of the equation by means of the cross-multiplication method, as suggested by Didis and Erbas (2015). The difficulties arise from a lack of relational understanding of the associated mathematics, as Vaiyavutjamai and Clements (2006) explain.

The last misconception identified in this study related to the reversal process using the given roots to find the equation. Skemp (1986) argues that when one network of ideas is recalled, it may lead to another network of ideas being recalled. In this context, if learners are unable to determine the roots of QE, it may be difficult for them to use the roots to determine the equation which is the reverse process. Most of the learners approached problems in sub-question 3.1 with the given roots 5 and -4 by first of all introducing the brackets as $(x+5)\ or\ (x-4)$ instead of interpreting the given roots as $x = 5\ or\ x = -4$ before they were able to continue with the reverse process to find the equation $x^2 - x - 20 = 0$.

It is suggested that once learners have memorised and practised steps to solve mathematical problems, they tend to have little understanding of the concepts that they find difficult to understand, they may have less understanding of the significance of and the reasoning behind those procedures according to Hiebert (1999). The finding agrees with the study of Pegg (2010) who points out that learners with limited capacity result in following incorrect procedures to solve problem.

CONCLUSION

The theoretical lens used to look in the findings of this study has provided the focus on understanding what to do and why use certain procedures to solve QE problems. The study has used two components of understanding which appear to be interrelated, merged relational and conceptual understanding, merged instrumental and procedural understanding to gain the insights of Grade 11 learners' understanding of QE.

The findings of this study revealed learners' difficulties and misconceptions when solving QE which confirmed literature sources used in this study (Prima Sari & Jailani, 2019; Didis & Erbas, 2015; Tendere & Mutambara, 2020; Vaiyavutjamai et al., 2005). Firstly, the difficulties and misconceptions revealed maybe the result of inability to describe the roots of QE viewed as the factors of the given equation $ax^2 + bx + c = 0$ written in standard form.

Secondly, learners' inability to answer the greatest common factor equations, inability to solve QE given in standard or non-standard forms may be due to lack of both instrumental and relational understanding. These findings coincide with the study of Didis and Erbas (2015), Kim How et al. (2021), Tall et al. (2014), Tendere and Mutambara (2020) and Vaiyavutjamai and Clements (2006) in their investigations into the difficulties and misconceptions learners had when solving QE.

Thirdly, the findings showed learners' inability to determine the root of the given QE when one root is given to determine the other root. Furthermore, learners had difficulties and misconceptions when determining the QE with two given roots to determine the equation. This concurs with the López et al. (2016) study that learners have difficulties with QE with one unknown. The difficulties may emanate from the lack of instrumental and relational understanding, pre-requisite knowledge of basic QE knowledge (Tendere and Mutambara, 2020; Vaiyavutjamai & Clements, 2006). Lastly, learners could not determine the unknown variables given in the QE using the two given roots. These difficulties and misconceptions may be the result of the lack of prior knowledge of basic QE or misapplication of rules and principles when solving equations, as Ojese (2015) suggested.

The study has shown that learners' lack of relational and instrumental understanding hinders the performance in determining the roots of QEs. My contention is that if learners lack an understanding of what has to be done and why, they tend to use incorrect procedures to determine the roots of QEs given any form, for example, either given in standard form or non-standard form, using one root to determine the other root, when QE has a leading coefficient and constant with more than one pair of factors, when the two roots to find the QE and when two roots of QE are given to determine the two unknown values in the same equation to two equations to be solved simultaneously.

The study suggests that the dichotomies from two complementary theories, that of Skemp and Kilpatrick et al., on the one end of the spectrum being instrumental or procedural understanding, and on the other end relational or conceptual understanding be given priority in the teaching and learning of QEs. In this study I concur with Skemp (1976) that learners should not only have instrumental/ procedural knowledge, but that this knowledge should be coupled with relational/ conceptual knowledge. Learners should know both what to do which is instrumental/ procedural understanding and the reason why certain procedures which is relational/ conceptual understanding, are used to solve QEs problems and other mathematical problems. While these dichotomies can be described with theoretically distinct characteristics, in practice they may merge.

The study further suggests that if mathematics teachers can teach QEs for relational/ conceptual understanding which is the understanding at rich and strongly interconnected end of the continuum, this can reinforce learners' using correct procedures with understanding. Mathematics teachers need to be aware of the difficulties and misconceptions to shape their instructions (Murphy & Alexander, 2004), when solving QE when given in standard and non-standard forms, when one roots is given to determine the other root, when two roots are given to determine the equation and when two roots are given to determine the unknown variables in QE, as misconceptions can be permanent obstacles to learning subsequent concepts (Yassin, 2017). Further studies can focus on how teachers can teach QEs for relational/ conceptual understanding either in grade 10 or grade 11.

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