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Diagrams support spontaneous transfer across whole number and fraction concepts

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Abstract

In mathematics, learners often spontaneously draw on prior knowledge when learning new ideas. In this study, we examined whether the specific diagrams used to represent more familiar (i.e., whole number division) and less familiar ideas (i.e., fraction division) shape successful transfer. Undergraduates (*N* = 177) were randomly assigned to demonstrate fraction division in a 3 (Diagram: Number Line, Circle, None) x 3 ("Warm-up" Example: Whole Number Division, Fraction Addition, None) between-subjects design. We hypothesized that transfer from whole number division would be greatest in the number line condition. When using number lines and warming up with whole number division, students generated more accurate conceptual models of fraction division. However, both number lines and circles supported transfer from whole number concepts to fraction concepts, whereas having no diagrams did not. Diagrams may play a critical role in helping learners make use of their vast prior knowledge.

Word Count: 146

Keywords: fraction learning; analogical transfer; diagram; conceptual understanding; number line

Diagrams Support Spontaneous Transfer across Whole Number and Fraction Concepts

1 Introduction

When engaging with new mathematical tasks, learners often have a wealth of prior knowledge of more familiar concepts that can shape their construal of, and approaches to, novel tasks. For example, by the time children are first introduced to fractions and their operations, they have considerable experience with prerequisite whole numbers, measurement, and division concepts (see NGA & CCSSO, 2010). Also, they have experience with the types of visual representations used to represent fractions, including shapes such as circles, rectangles, and lines (see NGA & CCSSO, 2010). A growing body of work demonstrates that learners often spontaneously transfer from prior knowledge, or draw on their understanding of previously-encountered problems and concepts, when tackling new problems. We define *spontaneous transfer* as learner-initiated transfer, in contrast to transfer that is explicitly directed by another (i.e., through hints, see Gick & Holyoak, 1980, or direct instruction, see Sidney, 2020). Spontaneous transfer shapes learning and problem-solving across the lifespan. For example, infants (Chen et al., 1997) and older children (Sidney & Alibali, 2017) apply strategies from previously-solved problems to new problems that immediately follow, even when subsequent problems are perceptually dissimilar. In doing so, they leverage prior experiences to solve later problems more efficiently. Similarly, when adults solve conceptually-similar, but perceptually-dissimilar problems, earlier strategies influence later ones (Day & Goldstone, 2011), even across a 24-hour delay (Schunn & Dunbar, 1996).

Several aspects of learners' prior experiences, and the relationship between prior experiences and the task at hand, can shape whether spontaneous transfer is likely to occur. Indeed, many studies of cognition and learning have demonstrated that features of the prior learning episode increase the likelihood of spontaneous transfer to new tasks. Opportunities to compare multiple exemplars (e.g., Gentner et al., 2003), using both concrete and abstract instantiations (e.g., McNeil & Fyfe, 2012), and using linguistic cues that highlight conceptual structure (e.g., Gentner & Hoyos, 2017) during the learning episode can contribute to the likelihood of transfer to later problems. Much less research in cognition and learning has focused on features of new tasks that support, or hinder, productive transfer from learners' prior experiences (see Sidney & Thompson, 2019). However, several researchers have proposed theoretical frameworks for transfer (Barnett & Ceci, 2002; Klahr & Chen, 2011; Nokes-Malach & Mestre, 2013) that suggest that the context of the transfer task itself can facilitate, or fail to facilitate, productive transfer. For example, Klahr and Chen (2011) proposed a 3-dimensional model of "transfer distance" to account for how similarities between familiar and novel tasks, similarities in the contexts and settings in which those tasks occur, as well as the intervening time between them all shape the likelihood of spontaneous transfer amongst learners.

The goal of the current study was to examine one aspect of context in mathematical tasks that may facilitate spontaneous transfer from learners' familiar prior knowledge: visual representation. Here, we examined whether the nature of the visual representations used to represent more familiar and less familiar mathematical ideas contribute to the likelihood of successful spontaneous transfer across them. First, we review the role of visual representations in mathematics learning, and fraction learning in particular. Then, we turn to a discussion of the role that visual representations may play in inviting transfer from learners' familiar whole number concepts to more challenging fraction concepts.

1.1 Visual Representations

External visual representations, such as diagrams, illustrations, and manipulatives, often support students' learning, performance, and transfer to new situations (Butcher, 2006; Cooper et al., 2018; Moreno et al., 2011; Sidney et al., 2019, see Mayer, 1999, 2005). In particular, diagrams that represent key mathematical relationships as spatial relationships can support learners' conceptual understanding (e.g., Hamdan & Gunderson, 2017; Kellman et al., 2008; Larkin & Simon, 1987; Moreno & Mayer, 1999; Moss & Case, 1999; Rau et al., 2014; Sidney et al., 2019). "Conceptual understanding" refers to a deep understanding of the meanings of symbols and the mathematical relationships between elements within a mathematics problem[1] (see Crooks & Alibali, 2014). For example, visual representations can improve learners' conceptual understanding of fraction division. Although few American adults can articulate what it means to divide by a fraction (e.g., Ball, 1990; Bentley & Bosse, 2018; Luo et al., 2011; Ma, 1999; Sidney et al., 2015; Yao et al., 2021), diagrams such as those in Figure 1 can help learners to conceptualize fraction division as the number of times the second operand "fits" into the first (Sidney et al., 2015; Sidney et al., 2019).

In the U.S., recommendations for instructional practice, such as the Common Core State Standards for Mathematics (NGA & CCSSO, 2010) and several practice guides from the Institute of Education Sciences (IES) aimed at improving math instruction (e.g., Fuchs et al., 2021; Siegler et al., 2010; Woodward et al., 2012), often include specific recommendations for using visual representations during teacher instruction and student practice. However, research in math and science learning demonstrates that not all visual representations are equally effective at supporting learners' understanding of conceptual relationships within a domain. Even very subtle perceptual differences between diagrams can elicit different mathematical processes. Children's (e.g., Alibali & Sidney, 2015; Shrager & Siegler, 1998; Siegler & Alibali, 2004) and adults' (e.g., Fazio, DeWolf et al., 2016; Sidney et al., 2019) mathematical thinking is inherently variable--people use multiple strategies within a domain. Problems with different features often elicit different ways of thinking and learning (e.g., Boyer et al., 2008; De Bock et al., 2011; Hurst, Shaw et al., 2020; Kaminski et al., 2008; Rau & Matthews, 2017; Schnotz, & Kürschner, 2008).

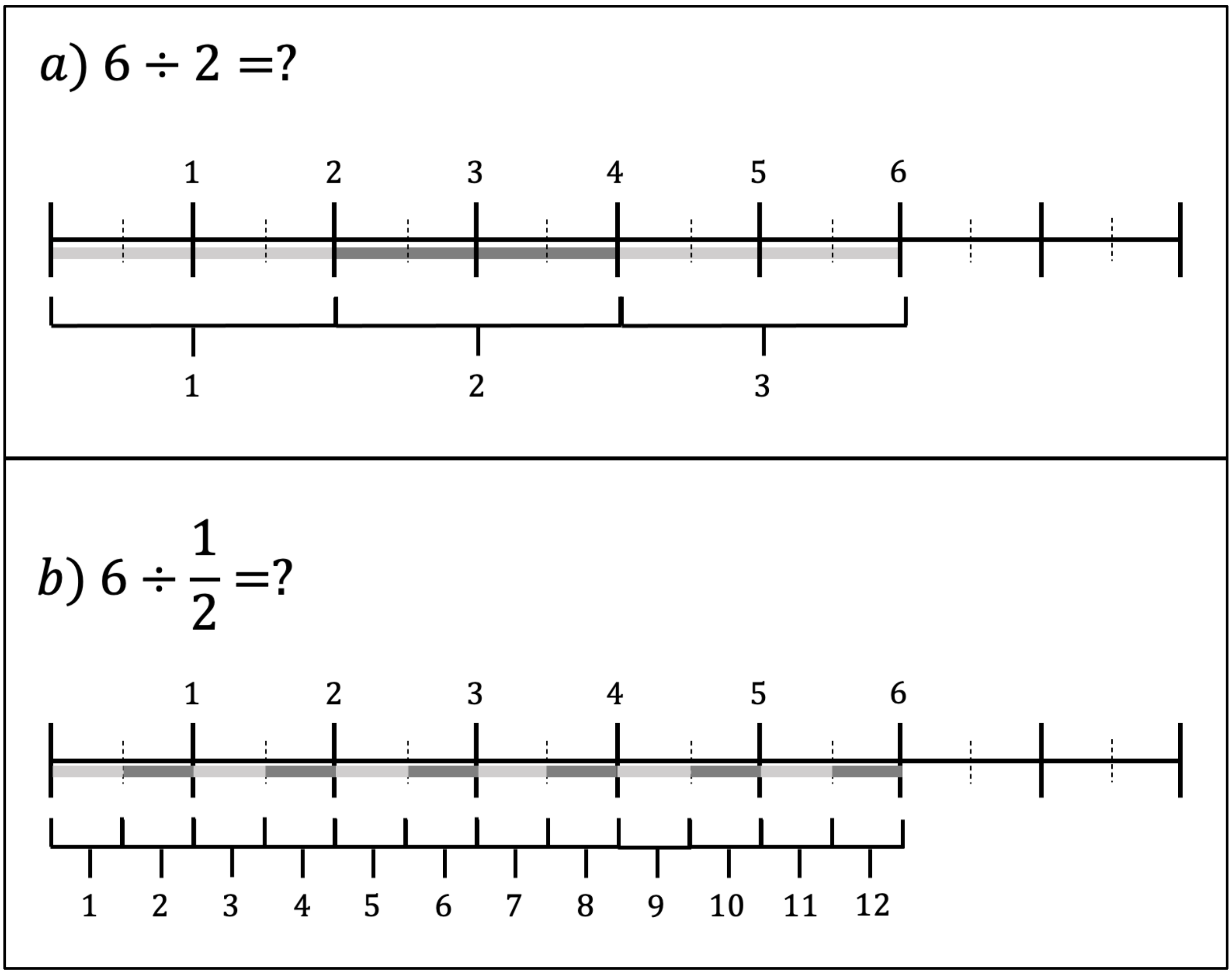

Figure 1. These number lines demonstrate quotative models of whole number division (*a*) and fraction division (*b*). In both cases, division is represented by the number of times the dividend, 6, can be partitioned into sections as large as the divisor, either 2 in panel (a) or ½ in panel (b). In the current study, participants in the Whole Number Division Example condition "warmed up" with an example problem similar to that shown in panel (a). Figure adapted from Sidney & Thompson (2019).

For example, Siegler and Ramani (2009) contrasted children's learning from number-line-like linear board games and circular board games that emphasized the relative magnitudes (i.e., size) of the number of whole numbers. Their prior research (Siegler & Ramani, 2008) demonstrated that linear board games improved low-income children's understanding of whole number magnitudes, leading them to consider whether the perceptual structure of the board game representation was a critical feature of the intervention. Siegler and Ramani found that indeed it was. Children who played linear board games were better able to precisely estimate the magnitude of whole numbers and had more success on later whole number arithmetic problems. Given that children in both conditions were instructed to play the number games in identical ways, differences in the perceptual structure of the representations themselves (i.e., linear vs. circular) likely caused these differences in learning.

Similarly, research on analogical transfer from an experimenter-taught "source" problem to a later, conceptually-similar "target" problem demonstrates that the nature of diagrams appears to matter. In one study, Gick and Holyoak (1983) found that including a diagram of the solution for the source problem did not support spontaneous transfer to a target problem among adult university students. In contrast, Beveridge and Parks (1987) demonstrated that both children and adults *were* more likely to spontaneously transfer across problems when the source problem included a visual representation than when it did not. Importantly, both studies taught learners the same classic analogy (see Duncker, 1945) but with different visual representations. This suggests that some visual representations may be more likely to facilitate spontaneous transfer to a later target problem than others.

These lines of research clearly demonstrate that diagrams can shape how learners engage with mathematical problems, and in some cases improve learning and subsequent transfer to new problems. However, the studies of learning and transfer reviewed thus far focus on the use of visual representations during the prior learning episode and observe transfer on an untrained task on which a diagram was not provided. Much less research has focused on when, and how, visual representations facilitate *transfer from* learners' existing prior knowledge when they are learning something new. In other words, although we know a great deal about how to use visual representations today that makes transfer more likely tomorrow, we know little about how instructors should use visuals in a learning context today to make spontaneous transfer more likely from what was learned yesterday.

A handful of studies have empirically examined learners' spontaneous and directed transfer from their own prior knowledge in mathematics. However, many include visual representations in *every* condition (e.g., Hattikudur et al., 2016; Richland & Hansen, 2013; Schwartz & Bransford, 1998; Sidney, 2020; Sidney & Alibali, 2015; Thompson & Opfer, 2010). Furthermore, researchers (Richland et al., 2007; Sidney & Thompson, 2019) have suggested that visual representations are a critical component of analogical instruction aimed at leveraging learners' prior knowledge. However, to our knowledge, no study has tested this prediction. Thus, the role that including visual representations versus not plays in supporting spontaneous transfer from learners' prior knowledge remains unclear. Theoretically, this issue is important for fully understanding the role of visual representations in transfer. Practically, this issue is important for making clear, evidence-based recommendations for how to best draw on learners' prior knowledge during instruction.

1.2 Visual Representations in Fraction Learning

We focus on the role of visual representations and transfer in learners' understanding of a challenging fraction concept, understanding the conceptual structure of fraction division.

Understanding fraction operation concepts is difficult for both children (e.g., Mack, 1990, 1995, 2001; Sidney & Alibali, 2015, 2017; Siegler et al., 2011) and adults (e.g., Ball, 1990; Bentley & Bosse, 2018; Luo et al., 2011; Ma, 1999; Sidney et al., 2015; Yao et al., 2021), despite being a critical aspect of children's development of deep understanding of mathematics. The National Mathematics Advisory Panel (NMAP) considers fraction understanding to be foundational for algebra learning (NMAP, 2008). Moreover, children's understanding of fractions predicts later algebra success and mathematics achievement more broadly, even when controlling for other individual differences in cognition (Bailey et al., 2012; Siegler et al., 2012).

Visual representations are a key component of recommendations for fraction instruction (e.g., Siegler et al., 2010). They are also a component of many empirically-tested interventions for supporting students' fraction learning (Cramer et al., 1997; Cramer et al., 2002; Fazio, Kennedy et al., 2016; Fuchs et al., 2013; Kellman et al., 2008; Moss & Case, 1999; Rau et al., 2014). However, many recent studies have demonstrated variation in learners' strategies (Hurst, Massaro et al., 2020; Sidney et al., 2019) and instructional effectiveness (Gunderson et al., 2019; Hamdan & Gunderson, 2017; Kaminski, 2018) when different types of visual representations are used during fraction instruction. In line with this prior research, here we focus on two types of diagrams commonly used to support learners' fraction concepts: circle diagrams and number line diagrams.

Critically, these two types of diagrams have different affordances for thinking about fraction concepts. Circle diagrams represent the magnitude of fractions, and other rational numbers, as *areas* (e.g., 1 as one whole circle, and ½ as half of the area of a circle), and have been shown to support children's part-whole thinking about fractions (e.g., thinking of ⅔ as 2 out of 3 parts in a whole; Cramer et al., 1997; Cramer et al., 2002; Cramer et al., 2008; Hurst, Shaw et al., 2020; Kieren, 1976; Pitsolantis & Osana, 2013). In contrast, number lines afford thinking about fractions as *measurements* (Kieren, 1976; Moss & Case, 1999) and can afford learners' thinking about the relative magnitude of two numbers represented on the same number line (Sidney et al., 2019; Siegler et al., 2011). Furthermore, several theoretical accounts of the development of children's rational number understanding (Moss & Case, 1999; Siegler et al., 2011) have emphasized the utility of number-line-like representations for understanding fraction concepts. Linear visual representations of number, such as number lines, have been shown to support children's whole number (e.g., Booth & Siegler, 2008; Siegler & Ramani, 2009) and fraction (e.g., Fazio, Kennedy, et al., 2016) reasoning. In addition, children's ability to place rational numbers on a number line strongly predicts their fraction arithmetic performance (Siegler et al., 2011; Siegler & Pyke, 2013). Given this evidence, Siegler and colleagues (Siegler et al., 2010; Siegler et al., 2011) have proposed that number lines are a critical visual representation for students' understanding of all rational numbers.

In line with these theoretical accounts, and recommendations from the IES practice guide (Siegler et al., 2010), several recent studies have demonstrated a number line *advantage* over circle diagrams for both simple and more complex fraction concepts. For example, Gunderson and colleagues (Gunderson et al., 2019; Hamdan & Gunderson, 2017) have demonstrated empirically that when children learn how to interpret fraction symbols while referencing a number line diagram, they are better able to reason about the magnitude of symbolic fractions when no diagram is present than those who learned with circle diagrams. Similarly, Sidney and colleagues (2019) demonstrated a number line advantage for reasoning about fraction division. Children who were asked to model fraction division problems with number lines were both more likely to generate a correct solution to problems and more likely to demonstrate conceptual understanding of the division relationship than those who modeled problems with circles, rectangles, or no diagram at all. Furthermore, qualitative analyses suggested that, as hypothesized by Siegler et al. (2011), number lines afforded representing both operands in a problem relative to a single endpoint, 0, allowing learners to better compare the relative magnitudes of both numbers. This may afford a key division concept: thinking about the quotient as how many times the second operand "fits" into the first.

1.3 Transfer for Fraction Learning

Although representing, understanding, and problem-solving with fractions pose challenges for many people, several recent studies have demonstrated that learners can leverage their more familiar whole number knowledge to bootstrap fraction learning. For example, when learning about the magnitudes associated with fraction symbols, Yu and colleagues (2020) presented third- through fifth-grade students with analogies to familiar whole number magnitudes (e.g., the location of 3/8 on a 0-1 number line is analogous to the location of 3 on a 0-8 number line). Children who saw the whole number and fraction number lines spatially aligned on the same screen made more precise fraction magnitude estimates than did those children who did not view aligned whole number and fraction number lines. These findings suggest that young learners can effectively draw on their prior knowledge of placing whole number magnitudes on number lines to increase the precision of their estimates of fraction magnitudes.

Furthermore, a growing number of studies (Sidney, 2020; Sidney & Alibali, 2015, 2017; Sidney et al., 2015) provide evidence for spontaneous transfer from learners' prior knowledge of whole number division concepts to their conceptual understanding of fraction division. In these studies, some learners practice with whole number division problems immediately before engaging in fraction division learning or problem-solving. Other learners either practice with fraction addition, subtraction, or multiplication (Sidney & Alibali, 2015, 2017) or do not practice with any previously-learned problem (Sidney, 2020). Across studies, when learners first practice with whole number division, their conceptual knowledge of fraction division is enhanced in comparison to practice conditions involving other operations. These findings suggest that when learners' relevant prior knowledge is activated immediately before new learning, spontaneous transfer from whole number division to fraction division is more likely.

Importantly, Sidney and Thompson (2019) have suggested that activities that include visual representations that are consistent with activities in the target instruction (e.g., Day & Goldstone, 2012; Sidney, 2020; Thompson & Opfer, 2010) further increase the likelihood of spontaneous transfer from prior knowledge concepts. In line with this argument, many studies of transfer have included external representations including diagrams and physical manipulatives. However, as discussed in section 1.1, the nature of diagrams used to invite transfer from more familiar concepts to less familiar ones vary across studies, including both number lines and circle diagrams.

Importantly, in proposing that number line representations support both whole number (e.g., Booth & Siegler, 2008; Siegler & Ramani, 2009) and fraction learning (e.g., Fazio, Kennedy et al., 2016; Gunderson et al., 2019; Hamdan & Gunderson, 2017; Sidney et al., 2019; Yu et al., 2020), Siegler and colleagues' (2011) *Integrated Theory of Numerical Development* implies that number line representations should also afford better integration of knowledge across whole number and fraction concepts. Furthermore, given that the physical actions of dividing with whole numbers of circles do not match the physical actions of dividing with fractions of circles (see Sidney & Thompson, 2019), circle diagrams could actively hinder transfer. If these predictions are borne out, they would have educational implications for how visual representations should be used in the classroom - that numbers lines, but not circles, should be used to make connections during instruction. Thus, in the current study, we test whether number lines are more effective at facilitating spontaneous transfer from a whole number division concept to a fraction division concept as compared to either circle diagrams or no diagram at all.

1.4 Current Study and Hypotheses

The primary goal of the current study was to examine whether number lines facilitate conceptual understanding of fraction division *because* they better allow learners to draw on their relevant, familiar prior knowledge of whole number division concepts than circle diagrams. Despite the importance of number lines for linking between whole number and fraction concepts (Siegler et al., 2011; Yu et al., 2020), to our knowledge, no study has examined whether number lines uniquely support learners' ability to effectively and spontaneously draw on their prior knowledge of whole numbers. Although Sidney and colleagues (2019) found that number lines facilitated conceptual understanding, they included a warm-up example of a whole number division problem in all conditions. Given that activating learners' prior knowledge makes spontaneous transfer more likely, it remains unclear whether the number line advantage was solely due to differences in how number lines and circles afford thinking about fraction division, or due to differences in how they afford *transfer* from whole number division. Thus, in the current study, we manipulate both the type of diagram and support for spontaneous transfer from learners' whole number knowledge. Furthermore, we examine the role of visual representations, in general, on transfer for learners' own prior knowledge. No study has empirically tested whether visual representations are necessary for transfer from learners' prior knowledge, despite recommendations to include visual representations to support such transfer (Richland et al., 2007; Sidney & Thompson, 2019). We hypothesized that number lines would support spontaneous transfer more often than circle diagrams or no diagram at all.

To test this hypothesis, we manipulated the type of diagram that learners used to demonstrate conceptual models for fraction division and whether or not learners' prior whole number knowledge was activated with a "warm-up" example. As in Sidney et al. (2019), students solved a set of fraction division problems. For some students, problems were presented with number line diagrams, some students solved problems with circle diagrams, and some students solved problems with no diagram at all. However, in contrast to Sidney et al. (2019), prior to solving fraction division problems, only one third of learners warmed up with a relevant whole number division example. One third of learners warmed up with fraction addition example, thus activating their prior knowledge of a less relevant fraction operation concept (see Sidney & Alibali, 2015, 2017). The last third did not warm up with any example, thus we did not activate any specific facet of learners' prior knowledge.

These manipulations allowed us to test the following hypotheses about learners' conceptual models of fraction division:

H1) We expected a main effect of diagram condition such that learners who demonstrated their conceptual understanding of fraction division with number lines would be more successful than those who used circle diagrams or no diagram at all.

H2) We expected a main effect of example condition such that learners who demonstrated their conceptual understanding of fraction division after their prior knowledge of whole number division was activated would be more successful than those who warmed up with fraction addition or did not engage in a warm-up activity at all.

H3) Most importantly, we expected an interaction between diagram and example condition. If number lines uniquely support transfer from learners' prior knowledge, then the effect of activating learners' prior knowledge of whole number division prior to demonstrating fraction division would be larger in the number line condition than the circle or no diagram conditions.

The study was designed to also shed light on two alternative hypotheses for *H3*. One possibility is that any visual representation would afford transfer from prior knowledge, in line with recommendations for analogical instruction (Richland et al., 2007; Sidney & Thompson, 2019). If so, we would observe an equally large effect of example condition in both the number line and circle diagram conditions but a smaller effect of example condition in the no diagram condition. A second possibility is that visual representations do not affect a learners' likelihood or ability to transfer from their prior knowledge of whole number division to fraction division. Instead, using visual representations and activating prior knowledge could affect conceptual understanding via distinct, non-overlapping mechanisms. In this case, we would expect main effects but no interaction between diagram and example condition.

Although much of the research reviewed above involves children (e.g., Sidney et al., 2019), we used an adult undergraduate student sample to test our hypotheses. Although children and adults have many differences, their understanding of fraction division has several similarities. Similar to children, few adults have robust conceptual understanding (Bently & Bosse, 2018; Sidney et al., 2015), even among preservice teachers (Ball, 1990; Luo et al., 2011; Ma, 1999; Yao et al. 2021). Given that undergraduate students do display transfer from recently activated prior knowledge (e.g., Day & Goldstone, 2011; Schunn & Dunbar, 1996) and learn from instruction connecting whole number division and fraction division (Sidney et al., 2015), we expected that adults would also demonstrate spontaneous transfer from whole number to fraction division concepts. Additionally, given that the number line is theorized to affect learning due to affordances of the representation rather than properties of the learner, we expected the number line advantage to hold for adults. However, the effects of visual representations are often moderated by learners' prior knowledge (e.g. Butcher, 2006; Cooper et al., 2018). Thus, circles may provide some advantages for undergraduate students to the extent that their math and diagram-relevant knowledge differs from children. Although understanding age differences is not a focal aim of the study, using an adult college student sample allows us to empirically examine whether key findings with children generalize to adult learners who often demonstrate many of the same misconceptions about fractions and fraction division.

Finally, although we were primarily interested in conceptual models, we also examined learners' fraction division problem-solving accuracy. There are many paths to accuracy in symbolic problem solving, including using symbolic procedures (e.g., invert-and-multiply) and generating a division model on a diagram. Number line diagrams likely support children's accuracy due to increased rates of generating correct conceptual models that include the correct quotient. In contrast to fifth- and sixth-grade children, undergraduate students are likely to know well-practiced symbolic procedures for solving fraction division problems (Ma, 1999; Sidney et al., 2015; Yao et al., 2021) that do not depend on conceptual understanding. Thus, we expected that simple accuracy may be similar across conditions for undergraduate students, but had no strong a priori hypothesis concerning accuracy rates.

2 Method

All aspects of this method were approved by the Human Subjects Institutional Review Board at the University of Kentucky under protocol #48308.

2.1 Power Analysis

To determine the necessary sample size to test our hypothesis that the nature of provided diagrams would moderate the effectiveness of warming-up with a relevant whole number concept, we conducted an *a priori* power analysis to ensure adequate power to detect the interaction effect. Using G*Power, we assessed the number of participants needed to detect an effect size of $\eta_p^2 = .08$ for a 4 *df* test, the test of the interaction. This reflects our expectation of a medium-sized effect based on prior research on the effects of diagrams on learning (e.g., Butcher, 2006; Moreno et al., 2011; Sidney et al., 2019) and effects of whole number warm-up activities on fraction division conceptual understanding (Sidney, 2020; Sidney & Alibali, 2017) that suggest that the use of effective diagrams and warm-up activities can account for approximately 6% to 10% of the unique variability in learners' transfer outcomes. Though our power analysis revealed that we would require $N = 143$ participants in our nine conditions, we planned to recruit $n = 20$ participants in each condition with the expectation that some participant data may need to be excluded.

2.2 Participants

Participants were $N = 177$ college students at a large public university in the southern United States recruited through their psychology courses to participate for partial course credit. The sample was primarily composed of non-Hispanic White women, as is typical of the university pool of Psychology participants (*M* age = 19.23*y, SD* = 1.42*y*; 82% women, 18% men; 76% non-Hispanic White, 11% Black, 6% Hispanic/Latinx, 3% Asian, 4% other/not specified). There was no missing data for the focal task, and no participants were excluded.

2.3 Design

As in Sidney et al. (2019), participants were assigned to solve and demonstrate fraction division problems in one of three diagram conditions: showing work with number line diagrams, showing work with circle diagrams, and showing work with no provided diagrams. Additionally, one-third of the participants were randomly assigned to view a whole number division example problem to activate learners' familiar prior knowledge as in Sidney et al. (2019). In contrast to Sidney et al. (2019), the remaining participants were assigned to view a fraction addition example problem or no example problem at all. Fraction addition was chosen as a comparison case following Sidney and Alibali (2015). Although activating knowledge of fraction magnitudes may prepare students to represent fractions using diagrams, it has not been found to support understanding of division (Sidney & Alibali, 2015; 2017).

Thus, participants experienced one of nine experimental conditions, in a 3 (Diagram: Number Line, Circle, No Diagram) x 3 (Example: Whole Number Division, Fraction Addition, No Example) between-subjects design (see Figure 2). Sample size for each condition can be found in Table 1. Below, we describe the manipulation of Diagram and Example in each of the two focal tasks; a full set of materials for each task is available on OSF: https://osf.io/nru2d/?view_only=8321f053d3fe4a06958878a7bfb4ad87 [this link will need to be updated upon publication].

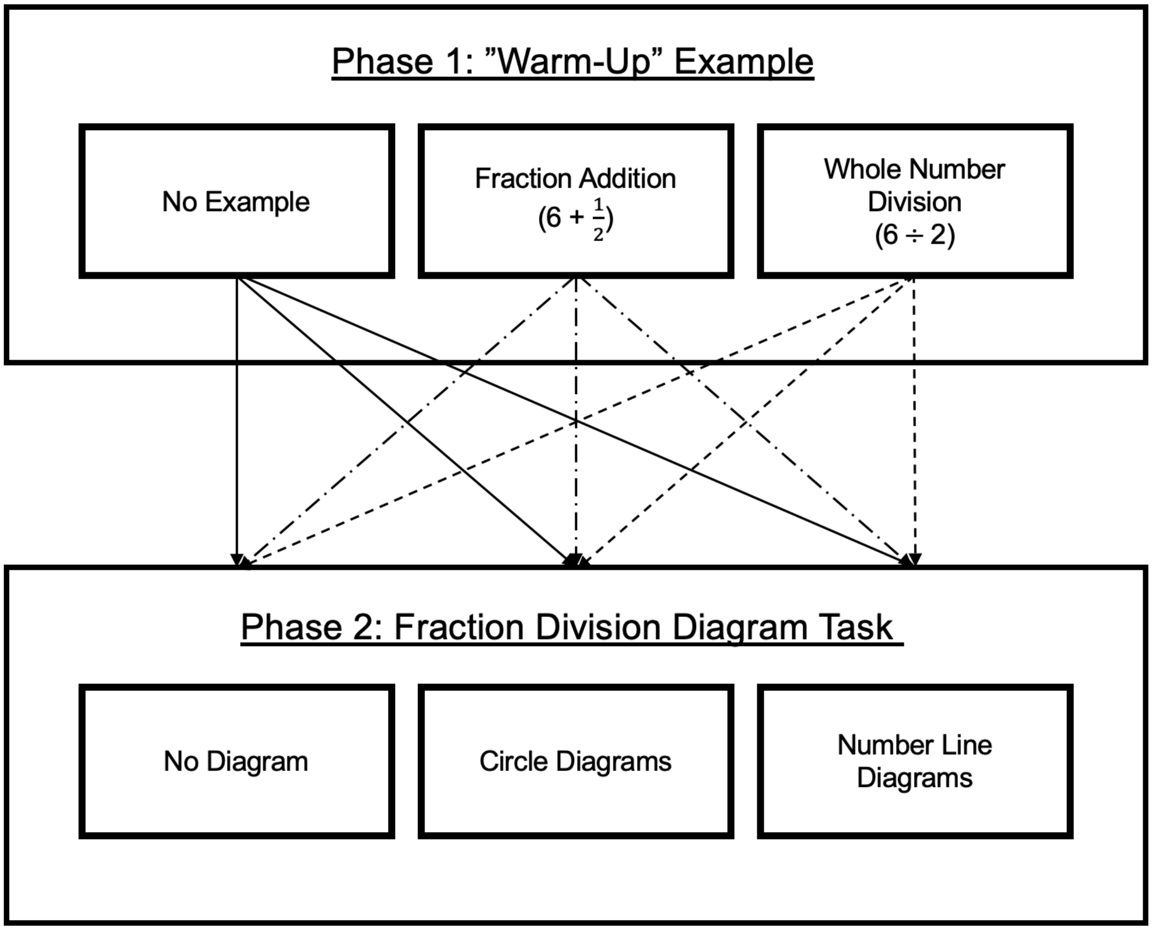

Figure 2. This figure represents the factorial design of the study. We manipulated both the example presented during the "warm-up" and the type of diagram used in during the task in a 3 (Diagram: Number Line, Circle, No Diagram) x 3 (Example: Whole Number Division, Fraction Addition, No Example) between-subjects design.

2.3.1 Fraction division diagram task. The set of fraction division problems included 12 problems drawn from the diagram task in Sidney et al. (2019). These problems were originally designed to represent the range of fraction division problems covered in 5th and 6th grade math, including problems with proper fraction, whole number, and mixed number dividends and unit fraction and proper fraction divisors (see Supplementary Material, Table SM1). For this study, we omitted the easiest items (i.e., a whole number divided by a unit fraction) and the items eliciting partitive division (i.e., a fraction divided by a whole number). Thus, the remaining problems included a larger dividend divided by a smaller divisor, eliciting quotative division (Sidney et al., 2019). Furthermore, accuracy on the problems was highly reliable across participants, *Cronbach's alpha* = 0.95. In the number line and circle diagram conditions, each problem was presented with a single diagram representing six whole units partitioned into the denominator units of the divisor (e.g., into eighths for $\frac{3}{4} \div \frac{3}{8} = ?$). Providing a helpful unit structure in the diagram eases the drawing demands of representing fraction magnitudes exactly, thus better allowing measurement of learners' understanding of the division relationship itself.

Each problem was presented on a separate page of a paper packet in one of two predetermined random orders. The structure of these pages was identical to the materials in Sidney et al., (2019). The fraction division problem was presented at the top of the page, and the diagram or an equivalently large blank space (in the *No Diagram* condition) was presented directly below the problem. On the bottom half of the page, participants were asked to report their confidence (*How confident are you that you solved the problem correctly?* 0% *definitely did not* - 100% *definitely did*) and perceived difficulty level of each problem (*How difficult was it to answer this problem? not difficult at all* [1] to *very difficult* [4]). Findings from the confidence ratings are not a focus of this manuscript, however, analyses of confidence rating data can be found on this project's OSF page.

2.3.2 "Warm-up" example problems. We introduced the fraction division diagram task using a separate example problem packet. The cover page of the packet for all participants included general instructions about the study, that they would "*be asked to show how to solve some math problems, and answer some questions about math*". For participants in the *Whole Number Division* and *Fraction Addition* example conditions, the packet demonstrated "*an example of how you can show your work*". Each participant viewed one example problem based on their assigned condition. Example problems for both example conditions consisted of four steps, with each step printed on a separate page. The whole number division example (6 ÷ 2), demonstrated quotative division and was modeled after Sidney et al., (2019). In step 1, participants were told that they could "*show how big six is*". In step 2, participants were told they could "*show how big two is*" and "*make a group of two*." In step 3, participants were told that dividing six by two is "*like asking how many times a group of two goes into a group of six*." Finally, in step 4, participants were told to count the total number of groups and the answer was stated. In the fraction addition example (6 + ½), step 1 was identical. In step 2, participants were told to "*make a group of one half*." In step 3, participants were told that adding was "*like asking how many we have all together*." In step 4, the answer was stated.

In the diagram conditions, a diagram matching the participants' assigned condition was used to demonstrate each step. On the bottom half of each page, a blank diagram was provided for participants to recreate the provided diagram. On the bottom of the final page, participants were told that they would "*be given some new problems. Please use the diagram to come up with the answers to the new problems*." In the *No Diagram* condition, each step was demonstrated with number symbols only, instead of modeling operands as an area or length, and in the whole number division problem the quotient was represented by writing “2” down three times. On the final page, participants were asked to “*use any method*”.

In all three *No Example* conditions, participants were not provided with any example. Instead, the cover page included the instruction to “*use the diagram*” or “*use any method*” to solve the problems, according to the participants’ assigned diagram condition.

2.3.4 Other tasks. Participants were asked to write down three numbers and completed a math attitude questionnaire. Data from these tasks are not a focus of this report; data from the attitude questionnaire are reported in Sidney et al. (2021).

2.4 Procedure

Students participated in the study in groups of one to five, $M = 3.61$, under the supervision of one or two experimenters. Groups were randomly assigned to conditions such that all students in a group experienced the same condition to avoid spill-over effects. Upon entering the room, participants were told to find a desk with a consent form, instructed to review the consent form, and provided the opportunity to ask questions about the study. As the experimenters collected consent forms, they distributed an example problem packet to each participant, and read aloud the instructions. Participants were instructed to raise their hand when they were ready for the next part of the study, at which point an experimenter collected the example packet and distributed the fraction division diagram problems. When the fraction division problems were completed, those were collected and participants received the math attitude questionnaire and demographic questions to complete. Each participant worked through both packets individually, though the experimenters were present to answer questions. Experimenters did not provide feedback, nor additional instructions, but did encourage participants to do their best and show their work. The entire study took approximately 30 to 40 minutes to complete.

2.5 Coding Participants' Work

Each participant's conceptual understanding and accuracy on each problem was coded by two independent coders following the coding scheme outlined in Sidney et al. (2019). To examine participants' conceptual understanding of fraction division, their written work (e.g., drawings or computations) was coded to characterize the conceptual model used to demonstrate the problem. Most importantly, participants' work was categorized as reflecting a quotative model of division, a partitive model of division, or no model of division. Quotative models were coded when participants represented division as the number of times the dividend could be partitioned into sections as large as the divisor (see Figure 3 for examples). Partitive division could be coded when participants represented the quotient as the magnitude of the whole unit where the divisor indicates the number of units within the dividend. Note that partitive models of fraction division tend to be less intuitive than quotative models (e.g., Fischbein et al., 1985) and were never used by our sample. No disagreements occurred for division model coding. Percentage of conceptually-accurate division models was calculated for each participant, and used as the outcome of conceptual understanding of fraction division.

Accuracy was defined as whether the participant had written the final correct answer anywhere on the page, and was coded with high agreement (agreement on 98% of trials). Percentage accuracy was calculated for each participant and used as the measure of problem-solving accuracy. All disagreements were flagged and discussed with the first author until 100% agreement was reached.

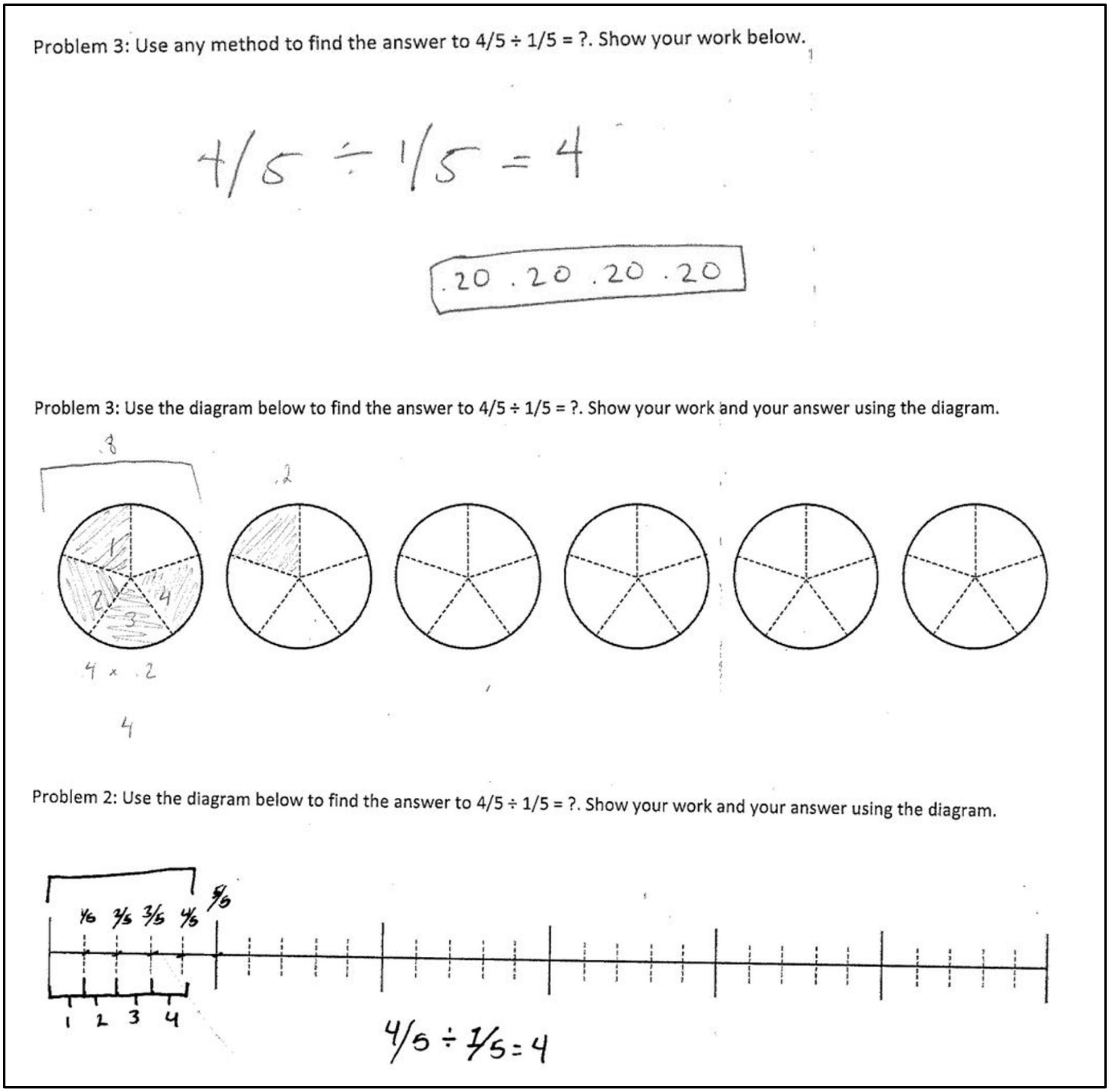

Figure 3. Examples of accurate conceptual models of quotative division from each diagram condition: no diagram (top), circle (middle), and number line (bottom).

3 Results

3.1 Condition Assignment

Prior to conducting the focal analyses, we examined whether participants' age and gender were systematically related to condition assignment using simple linear and logistic regressions. As expected, participants' age, $F(4,168) = 1.71$, $p = .150$, and gender, $X^2(4) = 5.62$, $p = .230$, were not associated with condition assignment, suggesting that random assignment was successful. Furthermore, problem accuracy, $F(1,175) = 0.67$, $p = .413$ and conceptual understanding, $F(1,175) = 0.01$, $p = .922$, did not differ by gender. These demographic variables were not included in the focal analyses.

3.2 Conceptual Understanding

On average, participants generated accurate conceptual models of division on $M = 25\%$ ($SD = 39\%$) of problems (3 out of 12) demonstrating undergraduate students' poor conceptual understanding of fraction division, in general. However, variability was considerable with some participants generating accurate conceptual models of division on all of the problems and others on none. To examine whether participants' fraction division conceptual understanding varied as a function of diagram and example condition, we conducted a linear regression on the percentage of conceptually-accurate division models across the 12 trials and included diagram condition, example condition, and their interaction as fixed factors. Note that this is equivalent to fitting a between-subjects 3 x 3 ANOVA. Regression was preferred for ease of testing and interpreting contrasts and simple effects.

In the regression model, diagram condition was represented by a set of centered Helmert contrasts comparing 1) any diagrams (number line and circle) to the no diagram condition and 2) the number line and circle conditions to each other. This allowed us to examine whether any diagram would support conceptual understanding, and transfer, and examine support for a number line advantage. Example condition was represented by a set of centered Helmert contrasts comparing 1) the whole number division example with the other two example conditions and 2) the fraction addition example with no example. This allowed us to examine whether spontaneous transfer from learners' whole number division knowledge occurred.

Models were re-run with dummy coded variables to describe and test individual contrasts between conditions. No covariates were entered into the model. Dataset and analysis script are available on OSF: https://osf.io/nru2d/?view_only=8321f053d3fe4a06958878a7bfb4ad87.

The regression model revealed significant main effects of both diagram condition, $F(2, 168) = 46.66$, $p < .001$, $\eta_p^2 = .36$, and example condition, $F(2, 168) = 15.74$, $p < .001$, $\eta_p^2 = .16$. The Helmert contrasts revealed that participants who showed their work with any diagram were better able to generate accurate conceptual models of fraction division than those who were not provided with a diagram, $M = 1\%$ of trials, $b = 0.38$, $t(168) = 8.33$, $p < .001$. Among those in the diagram conditions, those who modeled fraction division with a number line, $M = 54\%$, were more likely to generate accurate conceptual models than those in the circle condition, $M = 25\%$, $b = 0.28$, $t(168) = 5.17$, $p < .001$, see Figure 4, all means and *SE*s are presented in Table 1. Furthermore, participants who were shown a whole number division example problem, $M = 42\%$, were also more likely to generate accurate conceptual models than those in the other two conditions, $b = 0.26$, $t(168) = 5.54$, $p < .001$; the fraction addition example, $M = 19\%$, provided no benefit over no example at all, $M = 14\%$, $b = -0.05$, $t(168) = -0.98$, $p = .328$.

Table 1

Average Percentage of Accurate Conceptual Models by Diagram and Example Condition

	No Example	Fraction Addition	Whole Number Division
No Diagram	0% (6%), $n = 21$	2% (6%), $n = 21$	0% (7%), $n = 20$
Circle	13% (7%), $n = 20$	9% (7%), $n = 20$	53% (8%), $n = 20$
Number Line	32% (7%), $n = 17$	50% (7%), $n = 19$	78% (7%), $n = 18$

Note. The standard error of each mean is provided in parentheses.

These main effects were qualified by a significant interaction, $F(2, 168) = 4.93$, $p < .001$, $\eta_p^2 = .11$. The positive effect of the whole number division example on conceptual understanding was greater in the diagram conditions than in the no diagram condition, $b = 0.40$, $t(168) = 4.08$, $p < .001$. To further probe the simple effects of example condition, we re-coded diagram condition with each condition as the reference level, refitting the model each time. There was no simple effect of example condition within the no diagram condition, $F(2, 168) = 0.02$, $p = .981$. In contrast, the simple effects of example condition within the circle condition, $F(2, 168) = 13.88$, $p < .001$, $\eta_p^2 = .14$, and number line condition, $F(2, 168) = 10.88$, $p < .001$, $\eta_p^2 = .11$, were considerable. In both the circle and number line conditions, participants with the whole number division example had greater average conceptual understanding than the other example conditions. The positive effect of the whole number division example in the number line condition did not differ significantly from the effect of the whole number division example in the circle condition, $b = -0.05$, $t(168) = -0.47$, $p = .642$. These results suggest that number lines and circle diagrams both afforded spontaneous transfer from learners' prior knowledge of whole number division. Finally, we examined the contrast between the fraction addition and no example conditions in each diagram condition. As expected, there was no significant difference in conceptual understanding between the fraction addition example and no example within the circle condition, $b = 0.04$, $t(168) = -0.40$, $p = .687$, or the number line condition, $b = 0.18$, $t(168) = 1.85$, $p = .066$.

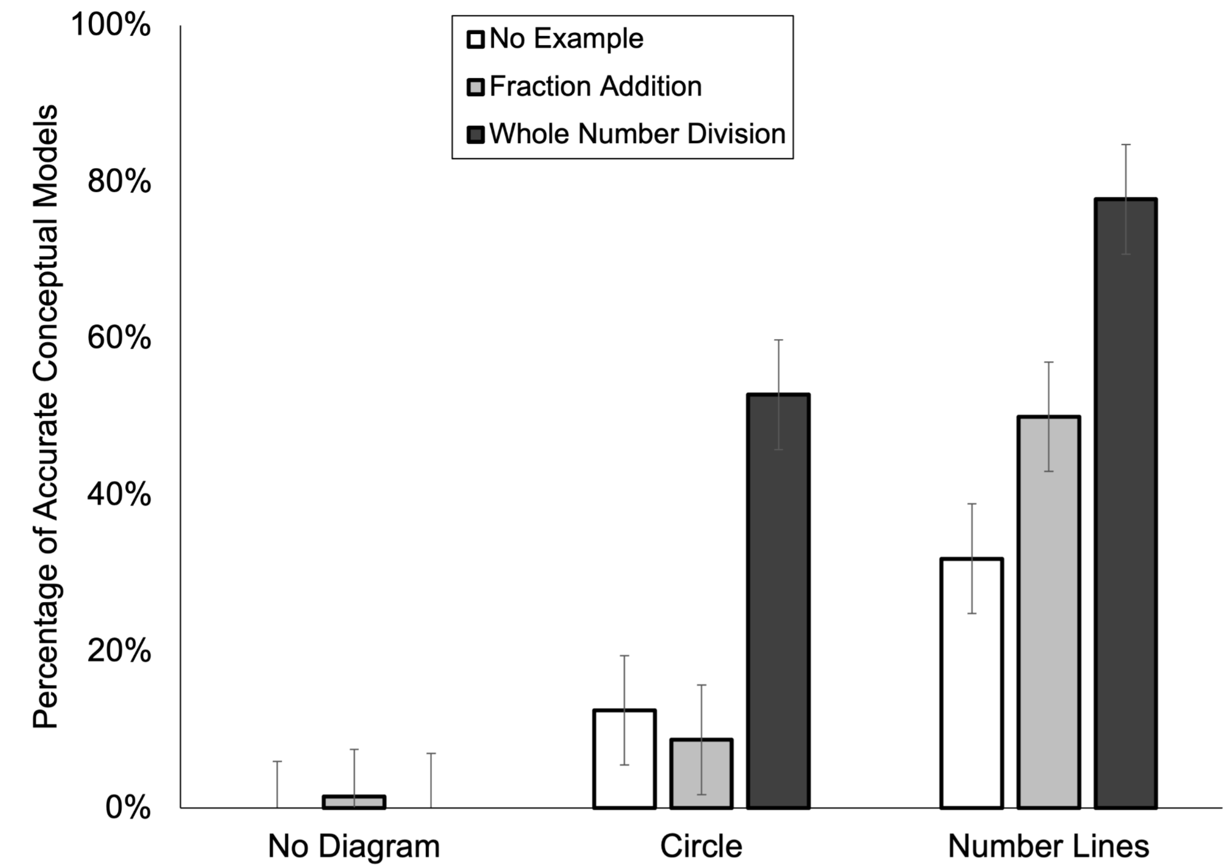

Figure 4. Average percentage of conceptual models of fraction division by diagram and example condition.

3.3 Accuracy

On average, participants generated correct answers on M = 64% (SD = 38%) of problems (between 7 and 8 out of 12), but variability in accuracy was considerable with some participants answering none correctly and some answering 100% correctly. As with conceptual understanding, to examine whether participants' fraction division problem-solving accuracy varied as a function of diagram and example condition, we conducted a linear regression on percentage accuracy scores including diagram condition, example condition, and their interaction as fixed factors, with Helmert contrasts specified as described above.

The model revealed a significant interaction between diagram and example condition on participants' problem-solving accuracy, $F(4,168) = 2.72$, $p = .031$, $\eta_p^2 = .06$. There were no main effects of diagram condition, $F(2,168) = 1.40$, $p = .250$, $\eta_p^2 = .02$, or example condition, $F(2,168) = 0.19$, $p = .830$, $\eta_p^2 = .00$, see Figure 5, Table 2. As with conceptual understanding, our Helmert contrasts revealed that the whole number division warm-up example had greater positive effects on accuracy in the diagram conditions than the no diagram condition, $b = 0.38$, $t(168) = 3.00$, $p = .003$.

Table 2

Average Percentage of Correct Answers (Accuracy) by Diagram and Example Condition

	No Example	Fraction Addition	Whole Number Division
No Diagram	74% (8%), $n = 21$	70% (8%), $n = 21$	50% (8%), $n = 20$
Circle	47% (8%), $n = 20$	57% (8%), $n = 20$	72% (0%), $n = 20$
Number Line	72% (9%), $n = 17$	61% (9%), $n = 19$	79% (9%), $n = 18$

Note. The standard error of each mean is provided in parentheses.

To better examine the interaction, we refit the model after dummy coding example condition with whole number division as the reference category and dummy coding diagram condition with no diagram condition as the reference category. This allowed us to examine how the whole number division advantage differed in the no diagram condition as compared to each diagram condition. First, we interpret the interactions between the contrast variables. Then, we interpret the simple effects of example condition within the no diagram condition.

In contrast to the no diagram condition, the whole number division example had more positive effects on accuracy relative to fraction addition in both the number line, $b = -0.37$, $t(168) = -2.20$, $p = .029$, and circle, $b = -0.34$, $t(168) = -2.06$, $p = .041$, conditions. In contrast to no diagram, the whole number division example also had more positive effects on accuracy relative to no example at all in the circle condition, $b = -0.49$, $t(168) = -2.91$, $p < .001$, though this interaction contrast was not significant in the number line condition, $b = -0.31$, $t(168) = -1.81$, $p = .07$. Interestingly, in the no diagram condition, the whole number division example led to lower accuracy as compared to the no example condition, $b = 0.24$, $t(168) = 2.06$, $p = .041$, though similar accuracy as compared to the fraction addition condition, $b = 0.19$, $t(168) = 1.69$, $p = .092$. Overall, the interaction contrasts and pairwise comparisons suggest that while the whole number division example may have been advantageous, or at least neutral, in the diagram conditions, the whole number division example *decreased* accuracy in the no diagram condition.

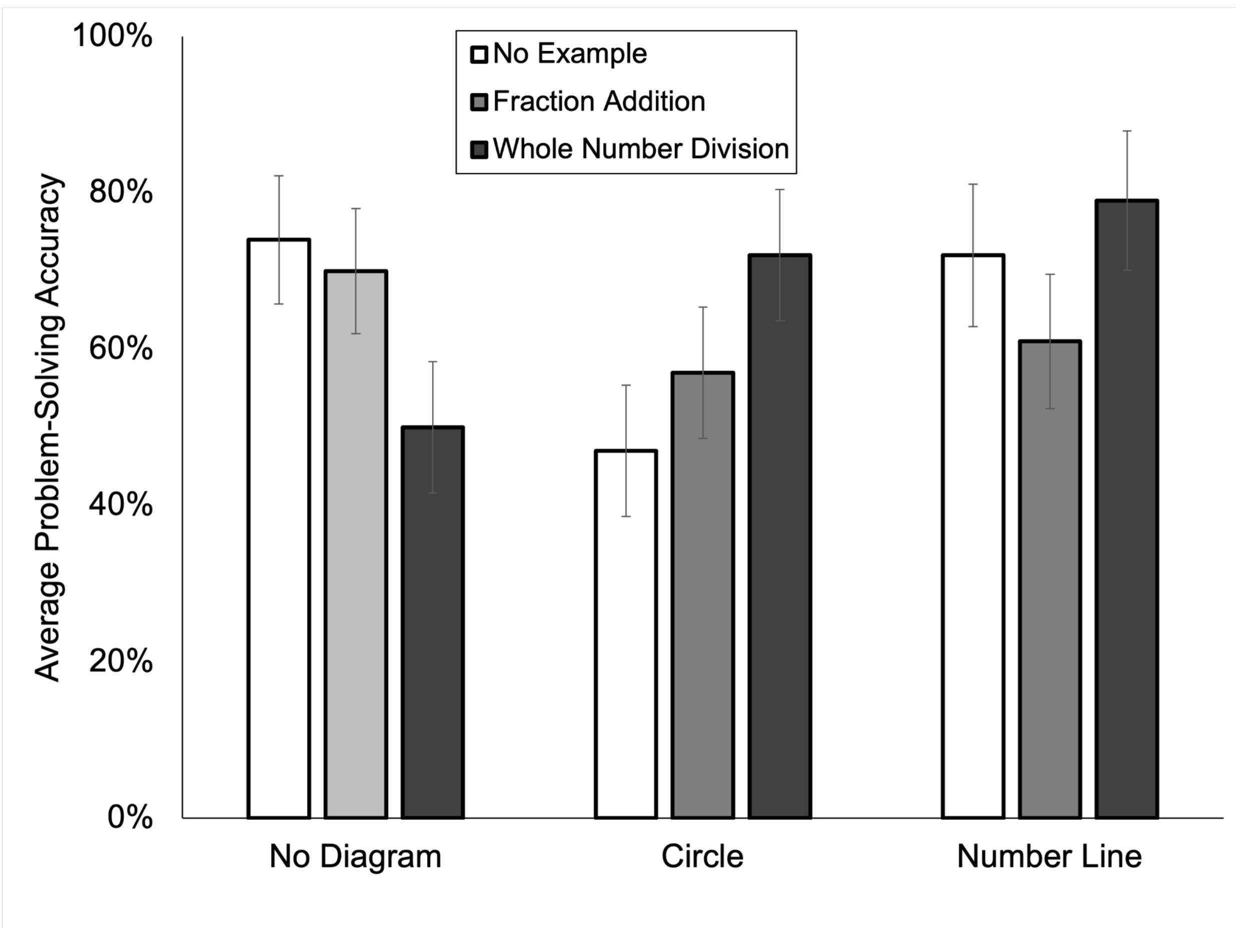

Figure 5. Average percentage of correct answers on fraction division problems by diagram and example condition.

Given the unexpected finding that the whole number division example decreased accuracy in the no diagram condition, we more closely examined the problem-solving approaches in the no diagram condition for each example condition. A thorough description of our qualitative analyses of participants' strategies and errors can be found in Supplementary Material, section 2. The strategy analyses suggested that when we attempted to activate participants' whole number division knowledge without a supporting diagram (no diagram, no example condition), participants less often relied on a known symbolic procedure (i.e., invert-and-multiply) for fraction division and instead sometimes attempted to draw on their whole number division knowledge. Participants' may have found it difficult to adapt the symbolic version of the "warm up" problem, and instead committed a variety of symbolic errors, some of which suggest negative transfer from whole number division.

4 Discussion

4.1 Overall Effects of Visual Representations and Activating Knowledge

In this study, undergraduate students were asked to solve fraction division problems with and without diagrams and after either warming up with relevant familiar concepts or not. In line with prior research (e.g., Butcher, 2006; Cooper et al., 2018; Sidney et al., 2019), undergraduates' work more often reflected conceptual understanding of fraction division when a diagram was present during problem-solving. Furthermore, we replicated the *number line advantage* (e.g., Hamdan & Gunderson, 2017; Sidney et al., 2019); those who used number lines were better at representing the conceptual structure of fraction division than those who used circle diagrams. Note that the overall effect sizes for diagram conditions are quite large. Without any diagram to support student work, learners almost never generated conceptual models for fraction division. Solving problems with circles provided some support, as students drew accurate conceptual models of fraction division on 25% of trials, averaging across example condition. More strikingly, the presence of a number line *doubled* the rate at which students were able to correctly conceptualize fraction division. Thus, a number line advantage for conceptualizing fraction arithmetic relationships is clear.

Also in line with prior research (Sidney, 2020; Sidney & Alibali, 2015, 2017), adults were more than twice as successful at representing the fraction division concept when their knowledge of an analogous, familiar whole number division concept was recently activated, as compared to either fraction addition or no warm up activity at all. Again, the size of this overall effect of activating prior knowledge is large. For example, in the circle condition, when learners did not practice with whole number division, the average percentage of accurate conceptual models was quite low at 13% and 9% for the other two example conditions. When learners in the circle condition did practice with whole number division first, they generated accurate conceptual models of fraction division on 53% of models, despite no instructions or other explicit support for transfer. Effects were similar in the number line condition though with higher rates of conceptual understanding due to the number line advantage. This finding suggests learners did spontaneously transfer from their prior knowledge of more familiar concepts when that knowledge was activated with a warm-up activity, and often did so successfully.

4.2 Visual Representations and Transfer

The primary goal of this current study was to examine the interaction between diagram use and activating learners' prior knowledge, and to test whether number lines *uniquely* facilitate learners' spontaneous transfer from their relevant prior knowledge of whole numbers when asked to demonstrate a challenging fraction concept. We expected that the effect of activating learners' relevant prior knowledge would be greatest in the number line condition. Instead, we found that both types of visual representations, circles and number lines, supported transfer from learners' prior knowledge of whole numbers. While there is an overall number line advantage, even across the conditions in which learners' prior knowledge of whole number division was activated, the number line likely confers its advantage through a different mechanism.

We did find support for an alternative hypothesis: the presence of visual representations does support spontaneous transfer from learners' prior knowledge of whole numbers in comparison to no visual at all. Moreover, the effect of including visual representations along with a warm-up example that activates learners' relevant prior knowledge was very large and practically meaningful. In the no diagram condition, we saw no evidence of successful spontaneous transfer from learners' whole number division understanding (i.e., that dividing is like asking how many times a group as big as the second operand goes into the first operand), despite providing identical linguistic information in all conditions. Whether a whole number division example was provided or not, no students generated accurate conceptual models of fraction division when no diagram was present. In contrast, when number lines or circle diagrams were present, activating learners' whole number division knowledge prior to engaging with fraction division had a large effect on conceptual understanding relative to the other example conditions. This suggests that the diagrams themselves facilitated spontaneous transfer from learners' prior knowledge.

Many related studies of spontaneous transfer (e.g., Day & Goldstone, 2011; Schunn & Dunbar, 1996; Sidney, 2020; Sidney & Alibali, 2017; Sidney et al., 2015) have included visual representations, and the use of visual representations has been recommended to support transfer from more familiar concepts to newer ones during instruction (see Richland et al., 2007; Vendetti et al., 2015). The current study is the first to provide empirical evidence that including visual representations during mathematics instruction may be necessary to facilitate such transfer. However, these findings cannot provide evidence for the specific mechanism of this effect. One possibility is that diagrams better serve to activate learners' existing conceptual knowledge, as spatial relationships in diagrams may aid in highlighting the key mathematics relationships (e.g., Hamdan & Gunderson, 2017; Kellman et al., 2008; Larkin & Simon, 1987; Moss & Case, 1999; Rau et al., 2014; Sidney et al., 2019). A second possibility is that diagrams facilitate transfer itself; that familiar actions on a diagram, or other external visual representation, can easily be applied to or adapted for less familiar problems (see Sidney & Alibali, 2017). A third possibility is that successful conceptual transfer occurred in all conditions, but providing diagrams better allowed us to measure conceptual understanding of fraction division.

The analyses of accuracy and strategy use in the no diagram condition provide reasons to be skeptical of the third possibility. If participants in the no diagram condition benefited from the whole number division example, but simply did not demonstrate conceptual understanding due to the absence of the diagram, we should have still observed a positive effect of the whole number division example on accuracy, which did not depend on drawing a diagram. Instead, the post-hoc qualitative coding of participants' strategies in the no diagram condition suggested that some participants in the no diagram condition may have attempted to spontaneously transfer from the whole number division example, but did so ineffectively (see Supplementary Material, section 2). Note that the propensity to rely on whole number knowledge while solving fraction problems is a common phenomenon called the *whole number bias* (Alibali & Sidney, 2015; Bently & Bosse, 2018; Bottge et al., 2014; Malone & Fuchs, 2017; Mohyuddin & Khalil, 2016; Ni & Zhou, 2005), and can result in either correct or incorrect reasoning depending on the nature of the task.

Unexpectedly, we did *not* find evidence that number lines uniquely supported spontaneous transfer. Even though, in line with evidence from children (Hamdan & Gunderson, 2017; Sidney et al., 2019), adults who used number lines were overall better able than those who used circles to demonstrate their conceptual understanding of fraction operations, number lines did not enhance the transfer effect as expected. These findings are in direct contrast to evidence from children (Sidney et al., 2019) showing no advantage of using circles over no diagram at all. Sidney and colleagues (2019) observed that the physical actions needed to generate an accurate quotative model of division for whole number and fraction division are closely analogous when representing problems on the number line (i.e., represent the first magnitude starting from 0, represent the second magnitude starting from 0, and iterate the second magnitude across the number line), but need to be adapted for circle diagrams (i.e, grouping circles in whole number division vs. partitioning circles for fraction division). These subtle differences may have caused children to have difficulty with spontaneous transfer with circle diagrams, but may not have posed as much difficulty to adults. Perhaps adults' conceptual understanding of quotative whole number division (e.g., "make groups as big as") is more abstract and easier to adapt across subtle perceptual differences. The nature of learners' prior knowledge likely matters considerably for the success of spontaneous transfer (see Sidney & Thompson, 2019), and it may matter for the role of diagrams in transfer as well; future research is needed. Alternatively, adults may be more familiar with using circle diagrams for fractions than children, due to the relatively recent emphasis on using a number line in early and middle grades (e.g., Siegler et al., 2010).

4.3 Accuracy

Although our primary goal was to examine spontaneous transfer of conceptual understanding, we also assessed problem-solving accuracy (i.e., calculating the right answer). As we noted, there are many paths to calculation accuracy including representing the conceptual structure and identifying the quotient as well as using symbolic procedures. Many undergraduate students are able to calculate quotients to fraction division problems without any conceptual understanding of the problems (see Ball, 1990; Ma, 1999; Sidney et al., 2015). Thus, as expected, there were few significant differences in accuracy across conditions. However, there were differences in strategy use across warm-up conditions when no diagram was present. Unexpectedly, participants in the no diagram condition correctly solved fewer problems when their familiar whole number division knowledge was activated using the example problem. Post-hoc strategy coding (see Supplementary Material, section 2) revealed that participants in this condition were less likely, compared to those in the fraction addition or no warm-up conditions, to use an otherwise common symbolic procedure, invert-and-multiply. In contrast, in the fraction addition and no-warm up conditions, participants were very likely to use invert-and-multiply, and as long as the students knew this procedure, they could have high accuracy despite poor conceptual understanding. The whole number division example appeared to prompt students to use a different strategy than the standard invert-and-multiply procedure, but without a diagram to support the analogy, participants failed to generate an equally good alternative strategy.

This finding is in line with observations from Sidney (2020) that sometimes learners explicitly attempt to apply strategies demonstrated on earlier problems to later ones despite having poor conceptual understanding of those strategies and knowing a correct symbolic procedure. Although activating learners' prior knowledge is often helpful for new learning, successful spontaneous transfer is not guaranteed, and continuing research is needed to further develop a comprehensive theory of how to support learners' use of relevant prior knowledge without inviting reliance on aspects of prior knowledge that do not enhance new learning and problem-solving.

4.4 Limitations and Future Directions

As noted in section 4.2, one limitation of the current study is that the findings from this study do not reveal the specific mechanisms of diagrams' effect on spontaneous transfer. One possibility is that diagrams better serve to activate learners' existing conceptual knowledge, as spatial relationships in diagrams may aid in highlighting the key mathematics relationships (e.g., Hamdan & Gunderson, 2017; Kellman et al., 2008; Larkin & Simon, 1987; Moss & Case, 1999; Rau et al., 2014; Sidney et al., 2019). Thus, we may have observed greater rates of successful spontaneous transfer due to more robust activation of learners' prior knowledge because we used a diagram in the whole number division warm-up example. A second possibility is that diagrams facilitate transfer itself; that familiar actions on a diagram, or other external visual representation, can easily be applied to or adapted for less familiar problems (see Sidney & Alibali, 2017). In this possibility, transfer may be facilitated by the match between diagrams used for the familiar and novel problems.

Future research is needed to further disentangle whether diagrams support conceptual transfer because they better serve to activate adults' conceptual understanding or because they better facilitate transfer itself. Instead of varying diagrams of both the familiar problem and novel problems in conjunction, varying them independently of one another may shed light on the specific mechanism. If diagrams serve to better activate prior knowledge, then using a diagram for familiar problems (compared to no diagram) may have a greater effect on conceptual understanding than using a diagram for novel problems (compared to no diagram). On the other hand, if diagrams facilitate transfer itself, then matching the diagrams between the familiar and novel problems (e.g., number lines for both as compared to numbers for familiar problems and circles for novel ones) may have the largest effect on conceptual understanding.

A second limitation is that our findings may or may not generalize to children who are still learning about fraction division. In some ways, the adults in our sample are similar to children in prior research on fraction learning (e.g., Hamdan & Gunderson, 2017; Richland & Hansen, 2013; Sidney & Alibali, 2015, 2017; Sidney et al., 2019). Both groups show very poor conceptual understanding of fraction division without intervention, number lines support conceptual understanding for both, and both benefit from activation of prior knowledge. However, one key difference from prior research with children (Sidney et al., 2019) is that adults in our sample benefitted from circle diagrams whereas children did not. Thus, additional research with children is needed to examine whether the number line would uniquely support transfer when prior knowledge of whole number concepts is less fully developed.

Finally, this study does not address other individual differences that might moderate the effectiveness of using visual representations during learning. For example, several researchers have proposed that learners' motivation plays an important role in their learning from external representations (e.g., Mayer, 2014; Moreno, 2005). For example, in one study of college students' trigonometry problem solving, students with greater interest in mathematics benefited more from diagrams than those with lower interest (Cooper et al., 2018). Additionally, Hegarty (2004) has argued that learners' spatial ability and working memory capacity may moderate the nature of the relationship between external representations and their internal representations. In other words, external representations may play different roles for different learners. As such, it is possible that spontaneous transfer may be more or less likely to occur.

4.5 Educational Implications

This study has several practical educational implications for college-level learners. First, this study adds to the literature documenting undergraduate students' widespread and persistent difficulties with fraction concepts, despite often knowing procedures for solving fraction problems (Ball, 1990; Bently & Bosse, 2018; Lee & Boyadzhiev, 2020; Luo et al., 2011; Ma, 1999; Sidney et al., 2015; Yao et al. 2021). As with children (Bailey et al., 2012), adult students' understanding of fractions predicts their success and persistence in college mathematics courses (Ngo, 2018). Observing poor conceptual understanding in undergraduate learners at a selective university is concerning though not surprising.

Encouragingly, this study provides evidence for dramatic improvement in understanding with a simple, easy-to-implement intervention. The overall level of conceptual understanding in the whole number example + number line condition in our study was astonishingly high (78%). Although many studies have examined children's thinking using number lines (e.g., Booth & Siegler, 2008; Fazio, Kennedy, et al., 2016; Moss & Case, 1999; Siegler et al., 2011), there are fewer with college-aged adult learners. In line with the number line advantage observed in this study, Schiller and colleagues (2021) have recently used a number line intervention to improve undergraduate students' understanding of equivalent fraction, decimal, and percentage magnitudes. This emerging evidence suggests that, like children, university students in developmental mathematics courses may benefit from learning about, and practicing, key rational number concepts using number line diagrams.

Furthermore, even though simple whole number division (e.g., $6 \div 2$) is highly familiar and well-practiced for adults, our findings suggest that in the absence of the example problem, adults were not likely to spontaneously, or successfully, draw on this knowledge to support their ideas about fraction division. This is in line with prior research with adults suggesting that, without domain expertise, they may not know what aspects of their prior knowledge are conceptually-similar to new topics (see Chi et al., 1981). To make transfer from students' prior knowledge more likely, university instructors should consider how they might draw on adults' familiar knowledge of related, earlier concepts and instantiate these using visual representations that are similar to those present in the main lesson.

4.6 Conclusions

When engaging with mathematics, learners often spontaneously draw on their prior knowledge to make sense of new mathematical ideas. This is especially important in mathematics, a domain in which earlier-learned topics are often directly related to later-learned topics, though sometimes related topics are separated by years of intervening instruction (see Sidney & Thompson, 2019). In line with common recommendations (e.g., Richland et al., 2007; Sidney & Thompson, 2019; Vendetti et al., 2015), we empirically demonstrated that visual representations, such as diagrams, play a critical role in helping learners make use of their vast prior knowledge. Thus, instruction that is aimed at drawing on learners' prior knowledge may be most effective when instructors activate the most relevant familiar concepts using "warm up" activities that include visual representations that are aligned with those used during target instruction.

Though number lines did not *uniquely* support spontaneous transfer across whole number and fraction concepts, using number lines did lead to more accurate conceptual reasoning. Although many researchers -- across developmental psychology, cognitive psychology, educational psychology, and mathematics education -- agree that number lines and other linear representations are critically important for the development of early number and arithmetic concepts, ongoing research is needed to further clarify the mechanisms by which using number lines benefits learners' fraction arithmetic learning and problem-solving.

Footnotes

[1]Note that researchers have defined "conceptual understanding" in many ways (see Crooks & Alibali, 2014). Here, we focus on the conceptual models learners' use to make sense of the relationships between numbers within a division problem.

[2]Both whole number division and fraction division can also be conceptualized with a partitive model of division (i.e., the first operand indicates the total amount, the second operand indicates the number of groups or segments, and the quotient represents magnitude of each group or segment); however, many children (Fischbein et al., 1985) and adults (Ma, 1999) educated in the U.S. appear to favor quotative models for fraction division.

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