CCNETWORK
Comprehensive Center Network

REGION 8
Indiana
Michigan
Ohio

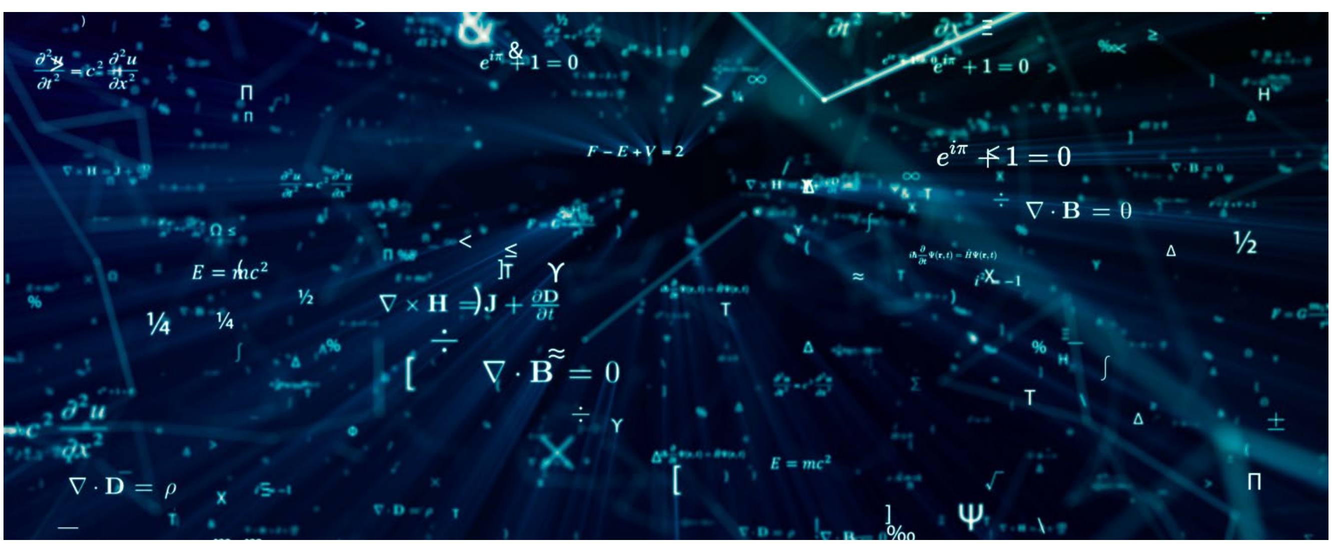
Michael Stevens

05 | 26 | 2023

Brief

Algebra I

Supports and Resources for Teachers



Preferred Citation: Stevens, M. (2023). *Algebra I supports and resources*. Region 8 Comprehensive Center, ICF International.

Mike Stevens is a math teacher and expert consultant living in New York's Hudson Valley. He taught all levels of high school mathematics at the High School for Law and Public Service in Washington Heights, NY, and holds a B.A. in Philosophy from the University of Florida and a master's degree in education from the City College of New York. Mike has extensive experience facilitating workshops and providing coaching for teachers, coaches, and leaders across the country. He is an expert consultant with a focus on formative assessment, curricular development, unpacking of content and practice standards, and practical strategies for effective instruction. Mike established Math Concepts Inc. in 2012 to provide expert consulting for the Gates Foundation's Math Design Collaborative as well as to support professional learning at a district level. He has been collaborating with the Region 8 Comprehensive Center since 2020.

TABLE OF CONTENTS

Background Information

The Region 8 Comprehensive Center (CC) is one of 19 Regional CCs in the CC Network that provides high-quality, intensive capacity-building technical assistance to clients from state, regional, and local educational agencies and schools in Indiana, Michigan, and Ohio. Region 8 CC staff serve clients by helping to identify, implement, and sustain effective evidence-based programs, practices, and interventions that support improved educator and student outcomes. Through these capacity-building services, Region 8 CC staff help agency staff improve educational outcomes for all students, close achievement gaps, and improve the quality of instruction.

This document contains a summary of the resources and supports that Michael Stevens has prepared addressing Algebra I courses for the Region 8 CC.

Purpose

The purpose of this resource is to help math teachers unpack, understand, and implement the current math content and practice standards. It describes the progressions of learning within each course and provides content supports that include broad ideas about effective instruction as well as practical instructional strategies. Math teachers, coaches, and leaders are encouraged to use these materials collaboratively to support ongoing instruction and the growth of individual teaching practice.

Organization

The content is organized by the following topics in Algebra I, including:

- Standards for Mathematical Practice
- Algebra I: Number and Expressions
- Algebra I: Functions
- Algebra I: Linear Relationships
- Algebra I: Systems of Linear Equations and Inequalities
- Algebra I: Quadratic and Exponential Relationships
- Algebra I: Data Analysis and Statistics

Each section includes: an overview and progressions of learning; content supports and strategies that vary and may include real-world scenarios, visual stimuli, open-ended questions and tasks, multiple representations, and formative assessments; and a section on addressing students who struggle, including language and communication, and vocabulary.

Standards for Mathematical Practice

The Standards for Mathematical Practice describe necessary academic and cognitive habits that help students access and understand the mathematical content at each grade level. The practices are based on proficiencies articulated in the National Council of Teachers of Mathematics (NCTM, 2000) process standards and the National Research Council's *Adding It Up* (2001).

While the content standards change for each grade and course, the eight Standards for Mathematical Practice are applied to all grade levels and intended to be developed continuously, year by year, as students progress from kindergarten through high school.

Standards for Mathematical Practice (MP)	
MP.1: Make sense of problems and persevere in solving them.	Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.
MP.2: Reason abstractly and quantitatively.	Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Standards for Mathematical Practice (MP)	
MP.3: Construct viable arguments and critique the reasoning of others.	Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen to or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.
MP.4: Model with mathematics.	Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Standards for Mathematical Practice (MP)	
MP.5: Use appropriate tools strategically.	Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.
MP.6: Attend to precision.	Mathematically proficient students try to communicate precisely with others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.
MP.7: Look for and make use of structure.	Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well-remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(x – y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Standards for Mathematical Practice (MP)	
MP.8: Look for and express regularity in repeated reasoning.	Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x^2 + x + 1)$, and $(x – 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Algebra I: Number and Expressions

This section will first present an overview of numbers and expressions and their progressions of learning. Next, it will describe five content supports and strategies: real-world scenarios, vertical planning, mental math, multiple representations, and daily formative assessment. Next, it will provide suggestions for students who struggle, including language and communication, and vocabulary.

Overview and Progressions of Learning

Elementary School

Prior to kindergarten, children are introduced to the most basic set of numbers, which we call counting or natural numbers. Their main purpose is to count objects in the real world. All the additional types of numbers that students interact with also have particular purposes and applications. The whole numbers, which include the natural numbers and the value 0, can be applied to contexts and problem-solving beyond the scope of the natural numbers. In kindergarten and the early grades, students begin to explore relationships between whole numbers in the form of basic operations and equivalence. In Grades 3–5, extending their understanding of multiplication and division, students use positive fractions to represent values beyond the realm of the natural and whole numbers.

Middle School

In the middle grades, students begin to work explicitly with integers and signed fractions to complete the set of rational numbers, and they also begin to explore irrational numbers within the context of square numbers and roots. In high school, students add the concept of imaginary numbers to the set of real numbers to form the complex number system.

As students extend their conception of number to include each new set (whole numbers, integers, rational numbers, irrational numbers, real numbers, and complex numbers), it is important to recognize that the four basic operations and their properties maintain their meaning and structure.[MP.7] The concept of place value and the relationships inherent in the base 10 number system are also consistent across number systems.[MP.7] The coherence in these structures helps students understand new notations, such as exponents and radicals, and navigate more complex applications of number as they progress through middle and high school.

Expressions are extremely important mathematical objects and the building blocks of algebraic equations and functions. An expression is a combination of numbers, symbols, and operations that represent a value, which may be known or unknown. Numerical and algebraic expressions may be simplified (written in a simpler but equivalent way) to express a slightly different meaning or application or evaluated (examined under given conditions to ascertain their value). Expressions cannot be solved as they have no inherent truth value until they are put into relationship with another expression within an equation or inequality. Students work with numerical expressions and informal algebraic expressions in elementary grades as they build fluency with basic operations and properties. In the middle grades, they begin to use formal algebraic expressions to interpret real-world contexts and solve real-world problems.

High School

A comprehensive understanding of high school algebra requires that students be able to recognize the operations and structure within an expression and purposefully alter that structure while adhering to the order of operations (for example, factoring a quadratic trinomial to identify the roots of a quadratic equation or simplifying rational expressions). A full understanding of expressions rests on all prior understandings of operation and number, and mastery of algebra and the mathematics beyond rests on a full understanding of expressions.

Content Supports and Strategies

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to number and expressions. The strategies include:

- Real-World Scenarios
- Vertical Planning
- Mental Math
- Multiple Representations
- Daily Formative Assessment

Real-World Scenarios

The complex number system and all systems within it allow us to interpret the world in different ways. Connecting different types of numbers to real-world contexts helps build students' conceptual understanding of numbers and gives meaning to the different procedures connected to each new set of numbers. Natural numbers are sufficient to count objects, but as we apply different operations, we discover the limitations of natural numbers, as when I have five (5) marbles and I take away all 5 (5-5=0). Suddenly, I need the whole numbers to address this problem. If I have $8 and I spend $10 (8-10), I need the set of integers to express the result of -2. Considering parts of a whole, as when three people share two pizzas (⅔), requires rational numbers that may include fractions and decimals.

Many applications of imaginary numbers occur in higher mathematics such as differential equations or in technical fields such as electrical engineering. The main application for imaginary numbers in high school is in solving quadratic equations. Some quadratic equations have solutions that cannot be expressed within the real number system. This connection should be made explicitly and repetitively so that students grasp the purpose and function of imaginary numbers within the math they are learning or will learn in the future. This is also true for any work around factoring quadratic expressions. The main purpose of this activity is for later work in solving quadratic equations and modeling with quadratic functions, and students should be made aware of this again and again to give purpose and meaning to their work as they learn to factor quadratics using different methods.

As with all applications of real-world scenarios, practicing number operations in terms of a context helps students reason about the operations and whether their answer makes sense, and this is particularly true about operations with rational numbers.[MP.4] Imagine being asked to add 1½+2⅝. A student who fully understands the meaning of fractions and has fluent procedures may be able to find an answer to this problem, but students who are still building that understanding would benefit from the question being posed within a context they could reason about, such as "The Domino family ate 1½ pizzas on Friday night and another 2⅝ pizzas on Saturday. How much pizza did they eat over the weekend?" The context increases engagement and purpose, encourages modeling and representation, and helps students reason about whether their answer makes sense.[MP.1]

Procedural fluency with linear expressions and an understanding of how they can be meaningfully applied to a context is essential for the functions and linear equation work within the Algebra I course. Many students entering high school are still building this understanding and remediation will be necessary. Application to real-world contexts is essential here, and students should be expected to translate from scenarios in words to algebraic expressions and vice versa. When modeling with algebraic expressions, students should be consistently expected to articulate what each part of the expression means within the given context.[MP.6] This type of work can be combined with the intentional use of whole numbers, integers, and rational numbers to generate assessment and give opportunities to build procedural fluency.

The study of square roots and operations with simple polynomials can be addressed within context problems about two- and three-dimensional objects.

- Find the side length of a square pool with a total area of $25x^2$.
- Find the area of a rectangular yard that is $x+4$ units wide and $2x+12$ units long.
- Find the volume of a gift box where the length is 3 more than the width and the height is twice the length.

Solving problems within a context helps students build conceptual understanding about the connections between algebraic and geometric expression of dimensions. A first-degree algebraic expression can be used to represent or measure one-dimensional space, a second-degree (quadratic) expression measures two-dimensional space, and a third-degree (cubic) expression measures three-dimensional space. Similarly, square roots relate to two-dimensional space and can be modeled geometrically with squares, and cube roots relate to three-dimensional space and can be modeled with rectangular prisms.

Vertical Planning

The teaching and learning in every domain within the algebra course needs to be planned and implemented with attention to connections between standards across domains. This can be thought of as "stacking" standards or teaching more than one standard at a time, and this is especially important for standards regarding number systems and expression. These standards cannot be effectively taught separately from the other content standards they relate to and doing so will only exacerbate students' incorrect view that the algebra curriculum is a fragmented list of unconnected topics.

The hierarchy of sets of numbers within the real number system is really addressed in every unit of the course, and teachers should use every learning task as an opportunity to be intentional about the types of numbers used to assess, remediate, and differentiate students' procedural fluency. Understanding of how imaginary and real numbers form the complex number system should be connected to work with quadratic equations and functions.

Simplifying algebraic rational expressions primarily prepares students to work with rational functions in later courses, but also connects to standards in the current course where students are building their basic skills and understanding around polynomial arithmetic. The properties and procedures involved in simplifying rational expressions also connect to work with linear equations with rational coefficients, solving simple quadratic equations, and manipulating literal equations.

Factoring quadratic expressions by various methods connects to algebra and functions standards as students solve quadratic equations and relate quadratic equations in different forms to key features of the graph. Again, it should be explicitly and repeatedly shared with students that factoring quadratic expressions does not stand alone as a worthwhile mathematical topic of study but is included in the curriculum for its practical application in solving quadratic equations and modeling quadratic functions.

As algebra students work with linear and nonlinear expressions, they are continuing to build key fluencies with polynomial arithmetic, which is the ability to relate terms of varying degrees using the basic operations. Students start to develop these skills in middle school as they learn to recognize parts of algebraic expressions and identify and combine like terms, but the work is mostly linear with limited inclusion of quadratic expressions. In high school algebra, students continue to develop these important skills as they manipulate and solve first and second-degree equations. To do this effectively, students need to attend to the degrees of different expressions, the operations between them, and the sign and role of the coefficients. Fluency with polynomial arithmetic will also support work with polynomial functions and other functions in the Algebra II course.

Mental Math

The consistent practice of mental math in classroom settings is a high-leverage strategy to build students' procedural fluency and number sense. Mental math can be practiced with the whole class, in small groups, in pairs, or individually, but it should be intentionally structured and should focus not only on answers, but on process. It can be done such that everyone has individual think time and then one student at a time shares thinking while others listen, or answers can be shared through choral response (everyone answers at once).

To make mental math a successful classroom structure, it is best to start early in the year, to begin with questions with broad access so students build confidence, and to do it regularly as a part of daily lessons. As with any discourse-based structure, trust needs to be built. Students need to see that the focus is not on whether they are right or wrong, but whether they are willing to think and share their thinking. They need to understand that their contributions will be valued regardless of correctness. To this end, it is productive when teachers do not validate correctness, but instead ask for justification and ask if the class agrees or disagrees.[MP.3] Also, do not let the discussion stop with the answer, but always follow up with something like, "Great; how did you get that?"

To make mental math a successful classroom structure, it is best to start early in the year, to begin with questions with broad access so students build confidence, and to do it regularly as a part of daily lessons.

As the classroom discourse develops, talk moves can be used to orient students toward the reasoning of peers.

Another benefit to mental math activities is that they expand students' procedural flexibility beyond standard algorithms while at the same time shedding light on the principles and properties that hold reliable algorithms together. As students' willingness to engage in mental math grows, take the time to focus discussion on the properties of operations that are employed in students' mental procedures. Students who are successful in algebra use mental math all the time as they compare quantities, solve equations, interpret graphs and tables, and identify patterns. Conversely, struggles with mental math and basic operational and number sense can bar the door to students' long-term success in algebra.

Multiple Representations

In addition to the different representations students will encounter as the number and expression standards are taught in conjunction with standards from other domains, some important representations to consider are number lines, area models, and Venn diagrams. Students often struggle to conceptualize the idea that while certain types of numbers are distinct from others, many include other types of numbers within them or are included within a larger set. The view of the complex number system as sets within sets can be expressed well with a Venn diagram and possibly with a tree diagram. These types of diagrams tend to be more effective if students have a large part in their design and creation, as opposed to simply adding words to prepared diagrams. If it is the structure of the diagram that expresses certain concepts, then students will benefit from interacting deeply with that structure. Teachers who utilize non-routine approaches to such activities often observe better outcomes in content knowledge as well as students' mathematical practice. Students could begin by writing out in words the set relationships of different types of numbers and then create and present a diagram which they believe expresses these relationships. They could also be challenged to place a given set of numbers in the appropriate places on a diagram or come up with their own examples and justify their choices.

Students have used number lines to express and compare values since the early grades. These can be very useful to gain assessment and have students practice comparing and identifying different types of numbers. Similar to the suggestions regarding mental math, the main focus here should not be on whether answers are correct or incorrect, but on student justifications for their responses and the discussions that follow.

Area models are simple rectangular representations that can be used in many flexible ways. They can activate learners who prefer to think geometrically, but they also deepen the understanding of any type of learner. Area models work well to practice basic polynomial math by simply assigning algebraic expressions of varied degrees to the dimensions of a square or rectangle and asking for area or perimeter. This gives students opportunities to create diagrams, attend to like terms, and use precise language to justify their operations. It also gives a lot of good assessment as to how students see the expressions and how they think about applying the given operations.

Also, ideas such as distribution, equivalent expressions, and factoring can be expressed very effectively in two-dimensional space and these models create a semipermanent record of student thinking that can be reflected on and discussed. Factoring a quadratic trinomial using an area model makes the structure of the process visible, and visibly represents the process of distribution and factoring (un-distributing) simultaneously.[MP.6,7]

Daily Formative Assessment

Many of the standards related to number and expressions are primarily procedural in nature, as noted by the verbs *simplify, factor, add, subtract,* and so forth. Other standards may lean more conceptual, as indicated by the verbs *explain* and *understand*. Because of the procedural focus of many of these standards, formative assessment needs to include a focus on connecting student procedures to the underlying concepts that justify those procedures. This can be seen as the "why" that justifies the "how." Many of the concepts that justify the procedures are connected to properties of operations, and some have to do with conventions and notation.

- Why is the product of opposite signed numbers negative?
- Why is the product of two negative numbers positive?
- Why is the square root of a negative number not a real number?
- What is a perfect square number?
- What is a square root? What is the inverse operation of taking a square root?
- What is *keep, change, flip* and why does it work?
- Why do we find a common denominator when adding or subtracting fractions?
- Why don't we need common denominators when we multiply or divide fractions?
- Why is the power rule of exponents not an example of the distributive property?
- What property are we using when we multiply two binomials?
- Why does the product of two binomials tend to give us a trinomial?
- What are two binomials that will give a product with only two terms and why is that happening?
- What is a factor? What does it mean to factor an expression? What is the opposite of factoring?
- What property or properties allows us to "cancel" common factors from a rational expression (multiplicative inverse and multiplicative identity)?
- What is the base of this exponent?
- Why don't we ever multiply an exponent by its base?

Good formative assessment questions give teachers important information about student misconceptions and gaps in learning, and they give students the opportunity to practice expressing and refining their understanding over time. For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not getting it.

For Students Who Struggle

Multiplication Charts

For students who struggle with procedural fluency or number sense in general, in addition to mental math activities and being intentional about the types of numbers in given problems, the use of multiplication charts can be productive. If they are solely used as tools for finding products, their impact is limited, but if students are repeatedly engaged in creating them and experiencing the properties and structures that hold them together the impact on fluency can be significant. As students create and build each chart, they experience important structures and patterns such as the commutative property of multiplication, the identity property of multiplication, the zero-product property, factors and multiples, and more.[MP.7] They can identify "friendly" rows like the 5s and 10s, and notice the unique diagonal populated by the perfect square numbers. One can also discuss the associative and distributive properties within the chart once it is created.

For students who struggle to understand and manipulate expressions, formative assessment can tell us what misconceptions are operating. Some students are not clear about basic operations and their relationships, or additional operations such as powers, exponents, and roots. Other students may struggle due to misunderstandings about notation. Still others may not understand the role of equivalence in relating and manipulating expressions or the basic structures in expressions.

- What is an expression? What do we use them for? What do we do to them?
- What does it mean for two expressions to be equivalent?
- Could you write this expression in a different way?
- What property or properties did you use to rewrite that expression?
- Why isn't subtraction commutative?
- How many terms does this expression have?
- What degree is this expression and how do you know?
- What operations do you see in this expression and how do you know?
- What is the base of this exponent?
- Can you combine these terms? Why or why not?
- What does it mean for two terms to be *like terms*?
- What is the difference between a minus sign and a negative sign?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including:

- Revoicing – "So you're saying that… ."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk.[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own

- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work, and actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are

- "Why?"
- "What does that mean?"
- "How do you know?"

These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessments. Also, they can be asked every day.

Vocabulary

Students benefit when they are expected to activate the names of the different number sets and use those terms on a regular basis. Numbers are likely used in every math lesson. That means every day is an opportunity to ask, "What type of number is this?" Also, the properties of operations need to be part of daily discussions in the math classroom. If these properties determine the operational structures of arithmetic and algebra, apply the entire complex number system, and justify many algebraic manipulations, they need to be discussed and referred to regularly in the math classroom so that students, over time, build a deep conceptual understanding of their meaning.

Important Vocabulary Terms for Numbers and Expressions

For standards involving numbers and expressions, important vocabulary terms that need to be defined, discussed, and consistently used include:

- Expression
- Term
- Coefficient
- Exponent
- Simplify

- Factor
- Multiple
- Commutative, distributive, associative, identity, and inverse properties
- Evaluate
- Equivalent
- Equation
- Solve
- Integer
- Rational Number
- Irrational Number
- Real Number
- Imaginary Number
- Complex Number

Algebra I: Functions

This section will first present an overview of functions and their progressions of learning. Next, it will describe five content supports: real-world scenarios, visual stimuli, open-ended questions and tasks, multiple representations, and daily formative assessment. Then, it will provide suggestions for students who struggle, including language and communication, and vocabulary.

Overview and Progressions of Learning

Functions are an extremely vital topic in high school mathematics and the primary intended destination of algebraic learning in elementary grades and more formal algebra standards in middle school. In the early grades, students explore informal algebraic reasoning as they compose and decompose numbers and examine equivalence through basic operations. In Grades 4 and 5, students work with shape and number patterns that describe relationships between two varying quantities.

In Grades 6 and 7, prior pattern work develops into the *Ratios and Proportional Relationships* domain where students become familiar with tabular and graphic representations, dependent and independent variables, and the multiplicative relationship between inputs and outputs of a proportional relationship. In Grade 8, students are formally introduced to functions, first as they analyze and represent proportional relationships and finally as they investigate applications of linear functions in general.

In high school, students continue to model with linear functions, also comparing them to exponential and quadratic models. The focus on additional function families, as students apply them within a context and utilize algebraic and non-algebraic representations, continues to be central to high school mathematics as well as college courses. In light of these progressions of learning, it is essential to establish a deep conceptual understanding of functions as well as fluency with related procedures to ensure successful student outcomes.

Content Supports and Strategies

Listed below are five strategies to help build students' conceptual understanding and procedural fluency relating to functions. The strategies include:

- Real-World Scenarios
- Visual Stimuli
- Open-Ended Questions and Tasks
- Multiple Representations
- Formative Assessment

Real-World Scenarios

It is essential to ground investigations of this domain in real-world contexts students can reason about.[MP.1,2,4] Students often analyze simple proportional relationships in their daily lives, and this understanding can be activated in the math classroom. By presenting open questions about real-world scenarios that include functional relationships and allowing students to think and conjecture about them, teachers can assess student understanding and practice while also connecting student ideas to the key concepts and vocabulary in the standards. Such contexts may include the total cost as a function of unit price, total pay as a function of hourly wage, total feet per number of animals, distance covered by a walker as a function of time, and so forth. Students benefit by examining function scenarios that are linear, such as a plumber with a visit fee and hourly rate, or nonlinear, such as the height of a bouncing ball or the growth of a cell by dividing in two.

Interpreting function notation in terms of a context helps students deepen their conceptual understanding and procedural fluency with function notation. For example, if the function h describes the relationship between the population of a country in millions and the calendar year after 1970, then the statement $h(5)=130$ means "in the year 1975 the population was about 130 million people." The statement $h(20)$ represents the population of the country in millions in the year 1990, or the answer to the question "What was the population of the country in the year 1990?" The solution to the equation $h(x)=225$ can be interpreted as the year in which the population reached 225 million, and the ordered pair $(7,134)$ can be interpreted by the statement "In the year 1977 the population was 134 million people." Notice that interpretation of function notation in terms of a context can be practiced, as above, without an explicit function rule because it is primarily about the meaning of the notation. Such interpretation can also be practiced within a context where the function rule is implied such as the function f, which relates the price of a tank of gas to the number of gallons in the tank such that $f(8)=18$.

It is important to note that the application to real-world scenarios both relies on and helps develop MP.2, Reason abstractly and quantitatively. MP.2 involves decontextualizing quantities and relationships, representing them with different symbols or notation, and contextualizing that symbology and notation in terms of the given context.

Visual Stimuli

The use of visual stimuli, including photographs or short videos, can engage students with different learning styles while still grounding their learning in real-world contexts. A picture of a vending machine can generate discussions about functions and non-functions, depending on whether the input is seen as the code punched in or the money inserted. A picture of a gas station price board or any grocery item can generate thinking about proportional relationships. A video of a walker or moving vehicle allows students to gather data and describe the distance covered as a function of time. The use of visual stimuli is non-routine for most learners and addresses the needs of students with varied learning styles.

Open-Ended Questions and Tasks

Choosing tasks with multiple correct answers or multiple solution paths encourages critical thinking and conceptual development. This could be simple structures such as "Which does not belong and why?," where the teacher presents four different items (e.g., four mapping diagrams, four equations or expressions, four different representations, four scenarios) and asks students to think on their own, then share reasoning and discuss. The purpose is not to elicit one correct answer but to allow students to construct arguments and share reasoning from different perspectives.[MP.3]

Open prompts such as "Tell me all you know about linear relationships" can give formative assessment information, encourage discourse, and allow students to connect prior knowledge with the current standard. As students work to develop conceptual understanding, prompts and tasks that keep the focus on student thinking are essential for authentic learning.[MP.1]

Multiple Representations

While scenarios are often presented in words, students should be encouraged to investigate them in multiple representations including tables, lists, graphs, diagrams, and so on.[MP.4] Most students prefer non-algebraic representations. While fluency with symbolic algebra is a goal for every student, many of the concepts in the functions standards can be well supported through varied representations. It is important to remember that the function itself is the relationship between the two sets of quantities (domain and range), which is distinct from any representation of that relationship (graphical, tabular, algebraic, verbal). The relationship itself stands on a different level than its representation.

A function in any representation consists of a domain, a range, a rule, and a solution set. Generally, in function notation, x represents the domain [including the x in $f(x)$], $f(x)$ represents the range, the expression represents the rule, and all possible ordered pairs that satisfy the equation represent the solution set of the function. Standard F.IF.1,

> F.IF.1 Understand that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range. If f is a function and x is an element of its domain, then $f(x)$ denotes the output of f corresponding to the input x. The graph of f is the graph of the equation $y = f(x)$.

This is largely conceptual, and if the underlying concepts are not in place then the symbology of the function notation (F.IF.2) will lack meaning. Students may also have misconceptions around the role of the parentheses in the notation $f(x)$. They may have connected parentheses to multiplication in the past as in the expression $2(5)$ and mistakenly interpret $f(x)$ as f times x or $f(3)$ as f times 3. Such misconceptions should be surfaced intentionally and addressed explicitly.

Daily Formative Assessment

Essential questions, such as "What is a function?," "How do you know this is not a function?," and "What two quantities are being related in this scenario and how would you describe their relationship?" should be asked again and again throughout this unit. Students benefit from answering from their own thinking and refining their expression over time, and teachers benefit from the subtle anecdotal assessment they receive. This type of informal formative assessment shapes ongoing instruction. Teachers who include open conceptual questions as part of daily lesson plans develop more effective questioning practice overall.

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not mastering the standard. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards?

If we can identify the missing concepts, we can begin to understand why they are struggling with the current standard. Students may begin to grasp some of the conceptual parts of the standards and yet struggle with the symbolic nature of the function notation.

For students struggling with standards related to functions, the first "step back" is to assess their understanding of linear relationships expressed algebraically. Do they understand what it means to solve an equation in s? Do they understand that a linear equation in two unknowns has a dependent variable that depends on an independent variable and that the input and output variables each represent an infinite set of numbers? If there is understanding here, however partial it may be, it can be used to connect to functions and function notation.

- What does it mean to solve an equation such as $y=3x+4$ or $x+y=8$?
- How many solutions are there and how do you know?
- What do the x and y represent in these equations?
- Which variable is dependent and why?
- What part of the equation $y=2x+8$ represents the input or domain?
- What part of the equation $y=3x+1$ represents the output or range?
- Why do we express solutions to linear equations in two unknowns as an ordered pair?
- How many ordered pairs do you think there are in the solution set of the equation $y=4x-3$? What do they all have in common?
- What does it mean to evaluate an algebraic expression?
- Write an expression with the variable x that will have a value of 9 when x is 3.

For students who still struggle, it can also be useful to investigate their grasp of linear relationships in multiple representations. Can they explain the connection between algebraic, graphical, and tabular representations of a specific linear relationship and can they create one type or representation from another? Do they grasp that the graph of a line is a picture of its infinite solution set?

- How does a graph connect to its equation?
- How does a table relate to its graph?
- Which representation do you prefer and why?
- What two characteristics describe a unique linear function?
- If two linear functions graph the same line, what two characteristics do they have in common?
- What connects a graph, table, and equation of a linear relationship?

Fluency across different representations can be assessed and developed by having students complete a graph, equation, and table of the same linear relationship where certain parts of each representation are given and some are left missing, but there is enough information to complete all of the representations by leveraging relationships between them.

For struggling students who are not able to express understanding of the concepts listed above, the next "step back" is to linear equations in one variable. Do they know what it means to solve a one-variable equation, and can they explain what makes the solution a solution? Can they solve by reasoning as well as by procedure?

- What does it mean to solve an equation such as $7=4x+1$ or $x+9=13$?
- Can you write a one-variable equation that has one solution? No solutions? Infinite solutions?
- What are some differences between the solution set of $9x=36$ and $x+y=12$?
- Can you graph the solution to $9x=36$ on a number line?
- How do number lines relate to a coordinate plane?

For students who struggle with function notation, the following assessment questions may be useful:

- Which part of the function $f(x)=5x-1$ represents the domain?
- Which part of the function $f(x)=3x2$ represents the range?
- How would you describe in words the function rule of $f(x)=2x+3$?
- What do all the ordered pairs that satisfy the function $f(x)=5x-7$ have in common?
- Name one ordered pair that satisfies this function and explain how you know it is a solution?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including:

- Revoicing – "So you're saying that… ."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk.[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work, and actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are

- "Why?"
- "What does that mean?"
- "How do you know?"

These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

For standard F.IF.1, while domain and range are clearly important vocabulary words, the use of input/output language is also important and helps students understand the dependency within each ordered pair or solution of the function and connect to their learning from prior grades. It is also necessary to ensure the application of key vocabulary within and across multiple representations.

Vocabulary Terms for Functions

Important vocabulary terms relating to functions that need to be intentionally developed, used consistently by students, and asked about by teachers include:

- Function
- Expression
- Evaluate
- Domain
- Range
- Input
- Output
- Rule
- Ordered Pair
- Solution
- Solution Set
- Dependent Variable
- Independent Variable

Regarding the question "What is a function?," for most students the answer "a function is a relation in which each element of the domain corresponds to exactly one element of the range" is insufficient. The follow-up question should immediately be "Great; what does that mean?"[MP.6] The concept of what a function is, as expressed in F.IF.1, deepens the conceptual understanding of the function notation required by F.IF.2. Function notation is accompanied by specific language that expresses and supports the utility of the notation.

Following MP.6, teachers need to be precise and intentional about their language and to actively uphold expectations of precision in student talk. Many students need to be explicitly shown how to express function notation in language. One common mistake is to refer to "the function *f(x).*"

This is imprecise because *f(x)* represents the range of the function, while the function itself is represented by *f* and is referred to as the function *f*. Also, for *f(x)* we say, "f of x," for *f(3)* we say, "f of three," and for *f(x)=3x+7* we say, "*f* of *x* equals *3x +7*." There may be misconceptions about what it means to evaluate a function for a domain value and how evaluation relates to simplifying and solving. These concepts should be addressed in class discussions. Evaluate means "to find the value of." So, to evaluate the function *f(x)=2x-7* for *f(4)* means to find the value of the function's output when the input is *4,* which is the same as finding the value of the expression on the right side of the equation when the input is *4.* Students need to understand that the notation *f* for a function is common. But students should also be exposed to functions notated as *g(x), h(x),* and otherwise. Similarly, they should begin to see that while the domain variable *x* is commonly used, functions with different input variables such as *f(m)* and *g(k)* are also used, and it is the location within the notation that indicates what the term represents.

Algebra I: Linear Relationships

First, this section will present an overview of linear relationships and their progressions of learning. Next, it will describe five content supports and strategies: real-world scenarios, multiple representations, big ideas, flexible procedures, and daily formative assessment. Then, it will provide suggestions for students who struggle, including language and communication, and vocabulary.

Overview and Progressions of Learning

Linear equations, and their connections to expressions and functions, form a central topic in high school algebra. While equations and formulas are but one representation among many that are used in high school modeling situations, the symbolic or algebraic representation is given a primary focus due to its elegance, explicit nature, and ability to generalize.

Two long progressions of learning come together to specifically support high school work with linear equations. These are the development of a deep understanding of number systems and a firm grasp of the conceptual and procedural aspects of algebraic thinking. This begins in kindergarten with the composition and decomposition of numbers and continues in the early grades in relation to addition and subtraction. In upper elementary grades, students practice multiplication and division, become familiar with both additive and multiplicative properties of operations, and deepen their understanding of equivalence and equality.

In the middle grades, the structures of number systems and the practice of algebraic thinking are formalized in the application of symbolic algebra to real-world contexts. Students learn to create and manipulate expressions and begin to see equations and inequalities as statements of relationship between expressions. They understand solving as a process of finding values that make the statements true. Students learn to do this both by reasoning about equations and inequalities and by using procedures based on operations and properties of operations. Students see solutions to equations and inequalities in one variable as sets of numbers that may be finite or infinite, and solutions to equations in two variables as sets of ordered pairs that can be recorded in a table or graphed on a coordinate plane. Finally, students apply all their understanding of linear relationships in multiple representations to investigate systems of equations and see the solution to a system as a set of ordered pairs that satisfies every equation in the system simultaneously.

Two long progressions of learning come together to specifically support high school work with linear equations. These are the development of a deep understanding of number systems and a firm grasp of the conceptual and procedural aspects of algebraic thinking.

These foundational concepts are extended through high school algebra. Some curricula sequence functions first as a broad concept, and then address algebraic representations as tools to handle functions, while others dive deeply into equations and expressions and culminate by addressing functions. In any case, it is clear that algebra and functions are deeply connected.

Student misconceptions can arise from seeing linear equations and linear functions as unconnected and separate and can also result from failing to see how equations and expressions are conceptually different from functions. As the practice of modeling becomes a major focus in high school, successful students rely on procedural fluency as well as conceptual understanding of the meaning of mathematical objects, processes, and relationships. In algebra and higher courses, student experiences with linear functions are extended to additional function families that allow for modeling within increasingly new and varied contexts. Like any major mathematical topic, the full understanding of linear equations, inequalities, and functions is arrived at through multiple grades and courses, and effective assessment and remediation must include a full grasp of those progressions of learning.

Content Supports and Strategies

Listed below are five strategies to build students' conceptual understanding and procedural fluency with linear relationships. The strategies include:

- Real-World Scenarios
- Multiple Representations
- Big Ideas
- Flexible Procedures
- Daily Formative Assessment

Real-World Scenarios

One aspect of rigor is the application of the mathematics that students are learning to real-world contexts.[MP.4] This is particularly important around linear equations, inequalities, and functions because this content includes so many important skills and procedures but also calls for deeper modeling experiences than in earlier grades. Consistent use of real-world scenarios can help create a meaningful bridge between the skills students need to master and the conceptual understanding that gives meaning to those skills.

Some standards relating to linear relationships lean toward the procedural as indicated by the verbs *solve*, *write*, and *calculate*, while other standards are more conceptual and require students to *understand*, *explain*, and *justify*. Several standards explicitly require students to represent real-world problems, but the consistent use of real-world scenarios to address all standards tends to encourage deeper and more lasting learning. Attending to context helps connect student thinking to underlying concepts. Instead of being asked to write an equation that includes the point (11, 160) and has a rate of change of 10, students could be challenged to write an equation to represent the following scenario:

> José gets paid a fixed amount every week and $10 per hour. Last week he worked 11 hours and made $160. Write an equation that relates the hours José worked to his total weekly pay.

The interpretation required and the multiple possible pathways in this problem help enrich discussion about the usefulness of different forms of linear equations. Similarly, learning how compound inequalities can be applied to the real world is essential to make sense of what they mean and why they are important.

For standards that require multiple representations, the use of real-world scenarios can help students conceptualize the different types of solution sets implied by one- and two-variable equations and inequalities. Generally, one-variable linear equations tend to have a single solution. One-variable linear inequalities have an infinite solution set that is a continuous or compound range of numbers. Two-variable linear equations have an infinite solution set made up of ordered pairs that graph a line, and linear inequalities in two variables have an infinite solution set made up of ordered pairs that graph a half-plane. Thinking about the solution sets in terms of a context and identifying possible individual solutions within those sets can help students solidify these important concepts and better understand the different mathematical objects involved.

Real-world scenarios also help students make meaningful connections between multiple representations. Through contexts, students can see how ideas such as rate of change or initial value are present in all representations, and also begin to think about the strengths and limitations of different representations.

Multiple Representations

The focus on linear relationships within the algebra course obviously requires attention to multiple representations, which are explicitly noted in the related standards. None of these representations are new to algebra students. All have been addressed in the middle grades and even earlier, so what are we to make of their continued presence in high school standards?

Consider this standard from Grade 6:

> 6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation.

Along with the use of variables and emphasis on the functional relationship, the attention to multiple representations is explicit. So, what do high school algebra standards require of students beyond what has been developed in earlier grades?

Two ideas to consider here are "integration" and "authorship." Integration points to the consistent simultaneous use of multiple representations to investigate various aspects of a linear relationship and the related context. While this may seem obvious, many algebra teachers still spend significant time focusing on one or another representation and its related procedures. Instead teachers should be focusing on intentionally integrating representations and using the connections between them, in light of key concepts, to deepen student understandings of individual representations.

At the high school level, use of multiple representations can also be somewhat less prescribed than in earlier grades, where students are given choice in the representations they utilize and the sequence in which they are created. Authorship refers to the idea that at this level, representations should increasingly be chosen by the mathematician in order to investigate or represent math in a way that serves their problem-solving goals. This includes some choice over how representations are constructed, what is included, labeling, scaling, and so forth. Over time, representations become tools the mathematician uses to express and extend their thinking as opposed to something they are told to do.

Of all the types of representation—including those that use symbols, graphs, tables, pictures, and manipulatives—there is one representation that is most often underutilized in the algebra classroom: verbal representation. While scenarios and math problems are often presented in words, verbal representations from students to communicate their thinking are often undervalued or dismissed. This is particularly important as related to linear relationships, because many students are developing algebraic thinking and can express their understanding in words, but struggle to express themselves clearly using symbols or equations.

We may present a scenario such as: "Tina went to the fair. It cost $5 to get in and $2.25 per ride. Write an equation that relates *r,* the number of rides Tina rode, to *c,* the total cost of going to the fair." Some students may struggle to construct a valid equation, but may be able to express their generalization verbally, as in: "I know that to find the total cost I just multiply the number of rides she took times $2.25 and then add $5, but I don't know how to write an equation … ." This verbal representation of the relationship is valid and complete and should be honored as such.

While it is true that we want students to be able to express such generalizations using equations, it is also true that for those who struggle with symbolic algebra, the clarity in the verbal representation can form the bridge between their current understanding and their next step. This is also true for students who feel comfortable thinking from tables or graphs, but struggle with equations. What connects all representations of linear relationships are the big ideas about linear relationships such as linearity, rate of change, and initial value.

Big Ideas

The standards related to linear relationships are generally high-priority standards for the course and should be given significant focus and time across different units of learning. They also include a lot of content, both procedural and conceptual, and it can be overwhelming to plan effectively. One strategy that can help is to plan instruction around the big ideas that dominate the mathematics. To grasp these ideas, students need consistent, intentional, and repeated exposure to them. Trying to understand linear relationships without understanding these big ideas is extremely difficult. Big ideas related to linear relationships include:

- Linearity
- Constant rate of change
- Rate of change
- Initial value
- What it means to solve an equation
- The difference between an expression, equation, and function

Students need to know what it means for a relationship to be linear. They need to be able to identify and justify linearity given any representation. This really comes down to whether the relationship has a constant rate of change, which is what creates a linear graph, and what is common to all linear relationships and linear functions. A rate of change is constant when, given a uniform increase or decrease in input, the change in outputs remains the same. Notice that it does not matter what the rate of change is, but only that it is constant. While this is not a new idea for students, it needs to be consistently addressed and asked about, and they need time to develop and generalize the idea. One way to assess this is by asking "Tell me all you know about linear relationships" at different times in the learning to get an idea about what students prioritize in this content. Another essential question to be asked again and again is "Is this relationship (or equation, graph, table, problem) linear? How do you know?"

When comparing linear relationships or identifying a specific linear relationship, there are two operative ideas: rate of change and initial value. These values create a sort of fingerprint for a linear relationship, and if two relationships have the same rate of change and initial value, they are in fact the same relationship. Identifying specific rates of change in multiple representations requires a deep understanding of the correspondence between domain and range values, and what governs their pairing. When graphing, the rate of change is referred to as slope because it is the character of a line, but in tables and scenarios the term rate of change is more precise. The initial value is simply the value of the output when the input is 0. On a graph, this is known as the *y* intercept, or the value where the line crosses the *y* axis. Students need to know that if they are creating, interpreting, or translating between representations of linear relationships, it is these two values that they always need to attend to and include in their thinking.

One essential question for this content is "What does it mean to solve an equation?" The more students are engaged in such discussions, the more they will connect the procedural aspects of solving to the conceptual meanings that underlie them. One way to differentiate learning about specific mathematical objects or processes is to ask students to define what they are.[MP.6]

It is also important to understand what actions we take on equations. Algebraic equations can be solved because the statement of equality between the two expressions creates a truth value, and members of the solution set (when substituted for the variables) make the statement true, while other values do not. If what it means to solve an equation is seen as a worthwhile big idea, students will also meaningfully consider different kinds of equations and understand the varied nature of the solution sets of equations and inequalities in one and two variables.

Flexible Procedures

One of the dangers regarding the content in this domain is that students may see each standard as a set of steps to memorize, and not establish the conceptual understanding outlined above. One way to address this danger is to intentionally support students to be flexible with their procedures. One reason students try to memorize "how to do math" is that it has been presented as a number of things to be done, each with a correct way to do it. The core value behind flexible procedures is that there is always more than one way to engage with a worthwhile math problem, and real problem-solving includes exploring and tinkering as much as the use of tried and true procedures.

In addition to having students choose and create representations to address a problem, the consistent and early use of various forms of linear equations will add to students' procedural flexibility. Some students mistakenly believe that an equation is only linear if it is in $y=mx+b$ form, which breeds deep confusion when addressing varied forms of linear equations. The concept that ties to this type of flexibility is the idea that equivalent equations have different forms, but the same solution set—just as equivalent expressions take different forms but give the same output when evaluated and can be represented with the same area models. It also ties back to the idea that linear functions are defined by their rate of change and their initial value. Allowing students to choose what representations to explore and what methods to use when writing equations given a point and slope, slope and y intercept, or two points, will help them understand the value and purposes of different forms of linear equations.

In the early grades, students are often told what values to evaluate expressions for and input into equations. Part of using flexible procedures in high school includes students choosing what values to use when investigating expressions and equations and keeping track of their findings through appropriate representations. Even when solving equations and inequalities, students should be encouraged to use flexible procedures, such as solving by reasoning about expressions, using inverse operations, and exploring solutions sets through trial and error. Encouraging students to use flexible procedures creates more differentiation in learning and can include struggle, confusion, and mistakes. If this can be seen as a vital part of the sense-making process and not as a problem to be fixed, students will be more likely to connect procedures to underlying concepts in a meaningful and productive way, thus growing their mathematical practice as well as their understanding of the content.

Encouraging students to use flexible procedures creates more differentiation in learning and can include struggle, confusion, and mistakes. If this can be seen as a vital part of the sense-making process and not as a problem to be fixed, students will be more likely to connect procedures to underlying concepts in a meaningful and productive way, thus growing their mathematical practice as well as their understanding of the content.

Daily Formative Assessment

For such high-priority content as linear relationships, daily formative assessment is essential to understand the learning needs of different students. There are many conceptual questions that may be essential questions for a student or group of students as they move through this content.

Essential questions should be intentionally planned and utilized, and asked again and again. They should meet students' current thinking and also cause them to deepen and extend their thinking. Essential questions also develop the students' ability to express their growing understanding.

Assessment questions should address student thinking about the procedures they are using and their understanding of key concepts as well as ways to productively connect the two.

- What does it mean to solve a one-variable equation?
- What does it mean to solve a one-variable inequality?
- How do you know when a real-world problem can be modeled with a one-variable equation? With a one-variable inequality?

- What does it mean to solve a compound inequality?
- What does it mean to solve a two-variable equation?
- Why can't we graph a one-variable equation on a coordinate plane?
- What is similar between the equation, graph, and table of a linear function? What is different in each representation?
- What does a graph show that the other representations do not?
- What does a table show that the other representations do not?
- What does an equation show that the other representations do not?
- What two values do you need to identify a specific linear function?
- How would you write an equation given rate of change and initial value?
- How could you find the rate of change and initial value given two points on the line?
- How could you find the initial value given the rate of change and one point on the line?
- How could you find the rate of change given the initial value and one point on the line?
- What needs to be identified in a real-world problem in order to write an appropriate equation?

Good formative assessment questions give teachers important assessment about student misconceptions and gaps in learning, and they give students the opportunity to practice expressing and refining their understanding over time. For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not getting it.

Students need to be exposed directly to all the concepts and skills connected to linear relationships, keeping in mind that they have already been exposed to nearly all these topics. One big question is what to do when they have been exposed to all the ideas and still are not proficient with the standards. Asking the types of questions suggested above and creating learning opportunities in which students can make sense of them will help teachers adapt instruction to student needs and help students make connections across standards and develop a more lasting understanding.

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including:

- Revoicing – "So you're saying that… ."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk.[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work, and actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are

- “Why?”
- “What does that mean?”
- “How do you know?”

These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

There is a significant amount of important vocabulary connected to linear relationships, and at the same time very little that students have not heard before, and sometimes for several years. The challenge on the high school level is not to find out if students have heard the words, but to dig deeply into what concepts students think the words represent and how they relate one term with another. This requires a lot of patient questioning about common terms that it would be easy to assume all students fully understand. It is amazing how some students will go right to work identifying y intercepts and slopes while not being able to define either term. Teaching also requires patient listening and building the habit of always asking students to define any terms they are working with, including entities such as expressions and functions as well as processes such as solving and transforming. This habit can be built by always adhering to this rule: You don’t really know what students are thinking until you ask.

Vocabulary Terms for Linear Relationships

Important vocabulary to be assessed, developed, and attended to regarding linear relationships includes:

- Expression
- Equation
- Inequality

- Function
- Input
- Output
- Dependent Variable
- Independent Variable
- Evaluate
- Simplify
- Solve
- Solution
- Solution Set
- Rate of Change
- Initial Value
- Slope
- *Y* Intercept
- Linear
- Equivalent

Supporting Student Use of Precise Language

Process Standard #6 calls for students to use precise language to communicate effectively. Teachers can support this practice by being very intentional about the language they use at all times. It is possible for a teacher who clearly understands the difference between expressions, equations, and functions to unintentionally interchange these words during instruction for lack of attention. It is also useful to be as specific as possible when referring to equations and inequalities by noting how many variables are in the equation or inequality. This helps students track the important distinctions between equations and inequalities in one and two variables.

Algebra I: Systems of Linear Equations and Inequalities

First, this section will first present an overview of linear equations and inequalities and their progressions of learning. Next, it will describe five content supports and strategies: real-world scenarios, conceptual and procedural balance, multiple representations, open tasks, and daily formative assessment. Then, it will provide suggestions for students who struggle, including language and communication, and vocabulary.

Overview and Progressions of Learning

Applications of systems of linear equations and inequalities are the natural follow-up to a mastery of linear functions in multiple representations. Systems can be seen as less of a separate topic and more as the final application of the long progression of learning around linear relationships that began in elementary school and are the main focus of middle grades.

In Grade 6, students begin to investigate and represent proportional relationships and equations, building on their understanding of number, operations, and algebraic thinking from the early grades. Throughout middle school, students build fluency with tables, graphs, and equations and they learn to focus on the constant of proportionality in proportional relationships as a multiplier that governs the functional correspondence between all inputs and outputs of the function. This focus supports the recognition of rate of change as an important feature of all linear relationships, as Grade 8 students move beyond proportions to linear functions more broadly. Students solve and represent simple inequalities in Grade 6 and address linear inequalities in Grades 7 and 8 as they use them to model real-world situations and represent solutions algebraically and on number lines.

High school algebra students build on students' vast learning from middle school as they use linear functions in multiple representations to model real-world problems. They also model with linear inequalities in two variables and represent those on a coordinate plane. The vast majority of the concepts allowing students to work productively with linear systems relate to standards that have already been addressed in prior grades or earlier units in Algebra I. The few new concepts required for systems are also extensions of significant prior learning, such as the concept of the solution set of a linear system being the intersection of the solution sets of equations and inequalities within the system. The new procedures required for systems, such as the algebraic manipulation used in substitution and elimination methods, are extensions of the significant manipulation students have already engaged in during their long study of linear relationships. In later high school courses, students will investigate systems that include nonlinear functions, and college courses such as linear algebra will extend applications of systems to solve more complex problems.

Content Supports and Strategies

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to systems of linear equations and inequalities. The strategies include:

- Real-World Scenarios
- Conceptual and Procedural Balance

- Multiple Representations
- Open Tasks
- Daily Formative Assessment

Real-World Scenarios

As with all important content, learning about systems of linear equations and inequalities needs to be grounded in real-world applications.[MP.4] Students have extensive experience modeling real-world problems with linear functions and inequalities, and now are able to use systems of equations to solve problems involving more than one linear relationship, and to compare and relate multiple linear relationships in a new and powerful way.

When there is a context, students can touch back on what the variables and relationships represent, what rate of change and initial values actually mean, and what the solution to the system means as they work through the problem and reflect on the validity of their answers. This builds the process of contextualization and decontextualization noted in MP.2, develops abstract and quantitative reasoning practice, and deepens conceptual understanding of linear functions and the processes used to manipulate and represent them.

As students explore and represent systems in context, they also learn to look at the whole picture of the system, which includes information beyond the solution set. They compare rates of change using graphs and tables and can solve optimization problems that look to maximize certain quantities or relationships. It is the context that drives the need for a system. This unit could be launched by showing students a real-world scenario that includes two linear relationships and asking them to make sense of what they see and consider what it might mean to solve the problem.

> Leo pays $15 dollars a month for his cell phone, plus $10 per GB of data.
>
> Ella pays $30 dollars per month for her phone, plus $8 per GB of data.
>
> Last month, the cost of Ella's bill was the same as Leo's. How much data did each of them use?

Students could investigate this scenario in many ways. They could create tables, graphs, or equations and they could also reason about patterns within each relationship to find a common output. If a solution is found, discussion could center on what the solution (7.5, 90) means and whether students think there are any other ordered pairs that could apply to both Ella and Leo.

Conceptual and Procedural Balance

The primary conceptual questions regarding systems of linear equations are "What is a system of linear equations?" and "What does it mean to solve a system of linear equations?" Students who can coherently answer these questions tend to be conceptually prepared to meet these standards, including transferring the procedures for solving systems of equations to systems of linear inequalities.

As students solve real-world problems with systems; represent them with graphs, tables, and equations; and find and justify solutions, part of the teacher's role is to help connect necessary procedures to underlying concepts. Students must consider what quantities are varying in the scenario, identify two distinct relationships between those quantities, note rates of change and initial values, decide which variable is independent, and consider which representations will be most helpful to find the solution. Teachers can support students by listening carefully to how students approach and navigate the task and encourage student reasoning by asking thoughtful questions that honor student thinking and challenge student misconceptions. The two questions noted above help students understand what such tasks are about and clarify the goals of their work.

> Jack solved a system of linear equations. His answer was 5.
>
> Jill said she knew his answer was incorrect without even looking at the system. Explain.

Once students understand what it means to solve a linear system, they practice with algebraic methods. There are many procedures involved here, but procedures alone will not build proficiency. Students need to understand that equivalent equations have the same solution set and can be generated by applying basic operations to the entire equation. Students who lack this concept may try to procedurally transform the equations so they can substitute or eliminate a variable but will not really know why they are doing it and will struggle more over time. An important concept at play here is the question of why we are trying to substitute or eliminate. Students need to understand that a linear equation in two variables cannot be solved algebraically because one variable will always be defined in terms of the other.[MP.7]

By using substitution or elimination to create an equation in one variable, we can identify the value of one variable and use that footing to define the other variable in the system, knowing that a unique solution to a linear system satisfies both equations. This concept needs to be addressed explicitly and asked about while students are working. "How does eliminating a variable help us solve this system?" "Why are we trying to substitute one variable for another?" When students are able to grasp different solving methods, it is useful to allow them to use these procedures flexibly and choose the solving method that makes the most sense to them.

Multiple Representations

The graphical representation is essential as students learn to classify solutions to systems of equations. If students do not make their own sense of the fact that two lines in a plane can only interact by crossing once, not crossing at all, or laying on top of each other (having the same rate of change and y intercept), they will not understand the possible nature of solutions to linear systems and will not apply this concept to later problem-solving.[MP.7] As they work on algebraic solutions, encourage the additional use of tables and graphs to clarify and verify their work. If they can determine the solution to the system through another representation, it may help them navigate the algebraic manipulation with more confidence, understanding, and perseverance.

Open Tasks

As with any content standards, there are elements related to systems that need to be shared through direct instruction, but once those lessons are taught students need time and space to construct lasting understanding through application, experiment, struggle, and mistakes. During this time, it is productive to use fewer tasks that are rich and rigorous as opposed to many tasks that are not. Rich tasks may include multiple representations, multiple solution paths, multiple entry points, discussion questions, justification, and possibly multiple correct answers. Consider the following problem:

> Solve this system:
>
> $3x+4y=19$
> $2x-2y=8$

This problem can be transformed or "opened up" by working backwards from the answer:

> Write a system of linear equations that has a unique solution of *(5,1).*
>
> Be prepared to justify your work and describe your process.

Open problems like this are challenging, require concepts and perseverance, work well as partner tasks, and give great assessment of student thinking and confidence.

Daily Formative Assessment

Two questions that can be asked continuously throughout this unit are "What is a system of linear equations?" and "What does it mean to solve a system of linear equations?"

Some students can replicate procedures well and solve problems correctly in class, but without the concepts indicated in the above questions they are much less likely to retain proficiency over time. Students should also be consistently asked to justify their procedures as they solve systems. This tends to be more productive when the problems they are solving are more open and allow for flexible procedures.

- What is a system of linear equations?
- What does it mean to solve a system of linear equations?
- Can a system of linear equations have two solutions? Why or why not?
- If you solve a system, how can you tell whether there might be more solutions?
- What are the possible types of solution sets for a system of linear equations?
- When using the substitution method, what are we substituting and why?
- When using the elimination method, what are we eliminating and why?
- Which method for solving systems do you like best and why? Can it always be used?
- What is the same about systems of linear equations and systems of linear inequalities? What is different about systems of linear equations and systems of linear inequalities?

For Students Who Struggle

For students who struggle to meet these standards, it is helpful to consider that if they truly understood the concepts and procedures around linear functions in multiple representations, then these standards are not asking significantly more of them. If students struggle to solve systems, they likely have misconceptions or incorrect procedures regarding linear functions. It can be time consuming and frustrating for teachers and students alike to continue to press current standards when remediating back to prior standards would be more engaging and productive. Ideally, daily instruction should include both remediation and current content and help students make connections between the two.

- What does it mean to solve an equation in one variable?
- What does it mean to solve a two-variable equation?
- Why can't we graph a one-variable equation on a coordinate plane?
- What is similar between the equation, graph, and table of a linear function? What is different in each representation?
- What does a graph show that the other representations do not?
- What does a table show that the other representations do not?
- What does an equation show that the other representations do not?
- What two values do you need to identify a specific linear function?
- How would you write an equation given rate of change and initial value?
- How could you find the rate of change and initial value given two points on the line?
- How could you find the initial value given the rate of change and one point on the line?
- How could you find the rate of change given the initial value and one point on the line?
- What needs to be identified in a real-world problem in order to write an appropriate linear equation in two variables?

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including:

- Revoicing – "So you're saying that… ."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk.[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work, and actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are

- "Why?"
- "What does that mean?"
- "How do you know?"

These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

Vocabulary Terms for Linear Equations and Inequalities

For systems of linear equations and inequalities, the following terms must be defined by students again and again and used with precision in daily work:

- System of Equations
- System of Inequalities
- Solution; Solution Set
- Intersecting
- Parallel
- Elimination
- Substitution
- Equivalent Equation
- Ordered Pair

It is particularly important to be intentional with language regarding solutions, solving, and solution sets. Over time, students have seen the nature of solution sets of equations change from being generally one number to including infinitely many ordered pairs, and now must understand the solution set of a linear system of equations as the intersection of two infinite sets of ordered pairs that may occur in three possible ways. These concepts and the progressions of learning they track are still being formalized for many students, so all language regarding solution sets and solving needs to be precise and include references to what is being solved.

Teacher questions and the consistent expectation that students justify their statements have a significant impact on discourse environments, student practice, and student understanding. As students solve systems, particularly using algebraic methods, it is important to explicitly address the idea of equivalent equations and how this idea connects back to solutions sets for linear equations. Equations are kept equivalent to maintain the integrity of the solution set. Attention to this idea helps keep students focused on the meaning of what they are doing as they manipulate and transform equations and inequalities.

Finally, it cannot be assumed that students fully understand parallel lines in a plane and how parallelism connects to slope and common solutions. Again, these ideas should be discussed explicitly and asked about continuously as students work. This will serve to confirm and verify correct thinking, and to highlight and correct any misconceptions.

Algebra I: Quadratic and Exponential Relationships

This section first presents an overview of quadratic and exponential relationships and their progressions of learning. Next, it will describe five content supports and strategies: real-world scenarios, vertical planning, big ideas, multiple representations, and daily formative assessment. Then, it will provide suggestions for students who struggle, including language and communication, and vocabulary.

Overview and Progressions of Learning

The purpose of functions as a mathematical tool is to describe, interpret, and represent real-world relationships in a way that allows us to understand them more deeply and solve related problems. Experiences in the middle grades build students' proficiency around modeling linear relationships in context and representing those relationships productively. In high school algebra, those understandings and fluencies related to linear functions are practiced, formalized, and deepened. This focus on linear functions is the highest priority content for the Algebra I course. Several other content areas (number, expressions, and data and statistics) either support or connect to this focus. Finally, topics such as systems of equations, quadratic relationships, and exponential models build on and extend the primary content focus of linear functions.

It is important to recognize that while deeper analysis and representation of quadratic and exponential functions is new content, understanding of functions in general and their representations is not. Learning related to this new content must rest on the conceptual understanding of what functions are and how they are analyzed and represented, including all the language and key concepts from earlier units, such as input, output, domain, range, rate of change, and so forth. Quadratic and exponential relationships extend students' modeling capabilities well beyond linear models and broaden the possibilities of applications to relationships with varied rates of change in the nonlinear realm. This new focus also demands that students apply their polynomial arithmetic and understanding of expressions from prior units to manipulate more complex expressions within modeling contexts. As students move into upper high school courses, their understanding of quadratic equations and functions grounds investigations of higher-degree polynomial functions, and their experience with exponential functions supports learning about logarithms.

Content Supports and Strategies

Listed below are five strategies to help build students' conceptual understanding and procedural fluency related to quadratic and exponential equations and functions. The strategies include:

- Real-World Scenarios
- Vertical Planning
- Big Ideas
- Multiple Representations
- Daily Formative Assessment

Real-World Scenarios

As students begin modeling with nonlinear functions, application within a context continues to be essential to help students understand the nature of quadratic and exponential relationships and to transfer the skills and concepts developed to handle linear functions into the new content.[MP.2,7] These include identifying varying quantities in a real-world problem, describing the nature of their relationship, discerning dependence, expressing the meaning of corresponding pairs, and constructing representations from generated data or to express observable trends.[MP.4] Much of the new conceptual learning is connected to recognizing and understanding nonlinear rates of change. Students need to compare linear and nonlinear rates of change and begin to distinguish between quadratic rates, which always rely on second-degree expressions and create symmetrical data (both increasing and decreasing) with a central turning point (the vertex), and exponential rates which always either increase or decrease and involve a much higher rate of change at the extreme.[MP.7] A classic way to consider the vast differences in linear and nonlinear rates of change is the following:

Would you prefer choice a or b?

a) I give you a penny on day one and double your payment every day for a month.

b) I give you $1,000 on day one and increase the daily payment by $100 every day for a month.

Many students are tempted by the $1,000 in choice b, and do not realize the astronomical results of the exponential growth in choice a. The shock that can be experienced upon realizing that choice a will lead to a payment of more than $5 million on day 30 and yield over $10 million dollars in total is called disequilibrium, which is a state in which the observed outcome of an event does not match the expected outcome. Disequilibrium is a significant condition for learning. Problems that cause disequilibrium create a state where the learner actively seeks new understanding to resolve the agitation of the disequilibrium. What can be learned in a problem like this is the power of exponential growth and how even a high constant rate is dwarfed in comparison over time. This problem is also sometimes presented such that choice b is to simply receive $1 million all at once. Note that choice a can be investigated with or without exponential equations as simple addition or multiplication will suffice.[MP.4]

As students begin to relate nonlinear rates of change to mathematical expressions, they begin to see that exponents play a key role. They also may notice that the constant term in a linear equation, also known as the initial value or y intercept, continues to operate in many quadratic and exponential equations and is used later to connect algebraic and graphical representations as with linear functions. Some real-world scenarios that can be modeled with quadratic functions include the path of a diver jumping off a diving board or after entering the water, a bouncing or thrown ball, an arch in a doorway or bridge, or a circular or elliptical orbit. Quadratics can also be applied to pattern problems such as the handshake problem and to any real-world application concerning the areas of squares or rectangles. Contexts that can be modeled with exponential functions include the temperature of a cooling object, bacterial growth, viral spread, or the use of new technologies.

Vertical Planning

Vertical planning is intentional planning around connections between standards in different domains within a curriculum. It can also be thought of as "stacking" standards or teaching more than one standard at a time. Connections between different domains can be addressed implicitly or explicitly, but it is best if both teachers and students are aware of the connections. If teachers are intentional about vertical planning, they will gather and respond to assessment with the later standards in mind and will naturally refer back to previously addressed standards when focusing on standards that come later. They will also be more likely to keep language, representations, and procedures consistent; make better use of instructional time; and build more effectively on prior learning.

As noted above, while there are new skills and concepts related to quadratic and exponential relationships, there is also a significant amount of transfer from earlier topics. Many of the concepts around the meaning of linear functions and how we represent them are utilized here, as they apply equally to linear and nonlinear functions. This indicates the opportunity to explore nonlinear functions and varied rates of change informally during instruction of linear functions. Certain standards related to number systems and expressions are also given context and application within the teaching of quadratic and exponential equations and functions, and should be intentionally and explicitly attended to along with quadratic and exponential standards.[MP.7]

Big Ideas

The learning around quadratic and exponential relationships relies heavily on concepts that were developed in previous units, including what a function is, why functions are important, and how to use them to model real-world problems. Success with this content depends on these understandings, but also requires the acquisition of new concepts. One strategy to clarify learning goals and optimize instructional time is to plan with the big ideas related to this content in mind. One such big idea is rate of change.

Students need to apply what they understand about linear rates of change to nonlinear rates, and also devise new skills and techniques to investigate them. Again, the use of real-world contexts can be essential here and is generally more effective in building a conceptual understanding of nonlinear rates of change. This includes discussions about why a constant rate of change results in a graph that is a straight line, and how a varied rate of change causes the graph to curve.[MP.7–8] The focus on rate of change will continue to be important as students consider additional function families and as they study higher math.

Another big idea is how students begin to understand functions as defined by the structure of expressions, specifically for quadratic and exponential expressions that have more complex structure than linear expressions.[MP.7] These are new structures for students, so they need to learn new skills while also applying prior knowledge about squares and roots and the properties of exponents in general. They utilize number sense, operational sense, and the properties of operations in new ways as they learn to manipulate nonlinear expressions in light of modeling contexts.

By now, students should be able to differentiate between expressions, equations, and functions, and the related processes we apply to them. A subtle idea that is new here is how we handle quadratic equations. Sometimes we take the quadratic function as a whole, considered in a context with two variables, but often we extract and manipulate just the expression on the input variable to reveal aspects of the function and its graph. Sometimes we set the expression equal to zero and use the zero-product property or other methods to identify "solutions" to the equation under these parameters. All of this can be very confusing to students who are used to handling linear functions, which we often use to define a y value for a given *x* value or vice versa, but rarely set equal to zero or manipulate the expression on the input variable. We also do not tend to focus on the *x* intercepts of a linear function, but suddenly with quadratics these points become extremely important. These subtle changes should be addressed explicitly and given purpose. Why do we suddenly make a habit of taking a quadratic in two variables, setting the expression on the input variable equal to zero, and solving it? Why are the solutions to this one-variable equation so important to the function? The particular focus on *x* intercepts for quadratic functions stems from the importance of these points in modeling contexts, such as where a ball hits the ground, where a diver hits the water, or finding the base points of an arch. This is also connected to the use of the zero-product property to solve a one-variable quadratic equation in factored form. The focus affects important vocabulary terms like solutions, roots, zeros and *x* intercepts as they are used in conjunction with different representations, but all refer to essentially the same object. Teachers or professional mathematicians may barely notice these subtle changes and conventions as they shift between linear and quadratic functions, and simply accept them as obvious and reasonable. Students need to have these ideas addressed so that the procedures they are expected to follow will have meaning and connect to deeper conceptual learning.

A third big idea concerning quadratic and exponential functions is the nature of the domain and range. Whereas a linear function always has an infinite domain and an infinite range, these new types of functions have limitations to the domain and range values. Quadratics have an infinite range in one direction but the range in the other direction does not extend beyond the vertex. Many students think the domain of quadratic functions is also restricted, but in fact the domain is infinite in both directions as the parabola continues to expand horizontally toward the extreme. Exponential functions generally do not enter quadrant three and four at all unless they are shifted by a constant term, so the range is always limited to positive numbers as determined by the structure of the exponential expression. Their domain is always infinite although, as with quadratics, it may appear to students to be restricted on the side of the graph that trends vertically. Exponential functions also include a horizontal asymptote, which the range approaches but never reaches. This is a new idea for students and requires both exposure to graphical representations as well as examination of the structure of exponential expressions.[MP.7] Asymptotes will play a key role as students study rational functions in future courses.

Multiple Representations

The standards relating to quadratic and exponential relationships explicitly call for the use of multiple representations, including tables, graphs, equations, and area models. Students apply their experiences in middle school and with linear functions in algebra to use these representations fluently and make sense of the connections between them. For quadratic and exponential functions, graphical representations are particularly important as they can be used to help students understand the rates of change and structures implied by the functions. A parabola is a visual representation of increase and decrease, and the importance of the vertex or turning point is plain to see. Similarly, the ramp in the graph of an exponential function is a visual indication of the exponential rate of change in terms of growth or decay as implied by the function.

One thing to consider, as with earlier function work, is the use of sketch graphs as students analyze real-world situations and try to articulate trends in the relationships between varying quantities. The sketch graph may or may not be labeled or scaled, and may or may not represent any known points, but it can be used to express a student's understanding of positive or negative trends, variations in growth, turning points, restrictions on domain and range, or the tightness of curves. Sketch graphs can also be useful when studying trends in linear functions and should be applied to work with both linear and nonlinear functions.

Due to the complex nature of the expressions in quadratic and exponential functions, the ways that different forms of those expressions interact with key elements of the graphs become essential to understanding the behavior of the functions. The factored form of a quadratic equation shows the solutions or x intercepts, the vertex form indicates the coordinates of the vertex point, and the standard form can be used to identify the y intercept and whether the parabola opens up or down. Big ideas such as the limitation of domain and range, the symmetry in a parabola, and the asymptotes in an exponential function should be explicitly discussed when interacting with graphical representations, and an effort made to connect those graphical features with the structures in the expressions. Through good questioning and attention to concepts, teachers can help students connect reliable procedures with the meaning of underlying concepts.

Daily Formative Assessment

Formative assessment questions can be used on a consistent basis both to assess new learning and also to assess how students are activating and transferring prior knowledge. Essential questions can also help to connect the many procedures used with quadratic and exponential functions to the concepts that drive those procedures and show how the behavior of quadratic and exponential functions connects to the very structure of quadratic and exponential expressions.[MP.7–8]

- What is a quadratic expression?
- What is a quadratic function?
- What is an exponential function?
- What causes a linear function to graph a straight line?
- What causes a quadratic graph to curve?
- Tell me all you know about the structure of a parabola.
- Why is a parabola shaped the way it is?
- Why is a parabola symmetrical?
- Why does a parabola have two inputs for every output, except at the vertex point?
- Why are quadratic functions used to model situations about area?
- What's the difference between linear and exponential growth?
- How does the factored form of a quadratic equation in one variable help us find solutions?

- Why does a quadratic equation in one variable generally have two solutions?
- Why do we call the *x* intercepts of a quadratic graph the solutions?
- Why don't graphs of simple exponential functions enter quadrants three and four?
- Why does the graph of a simple exponential function in the form $y=ab^x$ always go through the point (0, 1) when $a=1$?
- Why can't we solve an exponential equation for a given *y* value?

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not getting it. What is it that they do not understand? And how far back do their misconceptions go in the prerequisite standards? If we can identify the missing concepts, we can begin to understand why they are struggling with the current standards.

For quadratic and exponential relationships, remediation can involve concepts from other algebra topics, such linear relationships and functions, and may require revisiting standards from earlier grades. Students may also need time to grasp the inverse relationship between powers and roots and explore the patterns that emerge from different exponents. What is most important is to continue to ask students to explain how they are thinking about quadratic and exponential expressions, equations, and functions; and resist the temptation to reduce this vast new content to a series of steps and procedures. Like all math, this content is based on conceptual understanding, and it is also the doorway to all the function analysis students will do in subsequent courses through high school and beyond.

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including:

- Revoicing – "So you're saying that… ."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk.[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work, and actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are

- "Why?"
- "What does that mean?"
- "How do you know?"

These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

Vocabulary Terms for Quadratic and Exponential Relationships

Essential vocabulary related to quadratic and exponential relationships is extensive and includes terms from prior learning, new terms, terms whose meaning is expanding, and terms which are synonyms or homonyms.

- Expression
- Equation
- Function
- Evaluate
- Simplify
- Solve
- Linear
- Quadratic
- Exponential
- Rate of change
- Power
- Exponent
- Roots (as opposites of powers)
- Vertex
- Factor
- *X* Intercepts
- Solutions
- Roots (as solutions to quadratic equations)
- Zeros

- Parabola
- Line of symmetry
- Asymptote

To avoid confusion around vocabulary related to this content, class discussions and use of terms need to be planned and attended to, surfacing misconceptions as the learning progresses. The relationship between solutions, roots, zeros, and *x* intercepts—and in which context each is most appropriate—needs to be made explicit. Any discussion about solutions should include reference to what is being solved and how solutions can be justified. Attending to precision, the solutions to a quadratic equation in one variable are not the solutions to the function itself, as the solut45ion set to the function is the entire graph. Also, there are conventional differences in how the forms of quadratic equations are named, so these need to be made very clear and made consistent with any curriculum or other resources students may use.

Algebra I: Data Analysis and Statistics

This section first presents an overview of data analysis and statistics and their progressions of learning. Next, it will describe five content supports and strategies: real-world scenarios, estimation and prediction, planned debate, multiple representations, and daily formative assessment. Then, it will provide suggestions for students who struggle, including language and communication, and vocabulary.

Overview and Progressions of Learning

Students begin their study of data and statistics as early as kindergarten, where they represent and interpret data by classifying objects based on number, size, or other attributes. They continue to represent and interpret data (given, observed, or generated) through Grade 3, investigating situations with up to four categories and representing the data using picture graphs, bar graphs, and line plots. In Grades 4–5, students begin to formulate questions about data and make predictions. They display data using tables, frequency tables, line plots, and bar graphs, and collect data by experiment, observation, and survey.

In Grade 5, measures of center are used to interpret and describe data sets, and in Grade 6, students develop a formal understanding of statistical variability and use distributions described by center, spread, and shape. Box plots and histograms are utilized as data displays, and students are expected to formulate and investigate statistical questions.

By Grade 7, students build on their previous learning with univariate distributions as they begin to use measures of center and spread to compare two distributions and consider questions about their populations. They begin to use random sampling to generate data sets and deepen their understanding of the role of randomization in valid representation of statistical data. While probability has been operating informally within the statistics of lower grades, as students predicted long-term frequency of specific outcomes based on data, in Grade 7 probability is formally addressed and represented. Students develop probability models, identify sample spaces, make predictions, and relate experimental outcomes to theoretical findings. In Grade 8, students investigate patterns of association in bivariate data as they construct and interpret scatter plots. Much of the work centers on recognizing and modeling linear trends and fitting lines and equations to data to make predictions. Students also understand probabilities of compound events and represent those probabilities using lists, tables, and tree diagrams.

High school students continue to interpret and represent univariate data and use scatter plots and two-way frequency tables to interpret and represent bivariate data. They deepen their understanding of the role of randomization and bias in data collection and presentation and use probability models to make informed inferences and justify their conclusions. In algebra, there is a large focus on linear models. Students use graphical and algebraic representations, calculate correlation, and consider the role of causation between varying quantities. In upper grades, students explore bivariate data relationships involving quadratic and exponential models and investigate conditional probability. They also learn a new measure of variation called standard deviation and use it to investigate normal data distributions. As stated in the standards, the statistics we encounter in the world are often designed to elicit a particular response or tell a specific story, making the study of data and statistics not only important to extend mathematical understanding but an essential part of becoming an informed citizen.

Content Supports and Strategies

Listed below are five strategies to build students' conceptual understanding and procedural fluency related to data and statistics. The strategies include:

- Real-World Scenarios
- Estimation and Prediction
- Planned Debate
- Multiple Representations
- Daily Formative Assessment

Real-World Scenarios

Data and statistics are ever-present in our daily lives and yet many students see it as something that only exists in math class. One of the best ways to overcome this view is to ground the learning of data and statistics in real-world scenarios that students can reason about and discuss.[MP.4] As students build on their learning from middle school standards, part of the work in high school algebra is to assess student understanding and help students connect to a deeper and more formal understanding of statistical inference based on predicted and observed probability of outcomes. It is almost never necessary or desirable to engage with statistics that do not have a context application during the learning of these standards.

By high school, many students can procedurally find the mean or median of a set of numbers without a context, but for those still building conceptual understanding of these ideas, naked number problems do not develop understanding as effectively as problems with a context. It is important that the answer is not just a number that satisfies the problem, but that it has meaning that can make sense to students. For example, when asked to find the mean of the set 4, 8, 12, 6, 10, and 8 we determine the mean is 8 because it is the average of the 6 data values. But if we are asked to find the mean age of six students whose ages are 4, 8, 12, 6, 10 and 8, the mean of 8 has more meaning because it represents the average age of the group of students. So too with other measures of center and variability. The context adds meaning and encourages students to build conceptual understanding as well as procedural fluency.

When presenting data displays for students to interpret, it is often useful to discuss the context broadly first before analyzing the meaning of the data display. It is also productive to let students choose contexts they are interested in when generating questions, making predictions, collecting data, and presenting findings. This can be done simply by showing students a univariate set of data to be considered and having them choose a context that the data could represent, and can be delved into more deeply by probing the class for sets of data they would be interested in working with such as student interest in different kinds of music, point production of different sports teams, daily hits on web pages or memes, and so on. It is in light of contexts that students learn to generate statistical questions, build understanding of the importance of random sampling, and make sense of the differences between observational studies, surveys, and experiments.

One way to engage students in questions about the role of randomization and the effects of different data gathering methods is to present a vignette or story of a data collection event, including findings and possible debate, and have students respond to specific discussion questions. Again, it is the context in a good vignette that activates authentic student reasoning, enriches discussion, and builds conceptual understanding. The sense-making students engage in while considering real-world situations also deepens questions around correlation and causation. While definitions can be an important starting point, it is discussions and student sense-making in terms of a context that leads to lasting learning.

Estimation and Prediction

Estimation is an often-underutilized mathematical practice that can be productive in many different areas of study. In the realm of statistics, estimation can be used to prepare students to make informed predictions down the road. Estimation is what we do when we are not sure of the exact answer but wish to think in the direction of the answer. We often do it quietly to ourselves, but most students are uncertain about sharing their estimation publicly for fear that they might be "wrong." The irony is that an estimation at its best is often—if not always—wrong in some way.

In the math classroom, estimation must be practiced explicitly and consistently to become part of students' daily practice and honored not for correctness but for how it productively supports student sense-making.[MP.1] When studying univariate data, students could be asked to estimate the values of the mean or median before they are calculated. "What do you think a reasonable mean might be for this set? What would be your guess at a reasonable median for this set?"

Estimation is what we do when we are not sure of the exact answer but wish to think in the direction of the answer. We often do it quietly to ourselves, but most students are uncertain about sharing their estimation publicly for fear that they might be "wrong." The irony is that an estimation at its best is often—if not always—wrong in some way.

As students move toward making deeper inferences and justifying conclusions about studies, it is important that they first make predictions about possible outcomes and state a rationale for those predictions. This type of reasoning and looking ahead helps develop perseverance, argumentation, and problem-solving.[MP.1,3] Predictions can be reflected on and revised after the data is gathered and the study is complete. Students can also make predictions about the efficacy of different types of studies and the effects of randomization on the validity of statistics, reflecting on them and discussing them with peers after the findings are collected or displayed.

One specific classroom strategy that may be useful here is generally referred to as "first thinking and new group thinking." Students are presented with a problem or scenario, which could be a set of data, a statistical question, plans for a study or experiment, or a randomization process. Students are first asked to think on their own and write down their "first thinking" as a response to the prompt. Note that this time is not for problem-solving or "figuring it out," but a space for students to take in what is being presented and respond broadly with questions, statements, diagrams, wonderings, or ideas and procedures they think may be useful in addressing the problem. After students engage in individual think time and record their responses or first thinking, they are put into small groups where they each share their first thinking, discuss, and then collaborate to produce a new group thinking informed by their individual thinking but completer and more developed. This process teaches students to look ahead, to take risks, to consider different ways of approaching problems, and to value collaboration with peers.[MP.1] As with any classroom structure, repeated opportunities to engage in the protocol will increase its productivity.

Planned Debate

A discourse strategy, which can certainly be useful to support making inferences and justifying conclusions, is the practice of planned debate. The debate could be about whether a particular process of randomization will yield valid data collection, whether a correlation in bivariate data implies causality, or about a predicted outcome in an experiment. This could be arranged such that students pick the side they wish to argue, or students could be randomly selected to argue a particular side in the debate. The debate can be arranged in pairs, small groups, or with the whole class and can be public or private. A similar structure that could be used is called silent debate, where students are asked to debate a question silently by writing back and forth on one sheet of paper with a partner. Both of these strategies build justification and reasoning, support listening skills, and engage students in vital sense-making that is essential to the statistics standards.[MP.3] Debates also create a place where right and wrong are not the most important thing, but instead sharing your thinking, working to justify your reasoning, and convincing are valued more highly than whether your position can be proven correct or incorrect. Students could also be asked to debate whether a certain presentation of statistics was biased or presented in a neutral manner.

Multiple Representations

By high school, students have been exposed to a number of representations to display univariate and bivariate data including line plots, box plots, frequency tables, histograms, and scatter plots. In Algebra I they are also introduced to two-way frequency tables as a way to represent bivariate data. It is important to consistently discuss the structure and meaning of any displays presented to students before digging into an analysis of the data, and equally important to let students choose how to display collected data and to share their rationale for those choices.[MP.4] As students interpret linear models, discussions should be engineered to focus on how scatter plots, graphs, tables, and equations relate to one another in terms of the context. Some students may benefit from using hands-on tools to informally identify the line of best fit in addition to finding linear functions using technology.[MP.5] Such concrete experiences build conceptual understanding of linear trends, regression, and correlation.

Daily Formative Assessment

The topic of data analysis and statistics is broad and reaches back through progressions of learning that span many years. There are many conceptual questions that may be essential questions for a student or group of students as they move through the unit.

Essential questions should be intentionally planned and used and asked again and again. They should meet students' current thinking and also cause them to deepen and extend their thinking. Essential questions also develop the students' ability to express their growing understanding.

- What types of data displays have you used in the past?
- Why is a sample useful when making inferences about a population?
- Why is randomization important when selecting a sample population?
- What are some things you notice about this set of data?
- What are some important measures we use to describe sets of data?
- What does it mean when we talk about statistics and data being *biased*?
- What type of function would be best to model the trend in this data?
- What are the two things we need to know in order to identify any linear relationship?
- What is correlation and why is it important?
- Can you think of two quantities whose relationship expresses correlation and causation?
- Can you think of two quantities that have correlation but not causation?
- What is the difference between univariate and bivariate data?
- What is frequency and how can it be shown in a data display?

Good formative assessment questions give teachers important assessment information about student misconceptions and gaps in learning. Assessment questions also give students the opportunity to practice expressing and refining their understanding over time.

For Students Who Struggle

For students who struggle to meet the expectations of any standard, the operative questions are "Why?" and "What are they missing?" Only through consistent discourse and formative assessment can we begin to know why they are not getting it.

Formative assessment questions can also help students identify and grapple with big ideas in what they are learning. For example, it is essential that high school students understand that we describe data sets by measures of center, measures of variability, and shape. Asking questions about such ideas can often help students construct their own conceptual understanding more productively than simply telling the student about the idea.

- What are the different measures of center and what do they tell us about a data set?
- What is variability and why is it important?
- How would you describe the shape of this set of data? What other shapes are possible?

Formative assessment also helps inform instruction and assists in vertical planning. Vertical planning is intentional planning around connections between standards in different domains or clusters within a curriculum or can be thought of as "stacking" standards or teaching more than one standard at a time. Many data and statistics standards connect to learning around linear relationships and can be seen as an opportunity to continue to develop understanding of that high-priority content. Connections between different domains can be addressed implicitly or explicitly, but it is best if both teachers and students are aware of the connections. If teachers are intentional about vertical planning, they will gather and respond to assessment with the later standards in mind and will naturally refer back to previously addressed standards when focusing on standards that come later. They will also be more likely to keep language, representations, and procedures consistent; make better use of instructional time; and build more effectively on prior learning.

Language and Communication

Student discourse, including speaking and writing, is essential to the development of students' conceptual understanding. Teachers who develop discourse-based learning environments often use intentional talk moves including:

- Revoicing – "So you're saying that… ."[MP.6]
- Repeating – "Can you repeat Hayley's reasoning?"[MP.2]
- Reasoning – "Do you agree or disagree? Why?"[MP.3]
- Adding on – "Can anyone add on to that?"
- Wait time – Anytime a teacher restrains their own talking to encourage student talk.[MP.1]

Also, discourse environments benefit from daily use of planned talk structures where students are expected to:

- Think on their own
- Talk to a partner
- Talk in a small group
- Talk in the whole-class discussion
- Talk one-on-one with the teacher

Generally, teachers who are intentional about vocabulary acquisition generate better student outcomes. The use of a Frayer model or similar graphic organizer communicates that students are being asked to do more than just memorize a definition. Examples and non-examples are important, and for most students it is essential for them to express the definition of key terms in their own words. The full acquisition of productive academic vocabulary goes far beyond one day of focused vocabulary work, and actually develops over time if students are continually expected to use the vocabulary in their daily work as they refine their understanding of the concepts represented.[MP.6]

It is also important for teachers to consistently encourage students to verbalize their sense-making and to justify their statements.[MP.3] The expectation for justification and communication enhances the need for appropriate vocabulary and its acquisition. The best questions to encourage sense-making and justification are

- "Why?"
- "What does that mean?"
- "How do you know?"

These questions keep the focus on student thinking, allow students to practice and develop productive discourse, and give important formative assessment. Also, they can be asked every day.

When assessing students' prior learning as well as teaching current data and statistics standards, it is important to make "set thinking" explicit in discourse. Students need to understand that data collections are sets of numbers; can be looked at individually or in combination; may have a contextual meaning; and can be described using measures of center, variability, and spread.

Individual elements of these sets are generally referred to as data values, and while the set is often taken as a whole there are times when it is advantageous to focus on individual data values.

Vocabulary Terms

As noted above, key vocabulary terms need to be discussed and defined by students again and again, allowing them to build and refine their understanding of each concept and the connections between them.

- Measures of center
- Variability (and measures of variability)
- Spread
- Range
- Quartile
- Interquartile range
- Probability
- Inference
- Population
- Sample
- Random sample
- Bias
- Correlation
- Causation
- Frequency
- Univariate
- Bivariate

Data and statistics is an area of high school mathematics that is particularly dependent on literacy and language. While definition of basic terms is valuable, it is only through justification within a context that students can sufficiently grasp the concepts. Discussions will include confusion, slippery concepts, and the need to change one's mind as different views are expressed,[MP.1] so the intentional use of teacher-talk moves and talk structures as noted above will be necessary to navigate productive student discourse.

References

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* https://www.nctm.org/Standards-and-Positions/Principles-and-Standards/

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. The National Academies Press. https://nap.nationalacademies.org/catalog/9822/adding-it-up-helping-children-learn-mathematics

Resources

- The Illustrative Mathematics Resource Hub has a variety of free tasks and other resources for each of the domains in Algebra I.
- The Mathematics Assessment Project includes free resources on lessons, classroom challenges that teachers may use for formative assessments, professional development modules, summative assessment tasks, and prototype tests.
- National Council of Teachers of Mathematics has a variety of classroom resources on its site.
- The Phillips Exeter Academy Programs & Tools for Educators page includes math teaching materials.
- What Works Clearinghouse. (2019) *Teaching strategies for improving algebra knowledge in middle and high school students.*

  This practice guide provides three recommendations for teaching algebra to students in middle school and high school. Each recommendation includes implementation steps and solutions for common roadblocks. The recommendations also summarize and rate supporting evidence. This guide is geared toward teachers, administrators, and other educators who want to improve their students' algebra knowledge.