# Identifying and Assessing Students' Transition Barriers Between Additive and Multiplicative Thinking

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This paper reports on 73 Grades 3 to 6 students' written responses to equal groups, arrays, multiplicative comparison and Cartesian product word problems. It is part of a larger study relating to students' development of multiplicative thinking (MT). The potential of using multiplicative word problems as a diagnostic assessment to reveal students' transition barriers from additive thinking to MT was explored. The development of the assessment and some of the research findings are described here. Findings show that students experience different types of barriers during their transition from additive to MT. Some recommendations are given on how these barriers might be overcome during primary school years.

There has been a real and persistent barrier to many middle and upper primary school students' progress in MT (Hurst & Hurell, 2016). The transition from operating the single units to coordinating two composite units (that is, groups of equal size and the number of groups) simultaneously is a conceptual leap for many students which potentially constitutes an obstacle during the development of MT (Cheeseman et al., 2020). Transition from additive to MT is a process of cognitive change where transition barriers can be identified between developmental stages (Siemon et al., 2019). Many local and international researchers have been using various assessment styles including interviews and pen-and-paper tests to identify students' developmental stages of MT in order to support students' transition from additive to MT.

There has been extensive research (e.g., Downton & Sullivan, 2017; Cheeseman et al., 2020) into early and middle years students' responses towards various multiplicative problems, which mainly explore the complexity of early and middle stages of development in MT. However, there has been limited work exploring later stages of development in MT to reveal students' ability to discern multiplicative relationships between two quantities and their ability to extend to other pairs of quantities. This work is essential to identifying students' transition barriers and common errors based on their strategy choices. Therefore, an assessment is needed for middle and upper primary students to shed light on the following research question:

- To what extent can diagnostic assessment under four key situations—equal groups, arrays, multiplicative comparison and Cartesian product—reveal primary school students' transition barriers between additive and MT?

## Theoretical Framework

Evidence-based Learning Trajectories (LTs) have been used to inform students' progress over time by examining their knowledge and thinking through solution strategies to mathematical problems and students' level of understanding on a developmental continuum (Siemon et al., 2019). In a recent review based on studies of LTs, Siemon et al., (2019) highlighted that transition barriers are boundaries between two developmental stages during students' development. When students first learn the concept of multiplication, their strategies are often concrete and inefficient, relying on drawing equal groups and counting objects one by one to find the total. The transition barrier at this stage is failing to recognise an equal grouping structure (Siemon et al., 2019). Progressing to counting composites (skip counting and repeated addition) or even double count (that is, keeping a running total and keeping the track of the number of groups) (Steffe, 1992) becomes one of the key transition barriers during students' development in MT because of the relationship of many-to-one correspondences (Cheeseman et al., 2020). Once students learn to coordinate composite units, they

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might use a mix of additive and multiplicative strategies such as doubling/halving and/or partitioning/splitting strategies to solve problems (Siemon et al., 2006). This will be superseded by using multiplication facts and procedural based methods (Downton & Sullivan, 2017). However, procedural based learning could be a barrier which limits students to recognise multiplicative relationships and apply properties of multiplication (Hurst & Hurell, 2016). When students start using more abstract and efficient strategies such as properties of multiplication (Downton & Sullivan, 2017), some students still experience difficulty in dealing with two-digit by two-digit multiplications (Larsson, 2016).

In addition, LTs have been used for the design of rich assessment tasks to assess students' understanding of the areas of mathematics and their ability to apply their knowledge in unfamiliar situations and explain or justify their reasoning (Siemon et al., 2019). In this study, a framework of LTs in MT based on the literature was used to design 15 test items and coding categories for each item by identifying students' prior knowledge through their solution strategies ranging from counting all strategies to more sophisticated multiplicative strategies. The items ranged from familiar to unfamiliar multiplicative situations and from two single-digit factors to two two-digit factors to adjust the difficulty level of each item, aiming to help identify students' strategy choices including errors that underline students' performance.

## The Design of the Items

Researchers such as Greer (1992) distinguished four classes of multiplicative situations: equal groups, arrays, multiplicative comparison and Cartesian product, indicating that a wide range of contexts are needed to identify students' understanding of multiplicative relationships and transition barriers from additive to MT (e.g., Larsson, 2016; Downton & Sullivan, 2017). The test items in this study are situated in four situations and embedded with real life context.

Understanding a situation mathematically requires students to recognise relationships between given quantities. Each simple multiplicative relationship encloses one multiplication and two division operations involving three quantities where each has a particular role. Any one of the three quantities can be unknown, which could provide three mathematical problems sharing the same relationship. Including both multiplication and division operations offers a window to see how students would respond between the operations (Downton, 2013).

Some items were adopted with slight modification from previous research as shown in Table 1 and others were created based upon theoretical, literature-based analysis and on the interpretation of the performance that students exhibit from additive strategies to sophisticated multiplicative strategies. Since the items are designed for Grades 3 to 6, it is important that the items provide students with multiple entry points to demonstrate their level of thinking.

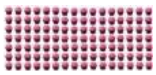
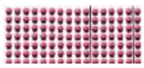
The test items in Table 1 were organised into four tasks: 1) Australian Coins, 2) Michelle's Bakery, 3) Beth's Cupcakes and 4) Sam's Outfits and sequenced based on students' familiarity of multiplicative situations, operations and size of numbers. The test items aim to reveal students' understanding of multiplicative relationships and potential barriers by analysing their solution strategies and tapping common errors in the transition from one stage to the next.

Representing equal groups, items 1a and 1b involve many-to-one correspondences where five 20c coins are matched with \$1. In this situation, the number of groups of five 20c coins stipulates the relation between the number of 20c coins and the dollar equivalent. In item 1b, the size of equal groups (5) is known and the number of groups is unknown. In item 2e, the size of equal groups is unknown and the number of groups (25) is known.

Representing arrays, the equation $6\times4=24$ in item 2b could be challenging for some students since it requires students to recognise the structure of the array as 4 rows of 6 without needing to fill in the unseen space (Downton & Sullivan, 2017). The equation $16\times7=102$ in item 3b is intended to assess students' understanding of partitioning and properties of multiplication. The use of a two-digit factor tests whether the size of numbers influence students' strategy choice across the same situation. Item 3c provides an opportunity to test student's understanding of the associative property of multiplication and to reveal a common transition barrier in moving beyond additive thinking (Squire et al., 2004; Larsson, 2016).

**Table 1**

*Multiplication and Division Word Problems Used in the Study*

| Multiplicative Thinking Assessment Items | Multiplicative Situations/References |
|---|---|
| 1a. The value of five 20c coins is same as one $1 coin. How many 20c coins are same as $4?  | Equal Groups/Downton (2013) |
| 1b. If you have 45 20c coins, how many $1 coins can you make? | Equal Groups/Downton (2013) |
| 2a. Michelle bakes 40 biscuits. She puts them in rows of 8 biscuits on a baking tray. How many rows of biscuits does Michelle bake? | Arrays/Downton (2013) |
| 2b. Michelle puts party pies on a baking tray like in the picture and fills the tray. How many party pies does Michelle bake?  | Arrays/Downton & Sullivan (2017) |
| 2c. Michelle bakes 18 pies. She also bakes 4 times as many sausage rolls as pies. How many sausage rolls does Michelle bake? | Multiplicative Comparison/Downton (2013) |
| 2d. Michelle sold 15 pies on Friday and 60 pies on Saturday. How many times as many pies were sold on Saturday? | Multiplicative Comparison/Larsson (2016) |
| 2e. Michelle needs 25 boxes to pack her 200 party pies. How many party pies are in each box? | Equal Groups/Hurst & Hurrell (2016) |
| 3a. Beth put cupcakes on the bench like in the picture. Can you work out the total number of cupcakes that Beth made?  | Arrays/Hurst & Huntley (2020) |
| 3b. In order to work out the total number of cupcakes, Beth divided the cupcakes into 3 sections like in the picture. How did Beth work out the total number of cupcakes?  | Arrays/Hurst & Huntley (2020) |
| 3c. Beth baked 12 rows of cookies with 15 cookies in each row. Three of her children tried to work out the total number of cookies. Sam did $12\times15=10\times17=170$. Tom did $12\times15=10\times10+2\times5=110$. Emily did $12\times15=6\times30=180$. Who do you think is correct? Why? | Arrays/Larsson (2016) |
| 4a. Sam has 4 jumpers and 3 shorts. If Sam chose a blue jumper, what might be Sam's choice of outfits? | Cartesian Product/Wright (2011) |
| 4b. Sam has 4 jumpers and 3 shorts. How many different outfits are there in total? | Cartesian Product/Wright (2011) |
| 4c. After Christmas, Sam has 5 jumpers and 30 outfits. How many shorts does Sam have? | Cartesian Product/Wright (2011) |
| 4d. Sam's Dad has 18 jumpers and 13 shorts so he has 234 different outfits. Sam's younger brother has 13 jumpers and 18 shorts. How many different outfits does Sam's younger brother have? | Cartesian Product/Squire et al. (2004) |
| 4e. Sam's older brother has 13 jumpers and 19 shorts. How many different outfits does Sam's older brother have? | Cartesian Product/Larsson (2016) |

Representing multiplicative comparison, item 2c tests students' understanding of "times as many". Item 2d involves reversibility of item 2c by asking students to find the unknown multiplier, aiming to see whether students' responses reflect barriers reported by Squire et al., (2004) and Larsson (2016) where "times as many" was confused with adding additional groups.

Cartesian product situations present another transition barrier for students (Wright, 2011; Downton & Sullivan, 2017) since repeated equal sets are not obviously presented. Item 4b aims to see whether students can construct repeated equal sets of different outfits by either using the jumpers or shorts. Item 4c involves reversibility of item 4b which requires students to share the total number of 30 outfits into 5 equal sets or to pair with 5 jumpers. Item 4d and 4e not only assess students' understanding of Cartesian product situation but also aim to identify students' understanding of the commutative and distributive properties in multiplication. In item 4e, students need to recognise the multiplicand or multiplier increasing by 1 so they can use derived facts (e.g.,$13\times19=234+13=247$) or distributive property to solve the problem efficiently. Item 4e also aims to identify another transition barrier in moving away from additive thinking (e.g., $13\times18=234$ so $13\times19=234+1=235$) as claimed by Squire et al., (2004) and Larsson (2016).

## Method

This study is part of a doctoral research study investigating the development of MT among primary school students, for which all necessary ethics approvals have been attained. As a part of iterative cycles of design-based research, prior to this study, the size of numbers, the use of language, format and the sequence of items within the task were evaluated during the focus group discussions with a panel of experts in the field. The use of language, the context and instructions of the test items were also discussed with a group of expert teachers through semi-structured interviews. The feedback from both groups was taken into account in the process of revising and refining the tasks shown in Table 1.

In August 2022, 27 Grade 3 students, 18 Grade 4 students, 18 Grade 5 students and 10 Grade 6 students in two different schools completed the assessment within 45 minutes during their regular maths class time. Students were asked to try a sample question first to ensure that they understood how to respond to the tasks in the assessment and individual students were provided with reading assistance if necessary. The first version of the test items was given to 48 students from Grades 3 to 6 at one primary school. Two weeks later, a slightly varied sequence of the same tasks was given to 25 students in a combined Grade 3/4 at a different school with the aim of seeking more responses on Cartesian product situations.

Students' solution strategies were coded according to the extent of multiplicative thinking shown in Table 2, drawing on the findings of the literature (Siemon et al., 2006; Downton & Sullivan, 2017), as well as necessary modifications based on students' actual answers from the study. The coding categories were flexible and accommodative of different possible strategies that students could use to solve a given test item. An example of the coding category is provided in Table 2 for item 2d. Two-digit codes were used to separate correct and incorrect responses with specific strategies. Students' correct responses were coded from a deficient response 10 through intervening stages to an optimal response 15. No response or irrelevant response or responses with no indication of strategies were coded 70. Incorrect responses resulting from adding or subtracting the two giving numbers were coded 71 as superficial strategies. Responses indicating additive strategies with errors were coded 72 and responses indicating a mix of additive and multiplicative strategies with errors were coded 73. Responses indicating multiplicative strategies with calculation errors were coded 74. Students' responses reflecting an error based on inappropriate generalisation of additive thinking were coded 75. Students' responses based on the coding categories were entered into a spreadsheet for analysis across the Grade levels, multiplicative situations, size of numbers and operations.

Two-digit coding (10, 11, 12, 13, 14, 15, 70, 71, 72, 73, 74, 75) as strategies listed in Table 2 were used to categorise the strategies students used to solve the problems. Based on the analysis of data, there are differences in students' strategy choices across four situations, size of numbers and operations. Findings suggest that students experience different types of transition barriers and overreliance on additive strategies seems hinder their transition from additive to MT. More students struggled with items involving two two-digit factors and multiplicative comparison and Cartesian product situations.

**Table 2**

*Coding and Solution Strategies for Item 2d in Michelle's Bakery Task*

| Coding | Strategies | Responses |
|---|---|---|
| 10 | Counting All | Relies on drawing to count by 1s. |
| 11 | Skip Counting | Shows skip counting by 15s 4 times. |
| 12 | Repeated Addition | Shows 15+15+15+15= 60 or 15(1), 30(2), 45 (3), 60 (4). |
| 13 | A Mix of Additive and Multiplicative Strategies | Uses a mix of additive and multiplicative strategies, e.g., using split strategies or showing doubling or halving strategies. |
| 14 | Multiplicative Operation | Uses 15×4=60 or 60÷15=4. |
| 15 | Holistic Thinking | Shows 10×4+5×4=60 or 30÷15=2 so 60÷15=4 |
| 70 | No Strategies | Shows no response or an irrelevant response. |
| 71 | Superficial Strategies | Incorrect response resulting from adding or subtracting two given numbers, e.g. 60-15=45. |
| 72 | Additive Strategies | Uses additive strategies with errors or an incomplete response. |
| 73 | A Mix of Additive and Multiplicative Strategies | Uses a mix of additive and multiplicative strategies with errors or an incomplete response. |
| 74 | Multiplicative Strategies | Uses multiplicative strategies with errors or an incomplete response, e.g., showing 60÷15. |
| 75 | Inappropriate Strategies | Shows overreliance on additive thinking, e.g., 4-1=3. |

## Results and Discussion

According to students' strategy choices based on the Grade levels in Table 3, the use of counting all strategy (10) in Grade 3 is 15%, the highest among the four grades, and gradually declining by grade level. The barrier for these students is failing to recognise an equal grouping structure (e.g., Steffe, 1992; Siemon et al., 2019). 22% of students (using strategies 11, 12 & 72) recognised the size of equal groups and used skip counting or repeated addition to solve the problems. Some used double count strategy, indicating a shift from operating a single unit to composite units but some responses (72) indicate students failed to recognise the number of equal groups (e.g., Cheeseman et al., 2020). Reliance on additive strategies (10, 11 & 12) to reach a correct solution remains almost unchanged (approximately 26%) across all grades.

The fact that students in Grades 5 and 6 fail to see many-to-one correspondence and still rely on additive strategies, which points to an important barrier. While 15% of responses show that students can use procedural based methods such as vertical multiplication or lattice method to get correct answers, for example, to 13×19, only 1% of students solved problems such as item 4e by applying the distributive property of multiplication to find 13×19 when the value of 13×18 =234 is given. Lack of knowledge of the distributive property of multiplication clearly remains a barrier for upper primary school students.

**Table 3**

*The Mean Frequency of Students' Strategy Choices Based on the Grade Levels (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Year Levels | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Grade 3 (n=27) | 15% | 7% | 4% | 6% | 7% | 0% | 32% | 15% | 7% | 2% | 2% | 3% |
| Grade 4 (n=18) | 10% | 11% | 4% | 8% | 9% | 0% | 28% | 12% | 10% | 2% | 3% | 4% |
| Grade 5 (n=18) | 9% | 9% | 8% | 6% | 20% | 1% | 25% | 9% | 6% | 1% | 3% | 3% |
| Grade 6 (n=10) | 5% | 8% | 11% | 3% | 38% | 1% | 19% | 5% | 5% | 0% | 4% | 2% |
| Mean (n=73) | 11% | 9% | 6% | 6% | 15% | 1% | 27% | 11% | 7% | 2% | 2% | 3% |

In Table 4, items involving equal groups situations provoked the highest frequency of using the additive strategies (10, 11, 12 & 72) of 53%. It seems that the overuse of equal groups situations may not assist students transition from additive to MT, a concern raised by Larsson (2016) and Cheeseman et al., (2020). On the other hand, array items produced a high frequency of using multiplicative strategies of 27% and the lowest of frequency of superficial strategies of 4%. Array items, unlike equal groups, provided students with a visual image encouraging students to recognise the composite units in the array structure (Hurst & Huntley, 2020).

**Table 4**

*The Mean Frequency of Students' Strategy Choices Based on Multiplicative Situations (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Multiplicative Situations | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Equal Groups | 18% | 16% | 10% | 6% | 18% | 0% | 12% | 7% | 9% | 2% | 1% | 0% |
| Arrays | 10% | 13% | 4% | 7% | 19% | 1% | 23% | 4% | 8% | 2% | 4% | 5% |
| Multiplicative Comparison | 3% | 8% | 10% | 13% | 15% | 0% | 8% | 27% | 5% | 3% | 3% | 3% |
| Cartesian Product | 10% | 1% | 3% | 3% | 9% | 0% | 50% | 14% | 5% | 0% | 2% | 3% |

Multiplicative comparison situations as shown in items 2c and 2d (Table 1) also require students to apply multiplicative relationships. In Table 4, 21% of students used additive strategies (10, 11 & 12) to reach a correct answer whereas 28% of students used effective multiplicative strategies (13 & 14) to solve the problems. However, 27% of students used superficial strategies by adding or subtracting two given numbers, indicating no understanding of the notion of "times as many". Clearly, multiplicative comparison situations are a barrier for nearly half of students and need attention.

In this study, Cartesian product situations showed the clearest barrier, producing the highest frequency 64% either showing no response (70) or using superficial strategies (71) by adding or subtracting two given numbers. Clearly, many students are unfamiliar with this situation since repeated equal sets are not clearly presented which constitutes a barrier for students to construct set of ordered pairs from two sets in Task 4. Wright (2011) and Downton and Sullivan (2017) also confirmed that Cartesian product situations present a barrier for many students.

Based on the results in Table 5, items involving two two-digit factors produced the highest frequency 67%–either showing no response (70) or using superficial strategies (71) by adding or subtracting two given numbers and the highest frequency12% using inappropriate strategies, indicating two-digit by two-digit multiplication is a barrier for students' development in MT. Some responses reflect errors of inappropriate generalisation of additive thinking (Squires et al., 2004; Larsson, 2016). For example, in item 3c some students wrote 12×15=10×17 by moving 2 to 15 or 12×15=10×10+2×5 by splitting 12 into 10 and 2 and splitting 15 into 10 and 5. These students use place value partitioning in addition for multiplication problems. However, some of these students successfully solved the item 2c, 18×4=72 by partitioning 18 into 10 and 8 so 10×8+4×8=72. This shows that these students do not fully understand the distributive property in two-digit by two-digit multiplication. In item 4e, some students wrote 13×19=234+1=235 because in item 4d 13×18=234 and 19 is 1 more than 18. It shows how some students attempted but failed to understand the multiplicative relationship between 13×18 and 13×19. In addition, items involving two single-digit factors provoked the highest frequency of 47% using additive strategies (10, 11, 12 & 72), indicating the need to move beyond single-digit multiplication and division problems.

**Table 5**

*The Mean Frequency of Students' Strategy Choices Based on the Size of Numbers (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Size of Numbers | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Two Single-digit Factors | 21% | 11% | 8% | 3% | 18% | 0% | 22% | 9% | 7% | 1% | 1% | 0% |
| Single & Two-digit Factors | 2% | 5% | 4% | 6% | 8% | 2% | 15% | 15% | 12% | 10% | 11% | 9% |
| Two two-digit Factors | 0% | 1% | 0% | 6% | 6% | 1% | 58% | 9% | 4% | 0% | 2% | 12% |

According to students' strategy choices based on operations in Table 6, the use of additive strategies (10, 11, 12 & 72) such as counting all, skip counting and repeated addition strategies in items involving division operations (37%) is higher than in items involving multiplication operations (29%). It is evident that there are more students using additive strategies in division than multiplication problems. For example, in items 1b and 2a, a typical strategy was to successively separate out groups of the specified size until the total number was reached and then to count the number of groups. Even though the group of equal size was created in the division problems, the calculation procedure does not reflect on the use of equal grouping structure which is the main barrier for the development of MT. There are significantly more students using superficial strategies in division items (19%) than multiplication items (7%) which indicates that students experienced more difficulty in understanding multiplicative relationships involving division operations than multiplication operations. This contrasts with the conjecture claimed by Downton (2013). Therefore, identifying an equal grouping structure in division items appears to be a barrier for students.

**Table 6**

*The Mean Frequency of Students' Strategy Choices Based on Operations (N=73)*

| | Strategies | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Operations | 10 | 11 | 12 | 13 | 14 | 15 | 70 | 71 | 72 | 73 | 74 | 75 |
| Multiplication | 9% | 8% | 4% | 6% | 15% | 1% | 32% | 7% | 8% | 2% | 3% | 4% |
| Division | 13% | 10% | 9% | 6% | 16% | 0% | 18% | 19% | 5% | 1% | 1% | 1% |

## Implications and Conclusion

This study investigated the extent to which diagnostic assessment under four key multiplicative situations can reveal primary school students' transition barriers between additive and MT. The data discussed above go a long way to answering this key research question. For example, in dealing with equal groups, many students appeared to not recognise and coordinate the size of equal groups and the number of equal groups, and so relied on additive strategies such as counting all, skip counting and repeated addition to answer the questions. This suggests that the overuse of equal groups situations may not assist students' transition from additive to MT. Instruction where multiplication is introduced as repeated addition or equal groups could be a contributor that leads students to rely on additive thinking. The use of arrays and multiplicative comparison situations showed clearer evidence of multiplicative strategies underlining the importance of using these kinds of tasks. However, understanding the notion of "times as many" in multiplicative comparison situations was a barrier for students who appeared to confuse "times as many" with "times more". This confusion clearly requires attention. Finally, Cartesian product situations presented the biggest barrier for many students because they could not identify repeated equal sets. Introducing two-digit by two-digit multiplication problems in arrays and Cartesian product situations also revealed that many students did not understand or use the distributive and associative properties of multiplication. Overcoming this barrier is an important step in the development of MT. This study highlights the need to move beyond single-digit multiplication and division problems for supporting students' transition from additive to MT. The use of procedural based methods, such as vertical multiplication and lattice methods, may limit students' ability to see the multiplicative relationships including the distributive and associative properties of multiplication between the pairs of quantities involved in the operation.

## References

Cheeseman, J., Downton, A., Ferguson, S., & Roche, A. (2020). Meeting multiplicative thinking through thought-provoking tasks. *Mathematics Education Research Journal*, 1–32.

Downton, A. (2013). Making connections between multiplication and division. In V. Steinle, L. Ball, and C. Bardini (Eds.), *Mathematics education: Yesterday, today and tomorrow. Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia* (pp. 242–249). Melbourne, Victoria: MERGA.

Downton, A., & Sullivan, P. (2017). Posing complex problems requiring multiplicative thinking prompts students to use sophisticated strategies and build mathematical connections. *Educational Studies in Mathematics*, *95*, 303–328.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276–295). New York: Macmillan.

Hurst, C., & Huntley, R. (2020). Distributivity, Partitioning, and the Multiplication Algorithm. *Journal of Research and Advances in Mathematics Education*, 5(3), 231–246.

Hurst, C., & Hurrell, D. (2016). Investigating children's multiplicative thinking: Implications for teaching. *European Journal of STEM Education*, *56*, 1–11.

Larsson, K. (2016). *Students' understandings of multiplication* [Doctoral dissertation, Stockholm University]. Stockholm.

Siemon, D., Barkatsas, T., & Seah, R. (2019). *Researching and using progressions (trajectories) in mathematics education. Brill.*

Siemon, D., Breed, M., Dole, S., Izard, J. and Virgona, J. (2006). *Scaffolding Numeracy in the Middle Years—Project Findings, Materials, and Resources*, Final Report submitted to Victorian Department of Education and Training and the Tasmanian Department of Education.

Squire, S., Davies, C., & Bryant, P. (2004). Does the cue help? Children's understanding of multiplicative concepts in different problem contexts. *British Journal of Educational Psychology*, *74*, 515–532.

Steffe, L. P. (1992). Schemes of action and operation involving composite units. *Learning and Individual Differences, 4*(3), 259–309.

Wright, V. J. (2011). *The development of multiplicative thinking and proportional reasoning: Models of conceptual learning and transfer* [Doctoral dissertation, University of Waikato]. Waikato.