Using a Triple Number Line to Represent Multiple Constructs of Fractions: A Task Design Process and Product

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This paper reports on a key representation, a triple number line, designed as part of the first author's doctoral study. The study sought ways to represent multiple constructs of fractions in the context of merging music and mathematics to support learners' understanding of fractions. A problem scenario was designed guided by Realistic Mathematics Education principles. Findings shared in this paper are based on the process of designing and implementing the tasks around the triple number line. Data for this qualitative, participatory dual-design experiment in task design were collected via formal and informal interviews in two micro-Communities of Practice. We conclude that the key representation of the triple number line can be a powerful tool for supporting learners in their fraction understanding.

The focus of this paper is the use of a triple number line (three number lines in parallel) as a key representation designed to support grade 4 and 5 learners (9 to 11 years old) in moving flexibly between different constructs of fractions (in particular, the part-whole, ratio, and measure constructs). We also share some of the process we went through in arriving at this key representation. The broader design-research study sought ways for integrating music and mathematics to develop and support fraction understanding. The research question we intend to answer in this paper is:

- How might one connect problem scenarios and music-mathematics integrated representations to deepen understanding across multiple constructs of fractions?

Guided by Realistic Mathematics Education (RME) (Freudenthal, 1991), an imaginary problem scenario was designed followed by a sequence of eight lessons in which music and mathematics representations were used. This paper hones in on the final two lessons of the sequence to show some of the ways in which the key triple number line representation was used to support learners in solving problems relating to fraction understanding. Data for the study were collected via two micro-Communities of Practice (micro-CoPs), adapting Lave and Wenger's notion of CoP (1991). A designer/researchers' micro-CoP and a researcher/teachers' micro-CoP were initiated. Analysis of these data provides insight into the design process that guided us towards implementing the triple number line. Although the broader study was designed and trialled in a South African context, we see it as having potential value for wider contexts given the universal challenges so often encountered in the teaching and learning of fractions (Cortina et al., 2015; Getenet & Callingham, 2021; Siemon, 2003; Streefland, 1991).

Literature Review

Multiple Constructs of Fractions

Fractions are a vital part of learning mathematics. Independently of their usefulness in everyday life situations, they contribute to developing proportional reasoning and algebraic thinking (Barbieri et al., 2020; Siemon, 2003). A challenge in teaching and learning fractions is that learners often have the misconception that whole number properties can be applied to fractions, for example, adding numerators and denominators as separate numbers, rather than considering a fraction as a number

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on its own (Cortina et al., 2015; Getenet & Callingham, 2021; Siemon et al., 2015; Streefland, 1991). Working with the multiple constructs of fractions (namely, fraction as measure, fraction as quotient, fraction as ratio, fraction as operator, and the part-whole fraction model) poses a further challenge in the teaching and learning of fractions (Behr et al., 1983; Getenet & Callingham, 2021; Siemon et al., 2015). These constructs (or meanings) of fractions, despite being referred to separately, are not discrete categories, but rather allow for multiple, interrelated ways of understanding and sense-making of the same situation (Siemon et al., 2015). Lamon (1999, p. 41) explains, the "meaning of fractions derives from the contexts in which they are used".

Often the sole focus for fractions at the primary school level is on the part-whole construct (dividing a pizza into equal parts, for example). This, however, is pedagogically insufficient (Barbieri et al., 2020; Getenet & Callingham, 2021). Much more advantageous is providing learners opportunities to work with the multiple constructs of fractions and to recognise the connections across them (Charalambous & Pitta-Pantazi, 2007; Shahbari & Peled, 2014; Siemon et al., 2015). For this reason we strove to find a problem scenario and representation that allowed for movement across the multiple constructs of fractions, creating opportunities to view a situation, and solve problems around it, using multiple meanings of fractions. As noted, we focused specifically on three constructs we see as connecting well with our intention to integrate music note values and rhythm into the teaching and learning of fractions: the part-whole construct—a set of "discrete objects or a continuous amount that can be divided into parts of equal size" (Shahbari & Peled, 2014, p. 373); the fraction as ratio construct—a comparison between two quantities (Charalambous & Pitta-Pantazi, 2007) conveying their relative magnitude (Shahbari & Peled, 2014); and the fraction as measure construct—a representation of the size of measured lengths as measured from a point on a number line (Cortina et al., 2015).

The Number Line as a Supportive Representation in Fraction Understanding

Number lines are well-recognised as having the potential to be a key representation for developing fraction understanding (Barbieri et al., 2020; Saxe et al., 2013; Soni & Okmoto, 2020). Barbieri et al. (2020) identify number lines as mathematically accurate ways for visually representing fractions. Monson et al. (2020) too, note that a number line is a useful model for demonstrating that a fraction is a single point on a number line, and therefore, a measure from 0 (thus emphasising the fraction as measure construct). Despite their usefulness for representing fractions, however, as Barbieri et al. (2020) note, number lines are seldom used in teaching and learning fractions due to various challenges. Barbieri et al. (2020) and Charalambous and Pitta-Pantazi (2007), for instance, caution that learners often count the partition markings of the number line instead of the spaces between them (for example reading quarters on a number line as fifths or thirds). Siemon and Luneta (2018) observe that learners might misconstrue the full number line as the unit that needs to be divided into equal parts rather than the distance 0 to 1 iterated a number of times. Recognising that learners may find it confusing to see equivalence on a single point on a number line, Siemon and Luneta (2018) recommend the use of fractions strips to introduce fractions on a number line.

Possible misconceptions notwithstanding, a single number line is a powerful means of supporting learning when working with equivalent fractions and fractions greater than one whole (Monson et al., 2020; Siemon et al., 2015). A double number line, where two lines run parallel and where there is a relationship between the scales of each line (Orrill & Brown, 2012), is a supportive visual representation when working with proportional reasoning and ratios. Double number lines are often used, for example, to solve problems relating to speed involving a scale of distance and time (Nabb, 2023). The value of a triple number line is highlighted in Cher's (2022) design of an electronic interactive triple number line to support learners in visualising solutions to equations.

In considering the literature on the challenges of teaching and learning fractions, the benefits of experiencing the multiple constructs of fractions, and the potential of single, double and triple number lines as key representations, we resolved to include them in our task design for integrating fractions and music. We believe our trialling of and reflecting on, in particular, the use of a triple number line to solve problems involving multiple, interrelated constructs of fractions can contribute both to the extant literature body and to practice.

Realistic Mathematics Education as a Starting Point

With the theoretical framing of RME (Freudenthal, 1991) we designed an experientially real starting point or problem scenario for the 8-lesson sequence. We found Cobb et al.'s (2008) three tenets of RME useful in the design of the integrated music-mathematics tasks: namely, that a meaningful mathematics task should (i) have an experientially real starting point, (ii) allow for informal reasoning and representing, which subsequently leads to (iii) formal representation and vertical mathematisation. The starting point need not be real-world. It should, however, be a *real experience* with which learners can engage. Even an imaginary fairy tale or folktale could be a meaningful starting point from which a need to use mathematics authentically emerges (van den Heuvel-Panhuizen, 2003).

For the start of the lesson sequence we contrived a folktale-type story (inspired by southern African wildlife and culture) to serve as our problem scenario: different animals crossing a river in different ways during a seasonal migration (see Lovemore, 2023). In this initial task, teachers guided learners in playing a game, imagining that they were different animals that had to make jumps on stones across a river, represented by a marked constant distance. A zebra, for example, would take four equal-sized jumps to cross the constant river-crossing; an ostrich would take two equal-sized jumps per river-crossing. This constant river-crossing unit became the unit that guided our follow-up tasks. With each jump the learners took to cross the imaginary river, the rest of the class would clap, thus resulting in rhythmical beats as the different animals crossed the river. Thus, the movement and clapping were a real experience for the learners from which to base follow-up tasks. Learners were then asked to informally represent their jumps per river-crossing unit. This was extended to continuing the jumps beyond one river-crossing unit, enabling representations and problem-solving with fractions greater than one whole. We recognised in our designer/researchers' micro-CoP the potential this problem scenario had as the starting point in our task design journey towards supporting fraction understanding.

Methodological Decisions

The broader study was a participatory dual-design experiment in task design, after Gravemeijer and van Eerde's "dual-design experiment" (2009, p. 259) whereby, through an intervention, both learners and teachers are afforded opportunities to learn something new. In the case of the present study, the dual-design involved the researchers' and teachers' learning. Researchers learnt through the design process, including feedback received from the participating teachers. In turn, the teachers were learning through their implementation of, and careful reflection on, the designed tasks. The designer/researchers and teachers operated within their respective micro-CoPs as co-researchers (Makar, 2021). The first author met with the second and third authors in the designer/researchers' micro-CoP for initial task design. (All three authors agreed to have their first names used in publications from the study). The first author then shared the task resources with the participating teachers in the researcher/teachers' micro-CoP. The two teachers discussed in the present paper were a Grade 4 teacher, Ms Savuka, and a Grade 5 teacher, Ms Clegg (pseudonyms). They both taught at the same independent school in the Eastern Cape Province of South Africa. Their participation in the study was wholly voluntary. Both were given the assurance at the outset that they were at liberty to withdraw from it at any stage. They were also given the assurance that their identities and that of their school would remain confidential. Their contribution throughout the study was to interrogate and reflect on the intended design of the tasks and to suggest possible adjustments ahead of their actual implementation and to then trial these tasks with their learners. Their post-implementation reflections were taken back to the designer/researchers' micro-CoP to allow for ongoing refinement of the task design. Data were collected mainly through unstructured discussions and interviews within the two micro-CoPs. The thematic analysis process for this paper was then achieved using NVivo software to deductively code for RME principles and for multiple constructs of fractions.

Findings and Implications

Discussion in this section is divided into two phases: Phase 1, dealing with the designer/researchers' process of designing the two-lesson task and the triple number line representation; Phase 2, dealing with the teachers' implementation of, and reflection on, the task.

Phase 1: The Task Design Process and Product

In our task design, learners could relate the animal jumps and musical claps to the fraction as ratio construct (for example, four zebra jumps/claps per river-crossing unit); the fraction as measure construct (distance and time of jumps); and the part-whole construct (equally dividing a whole river-crossing unit into smaller jumps). Having recognised the potential of the problem scenario for learners to experience using multiple constructs of fractions we then sought ways for learners to informally represent these constructs in the form of animal river-crossing jumps. Figure 1, below, shows a learner's informal representation of Zebra, Ostrich, Kudu (a South African antelope), and Monkey jumps, as told in the made-up folktale.

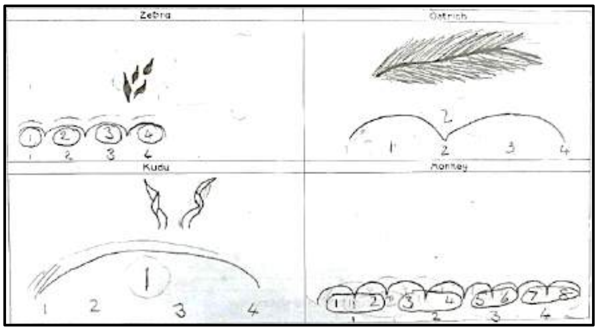

Figure 1. A Grade 5 learner's representation of animal river-crossing jumps.

The informal representations also held the opportunity to guide learners towards more formal representations on a number line, just as is suggested in RME theory (Cobb et al., 2008). In our designer/researcher micro-CoP we initially encountered an obstacle as to how we could represent the fraction as ratio and measure constructs on the same linear model. The following excerpt from the third author's reflection summarises our 'AHA-moment' when we realised that it was not necessary to merge the two constructs. Rather, we could use the problem scenario as a way to *align* the various fraction constructs as well as their musical and mathematical representations. We recognised that we could work simultaneously with the jumps per river-crossing (fraction as ratio) and the iterations of the jumps (fraction as measure of distance and time).

> Mellony: I think this is a big AHA, on exactly why these concepts are getting confused. Rate and fraction [as measure] are of course interrelated, but they're conceptually so different, and yet so similar, so one conflates… What you need to help teachers see is that we're working simultaneously with two fraction concepts, fraction as rate (jumps per river crossing). … but with this animal crossing thing, we can also link this to fraction as measure. [Designer/researcher micro-CoP, 2022-01-25].

We therefore built on the learners' informal representations of animal jumps per river-crossing by getting the learners to draw their animal jumps onto a river-crossing unit number line. To link the animal jumps and claps to a linear musical representation, we designed note value cards printed on transparency film that could be placed to fit exactly into the musical bars aligning with the river-crossing unit distance. (See more in Lovemore et al., 2022; Lovemore, 2023). This option mirrors Siemon and Luneta's (2018) recommendation of using fractions strips to introduce fractions on a number line. Our transparent music note value cards served as a form of fraction strip. Figure 2 below shows the alignment between the river-crossing unit animal jumps, the linear musical representation, and a subsequent matching task requiring learners to match animal fraction strips to both the musical note value strips and the animal river-crossing jumps.

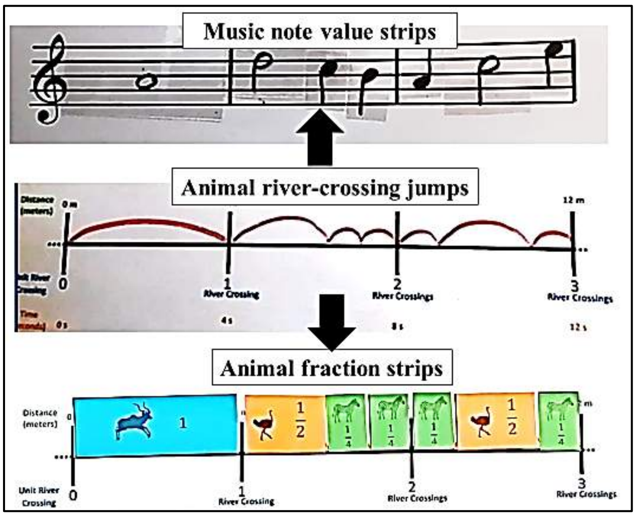

Figure 2. Linear musical and mathematical representations aligned.

The insights shared in Figure 2 led us to the realisation that in fact our problem scenario allowed for the creation of a triple number line (Figure 3). We saw the three parallel number lines as further facilitating the making of conceptual links between and across the fraction as ratio and fraction as measure constructs.

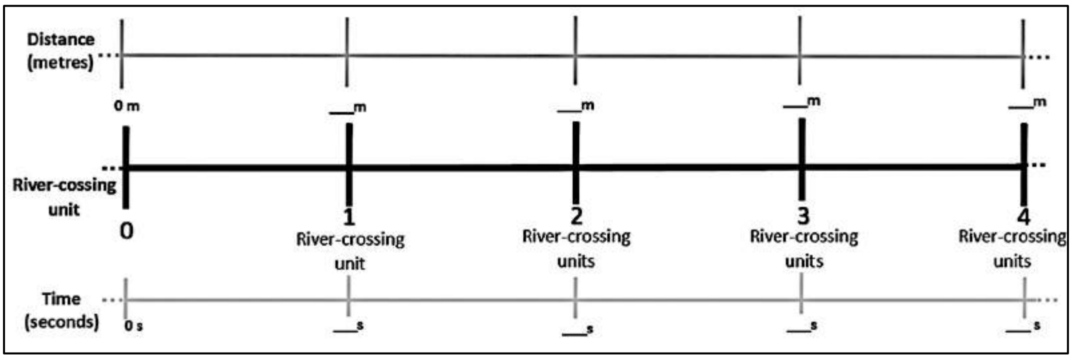

Figure 3. Triple number line.

In our triple number line representation, the river-crossing unit stays constant. The number line above the river-crossing unit indicates measurement of distance and the number line below it represents a timeline. The units of distance and time could indicate different variables parallel with the river-crossing unit, thus creating meaningful opportunities for problem-solving. Figure 4 below illustrates how the triple number line might be used to solve a problem exploring the speed at which the animals jump.

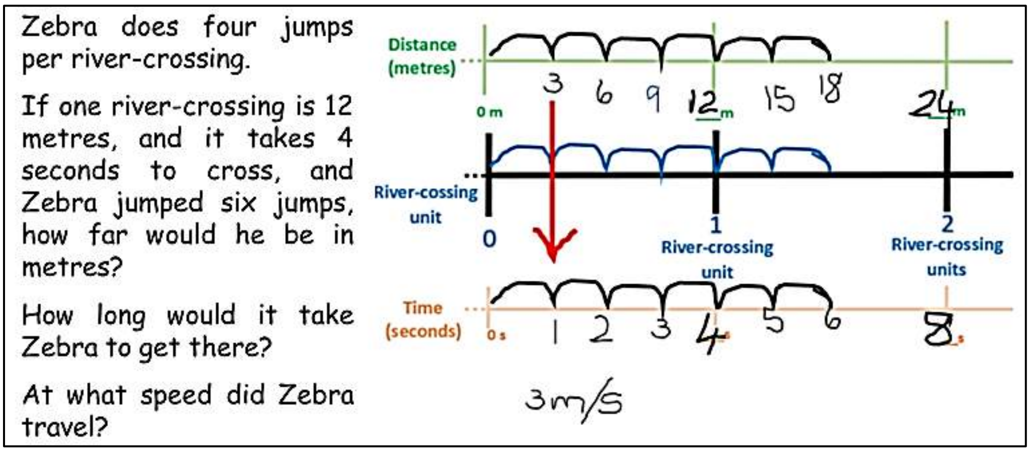

Figure 4. Designer/researchers' demonstration of using the triple number line to solve problems.

We anticipated that this triple number line representation had the potential to support learners in solving complex fraction problems. Below is an excerpt from our designer/researcher micro-CoP discussion on the value of using the constant river-crossing unit while varying the distance and time lines so as to allow flexibility in working with fractional understanding and proportion.

> Mellony: When we get to problem-solving, that's when we're really getting to the powerful fraction stuff, because we can change that. We will say, if the river is 10m wide... now what? So the river-crossing is the main unit, completely in the foreground in our head, and then the other two variables change.
>
> Tarryn: And we're getting to developing the deep conceptual understanding.
>
> Mellony: So the teachers have to know that the kids are not expected to answer any of those questions without the image of the triple number line... So speed is coming alive, ratio and proportion is coming alive. [Designer/researcher-CoP, 2022-02-01].

Phase 2: Teachers' Reflections on the Task Design Product

Within the second micro-CoP, the two participating teachers reflected on their implementation of the tasks, including the use of the triple number line. For the purpose of the current study, learners' work was not analysed, but rather teachers' feedback on their experiences of implementing the task. Both teachers reported on how their learners relied on the visual representation of the number line to support them in their understanding and in then solving the problems requiring fractional understanding. The Grade 4 teacher, Ms Savuka, noted that her learners relied heavily on the triple number line as a 'crutch' to solve the problems. The Grade 5 teacher, Ms Clegg, explained, by contrast, that there had been more variation in her class: some learners used the triple number line representations to solve the problems, others appeared not to need to do so. Shared below are some of Ms Savuka's and Ms Clegg's reflections on the value of the triple number line as a key representation.

> Ms Savuka: It's something that's new to them. So showing that not only [whole] numbers on the number-line but fractions too.
>
> Ms Clegg: I don't think they would have managed this without [the number line].
>
> Ms Savuka: They definitely needed it… I was a bit apprehensive. I thought, now introducing distance and time, how are they going to be able to answer problem-solving questions? And I was blown away… especially in Grade 4, problem-solving and word sums is something that so many of them struggle with. And I just saw the benefit of having a visual representation. That really helped them a lot.
>
> Ms Clegg: Maybe that's something that we can look at just in our general maths. Because I agree, where they go from very concrete, and then we should be semi-abstract, and suddenly we jump into, now they must read an entire paragraph, know what all the fancy maths words mean, and there's no pictures.
>
> Ms Savuka: Nothing they can use to answer the questions.

> Ms Clegg: Yes, whereas, having a basis like a number line, or a picture that forms a number line. Or something like that, that they can actually draw on. [Researcher/teacher micro-CoP, 2022-08-15].

These reflections highlight the teachers' recognition of the importance of visual representations for supporting learners' fraction understanding and their application in problem solving. Their responses align with the literature on the use of number lines in supporting fraction teaching and learning (Barbieri et al., 2020; Saxe et al., 2013; Siemen & Luneta, 2018; Soni & Okmoto, 2020), including using double number lines to aid proportional reasoning (Nabb, 2023; Orrill & Brown, 2012). In line with the RME principles guiding task design (Cobb et al., 2008; van den Heuvel-Panhuizen, 2003), Ms Savuka and Ms Clegg both indicated that they felt their learners were successfully guided from the starting point of the problem scenario, through to informal representations (animal river-crossing jumps and claps), and then on to the formal abstract representation of fractions on a number line, whereafter they were able to use a triple number line to solve complex problems requiring flexible movement between and across multiple constructs of fractions.

Conclusion

As noted, this paper shares a part of the first author's doctoral study which strove to integrate music into mathematics through RME-guided task design. The authors here report on how the problem scenario and designed triple number line allowed for the design of tasks requiring that learners move flexibly between the interrelated constructs of fraction as measure, fraction as ratio and part-whole. From the data, the designer/researchers' realisation that the multiple constructs could be aligned, yet not conflated, is an example of the process of a task design journey. The participatory nature of the journey included teachers' reflections on the triple number line as a key resource to support young learners in solving complex problems. Literature and the findings from this study show the value of using a number line, and particularly a triple number line, to support teaching and learning of complex fraction constructs. While this was a small-scale, qualitative study, future opportunities for research exist in, for example, the form of a quantitative study comparing pre- and post-test results to evaluate the effectiveness of using a triple number line as a key representation in solving complex fraction problems.

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