OPPORTUNITIES FOR REASONING-AND-PROVING IN MATHEMATICAL TASKS: A DISCURSIVE PERSPECTIVE

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In this paper, we offer a novel framework for analyzing the Opportunities for Reasoning-and-Proving (ORP) in mathematical tasks. By drawing upon some tenets of the commognitive framework, we conceptualize learning and teaching mathematics via reasoning and proving both as enacting reasoning processes (e.g., conjecturing, justifying) in the curricular-based mathematical discourse and as participation in the meta-discourse about proof, which is focused on the aspects of deductive reasoning. By cluster analysis performed on 106 tasks designed by prospective secondary teachers, we identify four types of tasks corresponding to four types of ORP: limited ORP, curricular-based reasoning ORP, logic related ORP, and fully integrated ORP. We discuss these ORP and the contribution of this framework in light of preparing beginning teachers to integrate reasoning and proving in secondary mathematics classrooms.

Keywords: Reasoning and Proof, Instructional Activities and Practices, Classroom Discourse, Preservice Teacher Education

Introduction

Reasoning and proving are mathematical *processes* such as identifying patterns, generalizing, conjecturing, and justifying, which are at the heart of mathematics (Ellis et al., 2012; Jeannotte & Kieran, 2017; Stylianides, 2008). The role of reasoning and proving in mathematics classrooms is to support students' sense-making and meaningful learning of mathematics (Hanna & deVillers, 2012; NGA & CCSSO, 2010; NCTM, 2009; 2014). This emphasis is embodied in the notion of Proof-Based Teaching (Reid, 2011) and in Buchbinder & McCrone's (in press) Teaching Mathematics via Reasoning and Proving (TMvRP) framework. The three guiding principles of TMvRP are: (a) integration of reasoning and proving within the mathematics curriculum; (b) emphasis on deductive reasoning for producing and validating mathematical results, and (c) use of precise mathematical language but within the conceptual reach of the students. These principles seem straightforward but require operationalizing. For example, how can reasoning and proving be integrated within the curriculum? And how an emphasis on deductive reasoning can look like in student learning? In particular, teaching mathematics via reasoning and proving requires instructional materials and tasks enabling teachers to engage students with reasoning and proving. Mathematical tasks constitute one of the main sources of student learning (Watson & Ohtani 2015). Teachers design learning opportunities for students by choosing, adapting, or creating mathematical tasks (Brown, 2009; Remillard, 2005), therefore opportunities for students to engage with reasoning and proving vary across tasks. This leads to a question: How can a task's potential for engaging students with reasoning and proving be assessed and characterized? Studies that examined the opportunities for reasoning and proving in tasks tend to focus on the processes students can enact while engaging with the task, such as identifying patterns or making conjectures (Davis, 2012; Thompson et al., 2012). Jeannotte and Kieran (2017) systematized these mathematical reasoning processes (e.g., generalizing, validating, proving) into a taxonomy, carefully defining each concept. However, when considering *how* students engage with such processes, research has identified persistent difficulties in understanding and applying deductive reasoning, which often deviates from reasoning outside mathematics (Harel & Sowder, 2007; Stylianides et al. 2017). Thus, it is important for teachers to expose students to deductive reasoning. This entails a different genre of mathematical tasks, whose learning opportunities for reasoning and proving are not captured by Jeannotte and Kieran's (2017) framework. This different genre of tasks aligns with the Pedagogical Framework for Teaching Proof proposed by Cirillo and May (2020), which delineates logic-related processes that focus on the nature of proof, its structure, rules of logic, the structure of theorems and their use.

These two frameworks provide valuable but different insights into important aspects of teaching mathematics via reasoning and proving, yet there seems to be no unified framework. Our objectives in this paper are two-fold. First, we offer a novel framework for analyzing, classifying, and characterizing tasks and the *Opportunities for Reasoning-and-Proving (OPR)* embedded in them. We rely on the tenets of the commognitive perspective (Sfard, 2008) to combine into a unifying *ORP Task Analysis Framework* the mathematical reasoning processes described by Jeannotte and Kieran (2017), and the logical aspects of deductive reasoning offered by Cirillo and May (2020). The discursive perspective of commognition allows characterizing ORP embedded in a task by identifying the type of discourse to which a task belongs: curricular-based mathematical discourse or meta-discourse about proof. Second, we illustrate the utility of *ORP Task Analysis Framework* by analyzing a corpus of tasks designed by prospective secondary teachers (PSTs) enrolled in a capstone course *Mathematical Reasoning and Proving for Secondary Teachers* (Buchbinder & McCrone, 2020a). We show the variation in ORP afforded by different types of tasks and discuss its potential application to broader contexts.

The Opportunities for Reasoning-and-Proving (ORP) Task Analysis Framework

The commognitive framework views learning as participating in discourse while learners communicate about objects (Sfard, 2008). Working within the commognitive perspective, Jeannotte and Kieran (2017) developed a conceptual model of mathematical reasoning for school mathematics. The model describes nine *mathematical reasoning processes* such as identifying a pattern, generalizing, conjecturing, justifying, validating, and proving. These meta-discursive processes "derive narratives about mathematical objects or relations by exploring the relations between objects" (Jeannotte & Kieran, 2017, p. 9). For example, generalizing is "a process that infers narratives about a set of mathematical objects or a relation between objects of the set from a subset of this set" (Jeannotte & Kieran, 2017, p. 9). The mathematical reasoning processes can be enacted on mathematical objects (e.g., numbers, equations, geometric figures, etc.), which are at the core of various mathematical discourses. For example, tasks in which students are asked to solve an equation and explain their answer focus on *algebraic expression* mathematical object and thus belong to the discourse on algebra (Caspi & Sfard, 2012). The object at the core of a task can be identified by examining what the task is about and what the task asks to find. As students engage with the task they operate on mathematical objects and enact different types of processes (routines in Sfard's terms), which are the "patterns of action that appear when people participate in discourse" (Lavie & Sfard, 2019, p. 6). For example, in the discourse on algebra, students formalize, distribute and group like terms. Identifying the objects at the core of a task and the processes students can enact while engaging with the task, can determine the discourses to which the task belongs. In this paper we define *curricular-based mathematical discourse* as a discourse that focuses on curricular-based mathematical objects. Students participating in a curricular-based mathematical discourse can engage with reasoning and proving by enacting mathematical reasoning processes (e.g., conjecturing, justifying, validating) on the curricular-based mathematical objects, such as linear functions or complementary angles.

However, there are other ways to provide students ORP. Justifying, validating, and other mathematical reasoning processes are rooted in deductive reasoning, which poses multiple difficulties to students (Stylianides et al., 2017). Therefore, it is important to provide students with opportunities to engage with the important elements of deductive reasoning such as analyzing the structure of a theorem or a conjecture, or discussing the generality of proof. These can involve activities such as identifying the hypothesis and the conclusion of a conditional statement, writing the converse of a conditional statement, and identifying or constructing counterexamples (Buchbinder & McCrone, 2020b; Cirillo & May, 2020). Such logic-related processes are not captured by Jeannotte and Kieran's model of reasoning processes since they are not directly related to the deriving of a new mathematical narrative about mathematical objects. Rather, these logic-related processes are enacted on logic-related objects such as conditional statements that are at the core of what we define as *meta-discourse about proof.*

The object at the core of the task and the processes students can enact while engaging with the task determine the type of discourse the task belongs to. By drawing upon the discourses to which the tasks belong, we conceptualize the ORP provided by the task. These ORP can be embedded in the meta-discourse about proof, where students can enact logic-related processes on logic-related objects, or in the curricular-based mathematical discourse, where students can enact mathematical reasoning processes. This classification is illustrated in Figure 1, which summarizes the *ORP Task Analysis Framework*.

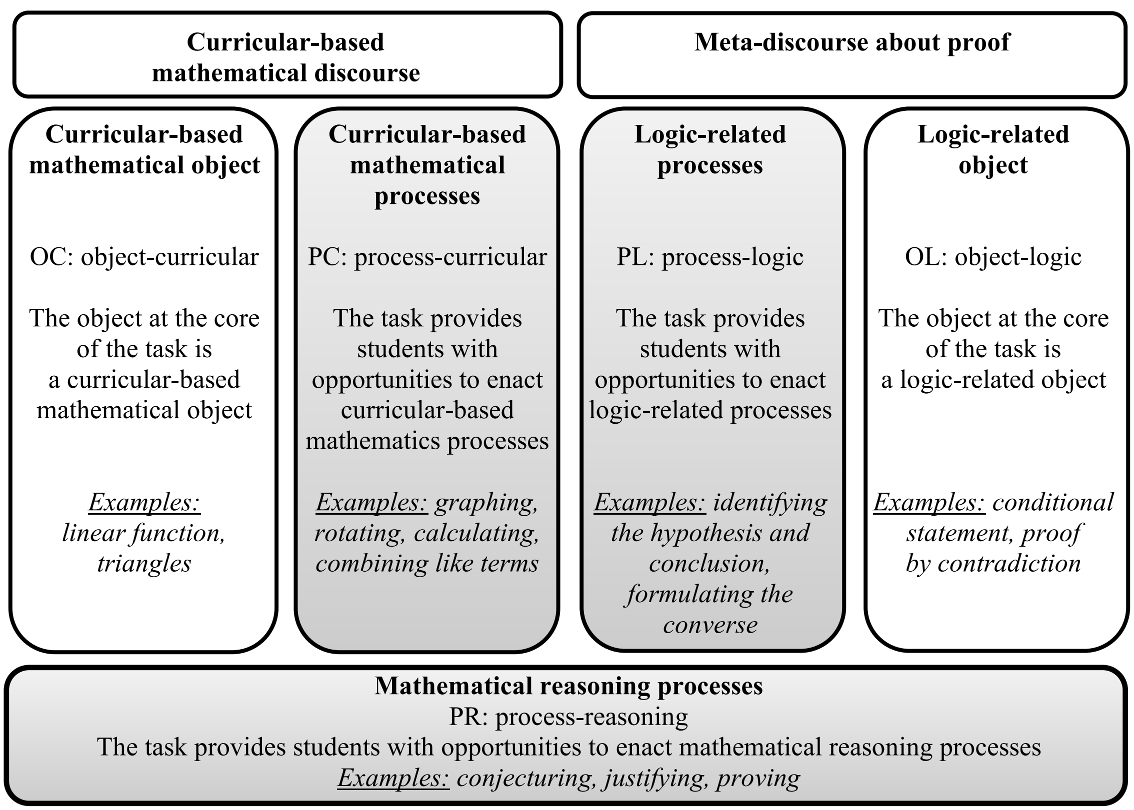

Figure 1: The OPR Task Analysis Framework

In what follows, we illustrate the application of the ORP Framework to identify types of ORP in tasks designed by prospective secondary teachers (PSTs) enrolled in a capstone course *Mathematical Reasoning and Proving for Secondary Teachers*. Our exploration was guided by the following research question: W*hat types of ORP can be identified in tasks designed by PSTs who learn how to teach mathematics via reasoning and proving?*

Application of the ORP Framework

Context: PSTs Learning to Teach Mathematics via Reasoning and Proving

This study is part of a larger research project which investigates beginning teachers' expertise to teach mathematics via reasoning and proving (Buchbinder & McCrone, in press). The first stage of this project focus on PSTs participating in a university-based capstone course: *Mathematical Reasoning and Proving for Secondary Teachers*. The course comprised four modules, which focused on: (1) direct proof and argument evaluation, (2) conditional statements, (3) roles of examples, and (4) indirect reasoning. Each module began with activities belonging to the meta-discourse about proof and contained activities engaging PSTs in integrating the proof themes with the regular curriculum. At the end of each module, each PST designed and taught in a local school a lesson that integrated a particular proof theme with the ongoing mathematical topic from the secondary school curriculum.

Data Corpus and Analysis

We analyzed 12 lesson plans designed by three PSTs who participated in the capstone course in 2021 Fall and consented to patriciate in the study. The analysis focused on mathematical tasks in these lesson plans. Since some tasks included one question while others had multiple questions, we chose the unit of analysis to be a single question – the smallest unit in which students were asked to come up with any type of answer. Overall, we analyzed 106 questions.

The analysis proceeded in several stages. First, we identified the object at the core of the question – namely, what is the question about? Questions that focused on curricular-based mathematical objects (e.g., isosceles triangles) were coded as OC (object-curricular), and questions that focused on logic-related objects (e.g., a counterexample) were coded as OL (object-logic). Second, we identified the different types of processes afforded by the question: curricular-based processes, e.g., graphing, formulating an equation, rotating (PC - process-curricular), logic-related processes, e.g., formulating the converse of a statement (PL - process-logic), and mathematical reasoning processes, e.g., justifying, conjecturing (PR - process-reasoning). For each question, we recorded its core object and the specific process.

We performed cluster analysis on the five variables: OC, OL, PC, PL, PR to identify types of opportunities for reasoning and proving. Cluster analysis is a statistical algorithm that groups a set of objects by identifying similar patterns, which are examined by the distance of the object from each other (Kaufman & Rousseeuw, 2009; Weingarden & Heyd-Metzuyanim, in press). Cluster analysis allowed us to identify similar patterns in the way questions were characterized by logic-related objects and processes, curricular-based mathematical objects and processes, and mathematical reasoning processes.

Types of ORP in Mathematical Tasks

The analysis revealed four clusters, which differ in the combinations of the five variables that characterize the questions designed by the PSTs. Table 1 describes the four clusters, the number of questions in each cluster and the frequencies of each variable in the cluster. The quality of this clustering according to Silhouette measure of cohesion and separation (Sarstedt & Mooi, 2014) was found to be good (between 0.5 to 1). In addition, a Chi-Square test between each of the five variables and the four clusters showed that there is a significant difference between the clusters (OC, OL, PL, PR: $\chi^2(3,106) = 106$, $p<0.001$. PC: $\chi^2(3,106) = 81.042$, $p<0.001$). The identified clusters correspond to four types of ORP that characterize the PSTs' questions (Table 1). In what follows, we describe these types of ORP, by depicting questions from each cluster.

Table 1: The four clusters describing the four types of ORPs

Variables	**Cluster 1:** Curricular-based mathematical discourse (OC-PC)	**Cluster 2:** Reasoning processes in the curricular-based mathematical discourse (OC-PC-PR)	**Cluster 3:** Meta-discourse about proof (OL-PL)	**Cluster 4:** Integrated curricular-based and logic-related discourse (OC-PC-PR-OL-PL)
	N = 37	N = 12	N = 36	N = 21
OC	**100%**	**100%**	0%	**100%**
PC	**100%**	**75%**	0%	**76.2%**
PR	0%	**100%**	0%	**100%**
OL	0%	0%	**100%**	**100%**
PL	0%	0%	**100%**	**100%**

Note. OC: Object-Curricular, PC: Process-Curricular, PR: Process-Reasoning, OL: Object-Logic, PL: Process-Logic

Cluster 1: Curricular-based Mathematical Discourse (OC-PC)

The first cluster includes questions belonging to a curricular-based mathematical discourse where both the object at the core of the question and the processes students can enact while engaging with the question are purely curricular-based.

During this activity, use the given quadrilateral and steps to reflect, rotate, or translate on the grid area.

1. Label the starting quadrilateral P
2. Rotate P'' 90 degrees clockwise about any corner, label your final shape P'''
3. Translate P', 2 inches in any direction, label the new shape P''
4. Reflect over line the x-axis, label the shape P'

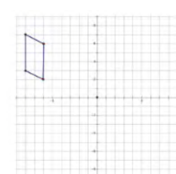

Figure 2: Questions that belong to curricular-based mathematical discourse

Figure 2 shows four questions from this cluster. The curricular-based mathematical object is geometrical shapes and the processes students are expected to perform are rotating, reflecting, and translating. These types of questions, as all questions in this cluster, provide only limited opportunities, if at all, for students to engage with reasoning and proving.

Cluster 2: Reasoning Processes in Curricular-based Mathematical Discourse (OC-PC-PR)

The second type of ORPs includes questions that belong to the curricular-based mathematical discourse but, in contrast to the questions in cluster 1, the questions in cluster 2 also involve mathematical reasoning processes (See Figure 3 for an example).

Create an equation that one can use to find the dimensions, area, number of smaller triangles for any given number triangle.

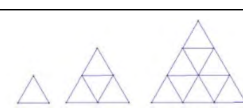

Figure 3: A question involving reasoning processes in a curricular-based mathematical discourse

Figure 3 shows the last question in a task which had students explore the growth of a triangular pattern. The four preceding questions were: (1) what would the next triangle look like? (2) What would its dimensions be? (3) How many triangles would be in the larger triangle? and (4) how many sticks would the next triangle be made up of? These questions belong to the curricular-based mathematical discourse (cluster 1) since the processes involved are drawing and counting sides and triangles, which are performed on the growing series of figures (mathematical objects). The last question of this task (Figure 3) involved curricular-based processes – creating an equation that describes the growing series of figures. However, this question also involved mathematical reasoning processes of identifying a pattern, generalizing, and conjecturing. Although the first four questions of this task provide limited ORP, they can be seen as scaffolding toward the last question (Figure 3), and specifically for the reasoning processes of identifying a pattern and generalizing.

Cluster 3: Meta-Discourse about Proof (OL-PL)

In contrast to the first two types of ORP that include questions that belong to the curricular-based mathematical discourse, the third type of ORP includes questions that belong to the meta-discourse about proof. Both the object at the core of the question and the potential processes provided are logic-related. Figure 4 presents two questions from this cluster.

> Using the given conditional statements, circle the given and underline the conclusion. At the end, write the form of these statements in terms of P and Q (These statements are in the form If P, then Q):
> - Tim knows that if he misses the practice before the game, then he will not be able to be a starting player in the game.
> - If a shape has 4 right angles, then it is a square.

Figure 4: Questions characterized by OPR in the meta-discourse about proof

The logic-related object of these questions is a conditional statement, where the logic-related processes embedded in these questions are identifying the hypothesis and conclusion in the conditional statement and writing a conditional statement in the form of "if-then." Such questions focus on the nature and the logical aspects of proof. Note that although the second question relates to squares and angels, in contrast to the first question that does not refer to mathematical objects at all, both questions focus on conditional statements and their structure rather than the content of the statements.

Cluster 4: Integrated Curricular-based and Logic-related discourse (OC-PC-PR-OL-PL)

The fourth type of ORP includes questions characterized both by opportunities for enacting reasoning processes in the curricular-based mathematical discourse (similar to cluster 2) and by logic-related objects and processes (as cluster 3).

> Now, let's make a conditional statement about this [an acute triangles that students were asked to construct in the former section] and equilateral triangles. (You can look back at your previous drawing for reference). Discuss with your partner what claim could be made that relates an equilateral triangle to an acute triangle.

Figure 5: A question characterized by curricular-based reasoning and logic-related ORP

As such, these types of questions provide rich opportunities for students to engage with reasoning and proving. See Figure 5 for an example. This question belongs to the meta-discourse about proof since it deals with a logic-related object – conditional statement ("let's make a conditional statement") and a logic-related process: writing a conditional statement. In addition, this question involves curricular-based mathematical objects – equilateral and acute triangles, and curricular-based processes, such as drawing triangles and recognizing the properties of different triangles. By integrating the two types of discourses, this question, like the other questions in cluster 4, provides students opportunities both to derive narratives about curricular-based mathematical objects, that of isosceles and equilateral triangles, while using mathematical reasoning processes such as conjecturing, and to deepen their deductive thinking about logic-related objects of conditional statements.

Discussion and Contributions of this Study

In this theoretical paper we offer a novel framework – *Opportunities for Reasoning-and-Proving (ORP) Task Analysis Framework*, for examining, characterizing, and classifying mathematical tasks according to the ORP provided by them. Cluster analysis performed on 106 questions revealed four different clusters corresponding to different types of ORP.

The first type represents *limited* OPR, as it corresponds to types of tasks focused solely on mathematical objects and includes curricular-based processes, as illustrated in Figure 2. These tasks belong to the curricular-based mathematical discourse without any specific demands on students to enact reasoning processes, such as conjecturing, justifying, or proving.

In contrast, *curricular-based reasoning* ORP corresponds to tasks that involve the enactment of reasoning processes on curricular-based mathematical objects (see example in Figure 3). Tasks characterized by this type of ORP are underrepresented in mathematical textbooks (Davis, 2012; Stylianides, 2009; Thompson et al., 2012), even in high-school geometry (Otten et al., 2014). However, curricular-based reasoning ORP are essential for supporting students' mathematical sense-making (Ellis et al., 2012) and for teaching mathematics via reasoning and proving (Buchbinder & McCrone, in press). Thus, teachers have an important role in designing tasks with curricular-based reasoning ORP, and teacher educators have an important role in preparing PSTs to design such tasks for students (Arbaugh et al., 2018; NCTM, 2014).

The third type of ORP – *logic related* ORP – appears in tasks characterized by logic-related objects (e.g., conditional statement) and can engage students with logic-related processes, such as identifying the hypothesis and conclusion, determining what needs to be done for proving or refuting a statement (see example in Figure 4). These types of tasks rarely occur outside high-school geometry chapter on proof (Otten et al., 2014), but they are essential for student engagement with reasoning and proving due to their focus on the nature of proof, the logical structure of theorems and how they are written and used (Cirillo & May, 2020).

The fourth type of ORP– *fully integrated* ORP – involves opportunities to enact reasoning processes on the curricular-based mathematical object with participating in the meta-discourse about proof. This type of task engages students with deductive reasoning and logical inferences, which are fundamental for learning mathematics (Harel & Sowder, 2007) while operating with mathematical objects (Figure 5). The fully integrated ORP can be contrasted with the 3rd cluster questions with logic related ORP where the mathematical objects often serve as a mere background for logic-related processes, such as identifying the structure of statements (see Figure 4 for an example). Tasks with fully integrated ORP require students to apply the logic-related processes on the mathematical objects. The questions in this cluster sensibly integrated both objects – the curricular and logic, so that students must operate on both, applying two types of processes: curricular-based and logic-related.

The four types of ORP described above came out of analyzing a specific corpus of data based on 12 lesson plans of three PSTs participating in a university-based course. Despite this methodological limitation of our study and the need to further validate its outcomes, our study offers several theoretical and practical contributions.

First, the *ORP Framework* combines two fundamental perspectives underlying learning and teaching mathematics via reasoning and proving (Buchbinder & McCrone, in press). One perspective focuses on reasoning and proving as a set of processes, such as identifying a pattern, conjecturing, and proving involved in deriving narratives about mathematical objects (Ellis et al., 2012; Jeannotte & Kieran, 2017; Stylianides, 2008). The second perspective focuses on the structure of theorems and the logical aspect of proof (Cirillo & May, 2020). The establishment of the *ORP Framework* strengthens previous studies that pursue the importance of engaging students with reasoning and proving and provides a unique tool for examining various discourses – curriculum-based and meta-discourse about proof – within a unified framework.

The second contribution of our study is identifying the four types of ORPs. While theoretical considerations suggest two types of objects at the core of the task and three types of processes (Figure 1), only some combinations of the five variables showed up empirically. Moreover, only two clusters, curricular-based mathematical discourse and meta-discourse about proof (clusters 1 and 3), contain tasks that fall under a single object-process characterization. Clusters 2 and 4 comprise tasks with multiple characterizations (Table 1).

Collectively, the *ORP Framework* and the OPR identified in this study contribute to further conceptualizing and operationalizing the notion of learning and teaching mathematics via reasoning and proof. While Buchbinder and McCrone (in press) suggested principles for *teaching* mathematics via reasoning and proving, we maintain that *learning* mathematics via reasoning and proving can be conceptualized discursively (Sfard, 2008) as students participating in the meta-discourse about proof and in a curricular-based mathematical discourse while enacting mathematical reasoning processes. This operationalization can also be used in the context of preparing PSTs to teach mathematics via reasoning and proving. PSTs need to develop expertise both in the logical aspect of proof (i.e. different types of proofs, valid and invalid modes of reasoning, the roles of examples in proving, logical relations), and in the pedagogical aspect of integrating reasoning and proof in curricular-based materials (Buchbinder & McCrone, 2020a). The *ORP Framework* can be used by teacher educators as a pedagogical learning tool for teachers, both to design tasks with various types of ORP and to discuss the potential of mathematical tasks in engaging students with reasoning and proving. The framework can enable teacher educators and PSTs to communicate about these OPR by operating with definite characteristics such as objects and processes. This may contribute to more explicit, and unambiguous communication among teachers and teacher educators about mathematical and pedagogical ideas (Weingarden, 2021).

In addition, the *ORP Framework* can be applied to various research settings which examine tasks and the ORP provided by them. These may include tasks designed by PSTs, beginning teachers, in-service teachers who participate in a professional development program, or tasks that appear in textbooks and other resources. In a related, ongoing study, we use the *ORP Framework* to examine how beginning teachers who participated in the capstone course integrate reasoning and proving in the tasks they design in their classroom. By promoting teacher competencies in designing tasks that embed rich ORP, teacher educators can contribute to the goal defined by NCTM (2014) as creating "systemic excellence" and providing "mathematics education that supports the learning of *all* students at the highest possible level" (NCTM, 2014, p. 2).

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