STUDENTS' INTUITIVE MEANINGS FOR INFINITE SERIES CONVERGENCE AND CORRESPONDING IMPLICATIONS

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This paper describes our work to determine the naturalistic images that first-time second-semester university calculus students possess for series convergence. We found that the students we interviewed most frequently determined whether a series converged by imagining a process of appending summands into a running total and examining whether this running total appeared to approach an asymptotic value. We provide examples and three corresponding implications of this "asymptotic running total" that informed students' actions while determining series convergence or the value of convergence. Our paper adds to the research literature by confirming students' meanings for limits reported for other topics (e.g., limit of sequence, function, Taylor series) apply to infinite series and proposing relationships between previously reported meanings for series convergence.

Keywords: Calculus, Cognition, Undergraduate Education, Advanced Mathematical Thinking

Introduction and Literature Review

Infinite series convergence is a central mathematical topic applied across many branches of mathematics and other disciplines such as physics and engineering (Azevedo, 2021). The topic of infinite series also incorporates many other pivotal topics in advanced mathematics, such as sequence, limit, and infinity (Martin, 2013). Research related to infinite series has focused on the state of curricula and instruction (e.g., González-Martín et al., 2011; Lindaman & Gay, 2012); effective interventions to promote productive student thinking about limit or convergence (Martin et al., 2011; Roh, 2008, 2010b, 2010a; Swinyard & Larsen, 2012); and how students conceive of series convergence, the sequence of partial sums or the symbolic forms of representation students utilize to express series (Barahmand, 2021; Eckman & Roh, accepted; Kidron, 2002; Kidron & Vinner, 1983; Martin, 2013; Martínez-Planell et al., 2012; Strand et al., 2012; Strand & Larsen, 2013).

Most of the reports about students' meanings for infinite series describe students who have previously received (or are currently receiving) instruction on this topic. The purpose of this study is to examine how first-time university second-semester calculus students consider the concept of series convergence before receiving formal instruction on sequences and series. Documenting students' naturalistic images for infinite series before they receive formal instruction can (1) assist instructors in better anticipating students' initial perceptions of series convergence and (2) help researchers better describe students' acquisition of rigorous and productive meanings for convergence. We summarize the goals of this study with the following research questions: (1) *What meanings for series convergence do first-time university calculus students conceive before receiving formal instruction on infinite series?* (2) *How do the implications of these students' meanings for convergence inform their actions for determining whether a series converges and the value to which a convergent series converges?*

Theoretical Perspective

In this study, we attempt to describe individual students' meanings for series convergence and their utilization of these meanings to reason about various series. We adopt a radical constructivist (Glasersfeld, 1995; Thompson et al., 2014) approach for the term *meaning*. Thompson et al. (2014) described a *meaning* as the space of implications resulting from assimilating a situation. We consider assimilation, in this case, to refer to an individual's ability to consider her situation as analogous to previous experience (Glasersfeld, 1995) such that her subsequent actions are selected from the implications elicited by the meaning in her mind.

The concept of *meaning* can effectively describe student thinking and actions for reasoning about series convergence. Suppose that a student, while examining the series $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} + \cdots$, conceives a dynamic image of adding consecutive summands and decides to track the value of the "moving sum" generated through this process. We call this "moving sum" a *running total*. An implication of the student's evoked *meaning* for the series might involve a belief that if the *running total* approaches a particular value, then the series converges. An alternative implication of the *meaning* might include the notion that if the *running total* does not approach an asymptotic value, the series does not converge. The student's subsequent actions to determine whether the *running total* approaches an asymptotic value are grounded in the implication evoked as part of the student's *meaning* for series convergence. In the results section, we propose an overarching meaning for series that the students in this study appeared to possess and the implications that informed the students' actions to determine convergence.

Research Methodology

In this report, we detail the meanings and implications that students exhibited for series convergence during a set of individual 90-minute clinical interviews (Clement, 2000) that simultaneously functioned as a selection and pre-test interview for a constructivist teaching experiment (Steffe & Thompson, 2000). The purpose of the teaching experiment was to offer calculus students specific tasks designed for developing productive meanings for series convergence while documenting the evolution of their corresponding symbolization for series convergence. The virtual clinical interviews were conducted during the Fall 2021 semester, prior to formal coursework on sequences and series, at a large public university in the United States. Our analysis and results focus on two students, Monica and Sylvia (pseudonyms), who were first-time second-semester calculus enrollees at the university level.

We presented the students with six infinite series in expanded form (see Table 1). We chose not to present infinite series using summation notation to mitigate students' potential difficulties interpreting this notation (Strand et al., 2012; Strand & Larsen, 2013). Series 1-4 reflect various behaviors of sequences of partial sums (i.e., monotone convergent, monotone divergent, oscillating convergent, oscillating divergent). Series 5-6 reflect series types that have been reported as problematic for students, such as the decimal expansion of an irrational number (Kidron, 2002; Kidron & Vinner, 1983) and variations of Grandi's series (Bagni, 2005; Martínez-Planell et al., 2012).

For each series, we asked the student two questions adopted from the sequences and series unit in Larson and Edwards (2015): (1) *Does the series converge?* and (2) *If the series converges, what value does the series converge to?* After the student responded to the two questions for each series, we presented each question in a generalized form to determine what students believed about convergence and the value of convergence for any series.

After compiling and transcribing the interviews, we identified and analyzed individual moments of the interviews in the spirit of grounded theory (Strauss & Corbin, 1998). We considered a new moment to begin when a student either (1) encountered a new series, (2) changed their thinking about whether a series converged, or (3) proposed a particular value to which they believed the series converged. For each moment, we recorded whether a student believed the series converged, the convergent value (if applicable), the meaning that students appeared to exhibit for series convergence, and how the students' actions to determine series convergence might be viewed as implications of their meaning. We found that the two students reported in this study appeared to have a similar meaning for series convergence and that three distinct implications of this meaning emerged as the students interacted with various series.

Table 1: The six series presented to students during the interviews

Series	Rule	Expanded Form	Series type	Sequence of Partial Sums	Value of Convergence
1	$\sum_{n=0}^{\infty} \frac{3}{\sqrt{n}}$	$\frac{3}{\sqrt{1}} + \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{3}} + \cdots$	p-series ($0 < p < 1$)	Monotone increasing divergent	
2	$\sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^5}$	$\frac{2}{1^5} - \frac{2}{2^5} + \frac{2}{3^5} - \cdots$	Alternating p-series ($p > 1$)	Oscillating convergent	≈ 1.94
3	$\sum_{n=1}^{\infty}\sum_{i=1}^{99} \left[10^{-2n-1} - 10^{-2(n+1)-1} i\right]$ $= \sum_{k=0}^{\infty} \frac{495}{10000}\left(\frac{1}{100}\right)^k$	$\frac{99}{10^3} + \frac{98}{10^3} + \cdots + \frac{1}{10^3} +$ $\frac{99}{10^5} + \cdots + \frac{1}{10^5} +$ $\frac{99}{10^7} + \cdots + \frac{1}{10^7} + \cdots$	Geometric	Monotone increasing convergent	$\frac{1}{20}$
4	$\sum_{n=0}^{\infty} \frac{(200-2n)(-1)^n}{n+1}$	$\frac{200}{1} - \frac{198}{2} + \frac{196}{3} - \cdots$	Alternating series	Oscillating divergent	
5	$\sum_{i=0}^{\infty} a_i$ (where a_i corresponds to the i^{th} decimal place of π and $a_0 = 3$.)	$3 + .1 + .04 + \cdots$	Decimal expansion of irrational number	Monotone increasing convergent	π
6	$\sum_{n=0}^{\infty} (.07) \cdot (-1)^n$	$.07 - .07 + .07 - \cdots$	Alternating series (Grandi's)	Oscillating divergent	

Results

The results comprise two sections. The first section explains the students' meaning of series convergence as analogous to a running total approaching an asymptotic value. The second section describes three distinct implications of this meaning that informed the students' actions to determine the convergence and convergence value of various types of series. We reflect on potential relationships between the implications in the discussion section.

Table 2: The evolution of students' responses to the question "*Does the series converge?*"

<table>
<tr><th colspan="2"></th><th colspan="3">Series 1</th><th colspan="2">Series 2</th><th>Series 3</th><th>Series 4</th><th colspan="4">Series 5</th><th colspan="2">Series 6</th></tr>
<tr><td rowspan="2">Monica</td><td>Converge</td><td colspan="2">Y</td><td>N</td><td colspan="2">Y</td><td>N</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td>Y</td><td colspan="2">N</td></tr>
<tr><td>Value</td><td colspan="2">Unsure</td><td></td><td colspan="2">Unsure</td><td></td><td>0</td><td>π</td><td>4</td><td>3.2 and 4</td><td>Any upper bound</td><td colspan="2"></td></tr>
<tr><td rowspan="2">Sylvia</td><td>Converge</td><td>Y</td><td>N</td><td>Y</td><td>N</td><td>Y</td><td>N</td><td>Y</td><td colspan="4">Y</td><td>Y</td><td>N</td></tr>
<tr><td>Value</td><td>4</td><td></td><td>Unknown value larger than 4</td><td></td><td>2</td><td></td><td>200</td><td colspan="4">π</td><td>0</td><td></td></tr>
</table>

Students' Meaning for Convergence as a Running Total Approaching an Asymptotic Value

Monica provided normative final responses for each series' convergence except series 3 and 4, while Sylvia provided normative responses for series 2, 5, and 6 (see Table 2; "Y" and "N" stand for "Yes" and "No"). However, in instances where both students believed that the series converged (i.e., series 2, series 4, series 5), the students proposed differing convergence values.

Both students exhibited similar meanings for series convergence, which appeared to involve imagining a dynamic *running total* approaching an asymptotic value as additional summands are calculated into the running total. We call this meaning an *asymptotic running total* meaning. Monica's response to the final interview question regarding general series convergence provides an image of her *asymptotic running total* meaning. Note that in each of the following transcripts, the ellipses (…) refer to omitted text.

Monica: OK. So for the first question [*How can I tell whether a series converges?]*. The first thing I thought of was, kind of like with the limits thing, where are we approaching with every new, like, next iteration of this series? Are we approaching a value or infinity?... [In] the one where we were adding and subtracting and then adding and subtracting [series 4], but the numbers were decreasing, you still saw like a general trend towards, a number, which in that case was zero. So that's what I would say when I'm thinking about, how can I tell whether [the] series converges… Like I'm interested in, what it's approaching [i.e., the running total] and how adding on each next piece of the series is getting it closer to that value.

Int: OK.

Monica: So then the last one [series 6] where it [i.e., subsequent summand] was like essentially undoing what the previous one had done. That was why I said it [series 6] wasn't converging. Because there was not like one overall direction it was going.

Int: OK. Now, when you say one direction "it" was going. Just to clarify, what do you mean by the "it" that's going somewhere?

Monica: The series as a whole, or like the sum of the series as a whole [i.e., running total]. So like, if I were to take…like one plus two plus three plus four. And on this like, huge number line and I just go up all the way to like a very large number, and that number that was getting big, closer and closer to infinity. And then I went to the 200th part of that series. I can be like, Yeah, that's still getting close to infinity, and I could go to the 1000th part and it would be like, yeah, it still goes to infinity.

In Monica's initial comment she states that a convergent series will exhibit a "general trend" or "overall direction" towards a particular value. When the interviewer asked Monica to clarify the "it" moving towards the asymptotic value, Monica stated that she envisioned instantiations of the *running total* after summing various numbers of summands (i.e., 200 or 1000 summands). However, there is no indication that Monica coordinated these instantiations of the *running total* with an index to form the sequence of partial sums. For this reason, we use the terminology *summand* and *running total* in place of normative terminology (e.g., *term, partial sum, sequence of partial sums*) to make clear that we do not consider either student to have constructed explicit sequences through their reasoning about series convergence.

Implications of Asymptotic Running Total Meaning for Determining Series Convergence

In this section, we report three distinctive implications of the students' *asymptotic running total* meaning which contextualize students' actions while determining series convergence. The three implications we describe are *decreasing summands convergence*, *monotone running total divergence*, and *running total recreation through grouping*.

Implication 1: Decreasing summands convergence. One implication of an *asymptotic running total* meaning is that a student might believe that if each consecutive summand in an infinite series is smaller than the previous summand, the running total will eventually trend toward one specific value, suggesting that the series will converge. We call this implication *decreasing summands convergence*. In conventional mathematics, the notion of decreasing summands is a necessary but insufficient property of a convergent series (the most famous example of this principle is the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$).

Both students' actions in various moments related to series 1 appeared to align with the *decreasing summands convergence* implication of the *asymptotic running total* meaning. We constructed series 1 to reflect a divergent p-series with a monotone increasing sequence of partial sums (see Table 1). We provide an example from Sylvia's reasoning about series 1 as evidence of the *decreasing summands convergence* implication.

Sylvia: OK, now I'm thinking that the series doesn't converge and like it [i.e., running total] keeps adding up and getting bigger and approaches like infinity or something like that.
Int: OK.
Sylvia: Wait…
Int: OK… can you say a little bit more about any of the stuff that you're thinking?
Sylvia: Um…if you picture like a perfect square root like three over the square root of nine is three over three, and that's one [$\frac{3}{\sqrt{9}} = \frac{3}{3} = 1$]. But then if you had like three over the square root of 81, that's one-third [$\frac{3}{\sqrt{81}} = \frac{1}{3}$]. So…each item in a series that you're adding on [*brings up hand and mimics placing summands of series sequentially*] is getting smaller. Yes, it is getting smaller. So it's going to… converge to something, but it's not going to be four. [*brief pause*] Yes, that's my view.

In Sylvia's initial conception of the convergence of series 1, she argued that series 1 does not converge because the *running total* will perpetually increase. She later determined that each subsequent summand is smaller than the previous summands, which lead to her declaration that the series converges to an unknown value. In this moment, Sylvia's asymptotic running total meaning implies that decreasing summands suggest convergence. However, we note that the *decreasing summands convergence* implication was insufficient for Sylvia to make a substantive claim regarding the value to which series 1 might converge.

Implication 2: Monotone running total divergence. Another implication of an *asymptotic running total* meaning is that a student might believe that since the *running total* perpetually increases (or decreases) in a monotone series, the *running total* will eventually surpass every potential upper bound (i.e., asymptotic value) that might indicate convergence. We call this implication *monotone running total divergence*.

Both students' actions in various moments related to series 3 aligned with the *monotone running total divergence* implication (Monica's actions related to series 1 also aligned with this meaning during certain moments). The authors constructed series 3, a convergent geometric series, by expanding each summand of the original geometric series into a finite arithmetic series that summed to the summand in the geometric series. Consequently, the expanded form of series 3 resembled a double summation (see Table 1). We provide an instance of Monica's reasoning about series 3 as an instance of the *monotone running total divergence* implication.

Monica: Um. [*pause*] So. In this case… we're adding very, very, very small fractions [i.e., summands], but they're always going to be positive numbers and then we're adding all of them together. This one [series 3]…the rate of growth would be significantly slower.
Int: OK.
Monica: But it would still, if you just did this forever [*makes a gesture to the right indicating extending series indefinitely*]…it [i.e., the running total] would just continue to increase.
Int: OK. So if I'm just summarizing what you've said, you're saying that this third series does not converge. And the reason why is once again, similar to the first series, all of the fractions [i.e., summands] are positive, they're all being added together. And so therefore, our value [i.e., running total] will never stop growing. And so as we continue on indefinitely, we'll approach infinity.
Monica: Right.

In Monica's comments, she noted that all of the summands in series 3 are positive and that although the rate at which the running total will grow is slow, the running total will never stop increasing. The interviewer then rephrased Monica's comment to determine whether Monica imagined that a monotone running total implies non-convergence, which Monica confirmed. In Sylvia's case, she recognized that although the summands for series 1 and series 3 both decrease in value, she claimed that series 1 converged and series 3 did not. Sylvia acknowledged the apparent contradiction in her reasoning but was unable to reconcile the issue.

Implication 3: Running total recreation through grouping. A final implication of an *asymptotic running total* meaning is that a student might believe that if she groups the terms in an alternating series to construct a uni-directional *running total*, the series will converge. We call this implication *running total recreation through grouping*.

Both students' actions related to various moments related to series 4 appeared to align with the *running total recreation through grouping* implication (Sylvia's actions related to series 2 were nearly identical to her actions for series 4). We constructed series 4 to reflect an alternating divergent series with a linearly decreasing numerator and linearly increasing denominator (see Table 1). Monica tenuously claimed that series 4 converged to zero and Sylvia confidently claimed the series converged to 200 (the value of the initial summand).

We provide Monica's rationale for claiming that series 4 converges to zero in the transcript below.

Monica: I think that if I were to say that it [series 4] does converge, then I would also say that it converges to zero.
Int: OK. Can you say a little bit more about that?
Monica: …So in this case, instead of seeing like each individual adding on a fraction is one thing [*places hands close together with small space between*], I'm kind of grouping them together, where we are subtracting and adding and that, I'm grouping that together in my head.
Int: OK, could you could you like, mark on the screen what it is that you're grouping together just so that I'm sure that I know?
Monica: Yes. So I would put these together [*draws a bracket above second and third summand*] and then I would put these together [*draws a bracket above the fourth and fifth summand*]… and I take the number 200 and I, do these two things do it [*cursor indicates second and third summands*], I'm going to have a number here [*moves cursor between third summand and minus sign separating third and fourth summand*] that's less than

> 200, but still very close to it. So basically, what I've decided is if you were to sum these two values [*moves cursor back to indicate second and third summands*], you would have a very small number and you subtract those… So that, that's what makes me think that this [i.e., the running total] is getting smaller and smaller and smaller and smaller.

In her response, Monica envisioned a *running total* that started at 200 (the value of the first summand) and noticed that by grouping each pair of consecutive summands after the first summand, she could construct a new series that had a distinctive pattern of $200 + (\text{small negative value}) + (\text{small negative value}) + \cdots$. Based upon her reconstruction of series 4 by grouping, Monica perceived a running total that approached an asymptotic value of zero.

In the transcript below, we provide Sylvia's rationale for claiming that series 4 converged to 200, which involved a different grouping action than Monica.

> Sylvia: So I'm going to say that this one [series 4] converges. And I think I'm going to follow the same logic that I did with [series 2,] that it kind of…drops some and increases a little bit less than it dropped [*draws concave-up curve starting from (0, 200) that stops before reaching* $y = 200$], if that makes sense. And then it, like each wave gets smaller [*draws more waves with decreasing amplitudes that progressively move closer and closer to horizontal line at y = 200*]. And I would say my guess is that it converges to 200.
> [omitted dialogue]
> Int: So, can you explain a little bit more to me about how you're…seeing that come about [convergence to 200]?
> Sylvia: So I guess like, if we start at 200, we subtract 99, we'll get 101, and then you add one ninety-six over three [$\frac{196}{3}$], and that number is smaller than 99. Yes. And so you're going to go back up and essentially cancel out some of the, the subtraction that you did. But not like all the way, like you're not going to get back up to 200.
> Int: OK.
> Sylvia: And then you're going to go down a little bit more, but not as great as just went up. And then, like, follow that pattern.
> Int: So there. So it's like you're imagining that every time we jump up, we're jumping up farther than we drop down. And so over time, we're slowly moving back up towards 200.
> Sylvia: Yes.

In her response, Sylvia's stated that the *running total* begins at 200, drops to 101, and then begins to move back toward 200 in progressively smaller "waves." Sylvia constructed her "waves" by recognizing that the sum of the third and fourth summand would be a positive value, and every subsequent pair of summands would yield positive—albeit decreasing—values. Sylvia then reconstructed series 4 as $101 + (\text{small positive value}) + (\text{small positive value}) + \cdots$ and envisioned that the *running total* would eventually approach the asymptotic value of 200, the initial summand of the series.

Discussion and Conclusion

Our findings provide unique and relevant contributions to the mathematics education literature. For instance, our description of the *asymptotic running total* meaning provides valuable insight into how students might initially perceive series convergence in university calculus courses. The three implications we described both emerge from and deepen the explanatory power of the *asymptotic running total* meaning. The first implication, *decreasing summands convergence*, emerges from an *asymptotic running total* meaning when a student imagines that if the summands in a series tend toward zero, the *running total* will eventually stabilize around a particular value. The second implication, *monotone running total divergence*, emerges when a student envisions that a never-ending string of non-zero summands imply that the *running total* will perpetually increase and eventually surpass all potential asymptotic values. The third implication, *running total recreation through grouping*, emerges when a student considers that she can regroup the terms in an alternating series to create a uni-directional *running total.* For the students in our study, recreating the series resulted in not only a uni-directional *running total,* but a plausible (in the students' minds) candidate for the asymptotic value to which the recreated *running total* might converge. In this manner, the three implications provide a connection between students' actions while considering the convergence of individual series and their overarching image of convergence as a *running total* approaching an asymptote.

Our findings both complement and extend work done by previous researchers. For example, the *asymptotic running total meaning* indicates that students consider series convergence as a dynamic process of approaching an asymptote, which has been previously reported about students' meanings for the limit of sequences (e.g., Roh, 2008) and functions (e.g., Swinyard & Larsen, 2012). Our findings also provide analogous results and additional insight into Martin's (2013) report on students' thinking for Taylor series convergence. For instance, Martin's (2013) *dynamic partial sum* image in the Taylor series case is analogous to the *asymptotic running total* meaning for the infinite series case. However, our report extends the findings of Martin (2013) by reporting new implications of the *asymptotic running total* meaning (e.g., *running total recreation through grouping*) and providing a meaning-implication relationship between images that were previously reported distinctly (i.e., *dynamic partial sum* and *termwise*).

Our data did not provide sufficient evidence to make rigorous claims about relationships between the three implications described in the results section. Still, we make two anecdotal comments about potential relationships. First, both students' actions tended to align with the *decreasing summands convergence* implication after constructing and reasoning about closed-form rules for the general summand in a series. In contrast, the *monotone running total divergence* implication often emerged during moments when the students could not construct a closed-form rule for the general summand or reasoned about the *running total* without referencing a closed-form rule for summand values. Second, the *monotone running total divergence* implication did not emerge during moments after students regrouped an alternating series. Rather, the students' responses indicated that although the regrouped series' running total would perpetually increase or decrease, the inherent oscillation of the original running total was sufficient to render the *monotone running total divergence* implication irrelevant.

We anticipate that our ongoing analysis of the teaching experiment data from which this report emerged will provide insight into effective instructional interventions to foster productive student meanings for series convergence. In particular, we hope to discern how to leverage students' naturalistic images of a series as a *running total* to construct a productive meaning for the sequence of partial sums, which Martínez-Planell et al. (2012) stated is pivotal to comprehending the formal definition of a series as the limit of its sequence of partial sums. We also hope to investigate possible relationships between the three implications we have described and uncover additional meanings and implications that first-time second-semester calculus students possess for series convergence.

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