CONCEPTUALIZING MATHEMATICS TEACHER EDUCATOR KNOWLEDGE: COMPARING AND CONTRASTING EXISTING FRAMEWORKS

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Over the past two decades, the landscape of research on mathematics teacher educators (MTE) has grown considerably. One particular area of interest has focused on the knowledge needed by MTEs for their work with preservice K-8 teachers (PTs). In an effort to understand this varied landscape, we conducted an extensive review of research on frameworks of MTE knowledge. This report explores the theoretical underpinnings of MTE knowledge and highlights similarities and differences among theoretical frameworks. By mapping the terrain of research on MTEs' knowledge, our goal is to identify aspects of MTE knowledge to inform the types of research that may be needed for its further development.

Keywords: teacher educators, teacher knowledge, mathematical knowledge for teaching

The issues of what mathematics teachers need to know (Ball et al., 2008; AMTE, 2017; CBMS, 2012) and how to prepare prospective teachers (PTs) as part of teacher preparation programs (e.g., Adler, 2010; Grossman et al., 2009; Dinham, 2013) have become a central focus in both research and policy arenas in many countries. The term Mathematics Teacher Educators (MTEs) refers to individuals who work with PTs in a variety of contexts to develop and improve the teaching of mathematics (Jaworski, 2008a), including mathematicians, graduate students, mathematics education researchers, and classroom teachers. As such, these individuals play a critical role in the mathematical preparation of PTs.

Over the past two decades, the landscape of research on MTEs has grown considerably. In an effort to understand this varied landscape, the authors conducted an extensive review of research on the nature of MTE knowledge. This paper reports the theoretical underpinnings of the existing frameworks for MTE knowledge identified in this review. We focus on MTEs who work with elementary teachers because of the currently limited number of studies focused exclusively on MTEs who teach secondary coursework. Much of the extant research on MTEs draws on the construct of mathematical knowledge for teaching (MKT; Ball et al., 2008), a model of teacher knowledge that is grounded in elementary school teaching. As such, research on MTEs is largely focused on MTEs who teach elementary PTs. We sought to answer the question: What frameworks for teacher knowledge are leveraged in conceptualizing MTE knowledge, and in what ways?

Conceptual Framework

Conceptualizing MTE Knowledge

Broadly defined, *knowledge* is the information and skills that teachers develop through experience and education. Researchers have long recognized the tension that exists for teachers between their general content-specific knowledge, developed in teacher education programs, and the *craft* knowledge that is developed through teaching practice (Shulman, 1986; Leinhardt et al., 1991). *Practice* refers to the things that teachers do consistently in their work with students (Lampert, 2010). Craft knowledge, for example, includes "know-how for teaching based on past experiences, empirical data, and well-reasoned arguments and predictions" (Hiebert & Morris, 2009, p. 476), the knowledge that is developed through experience working with students. Like others (Cochran-Smith & Lytle, 1999; Ball & Bass, 2000), we argue that teacher knowledge is intimately related to teaching practice – teachers' knowledge is further developed and enhanced over time as they teach.

Mathematical Knowledge for Teaching Teachers

Shulman (1986) introduced the term *pedagogical content knowledge* (PCK), which linked knowledge of teaching pedagogy with knowledge of the specific content being taught. Since then, a great deal of research has focused on identifying the knowledge needed for teaching mathematics (e.g., Ball et al., 2008; Davis & Simmt, 2006; Leinhardt et al., 1991; Ma, 1999; Rowand et al., 2005). Many of these studies argue that teachers need to understand the mathematics in the curriculum they teach in deep and connected ways that are specific to the needs of teachers. When combined with PCK, this knowledge base is commonly referred to as *mathematical knowledge for teaching* (MKT) (e.g., Ball et al., 2008).

Building on MKT research (Shulman, 1986; Davis & Simmt, 2006; Leinhardt et al., 1991; Ma, 1999), central to this review is an assumption that the knowledge needed by MTEs for their work with teachers differs from the knowledge needed by teachers for working with students. Researchers generally agree that the work of MTEs involves working with PTs and/or practicing teachers to develop their MKT (Jaworski, 2008a). As there is considerable diversity in the nature of MTEs' work, the range of expertise shared by MTEs is similarly diverse, involving varying levels of mathematical expertise, pedagogical expertise, and/or expertise derived from their experiences as teachers (Bergsten & Grëvholm, 2008). However, in addition to developing new content knowledge, MTEs need to help PTs' understand the ways in which the concepts they are learning connect to their future teaching of students. This requires MTEs to also understand what is involved in teaching mathematics to students. To simultaneously enhance different levels of PTs' awareness (as learners and future teachers), work that is different from what teachers do with students, MTEs need *mathematical knowledge for teaching teachers* (MKTT).

Method

Literature Search

We conducted a literature search using ERIC and Dissertation Abstracts using the following search terms: (1) *mathematics teacher educator(s)* AND *knowledge,* (2) *mathematics* AND *teacher educator* AND *knowledge*, (3) *teacher trainer* AND *knowledge*, and (4) *mathematics* AND *teacher trainer*. We also searched edited books and book series that were cited in many of the articles we reviewed, recent special issues of peer-reviewed journals, and conference proceedings of the Research in Undergraduate Mathematics Education (RUME) annual meetings, as these were not captured in the ERIC search results. The search covered articles written in English and published in peer-reviewed scientific journals or conference proceedings, as well as dissertations. Hereafter we use the term "articles" to refer to the resulting set of scientific studies, books, book chapters, dissertations, and conference proceedings. Because Shulman's seminal article about teacher knowledge was published in 1986 (Shulman, 1986) and the majority of the research on MTEs emerged in the early 2000s, we confined the search to 1986-2021. The literature search and initial review of abstracts for studies of MTE knowledge or its development (beyond just implications for MTE knowledge) resulted in 87 articles (including 6 dissertations). We then divided up the articles and individually reviewed each one in its entirety, summarizing each in terms of research goals, method(s), research questions, researcher's role, and findings, resulting in a final tally of 15 articles that proposed different frameworks for conceptualizing and characterizing MTE knowledge. Of these, we categorized 10 as proposing Complete Frameworks for MTE Knowledge and five as proposing Components of a Framework for MTE Knowledge. Additionally, we found one paper (Goos, 2009) that proposed a framework for *analyzing* MTE knowledge, which we will not discuss in this review.

Results: Frameworks for MTE Knowledge

Complete Frameworks for MTE Knowledge

We begin with Mason (1998) who presented one of the earliest frameworks for MTE knowledge. In particular, Mason conceptualized MTE knowledge not as different types of knowledge, but rather as different levels of awareness. Mason identified the key notions underlying teaching practice as the nature of awareness and the structure of attention (which encompasses the locus, focus, and form of attention moment by moment). Teaching teachers involves the refinement and development of a complex of awareness on all three levels, and this is manifested in alterations to the structure of attention. He proposed three levels of awareness: (a) *awareness-in-action* involves an awareness of one's actions and own learning, i.e., the development of one's knowledge of mathematical content; (b) *awareness-in-discipline* includes an awareness of one's awareness-in-action, i.e., the development of one's mathematical knowledge needed for teaching so that they can develop others' knowledge of mathematical content; and (c) *awareness-in-counsel* encompasses an awareness of how to develop awareness-in-discipline in others, i.e., the development of one's knowledge for teaching teachers so that they can develop other's mathematical knowledge for teaching. In this way, each level of awareness encompasses all prior levels. For Mason, MTEs develop awareness-in-discipline in PTs, as opposed to just awareness-in-action. Mason points to the unique nature of MTE knowledge and the ways in which it builds on, but is different from, teacher knowledge. His work provided a foundation for much of the research on MTE knowledge that followed.

Zaslavsky and Leikin (2004) expanded Jaworski's (1992) model for the practice of teaching mathematics to better understand the practice of teaching mathematics teachers. Jaworski's (1992) model, known as the "teaching triad of mathematics teachers," highlights the interactions among three important components of teaching: *challenging content for students* (mathematics), *the management of students' learning*, and *sensitivity to students*. Zaslavsky and Leikin created analogous terms for knowledge important to the teaching of teachers: *challenging content for mathematics teachers*, *the management of mathematics teachers' learning*, and *sensitivity to mathematics teachers*. In developing their "teaching triad of MTEs," Zaslavsky and Leikin considered Jaworski's "teaching triad for mathematics teachers" to be a subdomain of MTE knowledge, contained within the component of *challenging content for mathematics teachers*.

Similarly, Perks and Prestage (2008), reflecting on their work with PTs and their experiences as teachers themselves, proposed the "teacher-educator knowledge tetrahedron," positioning their entire "teacher knowledge tetrahedron" framework (Prestage & Perks, 1999) as a subdomain of MTE knowledge. Both frameworks highlight the interactions among four aspects of classroom practice: *teacher knowledge*, *professional traditions*, *practical wisdom, and learner knowledge*. However, in the teacher-educator knowledge tetrahedron, the learners in question are PTs and the teachers in question are MTEs. For Perks and Prestage, teacher-educator knowledge refers to the knowledge MTEs develop over time as they teach PTs; professional traditions refer to knowledge of teacher preparation coursework and research on mathematics teacher education; practical wisdom refers to the tasks and activities used with PTs; and, learner knowledge refers to the content that PTs need to understand. Notably, the teacher knowledge tetrahedron (Prestage & Perks, 1999) is entirely contained within the learner knowledge portion of their teacher-educator knowledge tetrahedron.

Reflecting on her work as an MTE, Chauvot (2009) conceptualized MKTT as consisting of different subdomains that encompass the varied responsibilities of a teacher educator who works with PTs. Specifically, she constructed a knowledge map consisting of subject matter content knowledge, PCK, and curricular knowledge that parallel knowledge domains for teachers described by Shulman (1986), as well as subsequent research on teacher knowledge (e.g., Ball et al., 2008). Chauvot's model expanded upon Shulman's work by placing all three of his domains of knowledge for teaching within the *subject matter content knowledge* of MTEs. She also extended several of the domains identified by Ball and colleagues (2008) and described how they are relevant to her work as a MTE-researcher (MTE-R), including knowledge of how to develop PTs' specialized and PCK and knowledge of how to engage PTs with content in ways that are connected to teaching. Chauvot also included the notion of *knowledge of context*, describing this knowledge subdomain for MTEs as including an understanding of contextual factors affecting the teachers with whom they work, e.g., standards and policies for teacher preparation programs, state teacher certification, and national accreditation. Like others, Chauvot's framework includes a teacher knowledge framework as one of the subdomains of MTE knowledge.

Shaughnessy and colleagues (2016) expanded upon the *instructional triangle* suggested by Cohen and colleagues (2003) by positioning the instructional triangle for teacher knowledge (i.e., interactions among teachers, students, and content) as the content knowledge needed by MTEs. Furthermore, they proposed analogous subdomains of MTE knowledge that parallel those of the MKT framework offered by Ball and colleagues (2008), arguing that MKT is a subdomain of MTE knowledge. These subdomains include *MTE common content knowledge* (e.g., knowledge of how to explain multi-digit subtraction that allows PTs to engage in this instructional practice); *MTE knowledge of content and students* (e.g., knowledge of common errors that PTs tend to make when engaging in instructional practices like regrouping using base-10 blocks); and *MTE knowledge of content and teaching* (e.g., knowledge of the types of tasks, representations, etc., that are useful in helping PTs learn an aspect of mathematics teaching).

Hauk and colleagues (2017) also expanded upon the MKT framework offered by Ball and colleagues (2008) in their conceptualization of MTE Knowledge, which they call Mathematical Knowledge for Teaching Future Teachers (MKT-FT). They argued that akin to the ways in which teachers require specialized mathematical knowledge for teaching students, MTEs require specialized knowledge specific to helping PTs develop MKT. The "subject matter" for MKT-FT is a combination of mathematics and mathematics education, as they conceptualize MKT becoming MTEs' common content knowledge. They then propose a framework for MKT-FT PCK, which builds upon a framework for teacher PCK proposed by Hauk and colleagues (2014). By adding a fourth domain, *knowledge of discourses*, connected to each of the three components of PCK in the framework offered by Ball and colleagues (2008), this model for teacher PCK becomes a tetrahedron with vertices that represent anticipatory, curricular, and implementation thinking. Hauk and colleagues (2017) expanded this tetrahedron model for teacher PCK to create an analogous tetrahedron to model MTE PCK, where the subdomain of knowledge of content and students for MTEs contains the entire framework for teacher PCK. The authors describe this subdomain as including knowledge of how to support PTs' development of MKT, as well as knowledge of how to engage PTs in learning to unpack mathematical ideas in ways needed for teaching students. Hauk and colleagues concluded by illustrating ways in which MTEs have used the MKT-FT framework in designing and implementing teaching-related tasks with PTs.

The conceptualization of MKTT that we have proposed (Castro Superfine et al., 2020; Olanoff et al., 2018; Welder et al., 2017) similarly extends from and is connected to the domains of MKT (Ball et al., 2008). We posit that akin to how MKT is composed of subject matter knowledge and PCK, MKTT is composed of subject matter knowledge for MTEs (including MKT) and PCK for MTEs (for facilitating PTs' learning of MKT). We conceptualize MTE-subject matter knowledge as being composed of three subdomains analogous to those comprising subject matter for teachers: *MTE common content knowledge* (which includes the entire framework for MKT), *MTE specialized content knowledge* (mathematical content knowledge that is specific to developing PTs' MKT), and *knowledge at the mathematical horizon for PTs*. We conceptualize MTE-PCK as being composed of three subdomains analogous to those comprising PCK for teachers: *knowledge of content and PTs* (i.e., MTE-knowledge of content and students), *knowledge of content and teaching PTs* (i.e., MTE-knowledge of content and teaching), and *knowledge of curriculum for PTs* (i.e., MTE-knowledge of curriculum). Our conceptualization of MKTT consists of not only the mathematical knowledge needed by teachers but also specialized knowledge of content that is unique to teaching PTs and knowledge of how to facilitate PT learning (i.e., relearning (Castro Superfine et al., 2020)). Like Beswick and Chapman (2012), we consider MKTT to be an elaborated extension of teacher knowledge that also includes domains of MTE knowledge that are characteristically different from teacher knowledge.

The most recently offered framework for MTE knowledge was published in a chapter of a book on the learning and development of MTEs edited by Goos and Beswick in 2021. Based on a review of research with and about mathematics teachers, Escudero-Avila and colleagues (2021) constructed a framework for MTE knowledge composed of seven subdomains, the last three of which the authors consider to be aspects of MTE PCK: 1) *mathematical knowledge*, which includes both knowledge of mathematics and MKT, 2) *knowledge about teachers' PCK*, which includes theories of teaching, key features in learning mathematics, and learning standards, 3) *knowledge about mathematics teaching practices and skills*, 4) *knowledge about professional identity*, 5) *knowledge of the features of the professional development of mathematics teachers*, 6) *knowledge of teaching the content of initial mathematics teacher education programmes*, and 7) *knowledge of the standards of mathematics teacher education programmes*.

In contrast to all of the frameworks discussed above, in the concluding chapter of a volume on MTE knowledge and practice that she edited, Jaworski (2008b) suggested that there are aspects of teacher knowledge that are unique to teachers, just as there are aspects of MTE knowledge that are unique to MTEs. In doing so, she conceptualized the relationship between MTE knowledge and teacher knowledge using a Venn diagram. Similar to Chavout (2009), Jaworski posits that unique to MTE knowledge is the professional and research literature related to mathematics teaching and learning, including knowledge of theories of learning and teaching and knowledge of methodologies of research focused on learning and teaching in educational systems. MTEs utilize their knowledge of methodologies used to study teaching and learning in schools as they work with PTs both in and out of school settings. According to Jaworski, knowledge unique to the needs of teachers, and not necessarily needed by MTEs, includes knowledge of school contexts and elementary mathematics curricula.

Components of a Framework for MTE Knowledge

Five articles do not include complete frameworks for MTE knowledge, but rather, they present components of a framework. Building on Shulman's (1986) subdomain curricular knowledge, in particular, Chauvot (2008) characterized what such knowledge entails for MTEs. Relevant to this review, she identified four components of curriculum knowledge for MTEs based on her experiences as an MTE and as a mathematics education researcher. These components include (a) *knowledge of programs and materials* (e.g., different models for teacher preparation, textbooks, and materials for use in courses for PTs), (b) *knowledge of indications and contraindications of curricula* (e.g., use of curricula or program materials in particular circumstances and effectiveness of curriculum programs), (c) *lateral curriculum knowledge* (e.g., knowledge of other courses PTs are enrolled in), and (d) *vertical curriculum knowledge* (e.g., knowledge of coursework that precedes and follows current courses in which PTs' are enrolled). Chauvot concluded by highlighting the use of MTE curricular knowledge in current studies of MTEs' professional learning, arguing for the centrality of curricular knowledge in MTEs' work.

In their studies of an MTE's practice, Chick and Beswick (2013; 2017) proposed a framework for the subdomain of MTE PCK, building their descriptions of PCK for MTEs from descriptions of the PCK teachers require. Analyzing the teaching practice of the first author, Chick and Beswick identified several components of MTE PCK, including knowledge of examples, curriculum, student thinking, and common misconceptions, among others. Using vignettes of practice from the first author's practice, they provided evidence of the existence of the various components, illustrating ways in which such knowledge is leveraged as an MTE teaches PTs. For each component, they identified ways in which such components of MTE knowledge are similar to and different from teachers' knowledge.

Olanoff (2011) used observations of and interviews with three MTEs teaching fraction multiplication and division concepts, to identify several components of MKTT. Unlike other authors, Olanoff examines the components of the domain of MKTT, with a particular focus on the knowledge components leveraged while teaching a particular concept to PTs. These components include knowledge of (a) multiple representations of the topics, how the representations relate to other topics, and which representations best support PTs in making connections, (b) how to set specific goals for student learning, and (c) how to design and use assessments effectively. Like Chick and Beswick (2013; 2017), Olanoff provided evidence of the existence of MTE knowledge components that are unique to MTEs.

Felton-Koestler (2020) proposed a framework for *knowledge for sociopolitical mathematics teaching* (KSMT), knowledge that teachers at all levels, including teacher preparation, need for addressing issues of equity and social justice by what he calls *mathematizing sociopolitical issues*. This framework extends MKT, which he uses as a blanket term to address the specialized knowledge for teachers and MTEs, to include knowledge of sociopolitical issues and knowledge of sociopolitical curriculum. To have and develop KSMT, teachers and MTEs must be aware of current sociopolitical issues and be able to turn a critical lens on how they are presented in the general discourse. This work goes beyond just understanding the effects of individual biases on current events to include institutional and structural forms of oppression. Although this framework is generalized to both teachers and MTEs, Felton-Koestler does identify a need for future work to address differences in KSMT for teachers and MTEs. Regardless of these potential differences, it is noteworthy to consider KSMT as a component of MTE knowledge.

Discussion

There is general consensus within the teacher education community that the knowledge MTEs require in their work with PTs includes not only the knowledge teachers need to know but also unique elaborations of that knowledge. In fact, many of the articles we reviewed conceptualize MTE knowledge as an extension of teacher knowledge. That is, these frameworks not only position teacher knowledge as a subdomain of MTE knowledge but also partition MTE knowledge into subdomains similar to those found in frameworks for teacher knowledge. In our previous work (Castro Superfine et al., 2020; Olanoff et al., 2018; Welder et al., 2017), similar to Hauk and colleagues (2017), we propose a fractalization metaphor to describe the ways in which various articles conceptualize MTE knowledge as the visualizations of many of these frameworks resemble part of a fractal. We use the term *fractalization* to refer to the process by which one component or subdomain is entirely contained within a larger subdomain, where the larger subdomain is analogous in structure to the smaller one. In many instances, a teacher knowledge framework becomes the content knowledge subdomain of a framework for the knowledge needed by MTEs. We refer to this fractalization metaphor throughout the discussion.

Three main themes emerged from our review of frameworks for MTE knowledge. First, many of these frameworks build on existing frameworks for teacher knowledge, and in many instances, represent fractalizations of teacher knowledge frameworks (e.g., teaching triad (Cohen et al., 2003), instructional triangle (Jaworski, 1992), teacher knowledge tetrahedron (Prestage & Perks, 1999)). Therefore, many frameworks for MTE knowledge included a teacher knowledge framework in its entirety as one of its subdomains. Only Jaworkski (2008b) proposed a Venn Diagram to suggest that teachers require additional knowledge (e.g., school context, elementary curriculum) not needed by MTEs in their work with PTs. Furthermore, researchers conceptualized certain subdomains (and related components) of MTE knowledge as being "meta" forms of analogous teacher knowledge subdomains (e.g., Chauvot, 2009; Hauk et al., 2017; Perks & Prestage, 2008; Zaslavsky & Leikin, 2004). For example, similar to Shaughnessy and colleagues (2016), we (Castro Superfine et al., 2020; Olanoff et al., 2018; Welder et al., 2017) have proposed MTE knowledge subdomains analogous to the MKT framework from Ball and colleagues (2008). These include MTE common content knowledge (which contains MKT), MTE knowledge of content and students, and MTE knowledge of content and teaching. In other words, as teachers of teachers, MTEs require similar types of knowledge needed by teachers, but MTEs need knowledge of these subdomains in ways that are specific to teaching PTs. Thus, we posit that fractalization can be a useful metaphor for conceptualizing MTE knowledge.

Second, despite some overall similarities, there are important differences in the subdomains of the frameworks we reviewed. Broadly speaking, all of the knowledge frameworks in our review include some or all of four main subdomains representing extensions of teacher knowledge domains (e.g., Ball et al., 2008; Shulman, 1986): knowledge of content, knowledge of curriculum and context, knowledge of PTs, and knowledge of ways of supporting PT learning. However, some frameworks included subdomains unique to MTEs, such as Zaslavsky and Leikin's (2004) sensitivity to mathematics teachers (PTs). Considering the uniqueness of PTs as a population of learners, this knowledge includes understanding that PTs often enter teacher preparation programs with limited conceptual understandings of mathematics. As such, the work for MTEs is to support PTs' relearning of mathematics, which involves PTs ultimately reconstructing their previously developed knowledge of mathematics (Author 2020; Zazkis, 2011). Further, Chauvot (2009) included MTEs' knowledge of mathematics education research as part of the knowledge needed to effectively prepare PTs. In fact, Chauvot posited that knowledge of research in mathematics education underlies all other knowledge subdomains MTEs require in their work with PTs (e.g., knowing the research on how children learn can inform MTEs' content course design). This subdomain echoes other researchers who describe conducting research in mathematics teacher education as a form of professional learning (e.g., Rowland et al., 2014). Notably, these different subdomains are unique to MTEs and arguably do not have analogous subdomains in a teacher knowledge framework. In addition, there are important differences in the grain size at which researchers conceptualized MTE knowledge. While the majority only identified subdomain levels using the four main subdomains described above, a few deconstructed their subdomains into components. For example, Chauvot (2008) described components of MTE curricular knowledge (e.g., lateral and vertical curricular knowledge); whereas we (e.g., Castro Superfine et al., 2020) specified components of MTEs' content-specific knowledge (e.g., specialized content knowledge, knowledge of content and teaching). Such variation is indicative of the fragmented research landscape on MTE knowledge.

A third theme that emerged is the process by which MTE knowledge has been conceptualized, which has largely been from a knowledge-in-practice perspective. Through an analysis of various artifacts of practice and reflections on MTEs' work with PTs, researchers applying a knowledge-in-practice perspective to the work of MTEs highlight the types of knowledge leveraged as they teach PTs, reinforcing the dynamic relationship between knowledge and practice. A majority of articles on MTE knowledge describe a self-study process wherein one or more authors reflect on and describe the types of knowledge they leveraged in their work with PTs (e.g., Masingila et al., 2018; Muir et al., 2017; Zazkis & Mamolo, 2018). Such a process is productive for understanding the types of resources (e.g., experiences, beliefs, knowledge) that impact MTEs' practice to support drawing on different types of knowledge in light of their expertise. However, more work needs to be done to explicate the analytic processes taken in research on MTE knowledge so that others can employ similar methods and contribute to the growing knowledge base on MTEs. Moreover, while self-studies (i.e., research *by* MTEs) provide unique insights into the nature of MTE knowledge, the research base would be strengthened by research *on* MTEs to corroborate and further specify the various MTE knowledge subdomains and components.

The goal of this review was to explore the theoretical underpinnings of the existing frameworks for MTE knowledge. We find that MTE knowledge frameworks that are fractalized versions of teacher knowledge frameworks tend to miss the same aspects of MTE knowledge that are missing from teacher knowledge frameworks. For example, Felton-Koestler's (2020) KSMT framework addresses MTE's knowledge of equity, a domain that does not explicitly appear in any of the fractalized frameworks. Additionally, none of the frameworks discussed the need for MTEs to be knowledgeable of classroom technologies. The COVID-19 pandemic and subsequent shifts to online courses demonstrated the need for MTEs to have knowledge of relevant technological tools, not only for their practices as teacher educators but also for preparing PTs to use such tools in their future classrooms. There exists a corpus of literature on *technological pedagogical and content knowledge* (TPACK) for mathematics teachers (e.g., Kohler & Mishra, 2009), but this work has not yet been integrated into frameworks for MTE knowledge. To address some of these missing aspects and further build a mutually agreed-upon knowledge base for MTEs, researchers might analyze MTEs' reflections of their teaching practices to understand their knowledge-in-practice and the types of knowledge MTEs draw on in their work with PTs. Such a knowledge base could inform the design and implementation of opportunities to improve the preparation and professional development of MTEs.

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