THE PRIME NUMBER THEOREM AS A MAPPING BETWEEN TWO MATHEMATICAL WORLDS

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This paper frames children's mathematics as mathematics. Specifically, it draws upon our knowledge of children's mathematics and applies it to understanding the prime number theorem. Elementary school arithmetic emphasizes two principal operations: addition and multiplication. Through their units coordination activity, children construct two mathematical worlds: an additive world and a multiplicative world. Understanding how children might map between their additive and multiplicative worlds provides insights into the prime number theorem. It also helps us appreciate the power of children's mathematics, constructed through the coordination of their own mental actions.

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Prime numbers fascinate mathematicians of all ages. When we learn to count, we learn that we can build numbers additively, without end, by continuing an iteration of 1s (e.g., 6 as 1+1+1+1+1+1). With prime numbers, we can build those same numbers multiplicatively, as a product of primes (e.g., 6 as 2×3). Confrey and Smith (1995) have described these constructions as two different worlds: an additive world and a multiplicative world. The prime number theorem relates those two worlds in a surprisingly simple way. It states that the number of primes less than or equal to any natural number, n, is approximately n/ln(n). In this way, the theorem relates the additive building blocks of number (n 1s) to its multiplicative building blocks (the π(n) prime numbers less than or equal to n). It states $\pi(n) \approx n/\ln(n)$.

Aside from its simplicity, the prime number theorem is surprising in that it describes the relative distribution of primes without discerning a pattern of primes. Attempts to generate primes algorithmically have never failed to fail (e.g., the first five Fermat numbers are prime but, as far as we know, none of the others are). In fact, our national security depends on such failures (though quantum computing might change the game). Moreover, simple conjectures about primes, such as whether every even number can be written as the sum of two primes (e.g., 24=11+13; Goldbach's conjecture), remain unproved.

The prime number theorem is also surprising because, in spite of the simplicity of the theorem itself, its proofs are notoriously complex. Attempts to understand the distribution of primes and prove the prime number theorem led to the invention of the Riemann zeta function—a function defined through an infinite sum of complex-valued terms. Indeed most proofs of the prime number theorem rely on that function. The apparent disconnect between the simplicity of the theorem and its proofs led mathematicians like Atle Selberg (1949) and Paul Erdös (1949) to generate "elementary proofs." Clever as they may be, these proofs are at least as convoluted as the complex ones, but we cannot blame them for trying. After all, prime numbers are natural numbers, and their definition relies on a sieving process that remains contained within the natural numbers. So, why should investigations of prime numbers appeal to complex numbers? One clue appears in the theorem itself, in the form of the natural logarithm.

Confrey and Smith (1995) framed logarithms as mappings between multiplicative and additive worlds. Logarithmic functions convert the formal operation of multiplication into one of addition, thus potentially transforming multiplicative building blocks into additive ones. Indeed, that's precisely what the prime number theorem implies. Formally, this implication appears in the claim (equivalent to the prime number theorem) that the second Chebyshev function—the psi function—approaches n asymptotically. We return to this function later.

Although this paper ostensibly concerns the prime number theorem, it is fundamentally about the nature of mathematics itself: an epistemology of mathematics that we can learn through interactions with children. In tracing the roots of mathematics to a coordination of actions, we can find in children's actions the primary source of all mathematics. We can learn mathematics from children, not only as a source of content knowledge for improving our instruction (Ball, Thames, & Phelps, 2008), but also as a trace of the historical development of formal mathematics. In the case of the prime number theorem, research on children's numerical development teaches us how mathematicians construct additive and multiplicative worlds and how they might navigate between those two worlds. This perspective stands in contrast to Platonist views that have become crystallized in formal mathematics (Lakatos, 1976). As such, this theoretical paper addresses three questions raised in this year's conference theme:

1. How does your work improve learning conditions for each and every mathematics learner?
2. How does your work challenge a settled mathematics learning status quo?
3. Whose voice does your work center in the mathematics learning process? What can be learned from reflecting on this question?

The purpose of this paper is to review research on the additive and multiplicative worlds that children construct and to theorize how our understanding of children's mathematics elucidates our understanding of mathematics as a whole. Along the way, the paper also theorizes why addition and multiplication appear as the two principal operations in formal arithmetic. The paper frames children's arithmetic as the construction, transformation, and coordination of units. In doing so, it privileges students' units coordinating activity as the psychological basis for formal arithmetic, complex analysis, and the prime number theorem. In addition to the framing offered by Confrey and Smith's (1995) mapping between worlds, the paper draws heavily from Ulrich's (2015, 2016) framing of additive and multiplicative units coordination, as well as Boulet's (1998) characterizations of multiplicative reasoning as a transformation of units.

Preserving Units in the Additive World

Researchers who build models of students' mathematics privilege students' mathematics over school mathematics. They explicitly recognize students' ways of operating as the focus of mathematics education: "If, as teachers, we want to foster understanding, we will have a better chance of success once we have more reliable models of students' conceptual structures, because it is precisely those structures upon which we hope to have some effect." (von Glasersfeld & Steffe, 1991, p. 102). Specifically, Steffe and colleagues began a research program to build models of the ways that students construct and transform units to solve arithmetic problems. These ways of operating become the basis for counting and arithmetic. Ulrich (2015, 2016) has exquisitely synthesized this knowledge gained from children in a pair of articles published in *For the Learning of Mathematics*.

According to Euclid, "a *unit* is that by virtue of which each of the things that exist is called one" (*Elements*, Book VII). This definition fits well enough but overlooks the human activity that goes into constructing units of 1. Children begin to construct units through their counting activity. At first, they count by pointing to figurative items (e.g., chips), building a one-to-one correspondence between their number words and their acts of pointing. A number word, say "five," might then stand in place of that activity (Steffe, 1992).

Arithmetic units emerge from the child's reflection on such activity, when they can take any of the counting acts as an identical unit of 1 (Steffe, 1992). The child can then iterate that unit of 1 to produce collections of 1's, such as a 5. This 5 might then become a composite unit, composed of five 1s, as well as a unit in itself (one 5). Reasoning additively amounts to partitioning composite units into other units—ultimately units of 1—and continuing an iteration of those units. This activity produces nested sequences of numbers (e.g., the explicitly nested number sequence; Steffe, 1992), wherein 5 contains 4, 4 contains 3, and so on.

Since Plato, Western philosophers have marveled at the absolute truth of mathematical equalities, such as 2+2=4. However, if we appreciate the human activity of constructing mathematical objects, like 2 and 4, the mystery fades away. If 2 is two iterations of 1, the addition of another 2 continues the iteration of identical units of 1, so that we have 1+1+1+1, also known as 4. One, two, and four do not have an existence apart from such activity, and that same activity establishes their various relationships. In the case of addition, we continue the iteration of a common unit.

Addition preserves units. If a and b are natural numbers, defined by the iteration of a unit of 1, then a+b is defined through the iteration of that same unit of 1. However, if a and b are fractions, we have to find a common unit for measuring each of them. Usually we refer to a common denominator, but implicitly this refers to a unit fraction that we could iterate to produce both a and b (Steffe, 2003). Moreover, the result, a+b, is expressed as an iteration of this same unit fraction; again, units are preserved.

Now consider the case of repeated addition, say determining the value of 4×5 through the addition 5+5+5+5. In this case, we have four composite units of 5, and we can think about the product as four iterations of an identical unit of 5. Alternatively, we can think about each 5 as five 1s, so that 4×5 becomes (1+1+1+1+1)+(1+1+1+1+1)+(1+1+1+1+1)+(1+1+1+1+1)=20. Either way, units (5s and 1s) are preserved. Ostensibly, we have completed a multiplication task via addition: four 5s is twenty 1s. However, multiplicative reasoning does not operate like additive reasoning. It does not preserve units; it transforms them. As we will see, additive and multiplicative reasoning are distinct, though related through units coordinating activity (Ulrich, 2015, 2016).

Transforming Units in the Multiplicative World

In another article published in *For the Learning of Mathematics*, Boulet (1998) set out to "formulate a uniform concept of multiplication that is not simply another arithmetical operation in disguise" (i.e. repeated addition; p. 12). Her approach stands in contrast to that of Vergnaud (1983) who required three subtypes to address multiplication across the domains of natural numbers, integers, and fractions. Following Davydov (1992), Boulet (1998) promoted multiplicative reasoning as a transformation of units. Boulet extended this perspective to include the multiplication of negative numbers, fractions, and irrational numbers. Consider the example illustrated in Figure 1.

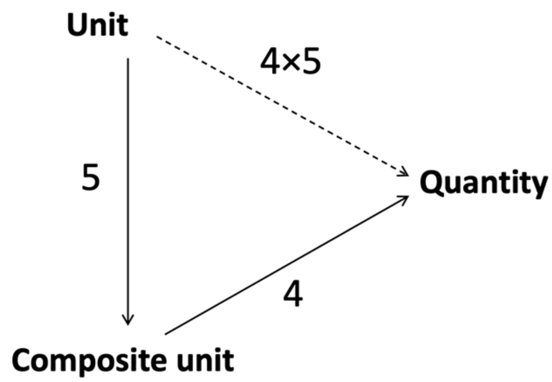

Figure 1: Multiplication as a Transformation of Units

When assimilated as a multiplicative task, 4×5 does not represent the repeated addition of 5s but, rather, a transformation from units of 5 to units of 1 (or vice versa). The product 4 times 5 generates a quantity that measures four units of 5, but we want to know its measure in units of 1, so we project five units of 1 into each unit of 5, transforming each unit of 5 into five units of 1. In terms of units coordination, we have distributed five units of 1 into each of the four composite units of 5 (Steffe, 1992). We could compute the result by counting by 5s (as in repeated addition), but the transformation itself produces the product. Rather than working within a single number sequence, we have transformed numbers across two number sequences: one measured in units of 1 and the other measured in units of 5, with a 5-to-1 transformation between those units.

Boulet's (1998) framing of multiplication places special emphasis on the distinction between roles of multiplicand and multiplier. The multiplicand is the composite unit; it is related to the unit of 1 through a unit transformation (i.e., a distribution of units). The multiplier describes the number of iterations of that unit. With repeated addition, we would take the composite unit (the multiplicand, say 5) and iterate it (say four times). With multiplicative reasoning, we take the four units of 5 and transform them into 20 units of 1, all at once, even if we have to determine their numerosity through skip counting. The focus of activity is on the multiplicand instead of the multiplier.

From this perspective, the fundamental theorem of arithmetic states that we can uniquely produce every natural number in either of two ways: through the iteration of units (a multiplier applied to a multiplicand), or through a transformation of units (transforming the multiplicand). The atoms of addition are units of 1; the atoms for multiplication are primes. Prime factorization of a number, then, represents a sequence of transformations of units, from units of 1, to units of n, where $n=p_1^{R1}p_2^{R2}\ldots p_k^{Rk}$. This idea is generalized in linear algebra, wherein vectors represent units, scalars represent iterations of a unit, and matrices represent unit transformations, transforming vectors (including the basis for a vector space) into other vectors. Additionally, matrix multiplication allows for a recursive transformation of units M×M without reference to any particular scalar (multiplier) or any particular unit (vector). In this way, we can understand the product (-1)×(-1) as a transformation from units of 1 to units -1, and back (as if composing two reflections; Norton, 2022).

Mapping between Worlds

Buttressed by a four-year longitudinal study of upper elementary school students, Lamon (2007) produced the most comprehensive review of literature regarding advanced numerical reasoning. Lamon concluded her report with a list of outstanding questions, beginning with the following: “what are the links between additive and multiplicative structures?” Ulirch (2015, 2016) responded in terms of composite units and ways that students operate with those units: iterating composite units, disembedding units from a composite unit, distributing the units of one composite unit across the units of another composite unit. Essentially, both additive reasoning and multiplicative reasoning rely on the construction of composite units, so they have a common cognitive basis, but as we have seen, students operate on composite units differently in additive and multiplicative reasoning. Specifically, we have made a distinction between ways students preserve (composite) units when engaged in additive reasoning, and ways they transform units when engaged in multiplicative reasoning.

Lamon’s (2007) review included Confrey’s (1994) concept of a split, through which students might transform a unit of 1 into a unit of n. Steffe (2002) proposed a similar concept, describing ways students could partition a whole into five equal parts and, reversibly, reconstruct the whole by iterating any one of those parts five times. For both Confrey and Steffe, splitting describes a multiplicative way of operating. However, in anticipating a mapping between additive and multiplicative worlds, Confrey (with Smith, 1995) included the possibility of recursive splits.

Confrey and Smith (1995) used “basic splits” to refer to splits by prime numbers (e.g., a one-to-seven split). They used basic splits recursively to consider how students might map between additive and multiplicative worlds. In the additive world, we begin at 0 and iterate by units of 1 to produce other natural numbers. In the multiplicative world, we being at 1 and perform basic splits to produce those same numbers. To map between the worlds, we rely on logarithms and exponential functions.

We can define exponential and logarithmic functions as mappings between additive and multiplicative structures. Formally, we say they are isomorphisms between the group of integers under the operation of addition, and the group of positive real numbers, under multiplication (e.g., $\log_2(1/4\times8)=-2+3$). Pedagogically, they relate children’s counting activity to their splitting activity. Following Confrey and Smith (1995), we can use the activity of paper folding to exemplify. In the example of $\log_2(1/4\times8)$ we might say that, starting from a folded sheet of paper, if we fold it in half twice more (-2; or 1/4) and then unfold it three times (+3, or ×8), we end up with a sheet that has twice the area we started with (+1, or 1/4×8).

Proofs of the Prime Number Theorem

Two and a half millennia after Plato, Platonist views of mathematics still dominate the field (Lakatos, 1976). When we make and prove conjectures about prime numbers, we tend to think about those numbers as pre-existing, lying in wait for their discovery; this despite the fact that Eratosthenes demonstrated, through his sieve, that prime numbers are the result of a process of filtering out multiples of units, beginning with multiples of 2. Indeed, the clearest existing explanation of the prime number theorem follows quickly from the fact that, to find primes up to n, we need only apply the sieving process up to the square root of n (Marshall & Smith, 2017). However, this proof is not generally accepted as rigorous because it relies on what professional mathematicians consider to be probabilistic arguments about the density of prime numbers. Neither will professional mathematicians trust empirical arguments, and rightly so, but again, such arguments suggest that the prime number theorem is not about prime numbers at all. Specifically, in computer simulations where a random sieve is applied to the natural numbers (randomly eliminating half of the numbers, taking the smallest remaining number, k, and randomly removing 1/k of the remaining numbers, and so on) the distribution of these “random primes” behaves the same way (Brown, 1978, citing a study by Hawkins, 1958). There are approximately n/ln(n) of them less than or equal to n, as n gets large.

To prove the prime number theorem, mathematicians are more comfortable moving into the complex plane, which comes complete with a geometry of numbers and all kinds of computational power, including infinite sums, complex roots, and calculus. The first proofs of the prime number theorem relied on the Riemann zeta function. Interestingly, and in line with more intuitive arguments introduced here, the infinite sum that defines the zeta function can also be expressed as an infinite product of terms involving distinct primes. However, the complexity of existing proofs has left mathematicians wondering whether a simpler proof could be found. After all, and to echo a question first raised by David Hilbert (Kreisel, 1984), why should the proof of a theorem about natural numbers rely on complex analysis?

Driven by such questions, Selberg (1949) and Erdös (1949) developed proofs that relied only on arithmetic functions. Paul Erdös was a Hungarian mathematician and a prolific mathematical collaborator. The subject of *The Man Who Loved Only Numbers* (Hoffman, 1998), Erdös believed in a Book of perfect proofs held by the supreme fascist (i.e., God), who withholds from us knowledge of the Platonic realm. No doubt, the convoluted proofs that he and Selberg developed—though clever—are not in the book. Without the computational power of complex analysis, they relied on several specialized functions, such as the Chebyshev functions, which were designed to investigate prime numbers. For example, Chebyshev's psi function ($\psi(n)$) takes a natural number, n, and produces the sum of natural logarithms of new prime factors introduced by numbers less than or equal to n (e.g., $\psi(6)=0+\ln(2)+\ln(3)+\ln(2)+\ln(5)+0$). Note that the value for the fourth term in the sum is ln(2) because 4 introduces a second 2 in its prime factorization; and the value for the sixth term is zero because both prime factors of 6 (2 and 3) were introduced earlier in the sum.

Learning Mathematics from Children

Children have a lot to teach us, especially if we want to teach them. For teachers, learning from children is critical because, if the goal is to promote the development of students' mathematical structures, we must first understand how their mathematical structures operate. Hackenberg (2010) has framed the issue as one of "mathematical caring relations." Children feel cared for when we attempt to harmonize with their thinking. Beyond teaching and learning, the present focus is on an epistemology of mathematics and what children might teach us about mathematics itself.

Consider the prime number theorem in light of what we know about the additive and multiplicative worlds that children construct. In the additive world, children produce natural numbers, starting from 0, through an iteration of 1s. When 1s become iterable for the child, each number contains all the numbers that precede it, as a nested sequence (e.g., as five units of 1, 5 contains 4, which contains, 3, and so on). In the multiplicative world, children produce numbers, starting from 1, through a transformation of units. As the multiplicative analogue of a nested sequence, the least common multiple of a sequence of numbers contains all the numbers that precede it. For example, 60 is the least common multiple of 2, 3, 4, and 5, and contains each of those numbers within its prime factorization: $60=2^2\times3\times5$.

As Confrey and Smith (1995) suggested, we can understand logarithms and exponentials as mappings between those additive and multiplicative worlds. The prime number theorem pertains to such a mapping, so we should not be surprised at the appearance of a logarithm within it. Still, we might wonder why it is the natural logarithm, in particular, that appears. The natural number, e, is defined by a limit. Corresponding to this limit, e^x is defined as follows, and from this definition, we can derive a definition for its inverse function, ln(x):

- $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$
- $ln(x) = \lim_{n\to\infty} n(\sqrt[n]{x} - 1)$

Thinking about units and unit transformations, we can read a lot into this definition of the natural logarithm. We have a multiplier, n, applied to the difference of two units: the nth root of x, and 1. In an additive world, we would produce n through n iterations of the unit of 1. In a multiplicative world, we would produce x through a sequence of n unit transformations, each with the n^{th} root of x as its ratio.

Now consider what happens when x is the least common multiple of the first k natural numbers. The prime factorization of x would be the union of the prime factors of 1, 2, 3, …, k (including multiples of the same prime factor), and ln(x) would be the sum of the logarithms of those prime factors. Because the logarithm function maps multiplicative worlds to additive worlds, we should expect this sum to be k: the multiplicative world containing the numbers 1 through k would get mapped to the additive world containing the numbers 1 through k. In fact, that is exactly what the natural logarithm function does as k gets large, and that fact is equivalent to the prime number theorem.

The least common multiple of the first k numbers is the product of the union of prime factors up to k. So, the natural logarithm of this least common multiple is the sum of the natural logarithms of those prime factors. This sum is Chebyshev's psi function, as Table 1 illustrates.

Table 1: The Prime Number Theorem

Iterations of 1	k	Product of Primes	New Unit Transformations	New Terms in Sum	Value of Sum
+1	1	p^0	×1	0	0
+1	2	p_1	$\times p_1$	$+\log(p_1)$	0.693
+1	3	p_2	$\times p_2$	$+\log(p_2)$	1.792
+1	4	p_1^2	$\times p_1$	$+\log(p_1)$	2.485
+1	5	p_3	$\times p_3$	$+\log(p_3)$	4.094
+1	6	p_1p_2	×1	+0	4.094
+1	7	p_4	$\times p_4$	$+\log(p_4)$	6.040
+1	8	p_1^3	$\times p_1$	$+\log(p_1)$	6.733
+1	9	p_2^2	$\times p_2$	$+\log(p_2)$	7.832
+1	10	p_1p_3	×1	+0	7.832
+1	11	p_5	$\times p_5$	$+\log(p_5)$	10.300
			$(p_1^{R1})\ldots(p_{\pi(k)}^{R\pi(k)})$	$\psi(k) \rightarrow \log(k)\pi(k)$	$\rightarrow k$

The left two columns in Table 1 show the production of natural numbers, k, through the iteration of a unit of 1, beginning from 0. The third column shows the production of each of those natural numbers through a transformation of units (a product of primes), beginning from 1. The fourth column shows new prime numbers introduced in the prime factorization of each natural number (×1 indicates no new primes). The fifth column shows the natural logarithms of new prime factors, which are the terms in the sum that define Chebyshev's psi function. The rightmost column shows this sum, up to each k. Note that because the iteration of 1s begins at 0 and the sequence of unit transformations begin at 1, the value in this right column would better approximate k if we added 1 to it.

The bottom line is that each number, from 1 to k, can be expressed as the product of primes contained within the least common multiple of those numbers. This is expressed in the bottom row of the third column as $(p_1^{R1})(p_2^{R2})\ldots(p_{\pi(k)}^{R\pi(k)})$, where Ri is the highest power to which each prime p_i can be raised without exceeding n. Note that there are π(k) factors, p_i^{Ri}, and as k gets large, each of the factors approximates k. So, the entire product is approximately k raised to the π(k). Correspondingly, ψ(k) is approximately π(k)×ln(k). As suggested by the definition of natural logarithm given above (as a mapping from nested products to nested sums), and as shown in the right column, ψ(k) is also approximately k, which implies the prime number theorem. Moreover, this definition of natural logarithm, along with the prime number theorem, suggests an alternate definition of e. As proved below, we can define e as the n^{th} root of the least common multiple of the first n natural numbers, as n gets large:

- $\Psi(k) = \ln(\mathrm{lcm}(1, 2, \ldots, k)) = \lim_{n\to\infty} n(\sqrt[n]{\mathrm{lcm}(1, 2, \ldots, k)} - 1)$
- In units of k, $\lim_{n\to\infty} nk(\sqrt[nk]{\mathrm{lcm}(1, 2, \ldots, k)} - 1) = \lim_{n\to\infty} nk(\sqrt[n]{\mathrm{lcm}(1, 2, \ldots, k)^{1/k}} - 1)$
- So, $\frac{\Psi(k)}{k} = \lim_{n\to\infty} n(\sqrt[n]{\mathrm{lcm}(1, 2, \ldots, k)^{1/k}} - 1) = \ln\sqrt[k]{\mathrm{lcm}(1, 2, \ldots, k)}$
- Because $\frac{\Psi(k)}{k} \to 1$, $\sqrt[k]{\mathrm{lcm}(1, 2, \ldots, k)} \to e$, so $e = \lim_{n\to\infty} \sqrt[n]{\mathrm{lcm}(1, 2, \ldots, n)}$.

Summary

To summarize the heuristic argument about the prime number theorem, if we knew that the natural logarithm mapped the nested product of the first n natural numbers (their least common multiple) to their nested sum (n), the prime number theorem would follow. The ideas of nested number sequences and logarithms as maps between multiplicative worlds were informed by research on children's mathematics (e.g., Steffe, 1992; Ulrich, 2015a, 2015b). Related constructs, such as units and unit transformations, provide additional insights into why we might call the natural logarithm natural.

Aside from arguments about the prime number theorem, this paper makes an argument about the epistemology of mathematics, particularly regarding questions related to the conference theme. The first question asks about improving learning conditions for all children. In response, we frame children's mathematics as genuine mathematics. This mathematics is important to teachers as they support children's continued development as mathematicians. This idea is in line with calls for teachers to develop pedagogical content knowledge (e.g., Ball, Thames, & Phelps, 2008), as well as Hackenberg's (2010) descriptions of mathematical caring relations as "harmonizing with students' schemes" (p. 242). Additionally, we argue that children's mathematics is mathematics that can inform research in mathematics, even at advanced levels.

In framing children's mathematics as mathematics, we answer the second question (regarding the status quo) by challenging Platonist perspectives that still the dominate the field. For instance, there is a tendency to think about prime numbers as pre-existing as if the prime number theorem pertains to discovering some kind of pattern in the fabric of the universe. However, our heuristic argument buttresses prior arguments (e.g., investigations of random primes; Hawkins, 1958) that suggest that the theorem pertains to processes of producing prime factors (e.g., sieving) and has little to do with a pre-determined subset of the natural numbers. In response to the third question, we privilege the voices of children, from whom we have much more to learn.

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