CONCEPTUAL REORGANIZATION, FROM COUNT-UP-TO TO BREAK-APART-MAKE-TEN: A CASE OF A 6TH GRADER STRUGGLING IN MATHEMATICS

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Through a constructivist teaching experiment, we studied how a 6th-grade student (Adam, pseudonym) struggling in mathematics may reorganize his available additive scheme (count-up-to) into a more advanced scheme involving the decomposition of composite units (break-apart-make-ten, or BAMT). First, we posed a task that led us to infer Adam was yet to construct the BAMT scheme at the anticipatory stage (solving a task without prompting). We thus turned to promote reorganization of his anticipatory, count-up-to scheme used to solve missing-addend tasks. Through reflection on the relationships between his goal, count-up-to actions, and effect of those actions, Adam independently brought forth what he called "easy number-pairs" (i.e., 10+X = X-teen). This seemed to afford his reorganization of count-up-to into the BAMT scheme. We discuss implications of this reorganization for theory building and practice.

Keywords: conceptual reorganization, concept of number, struggling student

We address the research question: How may a child, who constructed count-on for adding or count-up-to for solving a missing addend task (see Fuson, 1982, 1986) at the anticipatory stage (see below), advance to constructing the break-apart-make-ten (BAMT) strategy? The former two strategies indicate a concept of number that does not yet include disembedding units, whereas the latter does (Ulrich, 2015; see next section). Addressing this question is important in two ways. First, in additive reasoning, count-on (e.g., 8 + 5 = ?) and count-up-to (e.g., 8 + ? = 13) involve (a) recognition of the first addend as a unit of its own right and (b) *operating on 1s* of the second addend. In count-on, the child may use a sequential activity of concurrently uttering a number while putting up a finger for it (e.g., 8; 9-10-11-12-13). For count-up-to, the child may use a sequential activity of concurrently uttering numbers with fingers and stopping at the given total, then looking at their finger pattern to determine the answer. Critically, decomposing units by disembedding a part without "losing sight of a given total" is needed for subtraction to become the opposite of addition. For example, 13 is recognized as a unit made of two sub-units, 8 and 5, that could compose 13 or one unit be disembedded from 13 to figure out the other unit as the difference. Such a concept serves as a foundation for BAMT (e.g., 10 = 8 + 2 and 5 = 3 + 2, so "I can give 2 (from 5) to 8 to make a ten, then add the remaining 3 to 10).

Second, recent studies relating children's spontaneous additive strategies to multiplicative reasoning demonstrated that BAMT provides a stronger conceptual basis for the latter (Tzur et al., 2021; Zwanch & Wilkins, 2020). We further explain this linkage in the next section. Here, we only note that, to us, BAMT constitutes a direly needed 'conceptual key' to open the 'gate' for advancing any student, let alone those struggling in mathematics, to multiplicative reasoning. A study of how a child may construct BAMT as reorganization in their less advanced additive schemes can contribute to explaining, as well as promoting, this advance.

Conceptual Framework

Our study is informed by general and content specific notions of a constructivist theory. The former derive from Glasersfeld's (1995) scheme theory, which portrays the cognitive 'apparatus' a child uses in assimilation of mathematical problem situations (tasks). Drawing on the 3-part definition of a scheme (goal –> activity –> result/effect), Tzur & Simon (2004) postulated two stages in which a new scheme is being constructed. The first, *participatory* stage, is marked by the learner needing prompts to regenerate and use abstractions they have constructed. Without prompts, the learner brings forth and uses schemes available to them that are less advanced than the evolving (prompt-dependent) scheme. Available schemes are those at the *anticipatory* stage, which are marked by a learner's spontaneous use of those schemes to make sense of and solve given tasks (Tzur et al., 2021). To articulate how a learner reorganizes available schemes into more advanced ones we draw on Simon's (1995) notion of *hypothetical learning trajectory* (HLT). That is, we link goals specified for learners' construction with a hypothesized change process they may go through and with tasks that may foster that change process.

The content specific constructs revolve around Steffe and Cobb's (1988) core notion of *number as a composite unit*. They postulated a progression of schemes in children's numerical development. In the first, *Initial Number Sequence* (INS), a child anticipates a number to be the result of counting activities. The INS affords count-on because a child takes for granted both the first addend (e.g., uttering "8") and the second addend as shown in keeping track and anticipating where to stop the count (e.g., 9-10-11-12-13). However, a child reasoning with the INS is limited to operating on 1s and is yet to consider the nested nature of numbers – a scheme that evolves with the *Tacitly Nested Number Sequence* (TNS). Here, in activities that involve counting of 1s, a child can implicitly think of a given number (e.g., 8) as nested (embedded) within 9, or within 10, or within 13, etc. Whereas the TNS affords using count-up-to for solving missing addend tasks, a child reasoning with the TNS is yet to disembed composite units from a given number (e.g., 10) – a more advanced scheme known as the *Explicitly Nested Number Sequence* (ENS). Here, in situations that involve operating on 1s and composite units, a child can intentionally decompose given numbers into smaller numbers without losing sight of the given number. Thus, the ENS affords a child's use of BAMT, which requires intentionally decomposing the second addend in a way suitable for composing 10 with the first addend, then going back to the remaining part of the second addend (as explained in the previous section). Due to the new ability for composing/decomposing units at will, the child can also coordinate count-on and count-up-to with the use of "easy number pairs" in the second decade (i.e., $10 + 3 = 13$).

Methodology: A Constructivist Teaching Experiment

This study was part of a larger research and professional development project, focusing on teaching mathematics conceptually (grades K-8) and funded by a small school district in the USA southwest region (see Acknowledgment). We conducted a teaching experiment (Cobb & Steffe, 1983) with seven of the nine, struggling grade-6 students at the school who provided parent consent and a student assent. A teaching experiment is a qualitative methodology designed to build models of how learners construct (reorganize) schemes.

Participant and Context

Two main reasons led us to focus on Adam (pseudonym, age 11, grade 6, identified by teachers as struggling in mathematics), who worked with a partner (Gary). First, at the study start, Adam demonstrated spontaneous use of count-on and count-up-to (INS and TNS, respectively) considered a conceptual basis for constructing the BAMT strategy and thus advance to the ENS scheme. Second, despite our repeated efforts to promote BAMT during the first four episodes, using a task and manipulatives (see Double Decker Bus below) designed and successfully used in prior research, Adam was yet to construct BAMT. As our data show, BAMT was not yet available to Adam (or Gary) at the start of the episode reported in this study.

Accordingly, Adam can serve as a case for articulating an HLT from his anticipatory schemes to BAMT (and ENS). Consistent with literature on case studies (Creswell, 2013), we stress that the case is not Adam but rather the process of change we could infer from our work with him. That is, we designed our study to provide an explanatory account of the conceptual change process – not to demonstrate the extent to which it characterizes (many) other children's learning. We thus contend that the HLT explained here is likely to apply to other children who have an anticipatory count-up-strategy and spontaneously use "easy number pairs" as did Adam.

Data Collection

In a teaching experiment researchers serve as teachers of students whose schemes they intend to model. The lead researchers (denoted R1 and R2) conducted weekly teaching episodes with Adam and Gary, 20-40 minutes each, because they seemed conceptually near to one another. Due to the COVID-19 pandemic, we conducted the video recorded teaching episodes in a hybrid mode. R2 worked with the students on site. R1 participated virtually (Zoom) and created a back-up recording on that software. Our work required that we all wear masks, which limited the ability to observe the child's lip movements when operating silently. The presence of R2 allowed her to hear nearly everything the students whispered. To ascertain validity of our data, we thus constantly asked the students to first solve tasks as they wanted, then – using what we could observe – either repeat their work out loud or tell if a researcher's account of it is accurate.

In teaching Adam and Gary, we used a game called the *Double-Decker Bus* (DDB). The first author designed DDB to promote an advance to BAMT in children inferred to have an anticipatory stage of count-on or count-up-to. The students are introduced to the game with a picture of a double-decker bus and a rule made to constrain their operations on units so 10 becomes a special anchor. Each deck on the DDB has 10 seats; the rule is that all seats on the lower-deck must be filled before passengers can move to the upper-deck. In each round of the game, a task is posed using the following story line. The bus leaves main station empty, then a few passengers get on it at the first stop (e.g., 9) and at the second stop (e.g., 4). The child's goal is to figure out the total number of passengers on the DDB after the second stop.

To support the child's reasoning and reflection on their goal-directed activities, a two-row rekenrek, with 10 beads each (grouped by color as 5 + 5), is provided for them to use as they deem appropriate. In our example, a typical solution both Adam and Gary initially used was to move 5 + 4 beads on the lower-deck for 9 passengers, then count-on to add 1 more bead to that deck and up to the given four (hence, 3) to the upper deck. Then, they glean the answer from the manipulative as 10 + 3 = 13. The DDB can foster reflection on the decomposition of 10 into 9 + 1 as a step in the child's activity to find the total. For example, the researcher may ask students: "I asked you to add 4, so why are there only 3 on the upper-deck?" We note that numbers used in those tasks typically begin with 9 + x. Because 10 is the number after 9 (Baroody, 1995), a child is hypothesized to focus on decomposing just one unit of 1. Then, game variations proceed to 8 + x and to 7 + x. Children can reflect on their operation across different tasks (e.g., 9 + 3, 9 + 6, 9 + 4) to abstract the intended, invariant anticipation known as BAMT.

Data Analysis

To analyze our video records and field notes from each episode, we used three iterations. First, after each episode, we held an ongoing analysis (debrief) to discuss major events. These debriefs led to planning next episode tasks. Second, team members created and read transcripts of the episode on which we focus in this paper and the episode just prior to it, highlighting segments with critical events (Powell et al., 2003), such as changes in a student's strategy, or beneficial teaching moves. This iteration included hypotheses we raised about why Adam behaved in the ways he did. Third, we discussed (while re-observing) the highlighted segments to identify compelling evidence for our inferences and conceptual claims. We organized all segments in a story line (next section) that conveys the HLT from count-up-to into BAMT.

Results

We begin with data analysis indicating that, at the start of the episode in which Adam reorganized his concept of number, he was still using an anticipatory scheme involving count-up-to, that is, a concept of number characteristic of TNS students. Next, we shift to data analysis indicating his advance from that early numerical way of operating to the more sophisticated way of BAMT, characteristic of ENS students.

Adam's Anticipatory Count-Up-To Scheme (TNS)

During the first four episodes with Adam and Gary, they both appeared to use the count-on strategy spontaneously and independently. For example, when asked how many total candies they would have if adding 8 and 5, they both used their fingers to keep track of the second addend items (8; 9-10-11-12-13). During the episode prior to the one focused on in this paper, they also solved missing addend tasks using the count-up-to strategy (e.g., to solve how many more cubes are needed to create a tower of 12 if you already have 9 cubes, they both counted on their fingers: 9; 10-11-12 and then looked at the fingers they raised and respond, "3").

During those four episodes, we engaged Adam and Gary in solving Double Decker Bus (DDB) tasks. Using the manipulatives in activity, they showed some initial progress toward BAMT. For example, to add 9+4 they first moved one bead on the lower deck to complete the required number of 10 passengers. Then, they moved 3 more beads to complete the task and responded: "13" (pointing to 10+3 as the two "adjusted" addends). Furthermore, in reflection on their activity, they explained why the upper deck has only 3 passengers whereas the problem asked about 4 passengers added to 9 and responded: "Because you have to put 1 passenger here [lower deck] and 4 minus 1 is 3." In fact, at the fourth episode they also began solving tasks asking about 8 + 5 passengers (e.g., "we put 2 more on the bottom and 3 go here [upper deck]"). However, critically, when we presented a task at the start of each episode without any hints (e.g., How many in all are 9 cubes and 4 more cubes?), they reverted to spontaneously and independently using count-on to solve it.

After the 1.5-month winter break, we again presented a missing addend task for them to solve – not in the DDB context but rather in a context (towers and cubes) we never used with them before: "Pretend you made a tower of 9 cubes. Write 9 on your paper so you remember ... If I wanted to make the tower taller and have 16 cubes, how many more cubes would each of you need?" Excerpt 1 shows how Adam solved this task (R1 and R2 stand for the researchers).

Excerpt 1: Anticipatory count-up-to

R1: (After Gary explained his answer of 7) Adam, how did you get it?

Adam: I counted on my fingers; because 9; [then] 10 (puts up thumb on left hand), 11 (puts up index finger), 12 (puts up middle finger), 13 (puts up ring finger), 14 (puts up pinky), 15 (puts up thumb on right hand), 16 (puts up index finger; shows a full hand plus two more fingers). That's 7, so I know there's 7 [cubes needed].

Excerpt 1 indicates that, at the episode start, Adam has independently and spontaneously used the count-up-to strategy. As Tzur et al. (2021) explained, a spontaneous strategy for a task involving no hints serves as an indicator of a child's anticipatory scheme and, likely, the most advanced strategy (and concept) the child has available at the time. Because count-up-to involves operating on 1s while reasoning numerically about the missing addend, but not yet disembedding a composite unit, we attributed to them the TNS stage. We thus turned to further attempts of fostering their use of BAMT. Excerpt 2 presents a segment of those attempts. We emphasize that Adam's use of count-up-to was used as his answer-checking strategy, whereas his explanation of how he solved the task indicated a shift to BAMT.

Excerpt 2: Adam's shift to BAMT

R1: Now, you still have the tower of 9. [But] I said, you know what, 16 is too tall of a tower. I only want you to have 14. You already have 9, but you want to have 14, not 16. How many cubes would you need to ask for?

Adam: (First, immediately and with no indication of using his fingers, writes down "5" on the page. Then, seemingly to check his answer, he quietly uses count-up-to on his left-hand fingers, putting them up one at a time. After three times of checking his answer, he raises his hand to indicate, "I am done.")

R1: What is the answer?

Adam: Five (5).

R1: How did you get it?

A: I got it by taking away; because 9 + 1 is 10, then all you have to do is add 4 to it.

R1: This is really cool. I like it. (Turns to Gary) Did you understand what Adam said, or would you like him to repeat [his explanation]?

Gary: (To Adam) Could you repeat it?

Adam: I added 1 to 9, then I had 4 left; so, 14. [It is another] easy number-pair.

R1: Here's what I heard Adam say. I am at 9. I only need 1 more to make 10. Is that correct so far?

Adam: Yes.

R1: Now, if I take one to make 10, and I need 14, to get from 10 to 14, I need 4 more. Is that correct Adam?

Adam: Yes.

R1: Then, if I need to take 1 to make 10, and 4 more to make 14, the 1 and the 4, make 5. Adam, is that what you did?

Adam: (Nods head, "yes.")

R1: Adam, I then saw you also using your fingers like you did before. Did you do that to check?

Adam: (Nods head, "yes") Mm-hmm.

R1: So, first you did the 9+1 and 4 more, then you checked with your fingers?

Adam: (Nods head, "yes.")

R2: (A bit later) Adam, you said, "easy number-pair." Can you tell me what you mean by that?

Adam: Like 10+4, 10+3, 4+4, and different numbers like that.

R2: So, are you saying they're like easy numbers to add?

Adam: Yes.

R2: You're trying to create an easy number [to] add by making the 9 [into] a 10?

Adam: Yes.

R1: (A little later, asking about other easy number pairs) What if I wanted 19?

Adam: 10 + 9. Anything in the teens can be paired up with a 10.

R1: What about 13?
Adam: 10 + 3.

Excerpt 2 indicates the first time we witnessed Adam's use of BAMT independently, spontaneously, and not in the DDB context. Importantly, we neither taught Adam about "easy number pairs" nor have knowledge when and how he was taught to use it. Key here, however, is that for the first time he independently and spontaneously brought forth this idea in what, for him, seemed a novel situation. We explain his shift to BAMT as rooted in reflecting on his count-up-to strategy and its effect after solving the first task (9 + ? = 16).

Specifically, when solving and then explaining that first task, Adam repeatedly showed how he raised his thumb for 1 item added while uttering 10 as the number after 9. Upon completion of the count, he held up and looked at 7 fingers while uttering "16." We infer he noticed the effect of his count-up-to as a number in the teens, which brought forth his notion of easy number pairs. In turn, he likely also coordinated "6" in the easy number pair (10 + 6) as being 1 less than 7 given to him by the second addend (which he showed on his fingers). Our explanation resonates with how Adam explained to Gary why this (BAMT) strategy worked for finding how many more cubes were needed to make a tower of 14 cubes ("I added 1 to 9, then I had 4 left; so, 14. [Another] easy number-pair."). It was further corroborated by Adam's response ("Yes") to R2's question, "You're trying to create an easy number [to] add by making the 9 [into] a 10?" and to R1's follow-up question about 19: "10 + 9. Anything in the teens can be paired up with a 10."

We emphasize that Adam's work led us to explain his shift to BAMT differently than what would be a trajectory from count-on to BAMT. In such a typical shift, a child would have to first notice the need to decompose the second addend into two numbers, 1 and the number before the second addend (e.g., to add 9 + 7 they would decompose 7 into 6 + 1, then add 1 to 9 to make 10, and complete with adding 6). Adam's case presents a somewhat reversed order of the reflective process. That is, he did not begin by decomposing 7 into 6 + 1. Rather, he first reflected on his work to solve a missing addend task by noticing that a total in the teens (e.g., 16) could be thought of (decomposed) into 10 and another number in the easy pair (e.g., 6). In turn, this reflection seemed to lead to his consideration of that easy number pair addend as a constituent of the task's second addend (here, 7), which led to decomposing that given second addend (e.g., 7 = 6 + 1, and that 1 is used to "make 9 [into] a 10.") We contemplate that Adam's shift might be a more accessible one for students than the one requiring decomposing of the second addend because such a shift requires setting a sub-goal of disembedding 9 from 10 as a strategic first step (sub-goal) for a yet-to-be-foreseen decomposition of the second addend.

Following this realization of Adam's shift to BAMT, to further promote his and Gary's decomposition of 10 into composite units, we engaged them in creating a tower of 10 (each). Then, we asked one of them to "chop off" a few cubes and write an equation for their quantities. For example, we asked Adam to take the top 2 cubes and place them near the tower. To symbolize this situation involving his and Gary's cubes, he wrote: "10 = 8 + 2." Similarly, after Gary chopped off 3 from his tower of 10 cubes, Adam wrote the equation: "10 = 7 + 3." It is in those tasks that we realized the difference in the two students' reasoning. For Gary, writing an equation proved to be a tremendous challenge, mostly done after hearing Adam's explanation. For Adam, writing an equation symbolizing his decomposition activity seemed straightforward. We emphasize that his equations did not show an addition problem on the left and an answer on the right but rather what seemed to be an equivalence of the two quantities. At this point, we decided to turn back to tasks presented in the DDB context. The first task involved figuring out how many passengers need to get on the bus in the second bus-stop if there were already 9 passengers on it and no one went to the upper deck. Adam knew, and explained to Gary, why just 1 passenger would get on the bus at the second stop. We followed this task with one requiring to go beyond 10 passengers on the bus in all (Excerpt 3).

Excerpt 3: Adam's transfer of BAMT to the DDB context

R1: Let's go back to the start. First stop, 9 people get on the bus. Now, I want you to find the number of people who will get on at the second stop so in the end we will have 12 people on the bus in all. How many people should get on at the second stop, so we have 12 people?

Adam: (Immediately and with no hesitation uses the rekenrek to add 1 more bead to the lower level and 2 more to the upper level.)

R1: Adam, tell us how many you had to put in order to get to where you are.

Adam: Three (3), because there is one in the bottom deck. That makes 10, then 2 up here (points to the upper deck), makes 12.

R1: Why is it 3?

Adam: There's a rule saying the bottom deck has to be full before you go to the top one. So, 1 (shows moving again one bead on the lower level), 2 (moves a bead over on the top deck), 3 (moves another bead over on the top deck).

Excerpt 3 indicates that, in this new missing addend task, Adam transferred his evolving anticipation of the goal-directed activity in the towers and cubes context to a context (DDB) in which we never saw him using BAMT on his own. From his immediate actions we inferred that at this point he anticipated the activity sequence of first using an easy number-pair (12 = 10 + 2), leading to then concluding that 3 passengers would be added to 9 for that total to be accomplished because, for him, 1 would be necessary to compose 10 as the start of the easy number-pair. Based on this transfer, and on the fact that Adam spontaneously brought forth the easy number pair component of a BAMT strategy, we conjectured that he had constructed an anticipatory stage of an ENS concept of number, indicated by disembedding and decomposing two addends (i.e., 10 + x = x-teen and 10 = 9 + 1). We tested this conjecture with a no-hint task at the start of the following week's episode. It turned out we preemptively attributed the anticipatory stage to Adam. For example, he used count-on to solve the task: How many cookies are there in all if a baker placed 9 in one pan and 7 in another pan. We thus attributed to Adam an initial abstraction of BAMT at the participatory stage.

Discussion

In this study, we articulated an advance in a child's concept of number as a composite unit, from count-up-to (TNS) to the BAMT strategy (ENS). Being a case study of the phenomenon of such a conceptual advance, we contend that the reorganization articulated here is likely to pertain to other students at different ages, grade-bands, or mathematical aptitudes. In and of itself, this reorganization is key to the child's additive reasoning, that is, to decomposing and disembedding numbers as composite units when adding or subtracting whole numbers. For example, Adam's BAMT indicated he thought of 12 as a unit that could be decomposed in two related ways: (a) into 10+2 and (b) into 9+x, with x being composed of 1+(x-1). Such an understanding underlies thinking of "fact-families" (e.g., 9+3=12, 3+9-12, 12-3=9, and 12-9=3). Just as important, this conceptual advance opens the way to developing multiplicative reasoning with whole numbers, which are highly limited or impossible for INS or TNS students (Steffe & Cobb, 1988).

For research and theory, this study contributes a plausible HLT for advancing to BAMT. Importantly, the advance we studied did not begin with a child's use of the anticipatory count-on strategy (INS) and a corresponding attempt to promote decomposition of the second addend. In fact, our data indicated that this path, used with Adam and Gary during the first four episodes, failed to promote BAMT. Instead, capitalizing on what Adam taught us, our HLT builds on the child's count-up-to strategy (TNS) and focuses on disembedding and decomposing two addends that make up an "easy number-pair" in the second decade, or "x-teens" (e.g., 14 = 10 + 4). In cases of adding 9 + x (or 8 + x), the child would then bring forth a decomposition of 10 into 9 + 1 (or 8 + 2) and learn to use it strategically by subtracting the needed complement-to-10 (e.g., 1) from the given second addend (e.g., 4 – 1 = 3). In turn, the child would complete the process by adding that decomposed/disembedded unit to 10 (e.g., 3; so, 13).

For practice, our study can inform teachers' efforts to effectively foster what, arguably, is the most foundational concept in mathematics, namely, number as composite unit (Steffe & Cobb, 1988). The HLT we proposed would likely support children's construction of addition and subtraction of whole numbers as two sides of the same coin. Said differently, someone who can compose and decompose given units at will, could come to "see" that any whole number (e.g., 14) may be constituted as a part-part-whole relationships (14 = 10 + 4 = 9 + 5, etc.). Thus, this study supports a way of thinking about and carrying out teaching of BAMT to students, a concept that proved an asset in students' learning to reason multiplicatively (Tzur et al., 2021).

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