"MIRROR LOGIC": A PRESERVICE MATHEMATICS TEACHER'S THINKING ABOUT RADIAN IN THE CONTEXT OF LIGHT REFLECTION

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Integrated science, technology, engineering, and mathematics (iSTEM) education allow learners to utilize multiple disciplinary perspectives. However, the discipline of mathematics remains underrepresented in iSTEM curriculum. To explore the nature of mathematical thinking with an iSTEM curricular approach that emphasizes mathematics, we investigated the thinking of a preservice mathematics teacher, Alex (pseudonym), who engaged in a task-based digital activity involving radian angle measure in the context of light reflection. Findings suggest that Alex's ways of thinking comprise mathematical terminology, concepts, and processes, including mathematical ways of thinking about light reflection. The findings in this report suggest that emphasizing mathematics in this iSTEM context provided an opportunity for new ways of thinking about radian angle measure, and about how angle measure relates to light reflection.

Keywords: curriculum, geometry and spatial thinking, integrated STEM

Despite more than a decade of science, technology, engineering, and mathematics (STEM) reform initiatives toward integrated STEM (iSTEM) approaches (National Academy of Engineering [NAE] and National Research Council [NRC], 2014; National Council of Supervisors of Mathematics [NCSM] and National Council of Teachers of Mathematics [NCTM], 2018; NRC, 2013), the underrepresentation of mathematics in iSTEM education curriculum remains (English, 2016; Fitzallen, 2015). Given the need to develop iSTEM curriculum where mathematics holds equal importance as other disciplines (Baker & Galanti, 2017), it is incumbent upon educators to find ways of foregrounding mathematics within iSTEM experiences to better develop learners' understanding not only of core mathematics content and practices but also about how core mathematics content and practices meaningfully relate to other disciplines. English (2016) and Fitzallen (2015) called for approaches that address how mathematical concepts and practices contribute to the learning and understanding of other STEM disciplines in iSTEM instructional contexts. Additionally, Li et al. (2019) called for research that attends to how thinking in content-based approaches relates to thinking in other disciplines. We take up these calls (English, 2016; Fitzallen, 2015; Li et al., 2019) with purposeful attention to situating mathematics as pivotal in the iSTEM experience. We argue that iSTEM experiences that foreground mathematics can contribute to mathematical thinking, and we consider how mathematical thinking relates to other STEM disciplines. We specifically explore a preservice mathematics teacher's [PMT's] thinking about the mathematical concept of radian angle measure in the context of light reflection. The research question guiding this report is "What ways of thinking does a PMT demonstrate upon interacting with a digital task that involves radian angle measure and light reflection?"

Theoretical Framing

To answer the research question, we took a constructivist perspective (Schunk, 2012) on the construct of thinking, and spatial thinking in particular, which we describe in the following sections. Additionally, we clarify our perspective and definition of iSTEM curricular approaches.

iSTEM Curriculum

Curricular approaches that involve iSTEM have been defined in various ways in the literature (Navy et al., 2021), with teachers, administrators, and policy makers having different views on iSTEM education (Breiner et al., 2012; Holmlund et al., 2018). In this report, our perspective of iSTEM curriculum involves instructional activities with learning goals of content and/or practices from one or more of the STEM disciplines, as anchors, along with engineering and/or engineering design practices, as integrators. Additionally, iSTEM curriculum activities involve opportunities to emphasize twenty-first century skills in a real-world, authentic context, to be solved through collaboration, communication, and teamwork (Bryan & Guzey, 2020).

There are many challenges when it comes to implementing iSTEM curricular approaches (English, 2016; Fitzallen, 2015). One of these involves distinctions in the knowledge base between disciplines (Williams et al., 2016). For example, discipline-specific words have explicit definitions and are used in unique ways in that discipline (Morgan & Sfard, 2016) despite use and overlap of such words in other disciplines. For example, the light reflection principle (Figure 1) is understood as the equality of the angle of incidence and the angle of reflection relative to a perpendicular to the mirror known as the normal ($\alpha = \beta$). The light reflection principle can also be understood as the equality of the angles of incidence and reflection relative to the mirror ($\gamma = \delta$). In this context, the term *normal line* refers to perpendicularity in relation to the scientific phenomenon of light reflection. A person with a mathematical perspective might refer to the normal line in the context of light reflection, using its mathematical property of perpendicularity, rather than using the term itself.

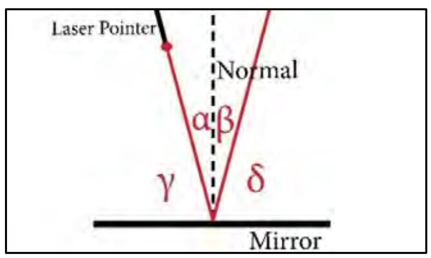

Figure 1. Demonstration of the principle of light reflection

Despite the challenges, research and reviews have reported the effectiveness of iSTEM education approaches on learners' engagement, motivation, interest in STEM, and increased mathematical achievement (Honey et al., 2014; Stohlmann, 2018; Zhong & Xia, 2020). However, little is known about learners' mathematical thinking as they engage in iSTEM instruction (Li et al., 2019). Hence, this study focuses on the ways of thinking that are involved in an iSTEM task that involves radian angle measure and light reflection.

Ways of Thinking

Our definition of ways of thinking builds on Harel's (2008) description of thinking as a learner's established cognitive characteristics, and Thompson et al.'s (2014) extension of Harel's (2008) definition, where thinking is the consistency in a learner's reasoning about mathematical situations. Building on these descriptions, we interpret ways of thinking as the thought patterns a learner demonstrates when reasoning about a particular concept given a specific situation that evokes such reasoning. For example, researchers demonstrated that PMTs think of radian angle measure as angles expressed in terms of π (Akkoc, 2008; Fi, 2003). Additionally, Moore et al. (2016) reported that PMTs' thinking about radian angle measure incorporates a unit circle diagram (Figure 2) to perform calculations. These studies suggest that through their prior coursework and experiences, PMTs may have developed a thought pattern to reason about radian angle measure. Such reasoning makes the use of special angles expressed in terms of π, and/or using calculational strategies an established way of thinking about radian angle measure.

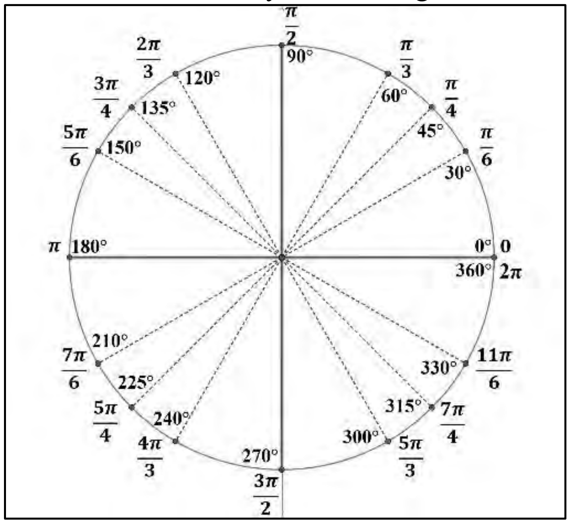

Figure 2. A typical diagram of the unit circle

Spatial Thinking

We characterize ways of thinking with particular attention to spatial ways of thinking. Commonly known as spatial reasoning, we use the term spatial ways of thinking to refer to thought patterns that include "the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects" (Bruce et al., 2017, p. 146), through "a collection of cognitive skills comprised of knowing concepts of space, using tools of representation, and reasoning processes" (NRC, 2006, 12). Spatial thinking is associated with various disciplines and is correlated with achievements in both mathematics (e.g., Clements & Sarama, 2009; Mix, 2019; Mulligan et al., 2018), and other STEM disciplines (e.g., Newcombe, 2010, 2013; Newcombe & Shipley, 2015; Pruden et al., 2011). However, there are few opportunities for students to engage in spatial thinking in school (Clements & Sarama, 2011; Sinclair & Bruce, 2015; Whiteley et al., 2015). Because spatial thinking is associated with achievements in mathematics in addition to achievements in other disciplines (Bruce et al., 2017), it is appropriate to investigate the ways of thinking involved in an iSTEM curricular approach with attention to spatial ways of thinking.

While spatial thinking is usually associated with visualization, Whiteley et al. (2015) suggested addressing and legitimizing broader spatial ways of thinking, including symmetrizing, comparing, decomposing-recomposing, situating, orienting, and scaling. We describe these spatial ways of thinking in the methods section (Table 1), however, we note that the spatial ways of thinking we mentioned do not represent all spatial ways of thinking, nor do they exist in isolation of each other and/or other ways of mathematical thinking (Davis et al., 2015). For example, Munier and Merle (2009) built on NCTM's (2000) recommendation to interrelate geometry and spatial thinking to provide 3-5 graders the opportunity to explore angle measure through physics-based teaching sequences, one of which included light reflection. Through an iterative process of spatial experimentation and geometric knowledge development, the 3-5 graders were able to discover light reflection principle, by attending to the symmetry between the angle of incidence and the angle of reflection relative to the mirror (γ = δ in Figure 1). This illustrates students' use of symmetrizing as a form of spatial thinking in conjunction with the notion of angle as a form of mathematical thinking to discover light reflection principle.

Methods

Research Design

We employed a qualitative case study design (Flyvbjerg, 2011) to examine a PMTs mathematical thinking during a lesson that was part of an iSTEM unit embedding the design of a periscope (Alyami, in-press). The lesson entailed a digital task that involved radian angle measure in the context of a light reflection scenario (described further in the following section).

Participant and Task

The PMT participating in this study was Alex (pseudonym), who was enrolled in a mathematics teacher preparation program at a large Midwestern university. Alex volunteered and was compensated for his time after the first author briefly presented the opportunity in his secondary mathematics methods course. While Alex likely encountered the concepts of radian angle measure and light reflection during his K-16 schooling, he was not offered a formal learning session about radian or light reflection prior to participating in this study.

A Desmos activity (i.e., *Radian Lasers*) comprised the task in this report, where Alex typed values of angle measure (in radian) to adjust a laser and one or two mirrors so the laser beam would successfully pass through three stationary targets at once (Figure 1). The angles of the laser and the mirror are relative to the horizontal and in standard position, where positive angle values are counterclockwise and negative angle values are clockwise. The task consisted of two warm-up activities to familiarize Alex with the functionality of the digital interface, followed by six challenges. A benefit of the *Radian Laser* task is that the angles needed to situate the mirror were not limited to the common special angle (e.g., π/6, π/3, π/2). For example, one way of solving Challenge 1 is by positioning the laser upwards at an angle that is 5π/6 radian, with the mirror angled at a 5π/12 radian, which is not a common special angle (Figure 4).

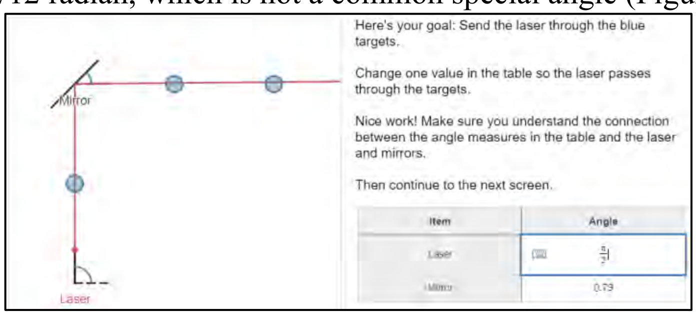

Figure 3. A challenge from the *Radian Lasers* activity

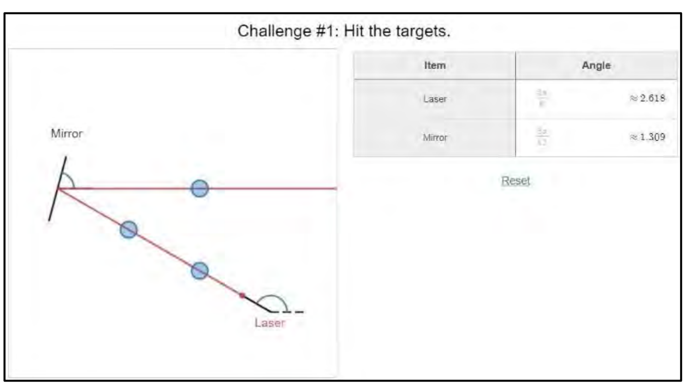

Figure 4. Challenge 1, where the angle of the mirror is not a common special angle

Following principles of structured, task-based interviews (Goldin, 2000), Alex engaged with the *Radian Lasers* task in a semi-structured, think-aloud interview setting, where he could use his own language to make sense of the task (van Someren et al., 1994). The semi-structured setting provided an opportunity to ask for elaborations (e.g., How do you know that? How would you represent your thinking?), which were informed by the responses Alex provided throughout the interview to encourage him to clarify his thinking.

Data and Analysis

The think-aloud, semi-structured interviews, led by the first author, took place virtually, were video recorded, and lasted approximately one hour. The interview video and time-stamped transcript comprise the data for this study. To analyze the data, the first author used a whole-to-part inductive approach for coding (Erickson, 2006), beginning with playing and watching the whole video without coding, stopping, or pausing. Then, NVivo software was used to code the media file, as described by Wainwright and Russell (2010). At this stage, the unit of analysis consisted of one or more sentences that formed coherent statements in which Alex described his thinking about how to reposition the angle of the laser and/or the mirror. To further analyze these statements, we used thematic analysis, which is a coding strategy that involves identifying themes in the data that are informed by the research questions, theoretical framework, and literature review (Saldaña, 2013). Specifically, we coded Alex's statements with attention to spatial ways of thinking described in Whiteley et al. (2015). As part of thematic analysis, we were open to the development of new categories that emerged from the data. Table 1 contains the codes that were evident in the data.

Table 1: Coding Scheme of All Spatial Ways of Thinking Utilized by Alex

Code	Description
Locating	Thinking about *where* objects are situated and/or positioned.
Orienting	Thinking of *how* objects are situated and/or positioned in relation to each other.
Comparing	Thinking about angle size in relation to itself or another angle (bigger, smaller, etc.)
Decomposing-Recomposing	Thinking of a whole as spatially broken into a specific number of parts, and/or spatially adding up parts to form a specific whole.
Symmetrizing	Thinking and applying properties such as congruence and symmetry with similar parts spatially facing each other around an axis.
Visualizing	Thinking visually of geometric objects and managing their characteristics. Includes: Managing both visible and imagined visual information.
Diagramming	Thinking of and managing geometric objects and patterns through drawing Includes: Gestures depicting semantic content (e.g., tracing angles as if to represent the angles on a typical unit-circle diagram) (Sinclair et al., 2018).

After coding all the transcript, the first author reviewed statements that were coded with multiple codes to provide a meaningful interpretation of the coded data "so that more can be gleaned from the data than would be available from merely reading, viewing, or listening carefully to the data multiple times" (Simon, 2019, p. 112). To answer the research question and provide evidence for our argument, the interpretation of the data focused on Alex's ways of thinking in relation to radian angle measure, and in relation to light reflection.

Findings

We describe in this section the mathematical and spatial ways of thinking Alex demonstrated upon engagement with the digital task, *Radian Lasers*. To argue that this iSTEM experience, which foregrounds mathematics, contributes to mathematical thinking and brings mathematics in relation to other STEM disciplines, we organize the two sections of the findings to start with Alex's ways of thinking in relation to radian angle measure. We then describe his ways of thinking in relation to making sense of light reflection.

Ways of Thinking about Radian Angle Measure Beyond the Special Angles

Since the angles needed in *Radian Lasers* are not limited to common special angle (e.g., π/6, π/3, π/2), Alex needed to think beyond the special angles commonly represented on a typical diagram of the unit circle (Figure 2). When a challenge required a noncommon angle, Alex estimated the measure of angle based on spatial comparisons and estimating the angles in between. For example, to reason about Challenge 1 (Figure 4), Alex initially situated the mirror at π/3 radian (left side of Figure 5), and noted that "the laser is hitting the mirror at a perpendicular angle … and that tells me that this angle needs to be slightly bigger." The previous statement suggests that Alex is observing the result of situating the mirror at π/3 radian angle, and then comparing the size of the resulting angle in relation to the angle that would lead the laser to hit the third target. Alex then entered π/2 for the mirror (right side of Figure 5), as he stated that "maybe π/2 would increase the angle," which caused the laser to reflect beyond where the third target is located. Upon missing the third target, Alex said "Okay, so I know it's between π/3 and π/2."

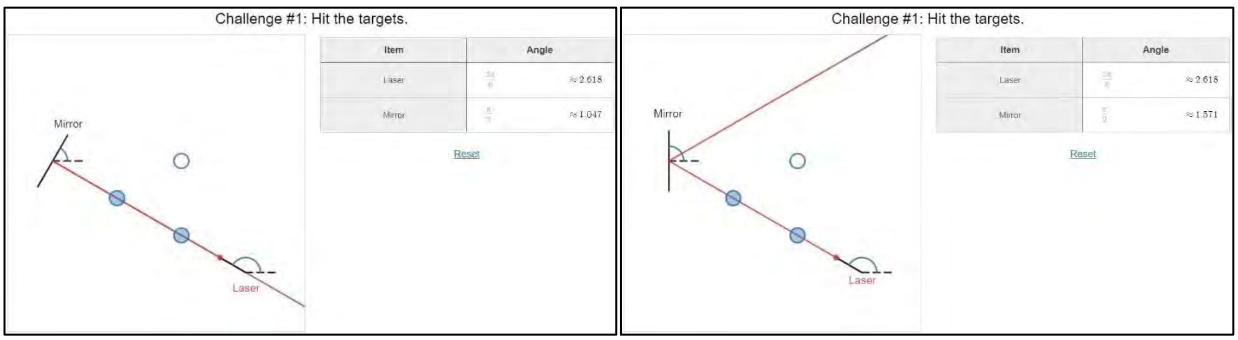

Figure 5. Alex's trials of π/3 & π/2, sending the laser respectively below & above the target

However, Alex was not familiar with a special angle that is between π/2 and π/3, and asked the first author if he could try an input such as 2.5π/6 for the angle. When he entered 2.5π/6, he observed the laser hit the third target. The interviewer asked Alex to explain why he questioned his ability to use the fraction, 2.5π/6. Alex explained:

> The 2.5π/6 was not like an option in my head because that's not, like one of the things that are usually mentioned or like associated with like radians. The reason why I got there is because I knew it was in between π/2 and π/3 but with the list of all, like the radians that I know, π/2 and π/3 are, like right next to each other, and there's, like no whole number π over anything in between those two numbers.

Alex's explanation is in reference to a typical diagram of a unit circle (Figure 2), where π/3 and π/2 are represented without depicting other angles between them. Alex recognized the need for an angle between π/3 and π/2, which led him to try the value between 2π/6 and 3π/6. He concluded that the angle would be 2.5π/6.

Mathematical Ways of Thinking about Light Reflection

In this section, we provide an analysis of Alex's ways of thinking as he makes sense of the scientific phenomenon of light reflection. We describe Challenge 3 of the *Radian Lasers* which could be solved by positioning the laser downward at an angle that is -π/6 radian, with the mirror being angled at a 5π/12 radian, to reflect the laser beam to the third target at the bottom (Figure 6).

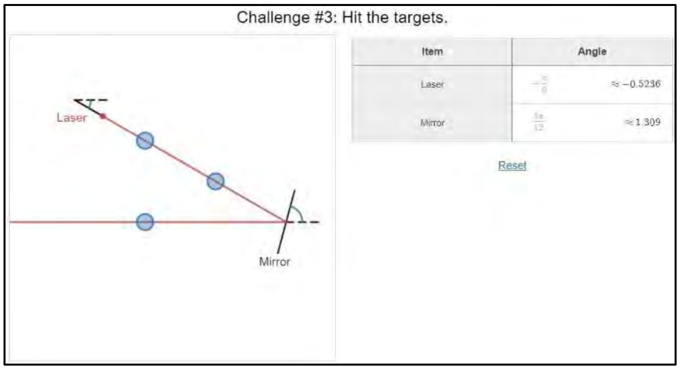

Figure 6. A possible solution for Challenge 3 from the *Radian Lasers* task

Alex used diagramming to represent his visualization of the challenge as alternate interior angles, then extended his diagramming with a focus on the mirror (Figure 7). Alex explained that "this entire thing [points to the mirror in Figure 7] is π, and so then I tried to find out what these two [the angles of incidence and reflection relative to the mirror] would be remaining, and I got … 5π/12 because there's two of them so I have to split the angle in half, because these two angles are equal." Alex's explanation of the mirror as a straight angle where its measure represents the "entire thing is π," suggests his thinking about the straight angle as a whole. Alex then decomposed the straight angle into π/6, which he concluded from the alternate interior angle theorem, and two angles that "would be remaining." Alex's elaboration demonstrates his attention to symmetry as he has "to split the angle in half because these two angles are equal."

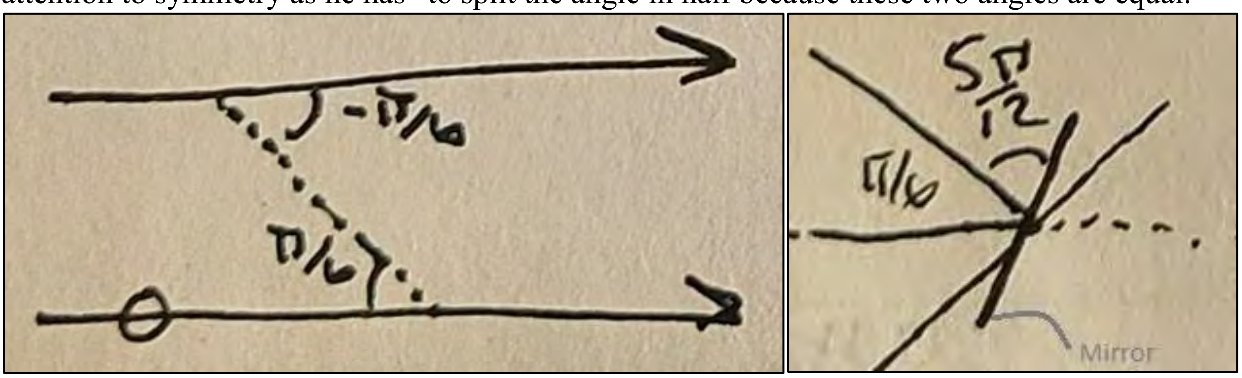

Figure 7. Alex's diagramming of Challenge 3 of the *Radian Lasers*

This example illustrates Alex's mathematical ways of thinking to make sense of the angle at which to situate the mirror. Alex's ways of thinking involved mathematical concepts (i.e., alternate interior angles), as well as various spatial ways of thinking (i.e., diagramming, decomposing, and symmetrizing). Alex went further to describe the light reflection principle from a mathematics perspective. Specifically, when Alex described his diagramming for Challenge 3 (Figure 7), he pointed at the angle the laser makes with the mirror upon reflection and stated, "this whole entire angle is π/6. I know the bisector, it, each angle would be like π/12. Then I know that this angle bisector is perpendicular with, you know, the mirror." Alex's description of the bisector of π/6 as "perpendicular" to the mirror illustrates his mathematical thinking about the science of light reflection, which he referred to as "adjust[ing] for mirror logic." The significance of the angle's bisector is because the angle the laser makes upon reflecting from the mirror (i.e., π/6) is the sum of the angles of incidence and reflection relative to the normal ($\gamma = \delta$ in Figure 1). However, Alex described the mathematical property of the normal line as perpendicular to the mirror, instead of using the science terminology. Alex's explicit description of the science of light reflection through his mathematical perspectives was not elicited by the interviewer. Alex utilized his mathematical ways of thinking to make sense of the principle of light reflection. While Alex did not explicitly use the terminology, "angle of incidence" or "angle of reflection," he was meaningfully incorporating mathematical thinking to make sense of the science of light reflection.

Discussion

To date, there are few iSTEM curriculum materials that emphasize mathematical concepts as the anchor discipline (English, 2016; Fitzallen, 2015), despite evidence of the benefits of iSTEM curricular approaches on mathematical achievement and development of mathematical understanding (Stohlmann, 2018). Additionally, there are few opportunities to engage in spatial thinking in schools (Clements & Sarama, 2011; Sinclair & Bruce, 2015; Whiteley et al., 2015), despite the role of spatial thinking in understanding both mathematics (e.g., Mix, 2019) and other disciplines (e.g., Newcombe & Shipley, 2015; Pruden et al., 2011). Our report illustrates a purposeful integration approach with a focus on mathematics in the context of science within an iSTEM unit. We argue that iSTEM experiences that foreground mathematics can meaningfully contribute to mathematical thinking, in addition to enhancing how mathematics relates to other STEM disciplines. The *Radian Lasers* as an iSTEM approach that emphasized mathematics provided Alex, a PMT, the opportunity to utilize mathematical and spatial ways of thinking about radian angle measure (e.g., alternate interior angle, perpendicular lines, angle bisector, visualization, diagramming, comparing), and to relate angle measure to light reflection principle.

Previous studies suggest that PMTs' thinking about radian angle measure is limited to special angles expressed in terms of π (Akkoc, 2006; Fi, 2003) and calculational strategies using the unit circle (Moore et al., 2016). Similarly, Alex initially referred to some of the special angles on the unit circle. However, the *Radian Lasers* task constrained these established ways of thinking as Alex was not able to only depend on few special angles. Alex used spatial comparison to think beyond the special angles that are associated with the unit circle. This suggests that the *Radian Lasers* as an iSTEM activity that focused on radian angle measure in a science context provided an opportunity for Alex to reason about radian angle measure beyond the special angles. Alex's use of multiple spatial ways of thinking reflects Davis et al.'s (2015) discussion that the spatial ways of thinking do not exist in isolation of each other and/or in isolation of other mathematical ways of thinking. Specifically, to reason about the functionality of the mirror, Alex used diagramming as a spatial way of thinking in relation to a mathematical concept (i.e., alternate interior angle theorem), and in relation to other spatial ways of thinking (i.e., Symmetry and Decomposing-Recomposing).

Alex's mathematical ways of thinking assisted him in not only applying mathematical content and processes (e.g., alternate interior angles, spatial thinking), but also in making sense of light reflection. The iSTEM activity in which Alex engaged brought mathematics to bear in a situation that represents a scientific phenomenon, which allowed for the construction of a relationship between a scientific phenomenon and a mathematical concept. Our findings align with English (2016) and Fitzallen's (2015) call for educators and curriculum developers to capitalize on iSTEM approaches that emphasize mathematics, as well as Li et al.'s (2019) call for research that explores thinking in iSTEM contexts. The report illustrates how iSTEM approaches that foreground mathematics have the potential to support learners' thinking of not only mathematical content, but also meaningful mathematical applications and connections to other STEM disciplines.

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