EXAMINING THE NATURE OF PEDAGOGICAL CONTENT KNOWLEDGE (PCK) WITH A VALIDATION ARGUMENT FOR THE PCK-FRACTIONS MEASURE

Maryam Zolfaghari
Kent State University
mzolfagh@kent.edu

Karl W. Kosko
Kent State University
kkosko1@kent.edu

Christine K. Austin
Kent State University
causti17@kent.edu

This study presents an extension of the validity argument for the PCK-Fractions measure. PCK-Fractions is designed to assess the effectiveness of professional experiences in facilitating teachers' pedagogical content knowledge (PCK) for children's fraction reasoning in grades 3-5. We examined data across 101 participants from two Midwest universities including non-education majors, education majors, and education majors with grade 3 to 5 math field experiences. Results provide additional validity evidence for the PCK-Fractions. Namely, evidence indicates differences in scores between groups of participants—preservice teachers grade 3 to 5 field experience had higher scores than those without, and all preservice teachers had higher scores than non-education majors.

Keywords: Mathematical Knowledge for Teaching; Rational Number.

Introduction

Mathematical knowledge for teaching (MKT) is a practice-based theory that investigates the nature of professional knowledge of mathematics that teachers use to make effective instructional decisions to help students' learning (Ball & Bass, 2002, Ball, Thames, & Phelp, 2008). MKT as a theory and set of constructs has been largely developed and disseminated with quantitative assessments across mathematics content. For instance, many scholars designed different kinds of MKT assessments in geometry, rational numbers, and number sense from grades K to 12 (Herbst & Kosko, 2014; Hill et al., 2008; Kazemi & Rafiepour,2018; Khakasa & Berger, 2016). MKT includes two primary domains of content knowledge (CK) and pedagogical content knowledge (PCK). While both domains are considered vital to effective teaching, most MKT assessments either target teachers' CK explicitly (Lo & Luo; 2012), or include both CK and PCK in the same assessment (Depaepe et al., 2015). There has been little attention to designing MKT assessments that focus on PCK as a domain worthy of investigation on its own (Copur-Gencturk et al., 2019; Hill et al., 2008; Zolfaghari et al., 2021). Rather, by incorporating PCK assessments within efforts to develop CK measures, many items designated as 'PCK items' have later been found to assess CK instead (Copur-Gencturk et al., 2019; Hill et al., 2008). This led to Zolfaghari et al. (2021) focusing on a PCK exclusive measure for fractions. Piloting items with a focus on the 'task of teaching' of assessing children's fraction reasoning resulted in a more accurate representation of the domain.

The current study is our continued effort to validate our framework for how PCK for Fractions develops, and our associated construct map for the PCK-Fractions measure (Zolfaghari et al., 2020, 2021). In this validation process, we hypothesize that PCK develops in a particular way that is useful in designing items of varying difficulty – described as *construct maps* within validity argument literature. We also found that exposure to coursework and field experience had a positive association with PCK scores. However, there was a need to better examine this phenomenon with a larger sample. Given prior results (Zolfaghari et al., 2020; 2021), we conjecture individuals' experiences inform the development of their PCK. In particular, we sought to understand the degree to which majoring in teacher education and having certain forms of field experience affected PCK for fractions.

Theoretical Framework

MKT is "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill et al., 2005, p.373). It involves abilities such as analyzing and interpreting students' reasonings and determining the related materials and information based on that. Mastering these skills promotes good instruction and, as a result, effective mathematics learning for students. The emphasis on knowledge of teaching a subject was primarily introduced by Shulman (1986) as PCK. Later, Ball and colleagues applied Shulman's theory to mathematics, thus framing MKT (Ball et al., 2008). MKT contains two primary domains of CK and PCK, with each of these domains consisting of several subconstructs (Hill et al., 2008b). To explore teachers' MKT, various assessments were designed within several mathematical topics, including geometry (Herbst & Kosko, 2014), rational numbers (Kazemi & Rafiepour, 2018), multiple mathematical topics at the secondary level (Khakasa & Berger, 2016) and elementary level (Hill et al., 2008), statistic subject (Siswono et al., 2018), and so forth. These various scholars explored several components of MKT, with many finding that MKT scores aligned with teachers' professional experience (Herbst & Kosko, 2014) and effective instruction (Hill et al., 2008).

MKT is professional knowledge; thus, various forms of professionalized experience have been found to affect and/or facilitate MKT. For instance, in examining the effect of types of experiences on teachers' MKT, Hill (2010) found characteristics such as grade taught, math content course, years of experiences, and math self-concept associated with teachers' MKT. Although these associations vary in terms of their strengths. Jakobsen et al. (2011) noted the grade level at which the teachers taught was related to their MKT scores. Similarly, in the study of mathematic teachers grade 3-7, Copur-Gencturk (2020) found that teachers with experiences teaching higher grades had stronger MKT scores. However, the number of years of mathematics teaching had a weak associate with teachers' MKT scores.

Research on MKT for fractions typically includes both domains of CK and PCK (Depaepe et al., 2015; Tirosh, 2000; Trobst et al., 2018). For instance, in studying PSTs' knowledge of teaching fractions, Tirosh (2000) found that most PSTs know how to solve fraction division (CK) but are unable to explain children's strategy or misconception (PCK). Similarly, examining secondary and elementary preservice teachers, Depaepe et al. (2015) noticed that PSTs with significant differences in demonstrated CK did not demonstrate differences in their PCK. A common premise across such studies is that CK is a prerequisite for higher PCK. Indeed, in describing PCK applied to assessing a student's error, Hill et al. (2008) noted that "teachers must be able to examine and interpret the mathematics behind student errors prior to invoking knowledge of how students went astray" (p. 390). Yet, Trobst et al. (2018) found that PSTs were able to increase their PCK despite limited CK. Coupled with Depaepe et al.'s (2015) findings of no statistical relationship between the constructs, there appears to be inconsistent evidence towards the common stance that CK is a prerequisite for PCK.

To be clear, there is a large body of evidence that supports a relationship between CK and PCK (Agathangelou & Charalambous, 2020; Depaepe et al., 2015; Hill et al., 2008). However, various findings in reports such as those described in the prior paragraph should not be disregarded. We believe the inconsistent findings in the literature point to an issue noticed by Hill et al. (2008) and expanded upon by Copur-Gencturk et al. (2019): development of PCK measures has focused more on CK than PCK and the resulting assessments have led to conflicting reports and an incomplete understanding of PCK as a theoretical construct. For this reason, PCK-Fractions was designed exclusively to focus on PCK (Zolfaghari et al., 2020; 2021), with an initial focus on the PCK sub-domain of knowledge of content and students (KCS). This targeted focus allowed us to decrease the risk to validity in designing unintended items that might measure CK instead of PCK.

Measuring PCK-Fractions

The intended purpose, or use, of the PCK-Fractions measure is to assess the effectiveness of professional experiences in promoting or facilitating teachers' PCK for upper elementary fractions concepts. The present version of PCK-Fractions focuses on the KCS domain, and represents one of several studies that collectively construct a validity argument for PCK-Fractions (Zolfaghari et al., 2020; 2021). Hill et al. (2008) define KCS as "used in tasks of teaching that involve attending to both specific content and something particular about learners" (p. 375). *Tasks of teaching* are the fundamental means Ball and colleagues have designed items for MKT, focusing on what a teacher must do in a specific aspect of the profession. For KCS, we focused on assessing children's reasoning as the task of teaching for fractions. Zolfaghari et al. (2021) initially adapted the sequence children learn certain fraction concepts but found that "student actions being assessed [were] a better explainer of why certain items have different difficulties" (p. 241). Table 1 presents a construct map developed from the initial pilot data (Zolfaghari et al., 2020; 2021). At Level 1, teachers are able to assess how children partition, fragment, or fair share fractional parts. At Level 2, teachers assess whether and how children coordinate parts to whole. Specifically, children may or may not have developed part-whole reasoning, but a teacher is able to distinguish between children's actions demonstrating such reasoning or not. Level 3 demonstrates an ability to assess how children compare and use different fractions. Level 4 focuses on a teachers' ability to assess how children coordinate non-unit fractions with the whole – actions often demonstrated in fraction multiplication and division.

Table 1: Construct map for PCK-Fractions.

Level	Description
Level 1	Assess children's creation and/or use of fractional parts.
Level 2	Assess children's coordination of parts and of the whole.
Level 3	Assess children's creation and use of non-unit fractions & comparison of fractions.
Level 4	Assess children's coordination of non-unit fractions with the whole & comparison of fractions and wholes.

Key in understanding the role of the construct map presented in Table 1 is that it is child-focused, not task-focused. Rather, a teachers' ability to assess children's reasoning begins with a focus on the child. As an example, consider item F44 in Figure 1. The mathematical task at-hand is fraction addition that requires the child to convert to a common denominator to obtain a correct answer. However, that's not what this child does. Rather, Tim counts the total shaded and unshaded and draws a representation illustrating that total. Thus, Tim is coordinating parts of the fraction, but without maintaining the whole. A teacher correctly assessing Tim's reasoning is, therefore, at Level 2 or higher. This focus on the student's reasoning within a mathematical task was key in aligning items with the construct map (Table 1). Additionally, all PCK-Fraction items were designed as multiple choice. Most items followed a common template in wording, as evidenced in responses for F44 (Figure 1). Specifically, response options were adapted from fraction diagnostic literature written for teachers (i.e., Battista, 2012; Hackenberg et al., 2016) and were listed in order of lower to higher level reasoning evidenced by the student depicted in the prompt. This design was purposeful, as prior data collection suggested randomization of such levels and changes in language decreased consistency in responses across items (Zolfaghari et al., 2020; 2021).

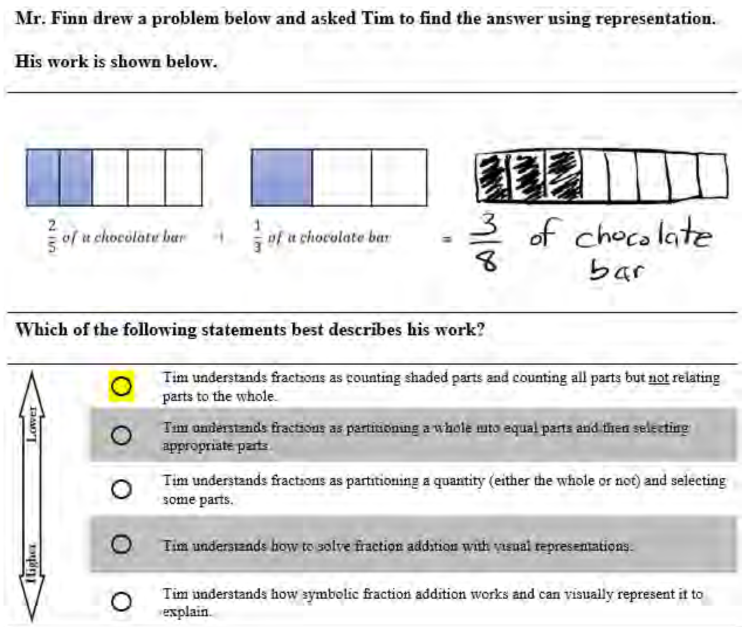

Figure 1: Example item (F44) from PCK-Fractions with correct response highlighted.

As noted previously, the present study is one of several papers that present evidence towards a validity argument for PCK-Fractions (Zolfaghari et al., 2020; 2021). Validity arguments consist of various claims regarding the validity of a measure, and evidence that warrants such claims (Kane, 2012). Further, such arguments are conveyed through "the accumulation of evidence from various sources for claims" (Krupa et al., 2019, p. 11) and "should occur over several studies in order to provide adequate warrants for claim of inference for a measure" (Kosko, 2019, p. 19). Prior validity evidence for PCK-Fractions suggests a unidimensional construct that aligns with the construct map presented in Table 1 (Zolfaghari et al., 2021). There is evidence that exposure to mathematics methods coursework facilitates PCK-Fractions, but prior samples also represented skew relative to higher scorers (Zolfaghari et al., 2020; 2021). Rather, Zolfaghari et al. (2021) identified a need for sampling participants with a much wider range of ability to better support the validity argument for PCK-Fractions. The present study focused efforts at recruiting a sample we believed would have a wider range in ability, but in doing so we also sought to align this range in ability with the construct map itself. Rather, we sought to better understand how participants with little to no exposure to teaching (outside of once being an elementary student) and those with more experience in such contexts would respond to our items. In doing so, we also sought to understand how responses aligned with our construct map. These efforts fulfill the purpose of the study which is to understand the effect of different types of professional experience on the PCK-Fractions measure by investigating the validity argument process. Particularly, we addressed the following research question:

How does the construct map for revised PCK-Fractions measure align with validity evidence for individuals with different professional teaching experiences?

Method

Participants included 101 undergraduate students enrolled in two Midwest universities. To ensure a sample with a wider range of PCK-Fraction scores, both high and low, we recruited undergraduate students enrolled in a Marketing 101 course (*n*=39) who were not, had not, and did not plan to major in education. Non-education majors primarily included financing, marketing, fashion design, and business. We also recruited participants from 36 early childhood (certification preK to grade 5) and 26 middle childhood majors (certification grades 4-8) across both universities (*n*=62). Because grades 3-5 (the focus grade band for PCK-Fractions) was included in the teacher licensure for both majors, these participants were asked to report whether, and in which grade levels, they had field experience. This resulted in 21 education majors reporting having grades 3-5 field experiences that included teaching mathematics (20.8% of total sample). Across the entire sample, almost all participants self-identified as white (97%). Also, 79.2 % of the participants self-identified as female, 18.8 % male and 2% nonbinary.

Measure

As previously described, PCK-Fractions items were designed to measure teachers' PCK for grades 3-5 students' fraction reasoning. Prior versions of the PCK-Fractions measure included 15 questions, which included several 'multiple-response' items that allowed participants to select 'all that apply' (Zolfaghari et al., 2020). However, following the pilot of the measure, such items were either revised into multiple-choice items or removed to reduce variance in response data. Previous Rasch modeling of the PCK-Fractions measure indicated sufficient item reliability of .90, but less than ideal person reliability of .41 (Zolfaghari et al., 2021). A primary reason identified for the poor person reliability was a negative skew in participants' scores with "75.3% of participants having a score above 0.00, or average ability" (p. 238). The current version of the measure includes 19 multiple-choice items (see Figure 1 for an example item). As noted earlier, these items were revised to further improve item reliability, reduce variance in responses, and better align theoretically to the improved construct map. Responses were all coded dichotomously (0 = incorrect, 1 = correct) for Rasch modeling.

Analysis and Findings

This paper examined validity evidence for revised PCK-Fractions and its construct by adopting the *Standards for Educational and Psychological Testing* (AERA et al., 2014) and building upon the prior validity evidence for PCK-Fractions measure (Zolfaghari et al., 2020; 2021). For the present study, we focused particular on validity evidence related to response processes andinternal structure. Evidence towards *response processes* focuses on how participant responses correspond with the intended theoretical design of the items. Evidence towards *internal structure* was used to provide study how response processes correspond to the conceptual framework for the revised PCK-Fractions' measure. The Rasch analysis and the data from one-way ANOVA allowed us to seek the validity evidence for revised PCK-Fractions as well as how participants' different level of experiences related their scores in PCK construct map.

Rasch Modeling

Rasch modeling was conducted to examine internal structure validities for PCK-Fraction assessment. Namely, we used item and person reliability aligned with unidimensionality and fit statistics to examine the validity of survey (AERA et al., 2014). The initial item analysis indicated the acceptable item reliability of .95, which exceeded the acceptable threshold of .90 (Linacre, 2021). Likewise, the item separation index showed a sufficient value of 4.23 (above 2.0 implies a good differentiation of item difficulty). In particular, this item separation value (4.23), indicates the items can be differentiated into four different tiers, which coincidentally aligns with the four levels of construct map (see Table 1). This result suggests that the hierarchy of difficulty of the items for PCK-Fractions would remain constant for different samples in similar contexts. Additional psychometric results from the average mean square for item infit (MNSQ= 1.00, Z=.00) and outfit (MNSQ= 1.01, Z= .10) provide another indicator that the data fit the model. These findings suggested that our revised PCK-Fractions measure represents a significantly wider range of item difficulties than the initial PCK-Fractions—improving from an item reliability of .90 and item separation of .30 (Zolfaghari et al., 2021) to item reliability of .95 and item separation of 4.23. Collectively, these results provide validity evidence that the measure is aligned to the construct map in Table 1.

Despite otherwise ideal psychometric data, the person reliability estimated for participants was .42 which is lower than the acceptable threshold of near or above .80. Further analysis showed that, although the person reliability was low, the average mean square for person infit (MNSQ= .99, Z=.00) and outfit (MNSQ= .99, Z= .00) aligned with Rasch Model expectation. This indicated that all participants' responses to the items behaved as expected (Bond & Fox, 2015). Two potential reasons for a low person reliability are: 1) the spread of item difficulty is too narrow or 2) the ability range of participants assessed is too narrow. In order to examine these issues, the Wright Map was investigated (see Figure 2). As indicated in the Wright Map, items are almost evenly distributed between -2.00 and 2.00 logits. Recalling that item reliability and separation index values were both ideal, this data suggests that the spread of item difficulty was not too narrow. Thus, we further examined the second most common reason for low person reliability.

Mean scores for participants were near the model average score of 0.00 (M = -.14, SD = .65), which did not clearly indicate an issue with skew in responses. To better understand whether, or how, the range of participants may have been too narrow, we visually examined the Wright Map, but distinguished participants into three groups: non-education majors, education majors without grades 3-5 experience, and education majors with grades 3-5 experience. We also juxtaposed a Box-and-Whisker plot to help explore such patterns. Notably, the third quartile is relatively smaller than the fourth quartile, suggesting some skew in the upper half of scores. In particular, this data suggests a need for more participants with higher scores to improve person reliability. The Wright Map in Figure 2 suggests that a targeted sampling of participants with grades 3-5 experience may be warranted in this regard. To test this conjecture, we compared the person reliability of the Rasch model ran with and without participants having grades 3-5 experience. Notably, excluding such participants reduced person reliability from .42 to .37. Note this reduction is not due to sample size but is due to a lower variance in scores. Thus, despite a sample of 101 participants with what visually appears to be a normal distribution (see Figure 2), results suggest a need for more teachers with grades 3-5 experience for our sample to be representative enough to reliably measure PCK across samples.

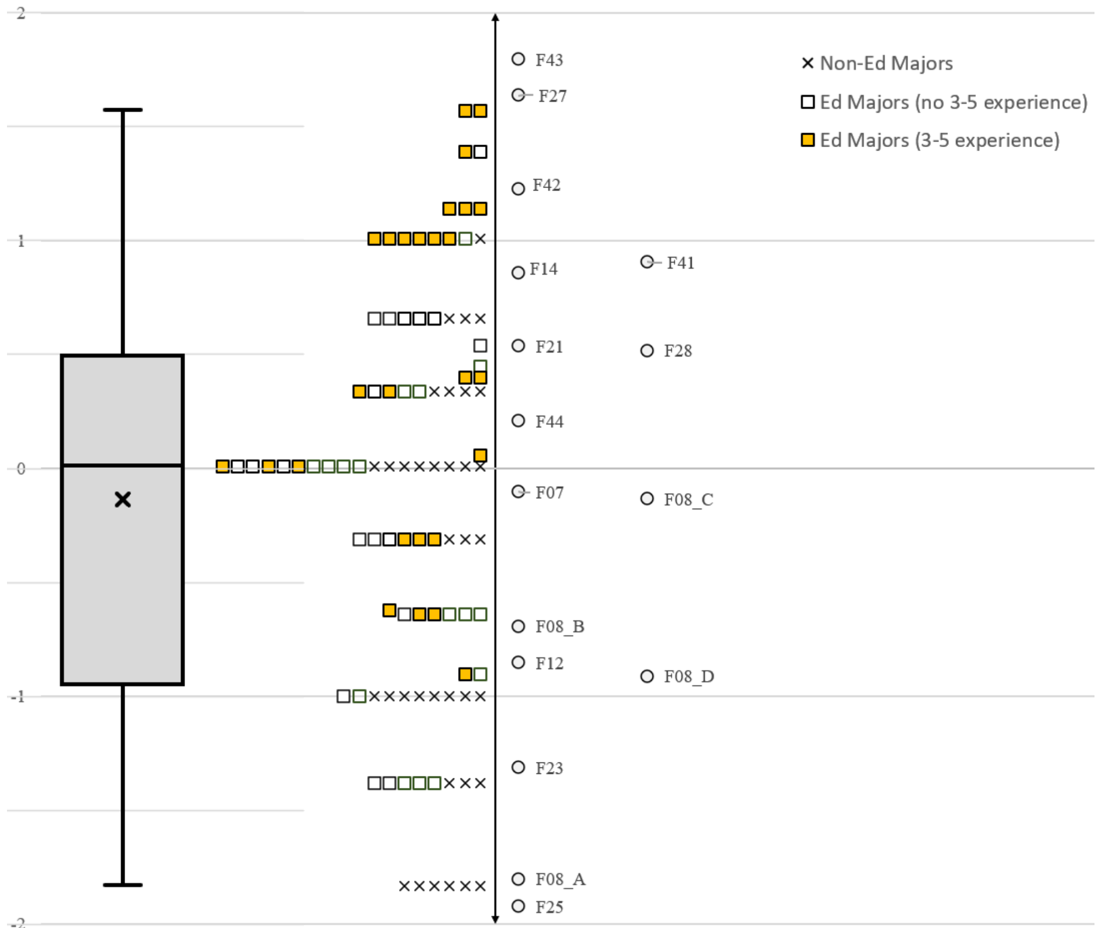

Figure 2: Wright Map for PCK-Fraction measure

ANOVA

The use of a one-way between subjects' analysis of variance (ANOVA) provided evidence to our claim in which differences in PCK scores are associated with participants' professional experiences. We compared Rasch PCK scores of non-education majors, education majors without grades 3-5 field experience, and education majors with such experience. Results from ANOVA found that there was a statistically significant difference between the three groups [F (2, 94) = 11.320, $p < 0.001$]. A Tukey HSD post hoc analysis indicates a statistically significant difference ($p < 0.001$) between the education majors with grade 3-5 experience ($M = 0.16$, $SD = 0.62$) and non-education majors ($M = -0.65$, $SD = 0.74$), as well as statistically significant difference ($p = 0.034$) between education majors with experience and education majors without grades 3-5 field experience ($M = -0.27$, $SD = 0.63$). Comparatively, the Tukey HSD post hoc analysis did not detect a statistically significant difference between non-education majors and education majors without 3-5 experience ($p = .057$). However, non-education majors did have lower PCK-Fraction scores than education majors without grades 3-5 experience, and the p-value was marginally non-significant at the .05 level. A slightly larger sample may have yielded a statistically significant result. Results indicate that differences in PCK-Fractions scores is associated with participants' experience with grades 3-5 students. However, the content and quality of such field experiences were not examined, and further investigation is warranted.

Discussion

This study explored differences in participants' PCK-Fractions scores considering varying professional experiences. Results suggest participants with field experience (particularly with grades 3-5) demonstrated higher PCK scores for fractions than participants without. Additionally, while marginally not statistically significant at the .05 level, education majors tended to have higher PCK scores than non-education majors. Prior studies indicated the importance of teachers having different types of professional experiences including coursework, higher CK, and experience teaching specific grade levels (Agathangelou & Charalambous, 2020; Copur-Gencturk, 2021; Hill, 2010). Our results support some of this prior literature (Copur-Gencturk, 2021; Hill, 2010) as we observed that PSTs' PCK may benefit from having upper elementary field experience. To our knowledge, this paper is the first to compare PCK scores of education and non-education majors. Although it is difficult to distinguish how much upper-elementary field experience may have interacted with some of our participants' coursework, there does appear to be some benefit to pedagogy courses prior to such field experience with education majors lacking grades 3-5 field having much higher PCK scores (M = -.27), than non-education majors (M = -.65).

Results from our psychometric analysis provided strong validity evidence that the PCK-Fractions measure is aligned well with our construct map (see Table 1). Recall that this version of our construct map was a result from analysis in our pilot study (Zolfaghari et al., 2021). Thus, in finding that items designed at specific levels appeared to ordinally fall in the sequence they were designed (see Figure 2), and that our item separation statistic allows for distinguishing four levels of items, we believe there is strong validity evidence for our construct map. This set of findings is non-trivial, as PCK in general is undertheorized. By successfully designing items as child-centered and not task-centered, we believe this study provides guidance to those constructing PCK measures in similar or other domains. Further, the construct map itself provides a framework for examining PSTs' PCK for fractions with or without our PCK-Fractions measure. For example, a PST's explanation of a child's mathematics when viewing a video of their own teaching could be examined in relation to the construct map in Table 1. Such research is needed to further theorize PCK for fractions, as well as other mathematical domains.

The validity evidence from the present study supports and expands that of the initial validation process for PCK-Fractions (Zolfaghari et al., 2020; 2021). Specifically, validity evidence suggests the items in the PCK-Fractions measure are strong and do not appear to need any further revision currently. However, there is a need to sample a wider range of teachers with varying levels of upper-elementary experience. This may include inservice teachers, as well as PSTs' with said experience. Future research in this area may also consider teachers' professional beliefs and other facets.

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