TYPES OF MATHEMATICAL REASONING EVIDENCED BY A MIDDLE SCHOOL TEACHER IN PATTERN GENERALIZATION

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The objective of this article is to describe types of mathematical reasoning evidenced by a middle school mathematics teacher, when answering two generalization questions in a figural pattern generalization task, related to quadratic sequences. Reasoning is delimited from teacher's arguments, reconstructed from a theoretical-methodological proposal that combines Peirce's definitions with some elements of Toulmin's argumentative model. The results show that the teacher evidenced abductive, inductive, and deductive reasoning, based on cognitive actions such as the decomposition of the figure, strategic counts, recognition of the behavior of the figural pattern, formulation, verification, and validation of conjectures.

Keywords: reasoning, argumentation, figural pattern, teacher.

Introduction

Reasoning is one of the fundamental cognitive processes in teaching and learning mathematics. Organizations such as the National Council of Teachers of Mathematics ([NCTM], 2000) conceive of reasoning as a skill or proficiency that must be developed at all levels. On this regard, teacher assumes an important role to promote it in students through different mathematical experiences in teaching conditions (Brodie, 2010). Research has reported that teacher have limited knowledge both the nature of reasoning and pedagogical knowledge to support development in their students (Clarke et al., 2012). On the other hand, the investigations on mathematical reasoning that have taken as a case study to the preservice teacher and in service have been of an exploratory nature (e.g., Bozkus & Ayvaz, 2018; Mata-Pereira & Ponte, 2017) and teaching proposals to develop it (e.g., Bragg & Herbert, 2018).

The study of mathematical reasoning has focused mainly on the central and classic forms of reasoning: abductive, inductive, and deductive (Arce & Conejo, 2019; Conner et al., 2014; Soler-Álvarez & Manrique, 2014), which, in their most have showed cognitive processes developed by middle school students and future mathematics teachers when solving tasks in arithmetic and geometry context. They connected the structures of Peirce and Toulmin in their analysis to show through the arguments, the reasoning they evidenced when solving them. However, the same has not happened with the study of these forms of reasoning about the in-service mathematics teacher. Patterns are one of the contexts that favor the study of reasoning because it contributes to the formulation and justification of generalizations (e.g., Barbosa & Vale, 2015; Rivera, 2013; Mata-Pereira & Ponte, 2017). Kirwan (2015) states that if students are expected to generalize and justify, then it is important to reflect on teachers' thinking about these concepts. The research is focused in describing the mathematical reasoning that is evidenced a middle school mathematics teacher in the framework of the generalization of quadratic patterns. Patterns are one of the contexts that favor the study of reasoning because it contributes to the formulation and justification of generalizations (e.g., Barbosa & Vale, 2015; Rivera, 2013; Mata-Pereira & Ponte, 2017). The research question of the study is, what types of mathematical reasoning does a mathematics teacher evidence when solving a quadratic pattern generalization task?

Theoretical Framework

Mathematical reasoning

In mathematics education there is no consensus on the definition of mathematical reasoning, because it is a polysemic term that encompasses a wide range of mathematical practices (Conner et al., 2014; Yackel & Hanna,2003). In this research, it is understood as any action or procedure that allows obtaining new information from: a) previous or known information, which corresponds to that provided by a statement as initial hypotheses; b) resolution of a problem or c) that derived from previous knowledge (Saorin et al., 2019; Torregrosa et al., 2010). In addition, it is related to other processes, such as inference, justification, and generalization (McCluskey et al., 2016).

Argumentation and argument

Argumentation is a sequential process that allows conclusions to be inferred from premises, through interactive communication between people (Toulmin, 1958/2003). Mathematical reasoning groups a set of arguments based on a series of propositions that implies a conclusion inferred from the data (Toulmin et al.,1984). The argument is a complex data structure that involves a movement from data (D) to a conclusion (C). The movement of the evidence to the conclusion is the certainty that the argumentative line has been carried out successfully, a movement or connection that is allowed by the warrant (W), which in turn has a backing (B), a modal qualifier (Q) that indicates the degree of strength or probability of the assertion and occasionally, objections or refutations may be presented (R). Toulmin´s model is considered to analyze the content of the arguments (Inglis et al., 2007) (see Figure 1).

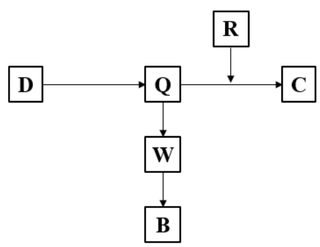


Figure 1: Toulmin's argumentative model (Inglis et al., 2007).

The study of arguments in this research is based on the core of the argumentative Toulmin´s Model (See figure 2). This structure allows to identify the typology or mathematical reasoning.

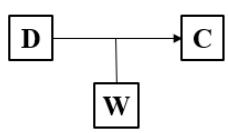


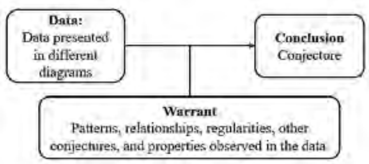
Figure 2: Toulmin's basic argumentative model.

Types of Reasoning

Mathematical reasoning is classified into three types: abductive, inductive, and deductive. *Abduction* and *induction* represent plausible and experimental reasoning, supported by the formulation and verification of conjectures, while *deduction* consists of validating the conjecture to warrant its veracity (Polya, 1966). We are based on the theoretical-methodological proposal of

Soler-Álvarez and Manrique (2014) (see Table 4) for the identification and description of the types of mathematical reasoning.

Table 1: General schemes of the types of mathematical reasoning (Soler-Álvarez & Manríque, 2014).

<table>
<tr><th>Abductive</th><th>Inductive</th><th>Deductive</th></tr>
<tr>
<td>
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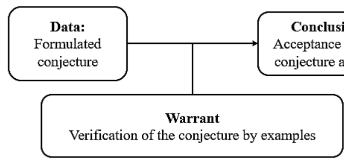
<td>
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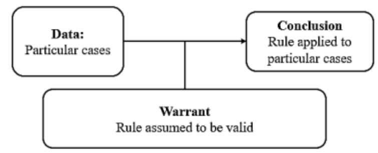
<td>
</td>
</tr>
</table>

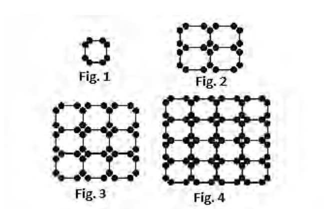
Generalization

When a conjecture expresses a plausible mathematical rule related to all the particular cases and it represents the mathematical pattern, we speak of generalization, which consists of the process of identifying a regular behavior in some particular cases of a sequence, in order to extend that identified regularity and construct an expression or general rule that represents and relates all the terms of the sequence (Radford, 2008).

Method

The research is a qualitative case study (Merriam & Tisdell, 2015). It was developed through a course-workshop (CW) in a virtual setting. Participants were 16 middle school mathematics teachers (MMT) from three Latin American countries. One of these teachers was selected for the analysis of the case study unit with the following criteria: a) to solve seven quadratic pattern generalization tasks and b) to participate in a semi-structured interview. For the purposes of this article, the analysis focuses on one of the tasks (see Table 2).

Tabla 2: Quadratic pattern generalization task.

<table>
<tr><th>Task of the points and sides</th><th>Characteristic</th></tr>
<tr>
<td>Considering the figures, answer the following questions and give a detailed argument for your answer.
a. Find a general rule for the total number of points in any figure.
b. Find a general rule for the total number of sides in any figure.

</td>
<td>Adapted from Rivera (2010).

The context of the task is a figural pattern. The explanation of the behavior of the pattern in any of its stages involved the construction of two general rules, one associated with the variable sides and the other with points.
Promotes the use of visual strategies.</td>
</tr>
</table>

Data analysis

The data analysis followed three stages: 1) reconstruction of teacher's argument from the task resolution process and the semi-structured interview. Toulmin's argumentative model (1958/2003) was useful in this process, 2) identification and description of the reasoning based on the typology of Soler-Álvarez and Manrique (2014) and 3) triangulation of data from previous stages, with expert researchers in argumentation, generalization and mathematical reasoning.

Results

The results are described by type of reasoning that the MMT evidenced when solving the task, which challenged him to build two general rules, one associated with the points variable and the other with the sides variable. In the construction of the general rule linked to the first type of variable, the teacher evidenced three forms of mathematical reasoning and two for the second.

Abductive reasoning

a) Reasoning in the points variable

In the construction of the general rule associated with the variable points, this way of reasoning appears when the MMT recognizes how it can explain the behavior of the pattern in each figure (see Figure 3). He identifies that the total number of points in the corners is a constant number, eight, that, on the sides, the number of points equals four times the figure number multiplied by three and in the center, it is four times the figure number minus one, squared. This fact constitutes a conjecture, which establishes three actions: a) to decompose the points of each figure in three parts: corners, sides and center, b) to count strategically the points in each of the parts, c) to establish a correspondence relation between the number of points of this decomposition, in relation to each figure.

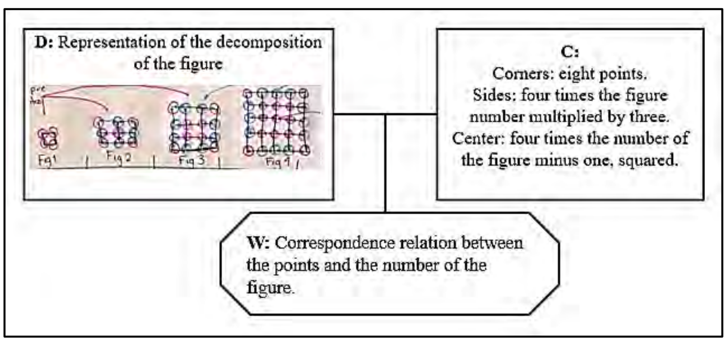


Figure 3: Abductive argument associated with the variable points.

The behavior identified by the teacher regarding the variable points when decomposed it into three parts, represents it through algebraic expressions where the figure number is written down in terms of n (see Table 3). The formulated conjecture is: $8 + 3[4(n-1) + 4(n-1)^2]$.

Table 3. Algebraic expressions about the points and the figure number.

Points	*Algebraic expressions*
Corners	8
Sides	$3[4(n-1)]$
Center	$4(n-1)^2$
Total (conjecture)	$8 + 3[4(n-1)] + 4(n-1)^2$

b) Reasoning in the variable sides

In the construction of the general rule associated with the variable sides, the abductive reasoning appears when the MMT recognizes that the increase of this variable from one figure to another has a quadratic type of behavior, that is, the number of the figure squared multiplied by two, plus twice the number of the figure. This fact constitutes a conjecture, which establishes four actions (see Figure 4): a) to count the total number of sides of each figure, b) to establish the first and second differences between the total number of sides of one figure and the next, c) to recognize that the first differences are not constant, and d) that the second differences are constant.

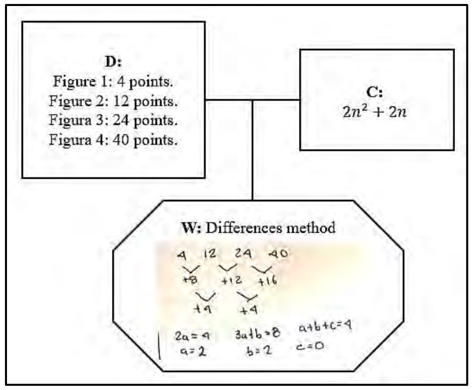


Figure 4: Abductive argument associated with the variable sides.

The behavior identified by the teacher with respect to the variable sides is based on the method of differences, which consists of determining the coefficients of the quadratic sequence. He represents them in terms of n. The formulated conjecture is: $2n^2 + 2n$.

Inductive reasoning

This type of reasoning was identified when the MMT evaluated the conjectures formulated by abduction within the framework of the variables points and sides of the figural pattern. Therefore, he relied on a double-entry table in which he recorded the total points and sides per figure. This evaluation consisted of substituting the total number of points and sides of four figures, from 1 to 4. These data are compared with those obtained by substituting the number of each figure (n) in the algebraic expression that he formulated by abduction (see Figure 5).

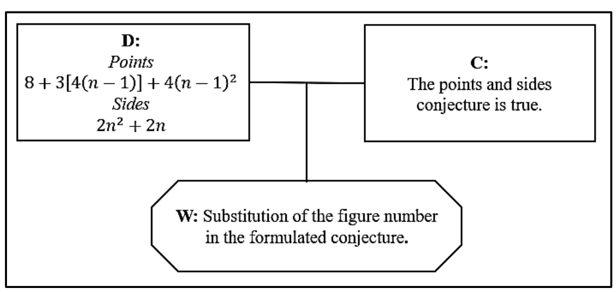


Figure 5: Inductive argument of the MMT in the context of the variables points and sides.

Deductive reasoning

This type of reasoning appears in the MMT when validating the conjecture associated with the points variable, obtaining the general expression. In addition, he used the method of differences to verify the general expression obtained (see Figure 6). The general expression obtained through the difference method is $4n^2 + 4n$, equivalent to $8 + 3[4(n-1)] + 4(n-1)^2$(see Table 3). He realized this by comparing them.

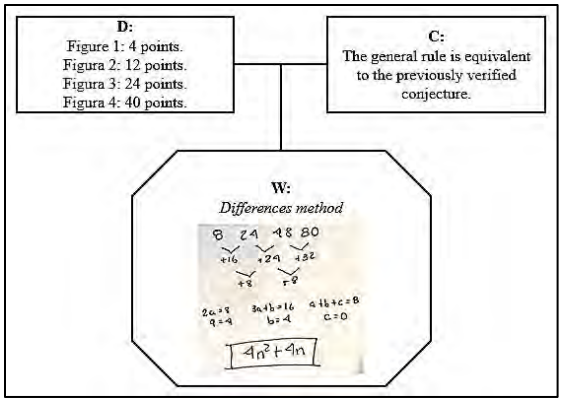


Figure 6: Deductive argument of the MMT on the variable of the points.

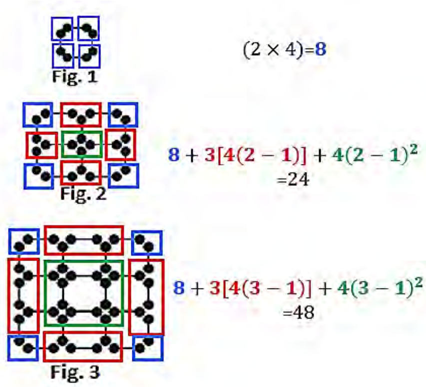
Discussion and conclusion

In this paper we described types of mathematical reasoning of a MMT when solving a task of generalization of figural patterns in the context of quadratic sequences. The mathematical reasoning evidenced by the MMT allows characterizing the thought process regulated by mental actions and relating it to the typology of reasoning (abduction, induction, and deduction). The way of reasoning abductively in the MMT begins with the observation of particular cases to formulate a conjecture. Then, inductively, he verifies the conjecture through the particular cases known, in order to verify its veracity. Finally, deductively, he validates the conjecture with known particular cases.

According to the characteristics and demands of the figural task, in the construction of the general rule associated with the points variable, the MMT evidences the three types of reasoning and in the sides variable, it reasoned abductively and inductively. In this sense, it is recognized that the types of mathematical reasoning can be presented in an ideal way and not necessarily in a linear way.

The context of quadratic pattern generalization contributed to the MMT evidencing different actions when solving the task in the figural context (see Table 4).

Table 4: Evidence of the reasoning by the MMT.

Type of reasoning	Cognitive Actions	
	Points	
Abductive	Decomposition of the figure, grouping and strategic counting of the points, identification of the regularity and formulation of the conjecture from the visual.  	Formulated conjecture $8 + 3[4(n-1)] + 4(n-1)^2$

<table>
<tr><td></td><td colspan="2">Sides</td></tr>
<tr><td></td><td>Total count of the sides by figure number, differences between terms, recognition of regularity and formulation of the conjecture from the numerical through the method of differences.

4 12 24 40
8 12 16 First difference
4 4 Second difference
Coefficients of the sequence in general terms $an^2 + bn + c$
$2a = 4$, $a = 2$ $3a + b = 8$, $b = 2$ $a + b + c = 4$, $c = 0$
$l_n = 2n^2 + 2n$</td><td>Formulated conjecture

$2n^2 + 2n$</td></tr>
<tr><td></td><td colspan="2">Points and sides</td></tr>
<tr><td>Inductive</td><td colspan="2">Total count of the points per figure and comparison with the results of the substitution in the points and sides conjecture.

<table>
<tr><td>Figure</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>Points</td><td>8</td><td>24</td><td>48</td><td>80</td></tr>
<tr><td>Sides</td><td>4</td><td>12</td><td>24</td><td>40</td></tr>
</table></td></tr>
<tr><td></td><td colspan="2">Points</td></tr>
<tr><td>Deductive</td><td>Difference method

4 24 48 80
16 24 32 First difference
8 8 Second difference
Coefficients of the sequence in general terms $an^2 + bn + c$
$2a = 8$, $a = 4$ $3a + b = 16$, $b = 4$ $a + b + c = 8$, $c = 0$
$p_n = 4n^2 + 4n$</td><td>The formulated conjecture has been justified by the method of differences, which are equivalent.
$8 + 3[4(n-1)] + 4(n-1)^2 = 4n^2 + n$</td></tr>
</table>

In the construction of the two rules of thumb in relation to the points and sides variables, the MMT involved his visual skills to recognize the behavior of the pattern. In the analysis of the points variable, the arrangements allowed him to establish correspondence relationships through decompositions of the figure, while for the sides variable, the MMT decided to work with the differences between the total number of points in each figure. According to Rivera (2010), working with figural patterns allows different interpretations of their behavior and the organization or configurations of the objects of some figural patterns are complex to interpret, even when their construction is well defined (Nuñez-Gutierrez & Cabañas-Sánchez, 2020).

In general, it is recognized that teaching experience and professional knowledge about mathematics, supported by the teacher's facts, conceptual images, and beliefs (Lithner, 2006), influence the choice of his strategies to formulate, verify and validate conjectures. Furthermore, it was evidenced that the mathematics teacher does not make his random choice of his actions but is supported by his knowledge of the subject and uses it to respond to the demands of the task, which, for his criteria, is the most accurate and quickest method.

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