EXPLORING TEACHERS' RESPONSIVENESS TO CHILDREN'S FRACTION THINKING AND RELATIONSHIPS TO FRACTION ACHIEVEMENT

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Identifying components of teaching that make a difference in children's learning is an ongoing challenge in our field. Focusing on teaching that is responsive to children's fraction thinking, we decomposed responsiveness into the instructional practices of questioning to support and extend children's thinking, noticing children's thinking, and anticipating children's thinking. We worked with 49 teachers in grades 3–5 in multiyear professional development and assessed their expertise in each of the practices. We also assessed their students' fraction achievement at the beginning and end of the school year. Correlational analyses revealed significant moderate relationships among teachers' expertise in the three practices, and a multilevel regression analysis revealed significant positive relationships for both expertise in teacher questioning and years of professional development with children's fraction achievement.

Keywords: Instructional Activities and Practices, Professional Development

An ongoing challenge in mathematics education is identifying components of teaching that make a difference in children's learning (Hiebert & Grouws, 2007). We take up this challenge guided by a vision of teaching in which children's mathematical thinking is centered and teaching is responsive to that thinking. By *responsive to children's mathematical thinking,* we mean teaching that elicits children's ideas and takes up and builds on those ideas as an integral feature of instruction (Richards & Robertson, 2016). This kind of responsiveness involves the continuous adjustment of decisions during instruction about what to pursue and how to pursue it in response to children's ideas.

A small number of studies have documented that teachers' instructional practices related to responsiveness to children's mathematical thinking are linked to children's achievement. For example, Webb and colleagues (2014) found that the more teachers engaged children in each other's thinking during instruction, the higher children's achievement was on a story-problem assessment. Bishop (in press) found that the more teacher discourse reflected uptake of children's ideas, the higher the gains were in children's proportional reasoning. (See also Fennema et al., 1996; Howe et al., 2019; Ing et al., 2015; Saxe et al. 1999).

We contribute to this body of work by presenting findings from our analyses of relationships between teachers' responsiveness to children's fraction thinking, which we decomposed into three instructional practices, and children's fraction achievement. The three practices were selected as a focus of our study because we considered them foundational to teachers' expertise in responding in the moment to children's fraction thinking. They include teacher questioning to support and extend children's mathematical thinking (Jacobs & Ambrose, 2008; Jacobs & Empson, 2016), teacher noticing of children's mathematical thinking (Jacobs et al., 2010), and teacher anticipating of children's mathematical thinking (Smith & Stein, 2018). Throughout the paper for the sake of brevity, we refer to these practices in shorthand as questioning, noticing, and anticipating, with the qualification that each focuses on children's mathematical thinking. We recognize other important aspects of teachers' responsiveness, such as to children's cultural, social, and linguistic identities (Parsons et al., 2018) but do not address them here.

Data for our study were drawn from a professional development (PD) design project in which we enrolled three successive cohorts of teachers to participate in up to three years of professional development (Jacobs, Empson, Pynes, et al., 2019). The PD was designed to support the development of teachers' expertise in responsiveness to children's fraction thinking, informed by research on the instructional practices and on children's fraction thinking (Empson & Levi, 2011). Our goals in this study were to explore relationships among the three practices comprising this expertise and between this expertise and children's fraction achievement. We collected data at the end of the third year of the project, in which teachers were enrolled in either their first, second, or third year of PD, to capture variation in expertise in questioning, noticing, and anticipating. Assessments given near the beginning and end of the school year documented children's fraction achievement at each point. These data allowed us to answer two questions: *(1) Do the instructional practices of questioning, noticing, and anticipating children's fraction thinking relate to each other? (2) Does teachers' expertise in questioning, noticing, and anticipating children's fraction thinking relate to children's fraction achievement?*

Conceptual Framework

Our conceptualization of responsiveness to children's mathematical thinking is informed by a theoretical view of teaching that foregrounds the work of teaching and its decomposition into instructional practices that are complex enough to authentically represent teaching but simplified enough to be accessible to teachers who are developing expertise (Grossman et al., 2009; Hiebert & Morris, 2012). The work required to enact a vision of teaching as responsive to children's mathematical thinking has been increasingly parsed by researchers (Boerst et al., 2011; Jacobs & Spangler, 2017; Franke et al. 2009; Munson, 2019). Although researchers have identified a number of potential practices, we selected three based on their connections to teachers' capacity to be responsive to children's mathematical thinking in the moment.

The first practice, *questioning to support and extend children's mathematical thinking,* involves making children's thinking visible during instruction and responding to that thinking in ways that consider children's existing understandings (Fraivillig et al., 1999; Jacobs & Ambrose, 2008; Jacobs & Empson, 2016). We have conceptualized the essence of this questioning as embodied by a teacher in conversation with children to explore their thinking about a mathematics problem—often a story problem—by posing questions to elicit children's thinking and pressing children for explanations of specific parts of their problem-solving processes (Jacobs, Empson, Jessup, & Baker, 2019). In these conversations, a teacher may also question to ensure children are making sense of a problem, link children's representations to the story context (if one exists), encourage children to consider other strategies, connect children's thinking to mathematical notation, or pose a related problem linked to children's understandings. Questioning is customized with respect to children's thinking and can be enacted during instruction in both one-on-one conversations with children as well as in conversations with groups of children, such as during whole-class discussions of children's strategies.

The second practice, *noticing children's mathematical thinking,* involves attending to and making sense of children's thinking in the moment. We have previously conceptualized noticing as a set of nested skills that are temporally and conceptually linked, which include attending to the details of children's strategies, interpreting children's understandings reflected in those details, and deciding how to respond on the basis of those understandings (Jacobs et al., 2010). Noticing is an invisible practice, in that it occurs prior to a teacher's observable response. Thus,

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teacher noticing is foundational for teacher questioning because, without noticing, teachers would not be able to question in ways that were customized with respect to children's thinking.

The third practice, *anticipating children's mathematical thinking*, involves envisioning how children might engage in solving a problem. We draw on a conceptualization of anticipating as teachers' consideration of the array of strategies that children would be likely to use for a problem prior to posing that problem. Anticipating orients teachers to possible conversations with children during instruction and can inform selecting and adapting problems, interacting during circulating, and planning for and facilitating discussions (Simon, 1995; Stein et al., 2008). Thus, teacher anticipating prepares teachers to notice and question.

This set of practices has three qualities which we argue are useful for enhancing teacher learning in PD and beyond. First, the practices are organized around a specific focus to lend coherence to the set. In our study, this focus was responsiveness to children's fraction thinking. Second, the practices are accessible to teachers as they are beginning to learn but also offer room for growth. They were therefore usable in all three years of our PD. Third, the practices are generative with respect to teachers' continued learning in that as teachers enact the practices in their classrooms, they have opportunities to not only support children's thinking but also improve their understanding of children's thinking and use this understanding to further develop expertise in the practices. Based on earlier findings about the generative nature of practices used by teachers to engage with young children's mathematical thinking (Franke et al., 2001), we conjectured that our practices would create similar opportunities for the teachers who completed our PD and continued to use the practices. In selecting a set of practices, we drew inspiration from a well-known precedent in mathematics education: the "5 practices," which are focused on the expertise needed to facilitate whole-class discussions of children's solutions to cognitively demanding tasks, and were also designed to support teacher learning (Stein et al., 2008).

Methods

Participants

The study was situated in three demographically diverse neighboring districts in a state in the southern region of the United States. A total of 49 teachers and 876 children were included in the analysis. The teachers represented a subset of the 92 teachers who were participating in the larger PD design project and were selected because they were working as classroom teachers in an upper elementary grade (3–5), were available to have one of their mathematics lessons observed, and had at least a third of the children in their classes who completed both the fall and spring administrations of the fraction assessment. Data were collected during one school year, when teachers were at the end of their first ($N = 15$), second ($N = 20$), or third ($N = 14$) year of PD.

The 49 teachers (42 females, 7 males) ranged in teaching experience from 2–36 years, with a mean of 12.4 years. The mean number of children per class who completed the fraction assessment was 18 (72% of the class) and ranged from 8–27 (33%–96% of the class).

Professional Development

The PD in which teachers were participating at the time of data collection was focused on teachers' responsiveness to children's fraction thinking, conceptualized in terms of the three instructional practices described above combined with research-based frameworks of children's fraction thinking (Empson & Levi, 2011). It included over 150 hours of face-to-face workshops over three years. Workshop activities involved working with children, analyzing children's written work, and discussing videos of math instruction focused on classroom instruction, small group instruction, and one-on-one conversations with children. These experiences provided teachers with opportunities to reflect on their teaching, explore new practices, and collaborate with colleagues (Jacobs, Empson, Pynes, et al., 2019).

Teacher Assessments and Scoring

We assessed teachers' expertise in each of the practices separately. The questioning assessment was based on a lesson observation and the noticing and anticipating assessments were written assessments. The questioning data were independently scored by at least two researchers and all disagreements were resolved through discussion, a process described as a consensus method for reliability (Goldsmith et al., 2014). Noticing and anticipating data were blinded so that teacher identities were hidden, and all data were at least double-scored. Interrater reliability was 80% or higher and discrepancies were resolved through discussion.

Questioning assessment. We asked teachers to plan a lesson that included at least one Equal Sharing story problem with a fractional answer (e.g., 6 children sharing 10 pancakes equally), using whatever lesson format they would normally use for story problems. All lessons were videorecorded by a member of the research team using a camera that followed the teacher, to capture all mathematical conversations between the teacher and children and as many details as possible of children's mathematical thinking. After the lesson, teachers were interviewed about the lesson. We used the videorecorded observations, supplemented by the interviews, to determine the level of responsiveness in teacher questioning. All parts of the lesson that focused on fraction story problems were considered, including launch, circulating, and discussion phases.

Rather than consider the lesson representative of teachers' typical instruction, we considered it evidence of teachers' capacity for questioning to support and extend children's fraction thinking. To indicate the extent of evidence of expertise in questioning in the midst of instruction, we assigned a holistic score of 1 ($N = 8$), 2 ($N = 18$), 3 ($N = 15$), or 4 ($N = 8$) to each teacher, with 4 representing the most evidence of expertise. We developed our scoring using an iterative process, which started by adapting prior research on teachers' engagement with children's thinking (Franke et al., 2001) and incorporating findings from earlier research on questioning (e.g., Franke et al., 2015; Jacobs & Ambrose, 2008; Jacobs & Empson, 2016).

Broadly, our scores reflected a continuum. We were not looking for "perfect" questioning but rather evidence that questioning made room for children's existing understandings and building on those understandings. At the high end, teachers actively explored children's thinking. Their questioning was customized with respect to the details of children's thinking and persistent in eliciting the details of that thinking. If there was a group discussion, children's thinking and talking predominated. When children shared their thinking, teachers followed up to support and extend that thinking, and children were regularly given opportunities to describe their thinking and engage with the thinking of others. In short, there was room for children to work from their existing understandings, and teachers positioned children as having authority for sensemaking. At the low end, teachers tended to question to evaluate the correctness of children's thinking and often took over children's thinking, especially when children had incorrect responses. If there was a group discussion, the teacher's thinking and talking predominated and children were provided few opportunities, if any, to describe their thinking or engage with the thinking of other children. In short, there was little room in these lessons for children to develop their existing understandings, and teachers positioned themselves as the authority for sensemaking.

Noticing assessment. Building on our earlier work on professional noticing of children's mathematical thinking (Jacobs et al., 2010), we assessed teachers' expertise in noticing with a written assessment structured around three instructional scenarios linked to solving fraction story problems. The scenarios were conveyed by strategically selected artifacts (video or children's written work), and for each scenario, teachers were asked to notice children's thinking and respond, in writing, to four categories of prompts. These prompts were related to the component skills of noticing: (a) attending to children's strategy details (*Describe in detail what you think each child did in response to this problem.*), (b) interpreting children's understandings (*Explain what you learned about these children's understandings.*), (c) deciding how to respond via follow-up questions (*Describe some ways you might respond to this child's work on the problem and explain why.*), and deciding how to respond via next problems (*What problem or problems might you pose next? What is your rationale?*).

We scored teachers' responses for the extent to which we had evidence for their engagement with children's fraction thinking. We then conducted a latent class analysis on the scores to empirically identify groups of teachers who displayed similar patterns of responses across the noticing assessment. This analysis yielded a 3-profile solution that was ordered in terms of overall noticing expertise. We assigned the profiles a score of 1 ($N = 25$), 2 ($N = 33$), or 3 ($N = 14$) for use in our multilevel model, with 3 representing the highest level of expertise. At the high end, teachers showed consistently strong expertise across the noticing component skills. They centered children's thinking in all their responses and the details of children's strategies were consistently visible. At the low end, teachers showed consistently weak expertise across the noticing component skills. They provided fewer details in their strategy descriptions and those details played a smaller role in other responses. Further, teachers sometimes privileged their own strategies over children's strategies by describing how they would funnel the children's thinking toward the teachers' preferred strategies (Wood, 1998). (See Jacobs & Empson, 2021, this volume, for more information on the noticing assessment and analysis.)

Anticipating assessment. In contrast to questioning and noticing, the construct of teacher anticipating was exploratory in that we had little prior empirical research on which to base our assessment. We created a written assessment with two open-ended items that asked teachers to anticipate a range of valid strategies that elementary-grades children might use to solve two fraction story problems. One was a Partitive Division (Equal Sharing) problem and the other was a Measurement Division problem, although neither was labeled as such for the teachers.

We scored each item on a 0–4 scale, for a total maximum score of 8. Teachers' scores on the assessment ranged from 2–8, with a mean of 5.8 ($SD = 1.7$). At the high end of the scale, teachers anticipated a variety of distinct strategies that were consistent with typical strategies children have been documented to use, spanning multiple levels of understanding and showcasing variety within those levels. At the low end, teachers tended to anticipate a smaller number of strategies that showed less variety and were sometimes accompanied by strategies that were inconsistent with research findings about children's fraction thinking and its development.

Fraction Assessment for Children

We assessed children's fraction achievement with a written assessment teachers administered to their students in the early fall and late spring. Teachers were told to allot 45 minutes but encouraged to allow extra time for children who wanted it. The assessment consisted of 7 items—5 fraction story problems and 2 fraction comparisons (see Table 1 for sample story problems). All items were open response and children were simply instructed to solve each problem. The story problems were designed to assess children's understanding of fraction quantities and relationships in story situations, whereas the comparison problems were designed to assess children's understanding of ordering relationships without the support of a story situation. The assessment was developed using an evidence-centered design approach and included protocols to ensure content-related validity (Mislevy & Haertel, 2006).

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There were two versions of the assessment—one for grade 3 and one for grades 4 and 5. The versions were parallel, with simpler numbers in the fraction story problems for grade 3. Due to the parallel nature of the assessments, they were treated as equivalent in the analyses. For all grades, the fall and spring forms of the assessment were identical.

Assessments were blinded for scoring. Scoring took place in teams of 3–5 researchers, who were trained on using a code book developed for the assessment. Total scores for the assessment ranged from 0–12, with the 5 fraction story problems each scored 0–2 and the two fraction comparisons each scored 0–1. When scoring the 5 story problems, we considered both the correctness of the children's answers and the validity of their strategies. When scoring the 2 fraction comparisons, we considered both the selection of the greatest number and children's rationales. Interrater reliability was at or above 80% for all items and the internal consistency of the assessment was adequate, as indicated by a Cronbach's alpha of .80. The grand mean of the 49 class means improved from fall (M = 2.2, SD = 1.9) to spring (M = 5.8, SD = 1.9), showing that learning did occur. The grand mean of 5.84 (out of 12) in the spring suggests that the assessment was challenging.

Table 1: Sample Story Problems from the Fraction Assessment for Children

Mathematical Focus	Grades 4/5 items (*Grade 3* number adjustments)
Equal Sharing	Mr. Lara gave 3 children 5 oranges to share so that each child got the same amount. If the children shared all of the oranges, how much orange did each child get? (*Grade 3:* 4 children, 9 oranges)
Multiplication	It takes 1/5 of a block of cheese to make a pizza. How much cheese do you need to make 17 pizzas? (*Grade 3:* 1/4 of a block of cheese, 6 pizzas)
Missing Addend	Allie has 1 6/8 sticks of butter. She needs a total of 5 1/8 sticks of butter to make cookies. How much more butter does Allie need so that she can make cookies? (*Grade 3:* 1 2/3 sticks of butter, 4 sticks of butter)

Findings

Research Question 1: *Do the instructional practices of questioning, noticing, and anticipating children's fraction thinking relate to each other?*

To explore the relationship among the instructional practices, we began by examining the three pairwise correlations, including the correlation between questioning and noticing (r (47) = .56), between questioning and anticipating (r (47) = .50), and between noticing and anticipating (r (47) = .54). All pairwise correlations were significant ($p < .05$) and of similar, moderate strength, suggesting that the practices are related, but distinct.

We were also interested in the relationships among the three practices when they were considered as a set. We therefore conducted a partial correlation analysis to identify the strength of the relationship between any two practices when all three were included but the effect of the third practice was removed. Again, all three partial correlations were significant ($p < .05$) and of moderate strength: .39 for questioning and noticing, controlling for anticipating; .29 for questioning and anticipating, controlling for noticing; and .36 for noticing and anticipating

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controlling for questioning. These findings suggest that the three practices together have interrelated features reflective of a coherent set of practices.

Research Question 2: ***Does teachers' expertise in questioning, noticing, and anticipating children's fraction thinking relate to children's fraction achievement?***

We conceptualized teachers' responsiveness to children's fraction thinking as the collection of our three practices—questioning, noticing, and anticipating children's fraction thinking. We were interested in the relationship between teachers' expertise in these practices, as captured by our three teacher assessments, and children's achievement. We used the children's spring scores on the fraction assessment as our measure of children's achievement and included children's fall scores on the identical assessment as a covariate. One-tailed tests were conducted because we hypothesized based on prior research that more questioning, noticing, and anticipating expertise should increase children's achievement (Carpenter et al., 1989; Jacobs et al., 2007).

We began by examining the three practices independently in unadjusted bivariate models. All three practices were significantly related to children's spring achievement ($p < .05$) and so were included in the multilevel model. We next constructed our multilevel model with the practices as independent variables, the children's spring achievement as the dependent variable, and the children's fall achievement as a covariate. The overall multilevel model (see Model 1 in Table 2) was significant (Wald χ^2 (4, $N = 876$) = 425.10, $p < .05$). Children's fall achievement was significantly related to their spring achievement, as expected. At the teacher level, questioning was the only practice that remained significantly related to children's spring achievement. Given that our PD was designed to help teachers develop expertise in the three practices, we were also interested in the relationship between the number of years of PD teachers had completed and children's fraction achievement. Further, we were interested in whether the relationship between teacher questioning and children's achievement would remain significant even when years of PD was included as a teacher-level variable. We therefore extended our multilevel model to include teachers' years of PD (see Model 2 in Table 2). As before, the overall model was significant (Wald χ^2 (5, $N = 876$) = 435.40, $p < .05$). In addition, the number of years of PD was significantly related to children's spring achievement, and again, teacher questioning remained significantly related to children's spring achievement.

Table 2: Models Relating Instructional Practices and Children's Spring Achievement

	Model 1		Model 2	
	Coefficient (*SE*)	*z*	Coefficient (*SE*)	*z*
Children's Fall Achievement	0.66 (0.03)	20.18*	0.66 (0.03)	20.26*
Teacher Questioning	2.59 (1.04)	2.50*	1.99 (1.02)	1.95*
Teacher Noticing	0.04 (0.37)	0.11	– 0.08 (0.35)	– 0.23
Teacher Anticipating	– 0.48 (1.15)	– 0.41	– 0.91 (1.11)	– 0.82
Years of PD			0.66 (0.29)	2.31*

*$p < .05$, one-tailed

In summary, increased expertise in teachers' questioning and increased years of PD were directly linked to children's higher achievement on the spring assessment, after adjusting for fall achievement. Expertise in teacher noticing and teacher anticipating were not significantly related to children's spring achievement after adjusting for fall achievement. However, as seen in the analyses for Research Question 1, expertise in these practices was significantly related to expertise in questioning. In other words, they appear to be necessary but not sufficient for teachers' questioning expertise and its direct link to children's achievement.

Discussion

Our first research question focused on investigating the relationships among teachers' expertise in three instructional practices representing responsiveness to children's fraction thinking. Our findings provide evidence that teachers' expertise in each practice is positively related to expertise in the others. We conjecture that these positive relationships may be due to the joint knowledge base on which the practices draw—namely, knowledge of children's thinking, which includes general knowledge of children's thinking and specific knowledge of individual children's thinking. However, each practice also captures a distinct piece of responsiveness, connected with how this knowledge is used in the work of teaching represented by each practice. For example, when teachers notice, they start with a specific instance of a child's thinking and connect it with what they know, but when teachers anticipate, they start with what they know and use it to generate possible instances of a child's thinking.

Our second research question focused on investigating relationships between teachers' expertise in instructional practices and children's fraction achievement. Our findings provide empirical support for the direct link between the two, and we add to a small but growing body of evidence of positive relationships between practices that are responsive to children's mathematical thinking and children's mathematics achievement (see, e.g., Bishop, in press; Webb et al., 2014). We highlight in particular the significant positive relationship in our final multilevel model between teacher questioning and children's achievement. Children in the classrooms of teachers with higher questioning scores tended to have higher fraction achievement, indicating their greater capacity to apply their understandings of fractional quantities and relationships to solve story problems and compare fractions. This finding provides evidence of the power of questioning to support and extend children's fraction thinking, which we conjecture resides in creating ongoing opportunities for children to articulate, consider, coordinate, and refine their fraction understandings in conversations with the teacher during instruction. We assessed teachers' capacity to create such opportunities and future research should directly assess the opportunities created over time in teachers' classrooms.

With this study, we identified three practices—questioning, noticing, and anticipating children's mathematical thinking—that comprise a set of related instructional practices for teaching that is responsive to children's thinking. We intentionally focused on a small number of practices because it offered a manageable way for teachers to engage with the complexity of responsiveness. Our findings suggest that teachers were indeed able to engage with this complexity. A focus on a manageable, but coherent, set of instructional practices defined by teachers' work with children's thinking thus offers a way to decompose teaching expertise so that it is accessible for teachers and can make a difference in children's learning.

Acknowledgements

This research was supported by the National Science Foundation (DRL–1712560) but opinions expressed do not necessarily reflect the endorsement of NSF. We thank the teachers for participating and Amy Hewitt, Naomi Jessup, Gladys Krause, Heather Lindfors-Navarro, D'Anna Pynes, and Cassandra Quinn for their contributions to data collection and analysis.

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References

Bishop, J. P. (in press). Responsiveness and intellectual work: Features of mathematics classroom discourse related to student achievement. *Journal of the Learning Sciences.*

Boerst, T., Sleep, L., Ball, D. L., & Bass, H. (2011). Preparing teachers to lead mathematics discussions. *Teachers College Record, 113*(12), 2844-2877.

Carpenter, T. P., Fennema, E., Peterson, P. L., Chiang, C. P., & Loef, M. (1989). Using knowledge of children's mathematics thinking in classroom teaching: An experimental study. *American Educational Research Journal, 26*(4), 499–531.

Empson, S. B. & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals*. Heinemann.

Fennema, E., Carpenter, T. P., Franke, M., Levi, L., Jacobs, V. R., & Empson, S. B. (1996). Mathematics instruction and teachers' beliefs: A longitudinal study of using children's thinking. *Journal for Research in Mathematics Education, 27*(4), 403–434.

Fraivillig, J. L., Murphy, L. A., & Fuson, K. C. (1999). Advancing children's mathematical thinking in everyday mathematics classrooms. *Journal for Research in Mathematics Education, 30*(2), 148–170.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing teachers' generative change: A follow-up study of professional development in mathematics. *American Educational Research Journal*, *38*, 653–689.

Franke, M. L., Webb, N. M., Chan, A. G., Ing, M., Freund, D., & Battey, D. (2009). Teacher questioning to elicit students' mathematical thinking in elementary school classrooms. *Journal of Teacher Education, 60*(4), 380-392.

Franke, M. L., Turrou, A. G., Webb, N. M., Ing, M., Wong, J., Shin, N., & Fernandez, C. (2015). Student engagement with others' mathematical ideas: The role of teacher invitation and support moves. *The Elementary School Journal, 116*(1), 126—148.

Goldsmith, L. T., Doerr, H. M., & Lewis, C. A. (2014). Mathematics teachers' learning: A conceptual framework and synthesis of research. *Journal of Mathematics Teacher Education, 17*(1), 5–36.

Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching Practice: A Cross-Professional Perspective. *Teachers College Record*, *111*(9), 2055–2100.

Hiebert, J. & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In J. Frank & K. Lester (Eds.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Charlotte: Information Age.

Hiebert, J. & Morris, A. (2012). Teaching, rather than teachers, as a path toward improving classroom instruction. *Journal of Teacher Education, 63*(2), 92–102.

Howe, C., Hennesy, S., Nercer, N., Vrikki, M., & Wheatley, L. (2019). Teacher-student dialogue during classroom teaching: Does it really impact on student outcomes? *Journal of the Learning Sciences, 28*(4–5), 462–512.

Ing, M., Webb, N. M., Franke, M. L., Turrou, A. C., Wong, J., Shin, N. & Fernandez, C. H. (2015). Student participation in elementary mathematics classrooms: the missing link between teacher practices and student achievement? *Educational Studies in Mathematics, 90*, 341–356.

Jacobs, V. R. & Ambrose, R. C. (2008). Making the most of story problems. *Teaching Children Mathematics, 15,* 260–266.

Jacobs, V. R. & Empson, S. B. (2016). Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM–The International Journal on Mathematics Education, 48*(1–2), 185–197.

Jacobs, V. R. & Empson, S. B. (2021, these proceedings). Profiles of teachers' expertise in professional noticing of children's mathematical thinking. *Proceedings of the 43rd annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Philadelphia, PA.

Jacobs, V. R., Empson, S. B., Jessup, N., & Baker, K. (2019). Follow up conversations: Inside or outside of children's strategy details? In A. Redmond-Sanago & J. Cribbs (Eds.), *Proceedings for the 46th Annual Meeting of the Research Council on Mathematics Learning* (pp. 156–163). Charlotte, NC: RCML.

Jacobs, V. R., Empson, S. B., Pynes, D., Hewitt, A., Jessup, N., & Krause, G. (2019). Responsive teaching in elementary math project. In P. Sztajn & P. H. Wilson (Eds.), *Designing professional development for mathematics learning trajectories* (pp. 75–103). Teachers College Press.

Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education, 38*(3), 258–288.

Jacobs, V. R., Lamb, L. L. C., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, *41*(2), 169–202.
Jacobs, V. R., & Spangler, D. A. (2017). Research on core practices in K–12 mathematics teaching. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 766–792). National Council of Teachers of Mathematics.
Mislevy, R. & Haertel, G. (2006). Implication of evidence-centered design for educational testing. *Educational Measurement: Issues and Practice, 25*(4), 6–20.
Munson, J. (2019). After eliciting: Variations in elementary mathematics teachers' discursive pathways during collaborative problem solving. *Journal of Mathematical Behavior, 56,* 1–18.
Parsons, S., Vaughn, M., Scales, R., Gallagher, M., & Parsons, A., Davis, S., Pierczynski, M., & Allen, M. (2018). Teachers' instructional adaptations: A research synthesis. *Review of Educational Research, 88*(2) 205–242.
Richards, J., & Robertson, A. D. (2016). A review of the research on responsive teaching in science and mathematics. In A. D. Robertson, R. E. Scherr, & D. Hammer (Eds.), *Responsive teaching in science and mathematics* (pp. 36–55). Routledge.
Saxe, G.B., Gearhart, M. & Seltzer, M. (1999). Relations between classroom practices and student learning in the domain of fractions. *Cognition and Instruction, 17*(1), 1-24.
Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.
Smith, M. S. & Stein, M. K. (2018). *5 practices for orchestrating productive mathematics discussions.* National Council of Teachers of Mathematics.
Stein, M. K., Engle, R. A., Smith, M. S., & Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical thinking and learning*, *10*(4), 313-340.
Webb, N., Franke, M. L., Ing, M., Wong, J., Fernandez, C., Shin, N., & Turrou, A. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning. *International Journal of Educational Research, 63,* 79–93.
Wood, T. (1998). Funneling or focusing? Alternative patterns of communication in mathematics class. In H. Steinbring, M. G. Bartolini-Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 167–178). National Council of Teachers of Mathematics.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.