REVEALING MATHEMATICAL ACTIVITY IN NON-FORMAL LEARNING SPACES

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We offer this synthesized framework as a tool to reveal mathematical activity in a non-formal makerspace. In particular, we connect research at different grain sizes to illustrate and explain how mathematics plays a crucial, if often implicit, role in this activity. We begin with describing the Approximate Number System and the Ratio-Processing System, explain how those systems connect to both embodied cognition and Thompson's (1994) conceptualization of quantities. We then examine how prediction and anticipation relate, with a particular emphasis on how social feedback guided the emergent mathematical activity. Finally, we synthesize across the two frameworks, in order to better reveal mathematical activity in low-notation environments, as the first step towards a framework for understanding mathematical learning in non-formal and low-notation contexts.

Keywords: Informal Education, Learning Theory, Technology

Identifying mathematical cognition in non-formal contexts where formal notation plays very little role can be a difficult proposition. In particular, mathematical notation-based performances are often taken as evidence of mathematical learning, and a tempting corollary is that mathematical learning is thus evidenced by mathematical notation. As a consequence, learners engaging in activities that have little or no formal notation can be seen as not engaging in mathematical learning, even when they may be experiencing a *mathematical* activity that merely lacks the explicit outward signs of such learning. In our research on mathematical play in a non-formal makerspace (Shokeen et al., 2020; Katirci et al., 2021), we have developed a new framework for identifying mathematical activity in a low-notation environment, and we share that framework here.

We build a theoretical argument that takes a multi-pronged approach: first, we develop a theoretical framework that builds from two primitive structures in the brain - the Approximate Number System and the Ratio-Processing System (Matthews et al., 2015) - tie those neural structures to Alibali and Nathan's (2012) embodied cognition view of perception and action, and interpret both of those frameworks through Thompson's (1994) conceptualization of *quantities*. Second, we describe a framework based upon prediction (Bieda & Nathan, 2009) and anticipation (Tzur, 2007) as components of mathematical learning, and tie those directly to our work on feedback and failure (Williams-Pierce, 2019). Lastly, we illustrate (or apply) the synthesis of these frameworks as a way to better identify and understand the mathematical activity that is taking place in the social context of an informal collaborative group.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

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Methodological Background

Our primary methodological approach for this paper is theoretical but we built our theory directly through observing and analyzing video data with the aim of examining it for evidence of mathematical play. In this section, we describe that video data. In the later sections, we provide illustrative examples of that data in order to illuminate how our comprehensive framework revealed mathematical learning and activity.

Our video data is composed of three video records of the same 20 minutes of a collaborative robotics activity with five fourth-grade students (2 M; 3F). Two of the video records were from the perspective of two students wearing GoPro cameras, while the third was a standing camera that captured the entire group's activity from a slight distance. The activity took place within the context of a physical classroom, although it was treated as a non-formal makerspace, and the participants were present voluntarily. The robotics activity had two phases: Phase 1, the group put masking tape on the floor to establish a path for a different group (who did not consent to be videotaped); and Phase 2, the group moved to the masking tape path established by the other group, and sought to measure the path and program a robot, Dash, to successfully travel it. Figure 1 illustrates the group putting down the masking tape path in Phase 1 (A), the iPad interface for programming Dash (B), and an image from the standing camera of the group measuring the path and watching Dash move in Phase 2 (C).

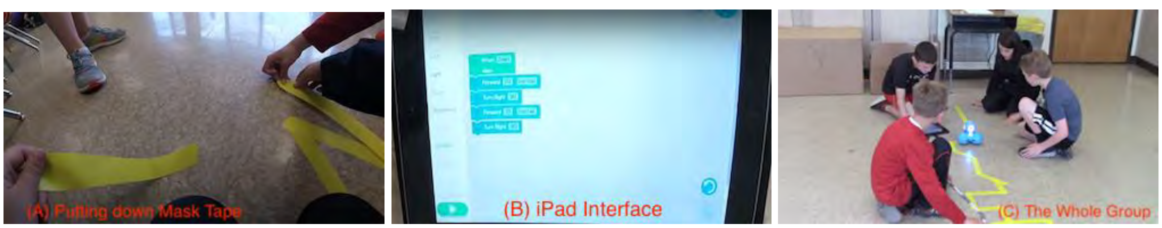

Figure 1: (A) Phase 1; (B) iPad interface; (C) Phase 2.

The research team who analyzed the data is composed of four regular members with varying areas of expertise. Two are experts in embodied cognition, in both physical and digital learning contexts; one specializes in mathematics learning in makerspaces (and originally collected the video data); one specializes in mathematical play. All four have considerable expertise with mathematics learning in both formal and informal contexts. The multidisciplinary nature of the team is how we developed our comprehensive framework over time, as our collaboration during analysis revealed both the need and the expertise for developing this framework.

Theoretical Background A: Approximate Number System to Quantities

In this section, we discuss on how perceptions, gesture, action, and the physical context relate to Thompson's (1994) conceptualization of *quantities*. We begin by describing the underlying neural systems that influence perception of magnitude (section A1); describe how perception, gesture, and action are complexly related in cognition (section A2); then describe how Thompson's quantities (1994) fit into that theoretical system (section A3).

A1: Underlying Neural Systems

Our physical bodies have perceptual systems that influence our cognition. Alibali and Nathan (2012) describe perception and simulations of perception: "When humans perceive objects, they automatically activate actions appropriate for manipulating or interacting with those objects (Ellis & Tucker, 2000; Tucker & Ellis, 1998). Thus, imagining an object can evoke simulations

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of perception (i.e., of the actions associated with perceiving the object) or of potential actions involved in interacting with the object" (p. 254). These perceptions and the perceptual systems that underlie them can have primitive neurological bases. For example, the Approximate Number System (ANS) ties estimation of a number of objects directly to certain animal neuron activation patterns, including humans (e.g., Dehaene, 1997; Matthews et al., 2015). A human adult, glancing at a set of three objects on a table, immediately subitizes: they know automatically and without conscious thought that there are three objects present (e.g., Miller, 1994). If that human adult is shown three objects repeatedly, the part of their brain responding to those three objects begins firing less actively as the perceiver becomes habituated to the number of objects being subitized. In such situation, if a fourth object is added, there is a small increase in relevant brain activity; whereas if three more objects are added (making six in total), a larger increase in relevant activity occurs. In other words, when the number of objects being perceived increases slightly, there is little increase in brain activity; but if the number increases considerably, so does the brain activity (e.g., Dehaene, 1997; Piazza et al., 2004).

Building upon the ANS, Matthews et al. (2015) describe the Ratio-Processing System (RPS) as a neural system in which we intuitively and immediately perceive and compare magnitudes of objects. (Although Matthews et al. (2015) describes quantities as an inherent quality of magnitude of an object or representation, we instead refer to that as *magnitude*, and reserve the term *quantity* for Thompson's (1994) definition.) With the ANS and the RPS as primitive structures that perceive and compare magnitudes, certain components of perception are built directly into our brains. Building upon those structures into more complex forms of perception (such as recognizing relevant tools in our environment, the social structures of a group, and so on), is more complex. Specifically, perception and action are reciprocal: our perception guides our action, and in turn our action reflects and guides our perception. These actions and perceptions are grounded in our physical environment, including the social, material, and structural aspects of our surroundings (Alibali & Nathan, 2012) *and* our neurological structures (Matthews et al., 2015).

A2: Perception, Action, and Gesture

Perception, whether based upon primitive numerical structures or otherwise, leads to action (such as gesture, physical movement upon the environment, or spoken language), and that action leads back into our perception. This feedback loop of perception, action, and imagining is described as mental simulation (Alibali & Nathan, 2012), and together compose the embodied nature of our cognition. This feedback loop can be evidenced through spoken or written language, physical movements that impact the physical world, or - often - can only be inferred by an outside observer through expression of gestures. These gestures are communicative acts that reveal perception and action in a variety of ways, such as through pointing (deictic) gestures that connect spoken language with objects or people in the physical environment or representational (such as iconic or metaphoric) gestures that directly reflect the state of perceptions and planned actions of the gesture. Consequently, we rely upon action and gesture as both composing and revealing perception, action, and their composite into cognition.

We now describe Thompson's (1994) conceptualization of *quantities*, then tie this conceptualization into Alibali and Nathan (2012) and Matthews et al. (2015) through illustrative examples of our data.

A3: The Role of Quantity in Perception, Action, and Gesture

Thompson (1994) specifically defines *quantity* as a conceptual entity - that is, quantity does not reside in the object, but rather in the perceiver. As noted above, our references to *magnitude*

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should be taken to refer to both Matthews et al.'s (2015) use of the term *quantity*, and to the *perceived* quality of an object or representation of taking up space (re: Thompson's (1994) definition).

Thompson (1994) goes on to define *quantity* as a schematic that involves "an object, a quality of the object, an appropriate unit or dimension, and a process by which to assign a numerical value to the quality" (p. 184). For example, a piece of masking tape that is "too big" (as stated by Peter; see Figure 2-A) indicates that the speaker perceives the magnitude of the tape, and compares it to some internal standard in order to determine that the piece needs to be shortened (See Figure 2-B&C). This perception and comparison of magnitudes (one physical, one imagined) occurs through Matthews et al.'s (2015) primitive structures. Then, the judgment of "too big" indicates that the speaker is perceiving the length of the masking tape as a *quantity* by Thompson's (1994) definition: the masking tape is the object; the length of the piece of masking tape is the quality they are considering; and the internal standard for magnitude is an appropriate unit or dimension. Although our participants did not have access to a measuring tape in order to assign a numerical value to the quality of length, they would have been able to do that measuring if the tool had been present (as they used such a tool in Phase 2). In other words, the speaker who says "too big" is using quantity as conceptualized by Thompson (1994), and that quantity is perceived and compared with a simulated perception (Alibali & Nathan, 2012) of appropriate unit or dimension. This perception and comparison of length is rooted in the speaker's ANS and RPS: although a lack of discrete or explicit measurement makes it difficult to determine how their ANS is contributing, the comparison of the physical length's magnitude with their imagined unit's magnitude can be directly attributed to their RPS.

Figure 2: (A) "Too big!"; (B) Shortened the piece; (C) Final decision- cut the tape

The speaker's comparison of the magnitude of the tape with their internal standard presents a communicative problem, as they must externalize their internal standard in some fashion for their group mates. One potential method of externalizing might be gesturing what "too long" is - while this does not externalize the internal standard, it indicates what magnitude the speaker is considering to be too much, which implies that the desired length of tape should be shorter. Another potential method was to engage the action of ripping the tape in half: this would serve to indicate what an appropriate length of tape would be, while requiring fellow perceivers to examine the magnitude of a resulting piece of tape in order to evaluate whether the new pieces are perhaps "too short." When a piece was too short, the choice of actions was different: they were crumpled up and thrown away, or used to extend a pre-existing length of tape already on the floor. While these actions and gestures may differ, they each indicate the same perception of magnitude, the quantification of that magnitude, and a comparison to an internal standard.

Now we shift to our second theoretical background.

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Theoretical Background B: Prediction to Social Feedback

Now that we've described how perception, action, and gesture relate to magnitudes and quantities, we will focus on detailing prediction (Bieda & Nathan, 2009) and anticipation (Tzur, 2007) as components of mathematical learning, and tie them directly to our previous work on feedback and failure (Williams-Pierce, 2019). We then focus specifically on how prediction, anticipation, feedback, and failure contribute to *social feedback* with illustrations from our data. We focus on social feedback in particular due to its crucial role in the collaborative mathematical activity present in our data. We introduce Theoretical Background B with as little reference to Theoretical Background A as possible, as we plan on focusing on that final step of synthesis in our Synthesis of Theories section. As the majority of the mathematical reasoning that occurs in this activity is grounded directly in perception and quantities, and much of it also involves mental simulation (i.e., aspects of Theoretical Background B), we give less mathematical examples here that do not require attention to quantities or mental simulation.

B1: Prediction and Anticipation

Bieda and Nathan (2009) describe prediction as looking at a pattern, and predicting a later instance of that pattern, whether near or far. The vast majority of our prediction examples in the data are intertwined with students perceiving and simulating quantities, but we present two examples of prediction that rely less upon quantities, both of which revolve around the teacher-facilitator warning the teams that they were running out of time. During Phase 1, the students changed their tape-laying pattern from trying to make the path 'zig-zag' (Shokeen et al., 2020, accepted), to simply placing a single long piece of tape to complete the path across the room. In other words, they were predicting that following their zig-zag pattern would not result in completing a path across the room, so they adjusted their activity accordingly to ensure they reached across the room within the allotted amount of time. A similar moment happened towards the end of Phase 2 when the teacher-facilitator gave a 45 second warning that the activity was almost over. One of the students from the other team was overheard by the target team saying, "We are not gonna make it" and Ryan responded across the room to them as he kept measuring the tape paths: "Neither are we." In that moment, Ryan was looking ahead in time, and predicting that if they continued programming Dash as their current speed, they would not be able to get Dash to the end of the tape path. As mentioned earlier, these are not particularly rich examples, mathematically, but the students are looking at the results of their activity thus far, comparing how much time that activity took, and predicting the results of continuing with exactly the same activity in the short amount of time that is left. In the first example, they modified their activity in order to achieve their goal of getting across the room; in the second example, there was no such modification available to similarly speed up their progress.

We now shift from prediction to Tzur's (2007) description of two stages in mathematical activity: participation and anticipation. During the participatory first stage, the learner has a mathematical understanding that emerges only when prompted by the activity at hand, and cannot be independently demonstrated without the contextual cues or tools. Tzur (2007) describes the "the well-known 'oops' experience" (p. 277) in the participatory first stage, where a student does something, notices a mistake as it manifests in their activity, and goes on to adjust it in the moment. During the anticipatory second stage, however, "the learner can independently call up and utilize an anticipated activity-effect relationship proper for solving a given problem situation" (p. 278) - in other words, they are able to use their mathematical understanding without engaging in the activity first. In our activity, participation and anticipation often manifested through prediction, feedback, and failure. We give specific examples about

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anticipation and its relationship to prediction in Section B3, as failure and feedback (Section B2) play a critical role in identifying Tzur's (2007) stages within this activity.

B2: Feedback and Failure

Our initial goal when we began analyzing this data was to identify how zones of mathematical play that emerged in concert with mathematical video games (Williams-Pierce & Thevenow-Harrison, 2021) might manifest in this new non-formal context. We began by attending particularly to feedback and failure, as Williams-Pierce (2019) defined failure and feedback as tightly paired in digital contexts and crucial to mathematical play. In particular, it is through failing and getting feedback that players, through their own actions, engage in learning the underlying mathematical content in the game (Williams-Pierce, 2019). This type of paired feedback and failure is instantaneous and often direct, in both videogames and the current activity. For example, in Phase 2, if the path in front of Dash was measured to be 80 cm and the programmer enters that measurement into the code, but Dash goes too far and ends up off the path, Dash's location manifests failure paired with feedback. The programmer then learns from the feedback (*Dash has gone too far*) to input a smaller distance into the program. This occurrence of failure and feedback is similar to that found in videogames, but we also found that *social* feedback played a crucial role in this non-formal collaborative context: students observing Dash's failure to stop at the correct spot on the tape often amplified the feedback of Dash's location - in this case, one student said, "That is a little far away" (*too far*). As a result, the programmer reprogrammed Dash to go 70 cm instead of 80, and Dash stopped at the desired location on the tape path. In short, in this type of situation, the paired feedback and failure may be direct - as in video games - may have social components, or may be fully social.

In fact, in Phase 1 the paired feedback and failure was often fully social. For example, at one moment, Peter was holding the roll of tape, and tearing off pieces to hand out to other students, who spontaneously formed a line to wait their turn for a piece of tape. Ryan, however, tried to cut in line immediately after they had just placed a piece of tape, but Peter did not let them, forcing them to go to the back of the line to wait their turn. This is an example of social feedback and failure: Ryan was essentially informed that they were performing a social activity that was not permitted within this community, and given feedback on how to actually get their next piece of tape in an appropriate fashion. As another example of social feedback, students would often disagree on how long a piece of tape should be, or what angle it should be placed relative to the path. However, this paired feedback and failure relies heavily upon perceiving or mentally simulating quantities, so we will discuss that further in the Synthesis of Theories section.

B3: The Relationships between Prediction, Anticipation, and Social Feedback

Prediction, anticipation, and social feedback have a complex but crucial relationship. For example, when Dash went too far in the Phase 2 example above, the students had input a centimeter measurement that they anticipated and predicted would lead Dash to the correct location on the tape. Consequently, when Dash stopped at the wrong place, the students received that paired feedback and failure, and amplified that feedback and failure through talking about it (e.g., social feedback). However, often feedback and failure are not clearly evident, because if what the students predicted would happen did, they had no need to remark upon it. In situations like this, where feedback and failure are missing, and the students move on to the next step, we concluded that they were content with their previous work. We also posit that this may be an indicator that students have shifted from *participatory* to *anticipatory*, because they have learned/internalized what to do or not to do, which results in no failure and often no social feedback. For example, in Phase 2, students are using a measuring tape and a pencil to measure

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a part of the path, and then write their measurement of that strip of tape on the path. After measuring and writing down the measurement, they move directly on to measuring the next part of the path without commenting, because they have successfully completed a step of the measurement. Measuring by itself is an activity that can be successful or unsuccessful in itself, even before Dash enacts the measurement – but the data showed no example of the students accidentally flipping the measuring tape to the inches side, or noticing any other potential measuring issues that could happen. This illustrates the other side of the 'oops moment,' because it is a 'we measured the path appropriately and are not surprised by it' moment. Sometimes, the students are successful but remark on their success, such as when Aaron coded Dash to traverse the first three lines and the angles within them, and after Dash ended up in the correct spot, Aaron said, "That's perfect!" We consider this to be an example of social feedback paired with success, rather than failure, and an illustration of the participatory stage rather than anticipatory, because they were at least mildly surprised that it worked (e.g., they lacked confidence in their prediction), unlike when using the measuring tape.

Synthesis of Theories

This section is the culminating synthesis of the theoretical groundings introduced above. In particular, we will describe how quantities and embodied cognition relate to prediction and anticipation, through the lens of failure and feedback in this context. We give two examples - one from each Phase - detailing exactly how the students were engaged in the activity, and conclude with another example that highlights the role of social feedback in particular.

In Phase 1, as students were placing down tape, they were using their perception of length and angle to guide their placement. At one point, Ryan places down tape at what he perceives to be and says is a "ten degree angle." The teacher-facilitator notices, and says it is "too tight" for Dash to traverse. Then Ryan pulls up that tape and re-places it, using his perception of quantities to increase the angle. This is an example of using perceived acceptable quantities of angle: Ryan uses his own perception of length and angle to mark 10 degrees, which he perceives as a perfectly appropriate angle for Dash to execute; then Ryan adjusted his understanding of an appropriate (perceived) angle quantity according to external guidance by the teacher, who knows Dash's limitations. This is an example of participatory first stage, where they have an oops! moment, but the shift to anticipatory second stage occurs immediately, as we see by a complete lack of other too-tight angles in Ryan and the group's remaining activity. Similar quantity and perception-based moments occur around the length of the masking tape, such as when Aaron is placing a long piece of tape, and Peter says, "But that's too long though." Aaron immediately adjusts the tape length by ripping some of it off, so that Peter's perception of an acceptable quantity of tape is respected. All of these interactions are based directly upon perceptions of quantity - whether of length or angle - and involve nuancing each students' view of what an appropriate quantity is for the task at hand.

In Phase 2, as students code a new length of the path for Dash, they are using quantities that have already been evaluated (measured) by their teammates with the measuring tape. However, in re-coding a length that they've already tested with Dash, they are using two different evaluated quantities alongside perceptual quantities. The centimeter measurement written on the path is one evaluated quantity, while the second – how far Dash was programmed to go – is another evaluated quantity, while the comparison between the two is purely perceptual. The two different evaluated quantities are both technically centimeters, but evaluated in two different ways: the first is 'centimeters as measured by the measuring tape'; while the second is

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'centimeters as enacted by Dash.' When those two evaluated quantities do not match up, the students must perceptually evaluate the difference between the two, and mentally simulate a comparison that supports them in re-programming Dash accurately. When Dash is programmed, the students are predicting that Dash needs to go the programmed distance in order to reach the correct spot; and they respond to the failure or success of that prediction accordingly, indicating their placement in the participatory or anticipatory stages.

The role of social feedback was particularly crucial, as there was a lack of mathematizing tools: each student had to use their own perception and mental simulation of quantities, as no more precise method was at hand. For example, at one point in Phase 1, the students decided that they wanted to lay the path underneath two chairs that are tucked under a table. As one student began laying the tape underneath the chairs, another student, Hannah, said something in a doubtful tone (not captured on audio), while tracing the floor under the chairs. Aaron says, "No no, that would work" and Ryan agrees, also tracing the floor under the chairs. As Hannah spoke, she was mentally simulating her perception of the size (quantities) of Dash, comparing that mental simulation with her perception of the space available underneath the chair, and visualizing a conflict between those two perception-based simulations such that Dash would run into the chair, rather than go smoothly underneath it. Aaron and Ryan, though, are either engaging in different mental simulations – one in which Dash fits under the chairs – or are merely thinking of Dash following the path (a participatory view), while Hannah was anticipating, and using that anticipation to predict that some issues would arise. Aaron and Ryan keep placing the tape, and then Peter joins to place the last piece of tape that brings the path out from under the chairs. As Peter finishes, he says, "We should move the chairs out, too, if it doesn't fit," and Ryan says, "Yeah." Then, when Phase 1 is ending, and the group is leaving their tape path for the other group to use, this group runs back to remove the chair from the path, indicating that the mental simulations of others (Peter and Hannah) have convinced the others that Dash probably will not fit - in other words, this is a moment of social feedback.

Conclusion

We offer this synthesized framework as a tool to reveal mathematical activity in a non-formal makerspace. In particular, we connected research at different grain sizes to illustrate and explain how mathematics plays a crucial, if often implicit, role in this activity. We began with describing the Approximate Number System and the Ratio-Processing System (Matthews et al., 2015), explaining how those systems connect to both embodied cognition (Alibali & Nathan, 2012) and Thompson's (1994) conceptualization of *quantities*. We then examined how prediction (Bieda & Nathan, 2009) and anticipation (Tzur, 2007) relate, with a particular emphasis on how social feedback guided the emergent mathematical activity. Finally, we synthesized across the two frameworks, in order to better illustrate the implicit mathematical activity in our data.

This theoretical framework is the first step in our efforts to better identify mathematical cognition in low-notation environments. We have connected multiple layers of research that emphasize mathematical cognition, and used it to reveal mathematical activity - and our next goal in this line of research is to connect that non-formal, low-notation mathematical activity with direct identification of the resulting learning. As yet, we do not claim a direct relationship between the revealed mathematical activity and learning, but rather focus on establishing the necessary groundwork for investigating that relationship. However, as increasing numbers of educators examine mathematical learning in informal environments such as ours, we consider the ability to identify the types of implicit, perceptual, and embodied mathematical cognition that

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emerge from these environments to be a necessary contribution to the field. Additionally, this identification requires using knowledge and frameworks from multiple fields examining different layers - from neurons to social interactions - in order to solidly ground each moment of mathematical activity. We offer this framework as the first step in this endeavor.

References

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the learning sciences*, *21*(2), 247-286.

Bieda, K. N., & Nathan, M. J. (2009). Representational disfluency in algebra: Evidence from student gestures and speech. *ZDM*, *41*(5), 637-650.

Dehane, S. (1997). *The number sense: How the Mind Creates Mathematics*. Penguin Press.

Katirci, N., Shokeen, E., Simpson, A., & Williams-Pierce, C. (2021). *Making with math: Extending a mathematical play framework to informal makerspaces.* Paper presented at the American Educational Research Association Annual Meeting and Exhibition. https://aera21-aera.ipostersessions.com/default.aspx?s=9F-63-14-D6-D5-77-20-39-F1-67-E8-FD-C9-9C-B8-97

Matthews, P. G., Lewis, M. R., & Hubbard, E. M. (2015). Individual differences in nonsymbolic ratio processing predict symbolic math performance. *Psychological science*, *27*(2), 191-202.

Miller, G. a. (1994). The magical number seven, plus or minus two: some limits on our capacity for processing information. 1956. *Psychological Review, 101*(2), 343–352. Retrieved from http://www.ncbi.nlm.nih.gov/pubmed/8022966

Piazza, M., Izard, V., Pinel, P., Le Bihan, D., & Dehaene, S. (2004). Tuning curves for approximate numerosity in the human intraparietal sulcus. Neuron, 44(3), 547–555. https://doi.org/10.1016/j.neuron.2004.10.014

Shokeen, E., Katirci, N., Bih J., Simpson, A., & Williams-Pierce, C. (2020). *Unpacking mathematical play within makerspaces using embodied cognition.* Extended Abstract in the 2020 Annual Symposium on Computer-Human Interaction in Play (CHIPLAY). Ottawa, Canada.

Shokeen, E., Simpson, A., Katirci, N., & Williams-Pierce, C. (accepted). *Use of zig-zag to represent mathematical thinking about angle.* In submission to the North American Chapter of the International Group for the Psychology of Mathematics Education.

Thompson, P. W. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational studies in mathematics*, *26*(2), 229-274.

Tzur, R. (2007). Fine grain assessment of students' mathematical understanding: participatory and anticipatory stages in learning a new mathematical conception. *Educational Studies in Mathematics*, *66*(3), 273-291.

Williams-Pierce, C. (2019). Designing for mathematical play: Failure and feedback. *Information and Learning Sciences, 120*(9/10), 589-610. https://doi.org/10.1108/ILS-03-2019-0027

Williams-Pierce, C., & Thevenow-Harrison, J.T. (2021). Zones of mathematical play. *Journal of the Learning Sciences.* DOI: 10.1080/10508406.2021.1913167

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