GESTURE INDICATES PRODUCTIVE STRUGGLE IN PROOF WRITING: CASE STUDIES FROM A BASIC TOPOLOGY COURSE

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Many students struggle with proof writing. However, struggle is not universally bad: researchers have distinguished between productive and unproductive forms of struggle and have identified productive struggle as essential for learning mathematics. Yet, in practice, recognizing when learners are engaged in productive struggle or unproductive struggle can be challenging. In this report, I argue that students' gesture production may indicate engagement in productive struggle. I observed three undergraduate students from an introductory point-set topology course, collaborating in pairs to complete proof tasks. I present evidence from the students' work on two proof tasks that undergraduate students' gesture frequently when they are engaged in productive struggle and that gesture is rare during engagement in unproductive struggle.

Keywords: Cognition, Communication, Reasoning and Proof, Undergraduate Education

Writing proofs is known to be challenging for mathematics students (Alcock & Weber, 2010; Azrou & Khelladi, 2019; Harel & Sowder, 1998; Iannone & Inglis, 2010; Leron, 1983, 1985; Moore, 1994). Hiebert and Grouws (2007) identified that allowing students to struggle with mathematics was an important feature of effective mathematics teaching; still, not all struggle is beneficial to students' learning. In this paper, I present evidence that undergraduate students' uses of gestures when working on proof tasks can be used as an indicator of engagement in productive struggle.

Vygotsky (1978) defined the *zone of proximal development*: "the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). To support students' learning of new material, educational tasks should be designed so that the concepts involved fall into the students' zone of proximal development: they should be challenging to students, but achievable through appropriate scaffolding and support from peers or a teacher. Hiebert and Grouws (2007), in a meta-analysis of effective teaching practices for conceptual understanding, echoed this idea, referring to the notion of *struggle*, defined as "effort to make sense of mathematics, to figure something out that is not immediately apparent" (ibid., p. 387). In this paper, I will refer to this kind of struggle as *productive struggle*, and distinguish it from *unproductive struggle*, or "needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems... [or] the feelings of despair that some students can experience when little of the material makes sense" (Hiebert & Grouws, 2007, p. 387).

Gesture use is known to be directly connected to cognition and perception (Alibali et al., 2014; Bernard et al., 2015; Goldinger et al., 2016; Hostetter & Alibali, 2008; Lakoff, 2012; Lakoff & Núñez, 2000; McNeill, 1992, 2005; Straube et al., 2011; Varela et al., 1993; Wilson, 2002). Research on undergraduate students' gesture use has shown that the use (or lack of use) of gestures influences strategy choices in problem solving (Alibali et al., 2011) and that gesture use can support recognition of important ideas in the construction of proofs (Gallagher, 2020; Pier et

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al., 2019; Williams-Pierce et al., 2017) and communication about ideas related to proof (Kokushkin, 2020).

In this paper, I present evidence that undergraduate students' use of gestures when working on tasks related to proof may be indicative of engagement in productive struggle.

Theoretical Framework

To frame this work, I utilize Sfard's theory of commognition as well as the notion of productive struggle.

Commognition is a portmanteau of the words *communication* and *cognition*; Sfard described it in the following way:

> Once we adopt the claim that thinking may be usefully defined as the individualized form of the activity of communicating, thinking stops being a self-sustained process separate from and, in a sense, primary to any act of communication and becomes an act of communication in itself, although not necessarily interpersonal. This self-communication does not have to be in any way audible or visible and does not have to be in words. In the proposed discourse on thinking, cognitive processes and interpersonal communication processes are thus but different manifestations of basically the same phenomenon. (Sfard, 2008, pp. 82-83)

The crux of the theory of commognition is that thinking and communicating are intrinsically linked. Rather than thinking of cognition as *preempting* communication or communication *following from* cognition, commognition adopts the perspective that these two actions are indeed one and the same. Furthermore, thinking can be conceptualized as *self-communication*; thus, commognition encompasses the practices of internal thought and "thinking out loud" as acts of communicating ideas with oneself.

In line with Sfard's assertion above, I assume that self-communication does not need to take the form of speech, and I include the production of gestures during self-communication as a form of commognition. Gestures are known to be produced spontaneously during thought, particularly when students are initially orienting to a problem or trying to communicate complex information (Alibali et al., 2014; Bernard et al., 2015; Hostetter & Alibali, 2008; Lakoff, 2012; Lakoff & Núñez, 2000; Straube et al., 2011).

With this in mind, in this work I associate gesture use with the concept of *productive struggle*, using the definition from Hiebert and Grouws (2007) given in the introduction to this paper. In the results that follow, I will show that gesture does not always occur spontaneously. Rather, I will argue that spontaneous gesture use occurs concurrently with productive struggle and can be used to distinguish productive struggle from unproductive struggle in undergraduates working on proof-related tasks.

Methods

Four undergraduates were recruited from a general topology course for a teaching experiment to gain insight into the ways undergraduates leverage examples, diagrams, and gestures when writing proofs in general topology. The author served as the researcher leading the teaching experiment. A total of 9 one-hour sessions comprised the teaching experiment, during each of which the students were asked to prove a true statement and disprove a false statement, although they only engaged with only one of these tasks during some sessions due to limitations on time. Each session was video recorded, and each video was transcribed. Videos and transcripts were then coded for instances of students engaging in productive struggle and unproductive struggle

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(operationalized definitions provided later in this section) and instances of students producing gestures.

A descriptive case study methodology (Cohen et al., 2013; Yin, 2003) was used to analyze the behaviors of Stacey, Tom, and Rachel, specifically focusing on when and how they used gestures while reasoning about proof tasks. The fourth student was excluded from this analysis, as he attended only one session, and participated very minimally in that session. As the students worked collaboratively to complete the proof tasks in a given session, and different groups of students were present during each session, I consider each session to constitute one "case" in this case study.

In this paper, I assume that the meaning of *struggle* is self-evident, but I distinguish between unproductive struggle and productive struggle. For the purposes of this paper, I claim that students are engaged in *unproductive struggle* any time they give visual or audible signs of focusing on the task under consideration but are not performing an action (such as drawing a diagram, considering an example, or writing notation) or proposing ideas or making conjectures (statements or questions like "I think I need to take the union of these sets" or "What happens if I take the intersection here?"). In other words, students are engaged in unproductive struggle when they appear to be thinking about a problem but seem to be unable to interact with the ideas involved in its statement or its solution. Most often, this is evidenced by students staring at the board in silence or expressing sentiments like "I'm not sure what to do here." In contrast, students are said to be engaged in *productive struggle* any time they are performing an action or proposing an idea or conjecture related to the task under consideration but seem to be uncertain about the usefulness or consequences of those actions, ideas, or conjectures. Examples of productive struggle include consideration of examples, drawing diagrams, attempting to write logical statements to move forward in a proof, and thinking aloud about the meaning of notation.

For this paper, I use the definition of *gesture* given by Rasmussen, Stephan, and Allen (2004) as "movement made by the hand with a specific form: the hand(s) begin at rest, moves away from the position to create a movement, and then return to rest" (p. 303). Gestures may be further divided into deictic gestures (pointing) and representational gestures (movements made to depict an idea, object, or action), though in this paper I do not consider these kinds of gestures separately.

Results

Throughout all nine sessions, instances of struggle were evident from all participants. I present results from two tasks: the prove task from Session 1 and the disprove task from Session 2. Stacey was present for all three (indeed, all nine) sessions; she was joined in Session 1 by Tom, and in Session 2 by Rachel.

Session 1

The students engaged in unproductive struggle when they were faced with notation they had used before but were unaccustomed to working with. In Session 1, Stacey and Tom struggled to get started on the following task, which was written on a chalkboard: *Let* $f: S \to T$ *be a function, and let* $\{U_i\}_{i\in I}$ *be a family of subsets of* T. *Prove that* $f^{-1}(\bigcap_{i\in I} U_i) = \bigcap_{i\in I} f^{-1}(U_i)$. After some initial thought, Stacey expressed the general proof strategy: "First, we have to prove that the first one is a subset of that [pointing from $f^{-1}(\bigcap_{i\in I} U_i)$ to $\bigcap_{i\in I} f^{-1}(U_i)$], and then we have to prove that this one is a subset of that one [pointing from $\bigcap_{i\in I} f^{-1}(U_i)$ to $f^{-1}(\bigcap_{i\in I} U_i)$." Tom suggested to start by proving the inclusion $f^{-1}(\bigcap_{i\in I} U_i) \subseteq \bigcap_{i\in I} f^{-1}(U_i)$, which he indicated by drawing the relation "⊆" in the air with his finger. Stacey wrote "Let $x \in f^{-1}(\bigcap_{i\in I} U_i)$" on the board.

The students then spent the next full minute in silence, both staring at the problem on the board, motionless. At the end of that minute, Stacey wrote "$\{1,2,3\}$" on the board and stated that "the intersection of all of those subsets would be the null set, 'cause there's nothing that would be common to every single one of them," a statement Tom agreed with. I interpreted this as Stacey's attempt to consider an example in which $T = \{1,2,3\}$, and that she has taken the family of subsets $\{U_i\}_{i\in I}$ to be the power set of T. She indicated that she knew the family of subsets did not need to contain *all* subsets of T, but she clarified that she was "just trying to figure out something to think about, I'm a little bit lost."

After another minute of silent consideration, Stacey turned to Tom and asked, "Do you understand... what an inverse of an intersection would even look like?" She proposed, as another attempted example, that if the intersection was the set $(2,3]$, then the "inverse" of that intersection might be "all of the other elements other than this?" Both students continued to stare at the board in silence.

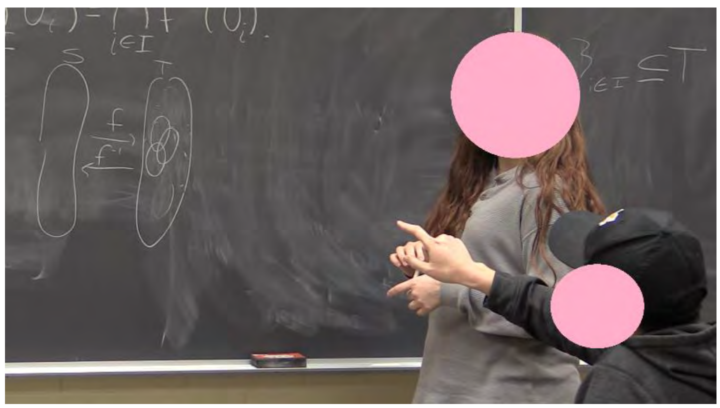

Figure 3: Tom pointing to Stacey's diagram.

I then prompted the students to draw a picture to represent the situation, and Stacey drew a standard set-theoretic diagram. Almost immediately, Tom pointed at the subsets of T in the diagram (Figure 1), claiming that "x is gonna be a point inside all three, in the intersection." Stacey considered this suggestion for a moment, then replied "... Is it?" In response, Tom began to explain his reasoning, but after pointing to the notation for $f^{-1}(\bigcap_{i\in I} U_i)$, he paused and second guessed his suggestion, pointing to the set S and saying "It's gonna be in this"; Stacey agreed, elaborating, "It's not in this [pointing to the text $\{U_i\}_{i\in I}$ in the problem statement], it's in the inverse of the intersection of that." Tom continued, explaining that for each U_i, $f^{-1}(U_i)$ represented a subset of S, first pointing to the notation $f^{-1}(U_i)$ and then tracing the outline of the corresponding subset of S with his finger in the diagram, and he noted that taking the intersection of those subsets would result in "only one area," tracing out a smaller region in the overlap of those sets. "Yeah, and x is *in* that area," Stacey concluded.

The remainder of the students' time spent on this task continued in a similar fashion, with Tom and Stacey pointing to notations from the problem statement and to regions of their diagram and using dynamic representational gestures to indicate elements being mapped between the sets S and T (Figure 2). Although they did not write a formal proof due to time constraints on the

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session, Tom and Stacey were able to articulate the key ideas of this proof and construct an oral and visual argument that appeared to convince both of them why this statement was true.

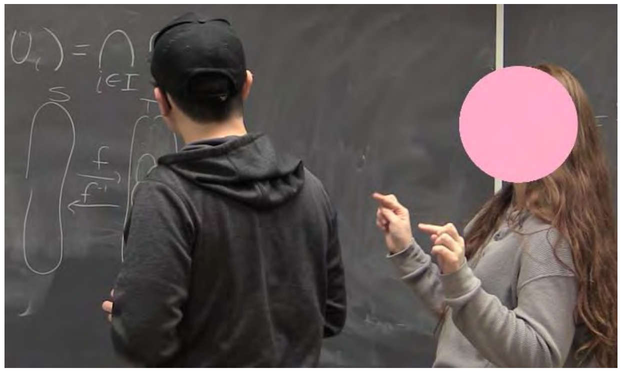

Figure 4: Stacey's dynamic representational gesture indicating a point mapping from T to S via f^{-1}.

Session 2

Stacey and Rachel worked together in Session 2 and did not experience the same immediate struggle that Stacey and Tom experienced in Session 1. Rather, they were able to discuss, relatively comfortably, the notions of symmetry, transitivity, and reflexivity that were necessary to work on the following task: *Disprove: Every relation* C *that is both symmetric and transitive must be reflexive.*

Upon reading the problem, Stacey immediately began by writing "$\{(1,2),(2,1),$'' at which point she paused and pointed with her index finger to the corresponding components of her writing as she read aloud, "So we have one-two... two-one... we have *a* related to *b*... it'd be one-one, if that was symmetric," and Rachel suggested adding $(2,2)$: "[pointing to where Stacey had written $(1,2)$ and $(2,1)$] I think you have to have both anyway, because it's 'for all.'" Stacey continued, "[pointing sequentially to each digit in $(1,2)$] We could do one to two, and then [writing] two to three, and then one-three, and that'd be transitive" (Figure 3).

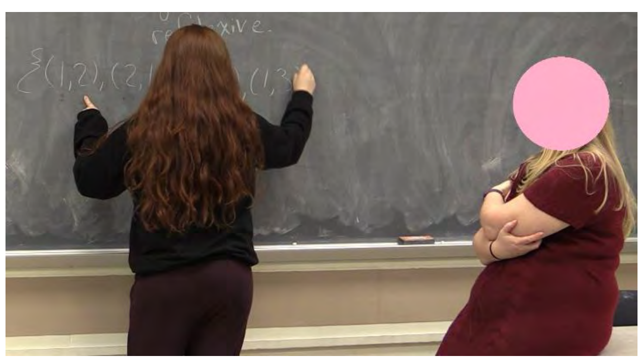

Figure 5: Stacey referencing element $(1,2)$ while adding element $(1,3)$.

After two minutes of work, Stacey and Rachel presented the relation $\{(1,2),(2,1),(2,3),(1,3),(3,1),(3,2)\}$ as their counterexample (the reader will note, however, that they did not specify a set on which to define this relation), at which point I informed them that, although their relation was symmetric on a certain set, it was not transitive, and thus could not serve as a counterexample to the given statement. Inspecting their work, Rachel pointed to the pair $(3,2)$, and Stacey pointed to the pair $(2,3)$ as she announced disappointedly "Then we'd need three-three, and we can't" throwing her hands into the air. Rachel replied, "No, we can have three-three," reminding Stacey that the reflexive property would not be satisfied unless their relation contained *all* of $(1,1)$, $(2,2)$, and $(3,3)$ – although they would later realize that all three of these pairs must be present for their relation to possess the transitive property (Figure 4).

Figure 6: Rachel realizing that $(1,1)$ must be included in the relation.

After adding $(1,1)$, $(2,2)$, and $(3,3)$ to their relation, both students backed away from the board and stared at their work, both silent and standing still. Rachel explained "We need to have one that's not like, one-one, two-two, or three-three, but it still satisfies symmetric and transitive, which I don't think that we can." Reading over the definition of the reflexive property, Rachel noticed that they had not specified a set for their relation, and she wrote $X = \{1,2,3\}$ under their relation. "We need something to not be in there, like one-one, two-two, or three-three... exactly where I'm stuck." Both students continued staring at the board, no longer writing nor gesturing. Near the end of this session, Rachel suggested a viable solution to their problem – but both students rejected it. She proposed, "I mean, if you threw a four into X... but then you'd just have to make more elements," referring to a misconception expressed by both students during this session that if $4 \in X$, then 4 would have to be related to the other elements in X, and thus would need to appear as a component of some ordered pairs in their relation. Stacey agreed, and they continued to stare at the board in silence (Figure 5).

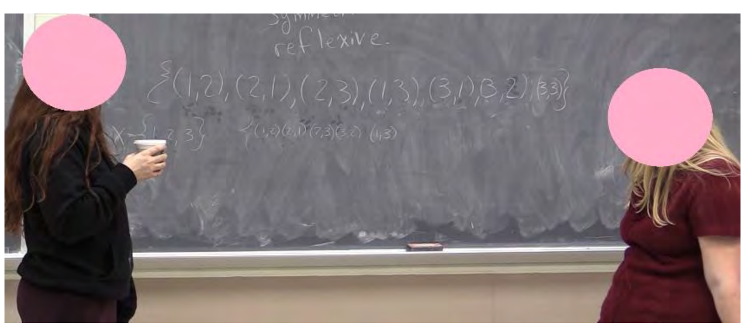

Figure 7: Stacey and Rachel near the end of Session 2, confused about how to prevent this relation from satisfying the reflexive property.

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Discussion

Throughout the data presented, and indeed, throughout the data corpus, the study participants produced significantly more gestures during times of productive struggle than during times of unproductive struggle. In fact, during times of unproductive struggle, students seldom produced gestures, which is in sharp contrast to periods of productive struggle, during which gestures were commonplace.

Sfard's theory of commognition treats thought and communication as two sides of the same coin. Taking this perspective and treating unproductive struggle and productive struggle as two distinct forms of cognitive activity, I note that, in these data, these corresponded to two distinct forms of communication. In addition to distinct modes of verbal communication – silence versus speech – these data also show distinct forms of nonverbal communication: stillness versus gesture. In this comparison, stillness may be thought of as a form of nonverbal silence.

Consider Stacey's and Tom's behaviors from Session 1. Stacey and Tom initially seemed unable to make any progress on the task, as evidenced by them taking little action and seeming to be confused by the terminology, notation, and concepts involved in the task. Although Stacey attempted to generate examples, those examples were inappropriate to model the conjecture, and they did not seem to provide Stacey or Tom with any advantages. However, when I suggested that Tom and Stacey draw a picture to represent the situation, they began to negotiate meaning for the various pieces of notation used in the statement of the conjecture and to develop intuition for the scenario it described, eventually gaining personal insights into why the statement was true.

Stacey and Rachel were not immediately hindered in Session 2. In fact, they were able to produce an equivalence relation on the set $\{1,2,3\}$ and competently discuss the concepts of symmetry and transitivity that were necessary to produce an appropriate counterexample. However, both students seemed to lack a complete understanding of the definition of reflexivity (or perhaps of relations more generally), which caused them to struggle as they tried to violate this property. Throughout their discussion, however, Stacey and Rachel produced numerous pointing gestures as they negotiated how to make their relation satisfy the symmetric and transitive properties and as they discussed why the relation they had chosen also satisfied the reflexive property. When they tried to identify a way to violate the reflexive property, they became "stuck," and their gestures ceased.

Conceptualizing gestures as a component of cognition gives a window into students' mental activities. These results show that the students in my sample produced gestures when they were engaging in a meaningful way with the content of a given proof task, and, conversely, that they did not gesture when they were not participating in such engagement. To be clear, I do not mean to imply that productive struggle will *always* be accompanied by gestures, but rather that *when* a student produces gestures, these may act as an indication that the student is engaged in productive struggle.

Conclusion

Struggle is essential in the process of learning mathematics. Unproductive struggle, however, prohibits learners from making learning gains and increases their frustration, leading to a decrease in motivation. Educators should strive to engage their students in productive struggle, as this is the part of the problem-solving process during which students grow their understanding, make connections, and feel like their efforts might be rewarded with success.

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In this paper, I framed students' gesture use as a way for teachers to discern whether students are engaged in productive struggle or unproductive struggle. With this tool, teachers can determine, at a glance, whether a task that has been set may be beyond the zone of proximal development for their students, and whether they may need to intervene to prevent students from losing motivation or simply let their students continue to work and develop their ideas.

However, as online instruction becomes more prevalent, researchers should attend to other means for distinguishing productive struggle from unproductive struggle, as gestures are not only more difficult to notice in the online environment, but may also be less frequent due to the inefficiency of pointing in such settings. Indeed, as reports from teachers and students indicate some students struggling to learn in the online environment, and as some classrooms transition back to in-person instruction, educators must be hypervigilant to notice signs of students struggling, and gestures serve this purpose well.

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