ISOMORPHISM AND HOMOMORPHISM AS TYPES OF SAMENESS

Rachel Rupnow
Northern Illinois University
rrupnow@niu.edu

Peter Sassman
Northern Illinois University
psassman1@niu.edu

Isomorphism and homomorphism are topics central to abstract algebra, but research on mathematicians' views of these topics, especially with respect to sameness, remains limited. This study examines 197 mathematicians' views of how sameness could be helpful or harmful when studying isomorphism and homomorphism. Instructors saw benefits to connecting isomorphism and sameness but expressed reservations about homomorphism. Pedagogical considerations and the dual function-structure nature of isomorphism and homomorphism are also explored.

Keywords: Advanced Mathematical Thinking, Undergraduate Education

Students' understanding of isomorphism in abstract algebra has been studied for over twenty-five years (Dubinsky et al., 1994), but research explicitly on students' understanding of homomorphism has begun more recently (e.g., Melhuish, et al., 2020; Rupnow, 2021). In an effort to position students' understanding, we wanted to learn more about how mathematicians position these topics in relation to notions of sameness. Thus, in this paper we address the following research questions: (1) What connections do algebraists see between sameness and isomorphism? (2) What connections do algebraists see between sameness and homomorphism?

Literature Review and Conceptual Framework

Prior work on students' understanding of isomorphism has shown associations between isomorphism and sameness. Leron et al. (1995) described a course in which students were taught to focus on sameness with isomorphic groups. Subsequent literature has confirmed references to sameness in the context of isomorphism by other groups of students and professors (e.g., Rupnow, 2021; Weber & Alcock, 2004). However, large-scale research has not verified whether this emphasis on "sameness" is normative across mathematicians.

Furthermore, small-scale research on mathematicians has revealed other types of language commonly used to describe isomorphism and homomorphism. Weber and Alcock (2004) highlighted algebraists' references to relabelings. Hausberger (2017) observed use of "structure-preservation" to refer to the homomorphism property. Rupnow (2021) observed renamings, relabelings, and structure-preservation as well as references to operation-preservation, disembeddings, and use of equivalence classes to describe isomorphism and homomorphism. However, the prevalence of these types of language among algebraists has remained unknown.

Our theoretical lens is conceptual metaphors (e.g., Lakoff & Nú ez, 1997), in which a source domain is used to structure understanding of a target domain. For example, "An isomorphism is an operation-preserving map" is a conceptual metaphor that describes the target domain (isomorphism) in terms of a source domain (operation-preserving map) to provide a way of thinking about isomorphism. In this case, the metaphor encourages focus on the homomorphism property, which guarantees a similar type of behavior in both structures (e.g., groups). In this paper, we build on Rupnow's (2021) previous isomorphism and homomorphism metaphors.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Methods

Data were collected from a survey sent to every 4-year college/university math department in the United States. This survey addressed how algebraists think about sameness in general and in specific mathematical contexts. Participants were 197 mathematicians from 173 institutions who had taught at least one abstract algebra or category theory course in the last five years.

The four survey questions relevant to this paper, numbered below, queried participants' beliefs about sameness related to isomorphism and homomorphism, and were the first reference to isomorphism or homomorphism in the survey text itself. These questions followed questions on the nature of sameness in math and about how similar particular objects were.

1. How might sameness be helpful when thinking about isomorphism/isomorphic structures? (Q1)
2. How might sameness be harmful when thinking about isomorphism/isomorphic structures? (Q2)
3. How might sameness be helpful when thinking about homomorphism? (Q3)
4. How might sameness be harmful when thinking about homomorphism? (Q4)

Responses to the isomorphism questions were grouped together for coding as were responses to the two homomorphism questions. Each paired response could receive multiple codes. To ensure coding validity, we used investigator triangulation with two members analyzing the data. Each member would independently code the data using the agreed codes; we then discussed any coding discrepancies and came to consensus on the final codes. These discussions included any modifications for future coding, such as refined code definitions or new codes for consideration.

The data were analyzed in accordance with thematic analysis (Braun & Clarke, 2006). First, we used versus coding (Saldaña, 2016) to identify different beliefs about sameness based on the help vs. harm contrast. However, after coding, we determined that these codes did not effectively capture all nuances in the data. We then revised codes, using descriptive coding (Saldaña, 2016) to supplement our initial coding. These second-round codes permitted clearer connections to our conceptual framework by incorporating codes based on Rupnow's (2021) prior work.

Results

We present mathematicians' responses about the helpfulness and harmfulness of sameness to considering isomorphism and homomorphism. Code frequencies and percentages are presented in Table 1. Participants largely viewed sameness as conceptually relevant to isomorphism. However, pedagogical issues and the context-dependent nature of isomorphism were noted as potential difficulties if using "sameness" as a substitute for isomorphism. In contrast, participants viewed sameness as needing to be qualified or viewed sameness as irrelevant to homomorphism.

Helpful or Harmful

Based on the question format, where we asked participants about how sameness might be helpful or harmful for thinking about isomorphism and homomorphism, our default expectation was for respondents to address both helpful and harmful aspects of sameness. This was the case for isomorphism, where 72% of respondents were coded as helpful/harmful. For example:

> [Helpful:] Isomorphism is a kind of sameness, so certainly you have to have some sense of sameness to understand the idea behind isomorphism. [Harmful:] Maybe thinking that sameness = identical in every aspect? At some point you always have to move away from

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

> intuition coming from English (and "sameness" is certainly not a mathematical concept) and rely only on mathematical definitions to make progress.

In this participant's view, isomorphisms are a type of sameness, but issues arise if one relies on a concept without a formal mathematical definition. Another participant had a different interpretation, focusing on the specific aspects that are and are not the same:

> [Helpful:] We all like to group things that are the same together, and it is useful to think that two very different looking objects (e.g., rings) that have the same algebraic properties should be put in the same group. We want to emphasize that algebraic objects should be studied based on their algebraic properties, not on the choice of names for their objects. [Harmful:] If a student starts to think isomorphic objects are the same as sets and mixes up elements of the objects, that could be harmful. If we use the wrong sort of sameness and think that the identity of a group must always be 0, for example, we could easily become very confused.

Table 1: Frequencies of Codes.

Category	Code	Isomorphism n(%)	Homomorphism n(%)
Help/Harm	Not harmful	18(9%)	5(3%)
	Helpful/harmful	142(72%)	72(37%)
	Not helpful	15(8%)	35(18%)
	Similar	0(0%)	38(19%)
	Not relevant	1(1%)	12(6%)
Pedagogical Considerations	Motivating instruction	15(8%)	7(4%)
	Leveraging intuition	33(17%)	8(4%)
	Misconceptions	25(13%)	20(10%)
	Imprecise language	29(15%)	14(7%)
Types of Sameness	Context-dependent	77(39%)	16(8%)
	Levels of sameness	11(6%)	6(3%)
	Generic identical	30(15%)	2(1%)
	Generic equal	20(10%)	6(3%)
	Isomorphism vs. homomorphism	1(1%)	76(39%)
Informal Sameness	Relabeling	10(5%)	0(0%)
	Matching	13(7%)	4(2%)
	Same behavior	48(24%)	36(18%)
	Same properties	7(4%)	7(4%)
	Structure preservation	9(5%)	12(6%)
	Operation preservation	5(3%)	20(10%)
	Disembedding	0(0%)	18(9%)
	Equivalence classes	0(0%)	3(2%)
Functions vs. Structures	Isomorphism vs. isomorphic	15(8%)	0(0%)
	Homomorphism vs. homomorphic	0(0%)	38(19%)
	Fundamental Isomorphism Theorem	0(0%)	12(6%)

This participant described isomorphisms as a way to classify objects into categories and viewed sameness as helpful for that grouping but emphasized that identification between objects was not required and could cause confusion for students (e.g., names of elements can differ).

In contrast, only 37% of respondents clearly highlighted both helpful and harmful aspects of homomorphism. For example:

> [Helpful:] It can be helpful, say to emphasize that it preserves some information, but not all. For instance, I like to say that homomorphisms from something complicated to something easier to work with or better understood are often the right approach (e.g., representations). Though one may lose info, by working with something easier you may still learn something new about the original. [Harmful:] Similar to the above, it should be emphasized that a lot of info can be lost, or that homomorphism is far from saying they are exactly the same, but is maybe a tool for extracting some information about sameness.

Note that the participant highlighted preservation of some aspects but a loss of some information, indicating utility but the need for care when discussing sameness with homomorphism.

Some participants only saw benefits to using sameness. For isomorphism, 9% of participants only expressed a helpful view of sameness: “Well, it’s the essential notion of isomorphism. In no way [harmful], but it is important that we understand sameness to mean sameness of underlying structure, not sameness of superficial characteristics, like labels.” Here we see the participant considered sameness to be the conceptual point of isomorphism, and thus did not consider sameness harmful. 3% of participants expressed an exclusively helpful view of sameness for homomorphism. For example: “[Helpful:] Same as isomorphism, except now we are only identifying a part of each of the two structures that behave the same algebraically. [Harmful:] Again, with carefully presented examples I don’t think there is harm per se.” Notice, even though the participant claimed sameness was helpful and not harmful for homomorphism, this sameness only referred to parts of structures instead of whole structures.

Although most isomorphism responses received a not harmful or helpful/harmful code, 19% did not. One participant was coded as not relevant: “When Isomorphism is being considered, isomorphism defines the sameness, and what makes the isomorphic objects “different” is to some extent obvious, but not really of interest. So considering sameness is neither a help nor a hindrance.” They saw the reverse connection of isomorphism giving some insight into sameness but did not consider this notion to be important for understanding isomorphism. Others saw sameness as relevant, but it was unclear whether they viewed sameness as helpful, harmful, or both: “[Helpful:] I like distinguish equality (for subsets of a given ambient object) and isomorphism. [Harmful:] The idea [of] flexible notions of equality or sameness is pretty subtle and counterintuitive.” While this participant typed distinct responses in the two boxes for the “helpful” and “harmful” responses, their response did not directly address how sameness might be helpful or harmful for understanding isomorphism, so it was not given any of those codes. Finally, 8% of respondents only expressed a harmful view: “Helpful? I don’t think it is. There’s nothing added to the concept of isomorphism by saying the word “same”. Well, homomorphisms also preserve something. Bijections also preserve something. So, talking about “same” is going to blur some distinctions.” This participant only saw a lack of clarity arising from sameness.

Participants expressed more skepticism to using sameness to discuss homomorphism, with 61% of responses *not* receiving a helpful-related code. 18% of participants considered sameness not helpful to describing homomorphism: “I think tying “sameness” to any homomorphism that is not an isomorphism is misleading at best. Not a fan.” Others were unwilling to use sameness but allowed similarity: “A homomorphism provides a notion of similarity.” 6% of participants considered sameness irrelevant to homomorphism: “Homomorphism is restricted version of the “sameness” defined by isomorphism. Usually when trying to show homomorphism exists, it is trying to show that a certain defined property holds and I do not see how sameness either helps or hinders.” Other participants acknowledged the relevance of sameness to homomorphism, but

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

whether they viewed it positively or negatively was unclear. For example: "[Helpful:] Can there be a connection between these two structures even though they are different? We are locating a connection that is not as deep as isomorphism. [Harmful:] Homomorphic structures may not be isomorphic." Here, the participant described some connections between structures and compared the relationship to isomorphism, but it was unclear what they meant by this connection.

Pedagogical Considerations

In addition to the notions of helpfulness and harmfulness explicitly prompted by the questions, a number of respondents focused on pedagogical implications of using sameness. Some specifically highlighted how sameness could be useful for motivating isomorphism or for connecting to students' intuition: "Different levels of "sameness" and different informal definitions of "sameness" can be used to motivate the formal definition of isomorphism." The participant here explained how formalizing sameness could provoke a need for isomorphism. Leveraging intuition was also described: "It might be helpful for students to think of sameness in a familiar context (e.g., geometry or linear algebra) in order to appreciate the notion of isomorphism in algebra." Here, the participant described how students' intuition and prior experiences with sameness in math could be used to help them understand isomorphism.

However, some respondents highlighted pedagogical concerns like student misconceptions or imprecision. Misconceptions often addressed difficulties with names of elements or objects: "Students often think that if two sets have different looking objects (integers vs matrices, for example), then they can't be "the same." This makes it more difficult for them to understand the more meaningful examples in class." Here the participant observed students could struggle with identifying superficially different objects. Another common concern was that using sameness may lead to imprecision in exercises and proofs: "Two objects can be isomorphic as groups under their additions, but not as rings, when both addition and multiplication are involved. The idea of sameness must be carefully used especially with students since they tend to forget the context." Here, the respondent worried that using sameness haphazardly could lead students to confuse different types of isomorphism and to not attend to context.

Although some participants described ways sameness could be helpful for teaching homomorphism (motivating instruction and leveraging intuition) this happened less than with isomorphism. Some motivated a specific aspect of homomorphisms: "Help understanding the importance of study of kernels." Others described how sameness can aid intuition: "If [a homomorphism] is injective, you could talk about how the structure of the domain is the "same" as the structure of the range, and again this informal notion could make the concepts accessible for students." Again, this participant did not make a blanket statement about sameness in homomorphism but qualified it as useful for considering injective (one-to-one) homomorphisms.

Pedagogical concerns about using sameness for homomorphism were similar to isomorphism concerns. Respondents often described misconceptions about the strength of sameness in homomorphism. For example: "Again, the wrong sort of sameness, as in equality of elements of sets, could be problematic if the student, for example, thinks all identities are actually the element 0." Participants also described issues with being imprecise, including difficulties that could arise when students wrote proofs: "If students get too comfortable expressing things "are the same" without being formal, their proofs can very quickly become incorrect." Although this participant had previously noted utility in thinking about sameness with homomorphism as it connected to the isomorphism theorems, they acknowledged dangers in using loose definitions.

Olanoff, D., Johnson, K., & Spitzer, S. (2021). *Proceedings of the forty-third annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Philadelphia, PA.

Types of Sameness

Other responses focused on the nature of sameness and specific types of sameness. Some detailed problems if using sameness for specific concepts by highlighting the context-dependent nature of sameness. For example: "The context and criteria for sameness need to be clear for isomorphism to be something that can be empirically verified as true." This highlights the necessity of describing the context in which sameness is used but gives few details. Others were more specific, describing different levels of sameness: "There are different "strengths" of sameness: equals, equivalent, related to, almost/weakly equivalent, etc. There is not a one size fits all to sameness." This provides a variety of types of sameness that might be placed on a continuum for strength comparison.

Similarly, the participants were specific about other concepts that could be confused with isomorphism, such as being equal or identical. Consider a confusion with equality example: "Isomorphic is not the same thing as "equals" as it does not imply a canonical identification. The word "same" can trip people up in that way." Here, confusion between different mathematical understandings of sameness, isomorphism and equality, are specifically highlighted. Similar issues arose with identical: "In common, nonmathematical parlance, same means identical, so when students hear the word "same" they may think identical." Note this participant's identification with sameness and identical, a strong type of sameness.

Many respondents compared isomorphism and homomorphism (39%), with a focus on the strength difference. For example: "Same has too strong a connotation in most students minds and they may interpret this to mean isomorphism rather than homomorphism." Implicitly, the participant seems to suggest that sameness implies a strong relationship, so students will identify the stronger concept (isomorphism) with sameness.

Informal Sameness

Participants used descriptive language of varied specificity to highlight sameness in isomorphism and homomorphism. Some participants highlighted shared defined properties or generally referred to same behavior. For instance, this participant was coded as same properties: "…Also, the facts that properties like cyclic and Abelian are preserved by isomorphisms." Notice, the respondent referred to defined properties that are shared by isomorphic structures. Other participants wrote generally of shared behavior for isomorphism or homomorphism: "The idea of an isomorphism is that two different sets of objects can behave the same in certain scenarios." And "With a homomorphism, the objects of the image will behave in "the same" way as the domain (or quotient based on the domain)." While highlighting the sameness of objects linked by a morphism, such responses did not provide details on the shared sameness.

Participants also used renaming/relabeling and matching language to describe isomorphism and, to a lesser extent, homomorphism. This participant described isomorphism in terms of a renaming of elements: "I like to emphasize to students that algebraists care about the algebraic structure and equations, and we don't care nearly so much about what we choose to name the elements in these structures." Note they highlighted the arbitrary nature of element names in keeping with Rupnow's (2021) distinction between renaming/relabeling and matching. Another participant described isomorphism in terms of matching: "For finite groups where Cayley tables are not too time-consuming either to make or to understand, one beneficial way is to see that they can be arranged to have the same overall pattern." Notice this respondent referred to rearranging Cayley tables to demonstrate a matching between appropriate elements in isomorphic objects.

Structure-preservation and operation-preservation were used to describe both isomorphism and homomorphism, but slightly more often for homomorphism. For example: "It gives a colloquial way of saying 'the algebra doesn't change' for particular structures. Things like order, dimension, and so on are preserved." Some explicitly connected structure-preservation to homomorphisms: "I teach homomorphisms as functions which preserve group structure. Homomorphic images are 'large scale structure' while subgroups are 'small scale structure' (at least in examples like symmetric groups and matrix groups)." We believe this participant means that homomorphisms reveal aspects of the domain group's structure by examining a simpler image. Most operation-preservation seemed focused on the homomorphism property: "A homomorphism preserves the operations of the algebraic structures. For example, it will take the identity element of one algebraic structure to the identity element of another algebraic structure." This explanation of operation-preservation foregrounds an identity connection, which highlights a specific type of shared structure.

Disembedding examples highlighted shared properties of the domain and codomain. This example highlighted how relevant shared structure could give insight into a group:

> Sometimes it is useful to think of a group as "sitting inside of" another group, even if in a literal sense the subgroup you are thinking of is not a subset of the bigger group. For example, one might think of some copies of the dihedral group D_4 sitting inside of the symmetry group of the cube…

Notice, although D_4 describes the symmetries of a square and a similar pattern of symmetries exist in the symmetry group of the cube, the underlying elements are not interchangeable, and we would not consider D_4 a subset of the symmetry group of the cube. Nevertheless, recognizing their shared structure could yield insight into the symmetry group of the cube.

Forming equivalence classes was used to discuss sameness of elements in homomorphisms:

> We often build new structures from old by a quotient structure which makes use of an equivalence relation. A homomorphism is one source of such an equivalence relation (but not the only example). I certainly believe that this is an immensely useful way to build structures. And the 'sameness' concept is at its root (in the quotient structure, elements are identified as 'the same' if they lie in the same equivalence class).

Observe that equivalence class language groups elements of a similar nature together into the same equivalence class, which highlights a similarity among these elements within the structure.

Functions vs. Structures

8% of respondents contrasted mapping (isomorphism) and structural (isomorphic) aspects of the concept of isomorphism. For example: "Two groups (for example) can be isomorphic, but the isomorphism may not be obvious….the groups $(C,+)$, and $(R,+)$, are isomorphic because they are isomorphic as Q-vector spaces, but it is fundamentally impossible to write down an explicit isomorphism!" Here the respondent emphasized that objects being isomorphic did not imply that an isomorphism specifying which elements act the same would be easy to find or define, despite such an identification being a likely criterion for considering objects the same.

More commonly (19%), responses detailed the difference between mapping (homomorphism) and structural (homomorphic) interpretations of homomorphism. For example:

> Students who are used to thinking about isomorphic = "the same" will want to think the same thing about homomorphism and will start taking about "G and H being homomorphic" without realizing that the concept is meaningless, and that when studying homomorphisms, we are typically more interested in the properties of the function itself rather than in what it tells us about the structures independently from the function.

Unlike isomorphism, where function and structural aspects are both commonly discussed, the participant here emphasizes that the mapping is the important part of homomorphism.

Some participants provided a way to interpret homomorphism structurally by involving isomorphism. Specifically, 6% of participants referenced the Fundamental Isomorphism Theorem to provide a way to connect sameness, isomorphism, and homomorphism. For example, one participant observed: "I guess the First Isomorphism Theorem should come to mind here. If you quotient out by the kernel, then you get the "same" group as the image, right?" Here we see how the concepts of homomorphism, isomorphism, and quotient groups are linked via theorem: the quotient and image are isomorphic, and the homomorphism defines the kernel.

Discussion and Conclusions

In this large-scale study, we confirmed some findings of prior small-scale studies. Specifically, relabeling/renaming, matching, structure-preservation, operation-preservation, and generic sameness metaphors like same behavior (Hausberger, 2017; Leron et al., 1995; Rupnow, 2021; Weber & Alcock, 2004) were all used by some mathematicians to describe isomorphism. Similarly, structure-preservation, operation-preservation, disembedding, and equivalence class metaphors (Hausberger, 2017; Rupnow, 2021) were used by respondents when describing homomorphism. However, none of these particular metaphors were used by more than a quarter of participants. Furthermore, though it appeared, only three mathematicians described homomorphism in terms of equivalence classes, although it was commonly used by one of Rupnow's (2021) algebra instructors. These differences may indicate that Rupnow's (2021) instructors used uncommon language for homomorphism or could suggest that examining language in instruction as well as out-of-class contexts is important to examine the breadth of language used for isomorphism and homomorphism. Future research should examine the prevalence of these metaphors in instruction for larger groups of mathematicians.

This study also shows a difference between mathematicians' perceptions of the relevance of sameness to isomorphism and homomorphism. This was demonstrated through limited resistance to the concept of sameness for isomorphism (81% of respondents coded with a partly helpful code), and resistance to "sameness" largely related to imprecision, not irrelevance. In contrast, a majority of respondents resisted or did not clearly relate sameness to homomorphism (39% of respondents coded with a helpful-based code), and the "sameness" in homomorphism related only to parts of structures, not whole objects. Differences were also emphasized through participants' portrayals of the function and structure aspects of these concepts (isomorphism vs. isomorphic and homomorphism vs. homomorphic) that highlighted whole object and partial object differences between isomorphism and homomorphism. While these results are not very surprising, they confirm that sameness is relevant to isomorphism and can be a conceptual base for making connections to other subjects as long as the reduction in precision is acknowledged.

Finally, the context-dependence of sameness was a clear theme in participants' responses. Distinguishing isomorphism from other, potentially "stronger" forms of sameness, like equality and being identical, as well as "weaker" forms like homomorphism relates to the importance of precision: what exactly or how much needs to be the same in a particular situation. Similarly, participants' concerns about misconceptions largely related to confusion about whether elements or groups need to look the same or what happens when intuition about sameness leads astray. However, content-dependence can also be viewed as a purpose for examining mathematical sameness. Considering how sameness appeared through equality in prior classes and relating that to isomorphism could create new connections for students and help them appreciate the subtleties of mathematical definitions, which we know are often problematic for students (Edwards & Ward, 2008). Future research should examine how many of these sameness connections are already made by students as well as examine how to help students make such connections, both to help future teachers appreciate how different notions of sameness have appeared in K-12 settings and to help math majors reexamine their prior learning.

Acknowledgments

This research was funded by a Northern Illinois University Research and Artistry Grant to Rachel Rupnow, grant number RA20-130.

References

Dubinsky, E., Dautermann, J., Leron, U., & Zazkis, R. (1994). On learning fundamental concepts of group theory. *Educational Studies in Mathematics, 27*(3), 267–305.

Edwards, B., & Ward, M. (2008). The role of mathematical definitions in mathematics and in undergraduate mathematics courses. In M. Carlson & C. Rasmussen (Eds.), Making the connection: Research and teaching in undergraduate mathematics (pp. 221–230). Washington, DC: Mathematical Association of America

Hausberger, T. (2017). The (homo)morphism concept: Didactic transposition, meta-discourse, and 348anderbilt348e348. *International Journal of Research in Undergraduate Mathematics Education, 3*(3), 417–443. Doi: 10.1007/s40753-017-0052-7

Lakoff, G., & Nú ez, R. (1997). The metaphorical structure of mathematics: Sketching out cognitive foundations for a mind-based mathematics. In Lyn D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 21–89). Mahwah, NJ: Erlbaum.

Leron, U., Hazzan, O., & Zazkis, R. (1995). Learning group isomorphism: A crossroads of many concepts. *Educational Studies in Mathematics, 29*(2), 153–174.

Melhuish, K., Lew, K., Hicks, M. D., & Kandasamy, S. S. (2020). Abstract algebra students' evoked concept images for functions and homomorphisms. *Journal of Mathematical Behavior, 60*.

Rupnow, R. (2017). Students' conceptions of mappings in abstract algebra. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, and S. Brown (Eds.), *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education Conference on Research in Undergraduate Mathematics Education,* (pp. 259–273), San Diego, CA.

Rupnow, R. (2021). Conceptual metaphors for isomorphism and homomorphism: Instructors' descriptions for themselves and when teaching. Journal of Mathematical Behavior, 62(2), 100867. Doi: 10.1016/j.jmathb.2021.100867

Saldaña, J. (2016). *The coding manual for qualitative researchers.* Thousand Oaks, CA: SAGE Publications, Inc.

Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*(2-3), 209–234.