THEO'S REINVENTION OF THE LOGIC OF CONDITIONAL STATEMENTS' PROOFS ROOTED IN SET-BASED REASONING

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This report documents how one undergraduate student used set-based reasoning to reinvent logical principles related to conditional statements and their proofs. This learning occurred in a teaching experiment intended to foster abstraction of these logical relationships by comparing the predicate and inference structures among various proofs (in number theory and geometry). We document the progression of Theo's emergent set-based model from a model-of the truth of statements to a model-for logical relationships. This constitutes some of the first evidence for how such logical concepts can be abstracted in this way and provides evidence for the viability of the learning progression that guided the instructional design.

Keywords: mathematical processes, reasoning and proof, advanced mathematical thinking

Teaching logic for the purpose of supporting students' apprenticeship into mathematical proving imposes fundamental challenges regarding how the content-general relationships of logic can be operationalized within students' reasoning about particular mathematical concepts. Scholars affirm that this requires that logic be understood in both its syntactic and semantic aspects (Barrier, Durand-Guerrier, & Blossier, 2009; Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012). In other words, students must be able to reason about the form of statements and arguments as well as the way they refer to mathematical objects. Previous studies find that logic taught syntactically often does not foster understandings that are useful in context (e.g., Hawthorne & Rasmussen, 2014), and textbooks downplay the referential aspects of logic (Dawkins, Zazkis, & Cook, 2020). How then are students to abstract logical relationships that generalize across contexts and yet interface with their meanings for particular concepts? How do such logical understandings become functional for comprehending mathematical proofs?

In our ongoing investigations of these questions (Dawkins, 2017, 2019; Dawkins & Roh, 2020), we have found that set-based reasoning can provide a unifying structure by which students abstract key logical relationships. Set-based reasoning is propitious for student thinking and it provides a clear means by which students can interpret statements about very different topics as being in some sense the same (Hub & Dawkins, 2018). We claim the two questions above can be answered by guiding students to formulate logical understandings by comparing interpretations and generalizing their reasoning about mathematical texts in particular contexts.

In this report, we share a case study that illustrates one student's pathway to reinventing some basic logical principles of conditional statements: proof by universal generalization, converse independence, and contrapositive equivalence. As we shall explore, Theo's construction of these new logical relationships depended on his coordination of two ways of thinking: properties defining sets of objects and proofs showing implications between properties as relating such sets. Our teaching experiment methodology (Steffe & Thompson, 2000) allows

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us to provide a detailed account of Theo's learning process, rooted in his meanings and activity (Piaget & Garcia, 1991; Thompson, 2013). This account of a student abstracting logical relationships is a novel contribution to the literature. We analyze the episode using the emergent models framework to document the emergence of a new mathematical reality (Gravemeijer, 1999), namely that of content-general logical structure rooted in set relationships. We highlight Theo's learning pathway to demonstrate the viability of the learning progression, which closely matched what we intended in the instructional design.

Conceptual Analysis of the Logic of Conditionals

In our prior teaching experiments (Dawkins, 2017; Hub & Dawkins, 2018), we guided students to reinvent logical principles by comparing their interpretations of mathematical statements of the same logical form. In this experiment, we extended this task sequence by asking students to read theorems paired with 2-4 proofs each and to determine whether each proof proved its associated theorem. We encouraged students to associate to each property the set of objects that makes it true (*reasoning about predicates*, Dawkins, 2017). This allowed them to formulate generalizable truth-conditions for the statements and generalizable interpretations of the proofs. In this section, we shall present a conceptual analysis (Thompson, 2008) of these set-based understandings to clarify what we intended students to learn.

Each theorem was a universally quantified conditional: "For any [$x \in S$], if [$P(x)$], then [$Q(x)$]." We use brackets since the statements/proofs that students saw always had particular objects and properties in these slots (e.g., "For every integer x, if x is a multiple of 6, then x is a multiple of 3" and "For all quadrilaterals $\blacksquare ABCD$, if $\blacksquare ABCD$ is a rhombus, then it is a parallelogram"). Each proof was either a direct proof, a proof/disproof of the converse, a proof by contraposition, or a proof/disproof of the inverse (see Table 1). No proofs contained errors. All the proofs (as opposed to disproofs) used universal generalization. The principle of universal generalization (Copi, 1954) states that a proof regarding an arbitrary particular justifies the claim for the whole set of such objects. Choosing such an arbitrary particular is conventionally expressed using the imperative "let" and assigning a property to an object. The argument that follows must depend only on that property and thereby the argument will carry to all objects with the property (Alcock & Simpson, 2002). Such proofs that "property P implies property Q" justify a subset relationship: the set of objects with P is a subset of the set of objects with Q. This is the truth-condition for such conditional statements, which we refer to as the *subset meaning* (Hub & Dawkins, 2018). Such statements are false when there is an object with property P and not property Q. To connect the subset meaning to proofs, students must relate chains of inference to the underlying sets of objects. Proofs establish implication relationships among properties, which are tantamount to containment relationships between the sets of objects with the properties. Counterexamples show lack of set containment and property implication.

Notice that the direct proof of a conditional and the proof of its converse (if both possible) deal with the same two sets of objects. They prove two facts about those sets: the Q's contain the P's and vis-versa. In this case the two sets are equal, meaning the exact same objects have the two properties. Since not every implication involves two equal sets, these two proofs are taken as independent (the converse proof does not prove the original theorem). However, the contrapositive proof is understood to prove the theorem as the contrapositive statement is equivalent to the original theorem (arguments for this will appear in the results section). Since we expected students to abstract these structures from reading statements and proofs, we

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purposefully maintained a parallel structure to the proofs. The disproofs are somewhat oddly stated, but we intended for them to match the same first-line/last-line structure to help students associate each proof to the statement it is normatively understood to prove or disprove.

Table 1: Forms of proof presented for comparison

Direct proof	Converse proof	Converse disproof	Contrapositive proof	Inverse disproof
Original theorem	"For any [$x \in S$], if [$Q(x)$], then [$P(x)$]."	"For any [$x \in S$], if [$Q(x)$], then [$P(x)$]."	"For any [$x \in S$], if not [$Q(x)$], then not [$P(x)$]."	"For any [$x \in S$], if not [$P(x)$], then not [$Q(x)$]."
Proof: Let x have property P. … Thus, x has property Q.	Proof: Let x have property Q. … Thus, x has property P.	Proof: Let x have property Q. x could be a. a does not have property P.	Proof: Let x not have property Q. … Thus, x does not have property P.	Proof: Let x not have property P. x could be a. a does have property Q.

Guided Reinvention and Emergent Models

Our instructional sequence was inspired by the Realistic Mathematics Education design heuristics of guided reinvention and emergent models (Freudenthal, 1991; Gravemeijer, 1999). Guided reinvention entails providing students with experientially real situations they can easily imagine and from which they might elaborate key mathematical ideas. The emergent models heuristic describes how students may first develop a *model-of* the situation. They then elaborate the model by applying it to new situations until the model becomes a new body of understanding apart from the situation(s) it interpreted. The model then becomes a *model-for* reasoning about new problems and concepts. The model's elaboration for mathematical exploration constitutes the establishment of a *new mathematical reality* for the student. The key distinction between model-of and model-for is the extent to which the structure of the model reflects the original situation or alternatively comes to take on its own internal meaning for the student.

To apply these tools to teaching logic to undergraduates, we first wondered what kind of experientially real activity would lead students to perceive questions about logical structure. Logic generalizes across language and proofs, which led us to engage students in comparative reading of statements and proofs of parallel form. By focusing them on set structure, students can develop a *model-of* how each statement refers to sets of objects (reasoning about predicates) and what it means for conditional statements to be true and false (in terms of set relations). By considering how this set structure repeats across various statements and proof texts, students may extend their *model-for* reasoning about content-general logical relationships.

Context independence is a key aspect of how we study students' models. Students often draw the contrapositive inference in a particular context. For instance, they may infer that since all multiples of 6 are multiples of 3, a number that is not a multiple of 3 cannot be a multiple of 6. While this relates to a logical principle, it is not a *logical understanding* for that student if they only apply it locally. We call an understanding *logical* to the extent it generalizes across contexts. Only content-general understandings will support students in reasoning about the logical relationship between *any* conditional statement and its contrapositive statement/proof.

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Methods

As part of a grant project developing constructivist models of students' learning of logical principles through guided reinvention, we conducted 8-12 session teaching experiments with pairs of undergraduate students recruited from Calculus 3 classes at two large public universities in the United States. The site for this study's data is a Hispanic Serving Institutions (HSIs). Students volunteered to participate and completed a screening survey to verify that they did not already know the target concepts to be taught (see Roh & Lee, 2018). The experiment featured in this paper was conducted remotely once per week over Zoom using OneNote as a shared space for reading and writing. The two participants chose the pseudonyms Theorem (which we abbreviate as "Theo" for clarity) and Phil. The lead author served as the teacher/researcher and the other three authors acted as outside observers (Steffe & Thompson, 2000). Each session lasted between 60 and 90 minutes and participants were compensated monetarily for their time.

This experiment consisted of an intake interview with a pre-test, nine instructional sessions, and an exit interview with a post-test. During the exit interview, we asked students to choose how they wanted to be identified in terms of their ethnic and gender identities and how those identities were significant for their mathematics learning at university. Theo identified himself as a white, non-Hispanic male. At the time of the study, he was in his first year of university as a finance and mathematics double major. He described himself as "passionate" about mathematics. Phil, an engineering technology major, identified himself as a Hispanic male. The two students worked productively and respectfully together, though they generally operated in parallel rather than interactively. We focus on Theo in this report because of the clear evidence of his progression toward our learning goals. Our models of other students' learning progressions will appear in other reports. Theo constitutes a clear existence proof for our intended learning path.

Consistent with teaching experiment methodology (Steffe & Thompson, 2000), the research team continuously made conjectures about the two students' understandings and tested those conjectures through questioning and iterative task design. The research team met once or twice between sessions to analyze and plan for subsequent sessions. All sessions were recorded on at least two or three screens: the interviewer screen that moved between pages in OneNote and two screens dedicated to capturing Theo and Phil's pages respectively. All main study sessions were transcribed. Our retrospective analysis drew upon field notes, transcripts, and compiled video.

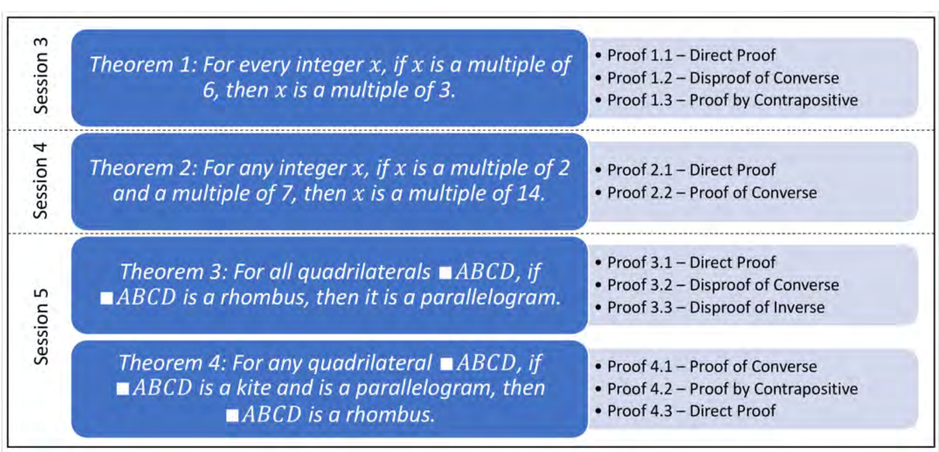

Figure 1: Sequence of proof reading tasks.

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Teaching Progression

In the first two instructional sessions of the experiment, Theo and Phil read sequences of universally quantified conditional statements and considered the relationships between the sets of objects that made the if-part true and objects that made the then-part true (hereafter the "if-set" and "then-set"). We intended them to formulate the *subset meaning* (see Conceptual Analysis) for such statements and the conditions for a counterexample. In the next three sessions, they read theorems and proofs as shown in Figure 1. Theorems 1-4 were chosen to intentionally vary the relationships between the underlying sets (proper subset in 1 and 3; set equality in 2 and 4) and to vary the mathematical context (number theory in 1 and 2; geometry in 3 and 4). In the sixth session, which is the last featured in this report due to space limitations, Theo reviewed all of the theorems and proofs and his decisions about which proved the associated theorems. We call this the Comparison Task. We sought for him to systematize the relationship between the logical form of the proof and whether it proved the given theorem (evidence of a model-for logical reasoning that generalizes across context).

Results

Developing Set-Based Meanings (Days 1-2)

In the intake interview, Theo read a direct proof, inverse proof, converse proof, and contrapositive proof of the claim "For any integer x, if x is not a multiple of 3, then $x^2 - 1$ is a multiple of 3." He affirmed the direct proof proved the theorem and denied that the other three did. His rejection of the converse proof was based on whether the middle section of the argument worked, not based on its reverse order from the theorem. Productively, he showed early evidence of associating an equation such as $x = 3k + 1$ to a set of values (*reasoning with predicates*).

On the first day of instruction, once Theo and Phil had assigned truth values to all the given conditionals, the interviewer asked the students to explain the relationship between the if-sets and then-sets. Theo initially drew a diagram showing the then-set as a circle nested within the circle for the if-set. He then thought about a specific statement (Theorem 1) and revised his answer to say, "Because if you think of it, one to 100, that'd'be more multiples of 3. So that's the larger set in these multiples of 6. That's a subset." Over those first two days, the pair came to confirm this interpretation of the set relationship for true conditional statements. They also agreed that a conditional was false whenever an element of the if-set was outside the then-set.

Theo generally represented complement sets using separate circles rather than the inside and outside of a given circle. In reasoning about Theorem 1 and its contrapositive statement, Theo drew three circles that corresponded to the equations $x = 3k$, $y = 3k - 1$, and $z = 3k - 2$. This reflects a common tendency to replace negative categories with a positive description (Dawkins, 2017). He then drew a smaller circle inside the x circle to represent the multiples of 6. This pattern of representing a partition by separate circles persisted throughout the experiment.

Reading Number Theory Proofs (Days 3-4)

During the third and fourth teaching sessions, Theo eventually adopted normative answers as to whether each proof proved the associated theorem based on his set-based reasoning developed in the first two days. At the beginning of Day 3, the interviewer asked Theo to summarize what he had learned the previous two days. He reported:

> It's true if the statement, if it exists inside then or is the same size as then… If the condition exists outside of the parameters of the then statement. Like if it goes beyond the bubbles or diagrams that we created, if it extends beyond it then that's when it's not true.

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We note two things about this explanation. First, Theo acknowledged that either the if-set may be contained in the then-set or they may be equal. Second, he referred to the parts of the statement as having physical extension in space and being contained by one another. This constituted his initial model-of interpreting statements by which he determined whether conditional statements were true or false. Each part of the statement corresponded to a group of objects and those objects could be imagined as taking up a region enclosed by a closed curve.

Theo affirmed that Proof 1.1 (direct) proved Theorem 1. He did so focusing on the steps within the proof, not the order from first to last line. He denied Proof 1.2 (converse), saying:

> I don't agree with this theorem [sic] because we're trying to say that if it's a multiple of six then it's a multiple of three, not if it's a multiple of three then it's a multiple of six. It kind of goes into what we were saying last week, if the condition falls outside of the realm of all possibilities and the then statement, then it doesn't hold up, it's not true.

In the first part of the quote, Theo restated Theorem 1 as what "we're trying to say" and contrasted it with what Proof 1.2 is addressing, which he articulated as the converse conditional. He thus attended to the order of the theorem and the first/last-lines of the proof in order to distinguished the meaning of the theorem from what the proof accomplished and to show conflict between the two. He then elaborated what the proof (which presents the counterexample 15) proved: that the if-condition for the converse "falls outside the realm" of the then-statement. He thus shifted back into the language of sets of objects as spatial regions.

Both Theo and Phil agreed that Proof 1.3 (a proof by contraposition) proved Theorem 1. They had read the theorem and contrapositive statement on Day 1 and then noted then that the contrapositive should be true based on the fact that all multiples of 6 are multiples of 3. It is worth noting that Proof 1.3 contains 19 lines and explores how a number having a remainder of 1 or 2 when divided by 3 means it has a remainder of 1, 2, 4, or 5 when divided by 6. The interviewer invited Theo to draw a diagram for how he understood the proof to ascertain how he was making sense of the complex case structure.

> Int: Okay. Can y'all try to use the diagrams that we were using the last two times we met? We have this kind of meaning for what the theorem says in terms of the group of multiples of 3 covering the group of multiples of 6. Can y'all try to explain to me how is it that Proof 1.3 proves it using that idea?
>
> Theo: I think you got to look at, it would be the pattern of all the non-multiples of three and you could be like, 1, 2, 4, 5, 7, 8. And you have that subset of numbers, and then you have the other subset that's three and obviously they're not in each other. However, the multiples of six does not exist inside the non-multiples of three. It only lives inside the multiples of three… It's talking about the subspace when x is not a multiple of three [see Figure 2], which is going to be this whole range of numbers on the left side. And basically, it proves that there exists no of this smaller subset that's on the right side, the blue circle that exists in the non-multiples of three, not even like a cross over even.

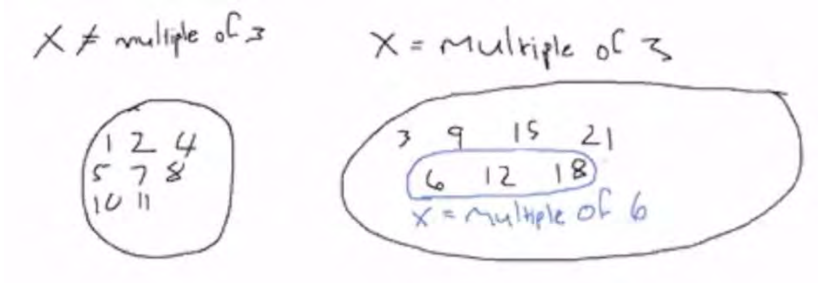

Figure 2: Theo's diagram for Proof 1.3

In this argument, Theo justified Proof 1.3 using what Hub and Dawkins (2018) called the *empty intersection meaning*, namely that "if x is not a multiple of 3, then x is not a multiple of 6" is true because there is no overlap between non-multiples of 3 and multiples of 6. While this is distinct from the subset meaning developed on the first two days, it supported Theo in perceiving symmetry between Proof 1.1 and Proof 1.3. He explained, "[Proof] 1.3 shows that the [multiple of 6] circle doesn't exist in the circle of non-multiples of three, while proof 1.1 would show it exists in the circle with multiples of three." By this point he was comfortable treating the complement of multiples of 3 as a category. However, his notation and reasoning in some sense expressed that x was *not in the set* of multiples, rather than saying it *had the property* of being a non-multiple. His empty intersection meaning similarly negated the "element of" relation, not the property of being a multiple of 6. We have found this preference common, meaning students often avoid treating a negative property as constituting a predicate (Dawkins, 2017). Theo's justification is similar to the arguments Yopp's students produced for how contrapositive proofs eliminate counterexamples (Yopp, 2017).

During Day 4, Theo affirmed Proof 2.1 (direct) and denied Proof 2.2 (converse). He did so using an analogy to Theorem 1 and Proof 1.2 (converse), desiring consistency. He explained:

> In this case, the if and the then are the same set. But, if you switch them around in a set where they're not the same, then it doesn't necessarily work out that way. In this example, it works out, but switching the if and then doesn't necessarily mean it will work out every time.

This argument marks a key development in Theo's thinking because his model-of the set structure allows him to make an analogy between Theorem 1 and Theorem 2 that determines how the proofs do or do not support the theorems. In Dawkins and Roh (under review) we discuss a prior study participant who similarly recognized the analogy, but she denied that it held force. That participant perceived that the difference between subset situations and equal set situations meant the proof relationships for Theorem 1 do not apply to Theorem 2. It is unclear why Theo took a different interpretation. Still, it shows how his set-theoretic model had become a model-for reasoning about more abstract relationships between theorems and proofs. However, we learned in the next sessions that his model still carried some contextual dependence.

Reading Geometry Proofs (Day 5)

Recall that our operating definition for a student's understanding as being logical is that it generalizes across semantic content. Theo's use of his set-theoretic model showed that to some extent he was attending to logical structure on the number theory tasks. In contrast with his prior reasoning, on Day 5 Theo affirmed that a Proof 4.1 (converse) proved the theorem and he denied that Proof 4.2 (contrapositive) did so. Initially Theo and Phil judged that Proof 4.2 was irrelevant to Theorem 4. Though Phil later developed an indirect argument for why Proof 4.2 supported the theorem, neither judged that it proved. Theo also had trouble applying his subset meaning to this theorem because he represented the if-set (kite and parallelogram) using two overlapping circles to show what is shared between the two properties. He then identified that the conclusion (rhombi) existed in the overlap, which led him to imagine the then-set as nested within the if-set. We hypothesize that the three-set structure and the more complex nature of negating the hypothesis kept Theo from structuring these theorem/proof pairs as he had on previous tasks.

Comparison Task (Day 6)

On Day 6, we presented Theo with all of the theorems and proofs from the previous three sessions along with his decision about whether each proof proved the theorem or not. We asked him to look for patterns among the proofs, his decisions, and how the proofs did or did not prove the associated theorem. He began by grouping all the direct proofs and affirming they all proved their associated theorem: “You start by saying like, okay, we have this if, it meets such criteria and then like, it continues on to conclude, okay, this criteria can be put inside this larger space.” He initially placed Proof 4.1 (converse) in this group, and then removed it since it started with “the then.” He then grouped the first three (dis)proofs of converse, omitting Proof 4.1. Later he decided to move Proof 4.1 into that group. When asked about his claim that Proof 4.1 proved the theorem, he decided to change his decision “to be consistent.” He explained the pattern among the proofs that proved the theorem: “We're going to have to start by looking at the smallest subspace, I’m saying for the other yeses (sic), excluding 1.3, because that’s doing it from the other way. But proving the converse doesn’t necessarily prove the theorem.”

Theo grouped all the rest of the proofs (inverse and contrapositive) together as not direct or converse. We explain this in terms of his set-based model, which operated with the structure of if-sets, then-sets, and everything else. Since his model did not include the complement of the if-set, he did not distinguish the order of inverse and contrapositive. Theo’s groupings demonstrate how the structure of the proofs reflected the structure of the sets, not the statements per se.

The interviewer asked Theo to explain again his argument for why Proof 1.3 proved, which he did in terms of his empty intersection meaning. The interviewer then asked him to apply the same argument to Proof 4.2. Upon considering he decided Proof 4.2 proved, explaining:

> Because I think when we do it like the same way in 1.3, we’re saying, okay, it has this property that it’s a non-rhombus. And if it’s not a rhombus, it either exists in the space that, like we don't have to look at it like is not a kite or is not a parallelogram… So it’s basically saying that the non-rhombi don’t have properties of both… Which is basically saying that the space where they have both of the properties is rhombi.

By adapting his empty-intersection argument, Theo began to construct contrapositive equivalence as a logical concept rooted in his set-based model-for reasoning about proofs.

Discussion and Conclusions

We proffer this account of Theo’s learning as an account of how logical understandings can emerge from set-based reasoning about the structure of conditional statements and their proofs. We argue that Theo’s ability to see necessity in logical relationships (e.g., converse proofs cannot prove for consistency) and to generalize logical arguments (e.g., applying his empty intersection argument from Proof 1.3 to Proof 4.2) as evidence that his set-theoretic model constituted a new mathematical reality for *reasoning about logic* (Dawkins & Cook, 2017).

To further illustrate what was involved in Theo’s learning, we highlight some shifts in Theo’s ways of talking about the statements and categories in the statements. First, he became

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comfortable talking about negative categories such as non-rhombus. Second, he shifted rather fluidly between using a) set language interpreted as spatial regions such as "smallest subspace," b) property language such as "meets such criteria," and c) syntactic/temporal order language of "if," "then," and "start." In this way, Theo coordinated quantification, property relations, and statement syntax to give meaning to these complex proof texts. What is more, these understandings allowed him to perceive theorems/proofs about number theory categories and geometric categories as the same, since they all shared set-theoretic structure. We conjecture that developing negative categories and exploring how properties stand for whole classes of objects are essential parts of his construction of a logic of conditional statements and proofs.

We began with questions about how students' understanding of logical relationships can interact with their content-specific reasoning. We claim that Theo's learning progression provides an actionable answer to this question. Specifically, logical concepts can be reinvented in context via the emergence of set-based models for the truth and falsehood conditions and the structure of mathematical proofs. Ongoing work is seeking to understand other students' pathway to these abstractions to create generalizable learning sequences for undergraduate students' introduction to mathematical proving.

Acknowledgements

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