ELEMENTARY PATTERNING PROBLEMS: VISUAL AND NUMERICAL STRUCTURING

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Research on elementary students' reasoning on patterning problems with pictorial representations has illustrated that students can visualize structure in patterns in different ways. In this paper, we offer a characterization of students' spatial structures and numerical structures and explain how the link between these two structures can support students' generalization of a pattern or prediction of a future value.

Keywords: Elementary School Education, Algebra and Algebraic Thinking

Reasoning about and with functions is a foundational topic in K-12 mathematics. Functional thinking in algebra can be defined as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (Smith, 2008, p. 143). Functional thinking builds from patterning in elementary grades to generalized algebraic equations in secondary mathematics. In grades 3-5, elementary students are expected to "describe, extend, and make generalizations about geometric and numeric patterns; represent and analyze patterns and functions, using words, tables, and graphs" and use equations to express mathematical relationships, inherently linking patterns, relations, and functions (NCTM, 2000, p. 158). To prepare students for functional thinking in later grades, Blanton and Kaput (2004) propose that elementary students should move beyond simple patterns in one variable to focus on problems in which two or more quantities vary simultaneously. Indeed, such complex patterning problems are often included in research studies with elementary students (e.g., Stephens et al., 2017; Wilkie & Clark, 2016) and on standardized assessments for elementary students, such as the National Assessment of Educational Progress (NAEP) mathematics assessment and Trends in International Mathematics and Science Study (TIMSS).

The mental activities used by students to generalize a pattern from a table, graph, or pictorial representation are of particular interest in studying students' functional thinking (Smith, 2008). Recent research on student thinking about patterning problems considers both functional thinking and spatial visualization. By analyzing student work on patterning problems from both an analytic and visualization perspective, researchers can understand the ways students reason with and about different function representations, including figures, tables, and generalized rules. While other studies have reported on students' spatial visualization when solving patterning problems with pictorial representations (Hershkowitz et al., 2001; Wilkie & Clark, 2016), in this paper, we identify both the spatial and numerical structures students use when solving a patterning problem and describe how linking a spatial structure with a numerical pattern structure can support a student's generalization of a pattern or prediction of a future value. "Spatial structuring is the mental act of constructing a spatial organization or form for an object or set of objects. Numerical structuring is the mental act of constructing an organization or form for a set of computations" (Battista et al., 2018, p. 211). *Spatial numerically-linked structuring* is a coordinated process in which numerical operations are performed based on a linked spatial structuring (Battista et al., 2018).

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Literature Review and Framework

Three modes by which researchers analyze student reasoning about functions and patterning problems with two varying quantities are recursive, covariation, and correspondence approaches (Blanton & Kaput, 2011; Stephens et al., 2017). The *recursive* approach describes the change within a sequence of values (Blanton & Kaput, 2011). It indicates how to obtain the next value in a sequence given the current sequence value. In a two-column table with two varying quantities, a student using a recursive approach would identify the change in one column independent of the other column and use this change to move from one value to the next within a column. The *covariation* approach describes how the change in two quantities is related (e.g., as x increases by one, y increases by 2) (Confrey & Smith, 1991). The *correspondence* approach describes a rule or mapping that relates any given x-value to a unique y-value (e.g., y = 2x + 3, indicating y-values are 3 more than twice the x-values) (Confrey & Smith, 1991).

While there are multiple learning progressions in the literature describing the ways elementary students may develop these different types of functional thinking (Blanton et al., 2015; Stephens et al., 2017; Wilkie & Clark, 2016), the progressions all provide evidence that students typically begin with recursive or covariational approaches and move toward more sophisticated correspondence approaches. In studies with students in elementary grades, researchers often present patterning problems by providing a series of figures or manipulatives that show a growing pattern in two variables (Stephens et al., 2017; Wilkie & Clark, 2016). This offers an opportunity for students to recognize the relationship between two variables. At times, tables are used to organize or display patterns and data (Schliemann et al., 2001). Standard questions include “far-prediction” problems or tables with a break in the sequence of values which have been used to encourage students to shift their approach from a recursive strategy to either a covariational or a correspondence approach or from a specific relationship between two items to a generalization for the whole set (Blanton et al., 2015; Blanton & Kaput, 2004; Schliemann et al., 2001; Stephens et al., 2017). In general, these studies have shown that young children are capable of functional thinking.

When functional relationships are represented pictorially, spatial thinking becomes an important part of students’ reasoning with functions. Students identify and visualize changes from figure to figure in a pictorial representation in many different ways (Hershkowitz et al., 2001). Visualization is the process involved in constructing and transforming visual mental images (Presmeg, 1997) and impacts the resulting spatial mental image that encodes properties such as location, size, and orientation (Sima et al., 2013). Battista (1999) defines spatial structuring as the mental process by which a person constructs an organization for a set of objects. The process of spatial structuring includes identifying the spatial components of the figure and organizing the components into composites with certain relationships between them. This is of particular interest for patterning problems with pictorial representations because the way a student sees the figure components, figure composites, and interrelationships between figures becomes a part of the student’s reasoning process. Visualization and the resulting spatial structures have the potential to enhance a student’s understanding of algebraic and function concepts (Boaler et al., 2016) and can sometimes influence the way in which a student generalizes a visual pattern or predicts future values. Wilkie and Clark (2016) found that students sometimes transition between multiple visualizations of a pattern while solving a single patterning problem and report that these visualizations likely lead to specific types of generalizations of the numerical pattern.

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Method

To explore students' thinking with pattern and relationship problems, clinical interviews (Ginsburg, 1981) were conducted with a convenience sample of three 4th grade students at a public elementary school in the United States. The students were all in the same mathematics class. According to the teacher, the three students represented the typical range of mathematical abilities in her classroom. The problem chosen for this study was the Pattern of Circles Item (**Figure 1**) of the 4th Grade 2011 TIMSS Questionnaire for which 75% of U.S. students (International 68%) correctly answered Part B, while only 47% of U.S. students (International 39%) correctly answered Part C (IEA, 2013). The problem provides opportunities for students to reason with both pictorial and table representations while predicting a future value. Students were asked to solve the problem while the researcher (second author) observed and asked the students to clarify their thinking. The video and audio recorded interviews were transcribed and reviewed by both authors, examining for evidence of the spatial structure students used when working with the pictorial representation, how they interacted with the table, and how they predicted the number of circles in future figures in the pattern. (Note: Figures in bold refer to the inserted figures in the paper. Figures not in bold refer to the Figures in the Pattern of Circles item of the 4th Grade 2011 TIMSS Questionnaire).

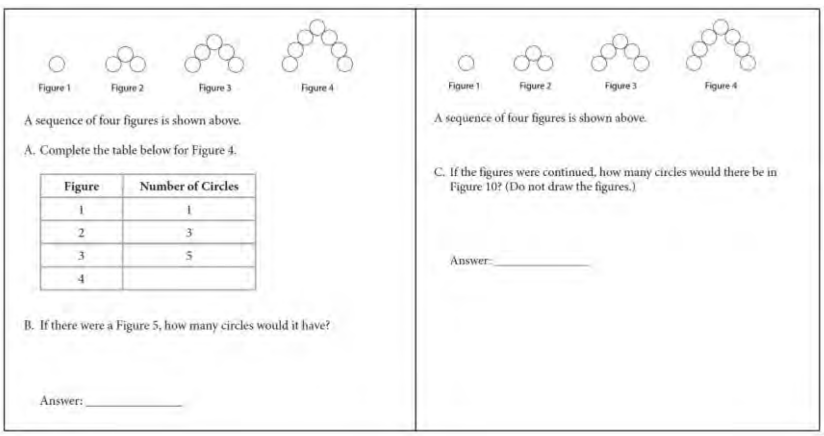

Figure 1 Figure 2 Figure 3 Figure 4

A sequence of four figures is shown above.

A. Complete the table below for Figure 4.

Figure	Number of Circles
1	1
2	3
3	5
4	

B. If there were a Figure 5, how many circles would it have?

Answer: ____________

Figure 1 Figure 2 Figure 3 Figure 4

A sequence of four figures is shown above.

C. If the figures were continued, how many circles would there be in Figure 10? (Do not draw the figures.)

Answer: ____________

Figure 1: Pattern of Circles Problem TIMSS 2011 Assessment (IEA, 2013)

Findings

Student 1: Dennis

Two students in the study, Dennis and Miles, used the same spatial structure (Battista, 1999) when describing the pictorial representation provided in the Pattern of Circles Problem (**Figure 1**). In Part A, Dennis stated, "I know the sequence, it's just adding on two [points to the circles at the bottom of each 'leg' as highlighted in **Figure 2**]." Dennis identified the way he saw the two additional circles in successive figures as the bottom two circles on each 'leg.'

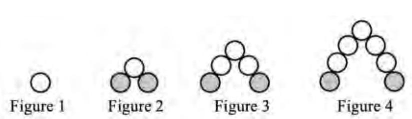

Figure 2. Dennis' Spatial Structure for the Pictorial Representation

While Dennis recognized the pattern in the pictorial representation of the problem, he stated "I don't really get this table." When the researcher explicitly asked Dennis what the numbers in the table might mean, he correctly related the first column to the figure number. The researcher further prompted him by asking what the second column in the table refers to and Dennis replied, "Oh, okay, … The numbers of circles that are in each triangle shaped thing." He then filled in the missing value in the table with the numeral "7".

To solve Part B, Dennis drew Figure 5 consistent with how he spatially structured the two additional circles in successive figures and counted to correctly conclude that there were nine circles in Figure 5. The order in which circles were drawn for Figure 5 is shown in **Figure 3**.

Figure 3: Dennis's Drawing of Pattern of Circles Figure 5

For Part C, Dennis attempted to count the number of circles in Figure 10 by tapping his pencil from left to right under each 'leg' of Figure 4 while counting aloud from circle seven: "8, 9; 10, 11; 12, 13; 14, 15; 16, 17 [see **Figure 4**]. So, I think it's 17." However, this is the correct number of circles for Figure 9, rather than Figure 10.

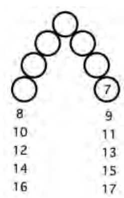

Figure 4: Dennis's Visualization and Counting of Additional Circles

When prompted to further explain his thinking, Dennis recounted the number of circles in Figure 10 using the same spatially structured counting method but was more explicit about the way he kept track of the figure numbers and the number of circles. Starting from circle seven in Figure 4 he stated, "So 8, 9, that would be one [figure more]; 10, 11, that would be two [figures more]; 13, 14 that would be three [figures more]; 15, 16 that would be four [figures more]; 17, 18, that would be five [figures more]." Two errors occurred when Dennis counted the second time. The first error was that he counted five figures from Figure 4, rather than six just as he did the first time he counted. The second error was skipping the number 12 when counting the circles. Coordinating the number of figures and the number of circles at the same time was challenging.

When asked how he knew when to stop adding circles, Dennis stated, "You only need to do five times two. Just need to do two five times. That's how you get your answer." While further explaining his thinking, Dennis corrected his counting error: "Because it says 10. Wait, six times [not five]. If you work with [Figure] four. Yeah, it's six times." Dennis again recounted the number of circles, using the same spatially structured counts illustrated in **Figure 4** and reached the correct number of circles, 19. He then generalized the counting process. "So yeah, you do two six times. Two times six, plus seven…Because there's only Figure 4, not Figure 5. If there was a Figure 5, then you only need to do five times, but there's no Figure 5."

Dennis correctly concluded he would need to add six sets of two circles to build Figure 10 from Figure 4, and extended his thinking in a way that would have facilitated starting with a different figure number. Dennis developed his generalization by imagining the changes from Figure 4 to Figure 10 using the spatial structure he described when looking at the figures (**Figure 2**), while explicitly stating the relationship between the addition of two circles and the successive figure in the sequence.

Student 2: Miles

In contrast to Dennis for Part A, our second student, Miles immediately wrote "7" in the table for the number of circles in Figure 4, making an unprompted connection between the pictorial and table representations.

> Miles: So, I saw one here [points to the one circle in Figure 1 and to the "1" in the output column], three here [points to the three circles in Figure 2 and to the "3" in the output column], five here [points to the five circles in Figure 3 and to the "5" in the output column]. So, I counted these [the circles in Figure 4], and I got an answer of seven, so I put that in the box.

When asked how he determined the number seven, Miles described the same spatial structure as Dennis (**Figure 2**).

> Miles: So, I saw figure one, and then I saw figure two, and right away I saw that it added two more circles [points to the bottom two circles on each 'leg' of Figure 2]. So then in figure three, I saw it add two more circles [points to the bottom two circles on each 'leg' in Figure 3]. And again, in figure four, I saw it add two more circles [points to the bottom two circles on each 'leg' in Figure 4]. So, I thought there was an addition of two from one going up to seven.

For Part B, Miles added two plus seven to correctly conclude that Figure 5 would have nine circles without producing a drawing. In explaining his reasoning, Miles stated, "I knew that there was a pattern of adding two [gestures from left to right over the figures]. So, I just add two to seven, if there was a figure five, and I got nine."

For Part C, Miles generalized from Figure 5 and stated that the answer would be 19 circles, because, "I knew that after each figure, two [circles] would be added. So, if there were five figures [from Figure 5 to Figure 10] and two were being added each time, I knew that it would be 10. So, I add 10 plus nine to get my answer of 19." Miles made this generalization without drawing or explicitly visualizing additional figures like Dennis did; rather Miles used the difference in figure numbers from five to ten to generalize the pattern.

Student 3: Margot

The third student, Margot, recognized the addition of two circles for each successive figure, but she saw the additional two circles in a different spatial structure than Dennis and Miles. After reading Parts A and B of the question, Margot initially analyzed the figures, saying, "Um, first you do like—so two [moves pencil across Figure 2 as shown in **Figure 5]**, two [taps Figure 3 as shown in **Figure 5**]. Um, two [moves pencil across Figure 4 as shown in **Figure 5**]. It would be like, 1, 2, 3, 4, 5, 6, 7 [counts Figure 4 as shown in **Figure 5**]." Rather than seeing the additional two circles in each figure added to the bottom 'legs' of the previous figure as Dennis and Miles did, Margot saw the two circles on one 'leg' of the figure. From her comments in later dialogue, we infer that she saw the number of remaining circles in the figure as equal to the number of circles in the previous figure, indicating that she may have recognized the recursive nature of the pattern.

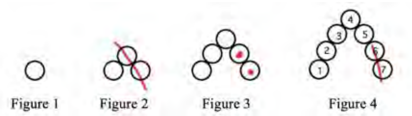

Figure 5: Margot's Spatial Structure for the Pictorial Representation

Initially when answering Part A, Margot incorrectly wrote "6" in the table as the Number of Circles in Figure 4. When the researcher asked her to explain her thinking, Margot responded, "Um—I just think—oh, now I know [erases the "6" and puts "7"]. Seven, because like, … I think it just matches. I don't know." Though Margot's second answer of seven was correct, she had difficulty explaining her reasoning within the table representation. The researcher then asked if she knew what the table was referring to, and Margot was prompted to relate the table to the figures. She verified her answer of seven in the table by recounting the number of circles in Figure 4, "Yeah, 1, 2, 3, 4, 5, 6, 7," (as illustrated in **Figure 5**). While Margot's reasoning with the numerical values in the table was imprecise, she ultimately relied on the pictorial representation to definitively and correctly state the number of circles in Figure 4.

When asked to solve Part B, Margot, like Miles, correctly predicted the number of circles in Figure 5 by simply adding two to the number of circles in Figure 4 without producing a drawing. However, she still indicated the additional two circles in each figure as shown below.

> Margot: Pretty sure I know it's nine…It's nine…Because there's—so one [points to the one circle in Figure 1], three [points to the three circles in Figure 2], because these are—and there's two more [gestures to the right 'leg' of Figure 2 as indicated in **Figure 6**] than each of them. Two [gestures to right 'leg' of Figure 3 as indicated in **Figure 6**], Two [gestures to right 'leg' bottom two circles in Figure 4 as indicated in **Figure 6**] more than each of these other ones, so I'm pretty sure it's nine. Because seven plus nine is, wait, seven plus two is nine.

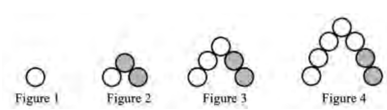

Figure 6: Margot's Gesturing of the Two Additional Circles in Each Figure

For Part C, Margot attempted to draw the figures up through Figure 10, but did not continue the pattern of circles following the spatial structure of adding two illustrated in **Figure 6**. Instead, she drew long, straight chains of circles to represent each figure but did not consistently draw the straight chains with an accurate number of circles (**Figure 7**). At times she added two circles to the next figure and at times she added three, ultimately leading to a series of figures that produced an incorrect answer.

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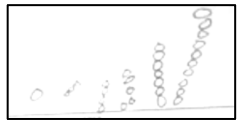

Figure 7: Margot's Figure Growth Drawings for Part C

Discussion

Two of the three participants in our study, Dennis and Margot, faced challenges when determining the number of circles in Figure 10 in Part C. By characterizing the spatial structures used by the students as spatial numerically-linked structures (Battista et al., 2018) or non-spatial numerically-linked structures, we provide insight for when a spatial structure may support students' reasoning about patterning problems.

The spatial structure utilized by Dennis and Miles (**Figure 2**) can be classified as a spatial numerically-linked structure (Battista et al., 2018) which has the potential to support student reasoning about far-prediction problems. By seeing the two additional circles at the bottom of the figure, Dennis and Miles were using a spatial structure that is aligned with a recursive numerical process. Numerically, a recursive pattern adds a value to a previous value; spatially, this can be thought of as adding objects to a previous congruent figure. Dennis' and Miles' spatial structuring organized the components of the figures, the circles, into composites: one part is the previous figure and one part is the two additional circles for each successive figure (**Figure 8**). This organization includes the geometric properties of symmetry within the figures and congruence between figure components. Even though neither Dennis nor Miles stated these geometric properties, these visually salient qualities may support imagining or visualizing future shapes.

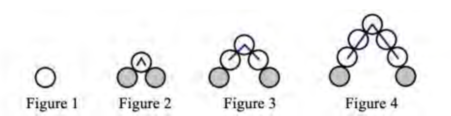

Figure 8: A Spatial Numerically-linked Structuring for the Pictorial Representation

A spatial numerically-linked structuring can provide a way to coordinate the two varying quantities in a patterning problem. The way in which Dennis saw the additional two circles added to each figure provided a way for him to coordinate the figure number and number of circles resulting in a numerical structure that provided an organization for his set of computations (**Figure 9**). Each time he imagined a new pair of circles being added, he moved to a new row and tapped his pencil adding the additional circles in an organized way. Even when he made two counting errors, he was able to recognize and correct those errors and generalize his process because his spatial structure and numerical structure were linked.

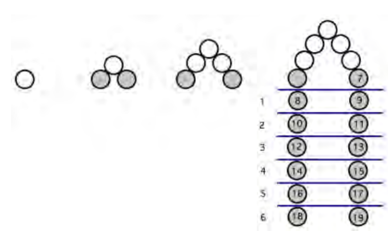

Figure 9: Dennis' Spatial Numerically-linked Counting

In comparison, Margot, did not reach the correct solution for Part C. One explanation for why Margot's visual structure did not support predicting the number of circles in Figure 10 is that the spatial structure was not linked to a numerical structure for adding on two circles in an organized way. Like Miles and Dennis, Margot also recognized that two circles were added in each successive figure. However, the way in which she saw the two additional circles within the figure (**Figure 6**) did not show the addition of the circles to a congruent previous figure or maintain the symmetry of the figures. Because the spatial structure was not linked to the recursive numerical pattern she verbalized, it was very difficult for Margot to imagine or draw the next figure even though she could explain the provided figures using her spatial structure. By observing her gestures and descriptions of the pattern, we hypothesize that she verified the pattern by recognizing that the collection of three white circles in Figure 3 were the three circles from Figure 2 in a different spatial arrangement (**Figure 10**). She used similar gesturing to verify that the collection of five white circles in Figure 4 is just a different spatial arrangement of the five total circles in Figure 3. But without a symmetric, congruent spatial structuring, creating a new figure, such as Figure 5, using the recursive relationship of adding two circles is very challenging, making a resulting numerical structure for computations to determine the number of circles in Figure 10 very difficult to coordinate with the pictorial representation.

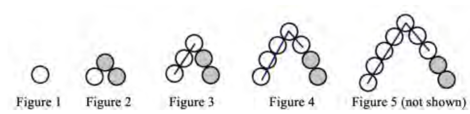

Figure 10: Margot's Spatial Structure

Indeed, for Part C, Margot did not attempt to draw using the same spatial structure. Instead, she drew long chains of circles to represent each figure (**Figure 7**). However, Margot's second spatial structuring was also not a spatial numerically-linked structure because it was not connected to the numerical recursive pattern of adding two. We hypothesize that Margot's re-arrangement of the figures into long strings of circles could be spatially linked to the numerical structure if the straight lines were maintained and the circles were congruent (**Figure 11**). This would offer the same spatial-numerical link as Miles' and Dennis' structure because the additional two circles would be added to a previously congruent figure. While symmetry within the figures is not as visually salient, the equal "heights" of the strings could have helped coordinate the additional two circles added to congruent strings. However, because Margot's drawings did not incorporate these features, it became very difficult for her to keep track of the total number of circles in each figure and to consistently add two circles to each string ultimately causing her to reach an incorrect solution.

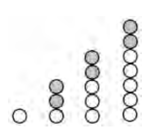

Figure 11: A Spatial Numerically-linked Structure Example

While spatial numerically-linked structures can help students reason about pattern problems with pictorial representations, it is certainly not required if other representations are utilized. A student could have been successful on the Pattern of Circles problem without such a structure by using the table representation to add two to the previous output value, extending the table to determine the number of circles in Figure 10. However, none of the students in the study explicitly reasoned with the table. Miles was the only student who did not draw additional figures while predicting future values. While it is possible that he used information from the table to support the development of his generalization, he also used the same spatial numerically-linked structuring as Dennis while reasoning about the problem.

Conclusion

Research has shown that students can think about pictorial representations with different visual structures, and we have offered evidence that these visual structures can support student thinking about patterning problems when the spatial structure and the numerical structure are adequately linked. By identifying features of spatial structures and numerical structures that are helpful for students when solving patterning problems, we can better understand how students' visualizations can facilitate the development of numerical generalizations and predictions.

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