INTEGRATION OF MATHEMATICS HISTORY INTO MODEL-ELICITING ACTIVITIES FOR MAKING SENSE OF NEGATIVE INTEGERS

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This study aims to explore seventh-grade students' understanding of negative integers as they engaged in mathematics history integrated model-eliciting activities in small groups. For this educational case study, we designed model-eliciting activities based on six design principles of the models-and-modeling perspective that incorporated history of negative integers. Both written data and video records of students were analyzed to elicit the facets of their models of negative integers. We found that students' thought that either daily life contexts or people's need drove the invention of negative integers. The findings also indicated students' reasoning on the evolvement of mathematics ideas by contribution of different culture, revealing the role of math history integration into the modeling process. In this sense, our study presents a unique approach in modeling literature.

Keywords: History of mathematics; models-and-modeling perspective; model-eliciting activities; negative integers

Negative integers have always been an interesting topic in mathematics education research, and related studies indicated that although students could perform the operations with integers, they struggled in making sense of negative integers (Lyte, 1994; Steiner, 2009). One of the major reasons for this struggle was the difficulty of connecting negative integers with real-life situations (Gallardo, 2002). Therefore, we approached to this phenomenon, making sense of negative integers, from the Models-and-Modeling Perspective that was centered around meaningful situations in developing a mathematical model (Doerr & Lesh, 2003). We, on the one hand, aimed to elicit students' understanding of negative integers through model-eliciting activities; and, on the other hand, incorporated mathematics history into model-eliciting activities. Hence, our study presents a unique approach in modeling literature by addressing the following research question: *What understandings do 7th grade students develop negative integers as they engage in mathematics history integrated model-eliciting activities in small groups?*

In the sections below, we briefly presented our theoretical framework involving history of mathematics and models-and-modeling perspective and presented our findings regarding 7th grade students' understandings of negative integers.

Theoretical Framework

History of Mathematics

The integration of history of mathematics into mathematics education has been on the agenda of many mathematics education researchers (Fenaroli, Furinghetti, & Somaglia, 2014). While some investigated the ways of including historical origins of mathematical concepts in teaching (Tzanakis & Arcavi, 2000), some explored the role of math history in teacher education (Clark, 2012; Fenaroli, Furinghetti & Somaglia 2014).

The use of history of mathematics in mathematics education was analyzed by Jankvist (2009) in terms of reasons (the whys) and integration ways (the hows) of history of mathematics in mathematics teaching and learning. The two main reasons of integration of the history of mathematics are (i) to assist mathematics instruction (i.e., use of history as a tool) and (ii) to learn the history of subject (i.e., use of history as a goal) (Jankvist, 2009). The first reason focuses on improving students' understanding in terms of cognitive and affective aspects of mathematics learning with the help of history of mathematics. The second addresses that history of mathematics encourages students to considering about the evolution of mathematics and role of humanity on the development of mathematics (Jankvist, 2009).

National Council of Teachers of Mathematics (NCTM) pointed out that mathematics is affected by different cultures and inherited to humanity, and students should be allowed to notice and perceive worldwide human effect on the field of mathematics (NCTM, 2000). With this in mind, Jankvist (2009) stated three basic approaches to include history of mathematics in mathematics education: (i) the modules refer to the integration of history of mathematics into a range of mathematics lessons related to topic, (ii) the history-based approach in which mathematics lessons are fully arranged taking the history and evolution of mathematics into account, (iii) the illumination refers to include some historical facts and information in mathematics lessons.

Several researchers mentioned about the benefits of integrating math history in mathematics education (e.g., Fried, 2001; Liu, 2003; Tzanakis & Arcavi, 2000). These benefits can be listed as follows:

- It encourages students to value mathematics as cultural and human product.
- It makes mathematics more interesting, understandable, and attainable for students by helping to perceive mathematical concepts, problems and their solutions.
- It facilitates learning activities by enhancing mathematical thinking ability.
- It affects students' affective dispositions towards mathematics.
- It guides teachers for the learning and teaching activities while asserting that the difficulties mathematicians encountered in the past helps teachers to identify and prevent the problems of students of today.

Regarding the last point, Jankvist (2009) claimed that historical development of a subject provides a parallel path to learn this subject within context revealing relationships between ideas, definitions, and applications: "To really learn and master mathematics, one's mind must go through the same stages that mathematics has gone through during its evolution" (p.239). Similarly, Savizi (2007, p.46) stated: "For students, issues of past real world are more tangible and understandable than today's problems or solving problems from real life by using human approaches may work better than application of complicated methods or offering high amount of information." That also improves students' self-confidence and encourages them to believe in their own abilities as human beings (Savizi, 2007). Moreover, recent studies on this field have indicated that students experiencing mathematical concepts within a meaningful historical context developed more positive attitudes towards concepts (Lim & Chapman, 2010).

When the mathematical concepts are presented as disconnected from real-life, students demonstrate difficulties in understanding the mathematical concepts, and in this vein the integration of history of mathematics enables students to understand the need for the concept (Gulikers & Blom, 2001). Since one of the distinctive features of the modeling perspective was the reality or meaningfulness of the context (Lesh, Hoover, Hole, Kelly, & Post, 2000), we considered the models-and-modeling perspective as a complementary strand of our theoretical frame.

Models-and-Modeling Perspective

The activities with meaningful contexts make students more willing to learn about the subject while they understand the importance of mathematics and real-life relevance of the concept (Lim & Chapman, 2010; NCTM, 2000). The models-and-modeling perspective proposed a problem-solving approach that involves problem-solvers' making sense of the real-life context mathematically, mathematizing the context, and developing a mathematical model that was expressed, tested, and revised iteratively until it provides a sufficient solution for the real-life problem (Lesh & Zawojewski, 2007). The term "mathematical model" refers to the conceptual systems that are built, defined, emphasized mathematically significant products, processes and mathematical reasoning (Doerr & Lesh, 2003). In modeling classrooms, teachers focus on students' understanding and processes of constructing, expressing, reasoning abilities while solving mathematically word problems rather than solely arithmetic computations (Lehrer & Schauble, 2000). However, eliciting students' models was not an easy task, and therefore Lesh and his colleagues proposed a genre of modeling activities called Model-Eliciting Activities (MEAs) (Lesh et al., 2000).

The MEAs involves real life situations in which students make meaningful mathematical explanations (Doerr & Lesh, 2003). To foster students' development of mathematically significant models, Lesh and colleagues (2000) identified six design principles of MEAs: (1) model-construction principle, (2) model-documentation principle, (3) reality (meaningfulness) principle, (4) self-assessment principle, (5) model shareability and reusability principle, and (6) effective prototype principle. Therefore, via MEAs, students produce mathematically significant, shareable and reusable model related to real-life situations. Moreover, these thought-revealing activities allow students assess their thinking and encourage working in groups to produce better models.

With these in mind, we, in this study, integrated history of negative integers into the MEA approach and conjectured that integration of math history would not only take students' interest but also provide them a deeper understanding of negative integers.

Students' Understanding of Integers

There have been many studies investigating how to advance students' understanding of integers by neutralization and number line models (Lyte, 1994). Whilst the neutralization model includes physical objects such as two-colored counters to represent negative and positive integers and operations with integers, the line model focuses on operation with integers considering the position and distance of integers by the direction of movement on the number line (Lyte, 1994).

The concept of negative integers and making sense of the use of negative integers in real-life was difficult for students because it was not as easy to grasp negative integers contextually as natural numbers (Whitacre et al., 2017). There have been several studies arguing that students had difficulty in understanding negative integers as they tried to accommodate their prior knowledge about natural numbers (Gallardo, 2002; Whitacre et al., 2017). This transition between natural numbers and integers led to difficulties in terms of number sense and making sense of the negative integers. In addition, the sense of negative integers and the idea of a number less than zero seemed nonsense for most of the everyday contexts from the viewpoints of students (Whitacre et al., 2017).

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Although students meet negative integers in their everyday life, after they encounter and focus on operational procedures in school, they do not make connection between outside-the-school learnings and school instruction (Steiner, 2009). The related studies showed that the real-life contexts, word problems and models including incomes and expenses, assets and debts, elevators, weather temperatures support students' understanding and reasoning about integers (Pettis & Glancy, 2015; Stephan & Akyuz, 2012). However, students might still struggle in comprehending situations involving opposites such as incomes and expenses, weather temperatures and elevators (Pettis & Glancy, 2015). Thus, it is important to encourage students to think within the context to improve their understanding of integers, for which we designed mathematics history integrated MEAs in this study.

Mode of Inquiry

To explore students' understanding of negative integers, we designed mathematics history integrated MEAs and carried out a qualitative educational case study with eight groups of 7th grade students (29 students in total). We explained characteristics of the case participants and the nature of data collection and analysis below.

Participants

The participants of this study were 7th grade students (15 male and 14 female) of a public middle school class in Istanbul, one of the metropolitan cities in Turkey. The students engaged in mathematics history integrated MEAs in small groups and randomly assigned to groups by the second author who was also the mathematics teacher of the classroom. Ten groups were formed in the classroom but only eight of them whose parents provided the consent for their participation in the study were included in the data set. A general view of the mathematics teacher for the participating students was that most of the students' prior knowledge and mathematics backgrounds were similar to each other because they attended the same classes during the primary and middle schools, and their mathematics achievement was average.

Data Collection Procedure

Students' group work was video recorded during the implementation of mathematics history integrated MEAs. There were eight groups containing 3-4 students per group; 29 students in total. The data set involves their written work in activity sheets and video records of their work during the implementation of the activities.

The mathematics history integrated MEAs were implemented with the aim of guiding students to achieve related objectives of middle school mathematics teaching program. The activities covered three dimensions of students' understanding on integers: (i) why negative integers were needed in mathematics, (ii) how to identify positive and negative integers, and (iii) how to use negative integers in real life contexts. In this proposal, we delimited our focus only on one MEA called "The Problem of Diophantus" that addressed the first dimension. The MEAs were designed considering the six design principles and implemented in a one lesson hour. Before the implementation of activities, any prior teaching about integers was not provided to the students. During the implementation of the activities, students were expected to reflect their understanding and making sense of negative integers. The essential principles followed during the implementation sessions were: (i) students should study as small groups and interact with each other, (ii) after they finished their studies, students should be encouraged to share their works and opinions with the guidance of teacher during whole class discussion, (iii) teacher should guide students when they needed without providing any right answer for the questions of the activities, and (iv) students should be allowed to reveal and reflect their own experiences by making connection with everyday life contexts.

The Problem of Diophantus. In this MEA, we aimed to take students' attention to the origins of negative integers. First part of the activity emphasized how people use the negative integers in daily life and why people needed negative integers in the history. The researcher intended to help students to consider and question necessity and need for negative integers not only for mathematical operations but also in everyday problems. The second part of the activity contains information about Diophantus, a mathematician, and is followed by a problem (i.e., 4×?+20=4) which is called as "absurd" by the Diophantus because of its' negative solution (Hettle, 2015). Students were expected to write a letter explaining their rationale for why mathematicians needed negative integers, their solution to the Diophantus's problem, and their reasoning for why Diophantus might have called the solution as absurd. Although not readable, the screenshot of the MEA (in original language) was given below to help readers make sense of the material that students received as a math history integrated MEA.

Figure 1: The Problem of Diophantus MEA-Part 2

Data Analysis Procedure

Students' performances on MEAs were recorded in written form on activity sheets and as video and audio records. These written data were coded through two cycles: (1) initial coding and (2) descriptive coding (Saldana, 2009). In the first cycle, the written data of each group first examined holistically, and then open codes were identified to make sense of students' conceptions. In the second cycle, these open codes were revised to create categories that were more descriptive of students' conceptions. Afterwards, the resulted codes were checked with the video and audio data. Specifically, audio and video records were not coded separately but used to make sure about students' conceptions written in the activity sheets.

The codes were then checked by another researcher, the first author, for the interrater reliability (Lincoln & Guba, 1985). Multiple sources of data helped to triangulate the findings. In addition, the second author, implementer of the MEAs, kept a research journal during both data collection and data analysis. Writing each step of the study transparently contributed to the credibility of the interpretations (Lincoln & Guba, 1985).

Findings

In this section, we present the findings including seventh-grade students' models of the negative integers that they developed during their small group engagement in math-history integrated MEAs. Lesh and Harel (2003, p. 150) defined the models as "conceptual systems that generally tend to be expressed using a variety of interacting representational media, which may involve written symbols, spoken language, computer-based graphics, paper-based diagrams or graphs, or experience-based metaphors." Hence, the students' models presented in this section were in the form of verbal descriptions and mathematical symbols and more importantly indicated their conceptual structure of negative integers.

Making Sense of Appropriate Contexts for Negative Integers

Seventh grade students' understanding of negative integers were associated with four contextual situations: (i) representation of weather temperature, (ii) representation of the debt and loss, (iii) representation of elevation, and (iv) an indication of floor numbers in elevators. Although the last two situations were related, students differentiated them. Group #1 and #5 identified a reference point and indicated that the interval below the reference point would be considered as negative. For Group #1 the sea level was a reference point (zero), and below sea level is represented with the negative integers. Similarly, Group #5 accepted the ground floor as a reference point (zero), and they represented the flats under the ground floor with negative integers. On the other hand, the fourth context, indication of floor numbers in elevators, referred to a static position. For instance, Group #2 and #4 stated that the buttons in the elevator included negative integers as symbolic representations of the levels of the floors.

Furthermore, students reasoned about the origins of negative integers and why people needed them. In this regard, Group #2, #4, and #5 stated contextual reasons and Group #1, #6, and #8 indicated that people such as mathematicians, scientists, folks needed and invented negative integers to illustrate the values less than zero. Groups considering contexts such as very cold weather, and debt and loss situations stated that people needed negative integers for their daily life requirements such as trading. For instance, one of the students from Group 5 stated that "one day, when the weather was too cold and snowy, people used negative integers to express the very cold weather." Students mentioning the scientists or mathematicians, on the other hand, stated that people needed to represent numbers less than zero: "A scientist might have invented negative integers to help his calculations with a scientific experiment" (A student from Group #1).

Making Sense of the "Absurd" Problem of Diophantus

In the second part of the MEA, groups were expected to write a letter to help a peer student's school magazine involving their thinking about the problem of Diophantus and possible reasons of why this problem might have called "absurd" by Diophantus. All groups of seventh-grade students thought that it was due to the lack of knowledge of negative integers by then. Regarding this, they had two slightly different aspects:

- A problem is called absurd when one does not have the knowledge of negative integers.
- A problem is called absurd when one cannot find a solution for a mathematics problem.

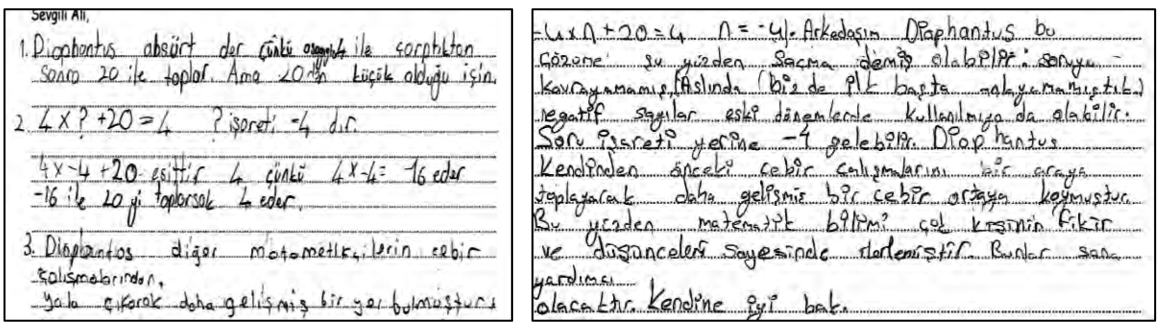
When students engaged in finding the value of the unknown shown with a question mark on the given problem (i.e., 4x?+20=4), the students used trial and error method in two different ways:

- Some groups stated that the unknown cannot be a positive integer, so they accepted the unknown as a negative integer. They tried negative integers respectively to find the value of unknown.
- Some groups tried zero and positive integers at first, but they didn't find a correct solution. Therefore, they tried negative integers to find the correct value for the unknown.

Moreover, groups discussed their interpretations about how mathematics evolved, considering the given information that Diophantus gathered the algebra studies before his era and developed his studies based on the prior work of other mathematicians. Students' discussion revealed three aspects:

- Mathematics is a continuously evolving field.
- A single math idea was developed by contribution of many peoples thinking and studies.
- Various mathematicians and cultures contributed to the field with their work evolved one after another.

These aspects along with the two ways of using trial and error method that were associated with how students made sense with the absurdity of the answer indicated the facets of students' models of negative integers. These facets were illustrated in the letters of the two groups given in Figure 2a and 2b below.


Sevgili Ali,
1. Diophantus absürt der çünkü 4 ile çarptıktan
sonra 20 ile toplar. Ama 20'den küçük olduğu için.
2. 4 x ? +20 = 4   ? işareti -4 dir.
4 x -4 +20 eşittir 4 çünkü 4 x -4 = 16 eder
-16 ile 20 yi toplarsak 4 eder.
3. Diophantos diğer matematikçilerin cebir
çalışmalarından,
yola çıkarak daha gelişmiş bir yol bulmuştur.

-4 x n + 20 = 4   n = -4. Arkadaşım Diophantus bu
çözüme şu yüzden saçma demiş olabilir: sonucu
kavrayamamış. (Aslında (biz de ilk başta anlayamamıştık.)
negatif sayılar eski dönemlerde kullanılmıyor da olabilir.
Soru işareti yerine -4 gelebilir. Diophantus
kendinden önceki cebir çalışmalarını bir araya
toplayarak daha gelişmiş bir cebir ortaya koymuştur.
Bu yüzden matematik bilimi çok kişinin fikir
ve düşünceleri sayesinde ilerlemiştir. Bunlar sana
yardımcı
olacaktır. Kendine iyi bak.

Figure 2a and b: The letter of Group #2 (on the left) and Group #7 (on the right)

As the letter of Group #2 stated, the problem was called absurd since the unknown number multiplied by 4 and added to 20 and somehow the result would be less than 20. Similarly Group #7 expressed that negative integers were not known in the past, and so the question did not make sense; that's why the given problem was called absurd. Students in Group #7 also stated that mathematics developed with the help of many people's opinions and Diophantus collected and improved the algebra studies based on the prior work, which was also observed in other groups' letters. Hence, with help of the math history integrated MEA, students could develop a rationale about the historical development of mathematics and improved their understanding of why people needed negative integers in daily life.

Conclusion and Discussion

We observed that seventh-grade students who encountered the integers formally for the first time with math history integrated MEAs found the topic interesting. The MEAs not only took their attention but also motivated them to understand why people needed integers in the history and what kind of mathematics equation would lead to a negative integer answer. Developing models of negative integers in small groups could reveal students' understanding. More specifically, although they used symbolic representation of a negative integer in their explanations, they also supported their statements with the contextual illustrations. Real-life contexts identified by the participating seventh-grade students included incomes and expenses, assets and debts, elevators, and weather temperatures, which were also observed in the related literature about integers (e.g., Stephan & Akyuz, 2012; Pettis & Glancy, 2015).

One of the major contributions of this study was integrating mathematics history into the modeling perspective, which has not been present in the related literature yet. This integration increased the motivational and attitudinal effect of MEAs as Savizi (2007, p.46) stated: "For students, issues of past real world are more tangible and understandable than today's problems or solving problems from real life by using human approaches may work better than application of complicated methods or offering high amount of information." To illustrate, in the Problem of Diophantus MEA, students studied on a problem called "absurd" by the Diophantus because of its' negative solution and noticed that Diophantus also did not make sense with the problems, likewise the students who sometimes do not make sense with math problems.

In this study, math history integrated MEAs activities brought real-life related mathematics problems from the past and the present together. After the implementation of the MEAs, most of the students expressed their wishes to continue mathematics lessons by working on similar modeling activities, which confirmed other researchers' claim that including history of mathematics could help students overcome their math anxiety (Liu, 2003; Tzanakis & Arcavi, 2000). Similar arguments regarding the affective benefits of modeling experiences were also exist in the models-and-modeling literature (English, Lesh & Zawojewski, 2003). In this sense, mathematics history integrated MEAs were beneficial tools to create a meaningful and real-life related learning environment in which modeling is significant not only for computing, but also for constructing, describing, mathematical reasoning and understanding (Doerr & Lesh, 2003). Comprehending the situations involving opposite directions such as incomes and expenses and cold and hot weather was not easy for students (Pettis & Glancy, 2015), but possible with encouraging students to think within the context (Whitacre et al., 2017). Furthermore, focusing on only operations with integers in school hindered students' understanding of situations involving negative integers in real-life problems (Gallardo, 2002). Our study showed that with the help of mathematics history integrated MEAs, students made sense of negative integers in historical situations.

This study involved three mathematics history integrated MEAs and in this proposal we focused only on one of them. Although the implementation of the activities was arranged considering the middle school mathematics teaching program and limited with the annual plan of mathematics lessons, more meaningful data about the students' understanding of negative integers might reveal if more time was spent and more activities were implemented. Another suggestion could be expanding the use of math history integration into different mathematics topics. In other words, we recommend a future research considering different mathematics topics in different grade levels for the integration of the mathematics history into the modeling perspective. Although the present study investigated the role of mathematics history integration into MEAs on students' understanding, these activities can also be used to improve mathematics teachers' education for their teaching repertoire. Thus, this study also suggests a professional development aiming to train teachers how mathematics history and modeling perspective can be used to enhance students' mathematical understanding.

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