HARMONY AND DISSONANCE: AN ENACTIVIST ANALYSIS OF THE STRUGGLE FOR SENSE MAKING IN PROBLEM SOLVING

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This work seeks to understand the emergent nature of mathematical activity mediated by learners' engagement with multiple artifacts. We explored the problem solving of two learners as they aimed to make sense of fraction division by coordinating meanings across two artifacts, one being a physical manipulative and the other a written expression of the standard algorithm. In addressing the question, "How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?" we took an enactivist perspective and used tools of semiotics to analyze the ways they navigated the dissonance that arose as they sought to achieve harmony in meanings across multiple representations of ideas. Our findings reveal the value of such tool-mediated engagement as well as the complexity of problem solving more broadly. Implications for learning mathematics with multiple artifacts are discussed.

Keywords: Problem Solving, Mathematical Representations, Learning Theory, Technology

Hiebert and Grouws (2007) synthesized evidence from a number of studies to argue that the conceptual learning of mathematics is associated with teachers' and students' "explicit attention to the development of mathematical connections among ideas, facts, and procedures" (p. 391). Much research has been done regarding the ways in which teachers can support students' engagement with multiple representations. What is less well understood is the process by which multiple representations of a concept can be leveraged and connected in order to contribute to learners' meanings of the referent of those representations.

Findings from an enactivist analysis of strategy development in mental mathematics contexts suggest that the nature of the processes at play are dynamic, emergent, and contingent on "an ongoing loop" (Proulx, 2013, p. 319) of interactions between the problem and the solver(s). Since sense making results from problem solving, and since problem solving is dynamic, emergent, and contingent (Proulx, 2013), it follows that sense making should be, as well. Moreover, sense making is inextricably linked to the material and symbolic tools that mediate its learning (Artigue, 2002; Verillon & Rabardel, 1995). Following this line of inquiry, we consider what an enactivist analysis might reveal about the processes at play in mathematical meaning making as it develops through the complex interplay of signs and meanings (Maffia and Maracci, 2019) associated with learners' engagement with multiple representations. Thus, this work seeks to address the following question: *"How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?"* We do so through an analysis of the mathematical activity of two learners as they aim to make sense of fraction division mediated by two representations: the flip-and-multiply algorithm for fraction division and a physical manipulative designed for learners' engagement with fraction concepts.

Theoretical Framework

This study is grounded in the enactivist theory of cognition, which asserts that: "1) perception consists in perceptually guided action, and 2) cognitive structures emerge from the recurrent sensorimotor patterns that enable action to be perceptually guided" (Varela, Rosch, &

Thompson, 1992, pp. 172-173). Thus, cognition, or active knowing, is not some "outward manifestation of some inner workings" (Davis, 1995, p. 4), but rather a dynamically co-emergent phenomena that arises and is brought forth (Maturana & Varela, 1987) through one's goal-directed, "embodied (enacted) understandings" (Davis, 1995, p. 4). In Davis's (1995) adaptation of Maturana and Varela's (1987) words, "*Knowing* is *doing* is *being*" (p. 7).

By viewing knowing in the interactivity of learners, the enactivist perspective offers an alternative to a view of knowledge as the static accumulation of facts and ideas that one may select in response to a problem at hand. Instead, "to know is to respond adequately; it is a situated doing that emerges through the interaction of the organism (e.g., a student, a researcher) and [their] environment" (Maheux & Proulx, 2015, p. 212). But fit is more than that. We use *harmony* to emphasize that fit is an internally "felt dimension of experience" (Petitmengin, 2017, p. 144) that drives problem solving. This drive toward a harmony of goals and actions is theoretically linked to the concepts of structural coupling and structural determinism.

Structural coupling is the process associated with the Darwinian concept of co-evolution, whereby an organism and its environment co-adapt through recursive and repeated inter-actions (Maturana & Varela, 1987). As they do so, the organism and environment experience mutual structural changes so that the fit between them is dynamic. Moreover, this fit is contingent upon unique histories of recurrent interactions and structural changes (Maturana 1988, as cited in Reid & Mgombelo, 2015, p. 175) that are determined by the organism's own structure, a phenomenon referred to as *structural determinism* (Maturana & Varela, 1987). Proulx's (2013) analysis of students' emergent problem-solving activity is committed to this concept as it assumes that a problem solver's strategies are determined by the solver's own way of making sense.

We take this enactivist perspective on mathematical activity as knowing-in-action to investigate the emergent problem solving of two learners as they aim to understand fraction division by finding harmony in meanings across what for them are recurring and competing interpretations in the various elements of two artifacts: 1) the flip-and-multiply algorithm for fraction division, and 2) a manipulative that one of them designed for engagement with fraction concepts. Maffia and Maracci's (2019) concept of *semiotic interference* is used to analyze these dynamic, emergent, and contingent (Proulx, 2013) interactions with the two artifacts. This concept, framed within the Theory of Semiotic Mediation (Bartolini Bussi & Mariotti 2008) relies on Peirce's (1998) triad of sign relations to analyze how meanings emerge from the translation of personalized signs into new signs and eventually into generalized mathematical signs.

According to Peirce, a *sign* is a triadic relationship among a *representamen* (the perceivable part of a sign), an *object* (what the sign stands for), and an *interpretant*, which Presmeg (2006) describes as follows: the "interpretant involves *meaning making*: it is the result of trying to make sense of the relationship… [between] the object and the representamen" (p. 170, emphasis added). Thus, semiotic interference becomes useful for analyzing the process of meaning making across multiple artifacts whenever "the interpretant of a sign whose object belongs to the context of [one] artifact is translated by a student in a new sign whose object belongs to the context of another artifact" (p. 3-58). That is, as the two learners aim to "make meaning" by negotiating their interpretations of signs across the orange and the algorithm, each of the artifacts affords them with differing semiotic potentials (Bartolini Bussi & Mariotti, 2008) for the emergence of a relationship between the personal use of the artifact and mathematical meanings associated with the artifact and its use. Semiotic interference provides a window into their chaining of signs (Presmeg, 2006; Bartolini Bussi & Mariotti, 2008) as they negotiate these interpretations in order to converge upon a meaning for fraction division. In this sense, meaning making is understood as emergent phenomena arising from this "complex interplay of signs" (Maffia & Maracci, 2019, p. 3-57). We thus frame the activity of problem solving from an enactivist perspective and leverage tools of semiotics to depict the evolution of meaning making to better understand how learners make meaning through the coordination of multiple representations of mathematical ideas.

As a critical point of clarification, "representation" in the Peircean sense is a thing perceived by a learner, and that is the meaning we will be using throughout the remainder of this paper. What the field of mathematics education terms a "representation" (e.g., tables, graphs, symbolic expressions) is what we will refer to as an "artifact."

Methodology

This project is part of a larger study that aims to test and refine the hypothesis that a pedagogically genuine, open-ended, and iterative design experience centered on the Making (Halverson & Sheridan, 2014) of a physical manipulative for mathematics learning would be formative for the development of practicing and prospective mathematics teachers' (PMTs') inquiry-oriented pedagogy. Data collection for this study took place across several semesters of a graduate-level mathematics course for PMTs at a mid-sized university in the northeastern United States. For the project reported here, we took a revelatory case study approach (Yin, 2014) in order to determine what an enactivist perspective might reveal about the phenomena involved in the problem-solving activity of "Dolly" and "Lyle" (both pseudonyms).

Dolly was a participant in this larger study; she is a participant-researcher on this project. She calls the tool she designed a "fraction orange" (Figure 1, left), and in designing it, she aimed to create a tool with affordances for the exploration of fraction concepts. The orange is a sphere partitioned into two hemispheres; one hemisphere is further partitioned into fourths, eighths, and sixteenths of the whole; the other into sixths and eighteenths.

Figure 1: The Orange and the Algorithm

The manipulative Dolly created and the thirteen-minute problem-solving interview she conducted with Lyle are artifacts of her participation in the larger study. They also constitute the data for this case study. Three researchers on this project, including Dolly, enacted interpretations of data both individually and in collaborative dialogue. Dolly's role as both participant and researcher offers validation by permitting a strengthening of the interrelationship between a research context and its participants.

We undertook the analysis by transcribing the recorded video and analyzing the "verbal utterances through line-by-line analysis of the transcripts; stud[ying] body language and intonation by viewing video tapes...; and in511andeng] mathematical forms and objects from the participants' actions, utterances and notations" (Simmt, 2000, p. 154). Specifically, we focused our analysis on the particular interactions where Dolly and Lyle aimed to coordinate meanings for fraction division in the manipulative and in an algorithm that presumably substantiates those meanings (Malafouris, 2013). As we take our learners' activity to be driven by an evolutionary imperative to maintain harmony through their problem solving, we used the enactivist concepts of structural coupling and structural determinism to analyze these inter-actions. And in order to analyze their emergent and recursive processes of meaning making across multiple representations, we employed Peirce's (1998) triad of sign relations and Maffia and Maracci's (2019) concept of semiotic interference to refine the analysis.

Results

Given the duration and non-linearity of Dolly and Lyle's problem solving, space constraints only permit us to share selected excerpts uniquely revealed by enactivist and semiotic lenses that elucidate critical moments in their emergent mathematical activity. As a note for the reader, Dolly and Lyle only make use of the hemisphere of the Fraction Orange that is partitioned into fourths, eighths, and sixteenths. In our analysis of their activity, unless otherwise indicated, all fraction pieces are named as Dolly and Lyle do, that is, as if that hemisphere of the orange is the whole.

Embarking on a path of problem solving

We set the stage for the presentation of these findings at the beginning of Dolly's interview with Lyle. Dolly poses the problem, ½ ÷ ¼, on paper alongside her fraction orange. Lyle chooses the pen and paper, performs the flip-and-multiply algorithm: $\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = \frac{2}{1}$, and declares his answer to be 2. We interpret this application of the standard algorithm as a structurally determined action informed by a lived history of structural coupling with traditional school mathematics, where a *knowing* of fraction division as the execution of an algorithm and the answer it yields was deemed *good enough* to "survive." It constituted what Lyle needed to do to achieve harmony within his mathematics learning environment.

Next, Dolly directs Lyle's attention to the orange and asks, "Can you show me with this?" With two artifacts affording them differing semiotic potentials, both Dolly and Lyle set off to navigate a complicated interplay of signs literally at (their) hand. As we will observe, they experience semiotic interference (i.e., meaning making through the enchaining of these signs) as they pursue a non-linear path of problem-solving activity punctuated by moments of what we refer to as either *harmony*[1], a pleasing fit, or *dissonance*, a displeasing conflict or lack of fit. The cognitive/affective underpinnings of these terms is intentional, because cognition from an enactivist perspective is synonymous with *effective* action.

First dissonance

This exchange captures the first moment of dissonance as Lyle responds to the task Dolly posed to him and as the two learners realize that their understandings of fraction division do not harmonize across the two artifacts.

> Lyle: A half divided by a quarter… <*removes what he considers to be a half piece*> a half divided by a quarter <*points to the fourth pieces inside of the half*> is four.
> Dolly: <pointing to the algorithm and the answer on the page> But that's not what you got.
> Lyle: Uh oh. <*Lyle pulls out the fourth pieces from the half pieces and looks back and forth between the paper and the orange. His gaze then shifts more rapidly between the two artifacts, and the timbre of concern in his voice grows as he continues*.> Uh oh. A half divided by a quarter. Why doesn't that work?

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In analyzing this excerpt, we first point out that we are able to observe Lyle's embodied knowings of mathematics precisely because those actions *are* his knowings. They are not inferences of *a priori* knowledge possessed internally; they are only "discovered in action" (Malafouris, 2013, p. 174). In our observations of his interactions with the orange – selecting, removing, gesturing, and communicating about pieces – we can see that the tool mediates new affordances for Lyle's actions. In this first moment, these new affordances evoke an emergent sense of dissonance, which is evident in Lyle's puzzled utterances and frantic glances – somatic markers (Damasio, 1996, as cited in Brown & Coles, 2011) of his negative affective response to seemingly conflicting interpretations of the same mathematical idea. We take these actions to indicate that his knowing of fraction division as expressed through the algorithm is discordant with his knowing of fractions and division as he perceives them in the fraction orange. This experience of semiotic interference between the two representamens (the orange and the algorithm) catalyzes an embodied drive to find harmonized meaning between them, an essential motivation for their problem solving.

The messiness of multiple representations

This next exchange features an extended moment of semiotic interference that is a particularly complicated one for Lyle and that we suggest speaks more broadly to the complexity that is characteristic of meaning making through the connections of multiple representations (Lesh et al., 1987; Hiebert & Grouws, 2007). Dolly and Lyle, motivated by a desire of sense making, strive for harmony in meanings between the orange and the algorithm as they evaluate the expression, ½ ÷ ¼.

> Dolly: Here's our half. <*She picks up the half piece and confidently places it next to the algorithm on paper. Lyle points to the piece and looks back to the paper.*> And how many quarters go into a half?
>
> Lyle: <*Looking at the orange*> Two. <*shifting his attention to the paper*> Four. <*shifting his attention back to the orange, and then again back to the paper*> Is that half of a quarter, though? It's half <*pointing to the ½ on the paper in the expression, "½ ÷ ¼"*> of a quarter. <*pointing to the ¼ on the paper*> It's not half of a whole thing. <*As he says, "whole thing," he circles the "4" of the* $\frac{4}{1}$ *in the flip-and-multiply part of the equation on his paper.*>
>
> Dolly: It's a quarter of a half, right? <*Lyle looks at the orange, back at the paper, and back at the orange*>
>
> Lyle: <*with uncertainty*> Yeah?
>
> Dolly: How many quarters of a half are there? <*pauses and laughs*> Why is this so hard?

Through Dolly and Lyle's varied interpretations of both fractions and fraction division in relation to the orange and the algorithm, we observe expressions of semiotic interference. Through their words and gestures, we see Dolly begin by enacting her knowing (interpretant) of "a half" (object) in the orange (representamen) and physically placing the piece on the paper, as if to propose a common meaning between the two by creating a physical bridge between the piece of the orange and the symbolic form of the fraction on paper. She interprets the posed problem, ½ ÷ ¼, as "How many quarters go into a half?" – an interpretation that is for Dolly both meaningful and actionable. Lyle, referencing the orange and evoking his own meanings of both one quarter and one half, determines that two quarter pieces fit into a half piece and (correctly) answers, "2." Immediately thereafter, however, he shifts his attention to the algorithm on the

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page, and possibly seeing $\frac{4}{1}$, he changes his answer to "4." Doing so provokes dissonance in the pair's meaning-making process, since the outcomes of what Lyle had enacted with the orange did not match what he had enacted with the algorithm. We conjecture that this shift from "2" to "4" was provoked by Lyle's prior knowing of fraction division as the execution of an algorithm, and as a result, he seems to privilege the algorithm over the orange as an anchor of certainty against which his own reasoning is measured.

Next, Lyle aims to resolve the dissonance he experienced as he produced two different solutions to the posed problem. Turning back to the dividend (½) and quotient (¼) in the problem, he seems set on finding a harmonious interpretation of the "whole thing" (object) across both artifacts and wonders yet again just what ½ ÷ ¼ means.

In our interpretation, Lyle's actions are directed at finding harmony across three instances of dissonance: 1) His expression, "Is that half *of* a quarter, though?" [emphasis added] corresponds to a (mis)interpretation of fraction division as one fractional part *of* another; 2) Lyle's ongoing endeavor to identify the whole in his interpretations of fractions – including the utterance, "It's not half of a whole thing" as he repeatedly circles the "4" on the paper – is an indication that he has yet to settle on what that whole is; and 3) His contemplative circling of the "4" could indicate that the number is a perceived point of both importance and confusion resulting from the actions of the flip-and-multiply algorithm. Dolly's utterance, "Why is this so hard?" is an expression of the messiness of engagement with multiple representations and what it feels like for her and Lyle to find themselves amidst spirals of semiotic interference across different *artifacts* (the orange, the algorithm), their wonderings about *objects* (e.g., What is a whole? What is division? What is $4/1$?), and the *relationships* between artifacts and objects across signs (e.g., What is the whole across these different representations?, What does ½ ÷ ¼, mean, and how does it relate to an enactment of "How many quarters go into a half?" with the orange?).

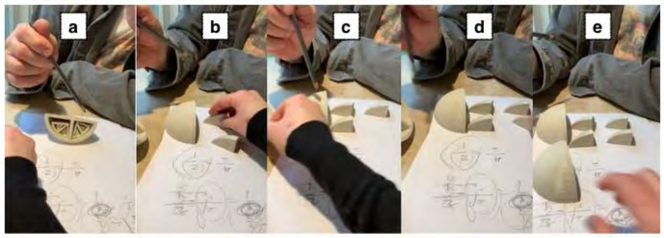
A crowning achievement

In this next excerpt, we present what appears to be a crowning achievement for Dolly and Lyle in their search for harmony in meanings for fraction division mediated by two artifacts. By enchaining signs across pieces of the orange and elements of the algorithm, more specifically by translating interpretations of parts of the orange to interpretations of quantities in the algorithm (i.e., $4/2$ and $2/1$), they have just made sense of those quantities. Next, they engaged in similar sense making in order to find interpretations for the ½ and ¼ in the posed problem, ½ ÷ ¼.

Dolly: <referring to the expression, ½ ÷ ¼> We wanna take a half of one and divide it by a quarter of one, right?
Lyle: Yes.
Dolly: Take a half of one and divide – oh, that's what it is!
Lyle: It's 2.
Dolly: We wanna take this <points to the half piece of the orange> and see how many of those *<now pointing to quarter piece>* fit in there *<points to the half piece again. Then, with confidence:>* And that's why our answer is 2.
Lyle: Yes.
Dolly: There's still two halves in a whole, 'cuz this *<the expression, ½ ÷ ¼>* is in regards to a whole. *<rephrasing>* This is in regards to 1. So a half of 1 divided by a quarter of 1 is 2, because 2 quarters fit into 1 half. Or *<returning to the expression,* $\frac{4}{2} = \frac{2}{1}$*>* 4 quarters fit into 2 halves.

Lyle: Yeah.

In this excerpt, we observe the meaning Dolly makes of the expression, ½ ÷ ¼, by enchaining interpretations of ½ and ¼ in light of the measurement meaning of division she and Lyle enacted earlier, as well as the meanings they enacted for $\frac{4}{2}$ and $\frac{2}{1}$ in the algorithm. Next, Lyle re-enacts the interpretation for himself.



Figures 2a - –e: Lyle re-enacts Dolly's understanding of "$4/2 = 2/1$."

Lyle: <*pointing to ½ on the page:*> So this is half of a whole <now pointing to ¼ on the page.> and this is a quarter of a whole. <*Next, he turns his attention to the orange (Figure 2a) and points to the half piece resting on the paper. He mutters quietly as if he's reassuring himself:*> Half of a whole. <*Next, he takes his pencil and points to each quarter piece in a sweeping motion of the pencil across each piece*:> Quarter of a whole <*Then, pointing to the two quarter pieces, he continues:*> is 2. <*Thus, he appears to be establishing that the number of quarter pieces he's identified – 2 – is the answer to the posed problem, ½ ÷ ¼*>.

Dolly: <*pointing to the 2 quarter pieces*> Yeah, 'cause there's two quarters of a whole.

Lyle: Yeah, that makes sense.

Dolly: 'Cause there's two of these <*She pulls out the quarter pieces and sets them next to the half piece (Figure 2b)*.> for every one of these <*she says as she touches the half piece*>.

Lyle: <*with a sigh, perhaps of relief*> Yes.

Dolly: Or there's four of these. <*She takes the quarter pieces out of the other half piece.* >

Lyle: <*points to the half piece and extends Dolly's thinking (Figure 2c*)>: For two of those.

Dolly: <*revoicing Lyle*> For two of those. <*As she speaks, she aligns all of the quarter pieces as well as the second half piece on the page (Figures 2d and 2e*).>

As if to establish his own meanings for fraction division and its coherence in representations across artifacts as Dolly has just done, Lyle uses the pencil in his hand to re-enact a physical bridge between the elements of the problem (½ ÷ ¼) and the pieces of the orange. He utters "half of a whole" as he points to the ½ on paper, and "quarter of a whole" as he points to the ¼. Then he repeats these phrases on the other side of the bridge he's establishing: "half of a whole" as he points to the half piece, and "quarter of a whole" as he points to the quarter piece. We interpret this activity as a matching of his interpretation of half of a whole and quarter of a whole in the symbolic representations (½ and ¼, respectively) to the representations he's identified in the orange (the half piece and the quarter piece, respectively). These embodied epistemic actions seem to reify the harmony that has finally emerged from recursive interactions that culminate in an enchaining of signs signifying the sense he and Dolly have made. This reification can be viewed as a newly coupled structure of fraction division for Dolly and Lyle, one that offers a

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stark contrast to the structurally determined response to fraction division that they enacted at the outset of their problem-solving activity. That is, rather than performing a rote algorithmic process as fraction division, they actually come to do (be/know) fraction division and enchain multiple mathematical signs in order to do so.

Concluding Discussion

This work set out to address the question, *"How do learners make sense of and coordinate meanings across multiple representations of mathematical ideas?"* We did so by analyzing Dolly and Lyle's sense making of fraction division through the complex interplay of signs and meanings that emerged from their engagement with multiple representations. In particular, we analyzed problem-solving interactions that were driven by an imperative to *make sense* of the complicated ideas of fraction division mediated by both an algorithm and a "Fraction Orange" manipulative. The course of their moment-to-moment activity beckoned us to leverage an enactivist framework for its stance on interactions *as* knowing, and for its appreciation of the doing of mathematics as a recursive, nonlinear, unfolding, embodied activity influenced by a system's lived history and its ongoing strive for *fit*.

In analyzing the iterative cycles of harmony and dissonance experienced by Dolly and Lyle, the analytic concepts of structural coupling, structural determinism, semiotic interference, and fit enabled us to discern valuable insights into learners' activity as they navigated multiple representations of mathematical ideas. In particular, structural coupling and determinism enabled a particular focus on the co-constitution that takes place between the individual and their environment through dialectic interactions that result in action-as-knowing. Dolly and Lyle's structural couplings with traditional school mathematics became apparent to us as they navigated felt experiences of harmony and dissonance throughout their drive for fit. For quite a while, they struggled to establish and maintain coherence in meanings across representamens (artifacts, symbols), objects (mathematical ideas), and interpretants (their own meanings of relationships between artifacts and ideas) at hand. Eventually, their dissonance gave way as they established harmony by enchaining meanings across signs through interactions with multiple representations of the complex network of mathematical ideas involved in fraction division. Ultimately, this harmony made way for deep (and felt) ways of doing/knowing mathematics.

The implications of this finding for practice are in recommendations for pedagogical and material resources that enable, support, and honor this sort of loosely structured problem-solving activity to occur in mathematics classrooms. On this point, we wish to re-emphasize that it was this activity that was fundamental to Dolly and Lyle's learning and not their assimilation of a path constructed by others. As Proulx (2013) reminds us, students' paths of problem solving emerge in interactions with the environment and are contingent on their particular mathematical structures and interactions. "Average" paths and tools presumed viable for sense making simply cannot be determined *a priori*. Rather, tools should be provided that are responsive to students' creative and agentive efforts at sense making as they lay down their own path while walking (Varela, 1987). And it is only in such walking that learners can define and refine their own authoring of mathematical ideas and meanings, and find confidence as a mathematical doer with membership in a classroom community.

Note

[1] We use the word *harmony* in a sense similar to Mariotti and Montone's (2020) concept of *synergy*, to denote "the emergence of a phenomenon of semiotic interference [that] fosters the

evolution of signs in an effective semiotic chain," which is an indication of a "deepening and weaving [of] the semiotic web" of mathematical meaning (p. 113).

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