ASSESSING THE QUALITY OF MATHEMATICS IN CAMEROON PRIMARY SCHOOL TEXTBOOKS AND ITS IMPLICATIONS TO LEARNING

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Mathematics textbooks for upper primary classes in the English Subsystem of Education in Cameroon were examined to determine the quality of mathematics in them and possible teacher knowledge fostered. The quality of mathematics in these textbooks is classified as medium and the dominant teacher knowledge fostered is common content knowledge. This is because the textbooks are full of accurate standard algorithms and mathematical definitions, yet lack the use of multiple strategies and representations. They also contain high proportion of mathematical explanations that are either partially accurate or accurate but incomplete. Textbooks with medium mathematical quality have high potentials of causing learners and teachers to be mathematically malnourished.

Keywords: Assessment, Mathematical Representations, Curriculum

In 2018, the Ministry of Basic Education in Cameroon introduced reforms in the Primary School curriculum for the English Subsystem of Education. Following this curriculum reform, Cameroon promulgated into law, for the first time, the one textbook policy, meaning only one textbook would be approved by the National Council for the Approval of Textbooks and Didactic Materials (NCATDM) for use in each class for each subject for a period of six years before the selection is reviewed. Following this policy, publishers of textbooks went into writing to submit materials for approval by the NCATDM so that primary school learners and teachers throughout Cameroon would use them for teaching and learning. A goal of the NCATDM is to select the textbook that covers the curriculum in the best possible way to ensure that learners learn appropriate content. This paper focuses on mathematics textbooks only.

Shulman (1986) argued that teachers need more than facts to adequately teach mathematics. A possible point where teachers could obtain knowledge for teaching is during pre-service teacher training programs. Ball, Thames and Phelps (2008) noted that subject matter courses in many teacher preparation programs fail to provide the much needed mathematics content for teaching as the emphasis seems to be on higher mathematics. Therefore, my hypothesis is that in such a case teachers, after being trained, actually encounter the mathematics they are to teach when exposed to textbooks designed for learners. Hence, mathematical knowledge for teaching seems to be encountered and developed as teachers use textbooks to teach.

A number of studies have investigated teachers' mathematical knowledge and its impact on student achievement as well as the quality of mathematics in classroom instruction. Hill, Rowan and Ball (2005) found that the stronger a teacher's knowledge of mathematics, the greater the learning exhibited by learners. Ball, Thames and Phelps (2008) identified the components of mathematical knowledge that the work of teaching demands on teachers. Hill, Blunk, Charalambous, Lewis, Phelps, Sleep and Ball (2008) investigated the quality of mathematics that teachers display in classrooms during instruction and found that there is a strong positive correlation between teacher knowledge and quality of mathematics exhibited in instruction. However, little has been investigated about the quality of mathematics provided in textbooks for

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Cameroon schools. This study investigates the quality of mathematics provided in primary school textbooks selected by NCATDM and attempts to answer two research questions. *What is the quality of mathematics in primary school textbooks approved for use by the English Subsystem of Education in Cameroon from 2020-2026? What types of mathematical knowledge for teaching might be promoted for teachers using these textbooks?*

This study has potentials to influence policy on textbook selection, focusing on the high quality of mathematics and the type of mathematical knowledge for teaching promoted. It may also be helpful to mathematics educators to examine the gap between what training of teachers offers and what teachers encounter in textbooks and fill in the space so as to adequately prepare teachers for teaching. In addition, this study can inform professional development on areas to focus so as to strengthen teacher learning.

Theoretical Perspective

Hill, Blunk, Charalambous, Lewis, Phelps, Sleep and Ball (2008) identified key aspects of high quality mathematics in classrooms including accurate mathematical explanations, mathematically accurate and intelligible definitions, accurate summary of mathematical ideas, reflection on explanations, conceptual discussion of procedures, accurate mathematical language, careful use of real world contexts, knowledge and use of multiple solution strategies, use of multiple representations and sequential construction of mathematics from one topic to another. Marshall, Superfine and Canty (2010) argued that multiple representations improve on the quality of mathematics taught in classrooms. Marshall, Superfine and Canty (2010) further argue that just using multiple representations is not enough but called for connections between or among the representations to ensure greater visibility to learners and therefore raise the quality of mathematics in instruction through reflection of the representations, create opportunities for learners to translate among representations. Connections should also be fostered between or among units in a textbook (Ball & Cohen, 1996) as this can help learners see mathematics as a connected subject and be able to pull learning from one unit to another to boost their understanding and sense making in the subject. Teacher's knowledge can also be supported as they use curriculum materials to teach. Ball, Thames and Phelps (2008) identified Common Content Knowledge (CCK), Specialized Content knowledge (SCK), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Students (KCS) as knowledge teachers need to teach.

Methodology

This study is part of a larger study investigating the quality of mathematics in textbooks approved by NCATDM for use in Primary Schools (classes one to six) of the English Subsystem of Education in Cameroon. Learners' textbooks for classes five and six were analyzed for this particular study.

Textbooks for this study. Textbooks approved by NCATDM for classes five and six are published by ASVA Education with titles *Foundation Primary Mathematics 5* and *Foundation Primary Mathematics 6*. Throughout these textbooks, each unit has sections for *let's observe, let's find out, let's retain* and *let's practice. Let's observe* contains demonstration of some methods pupils are expected to learn, *let's find out* contains questions that are presented for learners to reflect on the methods just observed, *let's retain* contains mathematical explanations or definitions of concepts learners are expected to understand as well as examples used to illustrate mathematical concepts learners are to learn and *let's practice* contains problems learners are supposed to engage with in order to reinforce the concepts learned.

Data sources. Data for this study were drawn from the *let's observe* and *let's retain* sections of each unit. This is because these are the sections where representations and mathematical explanations or definitions for concepts learners are expected to learn are provided. Simple random sampling was done and fifty percent of the units in each textbook was selected for analysis. This was to ensure greater coverage to adequately represent each of the textbooks. The following six units out of twelve were selected for analysis in *Foundation Primary Mathematics 5*: Unit 2-basic number operations, Unit 4-number and numeration, Unit 6-modulo arithmetic and number bases, Unit 8-money and shopping, Unit 10-speed, distance and time and Unit 12-graphs and statistics. For *Foundation Primary Mathematics 6*, six units out of twelve selected were: Unit 2-numbers and numeration, Unit 3-basic number operations, Unit 4-base system, Unit 5-fractions and decimals, Unit 6-modular arithmetic, and Unit 7-Rate, ratio and proportion.

Data analysis. In this analysis, mathematical explanations, solutions to examples, representations and definitions were coded. Mathematical sentence were coded using Figure 1.

Codes		**Descriptions**
1A	Accuracy	When all parts of the explanation are correct.
1B	Partially accurate	When some parts of the explanation are correct and other parts are not correct.
1C	Inaccurate	When all parts of the explanation are not correct.
1D	No explanation	When no explanation is provided.
1E	Incomplete explanation	When an incomplete accurate explanation is provided.
2A	Single method	Just one method is used in solving an example.
2B	Multiple methods	In more than one methods used in solving an example.
3A	Connections	Connections made between or among the methods.
3B	No connections	No connections made between or among multiple methods.
3C	Reference made in text	Reference is made about the solution in the text.
3D	No reference	No reference about the solution in the text.
4A	Single representation	Whether a single representation is used employed.
4B	Multiple representation	The use of more than one representation to explain a concept.
5A	Accurate representation	When all parts of the representations are correct, conveying conceptual aspects of the key mathematical ideas to be learned.
5B	Partially accurate representation	When some parts of the representations are correct, conveying conceptual aspects of the key mathematical ideas to be learned while other parts are not correct.
5C	Inaccurate representation	When all parts of the representations are not correct, conveying incorrect conceptual aspects of the key mathematical ideas to be learned.
5D	Reference to representation in the text	When explicit reference is made in the text to explain the representation used.
5E	No reference to representation in the text	When no reference is made in the text to explain the representation used.
5F	Connections between or among representations used	When explicit connections are made in the text to show relationships between or among representations used.
6A	Mathematically accurate and intelligible definitions	All components of the definitions are accurate with no limitations or ambiguity.
6B	Mathematically partially accurate definitions	Some parts of the definitions are accurate while others have limitations or ambiguity.
6C	Mathematically inaccurate definitions	All parts of the definition are not correct.
6D	Accurate but incomplete definitions	Definitions are accurate but incomplete.

Figure 1: Codes and Descriptions

Lastly, I deduced the quality of mathematics learners are likely to learn from using these textbooks as high (*accurate mathematics and representations used and connections among the representations*), medium (*mostly partially accurate mathematics and representations and sometimes connections among them*) and low (*mostly inaccurate mathematics and representations used and no connections among them*). Finally, from the mathematics embedded in these two textbooks, I inferred the dominant kind of mathematical knowledge teachers using them might possibly acquire over time. I coded the knowledge type as CCK (*mathematical knowledge common to other users of mathematics*), SCK (*mathematical knowledge specific to the teaching of mathematics*), KCS (*anticipating what students might think, the confusion/difficulties they might have*) and KCT (*knowledge of teaching and about the mathematics they are to teach, understanding the sequencing of topics, the design rationale of tasks or representations used*).

Results

		UNITS IN CLASS FIVE TEXTBOOK						**UNITS IN CLASS SIX TEXTBOOK**							
		2	**4**	**6**	**8**	**10**	**12**	**2**	**3**	**4**	**5**	**6**	**7**	TOTAL	%
	CODES														
1	1A	50	109	63	4	7	10	55	22	38	101	23	51	533	77.8
	1B	7	6	24	0	0	1	7	1	10	9	0	0	65	9.5
	1C	0	12	11	0	0	0	0	3	1	2	2	0	31	4.5
	1D	0	18	4	1	0	0	0	0	0	0	0	0	23	3.4
	1E	0	4	0	0	2	0	0	3	4	20	0	0	33	4.8
	TOTAL	**57**	**149**	**102**	**5**	**9**	**11**	**62**	**29**	**53**	**132**	**25**	**51**	**685**	**100**
2	2A	5	1	17	2	3	1	14	5	9	38	7	9	111	90.2
	2B	5	1	1	0	0	0	1	1	1	0	1	1	12	9.8
	TOTAL	**10**	**2**	**18**	**2**	**3**	**1**	**15**	**6**	**10**	**38**	**8**	**10**	**123**	**100**
3	3A	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	3B	5	1	1	0	0	0	2	0	1	0	2	0	12	100
	3C	5	0	0	0	0	0	1	0	0	0	0	0	6	50
	3D	0	1	1	0	0	0	1	0	1	0	2	0	6	50
	TOTAL	**10**	**2**	**2**	**0**	**0**	**0**	**4**	**0**	**2**	**0**	**4**	**0**	**24**	
4	4A	6	16	21	1	0	2	0	0	0	3	4	0	53	76.8
	4B	4	8	0	0	0	1	0	0	0	0	3	0	16	23.2
	TOTAL	**10**	**24**	**21**	**1**	**0**	**3**	**0**	**0**	**0**	**3**	**7**	**0**	**69**	**100**
5	5A	1	4	9	1	0	3	0	0	1	3	2	0	24	32.9
	5B	2	0	4	0	0	0	0	0	2	4	0	0	12	16.4
	5C	0	5	0	0	0	0	0	0	0	3	2	0	10	13.7
	5D	2	7	0	0	0	2	0	0	0	5	0	0	16	21.9
	5E	0	7	0	1	0	0	0	0	0	0	3	0	11	15.1
	5F	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	TOTAL	**5**	**23**	**13**	**2**	**0**	**5**	**0**	**0**	**3**	**15**	**7**	**0**	**73**	**100**
6	6A	11	5	0	8	5	7	8	0	0	3	0	3	50	92.6
	6B	0	2	0	0	1	1	0	0	0	0	0	0	4	7.4

	6C	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	6D	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	TOTAL	**11**	**7**	**0**	**8**	**6**	**8**	**8**	**0**	**0**	**3**	**0**	**3**	**54**	**100**

Figure 2: Coding results

Figure 2 shows that of the 685 mathematical explanations provided, 77.8% of them were accurate in all its parts. The accurate mathematical explanations were all standard procedures including steps that have to be executed by learners. For example, in the unit for *Base System*, conversion from one base to another is required. This textbook explains the procedure as follows "to convert numbers from one base to another other than base 10, we first change them to base 10, then, we change to the indicated base" (class 6 textbook, p. 39). Some of these mathematical explanations were simply facts that are to be learned, memorized and reproduced such as "not all prime numbers are odd" (class 5 textbook, p. 15) and "2 is a prime number but is also an even number" (class 5 textbook, p. 15).

Also, of all the 685 sentences providing mathematical explanations, 9.5% of them are partially accurate. In the class 6 textbook, it is explained that "to look for the cube root, first divide the number by all possible factors" (p. 25). This explanation is partially accurate in that we find the cube root of any number by dividing it by possible prime factors only not "all possible factors." The absence of "prime factors" in the textbook's explanation makes it partially accurate.

Of the mathematical explanations provided, 4.5% are inaccurate in all of its parts. For example, in expressing fractions as decimals, $\frac{1}{2}$ is used in the textbook and written as

$$\begin{array}{r|l} & 0.5 \\ \hline 2 & 10 \\ - & 10 \\ \hline & 0 \end{array}$$

together with the following explanation "1 cannot divide 2 so, we put a point above 1 and affix a zero behind 1 to make it 10, 10 divided by 2 is 5" (class 6 textbook, p. 66). This explanation is not correct in all its parts as the point is not put on 1. Note that every whole number has a decimal point after it. So, 1 can be written as 1.0. Now, since 2 cannot go into 1, we put a 0 above 1 and then put the decimal point above the decimal point and insert a zero (0) after the decimal point. Now 5 tenth multiplied by 2 gives 1.0 as shown to the right.

$$\begin{array}{r|l} & 0.5 \\ \hline 2 & 1.0 \\ - & 1.0 \\ \hline & 0 \end{array}$$

In the textbook's explanation, one wonders how we started with the dividend as 1 and ended up with it as being 10. Of the 685 mathematical sentences, 3.4% had no explanations.

In subtracting fractions, the textbook provides a problem as $\frac{3}{3}-\frac{2}{9}$. Then goes ahead to solve the problem as follows $\frac{3}{3}-\frac{2}{9}=\frac{3\times3}{3\times3}-\frac{2\times1}{9\times1}$, then $\frac{3}{3}-\frac{2}{9}=\frac{9}{9}-\frac{2}{9}$ and finally $\frac{3}{3}-\frac{2}{9}=\frac{7}{9}$ (class 5 textbook, p. 44). In this solution, the authors did not explain why the numerator and denominator of the fraction $\frac{3}{3}$ are multiplied by 3 and why that of $\frac{2}{9}$ is multiplied by 1. Without explaining why the multiplications were done, the learners and teachers are left with a thinking that the numbers were chosen arbitrarily, making their understanding flawed. Of the 685 mathematical sentences, 4.8% had incomplete accurate explanations. In explaining a mixed fraction, the textbook said "a mixed fraction is a fraction which has a whole number attached to it to the left side" (class 5 textbook, p. 41). This explanation is accurate but incomplete as the whole number is the quotient when a number is divided by another number. So, the complete accurate explanation could have been, "a fraction represented with its quotient and remainder is called a mixed fraction." In addition, learners are often confused about the operation between the whole number and the fractional part of the mixed fraction. Learners often see that operation as multiplication because ab means a ′ b. Therefore, emphasis could have been laid by the authors that the mixed fraction $a\frac{b}{c} = a + \frac{b}{c}$ to dispel this confusion and curb misconceptions that learners often have. This accurate complete explanation provided might cause a smooth transition between improper fractions and mixed numbers and fully explain the idea of mixed fractions.

Of the 123 solutions provided, 111 of them have just one strategy while 12 of them have at least two strategies. When solved using more than one method, no connections are made between or among the methods. This is a missed opportunity to have learners decide which approach or strategy they understand best and will be able to use. In 50% of the time, when more than one solution strategies are used, these are referenced in the text while in another 50% there is no reference about the solution in the text. When no reference is made about the solution in the text, learners are left with the option of struggling to understand what they actually mean. In 76.8% of the time, the authors used single representation to solve problems or demonstrate a concept while in 23.2% of the time, multiple representations are used.

The representations revealed that 32.9% of them were accurate in all parts, conveying conceptual aspects of the key mathematical ideas to be learned.

$$\begin{array}{cccccc} & 6 & . & 3 & 4 & 6 \\ & 0 & . & 0 & 3 & 5 \\ + & 8 & . & 5 & & \\ & 0 & . & 7 & & \\ \hline 1 & 5 & . & 5 & 8 & 1 \end{array}$$

In adding decimals, the authors accurately lined up all the decimal numbers using the place value table to the left and then calculating the sum (class 5 textbook, p. 53).

Of the representations used, 16.4 % are partially accurate, some parts of the representations are correct, conveying conceptual aspects of the key mathematical ideas to be learned while other parts are not correct.

Fractions	**Decimals**	**Percentages**
$\frac{1}{2}$	0.5	50%
$\frac{1}{4}$	0.25	25%
$\frac{1}{3}$	0.33	33%

In changing fractions to decimals and then percentages, the authors presented the table to the left which is not accurate in all its parts (class 5 textbook, p. 52).

The first two rows are both correct and accurate but the third row is not correct as $\frac{1}{3}$ is not exactly 0.33 as a decimal and $\frac{1}{3}$ is not exactly 33% as a percentage. This inaccurate representation of the third row can be very misleading to teachers and learners.

Of the representations, 13.7% are inaccurate in all parts, conveying incorrect conceptual aspects of the key mathematical ideas to be learned. The representation of equivalent fractions is incorrect, conveying misconceptions of the key mathematical idea. For example, $\frac{1}{2}$ is represented as equivalent to $\frac{2}{4}$ and also $\frac{3}{6}$ on two separate diagrams (class 6 textbook, p. 46). The emphasis in this textbook is on the generation and not on the understanding/meaning of equivalent fractions.

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As such the multiplication of numerator and denominator by the same whole number to generate equivalent fractions is emphasized and reinforced. Of the representations used, 21.9% of them are referenced in the text explicitly to explain teachers and learners the concept embedded in them, 15.1% of the representations are not referenced in the text and this has the potential of teachers and learners ignoring them for lack of understanding and none (0%) of the representations are connected explicitly or implicitly to show relationships between or among them and rationale for why they were used.

For mathematical definitions provided throughout the textbooks, 92.6% of them were accurate and having no ambiguity. For example, "proper fractions are fractions whose numerators are smaller than the denominators" (class 6 textbook, p. 45). Of the definitions provided, 7.4% are partially accurate. For example, "when an object is divided into equal parts, each part is a fraction of that object" (class 6 textbook, p. 44). This definition offered by the textbook is partially correct as it is not only when the parts are equal that it is a fraction of the whole. A part of a whole is a fraction whether they are equal or unequal. Also, fractions are formed by dividing *n* units into *m* equal parts $\left(\frac{n}{m}\right)$ and then collecting *n* of those equal parts. In addition, the book defines the calculation of speed or average speed as $\frac{\text{Distance}}{\text{Time Taken}}$ (class 5 textbook, p. 101). This definition is true and accurate for speed but not always for average speed. Average speed is calculated using $\frac{\text{Distance covered in an interval of time}}{\text{interval of time}}$ or $\frac{\text{increase in displacement in that interval of time}}{\text{interval of time}}$. Although speed and average speed might be the same at some point, this is usually not the case and should be clearly distinguished to the teacher and learner. Furthermore, none of the definitions are completely inaccurate or completely accurate; they are incomplete.

The results of this study revealed that the dominant kind of teacher knowledge that might be highly promoted is *Common Content knowledge* (CCK). Figure 2 indicate that majority of the mathematical explanations provided are accurate (77.8%). These explanations are mainly those that could be offered by mathematicians as well as other users of mathematics. Also, in the examples provided inside the textbooks, 90.2% of them were solved using a single method and when representations were used, only a single representation is used to explain a mathematical idea. The single solution methods provided are mainly standard algorithms. In addition, when definitions are provided, 92.6% of them are accurate and often these are standard mathematical definitions.

Discussion/Significance

Overall, the quality of mathematics presented in official textbooks for primary 5 and 6 of the English Subsystem of Education in Cameroon can be classified as medium. This is because in these textbooks, the proportion of partially accurate mathematics is significantly high; multiple solution strategies/representations are rarely used; when multiple solution strategies / representations are used, connections between or among them are rarely established; proportion of mathematical definitions that are inaccurate is significantly high. As such, these textbooks fall short of research recommendations for curriculum materials from which teachers can learn.

Davis and Krajcik (2005) recommended that curriculum materials should contain features to support teacher learning. These features include multiple ways learners might respond to a task or problem and together provide mathematical explanations embedded in these responses and representations that might be employed. In addition, connections between and among the strategies/representations used can be helpful in providing multiple access into the mathematical ideas learners are to learn. Marshall, Superfine and Canty (2010) have argued that when multiple representations are used, connections between or among them should be established in order to raise the quality of mathematics being taught. Ball and Cohen (1996) emphasized that these connections should be fostered by textbook authors. Therefore, establishing connections between or among multiple solution strategies/representations used can help to improve on the quality of mathematics learners learn. In addition to improving the quality of mathematics in textbooks, intentionally making connections in the mathematics textbook might enable learners to see the subject as connected and might be induced into making such connections so as to improve on the quality of their learning. The absence of these features in official textbooks selected for use in the English Subsystem in Cameroon seems to project these curriculum materials as creating very little opportunities for teachers and learners to learn appropriate mathematics and hence being mathematically malnourished.

Teachers are often mathematically malnourished when their learning is limited to a unique form of mathematical knowledge for teaching. The dominant teacher knowledge propagated in these textbooks is common content knowledge (CCK). This is because the percentage of mathematical explanations, single solution methods, single representation and mathematical definitions used in these textbooks are very high. In addition, their focus is laid on standard algorithms. The absence of other forms of teacher knowledge in these textbooks is a clear indication that the teachers using them might be limited in their mathematical knowledge for teaching as a whole and as such limited in teaching this subject to learners.

These findings reveal that mathematics textbooks approved for use in class 5 and 6 in the English Subsystem of Education in Cameroon are not fully providing and developing the needed mathematical proficiency in teachers and learners. The National Research Council (NRC, 2001) characterized mathematical proficiency as having five strands namely *conceptual understanding*, *procedural fluency*, *strategic competence, adaptive reasoning* and *productive disposition*. From the emphasis in these textbooks, one could deduce that only one strand, procedural fluency is promoted because of the heavy emphasis on standard methods and definitions. This study identified that mathematics textbooks selected for use by our learners and teachers fall short of the standard to support and develop their mathematical proficiency. Therefore, textbook authors can use the results of this study to develop materials that will support and develop the needed mathematical proficiency for both teachers and learners in Cameroon. The results will also help the NCATDM review their selection criteria for textbooks and focus on aspects that promote learning of both teachers and learners. The outcome of this study will also help professional development experts and teacher educators in Cameroon to focus on building teachers' capacities in areas identified as limited in these textbooks.

Although this study investigated textbooks in Cameroon, the quality of mathematics in many textbooks around the globe might not be promoting desired mathematical proficiency because the features to support improve this quality are highly limited. As such, the following question need further investigation: What combination of the features to develop mathematical proficiency in both teachers and learners is needed in textbooks to yield optimum learning outcomes? Answers to this question will enable textbook developers focus on using only those features whose interactions produce greatest learning outcomes rather than attempt to include all features that might be overwhelming to teachers.

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