EXPERIENCED SECONDARY TEACHERS' DECISIONS TO ATTEND TO THE INDEPENDENT VARIABLE IN EXPONENTIAL FUNCTIONS

Melissa Troudt
U. of Wisconsin – Eau Claire
troudtml@uwec.edu

Lindsay Reiten
U. of Northern Colorado
lindsay.reiten@unco.edu

Jodie Novak
U. of Northern Colorado
jodie.novak@unco.edu

We report our findings and perspective to document the knowledge exhibited by three experienced high school teachers in their instructional decisions for lessons on the equation of an exponential function. We describe the nature of the mathematical ideas and connections teachers promoted in discourse and the decisions that supported the emergence and connections of the mathematics. Despite similarities in the structure of the mathematical activities, differences existed in the ideas that emerged in the three teachers' discussions regarding the relationship between the exponent value and the independent variable. We describe links between collections of teacher decisions to their influences on the mathematics discourse.

Keywords: Teacher Knowledge; Classroom Discourse

Introduction and Background Literature

This study aims to contribute to understanding the nature and quality of mathematics teachers' decisions as a means of describing teachers' knowledge for teaching mathematics in practice. The field widely accepts that teachers' knowledge strongly relates to their effectiveness (e.g., Charlambous & Hill, 2008). Expanding on the work to document and assess a cognitive perspective of teachers' mathematical knowledge for teaching, a call exists to integrate conceptualizations of teachers' knowing and their actions in the classroom (Depaepe et al., 2013). Reviewing literature, Stahnke et al. (2016) categorized studies of teacher knowing in action by the situation-specific processes investigated, namely perception, interpretation, and decision-making (Blömeke et al., 2015). Stahnke et al. concluded decision-making is the most challenging for pre-service teachers (PSTs). Meanwhile, decision-making of experienced teachers is tacit, effortless, and based on sophisticated networks of schema (e.g., Shavelson & Stern, 1981). To inform preparation of PSTs, we sought to learn from experienced secondary teachers by inquiring into their decisions in teaching exponential functions topics.

Despite observations of more powerful ways of understanding exponential growth (e.g., Confrey & Smith, 1994), high school curricula often introduce exponential functions through tasks that facilitate making a correspondence between a quantity growing by repeated multiplication and another related quantity (Davis, 2009). Defining exponential growth by repeated multiplication provides a potentially useful entry point (Weber, 2002); however, the metaphor is insufficient for explaining the meaning of expressions such as $2^{2/3}$ (Davis, 2009). In action, learners may reason about changes in the y-value without attending to the y-value's relationship to the x-value (Ellis et al., 2016) and therefore struggle to connect the repeated multiplication to the closed form of an equation (Davis, 2009). The closed form of an equation, when developed, can represent a correspondence view of the function. That is, one builds a rule to represent the relationship between an x-value and its associated y-value in the form of an algebraic equation $y = f(x)$. We sought to describe how teachers work within their available resources and constraints (Schoenfeld, 2011) to facilitate students meeting teachers' learning goals for understanding of equations of exponential functions.

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Theoretical Framing

We view knowledge or knowing from an enactivist epistemology (Maturana & Varela, 1992). Enactivism stems from evolutionary biology and conceptualizes an organism interacting and co-evolving with its environment. An organism "knows" within the environmental context if it acts in a way that is fitting and effective for the context (Maturana, 1988). Therefore, knowing or cognition is not a *thing* a person holds but "acting in a world that emerges in the doing itself" (Maheux & Proulx, 2015, p. 212). Knowledge is the body of effective behaviors and the underlying cognition that engender one to perceive the situation and categorize which behaviors are effective (Varela, Thompson, & Rosch, 1991). Learning is "a reciprocal activity — the teacher brings forth a world of significance with the learners" (Towers et al., 2013, p. 425).

In classroom mathematics discourse, there is both "*doing* something (some thing) recognizable as *mathematics*, but also producing *mathematics* as this thing that we are *doing* when we do what we do" (Maheux & Proulx, 2015, p. 215). The mathematics is the "world of significance" that the teacher brings forth with the learners by implementing a plethora of decisions both to set up the environment and to respond to (and with) the students. The mathematical ideas (i.e., concepts, patterns, principles, procedures, relationships) that emerge are not isolated entities. They are connected to and built up from other ideas with forms of coherence and structure fitting for the doers of the mathematics. As Towers et al. (2013) indicate, enactivism prompts observing "the relationship between things in a mathematical environment (ideas, fragments of dialogue, gestures, silences, diagrams, etc.), rather than to what each of those things might mean or represent in their own right and for the individual generating them" (p. 425). We conceptualize the mathematics as the emerging ideas in the discourse of the mathematical activity and the connections made to build up and connect the new ideas from and to other ideas. We define *knowing for teaching mathematics* as the teacher decisions to perturb the learning environment and to participate with students to influence the emergence of mathematics in ways they deem effective for student learning.

Interested in describing experienced teachers' knowing for teaching exponential functions enacted in whole class discourse (WCD), we sought to describe the nature of the mathematical world that emerged as well as the teacher activity that supported its emergence. We describe the nature of the mathematics in terms of the emergent ideas and the connections, consistency, and justifications offered in the discourse. Our research investigated: With respect to the *equation of an exponential function,* what is the nature of the mathematical ideas promoted in WCD and what instructional decisions supported the mathematics to emerge and connections to develop?

Methods

As part of a larger study, we collected data from 16 high school teachers engaged in the teaching of exponential function topics in the courses of Algebra I, Algebra II, College Algebra, and Pre-Calculus. All teachers were identified as highly effective and experienced by their administrators or peers and had obtained master's degrees. The corpus of data included classroom observations and interviews regarding teacher instruction. This study focused on the WCD of three teachers, Gabe, Evelyn, and Abby who had 28, 19, and 24 years of teaching experience respectively. Gabe and Evelyn taught College Algebra while Abby taught IB Math 3 (equivalent to Algebra II). We focused on these three teachers because we perceived surface-level similarities in their lesson structures for introducing the equation of an exponential function or geometric sequence.

We transcribed the classroom observations and partitioned each lesson into smaller segments of episodes and sequences (Wells, 1996) based on transitions of classroom tasks. Transcribed interviews included images of documents the teacher referenced during the interview when appropriate. Using the classroom videos and transcripts we developed concept maps representing the mathematics in WCD, noting connections made between the mathematical ideas (Leinhardt & Steele, 2005). Looking across the concept maps in three teachers' classrooms, we identified three common themes: the role of the independent variable, the relationship between the recursive multiplication and exponential form, and the definition of exponential. We created narratives describing the emerging mathematics and contributing teachers' decisions for each teacher and theme (i.e., nine narratives in total). We used comparative methods to identify contributing decisions and teacher actions in cross case analyses of the three teachers.

Findings

These findings focus on the first theme that emerged during the analysis of the classroom discourse surrounding writing the equation of an exponential function; specifically, the relationship between the value of the independent variable and the exponent of the algebraic expression.

Gabe Narrative

To write each of the three equations, students were told to complete a table of values given at least four consecutive entries and $\Delta x=1$ (see Figure 1). Students only needed to determine the *y*-intercept (i.e., the value when $x=0$) for y_5, the table included the *y*-intercepts for y_6 and y_7. Once students identified multiplying by two to move down the entries in the table (y_5), Gabe reviewed using exponents by leading students through going from 1 to 16 in the table via repeatedly multiplying by two. Gabe asked students how 1(2)(2)(2)(2) could be re-written, thus encouraging them to recall their work with exponents. After writing $1(2)^4$ Gabe asked students for the exponent for y_5. In using an example (y_5) students were told and then reminded (in y_6) that the exponent represents repeated multiplication, so *x* was the exponent in the general equation. The following excerpt from the WCD highlights Gabe's implicit connection between using exponents in the equation for y_5 due to repeated multiplication and the exponent being *x* in the equation.

> T: So, this <the (2)(2)(2)(2) > would be two to what power? You said something power.
> S: Three... fourth.
> T: To the fourth power. <writes $1(2)^4$ under the expression 1(2)(2)(2)(2)>
> T: So, what we're doing each time is we're multiplying by two, what's our exponent going to be?
> S: x
> T: Just x. <writes $y_5=1(2)^x$ > So, that's the equation for the first one. [E1:S4:L13-18]

In discussing writing the equation for y_6 Gabe stated, "[n]ow when we write it in this form, it's what we're multiplying by each time because that's what the exponent represents, a series of multiplications" [E1:S7:L3]. When writing the equation for y_7, the exponent was written but not mentioned. During notes, when introducing $y=ab^x$, Gabe defined the exponent saying, "[a]nd then our exponent's the number of times that we're going to be doing it" [E2:S1:L1].

Due to the structuring of the task (i.e., having students write equations from a table of values void of context), defining the independent variable was not needed. Rather, *x* was implicitly defined as being the exponent because the exponent represents repeated multiplication. Additionally, a need did not exist for making an explicit correspondence between defining the independent variable and stating that *x* was the exponent. In moving from $1(2)^4$ to asking students what the exponent would be for y_5, Gabe focused exclusively on the y_5 column and did not discuss that the four in the exponent connected to the row corresponding to $x=4$. In fact, the *x* column of the table was not included on the note sheet that students were given (see Figure 1).

y_5	y_6	y_7
	486	
	162	1000
2	54	1100
4	18	1210
8	6	1331
16		

Write equations for functions 5-7.

Exponential functions:

$y = ab^x$ or $y = ar^t$ or $y = a(1+r)^t$

Compound Interest Formula $A = P\left(1+\frac{r}{n}\right)^t$

Figure 1: Re-creation of a portion of the note sheet Gabe created and gave to students

Evelyn Narrative

Evelyn introduced writing the equation of an exponential function in two lessons involving the discussion and then summary of three tasks (see Table 1). The prompts for the tasks provided the value of a quantity at a point in time and information that the quantity grew by a multiplicative factor over a set time period (i.e., each day or each year). In small groups, students determined the value of the quantity at other points of time, utilizing recursive multiplication or division. In WCDs, Evelyn oriented students to represent their computations as numerical expressions in a table and then generalize to an equation.

On the One Grain of Rice task, students worked to find how many grains of rice a girl would have on the thirtieth day if she started with one grain of rice and the number of grains doubled each day. As Evelyn predicted prior to the lesson, students reached different answers depending on whether they labeled the starting value Day 1 or Day 0. She considered having students compare the effects of labeling the staring value as Day 1 or Day 0, but in class she chose to have the class come to a consensus in choosing to state the girl received one grain of rice on Day 1, meaning the point (1,1) was in the data set. In WCD, Evelyn noted there could be another choice of creating that point as (0,1). The choice would affect the final answer but not their process. Rather than discussing the effect of the choice, the focus shifted to representing the situation and the students' computations in a table.

The tables created for the One Grain of Rice and Social Media WCDs captured the repeated multiplications used to calculate the values. Evelyn led students to rewrite the expressions as repeated multiplication and then exponential expressions. By recognizing a pattern down the right column of the table, the class generalized that to calculate the value for any point in time multiply the starting value by the growth factor some number of times.

To generalize beyond the table, Evelyn asked students to find expressions for larger values in the table (i.e., the number of users in year 2052). The class discerned a relationship between the exponents in the expressions for the dependent variable and the value of the time variable. For example, students noticed the exponent of the expression for a given year could be found by subtracting five from the years since 2000. The class looked across the two columns to generalize the relationship between the value of the independent variable and the computational exponential expression to find the number of users (or grains of rice) associated with the value of the time variable. Therefore, they developed equations to find the value of the dependent quantity in year x or day d.

Evelyn then presented the contexts for each of the three tasks and the equations they found for each situation reminding students how they defined the independent variable (Table 1) and then replaced each of these with "x." The class made observations that each equation involved a time period and that the exponent was some sort of time period. Evelyn then presented the general form, $y = ab^x$ and the class discussed the role of each parameter. The meaning of x was given as "some time period" and was not defined as the value of the independent variable.

Table 1. Three task summary provided by Evelyn

One Grain of Rice	Social Media	Fruit Flies
Started with one grain of rice and doubled each day.	Started with 3.2 million users and tripled each year after 2005.	Started with 5 flies and they quadruped each day of vacation.
Grains of Rice = 2^{day-1}	#Social Media Users = $3.2(3)^{year-2005}$	#Fruit Flies = $5(4)^{\#days}$

Abby Narrative

Before this observation, students spent time solving and presenting their solutions to the domino skyscraper task (http://threeacts.mrmeyer.com/dominoskyscraper/) which posed the question, "If you wanted to topple over a domino the size of a skyscraper, how many dominoes would you need?" Students were told, "a smaller domino can topple a domino that is up to 1.5 times larger in every dimension" and that the first domino was 5 mm tall. In four small groups, students generated solutions for several skyscrapers by guessing and checking, creating a table, and using an exponential equation. Abby began this class by shifting the conversation from the solution to the task to the equations the students generated.

Abby asked Group 4 to present their equation $y=5(1.5)^x$ and table for the domino task, telling them to define their variables and connect their table to their equation. They explained why their equation was $y=5(1.5)^x$, where x represented the domino number and y represented the height of the domino, and connected it to their table. After the presentation, the class worked in small groups to "make a very clearly defined table. Identifying your variables, alright? And matching it up with your equation, alright? You want to make sure your equation matches up" [E1:S2:L1].

In a small group discussion, Abby asked the students why their equation $u_n=5\cdot1.5^{n-1}$ differed from the one presented. The students offered that the difference of "minus 1" in the exponent was due to their choice to label the initial domino of height 5 the first domino instead of the zeroth domino in their table. The teacher engaged in a similar discussion with another group which had the equation $= \frac{450{,}000}{1.5^x}$. Although there is not more WCD on this point, Abby engaged with two of the four groups focusing on why their equation was different from the one presented. In these group conversations with Abby, students explained how they constructed their table, how it was different than the table presented and how that impacted their equation, with particular attention to how the independent variable was defined.

In WCD, Abby returned to this theme when developing with students the equation for the general term of a geometric sequence. She began by highlighting the equation $y=5(1.5)^x$ and its associated table. She connected it with their work on geometric sequences by noting that the equation generated a geometric sequence (the y column in the table), that 5 was the initial value, and 1.5 was the common ratio. She pointed out that this table labeled the initial domino as the zeroth term, but that the convention for geometric sequences was to label the initial term (domino) as the first term.

Abby highlighted the work of the group who produced the equation $u_n=5\cdot1.5^{n-1}$, indicating that their first domino was 5 and that it aligned with the initial term of a geometric sequence being called the first term. The class established that the difference in the exponents between the two equations reflected the difference between starting with x=0 and x=1. Abby connected this to transformations explaining the difference as one equation being the other shifted to the right one. Through an interactive discussion, the students connected the equation $u_n=5\cdot1.5^{n-1}$ to the general term of a geometric sequence of the form $u_n=u_1\cdot r^{n-1}$, where n is the term number, r is the common ratio, and u_1 is the first term. Abby concluded this episode with the comment, "These are equivalent. <Pointing to $y=5(1.5)^x$ and $u_n=5\cdot1.5^{n-1}$.> It's just a matter of defining your variable. Where your starting point is. But they are really equivalent equations" [E4:S6:L10].

Cross-Case Analysis

Differences existed in the ideas that emerged in the three teachers' discussions regarding the relationship between the exponent value and the independent variable in the development of the exponential equations. For Gabe, the idea of the connection between the x-variable and exponent was minimal. In both discussion of a specific function (y_5) and the general forms, Gabe's class indicated the value of the exponent corresponded to the number of multiplications by the constant multiplier and the exponent was x. No connection was made between x as the number of multiplications and the x column of the table. The x column was only referred to when finding the y-intercept as the value of y that corresponded to an x-value of zero and consistently indicated the place where they start doing the repeated multiplications.

For Evelyn, consistency existed in providing intellectual need and opportunity to make a connection between the expression that represents the y-value and the corresponding value of the time variable in each of three contextualized tasks. The class developed tables by thinking about changes in the independent variable by one, implicitly attending to the x-value but focusing on the relationship between values in the y-column. Evelyn asked students to write an expression for a large x-value, skipping values in the table. Thus requiring students to look between columns of the table and generate the relationship between the x-value and the exponent value. They used this observed relationship to write the final equation for the functions in question. These conversations emerged in the particular discussions of generating the equations from tables but did not emerge in the end discussions regarding the general form. In fact, Evelyn noted in her interview that finding the value of the exponent would depend on the specific problem context. Specifically, she said, "[u]m, so to see that form of the starting value, the base and then that the exponent relies on whatever the context of the problem is" [Pre-Int Obs2 08:35]. We did not see a connection made between the exponent value as a transformation of the independent variable and the exponent as counting the number of multiplications.

The student groups in Abby's class created their own equations to model the situation of toppling dominos. Consistently, Abby directed students to check their equations with their table; thus the exponents of the student equations could be modified to account for the values of the corresponding domino number (independent variable). Ideas emerged from the particulars of individual groups making different choices in their work on the same task. The students' equations differed based on how they defined the starting value (i.e., Domino 0 vs. Domino 1). Abby built from the varied approaches of the students to motivate the generalizations providing standard language and definitions as needed. The result was a final general idea that the exponent of the general term of the sequence differed based on the labeling of the term number. Abby connected the idea to a horizontal transformation of functions.

The notion that different equations exist dependent upon defining the independent variable emerged in both Evelyn's and Abby's class due to the opportunity for students to create their own tables modeling the situation. The contexts of the domino and grain of rice tasks did not specify that the starting domino or day corresponded to a specific value of the independent variable; therefore, students made different choices. Abby allowed student groups to develop their own equations and made sure students matched all representations of the situation (sequence, table, context, equation, and later graph). She chose to have multiple groups present their solutions; therefore, the class saw three equations meant to capture the same relationship. The different equations presented a need for Abby to provide some closure to the idea. Students in Evelyn's class did not create equations on their own, but they made different choices in how they labeled the first value. Evelyn chose to lead the WCD of creating the equation and opted to label the independent variable as most of the students did. Not all students saw how the equation might look different based on defining of the independent variable.

All three classes developed and filled in tables, initially, through applying repeated multiplication. To write the equation, Evelyn and Abby's classes attended to the relationship between the x-value and the y-value or exponents of the expression of the corresponding y-value. Evelyn facilitated attention to this relationship by asking students to skip values in the table to find the y-value associated with a large x-value. Abby asked students to check to see if their equation was correct by paying attention to input-output correspondence in the tables.

In Gabe's class, it is unknown if and how students attended to the x-column of the table to determine the equation due to the nature of the tables presented. The values of x in the tables presented were equal to the number of multiplications. Referring to the x-column was not necessary since the rows of the table increased in Δx values of one and the tables provided a row corresponding to $x = 0$. When Gabe asked students what the exponent of the equation should be and a student said, x, it was unclear if x referred to the corresponding input for an output in a single row of a table, if x was a generalization for counting the number of multiplications, or if the exponent was x due to prior knowledge that exponential equations have an x in the exponent.

All three teachers included WCDs toward the end of the lessons providing general forms of the equations of exponential functions (or geometric sequences). Table 2 summarizes the language teachers used for these WCDs. Gabe provided his informal language tied to the process of finding the equations in the table. Evelyn asked students to compare and notice similarities of the structures of the three equations generated. Students described the parameters using their own language which tied to the class's previous mathematical activity. Using the explored domino sequence and equations as examples of the parameters to introduce vocabulary, Abby provided formal definitions of term, term number, and common ratio prior to introducing the general term. Abby's generalizing discussion was the only one which described the exponent of the equation as a transformation of the independent variable.

Table 2. Language and origin of language when defining a general exponential form.

Gabe	Evelyn	Abby
$y = ab^x$ *a:* "a-riginal", y-intercept, value at 0 *b*: What you're multiplying by *x:* How many times you multiply	$y = ab^x$ *a:* starting value *b:* what you're multiplying by each time period *x:* some sort of time period *y:* total amount of stuff	$U_n = U_1 r^{n-1}$ U_1: First term *r:* Common ratio *n*: Term number U_n: nth term
Teacher-provided language	Student provided language	Teacher-provided language

Discussion and Conclusions

The enactivist lens prompted us to not only notice single ideas but the relationships and connections among utterances in the discourse which emerged from the nature of the mathematical activity in the room: the activity of creating equations to describe the data of the tables. While the structures of these lessons are similar, attending to connections among ideas as facilitated in the mathematical activity allowed us to notice if and how the thread of the role of the independent variable was integral to the activity. Viewing from an enactivist lens, the teachers' knowing is seen from the mathematical worlds they facilitate to emerge in that context.

Gabe provided tasks incorporating functions and representations of those functions where implicit minimal attention to the x-column or relationship between the x and y values was sufficient. Labeling the exponent as x worked for every equation. He presented the general forms of the equation in a way that aligned with the taught process which included simple procedures and easy-to-remember language. The mathematics of Gabe's classroom was largely characterized by closed-form mathematics, one-word responses, and narrow examples facilitating single ideas to emerge sequentially. We posit Gabe's facilitated instruction prioritizing a mathematics that might be easy for students to replicate without error.

Evelyn asked questions encouraging students to notice and test patterns (e.g., patterns in the relationship between the values of the independent variable and the exponent in the expression for the corresponding *y*-value). Connections made between ideas existed within the tables which were artifacts of the activity of the mathematical discourse. The inductive reasoning repeated at a higher level when the class compared the structures of the three generated equations and $y=ab^x$. The generalizations about exponential functions were tied to the idea of multiplying a starting value repeatedly to find a total rather than formalizing a relationship between the independent and dependent variable. Evelyn accepted the language and definitions offered by the students rather than providing formal language. Her decision-making prioritized students making and testing generalizations based on the inductive reasoning *inspired by* the collective mathematical activity.

In Abby's class, the attention to the relationship between the independent variable and the equation was grounded in the class practice of reconciling the multiple representations of the growing quantity (context, table, multiple versions of equations, and graphs). Prior to generating the general form of a geometric sequence, Abby provided vocabulary for the relevant parameters of the domino task which corresponded to finding the value of term *n.* Her decisions positioned students as doers of mathematics, enabling students to identify the pertinent mathematical concepts. The decisions positioned her to provide a shared formal language connecting the class' mathematical activities of representing the domino task, making sense of others' representations, and making generalizations about properties of geometric sequences.

We viewed teacher knowing through the decisions teachers make as they engaged in activity (including mathematical activity) with their students to promote the emergence of a mathematical world. Taking this lens freed us from concerning ourselves with the individual actions of the teacher and each student to notice the nature of the activity (pedagogical and mathematical) which seemed to facilitate the emergence and connections among ideas. This work suggests developing teachers as decision makers by engaging PSTs in considering and a deliberate analysis (Brown & Coles, 2011) of the mathematical worlds afforded by collections of teacher moves.

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