SUPPORTING STUDENTS' MEANINGS FOR QUADRATICS: INTEGRATING RME, QUANTITATIVE REASONING AND DESIGNING FOR ABSTRACTION

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We describe how we supported middle-school students developing meanings for quadratic growth by bringing together three theoretical framings and designing a task sequence grounded in that framework. Specifically, this framework was assembled from the research on students' quantitative and covariational reasoning, Realistic Mathematics Education, and the theory of designing for mathematical abstraction. We present data from a teaching experiment to highlight how this task sequence supported students as they imagined constant increases in the amounts of change in one quantity, a kind of change they later understood to be a defining characteristic of quadratic growth. We conclude with a discussion of our findings and how our integration of the three frameworks proved useful to design a productive task sequence.

Keywords: Algebra and Algebraic Thinking; Middle School Education; Design Experiments

Quadratic relationships are an important topic in middle-school through college mathematics. In this study, we build on prior work (e.g., Ellis & Grinstead, 2008; Ellis 2011a, 2011b; Hohensee, 2016) that provides accounts of middle and high school students developing productive meanings for quadratic relationships via their quantitative and covariational reasoning. Whereas students in these studies leveraged numeric quantitative reasoning, we were interested in designing a task sequence that supported them in both non-numeric and numeric reasoning to develop meanings for quadratic growth. Hence, our goal is to address the research question: *How can middle-school students leverage non-numeric and numeric quantitative and covariational reasoning to develop meanings for quadratic growth?*

In what follows, we outline our operationalization of quadratic growth grounded in students' quantitative and covariational reasoning, and then describe a task sequence we developed with principles of Realistic Mathematics Education (RME) and designing for abstraction in mind. We highlight how this task sequence was productive in supporting students developing productive meanings for quadratic growth. We conclude by highlighting how the three theories interacted to inform the task design and development.

Quantitative Reasoning, Covariational Reasoning, and Quadratic Change

As conceptual entities, meanings for quantities can and do differ from individual to individual. Therefore, it is critical to attend to a *student's* meanings for quantities. Accordingly, Steffe, Thompson, and colleagues' stance on quantitative reasoning (Steffe, 1991; Smith III & Thompson, 2008; Thompson 2008) underscores the thesis that as students construct quantities in order to make sense of their experiential world (Glasersfeld, 1995), teachers and researchers cannot assume that students maintain understandings of quantities that are compatible with teachers' and researchers' intentions.

Quantitative reasoning can involve numerical and non-numerical reasoning (Johnson, 2012), but the essence of quantitative reasoning is non-numerical (Smith III & Thompson, 2008). Building on these

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prior descriptions of quantitative reasoning, Carlson et al. (2002) described covariational reasoning as entailing a student coordinating two quantities with attention to the ways the quantities change in tandem. They specified mental actions that allow for a fine-grained analysis of students' activity. The mental actions include coordinating *direction of change* (area increases as base length increases; MA2) and *amounts of change* (the *change* in area *increases* as base length *increases in equal successive amounts*; MA3). Whereas researchers (Johnson, 2012; Moore, 2016; Paoletti & Moore, 2017) have described productive ways high school and college students can engage in reasoning compatible with Carlson et al.'s mental actions, few studies have explored the possibility of middle-school students enacting these mental actions in productive ways (e.g., Ellis, 2011a). We pay particular attention to MA3 as critical to students developing meanings for quadratic relationships, and we characterize a productive meaning for quadratic change to entail a student understanding that as one quantity changes by equal amounts, the amounts of change of the second quantity increase (or decrease). Further, and consistent with meanings described by others (Ellis, 2011a, 2011b; Lobato et al., 2012), these amounts of change increase (or decrease) themselves by a constant amount. Hereafter, we refer to these constant amounts of change of the first amounts of change as a constant AoC of AoC.

Ellis (2011b) provides evidence that middle-school students are capable of engaging in such reasoning. In her examination of middle-school students' ways of reasoning about heights, lengths, and areas of a growing rectangle, she describes how the students numerically and pictorially represented the first and second differences to identify a constant second difference, which later supported them in developing meanings for quadratic change. Building on and extending this work, we are interested in exploring whether we can support students in first identifying constant AoC of AoC non-numerically prior to reasoning numerically to develop meanings for quadratic growth.

Designing for Mathematical Abstraction with RME Principles in Mind: An Example Task

We begin by outlining important aspects of RME and designing for mathematical abstraction that informed the task development. Then we introduce the *Growing Triangle Task*.

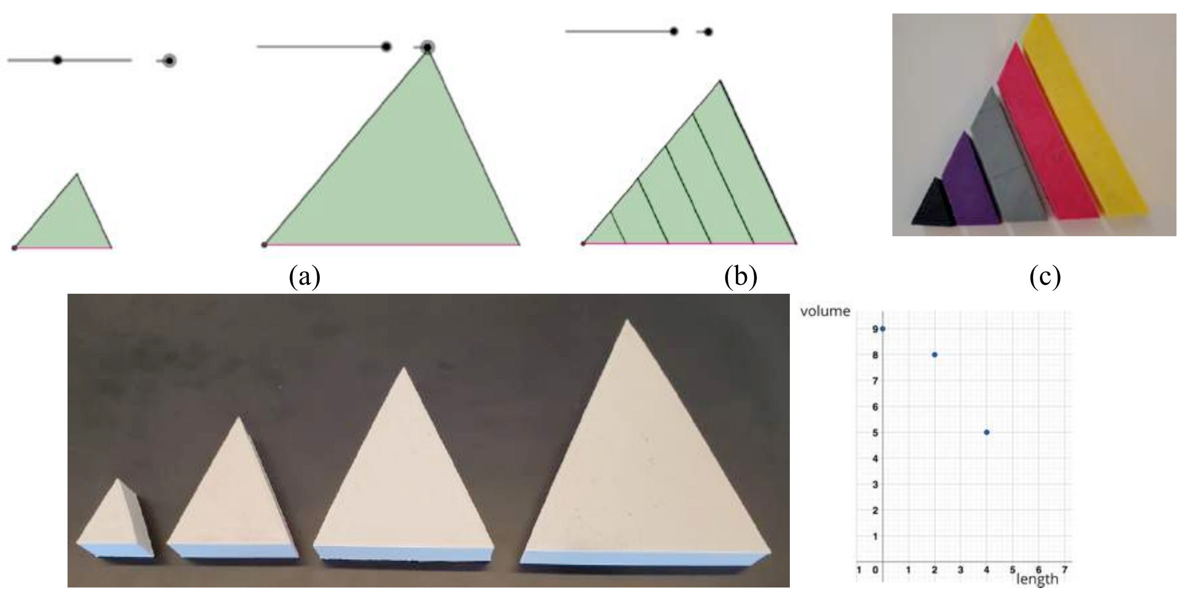
RME is an instructional theory that aims to find ways to connect what students already know to what they do not yet know (Gravemeijer, 2008). RME's emphasis is on opportunities for students to re-invent mathematics by organizing experientially real situations (Cobb et al., 2008; Gravemeijer, 2008). Tasks are experientially real to students if they can engage in personally meaningful mathematical activity; they need not refer to some 'real-world' situation or context.

By having students engage in an experientially real context, we intend to support their *horizontal mathematization*, a process that refers to a student's development of meanings for a specific context. A student's initial *model of* (Linchevski & Williams, 1999) the situation is specific to the context but should support her in developing informal strategies and representations that will be useful as she begins to generalize to other contexts. After beginning to mathematize the situation, a student can start the progressive process of *vertical mathematization,* which entails extending her informal mathematical representations or activity into more normative representations or activity. Thus, horizontal mathematization is preparation for vertical mathematization. Vertical mathematization may involve using conventional notations such as making a drawing, table, or graph (Cobb et al., 2008; Gravemeijer & Doorman, 1999). As students gather more experiences with similar problems, their attentions may shift towards mathematical relations and strategies which helps them develop further mathematical relations and a resulting shift from *models of* a context to *models for* a mathematical idea. This shift allows the students to use the model in a different manner (Gravemeijer, 2008) thereby becoming a *model for*. We describe in the Task Design section how we imagine this transition occurring in the context of quadratic growth.

Our operationalization of quadratic change emphasizes the importance of students constructing AoC of AoC as a quantity unto itself and then identifying constant AoC of AoC. We leveraged 3D-printing to design physical manipulatives we conjectured could support the students in both of these endeavors. As Greenstein (2018) noted, "The faithful mental representation of objects is critical, because conceptual thought proceeds from representational thought and representational thought proceeds from perception" (p. 3). Hence, leveraging principles of *designing for mathematical abstraction* (Pratt & Noss, 2010), we devised the physical manipulatives with the intention of making several quantities of interest, including AoC of AoC, available to students for abstraction through their sensorimotor engagement with those manipulatives (Piaget, 1970). We describe these manipulatives in the next section.

The Growing Triangle Task

We designed the *Growing Triangle Task* (https://bit.ly/2YTjwmj) with principles of RME, designing for abstraction, and theories of students' quantitative and covariational reasoning in mind. In this task sequence, students first interact with a dynamic GeoGebra applet showing an apparently smoothly growing scalene triangle (Figure 1a). Our intention is to provide students an experientially real context so that they could develop a *model of* the growing triangle situation. To support the students in attending to the triangle's area and base length (i.e., reasoning covariationally), we highlighted the base length of the triangle in pink and the area in green. The area and base length grow (apparently) smoothly as the longer slider increases (apparently) smoothly. With AoC in mind, we included a second smaller slider which allows students to increase the increment by which the pink length increases (e.g., to equal integer chunks versus apparently smoothly). We have the 'trace' option available so that students can visually identify the increasing AoC of area in the applet (i.e. the increasing size of the consecutive trapezoids shown in Figure 1b).



(a) (b) (c)

(d) (e)

Figure 1: (a/b) Several screenshots of the *Growing Triangle Task* shown in applet, (c/d) images of manipulatives and (e) new task for vertical mathematization

We conjectured that although the visual representation provided by the applet may support students in identifying the increasing AoC (MA3), it was unlikely to support them in identifying the constant AoC of AoC. Hence, with designing for abstraction (Pratt & Noss, 2010) in mind, we 3D-printed a set of manipulatives that consisted of four consecutive gray triangles (Figure 1d) to represent the growing triangle at four equal integer increases of the base length and five AoC blocks (one triangle in black, and four trapezoids, Figure 1c). Thus, these designs make these representations of increase in the amount added to each triangle to get to the next consecutive triangle available to students through their mediated engagement with them. Figure 2b demonstrates this potential for abstraction: by stacking the physical representations of change on top of each other, students can construct non-numerical interpretations of the constant AoC of AoC. After doing so, we will ask them to create a table of values for the relationship to explore if (and if so, how) they identify the constant AoC of AoC in this representation.

After developing a *model of* the Growing Triangle situation, the next step is to support students in extending this model by developing a *model for* constant AoC of AoC (i.e., by continuing the process of vertical mathematization). To do this, we presented them with several graphs, each with three points, and the instructions, "For each of the following, we know the differences in the amounts of change of volume are constant with respect to the length of a side. Complete each graph". In the ensuing activity, after students determined additional points using the constant AoC of AoC, they will be introduced to a normative definition for quadratic growth to further support their vertical mathematization: "Whenever the *amounts of change* change constantly (i.e. we have a second constant difference), the relationship is *quadratic*." Additionally, we consider that this activity may support students in developing more sophisticated meanings for other polynomial relationships. In such cases, these relationships and their related properties will be discussed (e.g., cubic growth has a constant third difference, etc.).

Methods, Participants, and Analysis

To examine the potential of supporting students in developing meanings for quadratic change, we conducted a teaching experiment (Steffe & Thompson, 2000) situated in a whole-class setting. The teaching experiment occurred in a middle school that hosts a diverse student population (over 75% of the students are of color; 75% are entitled to free or reduced-price lunch), in the northeastern United States. The class was selected from a convenience sample. The teacher of the school's only accelerated 8th grade geometry course[1] invited the research team to explore new activities with her students. We taught five class sessions, with each session scheduled for 76 minutes. We covered a variety of topics in these sessions (see Table 1), with the focus of this paper being the fifth session. The class had eight students who worked in three groups on large whiteboards during class time. We video- and audio-recorded two of the groups (one group of three and one pair), capturing their utterances, motions, and written work. For brevity's sake, we focus this report on the group of Neil, Aaron, and Nigel (pseudonyms).

Adopting a radical constructivist perspective (Glasersfeld, 1995), we contend that a student's mathematics is inaccessible to us as researchers. Hence, to analyze the qualitative data, we performed conceptual analyses (Thompson, 2008) to generate and test models of each student's mathematics so that these models provided viable explanations of his observable words and actions. With the goal of building viable models in mind, we analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998). Specifically, we watched all videos to identify instances that offered insights into each student's meanings. Using these instances, we generated tentative models of each student's mathematics, which we compared to researcher notes taken during on-going analysis. We tested these models for viability by searching for supporting or contradicting instances in his other activities. When evidence challenged our

[1] Most students in the school take algebra in 8th grade. The terminology is used to be consistent with the school's name for the course.

models, we revised hypotheses to explain each student's meanings and returned to prior data with these new hypotheses in mind to modify previous hypotheses. This process resulted in viable models of each student's mathematics.

Table 1: Whole class teaching experiment sequence

Session	Task	Content topics for students to develop
1 & 2	Faucet Task (see Paoletti, 2018)	Coordinate Systems, graphs, and graphing
3	Triangle Task (Part I)	Non-linear change
4	Triangle/Rectangle Task (Paoletti, Vishnubhotla, & Mohamed, 2019)	Comparing non-linear and linear change, Systems of Relationships
5	Triangle Task (Part II)	Extending non-linear change to quadratic change
6	Post-test	

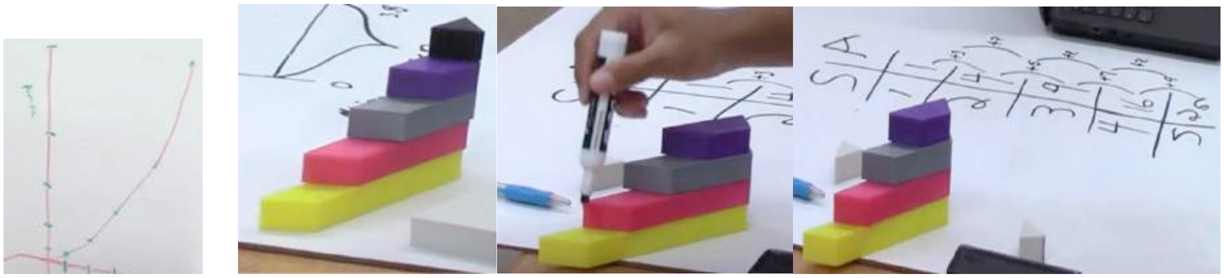
Results

In this section, we first highlight the students reasoning non-numerically about the quantities in the *Growing Triangle Task*. We then present the students' activity with manipulatives, which facilitated their identifying constant AoC of AoC (i.e., an instance of horizontal mathematization). We then characterize how the students develop a *model of* constant AoC of AoC when they are presented with a new task (shown in Figure 1: (a/b) Several screenshots of the *Growing Triangle Task* shown in applet, (c/d) images of manipulatives and (e) new task for vertical mathematization e), and how that *model of* transitions into a *model for* quadratic growth. We conclude with examples of student work on the post-test to characterize the extent to which students engaged in vertical mathematization.

The Growing Triangle Task: Horizontal Mathematization

Introducing the *Growing Triangle Task* on Day 3, the teacher-researcher (TR) distributed the manipulatives to the class intending to support the students in differentiating between the total area of the triangle and the changes in area for each increase in base length. To support the students developing meanings for area and change in area, the TR began with the first triangle (shown in Figure 1c) and added the first change. He asked the class, "When I add another unit of side length, is the amount of area I'm gonna add, is it the same amount, more, or less?" Neil responded, "More." The students were then asked to open up the applet on a laptop and begin making observations about the changes in the triangle. As they watched the applet play, the three students discussed how the changes were growing. When the TR asked, "So initially do we have a big jump, a small jump, medium jump?" Both Aaron and Neil simultaneously responded, "Small." As the TR asked what happened for the next two jumps, Neil responded, "It gets larger… even larger." In this activity, Neil's quantitative understanding of area was elicited as he described it growing by increasing amounts (MA3). After this interaction, Neil plotted points accurately representing this relationship between area and base length (Figure a). We infer Neil was reasoning covariationally as he leveraged non-numeric images of total area and AoC of area in the situation to graphically represent each quantity and the resulting relationship.

Whereas the focus of Day 3 was on supporting students in identifying increasing AoC, on Day 5 we returned to the triangle task with the intention of supporting students in imagining constant AoC of AoC in this context. Because the students had already identified increasing AoC of area with respect to side length, the TR asked the students to consider, "How do the amounts of change compare to one another?" The TR requested the students "play around with the manipulatives and explore." Neil and Nigel began to stack the trapezoidal manipulatives on top of one another in consecutive amounts (shown in Figure b). The following conversation ensued:



(a) (b) (c) (d)

Figure 2: (a) One group's graph; (b/c) students notice a constant second difference and use manipulatives to identify it; and (d) table displaying students' work

Neil: [*Places yellow trapezoid on bottom then places pink trapezoid on top*] Ooh! Look, look, it's a quadrilateral [*pointing to the piece at the end of yellow trapezoid not covered by the pink trapezoid, seen in Figure b*].

Aaron: Mmhmm. [*Agreeing*]

Neil: Right, look, if we do it again. [*Nigel places gray trapezoid on top of the pink trapezoid*] Same size [*pointing to quadrilaterals created by the difference in the yellow and pink trapezoids and pink and gray trapezoids to indicate these amounts are the same*]. Put that one on [*Nigel places the purple trapezoid on top of the stack*] Same size [*referring the quadrilateral formed by the difference in purple and grey trapezoids*].

TR: So what did you notice?

Neil: It's like, this quadrilateral [*pointing to quadrilateral shown by the differences Figure b*] keeps going I guess, it's added on to that [*pointing to each layer of the stack of trapezoids*].

TR: That, that piece we're adding on to the amounts of change is always the same? [*Nigel nods in agreement as TR speaks*]

Neil: [*as TR is completing his remark*] Yeah.

In this conversation, Neil characterized the amount being added to each trapezoidal AoC manipulative as equal (i.e. constant AoC of AoC). We conjectured from activity (e.g., adding consecutive trapezoids to the stack), agreement throughout the conversation, and their later activity, that Nigel and Aaron also understood that the relationship between area and side length exhibited constant AoC of AoC. We provide evidence for this shortly.

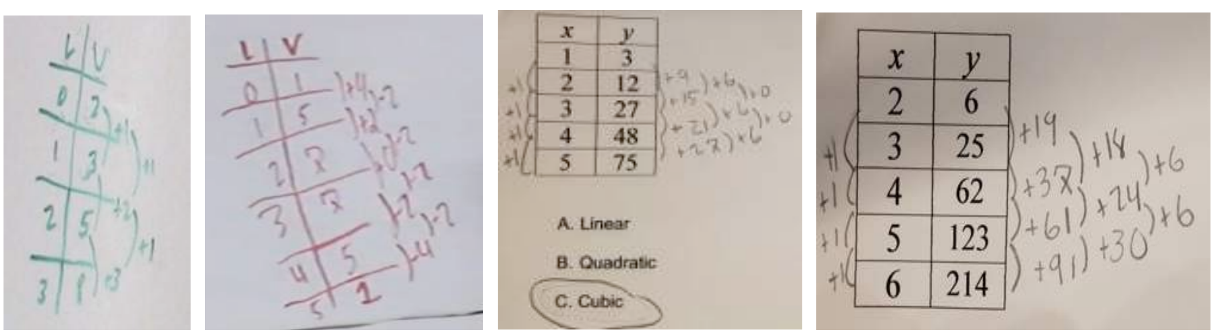
Beginning vertical mathematization for AoC of AoC

After identifying non-numeric constant AoC of AoC using informal representations, the TR prompted the students to create a table of values as a way to normatively represent such a relationship (i.e., an instance of vertical mathematization). His goal was for students to consider how the constant AoC of AoC would impact the numeric values in the table. The TR designated the smallest gray triangle as having an area value of one, and with this value of one, the students identified the purple trapezoid as having an area of three units and the total area of the resulting triangle as composed of four units. Similarly, when adding the pink trapezoid to the total area, Neil identified the area of the pink trapezoid as "seven," then immediately described the total area as "seven plus nine," with nine being the total area of the previous triangle. We infer Neil understood that in order to find a new total area he must first identify the AoC of area from the previous base length to the new base length and add that AoC-value to the previous total area. Using this reasoning, the three students completed their table (shown in Figure 2d).

To build on the students previously identifying non-numeric constant AoC of AoC, the TR asked the students how the values they had identified for the trapezoidal AoC manipulatives (i.e. first difference), and the differences in these (i.e. second difference), were represented in their table. As part of this process, the students identified the differences with Aaron notating with arrows and values (e.g., '+3', '+5') the AoC from one consecutive area to the next (seen in Figure d). As Aaron identified several AoC-values and notated them besides the corresponding total area values in the table, he spontaneously described how the constant AoC of AoC was represented to his peers. Pointing to the '+3' and '+5" written adjacent to the table, he said, "It's two" and then pointed to the quadrilateral made by differences of the AoC pieces (Figure c). We infer Aaron was connecting their informal identification of the constant AoC of AoC with the manipulatives to the changes in the AoC represented in their table.

Moving to a *model for* constant AoC of AoC

After each group identified the constant AoC of AoC in the table, we provided them with new tasks (example shown in Figure 1d) representing volume and length values for some new hypothetical situation. Indicative of the students engaging in extending their *model of* the *Growing Triangle Task* to a more general *model for* constant AoC of AoC, on the first task Aaron drew a table with the coordinate values, found the first AoC in the second quantity ('+1", "+2"), then identified an AoC of AoC value of '+1". Using the constant second difference, Aaron found the next first difference ('+3', shown in Figure 3a) and used this value to determine the next volume-value of eight. In the second task, Nigel engaged in compatible activity with a second difference value of '-2' (Figure 3b). We infer the students leveraged a meaning for constant AoC of AoC to calculate values of new points in a given relationship (i.e. a shift to a *model for* constant AoC of AoC). After engaging in this activity, we concluded the class episode by introducing the definition of quadratic (as well as cubic) growth as described above to extend students' understandings of a relationship that contains a constant second difference as quadratic.



(a) (b) (c) (d)

Figure 3: (a/b) Samples of students' vertical mathematization; post-test questions with students' work on (c) quadratic and (d) cubic relationships

Evidence of a *model for* quadratic growth

We present data from a post-test given on Day 6 as evidence of the three students extending their *models for* constant AoC of AoC to *models for* quadratic growth. The post-test included two tables (shown in Figure 3.c/d), with options linear, quadratic, cubic, or exponential. Given the first table (as shown with Nigel's work in Figure 3c), Neil and Aaron found a constant second difference and identified the relationship as quadratic. Nigel's work indicates he also successfully found a constant second difference, but then moved on to find a constant third difference of '+0' and concluded that the relationship was cubic. One possible explanation is that Nigel may not have noticed the constant second difference prior to finding the constant third difference; if this were the case, and Nigel had noticed the constant second differences, we conjecture he likely would have chosen quadratic. This conjecture is based in part on his response addressing the second table in which he and the other two students identified a constant third difference (shown in Figure 3d) to conclude the growth was cubic. Hence, we infer each student has at least begun to develop *models for* quadratic (and cubic) growth.

Discussion and Implications

To conclude, we highlight how we were able to draw from theories on students' quantitative and covariational reasoning (Carlson et al., 2002; Smith III & Thompson, 2008), designing for mathematical abstraction (Pratt & Noss, 2010), and RME instructional theory (Gravemeijer, 2008) to support students building *models for* quadratic growth (and potentially *models for* other polynomial growth as well). These findings extend previous examinations of middle-school students developing meanings for quadratic change (e.g., Ellis, 2011a) by describing how they leveraged their non-numeric quantitative and covariational reasoning to identify constant AoC of AoC before representing such relationships numerically.

We intend for our description of the task design and sequence to highlight ways in which these theories can work together to inform the design of a task sequence that is responsive to students' reasoning activity. For example, designing for mathematical abstraction supported us in developing manipulatives that we conjectured could support students constructing quantities, including total area, AoC of area, and AoC of AoC of area in ways compatible with our intentions. Further, these manipulatives supported students' non-numeric and numeric quantitative and covariational reasoning as well as their engagement in the mental actions described by Carlson et al. (2002). Connecting RME to the other theories, we note how the 3D-printed manipulatives were critical as the students transitioned to vertical mathematization as they identified constant AoC of AoC. Similarly, we highlight how the students leveraged their quantitative and covariational reasoning as they moved from *models of* the *Growing Triangle Task* to *models for* constant AoC of AoC, and further to *models for* quadratic growth.

We highlight the productive meanings for quadratic growth the three students described here developed as part of this study, and offer this research as both an existence proof and starting point for future researchers interested in exploring how to support larger populations of middle-school students in leveraging non-numeric and numeric quantitative and covariational reasoning to develop important mathematical ideas including quadratic and cubic growth. Integrating activities like these with activities described by Ellis (2011a), for example, may support students in developing more productive meanings for AoC and quadratic growth. Future researchers may be interested in exploring this possibility.

Acknowledgments

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