Mathematics Education Across Cultures

Proceedings

42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education

Editors:

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Revision

This PDF has been revised as follows:

- Minor corrections to the Table of Contents:
    - author names of two papers have been fixed;
    - corrections in the titles of two other papers.
- The paper “Understanding teachers’ professional development …” by Luc Trouche on p. 143 has been modified to its correct version.
- Corrected author information in paper “Disentangling the role of context and community …” on p. 2001
- Added missing transcript numbers on p. 2267

MATHEMATICS EDUCATION ACROSS CULTURES

MATEMÁTICA EDUCATIVA ENTRE CULTURAS

DIDACTIQUE DES MATHÉMATIQUES À TRAVERS LES CULTURES

Proceedings of the Forty-Second Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Mexico*

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PME came into existence at the Third International Congress on Mathematical Education (ICME-3) in Karlsrühe, Germany, in 1976. It is affiliated with the International Commission for Mathematical Instruction. PME-NA is the North American Chapter of PME. The first PME-NA conference was held in Evanston, Illinois in 1979.

Past PME and PME-NA conferences held in Mexico were:

- 1990 Oaxtepec: Joint Meeting PME 14 – PME-NA 12
- 1999 Cuernavaca: PME-NA 21
- 2006 Mérida: PME-NA 28
- 2008 Morelia: Joint Meeting PME 32 – PME-NA 30

Since their origins, PME and PME-NA have expanded and continue to expand beyond their psychologically-oriented foundations.

The major goals of the International Group and the North American Chapter are:

1. To promote international contacts and the exchange of scientific information in the psychology of mathematics education;
2. To promote and stimulate interdisciplinary research in the aforesaid area, with the cooperation of psychologists, mathematicians, and mathematics teachers; and
3. To further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

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PME-NA 42 Review Process & Reviewers

All submissions to the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, included in these proceedings, were strictly peer-reviewed in a double-blind process by three (sometimes four) colleagues.

PME-NA 42 Strand Leaders

The Local Organizing Committee is extremely appreciative of the following people for serving as Strand Leaders. They managed the reviewing process for their strand and made recommendations to the Local Organizing Committee. The conference would not have been possible without their efforts.

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PME-NA 42 Strands

Strand	Description	Related Keywords
Algebra, Algebraic Thinking and Number Concepts	This strand includes papers that focus on the teaching and learning of early algebra, algebra, and number concepts.	- Algebra and Algebraic Thinking - Number Concepts and Operations - Rational Numbers
Curriculum, Assessment and Related Topics	This strand includes papers that focus on curriculum analysis, development, implementation, assessment and evaluation, or on technology as curricular or assessment tools.	- Curriculum - Curriculum Analysis - Curriculum Enactment - Standards (Broadly Defined) - Assessment and Evaluation - Technology - STEM / STEAM - Equity and Diversity - Social Justice - Special education - Teaching tools and resources
Equity and Justice	This strand includes papers that focus on marginalization, systems of oppression, or other similar issues related to mathematics education at any level or in any context.	- Equity and Diversity -Social Justice - Policy - Students with Special Needs - Gender and Sexuality - Inclusive education - Rural education - First nations/Indigenous cultures - Cross cultural studies - Marginalized communities
Geometry and Measurement	This strand includes papers that focus on the teaching and learning of geometry measurement and spatial reasoning	- Geometry and Geometrical and Spatial Thinking - Measurement
Instructional Leadership, Policy, and Institutions/Systems	This strand includes papers that focus on teacher leaders, coaches, or teacher educators as the subjects of research.	- Instructional Leadership - Teacher Educators - Systemic Change
Mathematical Knowledge for Teaching	This strand includes papers that focus on teachers' subject matter knowledge in relation to teaching.	- Teacher Knowledge - Number Concepts and Operations - Geometry and Geometrical and Spatial Thinking - Rational Numbers - Algebra and Algebraic Thinking - Technology

Mathematical Processes and Modeling	This strand includes papers that focus on topics such as, but not limited to, problem solving, reasoning and proof, modeling, etc.	- Problem Solving - Reasoning and Proof - Advanced Mathematical Thinking - Modeling - Representations and Visualization - Communication
Miscellaneous topics	This strand includes papers on diverse topics, such as Neuroscience, Ethno-mathematics, Interdisciplinary studies; and Mathematics for sustainability (interdisciplinary approaches and complex socio-scientific issues within mathematics education, related to the environment and climate change, economic growth and poverty, etc.)	- Neuroscience - Ethno-mathematics - Mathematics fos sustainability - Interdisciplinary studies - Modeling - Equity and Diversity - Social Justice - Socio-scientific issues
Precalculus, Calculus, or Higher Mathematics	This strand includes papers that focus on the teaching and learning of Precalculus, Calculus and/or Higher Mathematics	- Precalculus - Calculus - University Mathematics - Advanced Mathematical Thinking
Statistics and Probability	This strand includes papers that focus on the teaching and learning of probability, data analysis and statistics	- Probability - Data Analysis and Statistics
Student Learning and Related Factors	This strand includes papers that focus on students' experiences and the influence of various factors (e.g., beliefs, diversity) on mathematical learning.	- Cognition - Metacognition - Affect, Emotion, Beliefs, and Attitudes - Embodiment and gesture - Gender and Sexuality - Technology - Informal Education - Equity and Diversity - Social Justice - Students with special needs - Representations and Visualization
Teacher Education - Pre-service	This strand includes papers that focus on the development of prospective teachers and their knowledge.	- Assessment and Evaluation - Instructional activities and practices - Equity and Diversity - Affect, Emotion, Beliefs, and Attitudes - Teacher Knowledge - Technology

Teacher Education - In-service / Professional Development	This strand is for proposals that focus on in-service teacher learning and professional development.	- Instructional Vision - Affect, Emotion, Beliefs, and Attitudes - Equity and Diversity - Assessment and Evaluation - Teacher Knowledge - Technology - Distance education -MOOC
Teaching and Classroom Practice	This strand includes papers that focus on analyzing the nature of classroom instruction and activity (e.g., discourse, culturally relevant pedagogy)	- Classroom Discourse - Culturally Relevant Teaching - Instructional Activities and Practices - Inclusive education - Technology - Representations and Visualization
Technology	This strand includes papers that focus on the use and development of technology for, and in, teaching and learning.	- Technology - Computational Thinking - Programming and coding - Communication - Modeling - STEM / STEAM - Teacher Knowledge - Teaching and assessment tools and resources - Learning Tools - Distance education - MOOC - Representations and Visualization
Theory and Research Methods	This strand includes papers that focus on the development of theory and/or research methods.	- Research Methods - Design Experiments - Learning Trajectories (or Progressions) - Doctoral Education

PREFACE

When we set out to organize the 42nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA 42) to take place in Mazatlán, Mexico, on 14-18 October, 2020, we never imagined, as no one did, that 2020 would freeze the world due to a pandemic. We therefore had to adapt to this world situation, postponing the conference to the summer of next year, and adapting it as a hybrid conference, both as in-person and virtual – with the virtual activities beginning around the 27 May, 2021 and the in-person conference planned for 2-6 June, 2021.

A history of PME/PME-NA conferences in Mexico

Mexico has hosted several past PME-NA conferences:

- 1990 Oaxtepec: Joint Meeting PME 14 – PME-NA 12
- 1999 Cuernavaca: PME-NA 21
- 2006 Mérida: PME-NA 28
- 2008 Morelia: Joint Meeting PME 32 – PME-NA 30

The PME-NA 42 conference was hoping to bring back, after twelve years, the conference to Mexico and have the honor to host our colleagues from Canada, the USA and other countries. Unfortunately, we will have to wait longer. But we hope to be able to welcome participants to Mazatlán in 2021, thirteen years after the last conference took place in Mexico. As of December, 2020, we have 527 registered participants for, at least, the virtual part of the conference.

Conference Theme

In accordance with the major goals of PME-NA (see PME-NA History and Goals section above) that include promoting international contacts, and stimulating interdisciplinary research, for the PME-NA 42 conference, we proposed as theme "*Entre Culturas* / Across Cultures". We consider that a way to promote the exchange and enrichment of mathematics education research is to look at its manifestations across different cultures, places and contexts. This is the focus of the PME-NA 42 conference, where we aimed to have an encounter of cultures, as well as forms of research in mathematics education. That is:

Across various cultures...

- of *languages, nationalities, communities, ethnicities,*
- of *genders*,
- of *indigenous people* and *migrants*;
- of different *abilities*;
- of different *generations*;
- of *teachers, researchers, policy-makers* and *parents;*
- of *school types*: urban vs. rural, multigrade, etc; and
- of *classroom cultures and modalities:* from traditional teaching to "new" classroom cultures: e.g., flipped classrooms, distance education, etc.;
- of *technologies and tools:* from pencil-and-paper, to the digital and connected;
- of *math and other disciplines* (multi-, inter- and trans-disciplinarity): e.g., STEAM, math and environmental sciences, etc.;

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- of *research*: different strands of PME-NA research, including cognition, neuroscience, ethnography, and new types of research.

In other words, we wanted an academic exchange that would reflect the ample diversity of ways of teaching and learning of mathematics, and of the tools and communities involved in mathematics education; as well as explore how the differences in cultures imply a need to consider how research results can be taken into account in varying contexts. Through all of the above, we hoped to honor the major goals of PME-NA.

Conference program

A multi-cultural program

We received proposals from participants not only from North America, but from 20 countries (Australia, Brazil, Canada, Chile, China, Colombia, Ecuador, France, Germany, Italy, Mexico, Norway, Peru, Puerto, Rico, Spain, Sweden, Turkey, UK and USA). In fact, 82 proposals were submitted in Spanish and two in French. But a requirement of PME-NA is for all accepted proposals to have an English version for the proceedings; thus, in order to ensure the high quality of published papers, we hired an English language reviewer to make sure that the accepted proposals that had been submitted in Spanish would be up to standard in their English versions. It is thus that there are many bilingual versions included in these proceedings; in those cases, the title of the paper or poster is given in two languages, and the alternate language version (Spanish or French) is included after the English one.

Research reports and poster presentations

The papers included in these conference proceedings are categorized according to 14 strands (please see the PME-NA 42 Strands section above). Seeking to expand the interdisciplinary aspect of PME-NA, for the PME-NA 42 conference we added to the lists of keywords, topics such rural education; first nations/indigenous cultures; cross cultural studies; marginalized communities, and mathematics for sustainability.

We received 490 proposals for research reports and poster presentations (225 for Research Reports, 165 for Brief Research Reports and 100 for Poster Presentations). All submissions were strictly peer-reviewed through a double-blind process (see Review Process section above). In the end, we accepted 80 as Research Reports, 190 as Brief Reports and 147 as Poster Presentations. Most of the authors with accepted proposals, though not all, registered for the conference and sent final versions that are included in these proceedings.

Working groups and research colloquia

In addition to the usual Working Groups, this year the Steering Committee added Research Colloquia, to focus on a research topic of substantial interest within the PME-NA community that has been developed during multiple previous PME-NA (or PME) working groups. Research Colloquia involve longer planned presentations to share what has been done in previous meetings within the group.

We received 11 Working Group and 2 Research Colloquia proposals, and a Steering Committee subcommittee accepted 10 Working Groups and 2 Research Colloquia.

We were happy to see that many Working Groups and Research Colloquia themes were in accord to the “Across Cultures” conference theme.

Plenary and special lectures

In accordance to our conference theme "*Entre Culturas* / Across Cultures", we invited plenary and special speakers that would represent different cultural contexts, as well as forms of research in Mathematics Education. They come from different countries, not only in North America, but also from Europe. They represent different areas of research and disciplines within Mathematics Education; and we include researchers from different generations. We hope that these lectures will reflect parts of the ample diversity of ways of teaching and learning of mathematics, and of the tools and communities involved in mathematics education. In that way, we hoped also to honor the major goals of PME-NA.

Special tribute to Eugenio Filloy

When planning the program for PME-NA 42, we had envisioned, since 2019, including a special live-person homage to Eugenio Filloy, co-founder in 1975 of the Mathematics Education Department of the Center for Advanced Studies and Research (Cinvestav) and a driving force of the Mathematics Education discipline at an international level from the early 1970s (see the Special Tribute section below). Sadly, Eugenio Filloy passed away in March, 2020; but the more reason to include this special tribute in his memory.

A tribute to the PME-NA members who were affected by the COVID-19 pandemic

As we were wrapping the edition of these proceedings, we heard on the 15th December, of the passing of our dear colleague, graduate of Cinvestav and member of AMIUTEM, César Martínez-Hernández of the Universidad de Colima, from COVID-19. César has two contributions included in these proceedings. May he rest in peace; he will be sorely missed.

We don't know how many other members of the PME-NA community have lost their lives, family members or friends, or been deeply affected by this pandemic in other ways, but we hope it is not too many; to those who have, we send our sympathies to them or to their families, friends and colleagues.

Looking forward

With the prospect of the SARS-CoV-2 vaccines, we are a bit more optimistic that maybe we will be able to host the in-person part of the conference, and welcome you in June in Mazatlán, Mexico. In any case, we look forward to welcoming you, either in person or virtually, for PME-NA 42.

The co-chairs of PME-NA 42,

Ana Isabel Sacristán
Cinvestav

José Carlos Cortés-Zavala
AMIUTEM /
Universidad Michoacana de San Nicolás Hidalgo

December, 2020

Prólogo

Cuando nos propusimos organizar la 42a Reunión Anual del Capítulo Norteamericano del Grupo Internacional de Psicología de la Educación Matemática (PME-NA 42) para realizarse en Mazatlán, México, del 14 al 18 de octubre de 2020, nunca nos imaginamos, como nadie lo hizo, que 2020 congelaría al mundo debido a una pandemia. Por lo tanto, tuvimos que adaptarnos a esta situación mundial, posponiendo la conferencia para el verano del próximo año y adaptándola como una conferencia híbrida, tanto presencial como virtual – con las actividades virtuales comenzando alrededor del 27 de mayo, 2021, y la conferencia presencial planeada del 2 al 6 de junio, 2021.

Historia de las conferencias PME / PME-NA en México

México ha sido sede de varias conferencias PME-NA anteriores:

- 1990 Oaxtepec: Reunión conjunta PME 14 - PME-NA 12
- 1999 Cuernavaca: PME-NA 21
- 2006 Mérida: PME-NA 28
- 2008 Morelia: Reunión Conjunta PME 32 - PME-NA 30

La conferencia PME-NA 42 esperaba traer de regreso, después de doce años, la conferencia a México y tener el honor de recibir a nuestros colegas de Canadá, Estados Unidos y otros países. Desafortunadamente, tendremos que esperar más. Pero esperamos poder dar la bienvenida a los participantes a Mazatlán en 2021, trece años después de que se llevó a cabo la última conferencia en México. A la fecha, tenemos 527 participantes inscritos para, cuando menos, la parte virtual de la conferencia.

Tema de la conferencia

De acuerdo a los principales objetivos del PME-NA (ver la sección *PME-NA History and Goals,* arriba) que incluyen promover contactos internacionales y estimular la investigación interdisciplinaria, para la conferencia PME NA 42, propusimos como tema "Entre Culturas / *Across Cultures*". Consideramos que una forma de promover el intercambio y el enriquecimiento de la investigación en educación matemática es observar sus manifestaciones en diferentes culturas, lugares y contextos. Este es el tema central de la conferencia PME-NA 42, donde pretendíamos tener un encuentro de culturas, así como formas de investigación en educación matemática. Es decir:

A través de varias culturas ...

- de *idiomas, nacionalidades, comunidades, etnias*,
- de *géneros*,
- de *pueblos indígenas* y *migrantes*;
- de diferentes *habilidades*;
- de diferentes *generaciones*;
- de *profesores*, *investigadores*, *autoridades*, *políticos* y *padres* de familia;
- de *tipos de escuelas*: urbanas vs. rurales, multigrado, etc; y
- de las *culturas y modalidades del aula*: desde la enseñanza tradicional hasta las "nuevas" culturas en las aulas: por ejemplo, aulas invertidas, educación a distancia, etc.
- de *tecnologías* y otras herramientas: desde el lápiz y el papel, hasta lo digital y conectado;

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- de *matemáticas y otras disciplinas* (multidisciplinaridad, interdisciplinariedad y transdisciplinariedad): por ejemplo, STEAM (CTIAM), matemáticas y ciencias ambientales, etc.;
- de *investigación*: diferentes líneas de investigación de PME-NA, que incluyen cognición, neurociencia, etnografía y nuevos tipos de investigación.

En otras palabras, queríamos un intercambio académico que reflejara la amplia diversidad de formas de enseñanza y aprendizaje de las matemáticas, y de las herramientas y comunidades involucradas en la educación matemática; así como explorar cómo las diferencias en las culturas lo que requiere considerar cómo los resultados de investigación pueden ser tomados en cuenta en diferentes contextos. A través de todo lo anterior, esperábamos honrar los principales objetivos de PME-NA.

Programa de la conferencia

Un programa multicultural

Recibimos propuestas, no solo de Norteamérica, sino de 20 países (Australia, Brasil, Canadá, Chile, China, Colombia, Ecuador, Francia, Alemania, Italia, México, Noruega, Perú, Puerto, Rico, España, Suecia, Turquía, Reino Unido y Estados Unidos). De hecho, 82 propuestas fueron enviadas en español y dos en francés. Pero un requisito del PME-NA es que todas las propuestas aceptadas tengan una versión en inglés para las memorias del evento; por tanto, con el fin de garantizar la alta calidad de los artículos publicados, contratamos a un revisor del inglés para asegurarnos de que las propuestas que habían sido evaluadas en español y que fueron aceptadas, cumplieran con el estándar en sus versiones en inglés. Así, estas memorias incluyen muchas versiones bilingües; en esos casos, el título del trabajo o cartel se presenta en dos idiomas, y la versión en idioma alterno (español o francés) se incluye después de la versión en inglés.

Informes de investigación y presentaciones de carteles

Los artículos incluidos en estas memorias de la conferencia, se clasifican de acuerdo a 14 áreas (consulte la sección *PME-NA 42 Strands*, más arriba). Buscando expandir el aspecto interdisciplinario del PME-NA, para la conferencia PME-NA 42 agregamos a las listas de palabras clave, temas como educación rural; primeras naciones / culturas indígenas; estudios transculturales; comunidades marginadas y matemáticas para la sostenibilidad.

Recibimos 490 propuestas para reportes de investigación y presentaciones de póster (225 para reportes de investigación, 165 para reportes breves de investigación y 100 para presentaciones de póster). Todas las propuestas fueron sometidas a un proceso estricto de arbitraje por pares doblemente ciego (consulte la sección *Review Process* más arriba). Al final, se aceptaron 80 reportes de investigación, 190 como reportes breves de investigación y 147 como presentaciones de póster. Casi todos los autores con propuestas aceptadas, aunque no todos, se inscribieron para la conferencia y enviaron versiones finales de sus escritos que se incluyen en esta memorias.

Grupos de trabajo y coloquios de investigación

Además de los Grupos de Trabajo habituales, este año el Comité Ejecutivo del PME-NA agregó la modalidad de Coloquios de Investigación (*Research Colloquia*), los cuales se centran en un tema de investigación de interés particular para la comunidad del PME-NA, y que ha sido desarrollado en grupos de trabajo durante múltiples reuniones del PME-NA (o PME). Dichos coloquios de investigación involucran presentaciones planeadas más largas que lo que se ha hecho en reuniones previas de los grupos.

Recibimos 11 propuestas para Grupos de Trabajo y 2 para Coloquios de Investigación; y un subcomité del Comité Directivo aceptó 10 Grupos de Trabajo y 2 Coloquios de Investigación.

Nos alegró ver que muchos temas de los Grupos de Trabajo y Coloquios de Investigación coinciden con el tema de la conferencia: “Entre culturas”.

Conferencias plenarias y especiales

De acuerdo con el tema de nuestra conferencia "Entre Culturas / *Across Cultures*", invitamos a conferencistas plenarios y especiales que representen diferentes contextos culturales, así como formas de investigación en Educación Matemática. Vienen de diferentes países, no solo de Norteamérica, sino también de Europa. Representan diferentes áreas de investigación y disciplinas dentro de la Educación Matemática; e incluimos investigadores de diferentes generaciones. Esperamos que estas conferencias reflejen partes de la amplia diversidad de formas de enseñanza y aprendizaje de las matemáticas, y de las herramientas y comunidades involucradas en la educación matemática. De esa manera, también esperábamos honrar los principales objetivos de PME-NA.

Homenaje especial a Eugenio Filloy

En la planeación del programa del PME-NA 42, teníamos previsto, desde 2019, un homenaje especial, y en vida, a Eugenio Filloy, cofundador en 1975 del Departamento de Matemática Educativa del Centro de Estudios Avanzados e Investigaciones (Cinvestav) y una fuerza impulsora de la disciplina de Educación Matemática a nivel internacional desde principios de la década de 1970 (ver la sección Homenaje Especial, más abajo). Lamentablemente, Eugenio Filloy falleció en marzo de 2020; pero esto es más razón para hacerle este homenaje especial, honrando sus contribuciones a la disciplina y su memoria.

Un homenaje a los miembros de PME-NA afectados por la pandemia COVID-19

Al cerrar la edición de estas memorias, el 15 de diciembre nos enteramos del fallecimiento, por COVID-19, de nuestro querido colega, egresado del Cinvestav y miembro de AMIUTEM, César Martínez Hernández de la Universidad de Colima. César tiene dos aportaciones incluidas en estas memorias. Descanse en paz; se le extrañará.

No sabemos cuántos otros miembros de la comunidad PME-NA han perdido su vida, de sus familiares o amigos, o han sido profundamente afectados por esta pandemia de otras formas, pero esperamos que no sean demasiados; a quienes lo han sido, les enviamos nuestro más sentido pésame, o a sus familiares, amigos y colegas.

De cara al futuro

Con la perspectiva de las vacunas contra el SARS-CoV-2**,** somos un poco más optimistas de que tal vez podamos ser anfitriones de la parte presencial de la conferencia y darles la bienvenida en junio en Mazatlán, México. En cualquier caso, esperamos darles la bienvenida, ya sea en persona o virtualmente, al PME-NA 42.

Los co-organizadores del PME-NA 42,

Ana Isabel Sacristán Rock
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José Carlos Cortés Zavala
AMIUTEM /
Universidad Michoacana de San Nicolás Hidalgo

Diciembre, 2020

SPECIAL TRIBUTE / *HOMENAJE ESPECIAL*

The PME-NA 42nd Conference and these proceedings are dedicated to the memory of Eugenio Filloy-Yagüe (1942-2020)

La 42a Reunión del PME-NA y estas actas se dedican a la memoria de Eugenio Filloy Yagüe (1942-2020)

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TRIBUTE TO EUGENIO FILLOY†: A PIONEER AND DRIVING FORCE OF MATHEMATICS EDUCATION AS A DISCIPLINE

HOMENAJE A EUGENIO FILLOY†: UN PIONERO IMPULSOR DE LA DISCIPLINA DE LA MATEMÁTICA EDUCATIVA

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Eugenio Filloy (1942-2020)

As we said in the Preface to these proceedings, when planning the PME-NA 42 conference in 2019, we had envisioned including a special live-person tribute to Eugenio Filloy, as the "father" of Mathematics Education in Mexico, with an extended influence in many countries in Ibero and Latin-America. Eugenio was one of the strongest supporters of PME-NA 42, and many contributions by his students and himself are included in these proceedings. Sadly, Eugenio passed away on the 23rd March, 2020. We do this tribute then, *in memoriam.*

Here is a brief sketch of his academic trajectory and his important contributions to the field of Mathematics Education:

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- Eugenio studied Mathematics and Theoretical Physics at the National Autonomous University of Mexico, graduating in Mathematics in 1965.
- He did a Master's degree in Mathematics at Cinvestav, graduating in 1966, and was also visiting scholar at UCLA in that same year.
- He received his PhD in Mathematics from the University of Chicago in 1970
- In 1971, he joined the Mathematics Department at Cinvestav, and did some postdoctoral work at the University of Geneva, Switzerland in 1972.
- From 1973 to 1977, he was president of the Mexican Mathematical Society.
- In 1975, he co-founded the "Section of Educational Mathematics" (*Sección de Matemática Educativa*-SME), part of the Department of Educational Research at Cinvestav, with two other mathematicians from the Mathematics Department –Carlos Imaz and Juan José Rivaud— with whom he had been collaborating in designing and writing the Mathematics textbooks of the Mexican National Program of Free Textbooks for Primary Schools launched in the late 1960s by Mexico's Ministry of Education (SEP). Ramiro Ávila-Godoy (2013) explains that:

> This experience led Dr. Filloy to become aware of the importance and complexity of the problematic of the learning and teaching of Mathematics and of the need to deal with it at all educational levels […] and seek solutions. […]
>
> … in the name "Educational Mathematics" was implicit the intention to deal with the problematic of the teaching and learning of mathematics, from mathematics itself. At the time, this approach was innovative and led many countries in Latin America to talk of *Matemática Educativa*

- From the onset, i.e., from 1975, the newly founded section - SME (today the Department of Mathematics Education –DME, at Cinvestav), offered a Master of Science in "Educational Mathematics" (*Matemática Educativa*), "with a strong content in mathematics as well as on the history and foundations of mathematics" (Trigueros, Sacristán & Guerrero, 2008, p.220); and from 1982, a PhD program. Eugenio himself was supervisor, active until the day of his passing, of close to 100 students of those programs (directing more than 60 Master's thesis and over 30 PhD dissertations); many of his former students have contributions included in these proceedings.
- In 1983, Eugenio founded the ambitious and far-reaching National Program for Training and Professional Development of Mathematics Teachers *(Programa Nacional de Formación y Actualizacion de Profesores de Mathematicas* - PNFAPM) which, in collaboration with a network of many Mathematics departments in over 30 universities and higher institutions across Mexico, offered a Master of Science in *Matemática Educativa*; as well as an undergraduate program in the teaching of mathematics.

> The PNFAPM also established academic links with international institutions, and many members of the SME participated in doctoral and post-doctoral studies abroad. Simultaneously to the PNFAPM, national and international conferences and meetings were launched so that researchers and teachers could share and discuss their experiences." (Trigueros, Sacristán & Guerrero, 2008, p. 221)

In fact, Eugenio continuously promoted collaborations with researchers worldwide. For example:

- He actively participated in international events, such as CIAEM (chairing, in 1980, the XXXII CIAEM, in Oaxtepec, Mexico), PME, and ICMI, and was instrumental in having Mexico be part of PME-NA; Teresa Rojano explains: "It was Eugenio Filloy who, in the 1980s, proposed that Mexico be considered as a member of PME-NA, arguing that geographically, Mexico is part of North America. A year later, the [PME-NA] Steering Committee added a spot for Mexican member."

- Trigueros, Sacristán & Guerrero (2008, p. 221) describe the international collaborations and academic links promoted by Eugenio for the SME-Cinvestav:

> From its inception, the SME-Cinvestav group [led by Eugenio] began studying what was being done internationally, and developed academic links with foreign researchers and institutions. The first links were done with Brousseau and Glaeser, and later with the *Instituts de Recherche sur l'Enseignement des Mathématiques* (IREM) in Bordeaux and in the *Université Louis Pasteur* in Strasbourg, France, as well as the *École des Hautes Études en Sciences Sociales* in Paris. This established an influence of the French school of *Didactique.[…]* the SME-Cinvestav group also took into account theoretical frameworks from other countries such as the USA (e.g. the work of Bruner and Skinner) the Soviet Union (e.g. that of Kruteski), the UK, as well as the work of Piaget. Other academic links took place with the University of London, UK, Cambridge University, UK, and the University of Toronto, Canada. By the 21st century, many other international links had been established with the DME-Cinvestav. In addition to the aforementioned ones, others include those with the Universities of Granada and Valencia in Spain; the *Université Joseph Fourier* in Grenoble, France; the University of Quebec in Montreal (UQAM), Canada; the Universities of Georgia, and of Massachusetts-Dartmouth in the USA; and the Universities of Nottingham and of Bristol, UK.

- In 1979, Eugenio also did postdoctoral work at the University of Strasbourg, France, and was invited as visiting professor to the Autonomous University of Barcelona, Spain.
- In that way, researchers such as François Pluvinage (who coincidentally passed away on practically the same day as Eugenio), Kat Hart, Carolyn Kieran, Luis Puig, Celia Hoyles, Ros Sutherland, Richard Noss and James Kaput, among others, became regular collaborators of the Department of Mathematics Education of Cinvestav
- And the result of both the Master's (and later PhD) program at Cinvestav, as well as that of the PNFAPM, is several hundred of graduates trained through Cinvestav, not only from Mexico, but from all over Ibero and Latin America that have had an influence in their regions in the field of Mathematics Education; as well as the launching of many programs of professional development for mathematics teachers, and/or of research in Mathematics Education (or *Matemática Educativa*).

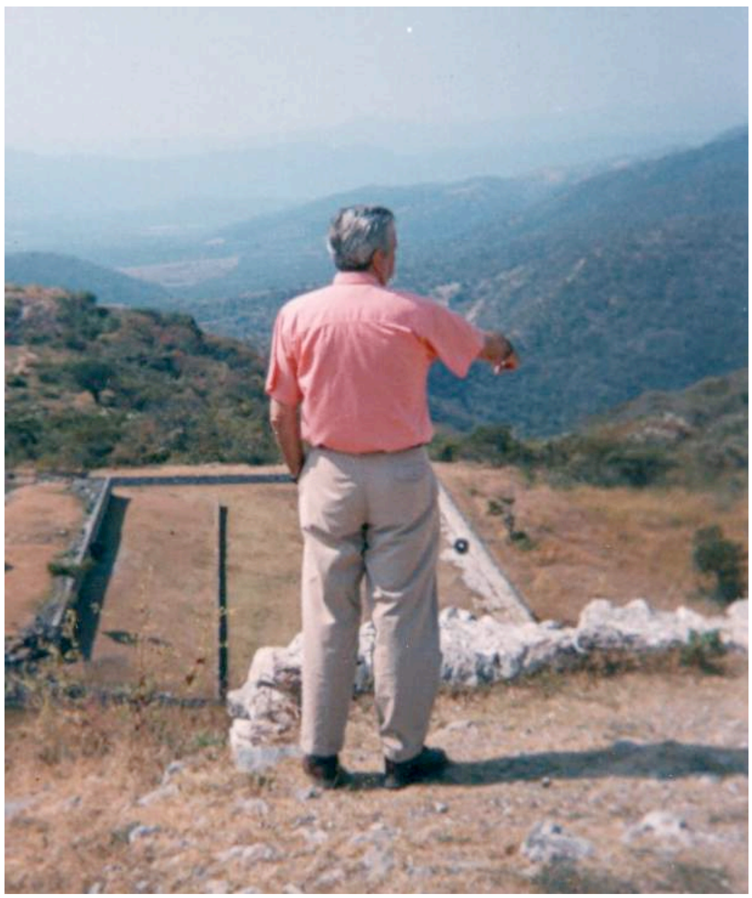
Eugenio not only strengthened the field of Mathematics Education through his professional development initiatives, but also through his research and the research approaches that he promoted (for further details, see Trigueros, Sacristán & Guerrero, 2008; and Solares, Puig & Rojano, 2020). For example,

- He promoted the analysis of the history and foundations of mathematics, as a research method in the field of mathematics education.
- He promoted the use and research of what used to be called "new technologies" (i.e., computers, audiovisual media, and other digital resources).
- In response to the need of carrying out controlled experimentation, he founded a school in which he could do that (the *Centro Escolar Hermanos Revueltas*).
- He recognized the limitations of general theoretical models and proposed the concept of Local Theoretical Models (see the book *Educational Algebra* by Filloy, Rojano & Puig, 2008).
- He produced close to 500 scientific publications in his areas of research: Didactics of Algebra and Geometry; History of Algebraic and Geometric ideas; Mathematical Systems of Signs; Curricular Design; Use of New Technologies;, and Local Theoretical Models.
- Eugenio was also a member of many Scientific and/or Editorial Committees, including of the journals *Educational Studies in Mathematics,* or *Recherche en Didactique des Mathématiques*.

- He received many distinctions, including being designed Professor Emeritus of Cinvestav (2002); being awarded an *Honoris Causa* Doctorate from the University of Sonora (2011); being granted the medal "*Mtro. Remigio Valdéz Gámez*" by the Mexican National Association of Mathematics Teachers (*Asociación Nacional de Profesores de Matemáticas A.C.)* for his transcendental trajectory in the field of Mathematics Education (2013); and receiving an honorific recognition from the Autonomous University of Guerrero for founding *Matemática Educativa* and having academic contributions that became international benchmarks (2016).

To all of that we add this tribute, in recognition of his profound legacy to the discipline of Mathematics Education.

Next we include some anecdote, memories and stories from some of his collaborators and former students:



"Eugenio will always be guiding us where to go forward." Luis Puig

Some happy memories of Prof Eugenio Filloy, by Celia Hoyles and Richard Noss

As far as we can remember, Celia first met Eugenio at an early PME conference – Grenoble 1981, or Antwerp 1982 we cannot be sure. Our relationship with Eugenio spanned many decades. It is full of so many wonderful memories, and we can only share their essence here.

Eugenio was remarkable in his deep commitment to mathematics and mathematics education, his devotion to Mexico and the promotion of mathematics education in the country. One only had to spend ten minutes in his company to recognise his charm, his sense of humour and perhaps most of all the twinkle in his eye. Not to mention his wonderful hospitality and his gourmet views on wine and food!

Celia and Richard with Eugenio in the 80s

In the mid-eighties, Eugenio and Celia set up the Anglo-Mexican Research Project, named “the Development of Mathematical Microworlds for Pupils and Teachers”. This was achieved with funding the British Council and Mexican Government and the work went forward in collaboration with *Programa Nacional de Formación y Actualizacion de Profesores de Mathematicas* or PNFAPM (Mexico´s National Program of Training and Professional Development for Mathematics Teachers) that Eugenio led. By the time Richard and later Teresa Rojano and Ros Sutherland joined the team in 1986, a steady stream of visits had been established in both directions along with research collaborations that lasted for many years.

A central feature of the project was to recruit PhD students whom we met in Mexico where they were subject to a selection process for working on an English PhD in Mathematics Education in London, with field work in Mexico.

This meant that there was, for us, an amazing ‘bonus’ to this project! We used to visit about once a year going to different parts of this amazing country with Eugenio as our guide and mentor. But that wasn’t all: Eugenio was not one to stay in a tent and eat in a roadside café! On the contrary, our visits were characterised by visits to wonderful restaurants, exotic hotels, and relaxed but fascinating seminars where we met several researchers who we now regard as friends and collaborators, Olimpia Figueras as just one example and notably one of Richard’s star PhD students on this program, Ana Isabel Sacristan who has become a leading figure in Mathematics Education in Mexico and internationally.

One incident stands out, one that is iconic in terms of Eugenio’s readiness to look at life – no matter how serious – with a twinkle in his eye. We were giving a talk in a seminar, somewhere in Mexico City. Celia was trying carefully to explain some key points of our research, and both of us were trying to communicate (shamefully, despite so many visits to Mexico, neither of us speaks Spanish well enough to give a talk)! Eugenio was kindly translating what we believed was an important and serious point, when the audience burst into laughter. Eugenio looked askance at us, a hint of a smile on his face. To this day we do not know what he said – or how closely – if at all – his 'translation' was to the real thing ☺

We miss Eugenio but at least have some consolation in the fact that our relationship with Mexico continues through collaborations with Ana and with Teresa that started with his initiative. We have again visited some of our favourite places to conduct workshops hosted this time by them. So we end with a photo of us with Ana taken at one of these workshops in the grounds of the lovely hotel Hosteria Las Quintas in Cuernavaca, which was introduced to us by Eugenio as the venue for one of his seminars and became one of our favourite places in the whole world ☺.

Richard and Celia with Ana Isabel Sacristan (left, who did her PhD in London as part of the collaboration set up by Eugenio Filloy) in the gardens of Hosteria Las Quintas

Celia and Richard in the gardens of Hosteria Las Quintas, Cuernavaca, Mexico

Memories of Eugenio, by Carolyn Kieran

Often, over the past several decades when I was invited to Mexico to give a talk to a group of teachers and young researchers, and Eugenio was requested to translate my oral presentation to the audience from English to Spanish, a strange thing happened: I would state a paragraph or so in English and then it would be Eugenio's turn to translate what I had just said. Inevitably, the translation took longer, but not only that, it often led to bouts of joyous laughter from the audience. Eugenio was an expert at injecting a note of humour into my otherwise straightforward and very serious presentations -- something that I was most grateful for and will always appreciate about Eugenio: his graciousness and keen sense of humour.

Memories of Eugenio, by Fernando Hitt

Eugenio was one of our Mexican "maîtres penseurs". He had a very broad vision of Mexico's problems in education and of our possibilities of contributing to the improvement of the teaching and learning of mathematics. When I was doing my doctoral studies abroad, he presented to the members of the Educational Mathematics Section a document on the evolution of the section and its members, which went down in history as the "Colorines document". In that document he made a projection of the future of the section. This document was used for several years as a guide to remind us of the objectives of the section.

At the beginning of the 80s Eugenio suggested that I reflect on a project related to the training of mathematics teachers to be presented to the office of Higher Education [of the Mexican Ministry of Education] shortly before a change of administration (in 1982). He proposed that we meet once a week at his house and that if the project wasn't approved, we would at least have a pleasant time discussing and having a couple of drinks... The result was that Jesús Reyes Heroles, who was appointed as Minister of Education, upon hearing of the project, mentioned that this was an excellent project and that many more of the same style should be implemented (for Spanish, Biology, Physics, etc.). This is how the *Programa Nacional de Formación y Actualizacion de Profesores de Mathematicas* or PNFAPM (Mexico´s National Program of Training and Professional Development for Mathematics Teachers) was born; the problem was that a short time later, Reyes Heroles died. Still, we were able to sustain the program for over 10 years.

Some memories of the PME conferences:

In a PME [of the late 80s or early 90s,] Eugenio went to a lecture on the use of new technologies. For the first time, Eugenio saw someone using a computer and beamer at a presentation. He explained that the image was a bit blurry and small. He told me that several minutes later a guy arrived, sat in the front near where the computer and the beamer were, got up, moved, did a couple of things and left a sharper and larger image. Eugenio told me that that moment was much more interesting on the usefulness of technology than the entire lecture.

At the PME in Asissi (Italy) in 1991, at the beginning of the conference, I saw him leave the hotel while I was in the cafeteria. I left about 5 minutes later and met him on the road. I told him that I thought he would be at the conference by now, and he replied that walking the Michael Jackson way took him longer ... ☺

Eugenio Filloy-Yagüe, a teacher who left me a legacy, by Miguel Díaz-Chávez

In my education, Professor Filloy was a very important person and the meetings were many; however I think the following are the ones I want to share.

My first meeting with Professor Filloy was in 1986 when I began my master's degree in *Matemática Educativa*, in the then section of the same name located in a beautiful house in Mexico City. In the first semester he was my teacher in a compulsory course on education and I think one of the first readings he left us was the book "Psychogenesis and history of science" by Jean Piaget and Rolando García. I read the sections he indicated and at the end I told him with great anguish that I did not understand and he very understandingly replied: someday you will understand. This course and his dissertations opened up the landscape of mathematics education for me.

A second memory that I have very much in mind was on my first trip to Europe in 1988 when I was surprised to find a book of his authorship in a bookstore in Madrid; this discovery showed me the importance of his work at an international level.

The last meeting I had with Professor Filloy was listening to him at an event in Mexico City about five years ago. At that time I was starting to do research on the free math textbook and I talked to him about the possibility of interviewing him about it, considering that he was one of the authors of those books in the 70's; as expected he agreed, but with his style he said: Yes, but hurry up. That interview never took place. To him, Professor Filloy, my infinite gratitude.

Some of my memories of Eugenio, by Ana Isabel Sacristán

When, in 1986, I began my Mathematics Education master's studies at Cinvestav, Eugenio Filloy was my teacher in the compulsory and introductory course in Education. I remember how intimidating his imposing presence was to me. For example, he was obsessive about punctuality and I was afraid of him for that: looking at his watch, he would tell me: "You were two minutes late"; or on one occasion: "You were 30 seconds late"!

On the other hand, in those days it was surprising to me how he valued more our reasoning than learning contents, and I remember how enlightening the study of the book "*Psychogenesis and history of science*" by Jean Piaget and Rolando García, was. Eugenio introduced me to an educational philosophy that was new to me then, but that I have followed since.

Some time after I finished my master's studies, one day in 1989, I ran into him by chance: At the outset he told me: "The English are here"; immediately, without my being able to utter a single word, or ask what he was referring to, he opened his agenda and ordered me to be there the next day to meet Celia Hoyles at a specific time. That was how short the meeting was, and I was not sure what it was about. But it was an interview for pursuing PhD studies at the University of London, England, with the support of the PNFAPM and the British Council. Thus it was that by a chance encounter, my future was sealed; and I will be eternally grateful to Eugenio, in addition to his teachings, for putting me on the path and supporting me on the way.

I am pleased to have been able to inform him, shortly before he passed away, that we would do this tribute at PME-NA 42. I think he was pleased.

Memories of Eugenio, by José Carlos Cortés

In 1991, I began my Master's degree in the Section of *Matemática Educativa* located at a house in Mexico City. Nearby was another building in which Dr. Filloy worked; at that time he was the head of the section. The first time I saw him, I was impressed by his strong tone of voice and his appearance. Since, in the first semester of my master's studies, I did not yet have a scholarship, one day Dr. Filloy approached me and asked me if I wanted to work at his school "Los Hermanos Revuelta" which I accepted with great pleasure and I will never forget that great experience. In the school "Los Hermanos Revuelta" directed by Maru, wife of Dr. Filloy, as teachers we were allowed to innovate; also, as masters and doctoral students, we were also allowed to carry out educational experiments (of course complying with all the protocols). Thus, Dr. Filloy was a pioneer in Mathematics Education research but he also put into practice through his school the new teaching trends in the field. Much later, Dr. Filloy participated with us at several AMIUTEM conferences of which we have very good memories. Best wishes to you, Eugenio, wherever you are.

Debating with Eugenio Filloy in Mexico City, by Armando Solares-Rojas (written in 2012)

I met Eugenio Filloy in the spring of 2000. The references I had of him made me place him as the founder of the Department of Educational Mathematics of Cinvestav, a member of the editorial committees of some of the most important research journals in the area, an international researcher famous for his contributions in the teaching of Algebra and as a very important figure in the educational reforms of the country. But there was still a lot to know about him...

The first work meeting we had was held at the offices of the *Sociedad Mexicana de Matemática Educativa* (Mexican Society of Educational Mathematics). I arrived promptly and we went to his meeting room. Three armchairs, a small table, several paintings, a collection of posters celebrating the 300th anniversary of the publication of Isaac Newton's Mathematical Principles of Natural Philosophy. We spoke very little, but in that short talk he invited me to a seminar on the philosophy of mathematics that was in progress, with Ignacio Garnica.

Together with Manuel Cruz, who was a fellow at the UNAM Institute of Mathematics and is now director of the Department of Mathematics at the University of Guanajuato, I plucked up the courage and spent long hours reading and discussing Gottlob Frege's text. With many ideas in mind, we launched into a seminar session in which we tried to answer questions such as: What are numbers?

What do expressions involving numbers refer to? Eugenio Filloy led the seminar to discuss theories about the meaning of mathematical texts and showed me the possibility of working, from mathematics, with epistemology, history and didactics. I entered the doctoral program in Educational Mathematics at Cinvestav in September of that same year.

As a formal student in the department, I continued to be part of the seminar whose sessions, intense and unforgettable, left me much more learning than I expected.

For example, the Wittgenstein readings. The long hours spent at the end of my bachelor's degree reading the Tractatus logico-philosophicus finally made sense! Today, language games continue to haunt my readings and, increasingly, they are incorporated into the texts that I write myself.

I also learned about Charles Sanders Peirce's semiotic theory and how Eugene used the triadic notion of the sign as a starting point to develop the Mathematical Systems of Signs, a theoretical notion that allowed him to describe the teaching and learning of algebra, thus articulating a new perspective on research that has given fruit to numerous studies and publications of which Dr. Rojano has already spoken extensively. I myself have had the fortune and honor to collaborate on some of them throughout the years of training and work with Eugenio and Tere: from presentations at the International Group for the Psychology of Mathematics Education (PME) Conferences, to the recent publication of the article "Problems of two unknown quantities and two levels of representation of the unknown" published in the *Journal for Research in Mathematics Education* in January of last year (2011).

From philosophy I went back to history. My previous readings on the history of calculus (Newton's De quadratura curvarum), of physics (Optiks, also Newton's) made sense also through my contact with Filloy!

Although I knew little about the history of the development of Algebra, like any mathematician in training, I knew the history of the short life of Galois, famous among young people for being bohemian, in love, anti-church, rebellious and… cool. But I knew nothing of the Indian astronomers and mathematicians Bhaskara and Brahmagupta, or of the Persian mathematician, astronomer and poet Omar Khayyám.

And that leads me to talk about a facet of Eugenio that characterizes him and that has had a profound influence on me and on many other people who have been close to him: his generosity. I think that the history and legacy of a man, of a teacher, can be described by the books he treasures, but even more by the books he offers and shares.

Entering the Filloy library is an invitation to navigate through the immense plurality of ideas that the history of mathematics grants us. There I came across different translations of Al-Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala (Concise Book of Calculation of Restoration and Opposition) and I was able to compare the versions and translations of Robert de Chester with that of Gerard of Cremona and with that of Frederic Rosen. Also the Arithmetic of Diophantus in several of its versions (See Eecke, P., J. Sesiano, Rashed, Tannery); the Liber Quadratorum 'The Book of Square Numbers' by Leonardo of Pisa ...

His generosity to his students goes beyond sharing his treasured library. An essential part of Filloy's didactic activity was supporting the student and teaching autonomy and self-sufficiency to write, participate in conferences, design and present research projects. I remember coming out of tutorial session and carrying a huge bundle made up of transcripts, videos, lessons, articles, books ... And hours of revisions and discussions on video-recorded interviews with laughter in front of the television; What did he say? What did he do? Where is it going? Sharing knowledge. Teaching. Learning.

I also received his generosity when becoming a teacher. Have you taught secondary school? Eugenio asked me. No… not yet, I replied. It was the middle of 2001 and by September of that year I was already preparing my math classes for second grade of middle school at Revueltas, the school with an alternative approach that Eugenio founded. How was I doing? Suffice to say that I came out sweating, hoarse, exhausted. Nothing to do with the classes that I had taught at the School of Sciences! More difficult? My friends asked me. Without a doubt!

That experience profoundly transformed my way of seeing the mathematics classroom, it was no longer just a matter of finding and designing good problems and putting them to the test in an interview situation, but of recognizing the complexity of the teacher's work, the diversity of tensions in those that build the mathematical activity of the classroom.

On the other hand, in Revueltas I saw how Eugenio, a researcher in educational mathematics, ran into students in the hallways who said: “high-five, Eugenio!" and they climbed on his back so that he would carry them "on horseback"; with administrators who were looking for him to say "teacher, the calculators and view-screens arrived"; and teachers who asked him for support for their classes "Eugenio, do we review what we are going to see in third grade next week?" Creating and sustaining a "living" research laboratory, as a school is, is one of the most demanding tasks I have seen, and in Eugenio it is also an example of the consistency between theoretical discourse and practice.

But Eugenio's generosity is not limited to books and mathematics… discussions, projects, articles, accompanied by a good meal and a good wine, taste even better!

What is the meaning and reference of the expression "good French wine"? Following Eugenio's teaching, we could say that his reference is located in the Montrachet vineyards, between the towns of Puligny and Chanssange, in the Côte-d'Or of the Burgundy region, in eastern France. Or perhaps in a bottle from a Château d'Yquem, produced in the Sauternes region of Bordeaux. The meaning in this case is even clearer, if you want to think about it like that. Sauternes is made with semillon, sauvignon blanc and muscadelle grapes that, affected by a fungus endemic to the region, are partially raisined, resulting in a higher concentration of sugar and wines with a distinctive aroma. The Château d'Yquem is also made in an artisanal way, processing grape by grape… by hand.

And just as he shared how much he knows about wines with me, I have had the opportunity to be close to him, to his family, to Maru, his wife. Close not only to his vision as a researcher but also to his vision of life, of what in Mexico and Latin America we understand of being a sybarite, that is, who enjoys life in all its vastness.

Today, from my own vision of closeness, I want to thank Eugenio for all that he has shared and still shares with me. A big hug and many congratulations, Eugenio.

Mexico City, November 2012.

Memories of one of Dr. Filloy’s last students, by María Leticia Rodríguez-González

Dr. Eugenio Filloy was a person with the ability to observe in order to know each and every one of us. He knew exactly what our strengths, weaknesses and especially feelings were. He was a man, with extraordinary universal knowledge, he always had a topic of conversation. He loved the narrative, weaving everyday life anecdotes into his stories, intermingling characters from Greek mythology and politics. Filloy was in love with life.

He loved it when we asked him things, although sometimes it was his students who were the most hesitant to ask him. The most interesting thing is that he wouldn’t give us the answer, instead he would provide a large number of books to research it. But the wonderful thing is that the material that he would share with us was unpublished in most cases.

However, it was sometimes very difficult to follow the logic of his speech; even though we were barely understanding the beginning, he was already coming to the end. Over time, I learned that all his ideas were mathematical and heuristic but above all scientific.

I dare to affirm that Eugenio Filloy was the scientist who gave legitimacy to mathematics education, breaking with the paradigm of finding problems in the way in which mathematics is taught and learned. He gave the arguments for the object of knowledge of Educational Mathematics to focus on mathematics itself, establishing a clear difference between Mathematics Education, Mathematics Didactics and Mathematics Psychology.

Eugenio Filloy, a visionary teacher that knew that the construction of scientific knowledge requires interaction between researchers, teachers and students through communication. Proof of this was his participation and in some cases founder of Conferences, Forums, Symposia, national and international such as the Mexican Mathematical Society, the National Program for Teacher Training (*Programa Nacional de Formación y Actualización de Profesores de Matemáticas*), the Central American and Caribbean Meeting on Teacher Training and Research in Educational Mathematics (*Reunión Centroamericana y del Caribe sobre la Formación de Profesores e Investigación en Matemática Educativa*), PME, PME-NA, CIAEM.

Thank you Dr. Filloy, even if you are no longer physically present, your presence will continue, starting with our Department of Educational Mathematics, which you founded together with Dr. Carlos Imaz. Your work is still present in your contribution to the research through the Local Theoretical Models.

Memories of Eugenio, by Ulises Xolocotzin

I will always be grateful that Dr Eugenio Filloy took the time to greet me and talk with me when we bumped into each other at our Department. In our talks, he either made me laugh, or made me think. He once said to me: "I entered the matter of mathematical thinking from the side of mathematics and I came out from the side of cognition. You are entering from the side of cognition, but who knows which side you will come out from!" He made me think and said goodbye to me smiling. I keep thinking and I hope he keeps smiling.

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HOMENAJE A EUGENIO FILLOY†: UN PIONERO IMPULSOR DE LA DISCIPLINA DE LA MATEMÁTICA EDUCATIVA

TRIBUTE TO EUGENIO FILLOY†: A PIONEER AND DRIVING FORCE OF MATHEMATICS EDUCATION AS A DISCIPLINE

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Eugenio Filloy (1942-2020)

Como dijimos en el prólogo de estas actas, cuando planeamos el programa del PME-NA 42, habíamos contemplado incluir un homenaje especial y presencial en vida a Eugenio Filloy, al que considero “el padre” de la matemática educativa en México, con una influencia que trasciende fronteras a otros países de Ibero y Latinoamérica. Eugenio fue de las personas que más apoyaron la realización del PME-NA 42 en México, y estas actas incluyen muchas contribuciones de sus estudiantes y suyas. Tristemente, Eugenio falleció el 23 de marzo de este año 2020. Hacemos este homenaje por tanto *in memoriam.*

Aquí bosquejamos su trayectoria académica y algunas de sus principales contribuciones al campo de la Matemática Educativa.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

- Eugenio obtuvo la Licenciatura en Matemáticas en la Facultad de Ciencias de la Universidad Nacional Autónoma de México, en 1965, habiendo también estudiado Física Teórica
- Obtuvo el grado de Maestro en Ciencias en Matemáticas, del Cinvestav, en 1966, y ese mismo año fue profesor visitante de la Universidad de California en Los Angeles.
- Obtuvo el grado de Doctor en Ciencias en Matemáticas de la Universidad de Chicago, en 1970.
- En 1971, ingresa como investigador adjunto del Departamento de Matemáticas del Cinvestav; pero también hace un trabajo posdoctoral en la Universidad de Ginebra, Suiza, en 1972.
- De 1973 a 1977, fue presidente de la Sociedad Matemática Mexicana
- En 1975, co-fundó la Sección de Matemática Educativa (SME), afiliada al Departamento de Investigaciones Educativas (DIE) del Cinvestav, junto con otros dos matemáticos del Departamento de Matemáticas –Carlos Imaz y Juan José Rivaud— con quienes colaboraba en el diseño y escritura de los libros de texto de Matemáticas para la escuela primaria, dentro del Programa Nacional de Libros de Texto Gratuitos, impulsado por la Secretaría de Educación Pública (SEP) a finales de los años 60s. Como explica Ramiro Ávila Godoy (2013):

> Esta experiencia del Dr. Filloy le permitió tomar plena conciencia de la importancia y complejidad de la problemática del aprendizaje y la enseñanza de las Matemáticas y de la necesidad de atenderla en todos los niveles educativos, en especial en el nivel básico; pero lo más trascendente no fue haberse percatado de la necesidad de atender dicho problema, sino el haber tomado la decisión de intervenir de manera directa en la búsqueda de solución al mismo. […]
>
> … en el nombre *Matemática Educativa* iba implícita la intención de enfrentar la problemática de la enseñanza y el aprendizaje de la matemática desde la matemática misma. En esa época, este enfoque resultó muy original, dando lugar a que en muchos países de Latinoamérica empezara a hablarse de Matemática Educativa como lo que en la actualidad es en la comunidad latinoamericana, *la disciplina dedicada al estudio de la problemática de la educación matemática*.

- Desde el inicio de la SME (hoy en día, el Departamento de Matemática Educativa –DME, del Cinvestav), i.e. desde 1975, se ofreció una Maestría en Ciencias con Especialidad en Matemática Educativa, “con un fuerte contenido tanto en matemáticas, como en la historia y fundamentos de las matemáticas” (Trigueros, Sacristán & Guerrero, 2008, p. 220); y a partir de 1982, un programa de Doctorado en Ciencias. Eugenio mismo (activo hasta el día de su fallecimiento) dirigió las tesis de casi 100 estudiantes de esos programas (más de 60 tesis de maestría y más de 30 de doctorado); muchos de sus ex alumnos tienen escritos en estas actas.
- En 1983, Eugenio fundó el ambicioso y trascendental Programa Nacional de Formación y Actualización de Profesores de Matemáticas (PNFAPM) que, en colaboración con una red de departamentos de Matemáticas de más de 30 universidades y tecnológicos regionales de México, ofreció una Maestría en Ciencias en Matemática Educativa; así como un programa de licenciatura en Enseñanza de las Matemáticas.

> El PNFAPM también estableció vínculos académicos con instituciones internacionales, y muchos miembros de la SME realizaron estudios de doctorado y posdoctorado en el extranjero. Simultáneamente al PNFAPM, se lanzaron conferencias y reuniones nacionales e internacionales para que investigadores y docentes pudieran compartir y discutir sus experiencias ”. (Trigueros, Sacristán y Guerrero, 2008, p. 221)

De hecho, Eugenio promovió continuamente las colaboraciones con investigadores de todo el mundo. Por ejemplo:

- Participó activamente en eventos internacionales, tales como los del CIAEM (presidiendo, en 1980, el XXXII CIAEM, en Oaxtepec, México), del PME e ICMI. Eugenio fue instrumental para que México formara parte del PME-NA; Teresa Rojano explica: "Fue Eugenio Filloy quien, en la década de 1980, propuso que se considerara a México como miembro del PME-NA, argumentando que geográficamente México es parte de América del Norte. Un año después, el Comité Directivo [del PME-NA] agregó un lugar para miembro mexicano."

- Trigueros, Sacristán y Guerrero (2008, p. 221) describen las colaboraciones internacionales y vínculos académicos promovidos por Eugenio para la SME del Cinvestav:

> Desde sus inicios, el grupo de la SME-Cinvestav [liderado por Eugenio] comenzó a estudiar lo que se estaba haciendo a nivel internacional y desarrolló vínculos académicos con investigadores e instituciones extranjeras. Los primeros vínculos se realizaron con Brousseau y Glaeser, y más tarde con los *Instituts de Recherche sur l'Enseignement des Mathématiques* (IREM) en Burdeos y en la *Université Louis Pasteur* en Estrasburgo, Francia, así como con la *École des Hautes Études en Sciences Sociales* en París. Esto estableció una influencia de la escuela francesa de *Didactique*. […] El grupo SME-Cinvestav también tuvo en cuenta los marcos teóricos de otros países como de los Estados Unidos (por ejemplo, el trabajo de Bruner y Skinner), de la Unión Soviética (por ejemplo, el de Kruteski), del Reino Unido, así como el trabajo de Piaget. Otros vínculos académicos tuvieron lugar con la Universidad de Londres, Reino Unido, la Universidad de Cambridge, Reino Unido y la Universidad de Toronto, Canadá. En el siglo XXI, se habían establecido muchos otros vínculos internacionales con DME-Cinvestav. Además de los mencionados, otras incluyen los de las Universidades de Granada y Valencia en España; la *Université Joseph Fourier* en Grenoble, Francia; la Universidad de Quebec en Montreal (UQAM), Canadá; las universidades de Georgia y Massachusetts-Dartmouth en los Estados Unidos; y las Universidades de Nottingham y de Bristol, Reino Unido.

- En 1979, Eugenio también lleva a cabo trabajo posdoctoral en la Universidad de Estrasburgo, Francia, y es profesor invitado en la Universidad Autónoma de Barcelona, España.
- De esa forma, investigadores como François Pluvinage (quien casualmente falleció prácticamente el mismo día que Eugenio), Kat Hart, Carolyn Kieran, Luis Puig, Celia Hoyles, Ros Sutherland, Richard Noss y James Kaput, entre otros, se hicieron habituales colaboradores del Departamento de Matemática Educativa del Cinvestav
- Y el resultado, tanto del programa de Maestría, y posteriormente del programa de Doctorado del Cinvestav, así como del PNFAPM, son varios cientos de graduados formados a través del Cinvestav, no solo de México, sino de toda Ibero y América Latina, los cuales han influido en sus regiones en el campo de la Educación Matemática; otra consecuencia es el lanzamiento de otros programas de capacitación para profesores de matemáticas y/o de investigación en Matemática Educativa.

Eugenio no solo fortaleció el campo de la Educación Matemática a través de sus iniciativas de desarrollo profesional, sino también a través de su investigación y los enfoques y métodos de investigación que impulsó (para más detalles, ver Trigueros, Sacristán & Guerrero, 2008; y Solares, Puig y Rojano, 2020). Por ejemplo,

- Impulsó el análisis de la historia y de los fundamentos de las matemáticas, como método de investigación en la disciplina de la matemática educativa.
- Promovió el uso y la investigación de lo que antes se solía llamar las "nuevas tecnologías" (es decir, computadoras, medios audiovisuales y otros recursos digitales).
- Ante la necesidad de realizar una experimentación controlada, fundó una escuela en la que podría hacerlo (el Centro Escolar Hermanos Revueltas).
- Reconoció las limitaciones de los modelos teóricos generales y propuso el concepto de Modelos Teóricos Locales (ver el libro *Educational Algebra* de Filloy, Rojano & Puig, 2008).

- Produjo cerca de 500 publicaciones científicas en sus líneas de investigación: Didáctica del álgebra y la geometría; Historia de las ideas algebraicas y geométricas; Desarrollo curricular; Uso de nuevas tecnologías en la enseñanza; Sistemas Matemáticos de Signos; y Modelos Teóricos Locales.
- Eugenio también fue miembro de muchos Comités Científicos y / o Editoriales, incluyendo de las revistas *Educational Studies in Mathematics* y *Recherche en Didactique des Mathématiques.*
- Recibió muchas distinciones, incluida la de ser nombrado Profesor Emérito del Cinvestav (2002); recibir un Doctorado *Honoris Causa* de la Universidad de Sonora (2011); ser otorgado la medalla "Mtro. Remigio Valdéz Gámez" de la Asociación Nacional de Profesores de Matemáticas AC (ANPM AC) por su meritoria y trascendente trayectoria en el ámbito de la educación matemática en México (2013); y un reconocimiento honorífico de la Universidad Autónoma de Guerrero por ser fundador en México y América Latina de la Matemática Educativa, aportaciones académicas de referente internacional (2016).

A eso le sumamos este homenaje, en reconocimiento a su profundo legado en el campo de la Matemática Educativa.

A continuación incluimos algunos, recuerdos e historias de algunos de sus colaboradores y ex alumnos:

"Eugenio siempre estará señalándonos hacia dónde seguir adelante." Luis Puig

Recuerdos felices del Prof Eugenio Filloy, por Celia Hoyles and Richard Noss

Celia y Richard con Eugenio en los años ochentas

Por lo que podemos recordar, Celia conoció a Eugenio en una de las primeras conferencias de PME: Grenoble 1981 o Amberes 1982, no podemos estar seguros. Nuestra relación con Eugenio duró muchas décadas. Está lleno de tantos recuerdos maravillosos, de lo que solo podemos compartir su esencia aquí.

Eugenio se destacó por su profundo compromiso con las matemáticas y la educación matemática, su devoción por México y el fomento de la educación matemática en el país. Uno solo tenía que pasar diez minutos en su compañía para reconocer su encanto, su sentido del humor y quizás sobre todo el brillo en sus ojos. ¡Sin mencionar su maravillosa hospitalidad y sus opiniones gourmet sobre el vino y la comida!

A mediados de los años ochenta, Eugenio y Celia pusieron en marcha el Proyecto de Investigación Anglo-Mexicano, denominado "El Desarrollo de Micromundos Matemáticos para Alumnos y Docentes". Esto se logró con la financiación del British Council y el Gobierno de México y el trabajo avanzó en colaboración con el Programa Nacional de Formación y Actualización de Profesores de Matemáticas o PNFAPM que lideró Eugenio. Cuando Richard y más tarde Teresa Rojano y Ros Sutherland se unieron al equipo en 1986, se había establecido un flujo constante de visitas en ambas direcciones junto con colaboraciones de investigación que duraron muchos años.

Una característica central del proyecto fue reclutar estudiantes de doctorado que conocimos en México, quienes fueron sujetos a un proceso de selección para realizar un doctorado en Educación Matemática en Londres, con trabajo de campo en México.

¡Esto significó para nosotros un gran "extra" de este proyecto! Solíamos visitar México una vez al año, yendo a diferentes partes de este increíble país con Eugenio como nuestro guía y mentor. Pero eso no fue todo: ¡Eugenio no era de los que se quedaban en una tienda de campaña y comían en un café al borde de la carretera! Por el contrario, nuestras visitas se caracterizaron por visitas a maravillosos restaurantes, hoteles exóticos y seminarios relajados pero fascinantes donde conocimos a varios investigadores a los que ahora consideramos amigos y colaboradores, como Olimpia

Figueras y en particular una de las estudiantes de doctorado estrella de Richard de este programa, Ana Isabel Sacristán quien se ha convertido en una figura destacada en Educación Matemática en México e internacionalmente.

Se destaca un incidente, uno que es icónico en términos de la disposición de Eugenio para mirar la vida, sin importar cuán seria sea, con un brillo en los ojos. Dábamos una charla en un seminario, en algún lugar de la Ciudad de México. Celia estaba tratando de explicar cuidadosamente algunos puntos clave de nuestra investigación, y ambos estábamos tratando de comunicarnos (vergonzosamente, a pesar de tantas visitas a México, ninguno de los dos hablamos español lo suficientemente bien como para dar una charla). Eugenio estaba traduciendo amablemente lo que creíamos que era un punto importante y serio, cuando el público se echó a reír. Eugenio nos miró de reojo, con un atisbo de sonrisa en su rostro. Hasta el día de hoy no sabemos lo que dijo, o qué tan cercana, si es que lo hizo, fue su "traducción" a lo que dijimos. ☺

Extrañamos mucho a Eugenio pero al menos nos da consuelo el hecho de que nuestra relación con México continúa a través de colaboraciones con Ana y con Teresa que comenzaron con su iniciativa. Hemos vuelto a visitar algunos de nuestros lugares favoritos para realizar talleres, en esta ocasión organizado por ellos. Así que terminamos con una foto de nosotros con Ana tomada en uno de estos talleres en el encantador hotel Hostería Las Quintas en Cuernavaca, que nos presentó Eugenio como sede de uno de sus seminarios y se convirtió en uno de nuestros lugares favoritos en todo el mundo. ☺

Richard y Celia con Ana Isabel Sacristán (izq., quien hizo sus estudios doctorales en Londres como parte de la colaboración iniciada por Eugenio Filloy) en los jardines de la Hostería Las Quintas

Celia y Richard en los jardines de la Hostería Las Quintas, Cuernavaca

Recuerdos de Eugenio, por Carolyn Kieran

A menudo, durante las últimas décadas, cuando me invitaban a México a dar una plática para profesores y jóvenes investigadores, y se le pedía a Eugenio que tradujera mi presentación al público del inglés al español, algo extraño sucedía: enunciaba más o menos un párrafo en inglés y luego era el turno de Eugenio de traducir lo que acababa de decir. Inevitablemente, la traducción tomaba más tiempo, pero no solo eso, a menudo provocaba ataques de risa en el público. Eugenio era un experto en darle un toque de humor a mis presentaciones que sin él eran sencillas y muy serias – algo por lo que siempre estaré muy agradecida y siempre apreciaré de Eugenio: su gentileza y gran sentido del humor.

Recuerdos de Eugenio, por Fernando Hitt

Eugenio era uno de nuestros "maîtres penseurs" mexicano. Tenía una visión muy amplia de los problemas de México en educación y de nuestra posibilidades de contribuir al mejoramiento de la enseñanza y del aprendizaje de las matemáticas. Cuando yo realizaba estudios de doctorado en el extranjero, presentó a los miembros de la Sección de Matemática Educativa un documento sobre la evolución de la sección y sus miembros, que pasó a la historia como el "documento de los colorines". En ese documento hacía una proyección de la sección hacia el futuro. Ese documento se utilizó durante varios años como guía para recordarnos sobre los objetivos de la sección.

Eugenio al inicio de los 80s me propuso reflexionar sobre un proyecto sobre la formación de profesores de matemáticas que se *debería* presentar a la Dirección General de Educación Superior Universitaria poco antes del cambio de sexenio (en 1982). Me propuso que nos reuniésemos un día cada semana en su casa y que si el proyecto no fuera aprobado, al menos pasaríamos un rato agradable discutiendo y tomando un par de copas... El resultado fue que Jesús Reyes Heroles al ser nombrado secretario de educación, al escuchar el proyecto por el Director General, mencionó que ese era un proyecto excelente y que se deberían implementar muchos más del mismo estilo (Español, Biología, Física, etc.). Fue así como nació el Programa Nacional de Formación y Actualización de Profesores de Matemáticas (PNFAPM), el problema fue que poco tiempo después murió Reyes Heroles. Aún así, pudimos sostener el programa por más de 10 años.

Algunas memorias de los PME:

En un PME de finales de los 80s o principios de los 90s, Eugenio fue a una conferencia sobre el uso de nuevas tecnologías. Por primera vez, Eugenio veía a alguien utilizando computadora y cañón en una presentación. Me explicó que se veía un poco borroso y la imagen pequeña. Me cuenta que varios minutos después llega un tipo, se sienta al frente cerca de donde estaba la computadora y el cañón, se levanta mueve realiza un par de cosas y deja una imagen nítida e imagen más amplia. Eugenio me dijo, ese momento fue mucho más interesante sobre la utilidad de la tecnología que toda la conferencia completa.

En el PME en Asissi (Italia) en 1991, al inicio del congreso lo ví salir del hotel estando yo en la cafetería. Salí unos 5 minutos después y me lo encontré en el camino. Le dije que pensaba que ya estaría en el congreso, y me contestó que caminando a la Michael Jackson le tomaba más tiempo... ☺

Eugenio Filloy Yagüe, un profesor que me dejó un legado, por Miguel Díaz Chávez

En mi formación, el profesor Filloy fue un personaje muy importante y los encuentros fueron muchos; sin embargo creo que los siguientes son los que quiero compartir.

Mi primer encuentro con el profesor Filloy fue en el año de 1986 cuando iniciaba mis estudios de maestría de matemática educativa, en la entonces sección del mismo nombre ubicada en el bello edificio de Dakota 379 en la colonia Nápoles. En ese primer semestre él fue mi profesor del curso básico de educación y creo que una de las primeras lecturas que nos dejó fue la del libro "*Psicogénesis e historia de la ciencia*" de Jean Piaget y Rolando García. Yo leí las secciones que nos indicó y al final le dije con mucha angustia que no entendía y él muy comprensivo me contestó: *Algún día lo entenderás*. Este curso y sus disertaciones me abrieron el panorama de la educación matemática.

Un segundo recuerdo que tengo muy presente fue en mi primer viaje a Europa en 1988 cuando me sorprendí de encontrar un libro de su autoría en una librería de Madrid, este descubrimiento me mostró la trascendencia de su trabajo a nivel internacional.

El último encuentro que tuve con él profesor Filloy fue escuchándolo en un evento en la ciudad de México hace aproximadamente cinco años. En ese tiempo yo iniciaba una investigación sobre el libro de texto gratuito de matemáticas y platiqué con él sobre la posibilidad de hacerle una entrevista al respecto, considerando que él fue uno de los autores de esos libros en los años 70's; como era de esperarse estuvo de acuerdo, pero con su estilo me dijo: *Sí, pero date prisa.* Nunca tuvo lugar esa entrevista. Para él, el profesor Filloy, mi gratitud infinita.

Algunos recuerdos de Eugenio, por Ana Isabel Sacristán

Cuando inicié, en 1986, mis estudios de Maestría en la Sección de Matemática Educativa del Cinvestav, tuve como profesor a Eugenio Filloy, en la materia que se llamaba "Básico de Educación". Recuerdo lo intimidante que me resultaba su presencia tan imponente. Por ejemplo, era obsesivo con la puntualidad y yo le tenía miedo por eso: mirando su reloj, me decía: "Llegaste dos minutos tarde"; o, en una ocasión: "Llegaste 30 segundos tarde"(!)

En contraparte, me resultó sorpresivo, en aquel entonces, cómo era más importante para él que razonáramos a que aprendiéramos contenidos, y recuerdo lo iluminante que fue el estudio del libro "*Psicogénesis e historia de la ciencia*" de Jean Piaget y Rolando García. Eugenio me introdujo a una filosofía educativa que desconocía entonces, pero que he seguido hasta ahora.

Tiempo después de haber terminado mis estudios de maestría, en 1989, me topé con él de casualidad un día: De entrada me dijo: "Los ingleses están aquí"; inmediatamente, sin que yo pudiera decir ni una sola palabra, ni preguntar a qué se refería, abrió su agenda y me ordenó estar al día siguiente para entrevistarme con Celia Hoyles a una hora específica. Así de corto fue el encuentro, y yo no sabía bien de qué se trataba. Pero era un entrevista para realizar estudios de doctorado en la Universidad de Londres, Inglaterra, con apoyo del PNFAPM y del *British Council*. De esa manera, por un encuentro que se dio de casualidad, mi futuro quedó sellado; y le estaré eternamente agradecida a Eugenio, además de sus enseñanzas, por ponerme y apoyarme en el camino.

Tengo la satisfacción de haberle podido informar, poco antes de que falleciera, que le haríamos este homenaje en el PME-NA 42. Creo que le dio gusto.

Memorias de Eugenio, por José Carlos Cortés

En 1991 inicié mis estudios de Maestría en la Sección de Matemática Educativa en el edificio ubicado en la calle Dakota en la Ciudad de México. En esa misma calle se encontraba también el edificio en el que trabajaba el Dr. Filloy, en ese entonces él era el jefe de la sección. La primera vez que lo vi me impresionó su tono fuerte de voz y su apariencia. Como en el primer semestre de mis estudios de Maestría no contaba aún con el apoyo de Conacyt, un día se me acercó el Dr. Filloy y me dijo que si quería trabajar en su escuela "Los Hermanos Revuelta" lo cual acepté con mucho gusto y nunca olvidare esa experiencia tan grande. En la escuela "Los Hermanos Revuelta" dirigida por Maru, esposa del Dr. Filloy, se nos permitía innovar a los profesores, también se nos permitía a los estudiantes de Maestría y Doctorado realizar experimentación educativa (claro cumpliendo todos los protocolos), es decir el Dr. Filloy fue pionero en la Investigación en Matemática Educativa pero a su vez ponía en práctica a través de su escuela las nuevas tendencias de enseñanza en Matemática Educativa. Mucho tiempo después el Dr. Filloy participó con nosotros en varios congresos de AMIUTEM y de lo cual tenemos muy Buenos recuerdos. Enhorabuena, Eugenio, en el lugar que te encuentres.

Discutiendo con Eugenio Filloy en Dakota 428, por Armando Solares Rojas (escrito en 2012)

Conocí a Eugenio Filloy en la primavera del año 2000. Las referencias que tenía de él me hacían ubicarlo como fundador del Departamento de Matemática Educativa del CINVESTAV, miembro de los comités editoriales de algunas de las revistas más importantes de investigación en el área, investigador internacionalmente famoso por sus contribuciones en didáctica del álgebra y como un personaje muy importante en las reformas educativas del país. Pero me faltaba mucho por conocer de él…

La primera reunión de trabajo que tuvimos se llevó a cabo en las oficinas de la Sociedad Mexicana de Matemática Educativa, en el número 428 de la calle de Dakota en la colonia Nápoles. Llegué puntalmente y pasamos a su oficina de reuniones. Tres sillones, una pequeña mesa, varias pinturas, una colección de posters celebrando los 300 años de la publicación de los Principios Matemáticos de la Filosofía Natural, de Isaac Newton. Hablamos poco, pero en esa corta charla me hizo una invitación a un seminario de filosofía de las matemáticas que estaba en curso, con Ignacio Garnica.

Junto con Manuel Cruz, quien era becario del Instituto de Matemáticas de la UNAM y es ahora director del Departamento de Matemáticas de la Universidad de Guanajuato, me armé de valor y dedicamos largas horas a la lectura y discusión del texto Sobre sentido y referencia de Gottlob Frege. Con muchas ideas en la cabeza nos lanzamos a las sesiones de un seminario en el que tratábamos de dar respuesta a preguntas como ¿qué son los números?, ¿a qué se refieren las expresiones que involucran números? Eugenio Filloy conducía el seminario para discutir teorías sobre el significado de los textos matemáticos y me mostró la posibilidad de trabajar, desde las matemáticas, con la epistemología, la historia y la didáctica. Ingresé al doctorado en Matemática Educativa en septiembre de ese mismo año.

Ya siendo formalmente alumno en el departamento seguí siendo parte del seminario cuyas sesiones, intensas e inolvidables, me dejaron muchos más aprendizajes de los que esperaba.

Por ejemplo, las lecturas de Wittgenstein. ¡Por fin cobraban sentido las largas horas dedicadas a finales de la licenciatura a la lectura del Tractatus logico-philosophicus! Hoy en día los juegos del lenguaje siguen rondando mis lecturas y, cada vez más, se incorporan en los textos que yo mismo escribo.

Conocí también la teoría semiótica Charles Sanders Peirce y cómo Eugenio utilizaba la noción triádica del signo como punto de partida para desarrollar los Sistemas Matemáticos de Signos, noción teórica que le permitiera describir fenómenos de la enseñanza y el aprendizaje del álgebra articulando así una nueva perspectiva de investigación propia que ha rendido frutos en numerosas investigaciones y publicaciones de la que ya ha hablado ampliamente la Dra. Rojano. Yo mismo he tenido la fortuna y el honor de colaborar en algunas de ellas a lo largo de los años de formación y trabajo con Eugenio y Tere: desde presentaciones en las Conferencias del International Group for the Psychology of Mathematics Education, hasta en la reciente publicación del artículo Problemas de dos cantidades desconocidas y dos niveles de representación de la incógnita publicado en el Journal for Research in Mathematics Education en enero del año pasado.

De la filosofía regresé a la historia. Mis lecturas previas sobre la historia del cálculo (el De quadratura curvarum, de Newton), de la física (la Optiks, también de Newton) tomaron sentido ¡también a través del contacto con Filloy!

Si bien sabía poco de la historia del desarrollo del álgebra, como todo matemático en formación, conocía la historia de la corta vida de Galois, famoso entre los jóvenes por bohemio, enamorado, anti-eclesiástico, rebelde y… genial. Pero nada sabía de los astrónomos y matemáticos indios Bhaskara y Brahmagupta, ni del matemático, astrónomo y poeta persa Omar Khayyám.

Y eso me lleva a hablar de una faceta de Eugenio que lo caracteriza y que ha tenido una influencia profunda en mí y en muchas otras personas que han estado cerca de él: su generosidad. Pienso que la historia y el legado de un hombre, de un maestro, puede describirse por los libros que atesora, pero aún más por los libros que ofrece y que comparte.

Entrar a la biblioteca de Filloy es una invitación a navegar por la inmensa pluralidad de ideas que la historia de matemáticas nos otorga. Ahí me encontré con distintas traducciones del Al-Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala (Libro conciso de calculo de restauración y oposición) y pude comparar las versiones y traducciones de Robert de Chester con la de Gerardo de Cremona y con la de Frederic Rosen. También la Aritmética de Diofanto en varias de sus versiones (Ver Eecke, P., J. Sesiano, Rashed, Tannery); el Liber Quadratorum 'El Libro de los Números cuadrados' de Leonardo de Pisa…

Su generosidad para con sus estudiantes va más allá de compartir su preciada biblioteca. Parte esencial de la actividad didáctica de Filloy es el apoyo al estudiante y la enseñanza de la autonomía y la autosuficiencia para escribir, participar en congresos, diseñar y presentar proyectos de investigación. Me recuerdo saliendo de asesoría y cargando un bulto enorme conformado por transcripciones, videos, lecciones, artículos, libros... Y horas de revisiones y discusiones sobre entrevistas video-grabadas con risas frente al televisor, ¿qué dijo?, ¿qué hizo?, ¿para dónde va? Compartir conocimiento. Enseñar. Aprender.

Recibí también su generosidad para formarme como maestro. ¿Has dado clases en secundaria?, me preguntó Eugenio. No… todavía, le contesté. Eran mediados del 2001 y para septiembre de ese año ya preparaba mis clases de matemáticas para segundo grado de secundaria en el Revueltas, la escuela con enfoque alternativo que Eugenio fundó. ¿Cómo me iba? Basta decir que salía sudando a chorros, afónico, exhausto. ¡Nada que ver con los cursos que me había tocado impartir en la Facultad de Ciencias! ¿Más difícil?, me preguntaban mis amigos. ¡Sin lugar a dudas!

Esa experiencia transformó profundamente mi manera de ver el salón de clases de matemáticas, ya no se trataba sólo de encontrar y diseñar buenos problemas y ponerlos a prueba en situación de entrevista, sino de reconocer la complejidad del trabajo del maestro, la diversidad de tensiones en las que se construye la actividad matemática del salón de clases.

Por otro lado, en Revueltas vi cómo Eugenio, investigador en matemática educativa, se topaba en los pasillos con alumnos que le decían "¡Chócalas, Eugenio!" y se subían a su espalda para que los llevara "de caballito"; con directivos que lo buscaban para decirle "maestro, llegaron las calculadoras y los view-screen"; y profesores que le pedían apoyo para sus clases "Eugenio, ¿revisamos lo que vamos a ver en tercero la siguiente semana?". Crear y sostener un laboratorio de investigación "vivo", como es una escuela, es una de las tareas más demandantes que he visto, y en Eugenio es también es una muestra de consistencia entre el discurso teórico y la práctica.

Pero la generosidad de Eugenio no se limita a compartir solamente sobre libros y matemáticas… las discusiones, los proyectos, los artículos, acompañados con una buena comida y un buen vino ¡saben aún mejor!

¿Cuáles son el sentido y la referencia de la expresión "buen vino francés"? Siguiendo las enseñanza de Eugenio, podríamos decir que su referencia se ubica en los viñedos de Montrachet, entre los pueblos de Puligny y Chanssange, en la Côte-d'Or de la región de Borgoña, en el este de Francia. O quizás en una botella de un Château d'Yquem, producido en la región de Sauternes, en Burdeos. El sentido en este caso es aún más claro, si quiere uno pensarlo así. El Sauternes se elabora con uvas sémillon, sauvignon blanc y muscadelle que afectadas por un hongo endémico de la región, quedan parcialmente pasificadas, de lo que resulta una mayor concentración de azúcar y vinos con un aroma distintivo. El Château d'Yquem se elabora, además, de manera artesanal, procesando uva por uva… a mano.

Y así como compartió lo mucho que sabe de vinos conmigo, he tenido oportunidad de estar en la cercanía de su persona, de su familia, de Maru, su esposa. Cerca no sólo de su visión de investigador sino también de su visión de vida, de lo que en México y América Latina entendemos por ser sibarita, o sea, quien disfruta de la vida en toda su vastedad.

Hoy, desde mi propia visión sobre la cercanía, quiero agradecer a Eugenio todo lo que ha compartido y comparte aún conmigo. Un gran abrazo y muchas felicitaciones, Eugenio.

Ciudad de México, noviembre de 2012.

Memorias de una de las ultimas estudiantes del Dr. Filloy, por María Leticia Rodríguez-González

El Dr. Eugenio Filloy era una persona con la habilidad de observar para conocer a cada uno de nosotros. Sabía exactamente cuáles eran nuestras fortalezas, debilidades y sobre todo sentimientos. Fue un hombre, con un extraordinario conocimiento universal, siempre tenía tema de conversación. Le encantaba la narrativa, anudando en sus historias relatos de la vida cotidiana, entremezclando personajes de la mitología griega y de la política. Filloy era un enamorado de la vida.

Le encantaba que le preguntáramos cualquier cosa, aunque algunas veces éramos nosotros sus alumnos quienes dudábamos en preguntarle. Lo más interesante, es que no nos daba la respuesta, sino qué nos proporcionaba un gran número de libros para investigarlo. Pero lo maravilloso, es que el acervo que nos compartía, eran materiales inéditos en la mayoría de los casos.

Sin embargo, a veces era muy difícil seguir la lógica de su discurso; pues aunque nosotros apenas estábamos entendiendo el principio, él ya estaba construyendo el final. Con el tiempo, aprendí que todas sus ideas tenían un sentido no sólo matemático y heurístico sino sobretodo científico.

Me atrevo a afirmar que Eugenio Filloy, fue el científico que le dio legitimidad a la matemática educativa, rompiendo con el paradigma de encontrar los problemas en la forma en que se enseñan y se aprenden las matemáticas. Él dio los argumentos para que el objeto de conocimiento de la matemática educativa se centre en las matemáticas mismas, estableciendo una clara diferencia entre Educación Matemática, Didáctica de las Matemáticas y Psicología de las Matemáticas.

Eugenio Filloy maestro visionario sabía que la construcción del conocimiento científico requiere de la interacción entre investigadores, maestros y alumnos a través de la comunicación, muestra de ello fue su participación y en algunos casos fundador de Congresos, Foros, Simposios, nacionales e internacionales como La Sociedad Matemática Mexicana, Programa Nacional de Formación de Profesores, Reunión Centroamericana y del Caribe sobre la formación de profesores e Investigación en Matemática Educativa, PME, PME-NA, CIAEM.

Gracias Dr. Filloy, aunque ya no estés físicamente, tu presencia seguirá vigente, empezando por nuestro Departamento de Matemática Educativa, el cual fundaste junto con el Dr. Carlos Imaz, Tu obra sigue presente con Tu contribución a la investigación a través de los Modelos Teóricos Locales.

Memorias de Eugenio, por Ulises Xolocotzin

Siempre agradeceré que el Dr Eugenio Filloy se diera el tiempo de saludarme y platicar conmigo cuando nos encontrábamos en nuestro Departamento. En nuestras platicas o me hacía reír, o me hacía pensar. Una vez me dijo: "Yo entré al asunto del pensamiento matemático desde el lado de las matemáticas y salí por el lado de la cognición. Tú estas entrando por el lado de la cognición, ¡Pero quién sabe por qué lado salgas!". Me puso a pensar y se despidió de mí sonriendo. Yo sigo pensando y espero que él siga sonriendo.

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REFLECTIONS ON DIGITAL TECHNOLOGIES IN MATHEMATICS EDUCATION ACROSS CULTURES

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In this plenary presentation paper, we reflect on issues related to the role, potential and extent of integration of digital technologies in mathematics education, and attempt to give our perspectives on these issues from across cultures and over time stretching over the past four decades. Three experts, from three different countries, give their reflections on how mathematics teaching and learning has changed and developed since digital technologies were introduced in schools. Their individual narratives are then complemented by a discussion of the differences in adaptation to the constant changes in technology, changes that arise from diverse socioeconomic, political and cultural visions of the role of digital technologies and their influence and on how the potential of these technologies can actually be harnessed.

Keywords: Technology; Instructional activities and practices; Culturally relevant pedagogy; Computational thinking

Introduction, by Ana Isabel Sacristán and María Trigueros

For this plenary presentation, we invited three renowned experts on the integration of digital technologies in mathematics education: Celia Hoyles from the UK, Carolyn Kieran from Canada, and Teresa Rojano from Mexico. We asked each of them to reflect on what has happened with the roles and potentials of digital technologies in mathematics education since their adoption in school systems, in order to develop a perspective of this topic across cultures and countries.

This paper presents their contributions complemented by our own reflections. The next sections present the experts' contributions. First, Hoyles presents a reflection on how mathematics teaching and learning has changed and developed since digital technologies in general and programming, in particular, were introduced into school mathematics education, specifically in England. Then Kieran, through the lens of the Canadian experience, presents a discussion of how issues related to the emergence of new technologies and the restrictions involved in their use by teachers in their classrooms created conditions that privilege some approaches; she also presents an interesting discussion on how computational thinking relates to mathematical practice and thinking. In the next section, Rojano begins by discussing some results of an Anglo-Mexican project that highlighted some differences in the use of technologies for algebra, then describes some national projects developed in Mexico, and how economic and political circumstances had an impact on their possibilities of generalization and success.

We finish by including our own reflections, based on their visions, on how digital technologies have played an important role in mathematics education and on how different cultural and economical conditions have a strong influence on how the potential of these technologies can actually be harnessed.

Reflections on the Role of Digital Technologies in Mathematics Education: insights from the past and ongoing research, by Celia Hoyles

The invitation to make this contribution at PME-NA has provoked me to look back on about 30 years of research in the area of digital technologies and mathematics education, in general and in particular around programming. My passion is and has been to help learners open windows to mathematical knowledge by using digital technologies in innovative, future-oriented and intellectually rigorous ways (Hoyles, 2018). This passion I have shared with Richard Noss with whom I have worked on numerous research projects over the years. The research questions we have posed, the technologies designed, and the research methods employed have all reflected this desire to widen access to learning through digital technology and to tease out the conditions necessary for this to happen successfully. Mathematics is central to the school curriculum, yet all too often mathematics does not engage learners who do not discern the point of the mathematics they are forced to learn. This is important at the individual level, but at the same time, the technology-based 'information society' needs model-based reasoners who can exploit mathematical ways of thinking to make sense of their world. This has become ever more apparent at the time of writing this paper when the world is enduring a global pandemic where everybody is assailed by data and graphs of the numbers infected by Covid 19. These representations and the models that generate them need to be interpreted. Thus we need to take seriously the design challenge to engage our students in school in mathematical thinking and application which as far as is possible is actually needed for them to achieve the goals that they find compelling.

Background

My work has been grounded in constructionism originating from the vision of Seymour Papert (Papert, 1980a), which asserts that one way to achieve sense-making for learners was for them to take the role (to some extent at least) of *producers* rather than *consumers* of digital tools, so they are better able to explain the *effects* of the tools (see for some example Confrey et al., 2009, p. 19). Our approach originated with a Piagetian basis but evolved to embrace "a hybrid social constructivist/sociocultural approach… with a vision of human–machine interaction and design for mathematical activity" (Monaghan, Trouche and Borwein, 2016, p. 10). In my view, fundamental to constructionism are two notions: *epistemological pluralism*, that is accepting the validity of multiple ways of knowing and thinking (following Turkle and Papert, 1992) and *designing for interaction in microworlds*.

I make a small diversion to give the reader a glimpse of the challenge Seymour Papert set us many years ago on this first issue of *epistemological pluralism*. As a member of a plenary panel at a conference in Newcastle, Australia, to commemorate the work of Jon Borwein, I was tasked to address the new ways people think, move and feel mathematically, thanks to the opportunities offered by digital technologies: the abstract went on to state:

> …the emergence of new digital technologies and new theories have helped researchers recognise the breadth and depth of that change and simultaneously provide a framework for the design and implementation of computational tools for learning mathematics. The possibility of putting mathematical objects into motion, for example, fundamentally changes the nature of these objects, how they are perceived and reasoned about; moving these objects changes the bodily actions and gestures of both learners and teachers; making the objects transform, collide and overlap, changes the stories that can be told about them. Research on the use of digital technology has also provided an extraordinary 'window' on mathematical meaning making, to use the metaphor provided by Celia Hoyles and Richard Noss, in part because of the visibility of thought, motion and feeling enabled in expressive digital technology environments. (Drijvers et al., 2016).

The example I gave at the panel was stimulated by Papert's keynote at 10th conference of PME in London way back in 1986 that I simply mention here to provoke the reader. Papert talked about what is generally known as the "alternate segment theorem" (p. 3) and argued how simple it was *if only one thought of oneself in the motion* and then "it is easy to see the total turn of a circular arc is the same as the angle at the centre of the circle"(end of second paragraph, Papert 1986, p. 3).

And what about *microworlds*? In Hoyles (1993), I described the evolution of the *microworld* idea from its genesis in the artificial intelligence community, in which it was used to describe a relatively simple and constrained domain where computational systems could solve problems, to a more broadly conceived environment that served as a concrete embodiment of a knowledge domain or structure. The structure comprises tools that are *extensible* (so tools and objects can be combined to build new ones), but also *transparent* so their workings are visible, and rich in different representations. There is a duality here: a successful microworld is both an epistemological and an emotional universe, a place where powerful mathematical ideas can be explored; but explored "'in safety"', acting as an incubator both in the sense of fostering conceptual growth, and a place where it is safe to make mistakes and show ignorance. And, centrally these days, it is a place where ideas can be effortlessly shared, remixed and improved (for an earlier discussion of these twin aspects of engaging through building and sharing, see Noss and Hoyles, 2006).

Programming and Mathematics

Now let us fast-forward to 2014 where, at least in England, programming or coding is widespread and moreover part of a *compulsory* curriculum for all students from age 6 to 16 years. It is important to recall now that teaching of computer programming is not new, with educational programming languages such as Logo and BASIC widely used in both primary and secondary education settings during the 1980s and 90s. However, Manches and Plowman (2015) highlighted that recent discussion around how to teach programming in schools often omitted the earlier research conducted during this time: some of which are reported in Hoyles and Noss (1992). We need to learn from these earlier studies. Research on the efficacy of programming had in fact produced mixed results (Clements, 1999; Voogt et al., 2015). From the point of view of learning programming per se, some of the key challenges identified were difficulties with programming syntax, dealing with error messages along with the severely limited access to technology within the classroom (Resnick et al., 2009; Lewis, 2010). Since then there have been hugely significant developments in novice programming languages that have overcome many of these challenges (though maybe raised others), and technology access has become increasingly commonplace within schools with resources readily available for sharing through the Web.

When computing became mandatory in English schools in 2014[1], the time was ripe to revisit research on the impact of programming on mathematics learning taking account of the new context and critically learning from the past that had shown the importance of design of microworlds and critical need to take account of the teacher's role. Richard Noss and I then embarked on a new research project, the ScratchMaths (SM) project, which set out to explore the potential of programming for the 9–11 primary1 age group (upper Key Stage 2 (KS2)) in light of the curriculum changes and the renewed enthusiasm and motivation for the teaching of programming in schools. The project consists of a design phase following which the intervention comprising detailed computer tools and curriculum materials is implemented over 2 years with the same pupils. Some factors that contributed to Logo and other early programming initiatives not fulfilling their potential have been mentioned earlier, but a key factor being identified by Noss and Hoyles (1996) as the importance of

[1] During the late 1990s programming faded from the English curriculum and became subsumed and eventually replaced entirely by the subject of ICT (Information and Communications Technology), which focused more on the use of technology than on its creation (Brown et al., 2014).

fostering a *sense of teacher understanding and ownership* of any programming innovation. In addition, there have been significant technological developments since this early teaching of computer programming, with a number of block-based languages such as Scratch now freely available and widely used. These environments have helped to address some of the difficulties of mastering programming syntax, but there remains the challenge of ensuring that teachers first appreciate why they are introducing programming as part of mathematics – and then have opportunities to develop appropriate skills to teach programming.

The ScratchMaths (SM) designed a 2-year intervention aiming to develop the mathematical knowledge of pupils (aged 9-11 years) through programming. The SM approach was to select and design activities around core computational ideas that would then be used as vehicles to explore specific mathematical concepts and promote mathematical reasoning. This approach enables parts of computing to be taught within or as a supplement to mathematics lessons. For a summary of the SM design research phases, activities and outcomes see (see Benton, Hoyles, Kalas & Noss, 2017, for a detailed account of the design of the study). The SM content was divided into six modules, or in terms for this paper, microworlds, three per year. In the first year for 9-10 year-old students, computational concepts (see for example, Wing, 2006) were foregrounded with mathematical ideas more implicit in microworlds titled *Tiling Patterns, Beetle Geometry* and *Collaborating Sprites*. In the second year, the same students (now 10-11 years old) were introduced to mathematical concepts and mathematical reasoning explicitly through a programming approach along with a set of new computational concepts in microworlds titled Building with *Numbers, Exploring Mathematical Relationships,* and *Coordinates* and *Geometry*. SM was intended to comprise approximately 20 hours teaching time across each of these two school years.

Given the challenge of implementing a new curriculum, the SM teachers were provided detailed guidance for navigation through support materials, which were themselves carefully structured and progressive. Prior to teaching each year of the SM intervention, teachers were offered two full days of professional development, spaced a few months apart. During these sessions, the teachers were introduced to Scratch and the SM curriculum content. The SM design was framed by constructionist theory whereby pupils would engage with the mathematical ideas by building programs to explore so the PD followed this approach (see Noss & Hoyles, 2017, 2019). It was then intended that this constructionist approach would be operationalized in the classroom through what we called the '5Es pedagogical framework' with teachers given the opportunity to participate in activities that incorporated exemplars of the different pedagogical strategies. The five unordered constructs of the framework are summarised below:

Explore: Pupils should have opportunities to explore different ways of dealing with constraints and ambiguity as well as investigating their own and others' ideas and debugging different types of errors.

Explain: Pupils should have opportunities to explain their own ideas as well as answer and discuss reflective questions from the teacher and peers.

Envisage: Pupils should predict outcomes of their own and others' programs with specific goals *prior* to testing out on the computer.

Exchange: Pupils should have opportunities to share and build on others' ideas as well as justify their own solutions.

bridgE: Pupils should be helped to make links between contexts beyond the Scratch programming environment by explicit re-contextualization and reconstruction within the language of mathematics, by for example unplugged activities.

The SM intervention was subject to cycles of iterative design research following which it was scaled out across England. The project was also evaluated through a randomised control trial conducted by another university (Boylan et al., 2018[2]). For the trial, 111 English primary schools (6300 pupils) were recruited and randomly assigned to control and treatment groups with the final quantitative outcome measures based on scores in first a test of computational thinking (designed by the evaluation team) to be applied in the first year of the evaluation, and second, scores on national standardised mathematics tests taken by all students at the end of primary school.

For completeness, I summarise the findings of the external evaluation of SM, which reported:

- a positive and significant impact on Y5 Computational Thinking skills, which was
  - particularly evident among disadvantaged pupils that is those who had or currently have free school meals
  - with no difference between girls and boys
- no evidence of impact on the national Key Stage 2 Maths test

Clearly what is interesting in these outcomes for researchers is to seek to explain the reasons for these outcomes. To be honest we are not sure and welcome research replications and adaptations that are underway, which throw light on these issues. I note for example the nationwide large scale study in Spain (INTEF, n.d.), which reported the following: "…the results show that it is possible to include programming activities in 5th grade in the area of mathematics, so that students not only learn to program and engage in computational thinking, but also improve the development of their mathematical competence greater than their colleagues who have worked in this same area using other types of activities and resources not related to programming."

Here I simply mention what I see as important contextual influences in England that might well have shaped the outcomes. First, why the significant positive effect of the SM intervention on CT scores as measured by the test used at the end of the first year of the trial. We cannot be certain, but simply point to the fact that the SM package is a systematic, progressive research-based curriculum that offers detailed support to teachers. The independent evaluators also remarked that SM was popular among the teachers who sustained their participation. Our surveys told us that fidelity to SM in terms of engagement in professional development, provision of SM curriculum time and coverage was very high in Year 5.

However, there was a dramatic drop in this fidelity in Year 6, along with huge variation in pedagogy. This was, we conjecture, a result of the negative impact of the high-stakes testing in mathematics in England at the end of Year 6. There is massive pressure on teachers who therefore might feel unable to engage with a new curriculum. In addition, our data indicated that many SM Y6 classes were taught by *teachers with little or no experience of SM,* either from teaching SM in Y5, or from engaging in the SM professional development (for more detail see Noss et al., in press). This was a matter of fidelity but also of the considerable teacher churn in schools – the average for the staying in the profession is 4-5 years. However, I would go so far to say that where professional learning was taken seriously by schools, as in the first year of the innovation, implementation tended to be successful. And, conversely without this it is very unlikely that an innovation like SM would operate in the classroom as planned.

Celia Hoyles' final remarks

I end by calling for more intensive and systematic classroom research and evaluation to explore the classroom implementation of SM or other computing initiatives, not least as computing is now embedded in school practice, and learners and teachers are more confident and competent in

[2] The evaluation report can be found at https://educationendowmentfoundation.org.uk/projects-and-evaluation/projects/scratch-maths/
The student and teacher materials are freely available from the UCL website http://www.ucl.ac.uk/scratchmaths (creative commons license)

programming. Such research would also need to reconsider methods of evaluation of research outcomes in terms of teacher and pupil learning.

Digital Technologies in Canadian Mathematics Education, by Carolyn Kieran

My contribution to the cross-cultural digital-technologies panel focuses on three aspects: i) how the digital-technology culture has evolved in the Canadian school system since the 1980s when the Logo movement began, ii) brief comments on the characteristics of computational thinking and how they relate to mathematical practice and mathematical thinking, and iii) Canadian research that illustrates the use of digital technologies for fostering mathematical thinking.

Evolution of the digital-technology culture in Canadian schools since the 1980s

When the Logo movement began in the 1980s, a corps of enthusiastic Canadian mathematics educators and teachers adopted MIT professor Seymour Papert's vision of having children use computers as tools to think with. The Logo programming language was at the heart of this movement. The mathematical connections associated with the Logo movement spilled off the pages of Papert's (1980a) book, *Mindstorms: Children, computers, and powerful ideas*. Mathematics education researchers across Canada, as well as in other countries, developed projects involving primary, secondary, and tertiary level students, which were aimed at exploring mathematical ideas in turtle geometry by programming with Logo.

However, the Logo movement in the public schools was hampered by a lack of funds – in other words, the absence of political will on the part of government to invest massively in any such endeavours – that would have allowed for obtaining the appropriate hardware and software. The promising results that were being highlighted in research reports and in presentations at various Logo conferences throughout the 1980s and early 1990s would unfortunately not lead to more widespread implementation in Canadian schools. By the end of the 1990s, many had let go of their interest in Logo. Handheld calculators with graphing capability, as well as Computer Algebra System (CAS) calculators, proved to be easier (and more economically feasible) to integrate into mathematics classes than the more expensive computers necessary for Logo.

Some time passed. Then by the end of the first decade of the new millennium, the situation with respect to digital technologies had begun to change in many ways. In 2006, Jeannette Wing, herself a computer scientist, wrote a brief paper titled *Computational thinking* where she argued that computer science was more than just programming; it also involved ways of thinking (Wing, 2006). The effect of this paper was probably more influential than Wing had expected or even dreamed. In contrast to what had not happened during the Logo movement, the idea of computational thinking began to have an effect on K-12 education. As in other countries, Canadian policy makers and curriculum leaders decided that our students needed to develop their technology skills within the school system.

Education in Canada is not a federal matter; there is no national curriculum. Every province is responsible for setting its own school programs. So far, five provinces have begun to develop new technology programs. The pioneers in this movement have been British Columbia, New Brunswick, Nova Scotia, Quebec, and most recently (in 2020) Ontario. But, in the main, these new programs are stand-alone technology programs – primarily for grades 6-8, with an emphasis on a range of technological tools, processes, and applications, especially coding. In contrast, Ontario opted to integrate coding within its Grades 1-8 mathematics curriculum.

With the exception of Ontario, one wonders if this new interest in computational activity will make its way into mathematics classes? Any optimism one could have might be tempered by the results of a recent survey of teachers of 5- to 14-year-olds in 23 countries around the world, carried out by Rich and colleagues (2019): While the teachers noted that their students loved the new coding activities, they also stated that they were more confident in their ability to teach students coding/computing as a stand-alone subject than they were with integrating it into other subjects. This finding suggests that, along with current curricular changes, an equally important component that cannot be neglected is the need for an ongoing form of professional development to allow teachers of mathematics to keep up with constant changes in digital technologies and to feel confident in their ability to integrate these technologies into the exploration and development of students' mathematical thinking.

Computational thinking, mathematical practice, and mathematical thinking

Papert (1980a) emphasized that when children use computers as tools to think with, they are also "talking mathematics" (p. 6) to these computers. It is upon his shoulders that the present computational thinking movement stands. So it seems appropriate to ask at this point how computational thinking has been characterized in this recent movement. Wing (2006), for example, states that she has drawn on ideas fundamental to computer science and asserts that "computational thinking is using heuristic reasoning to discover a solution" (pp. 33-34). And in a later paper, Wing (2014) writes: "The most important and high-level thought process in computational thinking is the abstraction process" (p. 1). Relevant to the questions addressed to this panel, she adds that computational thinking can be defined as "the thought processes involved in formulating a problem and expressing (with a linguistic representation) its solution in such a way that a computer – human or machine – can effectively carry it out" (p. 1). Interestingly, Andy diSessa (2018) – one of the two authors of *Turtle Geometry* back in 1981 – has taken issue with this point and has argued that non-computer scientists rarely map out exactly how a problem can be solved before actually doing the solving. But is he right?

In opposition to diSessa, and more in line with Wing, Al Cuoco (2018) in a paper on mathematical practice offers three examples. The first of these (see Fig. 1) relates to Wing's emphasis on the process of abstraction and her point about formulating a problem and expressing its solution in a way that a computing being or machine can carry it out.

- I take a guess, say 1200 miles.
- Check it:
  (∗) $\frac{1200}{60} = 20$
  (∗) $\frac{1200}{50} = 24$
  (∗) $20 + 24 \neq 36$
- That wasn't right, but that's okay— just keep track of your steps.
- Take another guess, say 1000, and check it, making the check more efficient:
$$\frac{1000}{60} + \frac{1000}{50} \stackrel{?}{=} 36$$
- Keep it up, until you get a "guess checker"
$$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} \stackrel{?}{=} 36$$
- The equation is
$$\frac{x}{60} + \frac{x}{50} = 36$$

Figure 1. Arriving at an equation from abstracting the regularity in numerical guesses (Cuoco, 2018, p. 3)

This example involves what Cuoco refers to as "the dreaded algebra word problem," where he insists that we think of the *answer to the algebra problem as an equation* rather than a number – in a method that involves abstracting from numerals. The problem is as follows: *"Mary drives from Boston to Chicago, travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. The total driving time takes 36 hours, how far is Boston from Chicago?"*

The method that Cuoco suggests builds upon students' ability to solve similar problems in middle school (note: they have already learned the relationship between speed, time, and distance) and is as follows: *Take a guess* – but the aim is not intended to get closer to the answer with each succeeding guess; rather it is to arrive at an equation, not a number. The idea is to carry out enough guesses so as to see the regularity of the calculations that allow for checking the guesses – in Cuoco's words: Develop "a generic 'guess checker' that is the desired equation". The processes of mathematical practice that are employed here, and which are ones that Cuoco declares he uses all the time in his own mathematical work, are:

1. Abstract regularity from repeated calculations, and
2. Use precise language (and algebraic symbolism) to give a generic and general description – the equation – for how you check your guesses. (Cuoco, 2018, p. 4)

The conclusion to be drawn from this example is that these two processes of mathematical practice fit well with the programming and thinking-like-a-programmer characteristics of computational thinking (Wing, 2006, 2014), and that students who are currently engaged in using digital technologies (e.g., laptops, robots) to code with visual (e.g., *Scratch*) or text-based languages are participating in mathematical practices. Nevertheless, other research (e.g., Bråting & Kilhamn, 2020) suggests that, while the representations used in programming languages may be similar to mathematical notations, the meanings of several concepts in the two domains differ. But that is a whole other story! In any case, digital technologies afford multiple varieties of mathematical activity that can offer experiences that involve coding but also those that do not.

Some Canadian research on the use of digital technologies to foster mathematical thinking

I take *mathematical thinking* to include the various processes that have been drawn upon by Wing and others to characterize aspects of computational thinking – but also more than this, for example, its conceptual aspects. While computational thinking is focused toward coding, mathematical thinking occurs within a host of activities that are not coding oriented, but which can clearly be engaged in within specifically-designed digital environments. However, the tricky thing about terms such as *computational thinking* and *mathematical thinking* is their overlap when referring to anything mathematical. Moreover, as Cuoco (2018, p. 2) has pointed out: "In real mathematical practice, it is rare that a piece of work employs only one aspect of mathematical thinking" – and, similarly, only one aspect of computational thinking. Despite the obvious intersection between the two terms, I find it helpful when discussing the use of digital technologies in mathematical activity to distinguish between *coding-related activity* and *non-coding-related activity.* In line with this distinction, I offer some examples that give a flavour of Canadian research that has focused on these two types of activity, both of which have successfully combined selected aspects of computational thinking and of mathematical thinking.

Digital Technologies in Coding-Related Mathematical Activity

***Scratch* coding on laptops.** My first example is drawn from the funded, multi-study research project of George Gadanidis and colleagues from across Canada, titled *Computational Thinking in Mathematics Education* – a project aimed at researching the use of computational thinking (via, e.g., digital tangibles such as circuits, programmable robots, and coding with *Scratch* on laptops) in mathematics education, from pre-school to undergraduate mathematics, and in mathematics teacher education (see ctmath.ca/about). In one of the publications from this project (Gadanidis et al., 2017), the initial activity engaged in by the Grade 1 students of a school in Ontario was the use of the block-based, visual programming language, *Scratch* (available at http://scratch.mit.edu), for exploring squares by drawing a set of squares rotated around a point (see Fig. 2; see also Gadanidis, 2015). One of the fundamental principles underpinning these study projects is connecting the digital technology work in classrooms to the math curriculum that teachers need to teach.

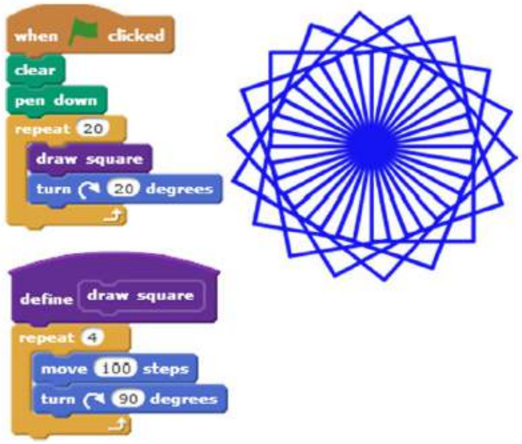

Figure 2: *Scratch* coding in Grade 1 (from Gadanidis et al., 2017, p. 81)

Figure 3. Programming a robot using loops (from Francis & Davis, 2018, p. 82)

Coding robots. Francis and Davis (2018) studied 9- and 10-year-olds' understanding of number, and the transition from additive to multiplicative thinking, in the context of learning to build and program Lego Mindstorms EV3 robots. The sequence of tasks focused on students' becoming aware of the architecture of robots, programming the robots to trace a triangle, square, pentagon, or hexagon; and building a robot that could find and douse a 'fire' in any of four rooms of a miniature model building. In one of the scenarios that Francis and Davis report on, a student learns how the number of sides and angles of a polygon connects to the number of repeats in a loop, which illustrates a developing shift from thinking additively in terms of a sequence of like actions to thinking multiplicatively in terms of a repetition of a single action (see Fig. 3). The authors argue that coding-related activity with digital technologies can co-amplify mathematics learning, as long as computer programming is seen as "something for" and is integrated into the existing curriculum with well-designed tasks, not as "something more" in a separate curriculum.

Digital Technologies in Non-Coding-Related Mathematical Activity

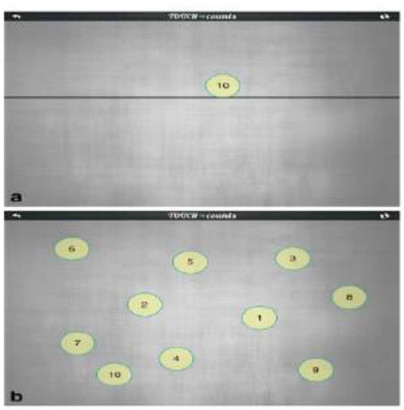

Figure 4. *TouchCounts* App: upper -- 10th tap; lower -- result of 10 single taps (Rodney, 2019, p. 169)

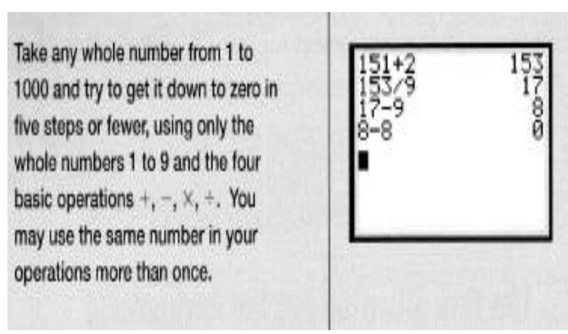

Figure 5. "Five Steps to Zero," with a starting number of 151 (adapted from Williams & Stephens, 1992)

***TouchCounts* – an iPad touchscreen App.** The *TouchCounts* application software, developed by Sinclair and Jackiw (2014), served as a window for the researcher Rodney (2019) to study how a 5-and-a-half-year-old, Auden, thought about number. Although Auden was able to say the number names initially, he seemed unaware that the written numeral '10' would appear right after '9' and that '10' also represented the number of taps made on the iPad screen (see Fig. 4). Auden's unsuccessful initial activity with the App revealed that his memorized number chanting needed the further support that *TouchCounts* could afford in order to reach a fuller understanding of counting and to begin to identify the relational aspect of numbers.

Calculators with multi-line screens. Calculators remain a staple in many mathematics classes. This resource, one with a multi-line screen, served as the digital tool underpinning a study that focused on the mathematical practice of seeking, using, and expressing structure in numbers and numerical operations (Kieran, 2018). The study (co-conducted with José Guzman†) involved classes of 12-year-old Mexican students on tasks adapted from the "Five Steps to Zero" problem (Williams & Stephens, 1992; see Fig. 5). Successfully tackling the designed tasks, and subject to the rules of the game, involved developing techniques for reformulating numbers (prime or composite) into other numbers in the same neighbourhood (not more than 9 away from the given number) that have divisors not larger than 9 so as to reach zero in five or fewer steps. Some of the most powerful structural explorations that occurred during the week of activity on the tasks involved the search for multiples of 9. For example, students became aware that "738 and 729 are two adjacent multiples of 9 and, when they are both divided by 9, the quotients are consecutive," and "in the 9-number interval from 735 to 743 inclusive, there is exactly one number divisible by 9." In trying to explain the often-surprising results produced by their digital tools, the students developed several mathematical insights that were new to them.

Carolyn Kieran's concluding remarks

My concluding remarks pick up on the interest shown by students in the use of digital technologies – be they coding-related or not. For example, Gadanidis et al. (2017) emphasize "learning experiences that offer the pleasure of mathematical surprise and insight" (p. 80). Similarly, Sinclair, Healy, and Noss (2015) speak of the "sense of delight" offered by digital technologies, but also of the need for a certain degree of "intellectual travel" (p. 2). In this latter regard, an early Logo study by Idit Harel (1990) is exemplary. Her 4th graders took up the challenge of designing and programming fractional representations that they thought would be helpful for younger children. This project led to significant gains in their understanding of both fractions and programming. As Harel points out: "the children's involvement in a rich, meaningful, and complex task, designing and programming a 'real' product for 'real' people, enhanced their understanding of Logo and their knowledge of how to use it" (p. 30). Clearly, embedding computational thinking into disciplinary contexts can be most productive and yields a strong lesson for policy-makers who advocate for stand-alone, coding programs in school.

The Use of DT in Mathematics Education: Experiences from Anglo-Mexican Collaborative Research And Implementation Programs in Mexico, by Teresa Rojano

Potential of the use of DT in the teaching and learning of algebra

There is abundant literature on research pertaining to the use of digital technologies for teaching and learning algebra, in which it can be observed a distinction between two types of tools, those developed expressly for this mathematical domain, such as *Computer Intensive Algebra*, *Cabri-Géomètre, Geometer Sketchpad*, *SimCalc*, and those that have been adapted for educational use, such as *Computer Algebra Systems* (CAS) and *Spreadsheets* (Sutherland & Rojano, 2014). Studies carried out with both types of programs have shown their strengths and limitations (Olive et al., 2010), and have highlighted the relevance of task or activity design in order to use them to create significant teaching environments (Donevska-Todorova et al, in press). A common denominator among many of these digital technologies, which is also one of its main potential strengths, is the connection between different representations, which allows for avenues to be created, where students can approach notions and learn novel and powerful algebraic methods, in an intuitive way (Zbeik & Heid, 2011).

Spreadsheets fall under the latter case, as the interconnection between the representation of numerical tables, the graphic representation and the use of algebra-like formulas, allows for the possibility of shifting between numerical and quasi-algebraic treatments, both for concepts like variable, unknown and function, and for word problem solving methods. Here I will specifically refer to my direct experience recreating intrinsic concepts and processes of algebraic thought, in this environment.

Spreadsheets and algebraic thinking. Outcomes from the *Anglo-Mexican Spreadsheets Algebra Project* (Sutherland & Rojano, 1993) carried out during the 90s, showed that students of different ages and school levels can work with algebraic ideas, taking a numerical approach, and using formulas whose syntax is similar to that of algebra. One of the studies was undertaken with two groups of students 10 to 11 years of age, one group in Mexico and the other in the UK, who worked with spreadsheet activities focused on the notions of function and inverse function, as well as on equivalent algebraic expressions and the solution of algebra word problems (Rojano & Sutherland, 1994). Figures 6 and 7 show examples of the type of worksheets used in the study.

The results of the pre-questionnaire applied to the students from both groups, before the experimental work, showed that the majority of the pupils did not think spontaneously in terms of a general object. Initially their mode of thought was on specific cases (for instance, on a particular line of a variation table) and the activities with spreadsheets helped them to move from focusing on particular cases to considering a general relation (see the worksheet in Figure 6). What is more, the sequence of activities on functional relations, the sequence on solving word problems helped the students to accept the idea of working with unknowns, to represent relations among data and the unknown of a problem, and to vary the numerical value of the unknown until they found the solution to the problem (see the worksheet in Figure 7). The experience with such activities enabled them to go from intuitive strategies for solving problems (such as trial and error, for example) to strategies in which they systematized their trials (trial and refinement) to finally encapsulate the relations between unknown and data in spreadsheet general formulas (Rojano & Sutherland, 1994; Sutherland & Rojano, 1993).

Function and Inverse Function

A) Copy the rule of the function that appears in the figure. Find the value of the function, when x = *30* ____________.

B) In B4, change the value of x to 100. Find the value of the function, for x = *115* ____________.

C) Make the x values vary in steps of 2. Copy down the function rule, when x is between *1* and *200*. Find the value of the function, when x = *150* ____________.

D) Try to vary the x values in steps of *0.5*.

	A	B	C
1			
2		x	Function of x
3			
4		1	6
5		2	7
6		3	8
7		4	9
8		5	10
9		6	11
10		7	12
11		8	13
12		9	14
13		10	15
14		11	16
15		12	17

Figure 6. Example of one of the worksheets provided for the teaching of function and inverse function.

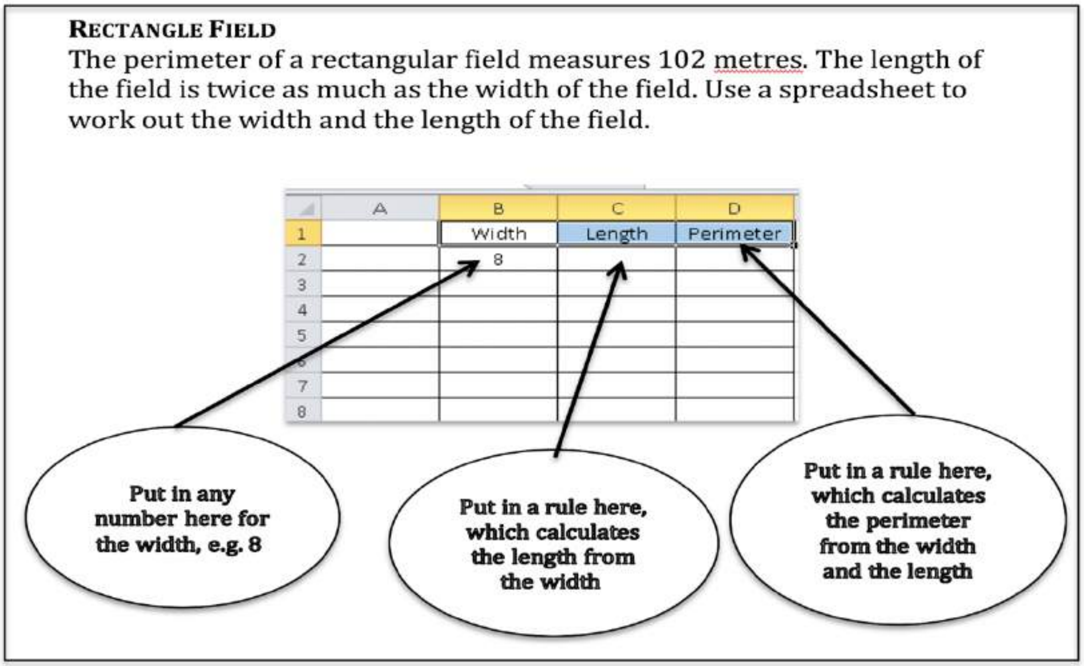

Figure 7. Rectangular field problem (Rojano & Sutherland, 1993).

In general, it is worth noting that despite the different experience in school mathematics that British and Mexican students could have had, given the differences in the mathematics teaching approaches in Mexico and the UK, there were no significant differences in the performance of the Mexican and British students, neither in the results of the pre-questionnaire nor during the work with the spreadsheets. The latter may be attributed to the connection that the students could have made between their own notions and intuitive strategies and the algebraic notions of function and unknowns, thanks to the combination of a numerical approach and the use of spreadsheet formulas, characteristic of the didactic design of the activities. In a second phase, the Anglo-Mexican project focused on working with 14 to 15 year-old students from both countries, students with a history of school failure in mathematics and who had already been introduced to the study of symbolic algebra. The results of the pre-questionnaire revealed that participating students were able to solve the problems using intuitive strategies and in some cases those strategies led them to the right solution. During the experimental work of this phase, activities with spreadsheets were used that were very similar to those of the study involving 10-11 year olds, in the end achieving the same effect –a connection between intuitive and non-formal approaches of the students, and notions and methods on the path toward algebraic thinking (Rojano & Sutherland, 1994).

Although spreadsheets were not developed for educational purposes, both the study undertaken with pre-algebraic students (10 -11 year-olds) and the study involving algebra resistant secondary school pupils show that use of that program accompanied by worksheets with an appropriate didactic design has great potential as a digital learning environment for exploring algebraic ideas and concepts.

The surprising work that participating students carried out in those studies did not prevent us from identifying the limitations of the environment that we used, which relate to what Zbiek et al. (2007) call 'mathematical fidelity'. In my interpretation, the distance between the syntax of Spreadsheet formulas and algebraic syntax may be a hallmark of weak mathematical fidelity; in the former case, formulas allow for representation and manipulation of generalizations, but they cannot be transformed with internal rules from that system of signs; while in the latter case, on the one hand, the analytical expressions of functions can be analyzed and transformed, under the rules of algebraic syntax so as to delve into the variation phenomena that they represent and, on the other, in advanced mathematics courses, those analytical representations can be treated as entities of more abstract mathematical structures. This continuity through the different educational levels is absent in the Spreadsheet environment. For its part, the spreadsheet method for solving word problems is mathematically and didactically pertinent to some families of problems, but it is far from the Cartesian method of solving problems, which is a general method and consists of getting the situation described in the text of the problem 'put into an equation'. In summary, finding the didactic connection between the versions of notions and methods used in the digital technologies and the 'paper and pencil' versions used in school mathematics is still a significant challenge for teachers, trainers and curriculum designers.

In addition to understanding its potential and limitations, one should recognized that, together with the first dynamic geometry programs, the adaptation of Spreadsheets in mathematics education can be considered as part of the background to open source software Geogebra, which indisputable widespread use has more recently led to a large number of reports of practical experiences and research that use this package and that, among many other things, highlight the didactic feature of being able to create connections between algebra and Euclidean, Cartesian and Analytical geometries.

Research experiences: Windows of mathematics school culture

From our first forays into collaborative research on technology and algebraic thinking in the late 1980s, Rosamund Sutherland and I found differences between the educational systems of Mexico and the United Kingdom, some of which appeared specifically in the presentation of the topics of algebra in the curriculum, as well as in the diverse various educational material to be used in class. Both the differences and the commonalities permeated the design of the tasks used in the studies we undertook. However, observation and analysis of the ways in which each of the two groups solved the same task served as a window that allowed us to glimpse the distinctive features of the students' mathematical practices, where multiple representations of the same situation or phenomenon play a central role.

Spreadsheets as a mathematical modeling tool. More recent versions of spreadsheets offer a suitable environment for mathematical modeling tasks using (hot-linked) graphical, symbolic and numeric representations of phenomena of the physical world. The activities in this environment correspond to a parameterized version of the behavior of the modeled phenomena, and knowledge of advanced mathematics is not necessary in order to explore and build the models. In the Mexican-British project, *The role of spreadsheets within the school-based mathematical practices*[3], research was carried out with two groups (one in Mexico and another in the UK) of pupils of 16-18 years of age, in which it was observed how the differences of school culture experienced by each of the groups influences both their mathematics practices and their mathematics modeling activities using spreadsheets (Molyneux, Rojano, et al., 1999).

In the pilot phase, important differences were observed in the preferences of students for certain external representations of situations and phenomena. For example, when answering questions about the long-term behavior of the phenomenon, UK students showed a clear preference for graphic representation, while students in Mexico were inclined to use algebraic representation. However, even though work with Spreadsheets, in the experimental phase, did not significantly change those preferences, in the end the students recognized the value of having a varied repertoire of representations. In fact, one of the relevant results of this research was that participants developed the

[3] The Anglo-Mexican Project was developed as a collaboration between the Institute of Education of the University of London and the Department of Mathematics Education of the Centre for Research and Advanced Studies (Cinvestav) in Mexico, funded by the Spencer Foundation of Chicago, Ill, Grant No. B-1493.

ability to move smoothly from one representation to another and to realize the advantages afforded by some of them in answering certain types of questions posed about the behavior of the modeled phenomena (Rojano & García Campos, 2017).

Notwithstanding the fact that the main purpose of our research was not to study or compare school culture contexts, when analyzing the data collected in the two countries differences emerged that could not be explained, other than on account of what was emphasized or valued in the mathematics classroom, that is, it could be said that the differences originated in the school mathematics culture.

Research and Practice. The use of DT in the Mexican Educational System

In Mexico, as in many other countries, it has not been easy to bridge the results of research on the role of DT in the teaching of mathematics and incorporating their use in the educational system. We have gone through government programs ranging from the use of TV (in the 1960s) to broadcast live or pre-recorded classes, to the *Teaching Mathematics with Technology* (EMAT), *Enciclomedia* and the *New Model for the Telesecondary System.* The EMAT project was conceived specifically for the subject of mathematics in middle school, based on results from research in mathematics education. An international team of researchers designed a constructivist pedagogical model and student-centered activities, with an exploratory approach that encourages bottom-up practices, rather than traditional top-down practices. The tools used were Spreadsheets, Cabri-Géomètre, the TI-92 Algebraic Calculator and Logo, and a gradual implementation was planned starting in 1997, that would expand the use of different tools in different states of the country. Despite the fact that said implementation was not carried out as planned, some teachers who participated in the initial stages have continued to work for several years with EMAT activities, managing to integrate use of the tool repertoire into their own long projects (Trouche et al., 2013).

The experience from the EMAT project was used in the design and implementation of the New Model for Telesecundaria (Lower secondary system of rural areas without access to regular schools), which main feature is the articulation of printed, video and digital interactive resources, which are still in use. It is worth mentioning that the results of a study carried out by the Ministry of Education revealed that since the New Telesecundaria Model was launched, this system, compared to that of regular secondary schools, showed more sustained progress in terms of improved student scores on the yearly national mathematics and language exams (SEP, 2016).

Other national programs were suspended shortly after they were started, however, together with EMAT, Enciclomedia and the New Model for the Telesecundaria, the infrastructure and diverse experiences left by the programs throughout the country have been used by some teachers who have adapted the activities and the use of different tools to the curricular changes derived from the educational reforms. Whereas other teachers have limited the use of technology to displaying Powerpoint material, video material and YouTube, as it has been recently documented by Salinas, Sacristán and Trouche (2018). Thus, the use of DT to fundamentally transform mathematical practices at school and outside of school continues to be a great challenge in this country.

Reflections on the integration of digital technologies for mathematics education, by Ana Isabel Sacristán and María Trigueros

Taking into account the previous authors' reflections on the way digital technology culture has evolved in different countries' curricular approaches, teachers' professional development and students' participation and learning, we present some additional thoughts and discuss also observed contrasts and what happens in societies with different socioeconomic and cultural backgrounds. We also consider to what extent the potentialities of digital technologies have been harnessed to enhance mathematics learning and to engage students.

The previous authors have given us a panorama of the digital technologies that have been used in mathematics education. To summarize these, we refer to the USA's NCTM Position Statement which states that, in addition to content-neutral technologies such as tools for communication and collaboration and Web-based digital media, mathematics' content-specific technologies that can support students in exploring and identifying mathematical concepts and relationships, "include computer algebra systems; dynamic geometry environments; interactive applets; handheld computation, data collection, and analysis devices; and computer-based applications" (NCTM, 2015). To that we can add other expressive technologies such as computer programming environments, as well as eBooks.

On the potentials and historical evolution of digital technologies for mathematical learning

As Hoyles discussed above, it is clear that today's technology-based society needs to develop students' abilities not only in the use of technology, but as reasoners to exploit their use. The question is how digital technologies can be integrated and exploited in schools for achieving that and enhancing mathematical learning.

As many researchers in mathematics education, the panellists and ourselves believe (and a wide corpus of research has demonstrated) that digital technologies have the potential to change education, the teaching and learning of mathematics (e.g., by opening windows to mathematical knowledge, as Hoyles suggested), the curriculum, and also rethink mathematics as a field. Also, as Hoyles also mentions, using digital technologies in innovative ways could potentially widen access to learning.

Both Hoyles and Kieran cite Seymour Papert's vision of having students use computers as tools to think with and of how mathematical understandings can be changed by the use of digital technologies. Papert's (1980a, 1971/1980b) vision was for students using and programming computers for *doing* mathematics, rather than learning *about* mathematics; it is interesting that more recently, Chevallard (2015) also called for students to do mathematics, rather than visit them as if looking at a monument.

As explained by Hoyles above, programming allows users to interact with and visualize mathematics in new ways; it also can be used together with other technological tools so that it is possible to construct relations among concepts that so far have been considered as different, avoiding compartmentalization of knowledge.

In any case, an important potential of technology is for *empowering* students, be it through computer programming or through providing mediums for them to express themselves as well as mathematical ideas, and to build their own products (which is why Papert, 1991, called this *constructionism*) – in Hoyles' words, where students are producers, rather than consumers, of technology. And as Hoyles also emphasizes, Papert's vision had two central ideas: epistemological pluralism and microworlds. Hoyles discusses epistemological pluralism above, but we could add that when one has multiple means and representations to engage, interact with, and express mathematical ideas, those ideas become less abstract (Wilensky, 1991) and can more easily be appropriated. As discussed by Hoyles, computer programming, in particular, helps students to interact with mathematical objects in different ways, enabling students to visualize and reflect on the factors involved in different types of interactions.

The other important idea is that of microworlds, which Hoyles above described as a place where powerful mathematical ideas can be explored 'in safety'.

In the advent of the digital age, the Logo programming environment provided an ideal medium and microworld for students to engage in this epistemological pluralism and develop mathematical thinking, through computer programming and computational thinking. As touched upon by Hoyles and Kieran, following the publication of Papert's 1980's book *Mindstorms,* Logo had a widespread influence on the use of digital technologies in (mathematics) education. It was thus that Logo was implemented in countries around the world during the 1980s (including Mexico); and it was a crucial turning point in countries such as the UK and Canada, as well as the USA.

And yet, for a couple of decades starting in the 1990s, there was a generalized abandonment of Logo and Papert's vision, which happened in most countries. Kieran mentions a lack of funds for Logo projects losing ground in Canada, but the general reasons for that abandonment around the world are much more complex, and involve a confrontation with what Sacristán (2017) calls the 'inertia of school cultures'. Agalianos, Noss & Whitty (2001; see also Agalianos, Whitty & Noss, 2006) explain this complexity by pointing to how technologies and their use in the classroom are socially contextualized and often appropriated in ways unanticipated by their developers – something that is still true today:

> To work with Logo in the way its developers had envisaged meant that teachers should make a fundamental shift in their relationship with pupils….This, however, was a lesson not learnt for many teachers who felt uncomfortable […] Logo (...) automatically constituted a disruption of the classroom's traditional social organisation. [...]
> Logo's introduction into mainstream US and UK schools in 1980 marked the beginning of **a struggle to integrate new forms of teaching and learning into old educational structures**. (p. 486-487, our emphasis).

On his part, Ruthven (2008; see also Ruthven, 2014) points to the conjunction of several influential factors that are needed for successfully uptaking an innovation in school mathematics, and discusses how difficulties related to each of those factors acted against the continuation of the use of Logo in schools (and, we argue, in general, of more innovative visions, including Papert's). Those factors are:

- *Disciplinary congruence* with an influential contemporary trend in scholarly mathematics.
- *External currency* in wider mathematical practice beyond the school.
- *Adoptive facility* in terms of ease of incorporation into existing classroom practice.
- *Educational advantage* through perceived benefits of use considerably outweighing costs and concerns. (Ruthven, 2008)

In terms of adoptive facility, in the 1990s other developments, such as graphing calculators, CAS, Excel, dynamic geometry, etc. became popular and, as Kieran points out, seemed to offer teachers a more direct, less costly and perhaps easier to adopt, use of technology for mathematics teaching (and learning?). Many producers of such technologies also pushed newer software and hardware as solutions for the introduction of digital technologies in classrooms, but there is a difference between having a technology and using it for meaningful learning: we argue that, very often, schools became consumers of commercial developments, due to political and administrative decisions, failing to integrate those technologies in ways that would improve learning.

In any case, due to those complex reasons, in the 1990s and early 21st century the use of programming, and its relation to the development of mathematical thinking, was abandoned. During that period, in some countries, such as Mexico, there were efforts to return to something closer to Papert's vision through the development of the EMAT national project presented by Rojano, above. That project highlighted the importance of design and showed a way to use popular software, such as Excel, as in the example given by Rojano, in a way that could help in making the transition from arithmetic to algebra smoother for students. However, the EMAT project, though successfully tested, succumbed to political changes (Trouche et al., 2013), as we discuss further below; although, as Rojano points out, some teachers have continued using the EMAT technologies and materials in their classrooms until today, to promote their students' learning.

It is only in the last decade that interest and the recognition of the importance of computer programming, and computational thinking, has resurfaced in developed countries (but, as discussed below, not so much in other countries). Computational thinking, popularized by Wing (2006), and 'coding' have now become educational trends. How programming and computational thinking relate to mathematical thinking had already been discussed by Papert (1980a) and others in the 1980s and 90s. But Kieran above gives a profound discussion of their relationship. She agrees that computational thinking and mathematical thinking overlap, and engaging in programming and computational thinking implies participating in some mathematical practices. She argues, however, that mathematical thinking is more than what is achieved through coding and computational thinking but that other specifically-designed digital environments can also be helpful in promoting it.

Digital technologies in mathematics curricula across cultures

In any case, because of the attention that coding and programming have received in the past decade, many developed countries around the world have integrated these into school curricula.

In the UK, Hoyles mentions above how programming has come to the forefront since 2014; and Kieran describes experiences in Canada where programming has also been included in the curriculum. The question is how this integration affects the mathematics curriculum. Hoyles describes how she and her colleagues in the UK developed ScratchMaths in a research attempt to integrate the computer science curriculum with mathematics. It is interesting that ScratchMaths has been adopted in other countries outside the UK, such as Australia (Holmes et al, 2018), to use coding to teach mathematics.

In Canada, Kieran describes how programming activities have been introduced early in school but are used together with other technological tools to foster opportunities to construct relations among mathematical concepts.

However, in other countries, programming and computational thinking have not been included in the curriculum, nor sometimes even considered. In fact, there is thus a huge gap in the recognition of the possibilities that the use of digital tools offer to mathematics students and teachers and how the use of technology has evolved in countries and cultures around the world. The political strategies vary in different parts of the world to foster the inclusion of technologies in the mathematics classroom and rethink the curriculum.

In the USA, according to the NCTM: "All schools and mathematics programs should provide students and teachers with access to instructional technology—including classroom hardware, handheld and lab-based devices with mathematical software and applications, and Web-based resources—together with adequate training to ensure its effective use" (NCTM, 2015). This statement indicates a public recognition of the necessity of integrating a wide range of digital tools for teaching mathematics at schools, as well as the importance of teacher training for an effective and strategic use of those resources in the classroom. However, it is striking that in that statement, there is no mention of programming and computational thinking, which, in other countries have been considered more prominently as central for the development of mathematical thinking.

In Mexico, as discussed by Rojano, although at the turn of the century the potential of digital technologies in the teaching of mathematics was recognized when different public administrations, between 1998 to 2006, launched national projects directed at middle and elementary school children – EMAT, the New Telesecundaria Model, and Enciclomedia (Trigueros et al., 2006) –, this is no longer the case. Unfortunately, when the government changed, those projects disappeared. Since then, only the use of calculators and office applications have been recommended, but not included, in the following curricular changes. (Only some teachers, who found them useful to promote students' mathematical learning continue to use the EMAT or Enciclomedia tools in their teaching).

Thus, in many countries, and perhaps for different reasons, schools and curricula have remained attached to the use of software and/or calculators that are aids to the set curriculum and are considered useful in mathematical problem solving and in helping students understand specific concepts. In other words, they are added on to existing curricula and used to do the same as before but with the technology add-on, instead of for innovative ways of interacting with mathematics.

How the learning potential of digital technologies is considered varies from country to country, but also in different regions inside countries (as in the case of Canada, discussed by Kieran). In some places, digital technologies are embedded in school culture and specifically in mathematics classrooms, while in others, this is not the case. Julie et al. (2010) had already presented perspectives from different parts of the world illustrating the diversity of access and implementation of digital technologies. Some of these differences persist, perhaps due to diverse policies but also because of socioeconomic differences and factors (also, in terms of how technology is used and perceived in schools, gender may also be an issue). More problematic is the fact that many parts of the developing world still have issues of lack of access to technologies, rely on very old hardware, if any is available at all (see, for example, Sacristán et al., in press), and teachers have insufficient or no training in their use and pedagogical integration.

Thus, economic and social problems result in a widening gap, not only between different countries, but also inside these countries since development, technological opportunities and even the possibility to have access to technology are not the same. As technology continues to quickly develop, the gap between students who have opportunities to use technologies for mathematical thinking and learning, and those who don't, widens, and inequalities among different populations increase.

This gap in opportunities along the whole schooling process, has an important impact in terms of equity and on students' future. Jobs will need more and more mathematical and computing abilities so students who have not had the opportunity to develop them in depth will find it difficult to find jobs, thus the social context in different countries is increasingly divided.

On integrating new technologies and adapting to their changes

When integrating digital technologies for learning in schools, there is a recognition of the importance to provide, in Hoyles' words above, "a framework for the design and implementation of computational tools for learning mathematics." The issue of design is paramount. In summarizing Papert's constructionism, Hoyles mentioned the two central ideas of epistemological pluralism and microworlds. The design of the learning and exploratory universes that microworlds are, need include epistemological pluralism and take into account different components, as described by Hoyles and Noss (1987): the student component, the pedagogical component, the contextual component and the technical component. EMAT was a project that designed a model that took into account all of those aspects (Sacristán and Rojano, 2009). But careful design is only as good as what can be taken up by the educational system. As in the case of why Logo and Papert's vision declined in the 1990s, there is also the issue of the difference between an intended design and how it is implemented. In projects like EMAT or ScratchMaths, a very careful research-based design involving many researchers in mathematics education, as well as some teachers, was carried out aiming at maximizing the (mathematical) learning possibilities. These designs included, not only mathematical tasks, but also pedagogical models, associated teacher training models and possibilities for scaling up in order to expand to cover all, or a majority of, schools at a national level. However, when implemented, many of the initial intentions are lost. In the EMAT project, teacher training was never achieved as designed, and the scaling up was suspended due to government changes. In ScratchMaths there was also the problem of fidelity that Hoyles referred to above (see also Hoyles and Noss, 2019): how faithfully teachers (and schools) adopt, or are able to do so, the intended design. This necessarily leads to results that are generally less than expected.

Also, technology designers look for ways to offer educational systems, software with more capability in terms of interactivity and in terms of what they believe is needed in schools. Research on the use of technologies is constantly adapting to better understand how teachers and students use new resources in order to incorporate change in teachers' training programs and to look at how and if implementation in the classroom works in terms of students' mathematical learning.

Results often do not meet designers' expectations and real use of technology at school do not match with them either. On the one hand teacher adaptation to new technologies takes time and on the other hand when teachers find out that some tools work well in terms of their student' learning they are unwilling to try something new which may or may not work. These problems, together with the lack of access to tools and the fact that teacher training programs do not reach all teachers and the need to convince many teachers who don't want to take risks of finishing curriculum on time or don't see how technology matches with official curricula, need to be taken into account if the use of digital technology is considered important for the learning of mathematics.

When analyzing the efforts in the past couple of decades of educational systems around the world to integrate digital technologies in schools (not necessarily for mathematics), the words of Healy (2006), although she was referring to the case of Brazil, are valid in the cases of many countries: the attempts may tend to emphasize the computer as a catalyst for pedagogical change, but "they fail to acknowledge the epistemological and cognitive dimensions associated with such change or the complexity associated with the appropriation of tools into mathematical and teaching practices" (Healy, 2006, p. 213).

Moreover, technologies change fast. Keeping up with those changes is difficult for teachers, particularly if they have to catch up by themselves and only highly motivated teachers who have had good opportunities to evidence positive changes in their students are willing to do the needed efforts to develop teaching plans for their students to work with technology. There are also institutional constraints that limit teachers' possibilities to fully use the potential of digital technologies. Sacristán (2017) discussed in depth some of the challenges for teachers (and schools) to meaningfully integrate digital technologies into teaching practices for mathematical learning. Hoyles above also mentions the pressure of preparing students for national examinations, but there are others, such as the time needed to cover specific curricular topics, or the way a topic is presented in the textbook. All these factors need to be taken into account when designing teachers training courses.

Projects that involve collaboration between university researchers and teachers have proved to play an important role in the possible success of implementation in schools and in the promotion of students' learning, as was the case of the example given by Hoyles as well as the EMAT project mentioned by Rojano. Collaboration offers teachers opportunities to discuss strategies to use technology to transform their practice and to develop technological skills to make decisions to guide students' mathematical thinking. This type of approach, together with other teacher training initiatives, are indispensable for teachers to overcome their difficulties.

Unfortunately, teacher training opportunities are unequally supported in different cultures. While in some places programs are offered continuously to teachers, in others teachers are left to develop their own strategies to introduce technology in their classrooms; thus teachers are in many occasions discouraged and leave technologies aside.

Another phenomenon that has been observed is that in most places research projects to introduce technology to the classroom create lots of activity in terms of groups of researchers and teachers working together to creatively devise ways of using new technologies to teach mathematics. However, when those projects end, this activity stops. Regrettably, the lack of continuity and the gap between research and school policies, is true everywhere: Even in developed countries, as is the case with many European projects, when the funding finishes, interesting innovations abruptly end, and researchers move on to different proposals. This cycle repeats over and over, leaving aside the possibility to develop long term proposals which could make an imprint in the educational system.

Only a handful of creative and motivated teachers continue using the materials developed for research projects or previous government initiatives, adapting them to their teaching, and developing strategies to use those proposed technologies with their students.

Teachers and school systems have difficulty adapting to the perpetual changes in projects, as well as in the technologies available. It is therefore important to develop long-term professional development strategies and programs so that teachers can develop knowledge to be able to cope with and adapt to the technological changes in a productive way.

Concluding remarks

Digital technologies have played an increasingly important role in changing the mathematics classroom. Research has come a long way in terms of how to harness the possibilities that technologies can offer in terms of developing students' mathematical learning, although schools today are very different than when digital technologies first became available for education. But we should not forget that in this information age where information and communication technologies dominate, there are also other content-specific digital technologies, such as expressive and computer programming environments, that are important to develop mathematical thinking and practices. In particular, the tools that are generally used (including, more recently, communication ones during the COVID pandemic), haven't fully led to a meaningful integration of digital technologies *in* and *for* mathematics education. Thus, despite the exponential growth and influence of digital technologies in society, innovative visions (such as Papert's) haven't fully come to fruition.

Many projects, developed around the world, have shown that students enjoy using digital technologies in the mathematics classroom and there is evidence of the potential of these technologies to promote students' learning. But, although we have clear evidence that they can be used to foster students' mathematical competencies and learning, the expected results of their use in the classroom are still far from those desired. There are some important areas, such as evaluation and assessment (including self-assessment), and activities outside school, for which the potential of technologies has not been harnessed to support students' learning.

An important area that needs attention is the need to foster teacher training programs to overcome an existing tendency of teachers to improvise in the classroom, and for developing creative ways to use technology, including for assessment and self-assessment.

Also, inequalities in different school systems around the world, as discussed above, can be bridged through the development of long term projects that foster equality in the access of technologies, rich teacher training programs and activities for students to promote their learning autonomy. As it is now, differences among educational systems are widening but also differences in terms of opportunities within specific systems are far from what would be expected in terms of mathematical learning for all. When social and economic conditions are considered at this particular moment in time when the COVID pandemia has strongly affected all countries, life has become dependent on the use of technology in many aspects; but the difference between children that have access to technology and those who don't (or only have access to state-produced educational TV programs, as is the case in Mexico) is expected to increase the education gap.

It is time to consider more thoroughly how to harness the technologies that are already widely available to students in most countries: in particular, mobiles and smartphones. These accessible technologies have spread quickly around the world and are used by many students, but their rich educational potential has not received enough attention from researchers and from policy makers. These technologies are powerful and it is shown that more people have access to these than to computers. These tools need to be taken into account to develop interesting ways for their users to develop mathematical thinking inside and outside the classroom, with diverse uses and applications. New developments need creative ways of thinking and of using the growing potential of digital technologies in order to provide opportunities for all students to have access to computational and mathematical thinking, thus providing them with a potentially better future in the technology-based society. But this also implies transforming school cultures – not an easy task, as the story of Logo and other innovations shows.

Nevertheless, it is necessary to focus on how to use the technology potential to help reduce access inequalities, by looking for creative ways to help children around the world develop their mathematical thinking potential, and contributing to the creation of a better world for all. At the same time, we should also continue to devote time and effort to consider what has been dubbed "Papert's 10%" (from his call in his keynote speech at the ICMI 17 Study in Vietnam 2006): how mathematics and mathematical practices can change due to the availability and access to digital technologies.

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EDUCATIONAL NEUROSCIENCE: PAST, PRESENT, AND FUTURE PROSPECTS

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This talk provides the speaker's perspective on how the fledgling new area of educational neuroscience has emerged from a disenchantment with brain-based education, through various multidisciplinary, interdisciplinary, and transdisciplinary initiates and collaborations involving educationists and neuroscientists. Specific examples and results pertaining to research in mathematics education will be presented. Beyond the current state-of-the-art, the speaker will conclude with some speculations on what might be anticipated as this area of research continues to unfold into the near and far futures.

Keywords: Educational Neuroscience; Embodied Cognition; Electroencephalography; Electrooculography

Good day. I'm pleased to be here to address an area of research that has occupied me for some time, one that has come to be known, somewhat ambiguously, as educational neuroscience. First, a few introductory comments. I have always been interested in the nature of consciousness, and how it is that we are able to experience the world in the way that we do. Oddly enough, back in the 1970's my industry experience in seismic imaging utilizing the most advanced computing technologies of the time, including one of the first CRAY I computers, led me to the study of philosophy and mathematics. How so? I had come to view seismic imaging as the social development of a new sense of perception. I blame Teilhard de Chardin for that.

During the 1980's I had the great pleasure to encounter a number of books that had a great influence on my subsequent interests and career development, ranging from Haugeland's edited volume *Mind Design* (1981), the epic volumes of Rumelhart, McClelland and the PDP Group on *Parallel Distributed Processing*, to Gardner's *The Mind's New Science* (1987) and Churchland's *Neurophilosophy* (1989). These books motivated a shift in focus away from imaging the Earth's interior to the knowledge engineering of intelligent software systems using neural nets and automated reasoning, and on to graduate studies in neurophilosophy.

I managed to escape the siren call of the oil and gas industry in the early 1990's and venture west from Calgary to Vancouver to study computing and education at Simon Fraser University, eventually settling into doctoral studies in mathematics education with Professor Rina Zazkis. During those days I was, as were many others at that time, drawn to Varela, Thompson, and Rosch's *The Embodied Mind* (1991). Subsequently, after half a decade at the University of California, Irvine, I returned to SFU and obtained funding to establish an educational neuroscience laboratory, the ENGRAMMETRON.

A good way to begin exploring the manner in which the area of educational neuroscience began is to chart its origins and development through its affiliated Special Interest Group of the American Educational Research Association called Brain, Neurosciences, and Education. That SIG, in its original incarnation in 1988, was referred to as the Psychophysiology and Education SIG. When Bill Clinton designated the 1990s as the "decade of the brain," the SIG was renamed the Brain and Education SIG, promoting "Brain-based Education." Despite growing popularity of brain-based education amongst educational practitioners, scholars and researchers became increasingly critical towards the point of dismissiveness, citing a number of "neuromyths" that were usually based on partial truths.

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Subsequently, in reaction against the growing uncritical educational use of brain-based metaphors, such as right and left brain learners, and appeals to multiple intelligences and VAK (visual, auditory, and kinaesthetic) learning styles (Geake, 2008), the Brain and Education SIG committed to being more rigorously grounded in neuroscience by including that term in 2003. The stated purpose of this newly-branded SIG "… to promote an understanding of neuroscience research within the educational community [with a] hope to achieve that goal by promoting neuroscience research having implications for educational practice and providing a forum for the issues and controversies connecting these two fields." Keynote speakers for the SIG shifted accordingly from champions of brain-based education to neuroscientists themselves with interests in educational problems, such as Bruce McCandliss and others.

This AERA SIG initiative wasn't as novel as it sounded, as an earlier funding initiative back in the 1980s between the *National Science Foundation*, the *Sloan Foundation*, and the *National Institute of Education* enabled scholars and researchers from neuroscience, cognitive science, and education to seek middle ground in what Rita Peterson aptly described as "the middle ground between those three points on a triangle." One term being bandied about for that disciplinary nexus was "pedagogical neuroscience" (McCulloch, 1989).

One of the main pioneers, perhaps *the* main pioneer, in bringing cognitive science, neuroscience, and education together and exploring that middle ground in a true disciplinary sense was the recently departed Kurt Fisher of *Harvard Graduate School of Education*. Professor Fisher founded the Mind, Brain, and Education program at the HGSE, and then went on to establish the *International Mind, Brain, and Education Society*, as well as serving as the founding editor of that society's flagship journal *Mind, Brain, and Education*.

This triangle of disciplines, and the vast areas of research and practice falling within it that they delineate, has come to be known as the new academic field of neuroeducation (Tokuhama-Espinosa, 2008). Within the broad purview of neuroeducation have emerged multidisciplinary, interdisciplinary, and transdisciplinary initiatives. The one that interests and concerns me most is referred to as educational neuroscience. This area of research can be considered in a variety of ways, distinguished perhaps most notably as to where one places the emphasis, educational *neuroscience* or *educational* neuroscience. Whereas contributions to the former come predominantly from cognitive neuroscientists, my focus has been on the latter.

Whereas neuroeducation is more broadly conceived, linking as much or more to educational practice, I see educational neuroscience as an area of *educational* research, and one that naturally draws on the neurosciences, especially cognitive neuroscience and psychophysiology. That is to say, I see *educational* neuroscience as an area of educational research that draws on, as in being informed by, theories, methods, and results from the neurosciences, but unlike educational *neuroscience*, arguably an applied cognitive neuroscience, is *not restricted* to them. This difference is important, as the focal point of *educational* neuroscience is the subjective experience of learners, not *just* their associated mechanisms.

In multidisciplinary initiatives where neuroscientists and educators collaborate, there is typically a strict separation between their respective philosophical frameworks and research methodologies, whereas interdisciplinary initiatives typically motivate collaborators to adopt more of a mixed methods approach. Educational neuroscience as a bona fide transdisciplinary activity, by definition, must entail the forging of new philosophical frameworks and research methodologies for bridging education and neuroscience, and especially, mind and brain (Campbell, 2011). Bruer famously referred to this as a bridge too far (1997).

Bridging mind and brain, and body more inclusively, is exactly the aim of Varela, Thompson and Rosch's *Embodied Mind* (1991), and Varela's initiatives in the area of *neurophenomenology*, to bridge what he referred to as the gap between the biological mind and the experiential mind via "reciprocal constraints" (1996, p. 343). In my view, this amounts to the hypothesis that any changes in subjective experience must in principle manifest objectively in some manner as changes in brain, body, and behavior, and vice versa (Campbell, 2011, p. 10). I have taken this hypothesis as both a foundational assumption and a necessary condition in striving toward a transdisciplinary view of educational neuroscience.

My approach to educational neuroscience in this transcendental sense has been to focus primarily on qualitative educational research rather than quantitative educational research, *per se*. That is because I am interested more in questions pertaining to ontology than epistemology. That is to say, I am more interested in the lived experience of learners of mathematics than I am, for instance, in how widespread their experience might be in the general population of learners.

That is not to say, however, that I do not draw on quantitative research from cognitive neuroscience, because I do, and it is central to my methodology that I do. As an exemplary case in point, consider the so-called "aha!" moment. Jung-Beeman and his colleagues in cognitive neuroscience (2004) have identified what they refer to as an "insight effect" in the right anterior superior temporal cortex detectable as a burst of electrochemical energy from neuronal activity in the gamma range (>30Hz) via electroencephalography (EEG), which they cross-validated using functional Magnetic Resonance Imaging (fMRI).

Knowing that such an "insight effect" had been identified enabled me to design an experiment using an instrument developed by Dehaene and his colleagues (2006) to explore "aha!" moments using an integrated methodology drawing upon audiovisual, eye-tracking, and EEG along with a palate of psychophysiological metrics including heartbeats and respiration. Figure 1 illustrates the setup in my laboratory (following excerpts from Campbell, in press).

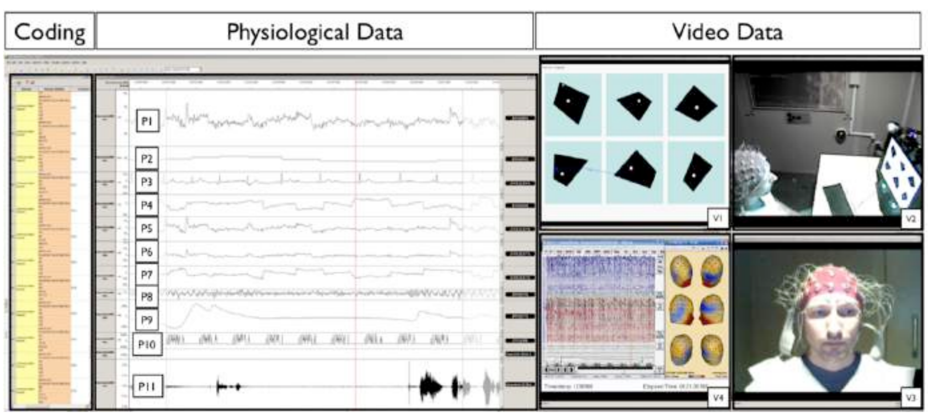

Figure 1: Integration of physiological and behavioral observations

The leftmost column is for coding for the observation. The physiological data includes P1, the central EEG channel; P2 is the heart rate in beats per minute; P3 are heart beats from which P2 was derived; P4 and P7 capture horizontal eye-movements, whereas P5 and P6 capture the vertical eye-movements (using electrooculography, EOG); P8 measures muscle movements (using electromyography, EMG) from the back of the neck; P9 is respiration; P10 the time code; and P11 is the voice channel. Video data: V1 screen captures the eye-tracking as the participant views a slide from Dehaene, et al's instrument; V2 and V3 video recordings of the participant; and V4 the full EEG data set.

Figure 2 below illustrates how EOG data can be used to identify gaze areas (d0 through d6) and movement intervals (1-10) and their associated times in the physiological data to integrate with the eye-tracking data (corresponding to and illustrated in Figure 5 below).

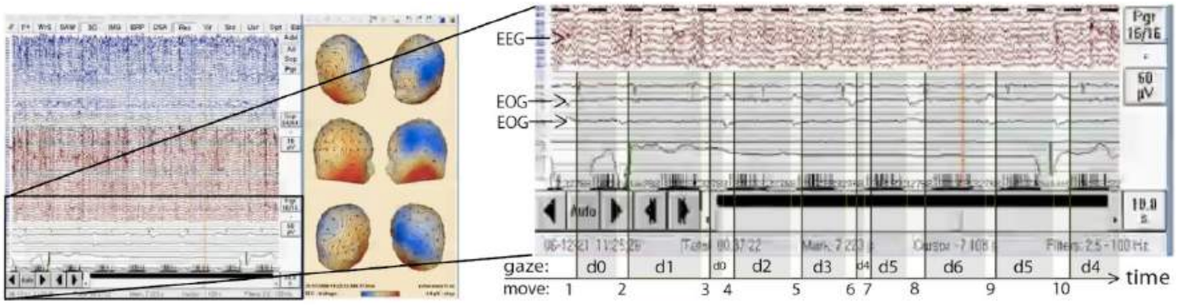

Figure 2: Time synchronizing physiological and behavioral data sets using EOG

A key premise of the approach I've taken to *educational* neuroscience is that theories, results, and methods of the neurosciences, cognitive neuroscience and psychophysiology in particular, can serve to augment and validate, *not replace*, traditional methods of educational research. These observations were part of a qualitative study in mathematics education research. The behavioral data provided by audiovisual data, coupled with the eye-tracking clearly indicated that an "aha!" moment occurred. So, then, what did the educational *neuroscience* tell us?

As noted above, a hallmark of the transcendental approach I've taken to educational neuroscience has been to draw upon methods, results, and findings from the neurosciences, and especially from the cognitive neurosciences. In this case, I drew upon results that clearly identify neural correlates of the "aha!" moment (Jung- Beeman, Bowden, Haberman et al, 2004; Bowden & Jung-Beeman, 2006). In two landmark experiments, Jung-Beeman, et al (2004) identified and located the neural correlates of an insight effect using EEG and fMRI. In the EEG experiment, the red line in the lower panel on the left side of Figure 3 below identifies an increase in gamma range power during moments of insight in contrast to the blue line where no insight was reported. Time zero on the horizontal scale designates the moment when participants reported the insight by pressing a button. The leftmost topographic maps of the right and left hemispheres show grand averages of EEG power distribution *prior* to the onset of the gamma burst (-1.52 to -.36ms), while the rightmost topographic maps show grand averages *during* the onset of the gamma burst, prior to the button press (-30ms to -.02ms).

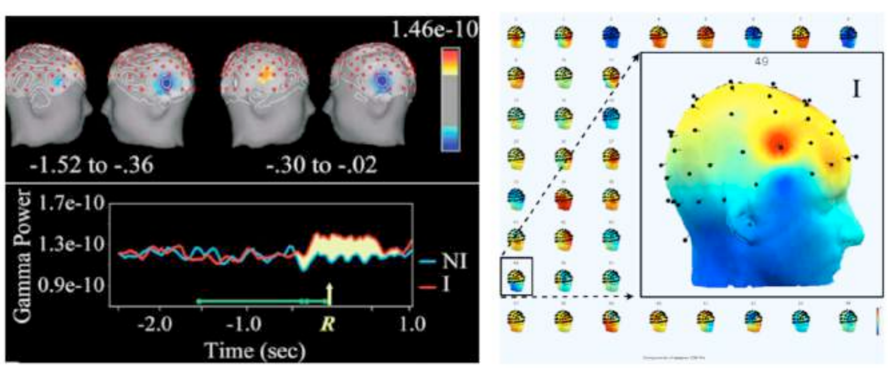

Figure 3: EEG Insight Effect, left hand side (after Jung-Beeman, et al., 2004, p. 505; Kounios & Beeman, 2009, p. 211). To the right, an independent analysis component of our EEG data corresponds to a phenomenon they identified with the EEG Insight Effect.

Jung-Beeman et al's fMRI experiment (upper panel on the left-hand side), cross-validated and located the effect in the anterior superior temporal cortex (ASTC). The question for us, given the behavioural evidence we had of an "aha!" moment, was whether an increase in EEG gamma power was evident in the vicinity of our participant's ASTC. Independent component analysis (Delorme &

Makeig, 2004; Makeig, Bell, Jung, & Sejnowski, 1996) was used to isolate different sources within our EEG data, and found the component illustrated on the right side of Figure 3.

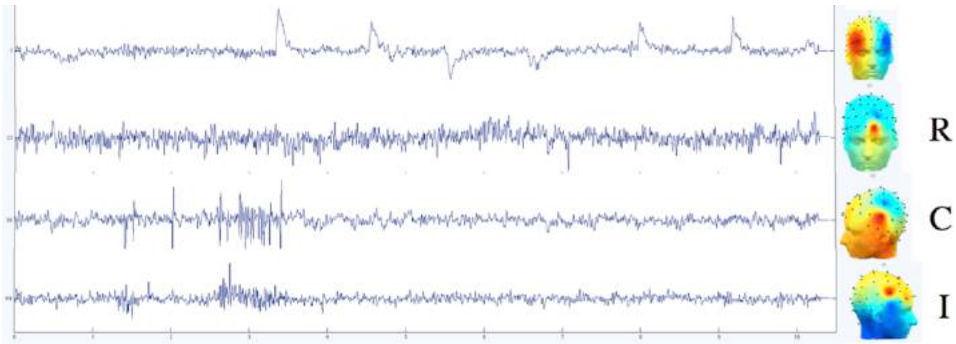

Figure 4: Independent components extracted from EEG data

Cutting to the quick here, Figure 4 (above) illustrates four independent components of EEG data acquired in my lab from the participant over this approximately 10 second interval are presented in Figure 3 below. From the top, the first component captures and isolates the participant's lateral eye movement. The spikes correspond to the vertical displacements from the EOG data (P7 in Figure 1). Second from the top in Figure 3 is an EEG component, labeled R, illustrating on-going activity in the participant's slightly left dorsolateral Prefrontal Cortex (dlPFC) associated with spatial reasoning, working memory (Knauff, Mulack, Kassubek, et al, 2002), and implicated in integrating verbal and spatial representations (Barbey, Koenigs, & Grafman, 2013). Third from the top, labeled C, is sourced in proximity to Broca's and Wernicke's areas in the left hemisphere, responsible for speech and comprehension respectively. Most germane here is the bottom component labeled I in which the burst of energy in the gamma range is evident in close vicinity to the ASTC, associated with the insight effect.

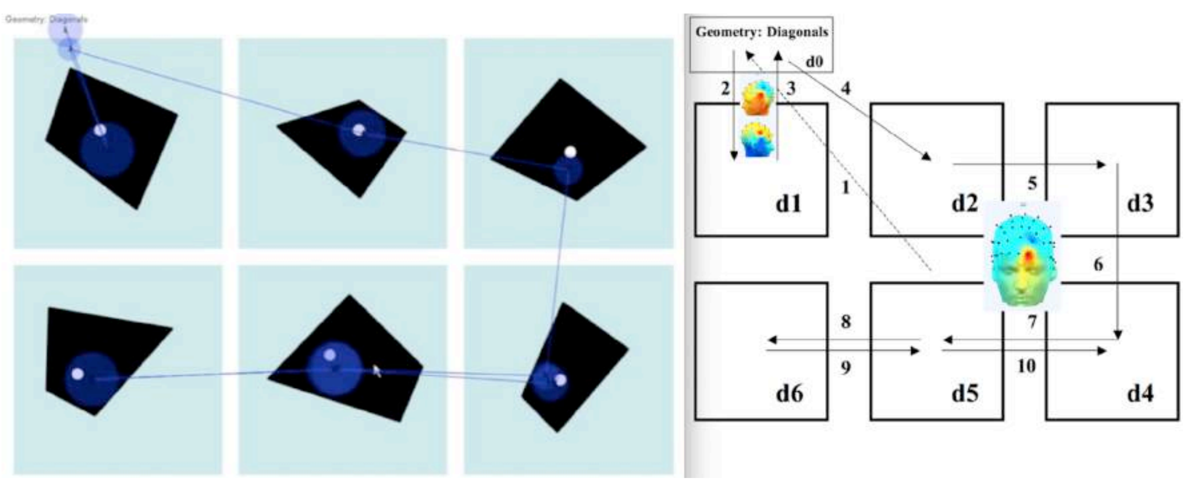

Figure 5: Detailed eye-tracking of the "aha!" moment. The behavioral eye-tracking data on the left side of this figure was time synchronized with the EOG data (Figure 2)

The left-hand side of Figure 5 (above) illustrates the eye-tracking data. After having been given two previously unsuccessful opportunities to identify the "odd ball" in this slide the participant's gaze was initially oriented toward the centre of the screen when this slide reappeared for his consideration for the third time (now with prompt terms revealed which had been previously masked). Hence, his first action was to immediately move his eyes directly toward the prompt phrase "Diagonals". The blue lines on the left-hand side of Figure 5 track his eye movements, while the blue circles indicate the locations where he held his gaze. The larger the circle, the longer the gaze interval. Eye movements and eye gazes are schematized on the right-hand side of Figure 5. The participant's eye movements (with higher frequency saccadic jitter filtered out) are sequentially indicated by the numbers 1 through 10, whereas the area of the prompt phrase is designated as d0, and the six diagrams d1 through d6, in the order in which they were first viewed by the participant.

It is clear that the participant read the word "Diagonals" silently at d0, then, as he shifted his gaze to d1 he took a breath (see P9 in Figure 1) and articulated the word "diagonals" (as evidenced in the voice recording P11 in Figure 1). As he did so, as the idea of diagonals associated with the word became the focal point of his intentional consciousness, his gaze returned to d0, presumably as confirmation (verification) at about the 2.5s mark into the onset of the presentation of the slide. He then continued quite systematically and relatively quickly to shift his gaze to d2, d3, and d4. When he came to d5, it is evident that his gaze lingered a little longer. He then continued to d6, then returned to d5, at which point he clicked his tongue (see P11 in Figure 1), took a breath (P9 in Figure 10), and then exclaimed "Ahhh!" as he continued back to d4. The participant went on to describe his insight as follows:

> Okay, yes, I see this quite differently now [than he was seeing this slide during his unsuccessful attempts to identify the oddball]. This, this one, um, in particular [referring specifically to this slide] I see very differently. I can see, if I mentally imagine a line [while looking at d1 and moving his cursor diagonally from the upper left corner to the lower right corner, then from the left corner to the right corner] connecting the diagonal edges [sic], I can see that this dot [while looking and pointing to the white dot in the centre of d1] is on that line. This dot [looking now at d2] is also on that line, this dot [d3] is on that line, this dot [d6] is on that line, this dot [d5] is not, and this dot [d4] is on that line. So here [d0], when I see the word "diagonals", it definitely prompted me for what to look for, and I clearly see that this one [d5] is the one, is the only one that the dot is not on the diagonal.

So, when exactly was his "aha!" moment? Clearly, as illustrated in Figure 4, there was an attunement of sorts for our participant regarding the C and I components of his brain activity as recorded by EEG. Given that the C component was a neural correlate of comprehension as he read the term "diagonals" and that the I component was a neural correlate of insight as he connected that term to the criterion he had been seeking to identify the oddball figure, the R component appears to correlate with his assessing of the validity of that insight.

Koestler notes: "The sudden activation of an effective link between two concepts or percepts, at first unrelated, is a simple case of 'insight'" (1967, p. 590). Does component I signal a spontaneous bisociative connection or link between, in this case, the participant's comprehension of 'diagonal' and the synthesis of that comprehension with the perception of the dot on the diagonal (d1 in Figure 5) coupled with his unfolding realization that 'diagonal' was indeed the criteria the participant had been seeking to identify the oddball (d5 in Figure 5)? The eye-tracking and audiovisual data, in tandem with results from the EEG data appears to support this interpretation.

There is widening acceptance and growing evidence that various modalities of consciousness, and mind more generally, are manifest within the dynamic fluctuations of the electromagnetic field generated by neuronal activity (e.g., Jones, 2013, 2017). Exactly how characteristics of mind, such as the binding of subjective experience into a coherent and stable whole, our sense of identity and privacy of thought, let alone how other matters of thought and perception, memory and foresight, creativity and insight, are so embodied remain to be satisfactorily resolved, and remain topics of on-going investigation.

As for the future of educational neuroscience, it seems more likely to me, after a number of years of promoting *educational* neuroscience in the transcendental sense that I have indicated above, whereby new philosophical frameworks are forged that are inclusive of lived human experience, that educational *neuroscience* will continue to prevail. That is, I see much of the past, present, and future of educational neuroscience unfolding as an applied cognitive neuroscience, elucidating biological underpinnings of mental processes.

Cognitive neuroscience, approached from a "hard" scientific orientation, has the luxury of focusing on various aspects of brain behavior in terms of neural structure, mechanisms, processes, and functions. On the other hand, neuroscience approached from a more humanistic orientation would have the luxury of not having to be concerned with trying to explain, or explain away, the lived experience of learners solely in terms of biological mechanisms or computational processes underlying brain behavior (Campbell, 2010).

I think educational researchers, at least those who think the brain actually does have something to do with informing our understandings of cognition and learning, would like to be informed by biological mechanisms and processes underlying learning, and perchance also have access to methods of cognitive neuroscience. As an educational researcher, however, my *primary* focus is not on the biological mechanisms and processes underlying or associated with cognition and learning. Rather, it is on the lived experiences of teaching and learning, along with the situational contexts and outcomes of those experiences.

The above considerations perhaps still hold out some hope for the possibility of a more humanist-oriented *educational* neuroscience, as a new area of *educational* research that is both informed by the results of cognitive neuroscience, and has access to the methods of cognitive neuroscience, specifically conscripted for the purposes of educational research into the lived experiences of embodied cognition and learning (ibid.).

One may speculate, if not fully anticipate, that at some point in the future, such matters will become sufficiently resolved to be of great *practical* significance for education. Consider the following possibilities: Dry electrodes arrays that can be comfortably worn by students like ball caps, capable of transmitting high spatial and temporal resolution EEG or MEG signals from each student in a classroom wirelessly to a central console, analysed for specific aspects of cognitive activity, and made available to the teacher in real time.

Although there are serious ethical issues associated with realising such a scenario, used in a responsible and sensitive manner, such a possibility could provide teachers with unprecedented insight into formative assessment and student learning. Moreover, such tools could provide invaluable information for the teacher regarding overall student engagement and effectiveness of their teaching in real time. Whether such a scenario will benevolently unfold as so envisioned, there can be little doubt that the neurosciences will continue to inform our understandings of cognitive phenomena such as insight and the "aha!" moment, along with many other aspects of cognition and learning at the nexus of mind and brain. How far into the future must we wait? Perhaps not too much longer.

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EYE-TRACKING MATHEMATICAL REPRESENTATIONS – FINDING LINKS BETWEEN VISION AND REASONING

COMPRENDIENDO LAS REPRESENTACIONES MATEMÁTICAS CON MÉTODOS DE RASTREO OCULAR – UNA APROXIMACIÓN BASADA EN LA TEORÍA

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Eye-tracking studies need adequate theoretical frameworks for producing insights about mathematics learning. The current study uses eye-tracking to investigate the effectiveness of tables and diagrams for supporting covariational reasoning amongst elementary students (n = 60). The theoretical framework emphasizes the cognitive functions of representations. Students showed more covariational reasoning around diagrams. The fixations showed that tables concentrated students' attention on the dependent variable data, whereas diagrams distributed students' attention evenly across the numeric and visual elements of the task. According to the theoretical framework, tables did not constrain a covariational interpretation of numerical data, whereas diagrams effectively constrained covariational interpretations, disrupting recursive tendencies and promoting the construction of a mental model of covariation.

Keywords: Representations and Visualization, Research Methods, Cognition, Algebra and Algebraic Thinking

Eye-tracking methods have much potential for studying mathematical thinking and learning. However, more work is necessary for developing conceptual frameworks to guide the design of eye-tracking studies and interpret eye movement metrics (Strohmaier et al., 2020). Here we report research that illustrates an attempt to make conceptual connections between low-level vision mechanisms and abstract mathematical reasoning. The study reported below investigated how external visual representations (VRs) might influence the reasoning of pre-algebraic elementary students while solving tasks that are widely used in the functional approach to early algebra, presented in tabular and diagrammatic formats. Conceptual frameworks of learning with multiple representations allowed us to formulate and respond to research questions with theoretically-driven interpretations of eye movement data.

Generalization and representation in Functional Thinking

The Functional Thinking (FT) approach to early algebra uses the function concept to articulate ideas such as variable, covariation, generalisation, and symbolic notation. FT research algebra relies heavily on tabular tasks to investigate and support generalization processes. The study reported below addresses a task that require students to define missing instances of a dependent variable, e.g., completing missing cells in a function table.

Learners might use at least three approaches for solving tabular tasks (Smith, 2008): (1) Recursive patterning involves attending to variation in sequences of values, (2) a covariational approach means analysing simultaneous change in two or more quantities, and (3) a correspondence approach emphasizes the relation between pairs of variables. The use of covariational and correspondence approaches indicate a notion of function based on covariational reasoning (Thompson & Carlson, 2017), which entails the conception of two quantities varying simultaneously with an invariant relationship between the values of the quantities, and every value of one quantity determines exactly one value of the other quantity.

Most elementary students struggle to transit from recursive approaches to covariational reasoning while working with tables (Tanışlı, 2011; Wilkie, 2016). The focus on recursion might result from an interaction between a natural the tendency to seek univariate patterns and the visual properties of tables. Consequently, other visual representations, such as diagrams, could ease covariational reasoning by disrupting recursive tendencies. However, research about representational factors in functional thinking is scarce, so the question remains open: How do different visual representations influence students' reasoning during functional thinking tasks?

Approaches to learning with multiple representations

We consider tabular tasks and diagrammatic tasks as cognitive tools that display information to achieve mathematical insight, and not necessarily to depict mathematical objects (Giardino, 2017). Therefore, we draw from the Design, Functions and Tasks (DeFT) framework to learning with multiple representations by (Ainsworth, 2006), which addresses the learning potential of multi-representational systems from a design perspective, considering representational features such as modality and number of representations as design parameters, as well as other dimensions to analyse the effectiveness of multiple representations, namely tasks and functions.

Tabular tasks and diagrammatic tasks are equal in design because both are in the visual modality and combine text with VRs. Tables and diagrams are also equivalent in the task dimension because these representations are common in the elementary classroom and, therefore, students know how to "read them". Pre-algebraic elementary students learn to relate tables or diagrams to the functional thinking domain while working in functional tasks. However, tasks and diagrammatic tasks are different in the functions dimension.

The DeFT framework outlines three cognitive functions of multiple representations. *Complementary functions:* In multi-representational tasks, the representations should complement each other by differing in the information each contains and the processes that each support. *Constraining functions:* Multiple representations help learning when one representation constrains the interpretation of a second representation. VRs can constrain text because text is ambiguous and VRs are specific (Schnotz, 2005); *Constructing functions:* Multiple representations support effective learning when learners integrate information from representations to achieve insights that could be difficult to achieve with only one representation. In tasks that include text and VRs, each representation is processed by parallel mechanisms resulting in complementary mental models that are mapped onto each other (Schnotz, 2005), thereby extending current knowledge and facilitating deeper understandings.

Representational functions of tabular tasks

Tables are semi-graphical representations that support learning by arranging information to exhibit facts or relations in a compact manner, and by directing attention to unsolved parts of a problem (Cox & Brna, 1995). In the tabular functional tasks reported in the literature (e.g., Tanışlı, 2011), graphic components such as cells, rows and columns, comply with the complementary function by representing mathematical relations that texts cannot represent. For example, columns represent variables, rows represent ordered pairs, and empty cells represent missing instances of the dependent variable. Tables comply with the constraining function by positioning numbers in a way that constrains their interpretation from a covariational reasoning perspective. The constructing function happens when the processing of numerical data produces a mental model of quantities, and the processing of the visual layout produces a mental model of covariation. The mapping of these mental models prompts insights about the invariant rule governing the relationship between quantities.

Representational functions of diagrammatic tasks

Diagrams represent objects with pictorial components that express conceptual relations spatially, and can be idealized or instantiated in some context (Belenky & Schalk, 2014). Diagrams give fast access to meaning, facilitate the comprehension of complex information, and elicit previous knowledge (Tversky, 2011). Diagrams effectively show physical layouts and how things work or are put together, organize information, make abstract ideas concrete, and allow the use of spatial skills (Winn, 1991). We have made explorations with diagrammatic functional tasks of shadow-casting phenomena (Xolocotzin et al., 2018). In these tasks numerical data is complemented by pictorial components representing a pole and its shadow, making an explicit representation of covariation and correspondence between the quantities pole height and shadow length. The pictorial components constrain a relational interpretation of numerical data. For example, the pictorial representations of the pole and its shadow are visually connected, facilitating the interpretation of numerical data from a covariational reasoning perspective. Diagrammatic tasks might comply with the constructing function by facilitating the integration of a mental model of quantitative properties extracted from numerical data, e.g., variation, with the mental model of shadow-casting phenomena, which is relational by nature.

Previous paper-based study

Before the eye-tracking study, we assessed the effects of diagrammatic and tabular tasks with a paper-and-pencil study conducted with 1145 students in Grade 4, Grade 5 and Grade 6, recruited from 16 public schools located in central Mexico. Because the schools are public, they must follow the official mathematics curriculum, which does not include algebraic content. We studied different representational versions of a functional task that required students to identify missing instances of a dependent variable. The task presents two number sets. The first set has 6 numbers of the independent variable. The second set has 3 known numbers and 3 unknown numbers of the dependent variable. Students must figure out the rule governing the relationship between the two variables to identify the missing numbers. There were four options to choose from: (1) functional, (2) recursive, (3) First instance, which is consistent with a rule that only applies to the first pair of data, thereby denoting lack of generalization, and 4) random, which presents three numbers defined randomly. Four versions of the tasks were generated by manipulating two factors: Representation (table or diagram), and context (with context or without context), see Fig. 1. The diagrammatic tasks, either with context of without context, generated more functional responses than tabular tasks. However, this effect was larger in the contextualized version of the diagrammatic task. We also observed that Year 5 students were the most sensitive to the effects of context. We considered these results as evidence that diagrams ease covariational reasoning.

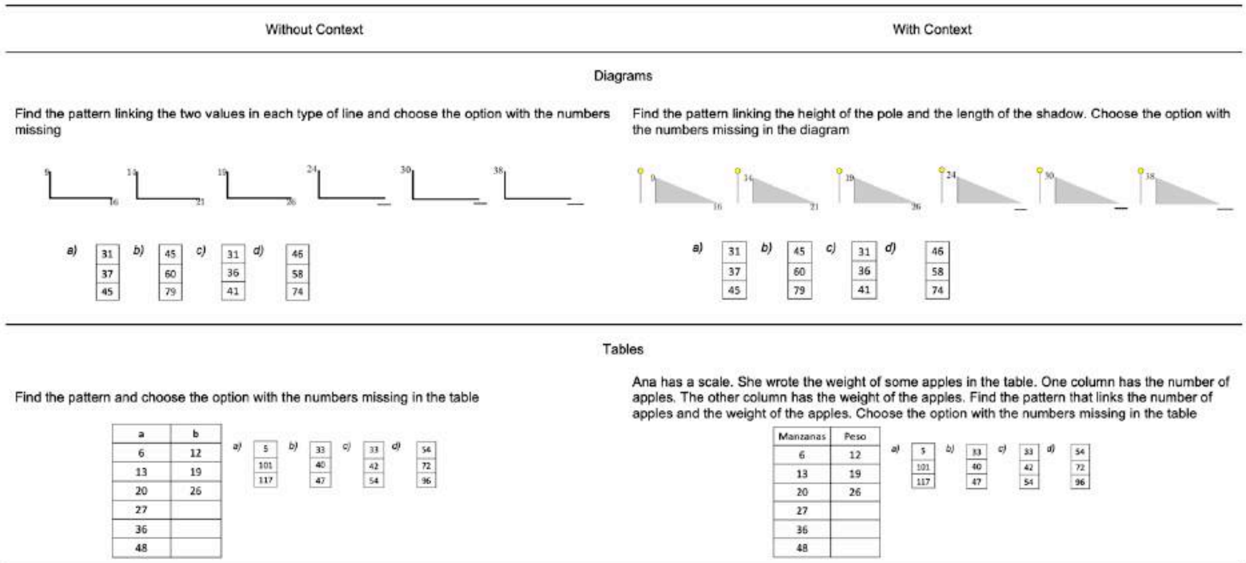

Figure 1 Examples of the tasks employed in the paper-based study.

Overview of the current study

The previous paper-based study suggested that diagrammatic tasks are more effective than tabular tasks at easing covariational reasoning. Albeit informationally equivalent, diagrams seemed to facilitate retrieval of functional information. In line with the DeFT framework, we hypothesized that diagrams are more effective than tables for complying with the functions of multiple representations, which opened our research question: How do diagrammatic tasks and tabular tasks comply with functions of multiple representations such as constraining and constructing? To answer this question, we analyzed students' eye movements to gain insights about the effects of tables and diagrams on students' attention.

Method

Participants

A total of 60 students in Grade 4 (n = 20), Grade 5 (n = 20) and Grade 6 (n = 20) from a public elementary school participated in the study. All students participated on a voluntary basis, with informed consent from parents and school authorities. Three participants failed to reach accuracy levels due to unforeseen circumstances, e.g., spectacles not allowing registration of the participants' eye. Therefore, their data were discarded, leaving a sample of 57 students.

Apparatus and stimuli

The data were collected with a portable eye-tracker Tobii Pro X2-30, with a 30 Hz sampling rate, 0.4 precision (binocular), and 0.32 gaze precision (binocular). Both eyes were tracked. This model allows robust detection of individuals' eye movements, even with unrestricted movement. The eye-tracker was mounted below the screen of a Dell Inspiron 5000 15 inch laptop, which display was set at 60 Hz refresh rate and 1366 x 768 resolution. The distance between the eye-tracker and the edge of the table was held constant at 60 cm.

The stimuli were a series of functional tasks presented in either tabular or diagrammatic format. The tasks were replicated from the "with context" of the previous study (See Fig. 1). Tabular tasks were grounded on a situation involving apples and their weight. The diagrammatic tasks were grounded in a shadow-casting situation involving the height of a pole and the length of its shadow cast. There were 12 tabular tasks and 12 diagrammatic tasks. In each type of task, there were four items involving sums, four items involving subtractions, and four items involving multiplications. The same as in the previous study, each task required the identification of a relationship between two quantities, and selecting one of four response options: functional, recursive, first instance, or random.

Experimental design

The experiment employed a factorial design involving the within-subjects factor representation (table/diagram) and the between-subjects factor Grade (4/5/6). The factor representation was counterbalanced within each Grade. The tasks were presented in fixed order: sums, substractions, and multiplications.

Procedure

Students were tested individually in the school IT suite. They were instructed as follows: We would like you to please help us solving a task. It is important that you know that the results do not have any relation with your grades. Do you have any question? the results are very important for us because we are studying how students solve some mathematics activities. Please, pay attention and do your best effort.

Results and discussion

The behavioural results replicated the previous study, that is, students chose the functional response more in diagrammatic tasks [F (1, 56) = 4.038, $p < .05$, $\eta^2 = .022$]. The eye tracking data, interpreted under the DeFT framework, allowed us to explain this result. Eye-trackers produce a range of eye-movement metrics. We wanted to know which areas components of tables and diagrams were more noticeable for students. Therefore, we used fixation time, which indicates difficulty in extracting information, or that the object is more engaging in some ways (Poole & Ball, 2006). We defined a series of analogous areas of interest (AOIs) corresponding with key components of the tasks, see Fig. 2. One set of AOIs contained data; A1 and A2 show contained the first and second half of the independent variable. A3 contained the first half of the dependent variable, and A4 contained the unknown second part of the dependent variable. A second set of AOIs contained contained the response options, functional, recursive, instance, and random.

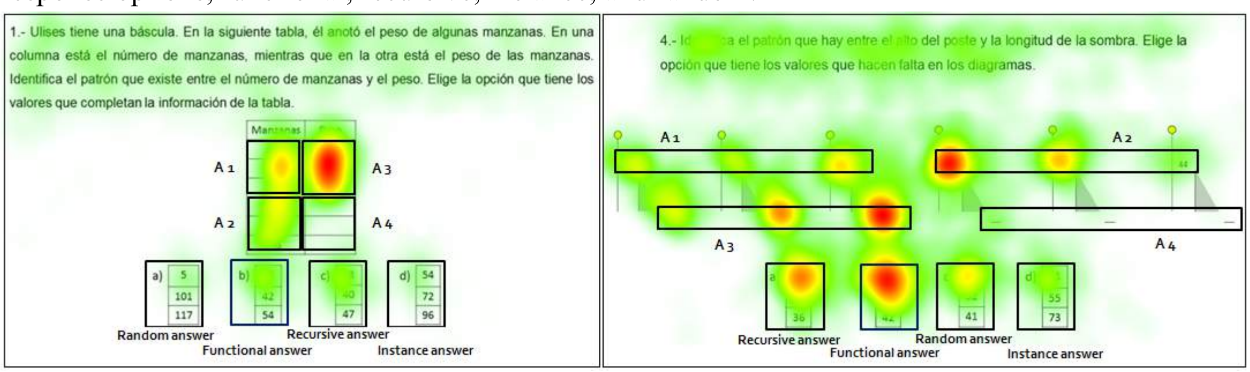

Figure 2 Ares of Interest in Tabular tasks and Functional tasks with an overlay heatmap of fixation duration

The analysis of the data AOIs revealed that students fixated more on AOI3 while solving tabular tasks, which contained the known numbers of the dependent variable, whereas in diagrammatic tasks students fixated evenly across the data AOIs [F (3, 162) = 20.541, $p < .001$, $\eta^2 = .085$]. As for responses AOIs, students fixated more on functional responses while solving diagrammatic tasks [F (3, 162) = 40.209, $p < .001$, $\eta^2 = .108$]. The DeFt framework allows an interpretation of these results. Figure 2 illustrates how students engaged more with AOI A3, which indicates the spatial layout of cells and columns, fails comply with its intended function of constraining a covariational interpretation of numerical data. Therefore, students are unable to integrate the mental model of quantitative properties extracted from the data, with a mental model of covariation. Therefore, they did not achieve the necessary insight for inhibiting the tendency to look for recursive patterns. This might explain why students engaged equally with recursive and functional responses. In contrast, diagrams disrupted the tendency to focus on recursion, and made students to expand the breadth of their attention, and engaged equally with data and probably other elements of the task. This might indicate that they considered all sources of pictorial and numeric information, moreover, pictorial components seemed to constrain a covariational interpretation of data. We argue that diagrammatic tasks allowed the construction and integration of a mental model of quantitative properties extracted from numerical data, and a mental model of a relational situation, extracted from the graphic elements of the task. In this way, students gained covariational reasoning insight and, therefore, engaged more with functional responses.

General conclusion

Eye-tracking research in mathematics education is growing steadily (Strohmaier et al., 2020). However, the release the full potential of these methods for gaining insights about mathematical learning, it is necessary to use theoretical frameworks that allow plausible interpretations of eye-movement data. The presented study aimed to illustrate the benefits of theory-driven interpretations of eye-movement data.

In our first paper-based study, we found that diagrammatic tasks were more effective at easing covariational reasoning than diagrammatic tasks. However, this result could not be explained from paper-based data. So, we had the output but were unable to empirically explain the process leading to such output. We addressed this issue with eye-tracking methods because the cognitive mechanisms involved in learning from visual representations cannot be observed directly. Moreover, these mechanisms rely heavily on unconscious vision processes which operation cannot be intentionally controlled by individuals.

The DeFT framework allowed us to make theoretically-informed accounts of the ways in which tables and diagrams are expected to support covariational reasoning. By analyzing tabular tasks and diagrammatic tasks under the DeFT framework, we identified that these representations were similar in the dimensions design and tasks, but different in the functions dimension. Therefore, we hypothesized that diagrams were more effective for supporting covariational reasoning in the first study because this representation complied more effectively with functions such as complementing textual information, constraining textual information, and constructing insights.

The behavioural results replicated the paper-based results, diagrammatic tasks were more effective for supporting covariational reasoning. The patterns of fixation duration confirmed our hypothesis. Diagrammatic tasks distributed the individuals' attention evenly across the visual display of the task, and directed their attention to recursive and functional responses evenly. In contrast, the tabular task concentrated individuals' attention on the first part of the dependent variable, and directed their attention to recursive only.

An interpretation of results from a cognitive load framework would have been problematic. Diagrams should have produced less functional answers because they have more information and require more cognitive resources than tables. The DeFT framework offered a more parsimonious interpretation of these results. The layout and structure of tables seemed unable to constrain a covariational interpretation of textual data such as task instructions and numerical data, thereby favoring a recursive interpretation of the data. In contrast, diagrams effectively constrained a covariational interpretation of data, disrupting the natural tendency to seek recursive patterns, and allowing the production and integration of a mental models of data's numerical properties with a mental model of covariation.

Acknowledgments

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SPECIAL LECTURES

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

MENTAL MATHEMATICS IN THE CLASSROOM: CONTENT, PRACTICES AND PAPERT'S *MATHLAND*

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This lecture reports on aspects of a larger research programme focused on studying mental mathematics in elementary and secondary mathematics classrooms. It specifically addresses an unplanned aspect that became salient through the work conducted in these classrooms. In this research programme, mental mathematics sessions are designed on a variety of mathematical topics (e.g., algebra, geometry, measurement, statistics, trigonometry, fractions), where students are given short amounts of time to solve tasks given orally and/or on the board, without the use of paper-and-pencil or any material aids. Whereas the central objectives centers on inquiring into the nature of the strategies students engage in to solve the tasks, more seem to be happening in these sessions. In particular, students' solutions and strategies to the task given in the mental mathematics context led to numerous questions, discussions, follow-up explorations, and so forth, by students, which in turn enabled the emergence of significant mathematical issues. This raised interest in investigating these (additional and unplanned) mathematical issues. This represents the core of this lecture, which focuses on the nature of the mathematics (in terms of content and of practices) that frequently unfolds in the mental mathematics sessions conducted. Using an illustrative extract from a mental mathematics session on analytical geometry in a Grade-10 classroom (15-16 years old), the analysis outlines how not only mathematical content is being worked on through these mental mathematics sessions, but also how mathematical practices are being enacted by students. This raises issues about the nature of the environment that these mental mathematics session plunge students into, one that could be tentatively, and boldly, aligned with Papert's concept of mathland.

Keywords: Mental Mathematics, *Didactique des Mathématiques*, Mathematical Practices, Problem Solving, Curriculum Enactment, Geometry and Geometrical and Spatial Thinking

> Being a mathematician is no more definable as knowing a set of mathematical facts than being a poet is definable as knowing a set of linguistic facts. Some modern mathematical education reformers will give this statement a too easy assent with the comment: 'Yes, they must understand, not merely know'. But this misses the capital point that being a mathematician, again like a poet, or a composer, or an engineer, means doing rather than knowing or understanding. (Papert, 1972, p. 249)

Preliminary note: the nature of my research work in *didactique des mathématiques*

My research work is in *didactique des mathématiques*. What does this mean and how does it impact on the nature of the work I conduct? As Douady (1984) expresses, research work in *didactique des mathématiques* investigates the processes and conditions for the production, transformation, communication and acquisition of mathematics, which is not to be reduced to the quest of finding effective teaching methods for mathematical notions. In other words, research work in *didactique des mathématiques* focuses on studying how mathematics happens and advances; which includes its teaching. Brousseau (1991) adds an important aspect to this, mainly that it is a "Science concerned with the production and communication of mathematical knowledge in how these productions and communications are *specific* to mathematics" (*my translation*). What comes out of this is that mathematics and its specificities are central to research work conducted in *didactique des mathématiques*. Hence, questions about mathematics education are addressed *through* mathematics,

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that is, where the *didacticien des mathématiques* is concerned with mathematical experiences and activities in how they are representative, specific and aligned with mathematics themselves. As an example, the interest in problem solving for a *didacticien des mathématiques* is not because doing problem-solving helps learn this or that mathematical concept or because it could contribute to better students' success in mathematics, but mainly because mathematics is defined as a problem-solving endeavor (e.g., Brown & Walter, 2005; Halmos, 1981; Papert, 1972, 1996; Polya, 1957). This is why *didacticiens des mathématiques* undertake studies on problem-solving or argue for its significance: because problem-solving is constitutive of mathematics as a discipline.

Research work in mental mathematics

My research programme is focused on studying mental mathematics in elementary and secondary mathematics classrooms. In this research work, sessions are designed and conducted on a variety of mathematical topics (e.g., algebra, geometry, statistics, measurement, trigonometry, fractions), where classroom students are given short amounts of time to solve tasks given orally and/or on the board, without the use of paper-and-pencil.

Mental mathematics can be defined along the existing research literature, e.g., following Hazekamp's (1986) view, as the solving of mathematical tasks through mental processes without paper-and-pencil or other material aids available. To this one can add that there are frequently time constraints to producing an answer, as well as the fact that questions are often asked orally. The mental mathematics sessions conducted usually follow the same structure, similar to what Douady (1994) suggests by carefully establishing a respectful climate that ensures thay students' share and listen to solutions:

(1) A task is offered orally or on the board;
(2) Students listen and solve the task mentally;
(3) When time is up, students are asked to explain their answer (adequate or not) in detail to the classroom, taken in note on the board (and in some cases students themselves come to the board to explain it);
(4) Other students who solved differently (or thought of solving differently) are invited to offer their answers; once all is said and done, another task is given.

It is often reported that the strategies used to solve mental mathematics tasks differ from those usually referred to in a paper-and-pencil context. Butlen and Pézard (1992), for example, report that the practice of mental mathematics can enable students to develop new and economical ways of solving arithmetic problems that traditional paper-and-pencil contexts rarely afford, because the latter are often focused on techniques that are too time-consuming for a mental mathematics context. These economical ways of solving are said to have the potential to open varied and alternative mathematical routes for handling the concepts under study (e.g., Alain, 1932; Murphy, 2004; Plunkett, 1979; Reys & Nohda, 1994; see also Proulx, 2019). Thus the central objectives of this research on mental mathematics is to inquire into the nature of the mathematical activities (strategies, ways of solving, ideas, reasoning, etc.) that students engage in to solve these tasks.

This said, as these mental mathematics sessions were conducted in classrooms, it became quite apparent that much more than strategies and solutions was happening in these sessions. In effect, the answers given by students and the strategy shared to arrive at them becomes some kind of natural occasion for other students to question or comment them, if they are not convinced or do not understand them. This leads to numerous interactions between students and the Principal Investigator (PI) (and the regular classroom teacher), where students ask important questions about the mathematics at play, which in turn would often lead students to engage in subsequent investigations about these issues (through questions, discussions, follow-up explorations, etc.; see also Cobb et al., 1994, on this). In addition, the sharing of numerous strategies leads invariably to discussions about these strategies, where the various strategies and their answers are compared and discussed by the PI and students concerning their effectiveness, links, (dis-)advantages, possible extensions to other tasks, and so forth. Even if from the outset this was not the scientific objective of the research, this phenomenon became intriguing. And from a *didactique des mathématiques* orientation, some attention was given to how all this was contributing to the advancement of mathematics with students.

Advancement of mathematics: content and practices

The advancement of mathematics can be addressed along two dimensions. The first is relative to the advancement of *mathematical content*. Mathematics is filed with content, from number to geometry and algebra, to name a few, through various algorithms, formulas, procedures, methods, definitions, theories and theorems about them. Analyses of the advancement of content in classrooms focuses on the development of this content with students, that is, on their understandings and reasoning relative to this mathematical content. Having said this, as Papert's above quote insists, mathematics is not only about its content; it is an activity that is done and takes shape in action (see also Brown & Walter, 2005; Hersh, 2014; Lockhart, 2009; Schoenfeld, 2020). Mathematics is about doing mathematics; mathematics is a practice. Another dimension thus concerns *mathematical practices*. This second dimension of the advancement of mathematics in classrooms is about the development of mathematical practices in students, that is, how these emerge, unfold, progress, and so forth, as mathematics is being explored and produced.

In other words, mathematics is composed of content and practices, where this mathematical content is explored and engaged with. Intertwined with the advancement of content, the emergence and development of mathematical practices thus acts as a fundamental dimension to consider in relation to mathematics. It is also along these lines that Lampert (1990a) raises the relevance of working on a double agenda, that is, simultaneously on *of* and *about* mathematics:

> This meant that I needed to work on two teaching agendas simultaneously. One agenda was related to the goal of students' acquiring technical skills and knowledge in the discipline, which could be called knowledge of mathematics, or mathematical content. The other agenda, of course, was working toward the goal of students' acquiring the skills and disposition necessary to participate in disciplinary discourses, which could be called knowledge about mathematics, or mathematical practice. (p. 44)

Both these dimensions of content and practices have been salient in the mental mathematics sessions conducted. This research, strongly grounded in Papert's work (e.g. 1972, 1980, 1993, 1996; see also Barabé & Proulx, 2017), compelled investigations of mathematical practices. Papert is indeed quite adamant on the importance of the development of mathematical practices, where mathematics is not something given and fixed, but is alive and a source of ongoing investigations in order to enrich students' experiences and culture in mathematics (see, e.g., 1993, 1996). This idea also relates to Bauersfeld's (1995, 1998) notion of plunging students into a "culture of mathematizing", where mathematical practices unfold and take shape through interactions and investigations.

Participants in a culture of mathematizing are seen as authors and producers of mathematical knowledge, understandings and meanings. In the establishment and development of such a culture, where mathematical practices unfold and concepts and methods are explored and worked on, students are encouraged to generate ideas, questions and problems, to make explicit and share understandings and solutions, to develop explanations and argumentations to support the solutions and strategies put forth, to negotiate proposed meanings, to share and explore various ways of understanding problems, concepts, symbolism, and representations, and to assess and validate other's understandings and ways of doing (see e.g. Bartolini Bussi, 1998; Bednarz, 1998; Borasi, 1992, 1996; Brown & Walter, 2005; Cobb & Yackel, 1998; Lampert, 1990a; Schoenfeld, 2020; Voigt, 1985, 1994). From these practices, a number of elements can be outlined to characterize and analyze the advancement of mathematics.

- *The emergence of a community of validation*. Central to a mathematics-producing practice are participants who are engaged in explaining, discussing, arguing, and validating mathematical understandings and meanings (Boaler, 1999; Borasi, 1992; Hersh, 1997; Krummheuer, 1995; Lakatos, 1976; Lampert, 1990a).
- *The role, relevance and development of mathematical languages, symbolisms and conventions*. Mathematical symbolism, languages and conventions, and their development, are used to express mathematical understandings, explanations, arguments, etc., and play a major role in the emergence of mathematics and mathematical thinking (Bednarz et al., 1993; Byers & Erlwanger, 1984; Byers & Herscovics, 1977; Lampert, 1990b; Lockhart, 2017).
- *The role given to errors and how they are handled*. Errors play and have played a fundamental role in the emergence of mathematical thinking and understanding. The way they have been handled has enable new ways of seeing and understanding mathematics, leading to unpredicted or as yet not thought of avenues (Borasi, 1996; Hadamard, 1945).
- *The solving and posing of problems*. Doing mathematics is an activity of posing and solving problems of many kinds (Bkouche, Charlot & Rouche, 1992; Brown & Walter, 2005; Hersh, 1997; Lang, 1985; Polya, 1945), where explorations of mathematical content have contributed to the development of additional mathematical content.
- *The authorship, ownership and responsibility in mathematics*. Doing mathematics imposes an active engagement. People doing mathematics do not conceive of themselves as mere consumers or receivers of mathematics, but as producers and even authors of mathematics (Papert, 1996; Povey & Burton, 1999; Schoenfeld, 1994). Mathematics confers a double sense of responsibilities (Borasi, 1992, 1996), where people doing mathematics are responsible for the mathematics they produce and also responsible for producing mathematics.

As scientific interest arose about these dimensions relative to the advancement of mathematics in the mental mathematics sessions conducted, the following question oriented the inquiry: *In what ways is mathematics advancing in the mental mathematics sessions, under both its mathematical content and practices dimensions?* As a way of showing how the advancement of mathematics happened in the sessions, an extract taken from one session is presented. This extract is then looked into in relation to how mathematics content and practices advance, as a way of offering an initial illustration of what it can mean to analyze the advancement of mathematics in these mental mathematics sessions.

Extract from a mental mathematics session

The extract is taken from a session led by the PI in a Grade-10 classroom of about 30 students, who were working on analytical geometry in relation to distances (points, midpoints, lines, etc.) and had been initiated to usual algebraic formulas. One of the tasks given to students was "Find the distance between (0,0) and (4,3) in the plane" (given orally, with points drawn on a Cartesian plane on the front board); they had 15 seconds to answer without recourse to paper and pencil or any other material. When time was up, students were invited to share and justify their solutions to the group. The following is a synthesis of the strategies engaged in and the discussions, questions, and explorations that ensued.

The first strategy referred to applying the usual distance formula ($D=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$), leading to 5 as a distance. A second strategy suggested drawing a triangle in the plane, with sides 3 and 4, for then finding the hypotenuse by using Pythagoras (Figure 1a).

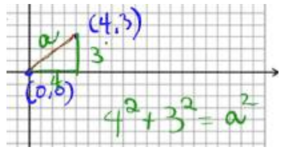


Figure 1a – Drawing the right triangle

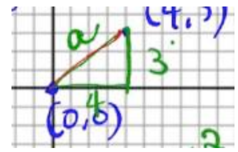


Figure 1b – Close-up on the triangle

Another student then suggested a third strategy, coming to the board to trace a red segment to count directly on it from (0,0) to (4,3) as in Figure 1b. Starting from (0,0), she counted "the number of points" to arrive at (4,3), counting the number of whole-number coordinate points from (0,0) to (4,3). While doing this at the board, she suddenly stopped and mentioned that her red segment did not go through the points she envisaged, which made the counting difficult. The PI then traced another segment going through square diagonals linking two separate points, which could enable counting the number of (whole-number) coordinate points from one point to the next (giving 4 as a distance, Figure 2). The student agreed that for this case, it would work.

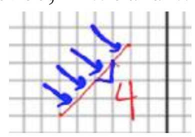
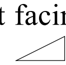


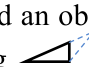
Figure 2 – Line drawn through square diagonals

The PI then asked if the measure obtained with square diagonal lengths was identical to that obtained with the side of the square (drawing  on the board).

> One student asserted that both lengths were not identical, because the diagonal of the square was not of the same length as the square's side. Another explained that both lengths were different, because the hypotenuse is always the longer side in a triangle. Finally, a student claimed that the diagonal was longer because it faces the wider angle.

The PI then asked if that last assertion about facing the wider angle was always true, and if so why (drawing on the board a random right triangle ).

> One student, pointing at the triangle, stated that it was indeed the case in this drawn triangle. Another student explained that in a triangle the bigger the angle the longer the opposite side, mentioning that if the side-hypotenuse had been longer, the opposite angle would have been wider. And, because the sum of the (measures of the) angles in a triangle is 180^o, then the 90^o angle is always the wider one, the other 90^o being shared between the remaining two angles.

Using the drawing of the triangle, the PI simulated the variation of the right angle toward an obtuse one and traced the resulting side obtained, showing how it would become longer (drawing  on the board). He then moved it toward producing an acute angle, asking students if their "theory" about opposite side of the angle worked for any angle, like acute ones.

> One student asserted that it works for isosceles triangles, with equal sides facing equal angles, and another mentioned that it is the same for the equilateral triangle, because it is "everywhere the same" with same angles and same side lengths.

The PI explained that these ideas about the diagonals being longer than the side underlined the fact that this initial strategy amounted to counting diagonals, that is, the number of diagonals of a unit square. And, that this offered a different sort of measure for the (same) distance between the two points: one in terms of units and one in terms of diagonals. A student added that if one knows the value of the diagonal (e.g. 1.2 or else), then one could find the number of unit squares for the diagonal-segment by multiplying by that factor.

One student offered a fourth strategy to find the distance, suggesting using the sine law with angles of 45^o. The PI asked the student how he knew that both angles were 45^o in the triangle. As skepticism grew in the classroom, the PI suggested that students inquire, in small groups or individually, if the triangle's angle were 45^o or not, and to be able to convince others. After 5-6 minutes of exploration, students were invited to share their findings.

> One student explained that on her exam checklist there is an isosceles right triangle with 45^o angles. Thus with this triangle of side length of 4 and 3, one cannot directly assert that it is 45^o because it is not an isosceles triangle as its sides are not equal. Another student illustrated on the board that if one "completes" the initial triangle into a rectangle ( , see Figure 3a), then the hypotenuses of both triangles are the rectangle's diagonal which cuts it in two equal parts and thus cuts its angle in two equal 45^o parts.

As the PI highlighted that the two arguments were opposed, one student replied not agreeing with the last argument, drawing on the board a random rectangle with its diagonal (Figure 3b), and asserting that in this rectangle *it was not certain* that the angle was divided into two equal parts. Another student added that because the sides of the triangle were not identical (of 3 and 4), then the diagonal would not necessarily cut the 90^o angle in two equal parts of 45^o.

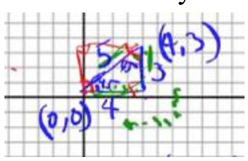


Figure 3a – The "completed" rectangle

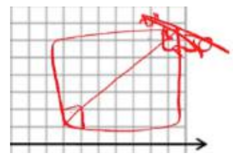


Figure 3b – The "counter" rectangle

The PI highlighted that this last argument reused aspects of the precedent "theory" that the longer side faces the wider angle in the triangle. Hence, following this, a longer side needed to face a wider angle. Then a counter-example was offered to the group.

> The student who made reference to the checklist asserted that sometimes in their exams right triangles did not have 45^o angles, for example, one with 32^o and 58^o; coming to the board to draw it (Figure 4). She completed her drawing to create a rectangle, explaining that the diagonal cuts as well this rectangle in two parts, but that the angles obtained are not of 45^o.

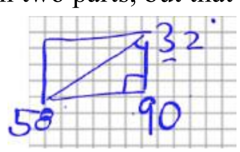


Figure 4 – The triangle counter-example with angles of 32^o and 58^o, and the rectangle

The PI asserted that this offered a counter-example, with a type of right triangle frequently met that did not have angles of 45^o.

> One student added that because all sides were different, then their associated angles would be different, the longer side needed to face a wider angle, which would lead to different angles.

The PI then highlighted the work of one student who drew a square in his notebook to assess the 45^o situation. Drawing a triangle of sides 3-4-5, he extended the cathetus of 3 toward one of 4 to create a 4x4 square. Then, because in the previous unit-square the angles were of 45^o, in this 4x4 they were 45^o as well (Figure 5). Comparing hypotenuses of both triangles, it illustrated that in the initial 3-4-5 right triangle, the angle is smaller than the right triangle of side 4 and 4. All this led students to appear to agree that the angle was not 45^o, ending the explorations (and leading the PI to offer another task for the students to solve).

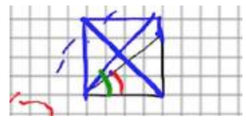
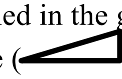


Figure 5 – Comparing triangles within a square

Analysis of the advancement of mathematics

A *didactique des mathématiques* analysis of this extract in relation to the advancement of mathematics underlines issues of mathematical content and of mathematical practices. First, mathematical content is significantly present in this extract through the explorations undertaken. Some mathematical content is engaged with more superficially or in an isolated way, without requiring subsequent exploring and mostly being referred to: Pythagoras' relation, distance formula, hypothenuse, angles (acute, obtuse, right), triangles (various types, and isosceles and equilateral). These are not explored in depth, but are mobilized during the session and play an important part in it. Other mathematics content takes a more important place, enabling or representing some mathematical advances in the session through deeper explorations than the former: the sum of the measures of the angles of a triangle is 180^o, the possibility of having two different measures for the same distance, the relationship between the rectangle's diagonal, and the bisector of its angles. Finally, some content appears at the heart of the explorations in the session, thought of and recurrently being engaged with by students: the difference between the (measure of the) square diagonal and (the measure of) its side, and the relation between and variation of one side of the triangle and its opposite angle. There would obviously be more to outline, and along much subtler lines, but what is significant is the magnitude of the mathematical content worked on, mobilized, and continually explored with the students.

Second, students are enacting a variety mathematical practices, which participate in the environment where the mathematical content is taking shape. In sum, the mathematical contents engaged with in the session are grounded in these mathematical practices:

- *The emergence of a community of validation*. The investigation of the 45^o angle is an example of how a community of validation was established, in which students offered conjectures, argued and counter-argued on the ideas suggested, justified their claim, developed elements to prove it, engaged in reflections to establish what works and does not, and why, etc. The mathematical "truths" were not passively received from outside, from an external authority, but were debated and worked on to develop consensus.
- *The role, relevance and development of mathematical languages, symbolisms and conventions*. Although complex to analyse from a short extract, it is possible to seize some of the symbolisms and representations that took shape in it. For example, the manner of drawing rectangles and triangles with a "cut" to argue about the value of their angles is representative of a strong symbolization that became established in the group, that evolved, and that was used throughout the session. Thus, from a triangle (), students were led to "complete" it to make a rectangle (), enabling them to discuss and explore what happens with the rectangle's and triangle's angles. It is this specific symbolic representation that is used in Figures 3b and 4 to argue and counter-argue about the rectangle's diagonal and the division in half of the 90^o angle. This "invented" representation to symbolize the relationship between rectangles and triangles regarding their angles contributed to the mathematical understandings, and was often reused by students in the session.
- *The role given to errors and how they are handled*. Errors have played a productive role in the session, provoking additional questions and explorations. For example, the third strategy about measuring the distance between the points through the diagonal of the unit-square has

unleashed important questioning on the difference between the diagonal and the side of the square, and has led to the idea that it is possible to have different measures for the same distance. The suggestion that the triangle had a 45^o angle also provoked the investigation about triangles' angles and sides, as well as rectangles' sides and diagonals. None of these assertions, even when erroneous, were criticized and all were taken seriously: they were respected as authentic mathematical productions and enabled deeper understandings of the mathematics at play.

- *The solving and posing of problems*. Throughout the session, questions were asked and sub-problems emerged, unpredicted and contingent on the ongoing explorations undertaken (e.g. diagonal of the unit-square; the 45^o angle; the diagonal splitting the 90^o angle into two equal parts). Students raised and engaged intensely in these questions and sub-problems. It is through these questions and problems that the main part, if not the entirety, of the mathematical content was explored and deepened.
- *The authorship, ownership and responsibility in mathematics*. Students took an active part in the investigation through a number of mathematical assertions and proposals (through strategies, answers, questions, disagreement, explanations, etc.). In this sense, they took ownership of the ideas produced and were engaged in producing them. This double-responsibility took place as students were not passive in the session, but contributed to it with their own ideas. As an example, students' spontaneous use of the front board shows how they felt compelled to share their ideas and participate in the explorations to reach a consensus: they show ownership over this consensus and do not appear to wait for someone else to reach it for them, interacting with others and the PI, raising issues, arguing, questioning, responding, etc.

Another mathematical practice also comes out of this extract, and one considered of significance in mathematics. It is related to what Papert (1980, 1993) calls *theorizing*. In the discussions about the difference between the measure of the diagonal and the side of the square, an important theory was suggested by students: the bigger side of the triangle faces its bigger angle. First, this theory was mainly an assertion, some sort of conjecture. But, after some questions raised by the PI (*Does it work all the time? / What happens if the angle changes? / etc.*), it was increasingly confirmed by and through students' justifications. This theory was then used by others, and as much by the students than by the PI, to address the issues about the 45^o angle: if the measure of one side of the triangle is not the same as another, then neither can be the opposite! Throughout the session, this theory took shape and strengthened, giving rise to a number of side assertions, in the form of corollaries, like the following:

Corollary 1: In a triangle, the smallest angle is always opposed to the smallest side.
Corollary 2: In a triangle, the smaller an angle is, the smaller its opposite side is.
Corollary 3: In an isosceles triangle, both equal angles are opposed to both equal sides.
Corollary 4: In an equilateral triangle, angles are the same, linked to sides of same length.
Corollary 5: Since the sides are not equal, its angles are not equal either.
Corollary 6: Since the sides are of different length, they opposed angles of different size.

And the list could go on. Without always being stated explicitly, the arguments and explanations related to the initial theory, that justified it, underlined these ideas and strengthened them. This made the theory increasingly accepted by students and the PI, to the point of being used itself as an argument. It is in this sense that this theory, and its corollaries, became established during the session, and became "proven". It can be seen as some kind of proof by use, which is shown to be truthful through its efficient functionality and recurrence (Hersh, 2014). *The proof of the pudding is in the eating!* The establishment of theories thus acts here as an additional mathematical practice being put forth in the session.

Concluding remarks

The above analysis could be deepened and refined. However this sketch, albeit rapid, is significant: it illustrates how mathematics not only advanced in relation to its content, but also relative to its practices, and how both content and practices are intermingled in this advancement, going hand in hand, participating in the unfolding of the other. Content arises through mathematical practices, which in turn are geared toward specific contents. The need to talk about triangles and their angles as content gave rise to a specific *symbolisation* to represent it, which in turn helped to make sense of triangles, rectangle and their angles. The need to understand the 45^{o} angle as content, and the skepticism that it caused, led the *community of validation* to take shape, helping in return to give stronger meaning to the 45^{o} angle. The notion of the measure of the side of a square and its diagonal made emerge a question about their difference, becoming a *sub-problem* to inquire into, which led not only to understandings about their difference in the square, but also gave rise to the *theory* of the triangle's angle and its relation to its opposite side. And the list could go on, for each dimension of mathematical practices outlined, each linked to aspects of mathematical content covered in the session.

This extract is only a short glimpse into the nature of the work conducted regularly with groups like these in mental mathematics settings. As these mathematical practices continually unfolded, sessions after sessions and with different group of students, one cannot but be seized by how students plunged deeply into aspects at the heart of Bauersfeld's (1995, 1998) culture of mathematizing. The mathematical ideas emerge, are alive and flow dynamically. The students are strongly engaged, compelled to contribute, enthusiastic in responding to one another and to the ideas shared, and so forth.

However, above all, this was not staged nor planned. Mental mathematics sessions are usually designed to gather and then analyse students' strategies about various mental mathematics tasks. But classrooms are what they are, and students are who they are: asking them to solve mental mathematics tasks made emerge lots of questioning from them, and between them, about the mathematics. The tasks then became springboards for inquiry or "seeds" for explorations (Borasi, 1992, 1996; Schoenfeld, 2020), as opportunities for developing not only mathematical content but also mathematical practices. This is why the mental mathematics environment that students seem to be plunged into appeared to be worth reflecting on.

Although Papert never profoundly developed this concept, one is compelled to wonder if this environment of exploration happening in the mental mathematics sessions could represent, at least a little, what he had in mind with his *mathland*. Here, for example, is one quote taken from *The Children's machine*:

> It is thoroughly embedded in our culture that some of us have a head for figures while most don't, and accordingly, most people think of themselves as not mathematically minded. But what do we say about children who have trouble learning French in American schools? Whatever the explanation of their difficult, one certainly cannot ascribe it to a lack of aptitude for French – we can be sure that most of these children would have learned French perfectly well had they been born and raised in France. […] In the same way, we have no better reason to suppose that these children who have trouble with math lack mathematical intelligence than to suppose that the others lack "French intelligence". We are left with the question: What would happen if children who can't do math grew up in a Mathland, a place that is to math what France is to French? […] while what happened in the regular math class was more like the learning math as a foreign language. […] In the math class, where knowledge is not used but simply piled up like the bricks forming a dead building, there is no room for significant experimenting. (Papert, 1993, p. 64)

However bold, asserting that the mental mathematics environment in which students are plunged, for which the Grade-10 extract is an illustration of, could be aligned with a form of *mathland* has a nice ring to it. And, this ring leads one to become attentive to the strength of the engagement and the richness of the explorations undertaken. It seems to orient the focus, as Papert insisted, on doing mathematics more than on knowing mathematics. In this sense, doing mental mathematics becomes more about inquiring than about knowing facts (see PME-NA research report in Proulx, 2014, 2015a; or others e.g. in Proulx 2013, 2015b, 2019).

Although at first a curiosity, inquiring into the environment of the mental mathematics sessions seemed to help draw out both these content and practices dimensions, and their intertwinement in the advancement of mathematics in the sessions. And it might be where Papert's *mathland* fits in well, that is, in an environment where mathematics grows as much in terms of content as in terms of practices.

Acknowledgments

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CONSTRUCTION OF ARITHMETIC-ALGEBRAIC THINKING IN A SOCIO-CULTURAL INSTRUCTIONAL APPROACH

CONSTRUCTION D'UNE PENSÉE ARITHÉMICO-ALGÉBRIQUE DANS UNE APPROCHE SOCIOCULTURELLE DE L'ENSEIGNEMENT

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We present the results of a research project on arithmetic-algebraic thinking that was carried out jointly by a team in Mexico and another in Quebec[1]. The project deals with the concepts of variable and covariation between variables in the sixth grade at the elementary level and the first, second, and third years of secondary school – namely, children from 11 to 14 years old. We target secondary students (first year or K7) in this article. Our objective relates to the development of a gradual generalization in arithmetic-algebraic thinking in a socio-cultural approach to the learning of mathematics. We experimented with investigative situations using a paper-and-pencil approach and technology. We analyze the emergence, in this context, of a visual abstraction, the production of institutional and non-institutional representations, a sensitivity to contradiction, and, finally, the concepts of variable and of covariation between variables.

Key words: gradual generalization, socio-cultural approach, arithmetic-algebraic thinking.

Introduction: Steps of the Project

The project presented herein has been ongoing since 2008, carried out jointly by a team in Mexico and another in Quebec. The experimentation was done at the primary and secondary levels as well as in a pre-service teacher education program.

- Step 1: studies of the concept of function (Hitt, Gonzâlez & Morasse, 2008; Hitt & Gonzâlez-Martîn, 2015; Hitt & Quiroz, 2019; Passaro, 2009) among students in Secondary 2 and 3 (aged 13-15 years, K8 and K9 equivalents)
- Step 2: a study of the generalization of the concepts of variable and of covariation between variables in relation to arithmetic-algebraic thinking among Secondary 1 students in Quebec (aged 12-13 years, K7 equivalent) (Hitt, Saboya & Cortés, 2017, 2019a, 2019b) and among Secondary 3 students in Mexico.
- Step 3: studies of the concepts of variable and covariation between variables and of the generalization (in the transition from primary to secondary levels) related to arithmetic-algebraic thinking among 6th grade elementary students with learning difficulties in Mexico (11-12 years of age, K6 equivalent) (Hitt, Saboya & Cortés, 2017a, 2017b; Saboya, Hitt, Quiroz & Antoun, 2019).

Páez's (2004) doctoral thesis worked on teacher training with a teaching method based on collaborative learning, scientific debate, self-reflection, and the process of instutionalization (ACODESA) (see Hitt, 2007).

In order to use the same method in our project, which targeted elementary and secondary students, we had to use, in Step 1 of the project, the results obtained among Secondary 2 and 3 students to create theoretical tools which would allow us to better analyze students' spontaneous representations and their role in the resolution of non-routine situations.

[1] Joint project: Carlos Cortés (UMSNH); Samantha Quiroz (UAC); Fernando Hitt; and Mireille Saboya (UQAM).

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Step 2, which is the focus of this paper, will allow us to better understand the processes of abstraction[2] that trigger a generalization among students (in Secondary 1 in Quebec) in the transition from elementary to secondary levels as well as the construction of a cognitive structure related to arithmetic-algebraic thinking (which we elaborate further below).

We are currently in the process of analyzing the results of Step 3.

Theoretical framework: Socio-Cultural Approach to Learning

Our approach to the construction of knowledge is based on the notion of activity from Leontiev's (1978) activity theory. According to Leontiev, activity, mediated by mental reflection that situates a subject in the objective world, follows a system of social relations. Leontiev holds that an individual's activity depends on their place in society and their life circumstances (idem, p. 3). Further, activity is intimately related to a motive: "different activities are distinguished by their motives. The concept of activity is necessarily bound up with the concept of motive. There is no such thing as activity without a motive" (idem, p. 6). Hence, the activity of an individual in a society has a central role in the "subject-activity-object" relation (known as Leontiev's triangle) which, in turn, is part of a system of relations within the given society.

> It stands to reason that the activity of every individual depends on his place in society, on his conditions of life… The activity of people working together is stimulated by its product, which at first directly corresponds to the needs of all participants. (p. 3-6)

Engeström (1987, 1999) analyzes Leontiev's triangle as a model of the relation between subject, object, and artefact-mediation, and concludes that Leontiev's triangle does not capture all elements and relations of a system:

> I am convinced that in order to transcend the oppositions between activity and process, activity and action, and activity and communication, and to take full advantage of the concept of activity in concrete research, we need to create and test models that explicate the components and internal relations of an activity system… To overcome these limitations, the model may be expanded. (p. 29-30).

Voloshinov's (1929/1973) ideas about the construction of sign emphasize the importance of the collaborative work that enriches Leontiev and Engeström's theoretical approach: "[t]he reality of the sign is wholly a matter determined by that communication. After all, the existence of the sign is nothing but the materialization of that communication. Such is the nature of all ideological signs" (p. 13)

Building on the ideas above, we adapted Engelström's model (see Figure 1) while adhering to the ACODESA teaching method (Hitt, 2007). The mathematics classroom is viewed as a microsociety whose various members are the teachers, the students, the institution, and the tools used in the co-construction of knowledge through the resolution of investigative situations (physical materials, school textbooks, computers, etc.).

Collaborative work, communities of practice, and even societies, according to Engeström (1999), Legrand (2001), Leontiev (1978), and Wenger (1998), among others, involve a motive, rules, a division of labour among members, a mediation of artefacts, and interaction among the various actors (see Figure 1). In our case, given a mathematical task, we are interested in the co-construction of students' knowledge through the evolution of their representations in the context of an ACODESA method of teaching.

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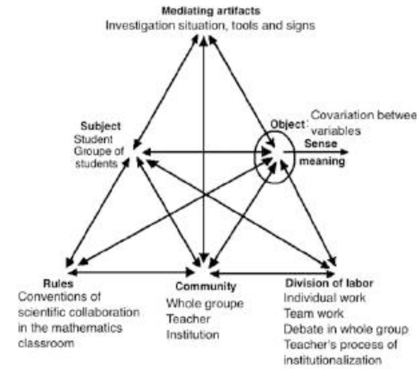
[2] The following is a translation of the definition of abstraction in the Larousse dictionary: an intellectual operation which consists in isolating by thought a characteristic of an object and considering it independently of the other characteristics of that object.



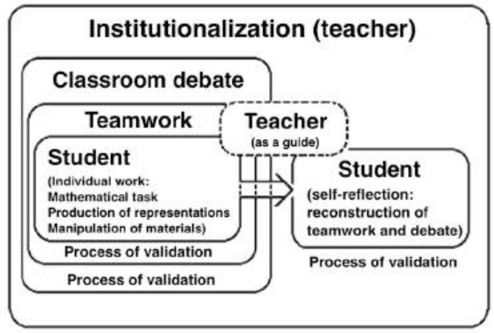


Figure 1. Engeström's (1999) model adapted to have covariation between variables as object; the various phases of the ACODESA method of teaching

Local Theoretical Framework and First Elements of Arithmetic-Algebraic Thinking

Given that we are primarily interested in the co-construction of knowledge, we searched for theoretical elements specific to moments of understanding or to the epistemic actions of Pontecorvo & Girardet (1993):

a) Higher-level methodological and metacognitive procedures; and

b) explanation procedures used for the interpretation of particular elements of the task.

To better understand the epistemic actions taking place during the resolution of a mathematical task, we use Rubinshtein's (1958) notions (cited in Davidov, 1990, p. 93-4) about the distinction between "visual empirical thought" and "abstract theoretical thought." In our project, just as in Rubinshtein and his group's, we are interested in the *gradual generalization* that occurs in a collaborative process of learning. For Davidov (idem), generalization is a process: "[i]f we mean the *process* of generalization, then the child's transition from a description of the properties of a particular object to finding and singling them out in a whole class of similar objects is usually indicated" (p. 5).

In the previous century, research about the transition from arithmetic to algebra focused on the concepts of epistemological obstacle (Vergaud, 1988), cuts (Filloy & Rojano, 1989), and gaps (Herscovics & Linchevski, 1994). Today, a change of paradigm purports that cognitive difficulties can be overcome (by a majority of students) with appropriate teaching. The discussion is one of a *continuum* rather than a *rupture* (Hitt, Saboya, and Cortés, 2017a). In this new paradigm, three types of approaches have emerged:

- "Early Algebra," which is based on a functional thinking approach with "an early inclusion of algebraic symbols as a valuable tool for early algebraic thinking" (Carraher, Schliemann, & Brizuela, 2000; Kaput, 1995, among others);
- "Algebraic nature of arithmetic" (Fujii 2003, among others); and
- a "development of algebraic thought" which acts as a support from which to delve deeper into arithmetic (Davidov, 1990; Kilpatrick, 2011; Radford, 2011a, 2011b, among others).

The Early Algebra approach prioritizes the use of institutional algebraic symbols to express covariation between variables and functions (tables of values and algebraic notations of the type $n \rightarrow n + 3$, for example). The second approach is similar to the first, albeit with a broader focus on the use of algebraic symbols in classical arithmetic tasks (see below in Section 3.2). However, the third approach relates to the use of general mathematical notions such as intuition, abstraction, and generalization in a socio-cultural learning of mathematics.

We situate ourselves in this third, socio-cultural type of approach (Engeström 1987, 1999; Leontiev, 1978; Voloshinov, 1929/1973) to the learning of mathematics (Radford's Theory of Objectification, 2011b). We propose the development of complex intuitive ideas by considering, for example, mathematical visualization (including "visual empirical thought" and "abstract theoretical thought," Rubinshtein, 1958), generalization (Davidov, 1990; Radford, 2011a), and the promotion of sensitivity to contradiction (Hitt, 2004) in mathematical activity. We worked on these general notions in elementary and secondary schools; specifically, we worked on the notions of variation and covariation between variables with the aim of developing arithmetic-algebraic thinking in students.

The notion of arithmetic-algebraic thinking is related to the development of a cognitive structure that we wish to promote in students, a structuring structure (a *habitus*) in the sense of Bourdieu (1980): the conditioning associated with a particular class of living conditions produces habitus, systems of durable and transposable dispositions, structured structures predisposed to function as structuring structures (p.88-89).

In our project, we attempt to show how to develop a structuring structure related to arithmetic-algebraic thinking in a mathematics classroom that is viewed as a microsociety.

Co-construction of Knowledge and a Sensitivity to Contradiction in the History of Mathematics

Szabó's (1960) studies of the history of mathematics detail elements that, during the Golden Age of the Greek civilization, contributed to the transformation of an empirical-visual mathematics into a definition-based on an axiomatic deductive science. We highlight the following elements:

a) The socio-political progress of the Greeks that allowed for the development of the art of rhetoric, polemical discussion, and critical thinking;

b) the influence of the philosophy of Parmenides of Elea and his disciple, Zeno of Elea (and, in particular, his paradoxes), on the Pythagoreans, who had an interest in mathematics; and

c) a "sensitivity to contradiction" when confronted with mathematical results developed by the Babylonians and the Egyptians, which did not always agree (e.g. the area of the disc).

Indeed, Szabó (idem) shows that Thales of Miletus' results were obtained in an empirical-visual manner. Szabó (idem) also gives the example of Plato's (4th century B.C.) Socratic dialog, Meno, which deals with the doubling of the area of a unit square. At the end of the dialog, a slave builds a square on the diagonal of the original unit square. It is easy, visually, to see that the surface area of the new square is double that of the first.

Parmenides' philosophy on the existence of being excludes non-being and provides the first reflections on logic and on the law of excluded middle. Szabó believes Parmenides influenced the Pythagoreans and that they, in turn, influenced mathematics, creating not only critical thinking but also a sensitivity to contradiction in mathematics. Szabó states:

The earliest Greek mathematicians, the Pythagoreans, borrowed the method of indirect demonstration from the Eleatic philosophy; consequently, the creation of deductive mathematical science can be attributed to the influence of the Eleatic philosophy. (p. 46)

Unfortunately, many of the Greeks' documents have been lost. Nevertheless, historians point to Euclid's Elements, which record the content of the Pythagoreans' books (Books VII, VIII, IX, and X). In Euclid's Elements, it is common to find theorems proved by contraposition. Vitrac (2012) confirms that indirect demonstrations (known as reduction to absurdity) are not uncommon in Euclid's Elements; they appear in a hundred or so propositions (p. 1).

One of Szabó's main assertions is that the transformation of mathematics into a deductive science (from the 5th century B.C. to the 3rd century B.C.) was accompanied by a transformation of mathematics into an anti-illustrative science. The visual demonstration of the duplication of the surface area of a unit square did not have a place in the new approach in Euclid's Elements. In Euclid, the illustration did not play a role in the visual demonstration process, but rather as an aid to the formal demonstration.

Historians report that the birth of algebra as a discipline was developed by the Persian al-Khwarizmi (790-850). Hence, while algebra did not originate with the Greeks, they did lay the groundwork for critical thinking, mathematical logic, indirect proof, and a sensitivity to contradiction. This type of thinking is, historically, an important precursor to the development of algebra.

How can we draw inspiration from the history of mathematics in the classroom? How can these historical elements of different cultures be integrated into the mathematics classroom?

Sensitivity to Contradiction in the Construction of Arithmetic-Algebraic Thinking

Research from the 1980s offers a glimpse into students' difficulties in solving algebraic problems. We consider, as an example, Fujii's (2003) study of the success rates among elementary and high-school students in the United States and in Japan in solving the following two problems:

Problem 1. Mary has the following problem to solve: "Find value(s) for x in the expression: x + x + x = 12" She answered in the following manner. a. 2, 5, 5; b. 10, 1, 1; c. 4, 4, 4 Which of her answer(s) is (are) correct? (Circle the letter(s) that are correct: a,b,c)	Problem 2. Jon has the following problem to solve: "Find value(s) for x and y in the expression: x + y = 16" He answered in the following manner. a. 6, 10; b. 9, 7; c. 8, 8 Which of his answer(s) is (are) correct? (Circle the letter(s) that are correct: a, b, c) State the reason for your selection.

> It is also important to note that it is rare for students to get both problems correct, which was also consistent with the data for both countries [USA and Japan]. Let me select the Athens (GA) 6th, 8th and 9th graders from the American data, simply because these students have a common educational environment. The percentages of correct answers for 6th, 8th, and 9th grade are 11.5%, 11.5% and 5.7% respectively. For Japanese students, the correct response from 5th, 6th, 7th, 8th, 10th and 11th grades are 0%, 3.7%, 9.5%, 10.8%, 18.1% and 24.8% respectively (Fujii, 1993).

These problems help distinguish between students with a *conception* of the role of a variable in an algebraic expression and those who had formed the *concept* of a variable.

By analyzing the tasks Fujii (2003) proposes, we see they had been designed as assessment tools (to detect the conceptions students had formed). The design of a task meant to promote learning based on students' conceptions, however, is a whole other matter. In what follows, we present two examples of sensitivity to contradiction.

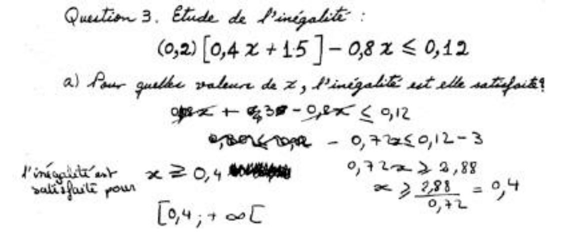
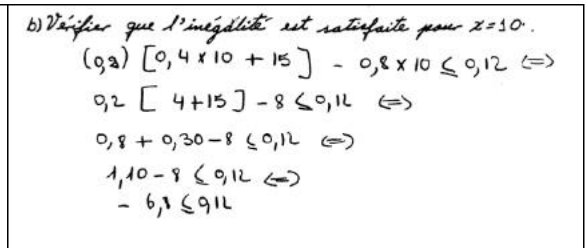
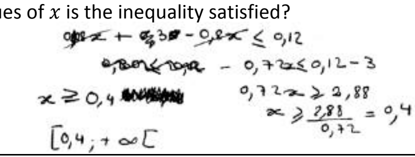
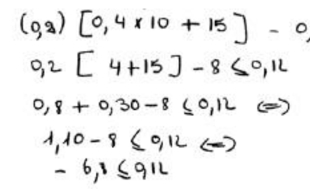
First example. Sensitivity to contradiction in the process of solving the following:

a) Solve this inequality: $0.2(0.4x + 15) - 0.8x \leq 0.12$
b) Verify that $x = 10$ is an element of the solution set.

In designing this activity, we took into account Brousseau's (1997) notion of epistemological obstacle in the learning of decimal numbers: an error that results when knowledge that, in other situations, had been valid and effective proves to be erroneous in a new situation. In this case, an error occurs when knowledge about multiplication of natural numbers is applied to multiplication of decimal numbers. We take advantage of this error to promote a richer mathematical structure: a sensitivity to contradiction. Here is an example of a student's work:

$$0{,}2 \times 0{,}4 = 0{,}8$$

$$0{,}2 \times 15 = 0{,}30$$

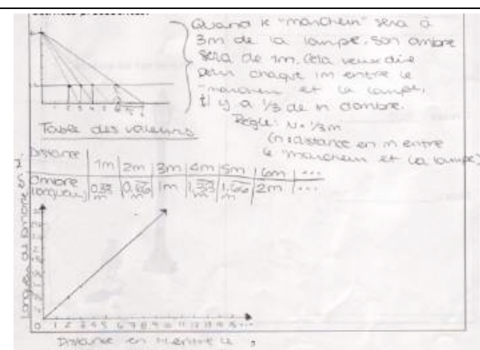
	
Question 3. Study of the inequality: $(0.2)[0.4x + 1.5] - 0.8x \leq 0.12$ a) For which values of x is the inequality satisfied? The inequality is satisfied for 	b) Verify that the inequality is satisfied for $x = 10$: 

We note the student made the mistakes anticipated by the researcher. The student had proposed the solution "st [solution] = 0,12," but after addressing question b), the student noticed the contradiction. The student retraced his steps to resolve the contradiction in part a). He spotted and overcame the cognitive contradiction, even if, formally, the contradiction remained in item b). This shows the student is sensitive to contradiction.

Second example. The *Shadow Situation* was one of five situations proposed in a month-and-a-half-long experiment with students in their third year of high-school. The five (sequential) situations were worked on in connection with the ACODESA method and with the goal of developing the concepts of covariation between variables and of function (Hitt & González-Martín, 2015; Hitt & Morasse, 2009). The following is a translation of the Shadow Situation given to students:

> Suppose we have a source of light with a height of 6 meters (a street light). We consider the shadow formed when a when a person who is 1.5 meters tall walks down the street. We are interested in the relationships between the quantities involved.
>
> 
>
> *Are some of the quantities dependent on one another? Which ones?*
> Select two quantities that depend on one another and describe the phenomenon with the various representations you used in previous activities.

Phase 1: Individual work. Two girls work on their own[3] to understand the task. One of them represented the situation through a proportional drawing. Starting with an empirical-visual thought, she found a relationship between the quantities "distance travelled by the person" and "length of the shadow."



Phase 2: Teamwork (Prusak, Hershkowits, & Schwarts, 2013, suggest groups of two to three). The two girls produce a verbal description of a relationship, an algebraic expression, and a graphical representation of the situation.

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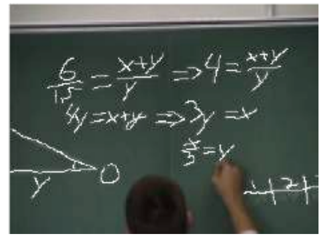
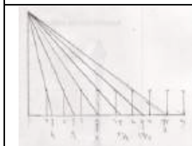
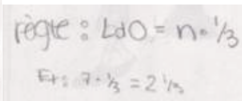
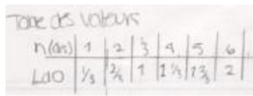
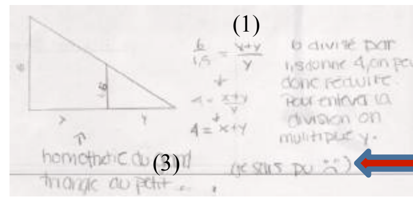
[3] Translation of text in top right corner of image: [w]hen the "walker" will be at 3m from the lamp, his shadow will be 1m. This means that for every 1m between the "walker [sic] and the lamp, there is 1/3 of a m of shadow.
*Rule: N * 1/3m (n [*sic*]: distance in m between the "walker [*sic*] and the lamp)*
Table of values, first row: distance; table of values, second row: shadow (length)
*Graph, x-axis: distance in M [*sic*] between [*sic*]; graph, y-axis: length of the shadow in M [*sic*]*



Phase 3: Classroom debate. One group of students failed to find an answer due to algebraic errors. Upon seeing the two girls' results, this group manages to construct an algebraic approach by using similar triangles.

Phase 4: Self-reflection. The instructor collects everything the students produced and re-assigns them the situation as homework, this time with the instruction to re-create the work done in class. The following is what one of the girls (mentioned above) produced as a reconstruction of what had been discussed in class:

Reconstruction of work done in teams	Reconstruction of the classroom debate
  [4]  [5]	(1) (2) (3)  [6]

She reconstructed with no difficulty what she had done numerically and visually with her team-mate. Unfortunately, when she wanted to reconstruct the boys' algebraic process, she made a mistake and failed to come up with a solution. In her drawing (the one on the right), she expressed a feeling of unease in the face of a contradiction she couldn't overcome. This shows she had formed a sensitivity to contradiction. From a cognitive standpoint, *a sensitivity to contradiction is an awareness of contradiction accompanied by a sense of unease, and its resolution by a sense of happiness.*

These examples demonstrate the importance of students' spontaneous representations. Given these findings on students' spontaneous representations, Hitt and Quiroz (2019) proposed the notion of ***socially-constructed representation***, one which materializes through the evolution of students' functional-spontaneous representation as it emerges in individual work and is then discussed in a team, in large groups, and in self-reflective work. According to Hitt and Quiroz (2019, p.79),

> [a] socially-constructed representation is one that emerges in individuals when given a non-routine activity; the actions in the interaction with the situation have functional (mental, oral, kinesthetic, schematic) characteristics and are related to a spontaneous (external) representation. The representation is functional in the sense that the student needs to make sense of the situation, and it is spontaneous because it naturally occurs in an attempt to understand and solve the non-routine situation. [Translation]

The Investigative Situation (the Task): Key Element in the Co-Construction of Mathematical Knowledge

The theories of didactical situations (Brousseau, 1998), of "problem solving" (Mason, Burton, & Stacey, 1982; Schoenfeld, 1985), and of Realistic Mathematical Education from Freudenthal (1991) have prompted changes in curricula worldwide. There is a break from the classical approach – that is, from "definition-theorem-exercises and problems" instruction. Situational problems, problems in general, and contextualized problems have a fundamental role to play in the new approach. In light of these theories, task design is viewed as central for overcoming cognitive barriers. A new era came for

[4] Translation: Rule: LoS [Length of Shadow] = n * 1/3

[5] In this table of values, the entries in the first row correspond to "n (dis) [distance]" and those in the second row to "LdO [Longueur de l'Ombre, or Length of Shadow]."

[6] Translation of items (1), (2), and (3) in the image: (1) 6 divided by 1.5 gives 4, so we can reduce. To get rid of the division, multiply y. (2) Homothety of the big triangle to the small one. (3) (I don't know anymore ☹)

the organization and role of the task in mathematics instruction, in situational problems related to creativity, and in mathematical modelling (Blum, Galbraith, Henn, & Niss, 2007; Hitt & González-Martín, 2015; Hitt, Saboya, & Cortés, 2017; Hitt & Quiroz 2019; Lesh & Zawojewski, 2007; Margolinas, 2013).

The activities we designed are related to the ACODESA teaching method in a socio-cultural approach to mathematics instruction. We call our activities "investigative situations":

> An **investigative situation** consists of different tasks that follow the steps of the ACODESA method. The tasks attempt to promote, first and foremost, the emergence of non-institutional or institutional representations, empirical-visual thinking related to diversified thinking (that is, divergent thinking), conjecture, prediction, and validation. In second and third stages (teamwork and classroom debates), we try to promote abstract thinking that includes sensitivity to contradiction as well as an evolved version of the representations and characteristics formed in the first stage. In a fourth stage, students re-construct what had been done in class so as to solidify the knowledge they had formed. Finally, the teacher reviews students' various solutions and presents the institutional position vis-à-vis the content considered in the situation. [Translation]

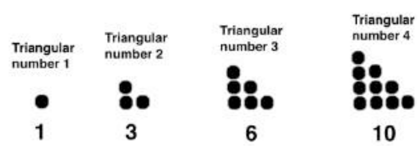
The design of investigative situations follows an organization such as that outlined in Hitt, Saboya, and Cortés (2017b).

Variation and Covariation between Variables: an Example with Polygonal Numbers

We now present the first step of an investigative situation that involves polygonal numbers and which is targeted towards students in their first year of high-school. This step consisted of five questions to be solved with paper and pencil. The second step had students use technology to validate their conjectures. In total, the situation was eight pages long. The following is a translation from French:

Step 1 (Individual work, followed by teamwork; paper-and-pencil approach)

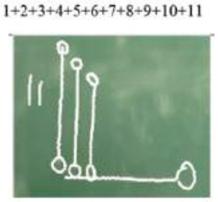
A long, long, long, long time ago (around 520 B.C.), a mathematician called Pythagoras founded a school on an island in ancient Greece. He and his students were fascinated by both numbers and geometry. One of their ideas consisted of representing numbers by geometric figures. They called these *polygonal numbers*. For example, they noticed that certain numbers could be represented by triangles. Thus, 1, 3, 6, and 10 are the first four triangular numbers since they can be represented by points arranged in triangles as follows:



1) Observe these numbers carefully. What is the fifth triangular number? Represent it. Explain how you did this.
2) How do you think a triangular number is constructed? What do you observe?
3) What is the 11th triangular number? Explain how you found its value.
4) You must write a SHORT email to a friend describing how to calculate the triangular number 83. Describe what you would write. YOU DON'T HAVE TO DO ANY CALCULATIONS!
5) And how would you calculate any triangular number? (We want a SHORT message here as well.)

Teams' Responses to Questions 1, 2, and 3

1+2+3+4+5+6+7+8+9+10+11

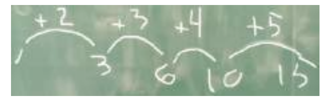


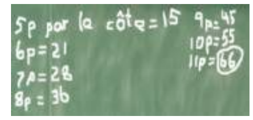
11+10+9+8+7+5+4+3+2+1

In this first step, we wanted to promote empirical-visual thinking (Rubinshstein, 1973) and generalization (Davidov, 1990; Radford, 2011). Students (in teams G1 and G3) naturally shifted from a visual approach to an arithmetic procedure (an epistemic action). For example, to calculate T_{11}, they wrote 1+2+3+4+5+6+7+8+9+10+11.

Team G2 first moved from a concrete visual approach to a more general visual approach and then to an arithmetic procedure (an epistemic action). Hence, for T_{11}, they wrote 11+10+9+8+7+6+5+4+3+2+1.

We note the abandonment of the iconic representation by one student (from team G4) who, during the classroom debate, switched from a detached visual approach to the polygonal configurations to an iterative calculation which he had not discussed with his teammates (what Rubinshtein would term theoretical abstract thinking). Team G4 used this final strategy, along with Excel, to tackle the fifth question of the second step of the investigative situation.





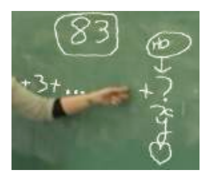
This shift shows the importance of teamwork and of Yan's reflection as he organized his thoughts (in what Vygotsky, 1932/1962, would call inner speech) so as to communicate them to the group (Voloshinov's construction of sign). Yan needed to make himself understood by the rest of the class.

Team Responses to Questions 4 and 5 and First Classroom Debate

The following are the responses given by each team in the first classroom debate:

	Team's response to question 4	Team's response to question 5
G1	We add all the numbers from 1+2+3... all the way to the number of points on the side.	We add all the numbers from 1+2+3... all the way to the number of points on the side.
G2	Add up the numbers from 1 to 83.	You add up the numbers from there?
G3	You have to do 83+82+81… all the way to 1.	Calculate the last diagonal column and calculate by doing -1 to the number. E.g. 15th, 15+14+13… etc.
G4	You have to do; 1+2+3+4+5+6+7+8… +83 and this will give you the answer.	You put the same number on the other sides and then you add up 1+2+3+4+5+6… until you get to your number and your answer is the triangular number.

During the classroom debate, the researcher asked what answers had been written in response to question 5 (see responses above), which asked for a short message describing how to find any triangular number. The students first suggested the sum "1+2+3 all the way to your number." The researcher intervened: how can I write a number I don't know? Different proposals emerged. The first was to write "?" ; afterwards, they proposed "x" or "y." The teacher asked whether a heart could be used: "♥." One student replied that they could use anything that wasn't a number.



The students transitioned from empirical-visual thinking to abstract arithmetic-algebraic thought. The variable was first expressed in words: "all the way to your number." Then, it was expressed as "?," then, as "x" or "y," and, finally: "we can use anything that isn't a number."

Output produced by team G4 (during teamwork and during the classroom debate)			
Teamwork and spontaneous generalization	Surprise at finding a decimal number as T_{100} and team discussion towards generalization (with Excel)	General computation of a triangular number presented to the rest of the class	Generalization obtained through classroom debate
(83+1)÷2= 42 42x 83 =	100 ((100+1)÷2=5x X x 100 =	(46+1)÷2 = 47 ÷ 2 =23,5 46x23,5=1081	(x+1)÷2=y y·X=

We note that each abstraction came with a certain type of generalization. The processes of abstraction were of the following types: *visual abstraction, arithmetic abstraction, emergence of the concept of a variable, emergence of the concept of covariation between variables*.

45 Days Later: Phase of Self-Reflection (Reconstruction)

During this step, Yan, the student who found an algebraic expression for triangular numbers, tried to remember his formula but got it wrong. He wrote: ***(Row*2)-1=y*** and ***(Row*y=triangular number***. During the pentagonal number activity which we had given him as a challenge, he wrote that ***Row*(Row + (Row * 0.5 – 0.5)) = pentagonal number***. He found this expression by using the same strategy he had used 45 days beforehand to deal with triangular numbers. This expression is equivalent to the institutional one: $P_n = \frac{n(3n-1)}{2}$.

During this process of self-reflection, another student obtained the following in response to the question about triangular numbers: ***Odd number: (row + 1) ÷ 2 * row = triangular number.*** This expression is equivalent to $T_n = \frac{n(n+1)}{2}$ (when restricted to odd numbers).

Conclusions

In this paper, we wanted to show the various elements needed for the construction of arithmetic-algebraic thinking. Building on a few ideas from the history of mathematics, from a socio-cultural theory of learning, and from the ACODESA teaching method, we have shown that for the construction of arithmetic-algebraic thinking, various elements of the mathematics classroom need to be taken into account: the role of the task (investigative situations) in the acquisition of knowledge, communication in the classroom, mathematical visualization, the role of non-institutional and institutional representations, generalization, conjecture, sensitivity to contradiction, validation, and proof.

Our approach seeks to develop and enrich an association between arithmetic and algebra (*a habitus*) so as to promote the construction of a structuring structure, in the sense of Bourdieu (1980), that is related to arithmetic-algebraic thinking and which supports not only algebra, but also an enrichment of the cognitive structure of arithmetic tasks.

Further, we observed the emergence of the concepts of variable and of covariation between variables through the process of co-construction of knowledge.

The results of our studies have encouraged us to experiment new investigative situations (following the ACODESA method) in grade 6 classrooms. So far, we have suggested five investigative situations of different types and which require electronic tablets: Marcel's Restaurant, The El Dorado Jewelry Shop, Windows, The Garden and the Pumpkins, and Rectangles and Disks. We are currently analyzing the results.

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CONSTRUCTION D'UNE PENSÉE ARITHÉMICO-ALGÉBRIQUE DANS UNE APPROCHE SOCIOCULTURELLE DE L'ENSEIGNEMENT

CONSTRUCTION OF ARITHMETIC-ALGEBRAIC THINKING IN A SOCIO-CULTURAL INSTRUCTIONAL APPROACH

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Dans cet document, nous présentons les résultats d'un projet de recherche sur la pensée arithmético-algébrique, qui a été réalisé conjointement par une équipe au Mexique et une autre au Québec[7]*. Le projet porte sur le concept de variable et de covariation entre variables dans les classes de 6e année du primaire, 1re, 2e et 3e du secondaire, à savoir les enfants de 11 à 14 ans. Nous allons ici cibler les élèves de 1re secondaire. Notre objectif porte sur le développement d'une généralisation graduelle liée à la pensée arithmético-algébrique dans une approche socioculturelle de l'apprentissage des mathématiques. Nous avons expérimenté avec des situations d'investigation dans une approche papier-crayon, puis avec la technologique. Dans ce contexte, nous analysons : l'émergence d'une abstraction visuelle, la production de représentations institutionnelles et non institutionnelles, une sensibilité à la contradiction et, enfin, l'émergence de la notion de variable et de covariation entre variables.*

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Mots-clés : Généralisation graduelle, approche socioculturelle, pensée arithmético-algébrique.

Introduction (étapes du projet)

Le projet que nous vous présentons ci-dessous a été réalisé conjointement par une équipe du Mexique et un autre du Québec dès 2008. L'expérimentation a été faite au primaire, au secondaire et à la formation des maîtres.

- 1re étape : Étude du concept de fonction (Hitt, Gonzâlez & Morasse, 2008; Hitt & Gonzâlez-Martîn, 2015; Hitt & Quiroz, 2019; Passaro, 2009) chez les élèves de 2e et 3e secondaire (13-15 ans équivalentes à K8 et K9).
- 2e étape : Étude de la généralisation liée à la notion de variable et de la covariation entre variables en lien avec la pensée arithmético-algébrique chez les élèves du 1re secondaire au Québec (12-13 ans équivalente à K7) (Hitt, Saboya & Cortés, 2017, 2019a, 2019b), et 3e secondaire au Mexique.
- 3e étape : Étude des notions de variable, de covariation entre variables et de la généralisation (transition primaire-secondaire) en lien avec la pensée arithmético-algébrique chez des élèves en difficulté d'apprentissage de la 6e année du primaire au Mexique (11-12 ans équivalente à K6) (Hitt, Saboya & Cortés, 2017a, 2017b; Saboya, Hitt, Quiroz et Antoun, 2019).

Dans sa thèse de doctorat, Páez (2004) a travaillé à la formation des maîtres avec une méthode d'enseignement basée sur l'apprentissage collaboratif, le débat scientifique, l'autoréflexion et le processus d'institutionnalisation (ACODESA) (voir Hitt, 2007).

Afin de pouvoir utiliser la même méthode dans notre projet, mais cette fois avec des élèves au primaire et au secondaire, nous avons dû, pendant la 1re étape du projet, construire des outils théoriques avec les résultats obtenus auprès d'élèves de 2e et 3e secondaire. Ces outils nous permis par la suite de mieux analyser les représentations spontanées des élèves et leur rôle dans la résolution de situations non routinières.

La 2e étape, celle à laquelle nous allons nous attarder, va nous permettre de mieux comprendre les processus d'abstraction[8] qui d'déclenchent une généralisation chez les élèves (1er secondaire au Québec) dans la transition primaire-secondaire et sur la construction d'une structure cognitive liée à la pensée arithmético-algébrique (nous allons préciser plus loin).

Nous sommes présentement en train d'analyser les résultats de la 3e étape.

Cadre théorique général (approche socioculturelle de l'apprentissage)

Notre approche sur la construction des connaissances est basée sur la notion d'activité de Leontiev (1978) liée à sa théorie de l'activité. Selon Leontiev, l'activité, médiatisée par la réflexion mentale, qui a comme fonction d'orienter le sujet dans le monde objectif, obéit au système des relations dans la société. Pour Leontiev, l'activité de chaque individu dépend de sa place dans la société, de ses conditions de vie (idem, p. 3). Selon Leontiev, l'activité est intimement liée à un motif : « different activities are distinguished by their motives. The concept of activity is necessarily bound up with the concept of motive. There is no such thing as activity without a motive » (idem, p. 6). Ainsi, l'activité d'un individu dans une société, a un rôle central dans la relation « subject-activity-object » (connue actuellement comme le triangle de Leontiev) et celle-ci est une partie d'un système de relations dans cette société.

> It stands to reason that the activity of every individual depends on his place in society, on his conditions of life… The activity of people working together is stimulated by its product, which at first directly corresponds to the needs of all participants. (p. 3-6)

[8] La définition d'abstraction du dictionnaire Larousse est : Opération intellectuelle qui consiste à isoler par la pensée des caractères de quelque chose et à le considérer indépendamment des autres caractères de l'objet.

Engeström (1987, 1999) analyse le triangle de Leontiev comme un modèle de la relation entre le sujet, l'objet et les artefacts de médiation, et conclut que le triangle de Leontiev ne montre pas les différents éléments du système et ses relations :

> I am convinced that in order to transcend the oppositions between activity and process, activity and action, and activity and communication, and to take full advantage of the concept of activity in concrete research, we need to create and test models that explicate the components and internal relations of an activity system… To overcome these limitations, the model may be expanded. (p. 29-30).

Les idées de Voloshinov (1929/1973) sur la construction du signe : "The reality of the sign is wholly a matter determined by that communication. After all, the existence of the sign is nothing but the materialization of that communication. Such is the nature of all ideological signs" (p. 13), accentuent l'importance du travail collaboratif qui enrichit l'approche théorique de Leontiev et d'Engeström.

En nous appuyant sur les idées précédentes, nous avons adapté le modèle d'Engeström (voir Figure 1) tout en respectant la méthode d'enseignement ACODESA (Hitt, 2007). La classe de mathématiques est considérée comme une microsociété dont les différents acteurs sont les maîtres, les élèves, l'institution et différents types d'outils à utiliser pendant la co-construction des connaissances à travers la résolution de situations d'investigation (matériel physique, manuels scolaires, ordinateurs, etc.).

Selon Engeström (1999), Legrand (2001), Leontiev (1978), Wenger (1998), entre autres, dans un travail collaboratif, ou dans une communauté de pratique, ou même dans une société, il doit y avoir un motif, des règles, une division du travail entre les membres, la médiation des instruments, et une interaction entre les différents acteurs (voir Figure 1). Dans notre cas, étant donné une tâche mathématique, nous sommes intéressés à la co-construction des connaissances des élèves à travers l'évolution de leurs représentations en suivant la méthode d'enseignement ACODESA.

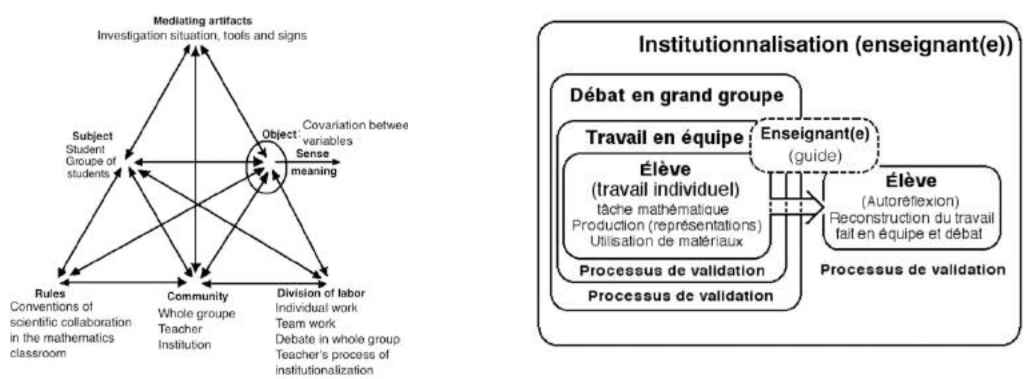


Figure 1. Adaptation du modèle d'Engeström (1999) ayant comme objet la covariation entre variables et les différentes phases de la méthode d'enseignement ACODESA

Cadre théorique local et premiers éléments de la pensée arithmético-algébrique

Étant donné que nous nous intéressons particulièrement à la co-construction des connaissances, nous avons cherché des éléments théoriques spécifiques sur les moments de compréhension ou d'actions épistémiques de Pontecorvo & Girardet (1993) :

a) [A] higher level methodological and metacognitive procedures,
b) Explanation procedures used for the interpretation of particular elements of the task.

Pour mieux comprendre les actions épistémiques lors la résolution d'une tâche mathématique, nous allons utiliser les notions de Rubinshtein (1958) (cité par Davidov, 1990, p. 93-94) sur la distinction entre « visual empirical though » et « abstract theoretical though ». Dans notre projet tout comme Rubinshtein et son équipe, nous sommes intéressés *à une généralisation graduelle* dans un processus d'apprentissage collaboratif. Pour Davidov (idem), la généralisation c'est un processus : If we mean the *procces* of generalization, then the child's transition from a description of the properties of a particular object to finding and singling them out in a whole class of similar objects is usually indicated. (p. 5)

Au siècle dernier, les recherches portant sur la transition de l'arithmétique à l'algèbre étaient centrées sur la notion d'obstacle épistémologique (Vergaud, 1988), coupure (Filloy & Rojano, 1989) ou écart (Herscovics & Linchevski, 1994). Aujourd'hui, il a eu un changement de paradigme qui supporte l'idée que les difficultés cognitives peuvent être surmontées (par une grande majorité des élèves) par un enseignement approprié. On parle d'un *continuum* au lieu d'une *rupture* (Hitt, Saboya et Cortés, 2017a). Trois types d'approches se sont manifestés dans ce nouveau paradigme. Ce sont :

- « Early Algebra », basée sur une approche de la pensée fonctionnelle avec « un early inclusion of algebraic symbols as a valuable tool for early algebraic thinking » (Carraher, Schliemann & Brizuela, 2000; Kaput, 1995, entre autres) ;
- « Algebraic nature of arithmétique » (Fujii 2003, entre autres);
- « Développement d'une pensée algébrique » qui donne un support pour approfondir l'arithmétique (Davidov, 1990; Kilpatrick, 2011; Radford, 2011a, 2011b, entre autres).

L'approche Early Algebra donne une priorité à l'utilisation de symboles algébriques institutionnels pour exprimer la covariation entre variables et les fonctions (table de valeurs, notations algébriques de la forme $n \rightarrow n + 3$ par exemple). La 2^e^ approche est similaire à la 1^re^, mais avec un domaine plus élargi sur l'utilisation de symboles algébriques dans les tâches classiques de l'arithmétique (voir plus loin, section 3.2). Par contre, le 3^e^ est liée à l'utilisation des notions mathématiques globales comme l'intuition, l'abstraction et la généralisation dans une apprentissage socioculturelle des mathématiques.

Nous nous situons plutôt dans cette 3^e^ approche de type socioculturel (Engeström 1987, 1999; Leontiev, 1978; Voloshinov, 1929/1973) de l'apprentissage des mathématiques (théorie de l'objectivation de Radford, 2011b). Nous proposons le développement d'idées intuitives complexes, en prenant en considération, par exemple, la visualisation mathématique (en incluant « visual empirical thought » et « abstract theoretical thought », Rubinshtein, 1958), la généralisation (Davidov 1990; Radford 2011a) et la promotion d'une sensibilité à la contradiction (Hitt, 2004) dans l'activité mathématique. Ces notions générales, nous les avons travaillées à l'école primaire et secondaire, spécifiquement avec les notions de variation et de covariation entre variables, avec l'objectif de développer chez les élèves une pensée arithmético-algébrique.

Cette notion de pensée arithmético-algébrique est liée au développement d'une structure cognitive que l'on veut promouvoir chez les élèves, structure structurante (un *habitus*) dans le sens de Bourdieu (1980):

> Les conditionnements associés à une classe particulière de conditions d'existence produisent des habitus, systèmes de dispositions durables et transposables, structures structurées prédisposées à fonctionner comme structures structurantes… (p. 88-89)

Notre projet essaie de montrer comment développer cette structure structurante dans la classe de mathématique vue comme une microsociété en relation à la pensée arithmético-algébrique.

La co-construction des savoirs et la sensibilité à la contradiction dans l'histoire

Les recherches de Szabó (1960) sur l'histoire des mathématiques fournissent les éléments qui ont participé à la transformation pendant la période d'or des Grecs d'une mathématique empirico-visuelle en une science déductive basée sur des définitions et des axiomes. Nous retiendrons :

a) L'avancée sociopolitique des Grecs qui leur a permis de développer l'art de la rhétorique, la discussion polémique et la pensée critique,

b) L'influence de la philosophie de Parménide d'Élée et de son disciple Zénon d'Élée (avec ses paradoxes) sur les pythagoriciens qui s'intéressaient aux mathématiques.

c) La « sensibilité à la contradiction » face aux résultats mathématiques développés par les Babyloniens et les Égyptiens, et qui ne concordaient pas toujours (par exemple l'aire du disque).

En effet, Szabó (*idem*) nous montre que les résultats de Thalès de Milet avaient été obtenus de manière empirico visuelle. Szabo (*idem*) nous donne aussi l'exemple du dialogue de Ménon de Platon (IV^{e} siècle av. J-C.) traitant de la duplication de l'aire d'un carré unitaire. À la fin du dialogue, un esclave construit un carré sur la diagonale du carré unitaire original. On peut facilement constater visuellement que l'aire de ce nouveau carré est le double de l'aire du premier.

La philosophie de Parménide sur l'existence de l'être exclut le non-être, et fournit les premières réflexions sur la logique et le principe du tiers exclu. Szabo pense que Parménide a eu une influence sur les pythagoriciens et qu'ils ont à leur tour influencé les mathématiques, créant non seulement une pensée critique, mais aussi une sensibilité à la contradiction en mathématiques. Szabó affirme:

> The earliest Greek mathematicians, the Pythagoreans, borrowed the method of indirect demonstration from the Eleatic philosophy; consequently, the creation of deductive mathematical science can be attributed to the influence of the Eleatic philosophy. (p. 46).

Malheureusement, de nombreux documents des Grecs ont été perdus; cependant, les historiens nous informent que dans les Éléments d'Euclide se trouve le contenu des livres conçus par les pythagoriciens, qui ont été transformés par Euclide (livres VII, VIII, IX et X). Dans les Éléments d'Euclide, il est habituel de trouver des théorèmes prouvés par contraposition. Vitrac (2012) affirme: Les démonstrations indirectes (dites par réduction à l'absurde) ne sont pas rares dans les Éléments d'Euclide; ils apparaissent dans une centaine de Propositions (p. 1).

L'une des principales thèses de Szabó est celle que la transformation des mathématiques en science déductive (du V^{e} siècle av. J-C., au III^{e} siècle av. J-C.), s'est également transformée en science anti-illustrative. La démonstration visuelle sur la duplication de la surface d'un carré unitaire n'avait plus sa place dans la nouvelle approche dans les Éléments d'Euclide. Avec Euclide, la figure a joué un rôle d'aide à la démonstration formelle et non aux processus visuels de démonstration.

Les historiens nous signalent que la naissance de l'algèbre en tant que discipline a été développée par le perse al-Khwarizmi (790-850). Cela nous montre que bien que l'algèbre ne soit pas originaire des Grecs, ils ont jeté les bases de la pensée critique, de la logique mathématique, de la preuve indirecte et d'une sensibilité à la contradiction. Ce type de pensée est historiquement important avant le développement de l'algèbre.

Comment s'inspirer de l'histoire de la mathématique dans la classe? Comment intégrer ces éléments historiques des différentes cultures dans la classe de mathématiques?

Sensibilité à la contradiction dans la construction de la pensée arithmético-algébrique

Les résultats des recherches des années 1980, nous ont donné un aperçu des difficultés des élèves face à la résolution de problèmes de type algébrique. Nous allons prendre comme exemple Fujii (2003) qui montre les pourcentages de réussite des élèves du primaire et du secondaire des États-Unis et du Japon par rapport à deux problèmes :

Problem 1. Mary has the following problem to solve: "Find value(s) for x in the expression: x + x + x = 12" She answered in the following manner. a. 2, 5, 5; b. 10, 1, 1; c. 4, 4, 4 Which of her answer(s) is (are) correct? (Circle the letter(s) that are correct: a,b,c)	Problem 2. Jon has the following problem to solve: "Find value(s) for x and y in the expression: x + y = 16" He answered in the following manner. a. 6, 10; b. 9, 7; c. 8, 8 Which of his answer(s) is (are) correct? (Circle the letter(s) that are correct: a, b, c) State the reason for your selection.

It is also important to note that it is rare for students to get both problems correct, which was also consistent with the data for both countries [USA and Japan]. Let me select the Athens (GA) 6th, 8th and 9th graders from the American data, simply because these students have a common educational environment. The percentages of correct answers for 6th, 8th, and 9th grade are 11.5%, 11.5% and 5.7% respectively. For Japanese students, the correct response from 5th, 6th, 7th, 8th, 10th and 11th grades are 0%, 3.7%, 9.5%, 10.8%, 18.1% and 24.8% respectively (Fujii, 1993).

Ces problèmes permettent de différencier les élèves qui ont une *conception* du rôle de l'inconnue dans une expression algébrique de ceux qui ont construit le *concept* d'inconnue.

En analysant les tâches proposées par Fujii (2003), nous voyons que celles-ci ont été conçues pour l'évaluation (détection des conceptions). Construire une tâche pour promouvoir l'apprentissage à partir des conceptions des élèves est tout autre chose. Voici deux exemples sur la sensibilité à la contradiction.

Premier exemple. Sensibilité à la contradiction dans un processus de résolution de :

a) Résoudre l'inégalité: $0,2\ (0,4x + 15) - 0,8x \leq 0,12$
b) Vérifier que $x = 10$ est un élément de l'ensemble solution.

Dans l'élaboration de cette activité, nous avons tenu compte de la notion d'obstacle épistémologique de Brousseau (1997) sur l'apprentissage des nombres décimaux. L'erreur est conçue comme une connaissance qui a été efficace, valable dans d'autres situations, mais qui s'avère erronée dans une nouvelle situation. Dans ce cas, c'est la connaissance de la multiplication des nombres naturels qui appliquée aux nombres décimaux va entrainer une erreur. Nous profitons de l'erreur pour promouvoir une structure plus riche de la mathématique : la sensibilité à la contradiction. Voici un exemple d'une production d'un élève :

$$0,2 \times 0,4 = 0,8$$

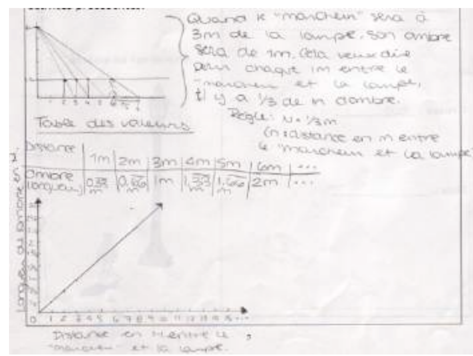
$$0,2 \times 15 = 0,30$$

Question 3. Étude de l'inégalité : $(0,2)[0,4x+15] - 0,8x \leq 0,12$ a) Pour quelle valeur de x, l'inégalité est elle satisfaite? $0,8x + 0,30 - 0,8x \leq 0,12$ $- 0,72x \leq 0,12 - 3$ $0,72x \geq 2,88$ $x \geq \frac{2,88}{0,72} = 0,4$ l'inégalité est satisfaite pour $x \geq 0,4$ $[0,4 ; +\infty[$	b) Vérifier que l'inégalité est satisfaite pour $x=10$. $(0,2)[0,4 \times 10 + 15] - 0,8 \times 10 \leq 0,12 \Leftrightarrow$ $0,2 [4+15] - 8 \leq 0,12 \Leftrightarrow$ $0,8 + 0,30 - 8 \leq 0,12 \Leftrightarrow$ $1,10 - 8 \leq 0,12 \Leftrightarrow$ $- 6,1 \leq 0,12$

On peut remarquer que l'élève a commis les erreurs anticipées par le chercheur. L'élève est arrivé à proposer la solution St [Solution] = 0,12, mais à la question b), l'élève a remarqué la contradiction. L'élève est revenu sur ses pas pour résoudre la contradiction en a). Il est sorti de la contradiction cognitive, contradiction qu'il a repérée, même si formellement la contradiction continue dans l'item b. Cela montre que l'élève est sensible à la contradiction.

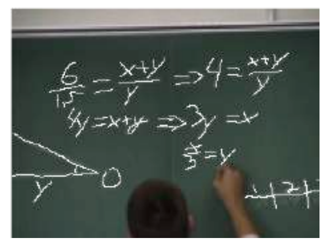
Deuxième exemple. *Situation sur les ombres* : Cette situation était l'une des 5 situations proposées lors d'une expérimentation d'une durée d'un mois et demi, avec des élèves de 3^e^ secondaire. Les 5 situations (enchainées) ont été travaillés en lien avec la méthode ACODESA, avec la finalité de faire développer les concepts de covariation entre variables et de fonction (Hitt et González-Martín 2015, Hitt et Morasse 2009) :

> Supposons que nous avons une source lumineuse d’une hauteur de 6 m (un lampadaire). Nous observons l’ombre sur le sol lorsqu’une personne de 1,5 m de hauteur marche dans la rue. Nous nous intéressons aux relations entre les grandeurs en jeu.
>
> 
>
> *Existe-t-il des grandeurs qui sont dépendantes les unes des autres? Lesquelles?*
> Sélectionne deux grandeurs qui sont dépendantes l’une de l’autre et décris le phénomène à l’aide des différentes représentations que tu as utilisées dans les activités précédentes.



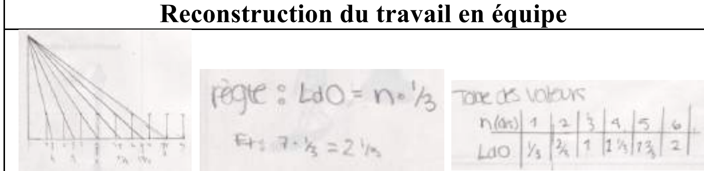
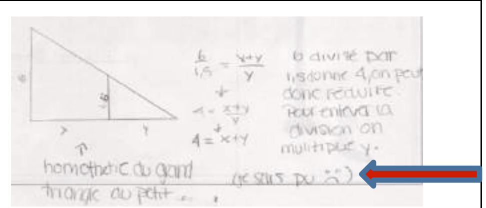
Phase 1 : Travail individuel. Deux filles travaillent individuellement pour comprendre la tâche. L’une d’elles a représenté la situation par un dessin proportionnel. À partir d’une pensée empirique visuelle elle a trouvé une relation entre les grandeurs « distance parcourue par la personne » et « mesure de l’ombre ».

Phase 2 : Travail en équipe (Prusak, Hershkowits et Schwarts 2013, suggèrent des équipes de deux ou trois personnes). Les deux filles parviennent à construire : une relation verbale, une expression algébrique et une représentation graphique de la situation.



Phase 3 : Débat en grand groupe. Une équipe d’élèves n’est pas parvenue à trouver la réponse à cause d’erreurs algébriques. À la vue du résultat des deux filles, ils ont réussi à construire une approche algébrique en utilisant des triangles semblables.

Phase 4 : Autoréflexion. Le professeur collecte alors toutes les productions des élèves, puis leur remet la même situation à faire en devoir avec pour consigne de reconstruire tout le travail réalisé en classe (4[e] phase). Voici la reconstruction de ce qui avait été débattu en classe par l’une des filles dont on a parlé.

Reconstruction du travail en équipe	Reconstruction du débat en grand groupe
	

Elle a reconstruit, sans difficulté, ce qu’elle avait fait de façon numérique et visuelle avec sa coéquipière. Malheureusement, quand elle a voulu reconstruire le processus algébrique des garçons, elle a fait une erreur sans arriver à la solution. Dans son dessin, elle a exprimé (voir dessin à droite) un sentiment de gêne face à une contradiction de laquelle elle ne pouvait pas se sortir. Cela montre que cette fille a développé une sensibilité à la contradiction. D’un point de vue cognitif, *la sensibilité à la contradiction est la prise de conscience d’une contradiction, accompagnée d’un sentiment de malaise, et son dépassement d’un sentiment de bonheur*.

Les exemples nous montrent l’importance des représentations spontanées des élèves. À partir de ces résultats sur les représentations spontanées des élèves, Hitt et Quiroz (2019) ont proposé la notion de ***représentation socialement construite***, qui est liée à l’évolution de la représentation fonctionnelle-spontanée chez les élèves quand celle-ci a émergé dans le travail individuel, puis a été discutée à l'intérieur d'une équipe, en grand groupe et dans un travail d'autoréflexion. Selon Hitt et Quiroz (2019, p. 79) :

> **Définition.** Une RF-S est une représentation qui émerge chez les individus dans la pratique, face à une activité non routinière : les actions liées à l'interaction avec la situation ont des caractéristiques fonctionnelles (mentales, orales, kinesthésiques, schématiques) et sont liées à une représentation spontanée (externe). La représentation est fonctionnelle dans le sens où l'élève a besoin de donner un sens à la situation et qu'elle est spontanée, car elle s'exprime naturellement dans l'action quand on essaye de comprendre et de résoudre la situation non routinière.

La situation d'investigation (la tâche) comme élément clé dans la co-construction du savoir mathématique

La théorie des situations didactiques (Brousseau, 1998), celle du « problem solving » (Mason, Burton & Stacey, 1982; Schoenfeld, 1985) et de la mathématique réaliste de l'école de Freudenthal (1991) ont provoqué un changement dans les curriculums du monde entier. L'approche classique, à savoir l'enseignement « définition-théorème-exercices et problèmes » a été brisée. Les situations-problèmes, les problèmes en général et les problèmes en contexte ont pris alors une place fondamentale dans cette nouvelle approche. À partir de ces théories, on retiendra l'élaboration de la tâche comme élément central pour le dépassement d'un obstacle cognitif.

Une nouvelle ère est apparue pour l'organisation et le rôle de la tâche dans l'enseignement des mathématiques, les situations problèmes liées à la créativité et à la modélisation mathématique (Blum, Galbraith, Henn & Niss, 2007; Hitt et González-Martín, 2015; Hitt, Saboya et Cortés, 2017; Hitt et Quiroz 2019; Lesh & Zawojewski, 2007; Margolinas, 2013).

Dans notre cas, les activités que nous avons développées sont liées à la méthode d'enseignement ACODESA, dans une approche socioculturelle de l'apprentissage des mathématiques. Nous les avons appelées les « situations d'investigation ».

> **Situation d'investigation**. La situation est constituée de différentes tâches que suivent les étapes de la méthode ACODESA. Les tâches essayent de promouvoir premièrement, l'émergence des représentations non institutionnelles ou institutionnelles, une pensée visuelle empirique liée à une pensée diversifiée (pensée divergente), à la conjecture, à la généralisation, à la prédiction et à la validation. Dans un deuxième et troisième temps (travail en équipe et grand groupe), on essaye de promouvoir une pensée abstraite où la sensibilité à la contradiction est une partie ainsi comme une évolution des représentations et caractéristiques de la première étape. Un quatrième temps est envisagé pour promouvoir une connaissance plus stable avec une reconstruction de ce qui avait était fait en classe. Finalement, l'enseignant(e) fait un retour sur les différentes solutions des élèves et présente la position institutionnelle par rapport au contenu envisagé dans la situation.

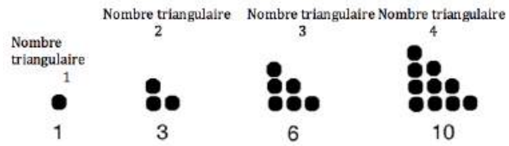
L'élaboration des situations d'investigation suit une organisation comme celle signalée dans Hitt, Saboya et Cortés (2017b).

Variation et covariations entre variables (un exemple avec les nombres polygonaux)

Nous allons présenter ici la 1re étape d'une situation d'investigation portant sur les nombres polygonaux et destinée à des élèves de la 1re année du secondaire. Cette étape était composée de 5 questions à résoudre papier-crayon. Dans la 2e étape était envisagée l'utilisation de la technologie pour valider leurs conjectures. Au total la situation comptait 8 pages.

1re étape (Travail individuel, suivi de travail en équipe, approche papier-crayon)
Il y a très, très, très, longtemps (vers l'an 520 av. J.-C.), un mathématicien du nom de Pythagore fonda une école dans une île de la Grèce antique. Ses élèves et lui étaient fascinés à la fois par les nombres et par la géométrie. Une de leurs idées consistait à représenter les nombres par des figures géométriques. Ils

appelèrent ces nombres : les nombres polygonaux. Par exemple, ils s’aperçurent que certains nombres pouvaient être représentés par des triangles. Ainsi, 1, 3, 6 et 10 sont les quatre premiers nombres triangulaires parce qu’on peut les représenter par des points disposés en triangles comme ci-dessous :

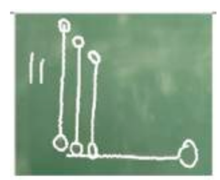


2) Observe bien ces nombres. Quel est le cinquième nombre triangulaire ? Représente-le. Explique la façon dont tu as procédé.
2) D’après toi, comment construit-on un nombre triangulaire ? Qu’observes-tu ?
3) Quel est le 11e nombre triangulaire ? Explique comment tu fais pour trouver sa valeur.
4) Tu dois écrire un courriel COURT à un ami pour lui décrire comment procéder pour calculer le nombre triangulaire 83. Décris ce que tu lui écrirais. TU N’AS PAS À FAIRE LES CALCULS!
5) Et pour calculer n’importe quel nombre triangulaire, comment ferait-on (on veut encore ici un message COURT).

Production des équipes aux questions 1, 2 et 3

1+2+3+4+5+6+7+8+9+10+11

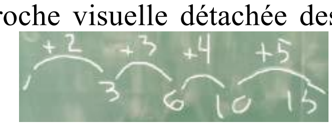
Dans cette 1re étape, nous voulions développer la pensée empirique visuelle (Rubinshtein, 1973) et la généralisation (Davidov, 1990; Radford, 2011). De façon naturelle, les élèves (équipe G1 et G3) sont passés d’une approche visuelle à une procédure arithmétique (action épistémique). Par exemple, pour le calcul de T_{11} ils ont écrit 1+2+3+4+5+6+7+8+9+10+11.

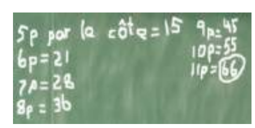


11+10+9+8+7+5+4+3+2+1

L’équipe G2 est passée d’une approche visuelle concrète à une approche visuelle plus générale, et ensuite à une procédure arithmétique (action épistémique). Ainsi pour le T_{11} ils ont écrit 11+10+9+8+7+6+5+4+3+2+1.

Nous pouvons noter l’abandon de la représentation iconique par un élève (équipe G4) qui, pendant la présentation en grand groupe, est passé d’une approche visuelle détachée des configurations polygonales à un calcul itératif qu’il n’avait pas discuté avec les membres de son équipe (pensée théorique abstraite selon Rubinstein). L’équipe G4 a utilisé cette dernière stratégie pour résoudre la 5e question de la 2e étape avec Excel.



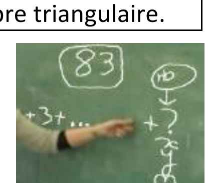


Ce passage montre l’importance du travail en équipe et la réflexion de l’élève Yan pendant qu’il organisait ses idées (dialogue intérieur selon Vygotsky 1932/1962) pour les communiquer à tout le groupe (construction du signe de Voloshinov). Yan avait besoin de se faire comprendre par le reste de la classe.

Réponses en équipe aux questions 4 et 5 et premier débat en grand groupe

Voici les réponses (travail en équipe) donnés par les différents groupes.

	Réponses à la question 4 par équipe :	Réponses à la question 5 par équipe:
G1	On additionne tous les nombres de 1+2+3… jusqu’au nombre de points sur le côté.	On additionne tous les nombres de 1+2+3… jusqu’au nombre de points sur le côté.
G2	Additionner des nombres de 1 à 83.	Tu additionnes les nombres de là ?
G3	Tu dois faire 83+82+81… jusqu’à 1.	Calcul la dernière colonne diagonale et calcul en faisant -1 au nombre. Ex. 15ème, 15+14+13… etc.
G4	Tu dois faire; 1+2+3+4+5+6+7+8… +83 et ça va te donner la réponse.	Tu mets le même nombre sur les autres côtés et ensuite tu additionnes 1+2+3+4+5+6… jusqu’à ce que tu arrives à ton nombre et ta réponse est le nombre triangulaire.



Dans une discussion en grand groupe, la chercheure demande les messages écrits pour répondre à la question 5 (voir réponses plus haut) : Message court pour n’importe quel nombre triangulaire. Les élèves ont d’abord proposé l’addition

« 1+2+3 jusqu'à ton nombre ». Intervention de la chercheure : Comment écrire le nombre que je ne connais pas? Différentes propositions ont émergé. La première était d'écrire « **?** », après ils ont proposé « **x** » ou « **y** ». La chercheure a demandé si un cœur pourrait être écrit : « ♥ ». Un élève a répondu : « On peut mettre n'importe quoi qui n'est pas un chiffre ».

Ce qu'on peut souligner c'est que les élèves sont passé d'une pensée empirique visuelle à une pensée abstraite arithmético-algébrique. La variable a été exprimée premièrement en mots : « jusqu'à ce que tu arrives à ton nombre », après comme « **?** », ensuite comme « **x** », ou « **y** », et finalement : « On peut mettre n'importe quoi qui n'est pas un chiffre ».

Production de l'équipe G4 (travail en équipe et présentation en grand groupe)			
Travail en équipe et généralisation en acte	Surprise d'obtenir un nombre décimal avec T_{100} et discussion en équipe vers la généralisation (Excel)	Présentation en grand groupe du calcul général d'un nombre triangulaire	Généralisation dans le débat en grande groupe
(83+1)÷2= 42 42x 83 =	100 (100+1)÷2=X X x 100 =	(46+1)÷2 = 47 ÷ 2 =23,5 46x23,5= 1081	(x+1)÷2=y y·x=

On peut remarquer que pour chaque abstraction, il y a un certain type de généralisation. Les processus d'abstraction ont été du type : *Abstraction visuelle, abstraction arithmétique, émergence de la notion de variable, émergence de la notion de covariation entre variables.*

Après 45 jours, à eu lieu la phase d'autoréflexion (reconstruction)

Lors de cette étape, Yan, l'élève qui avait trouvé l'expression algébrique pour les nombres triangulaires, a essayé de se rappeler de sa formule, mais s'est trompé. Il a écrit : ***(Rang*2)-1=y*** et ***Rang*y=nombre triangulaire***. Dans l'activité sur les nombres pentagonaux que nous lui avons ajoutée comme défi, il a écrit : ***Rang*(Rang + (Rang * 0,5 − 0,5)) = nombre pentagonal***. Expression qu'il a trouvée en utilisant la même stratégie que celle utilisée 45 jours avant pour les nombres triangulaires. Expression équivalente à l'expression institutionnelle : $P_n = \frac{n(3n-1)}{2}$.

Dans ce processus d'autoréflexion, une élève a obtenu à la question sur les nombres triangulaires : « ***Nombre impair : (rang + 1) ÷ 2 * rang = nombre triangulaire***. » Expression équivalente (avec restriction aux nombres impaires) à : $T_n = \frac{n(n+1)}{2}$.

Conclusions

Dans ce document, nous avons voulu montrer les différents éléments nécessaires pour la construction d'une pensée arithmético-algébrique. Basés sur quelques idées de l'histoire des mathématiques, d'une théorie socioculturelle de l'apprentissage et d'une méthode d'enseignement ACODESA, nous avons montré la nécessité de prend en compte différents éléments dans la classe de mathématiques pour une construction d'une pensée arithmético-algébrique; à savoir : le rôle de la tâche (situations d'investigation) dans l'acquisition des connaissances, la communication dans la classe, la visualisation mathématique, le rôle des représentations non institutionnelles et institutionnelles, la généralisation, la conjecture, la sensibilité à la contradiction, la validation et la preuve.

Notre approche cherche à développer et enrichir une articulation entre l'arithmétique et l'algèbre (*un habitus*) avec l'intention de promouvoir la construction d'une structure structurante dans le sens de Bourdieu (1980) liée à la pensée arithmético-algébrique qui donne non seulement un support à l'algèbre, mais aussi un enrichissement de la structure cognitive de l'arithmétique.

Nous avons aussi pu constater dans ce processus de co-construction des connaissances, l'émergence de la notion de variable et de la covariation entre variables.

Les résultats de nos recherches nous ont amenés à vouloir expérimenter des nouvelles situations d'investigation (en suivant la méthode ACODESA) en 6e année du primaire. Nous avons proposé 5 différents types de situations d'investigation avec l'utilisation de tablettes électroniques : Le restaurant de Marcel ; La bijouterie El Dorado ; Les fenêtres ; Le jardin et les citrouilles et Rectangles et disques. Nous sommes en train d'analyser les résultats.

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UNDERSTANDING TEACHERS' PROFESSIONAL DEVELOPMENT THROUGH THEIR INTERACTIONS WITH RESOURCES: A MULTILINGUAL PROJECT

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The Documentational Approach To Didactics (DAD) aims to study teachers' professional development through their interactions with their resources for/from teaching. It has been introduced in the French community of didactics of mathematics in 2007, then extended at an international level. It has been introduced as an entry in the Encyclopedia of Mathematics Education in 2020. The DAD-Multilingual project (2020-2021) is dedicated to gather and confront the translations of this entry towards 14 languages. The project main goals are: making available – for students as well as for researchers - a presentation of DAD in various languages; deepening the DAD concepts themselves in thinking their possible instantiations in different languages; questioning the translation processes; and questioning the notion of resource itself, resource for/from teaching. The lecture presents this project, and draws some lessons from its first steps.

Keywords: Documentational approach to didactics; Cross-cultural studies; Teacher Education – In-service / Professional Development; Teacher Knowledge; Teaching Tools and Resources.

In this lecture, we want to present an on-going project dedicated to better understand mathematics teachers' professional development through the lens of their interactions with a diversity of resources. In the first part, we will introduce the so-called Documentational Approach to didactics (DAD), which has been developed for about 10 years. In a second part, we will present the DAD-Multilingual project, aiming to deepen this approach through its adaptation towards different social, curricular and linguistic contexts. In the third part, we will present the feedback of the scientific committee of this project, allowing to better situating the scope of the project and the ways for its development. In the fourth part, we will present the preliminary results, and will conclude in drawing some perspectives.

DAD, towards a 'resource' approach to mathematics education.

The documentational approach to didactics (DAD) has been introduced by Ghislaine Gueudet and Luc Trouche (Gueudet & Trouche, 2009), and has been developed further in joint work with Birgit Pepin (Gueudet, Pepin & Trouche, 2012). We will just introduce here the main concepts of this approach; more information may be found in the DAD entry (Trouche, Gueudet & Pepin, 2020) of the second edition of the Encyclopedia of Mathematics Education edited by Stephen Lerman.

DAD is originally steeped in the French didactics tradition in mathematics education (Artigue et al., 2019), where concepts such as *didactical situation*, *institutional constraint* and *scheme* are central. At the same time it also leans on socio-cultural theory, including notions such as *mediation* (Vygotsky, 1978) as constitutive of each cognitive process. Moreover, the approach has also been developed due to the emerging digitalization of information and communication, which asks for new conceptualizations. This digitalization and the development of Internet had indeed strong consequences: ease of quick access to many resources and of communication with many people. This necessitated a complete metamorphosis of thinking and acting, particularly in education: new balances between *static* and *dynamic* resources, between *using* and *designing* resources, between *individual* and *collective* work (Pepin, Choppin, Ruthven, & Sinclair, 2017). Taking into account these phenomena, DAD proposed a change of paradigm by analyzing teachers' work through the lens of "resources" for and in teaching: what they prepare for supporting their classroom practices, and what is continuously renewed by/in these practices. This sensitivity to *resources* meets Adler's (2000) proposition of "think[ing] of a resource as the verb re-source, to source again or differently" (p. 207). Retaining this point of view, DAD takes into consideration a wide spectrum of resources that have the potential to resource teacher activity (e.g. textbooks, digital resources, email exchanges with colleagues, or student worksheets), resources *speaking to the teacher* (Remillard, 2005) and supporting her/his engagement in teaching.

During the interaction with a particular resource, or sets of resources, teachers develop their particular *schemes of usage* of these resources. The concept of "scheme" (Vergnaud, 1998) is central in DAD. It is closely linked with the concept of "class of situations", which are, in our context, a set of professional situations corresponding to the same aim of the activity (for example, introducing a given mathematical property for a given grade). For a given class of situations, a teacher develops a more or less stable organization of his/her activity, that is a scheme. A scheme has four components:

- The aim of the activity;
- Rules of action, of retrieving information and of control;
- Operational invariants, which are elements, often implicit, of knowledge guiding the activity;
- Possibilities of inferences, meaning of adaptation to the variety of situations.

Over the course of his/her activity, a teacher enriches his/her schemes, e.g., integrating new rules of actions, or s/he can develop new schemes. Schemes are likely to be different for different teachers, although they may use the same resources, depending on their dispositions and prior knowledge.

The resources and the scheme, developed by a given teacher for facing a given class of situations, make up a document. The process of developing a document has been coined *documentational genesis* (Figure 1). The 'use' of resources is an interactive and potentially transformative process. This process works both ways: the affordances of the resource/s influence teachers' practice (that is the *instrumentation* process), as the teachers' dispositions and knowledge guide the choices and transformation processes between different resources (that is the *instrumentalisation* process). Hence, the DAD emphasizes the dialectic nature of the teacher-resource interactions combining instrumentation and instrumentalisation. These processes include the design, re-design, or 'design-in-use' practices (where teachers change a document 'in the moment' and according to their instructional needs).

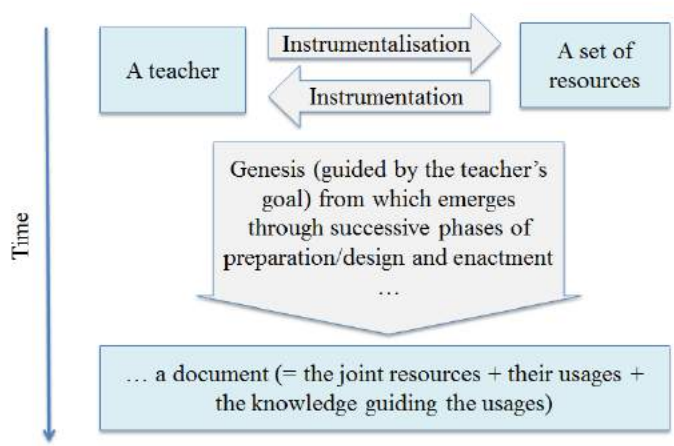

Figure 1. A representation of a documentational genesis

The set formed by all the resources used by the teacher is named his/her *resource system*. These resources are associated with schemes of usage, forming documents. The documents developed by a teacher also form a system, called the *document system* of the teacher. Its structure follows the structure of the class of situations composing the professional activity of the teacher. When teachers share their documentation work, for example in a group preparing lessons collectively, they may also develop a shared resource system (Gueudet, Pepin & Trouche, 2012). Nevertheless, the different members of the group can develop different schemes for the same resource, resulting in different documents.

We have then presented the main concepts grounding DAD. Since its introduction, this approach has been used in a variety of contexts, in Ph.D. and research projects. The Re(s)source international conference, held in 2018 in Lyon (https://resources-2018.sciencesconf.org/), gathering 130 people from 30 countries, gave a good image of the extension of the French original field. This cultural and linguistic diversity was understood as a potential richness for deepening the concepts at stake:

- One of the sessions of the young researchers workshop, held during this conference, was dedicated to « Naming systems1 used by secondary school teachers to describe their resources and their documentation work », meaning the structured set of words used by teachers, in their own language, for describing their resource systems;
- And, in my final conference (Trouche, 2019), among the 10 research programs that I proposed for developing DAD, two of them addressed linguistic issues: the first one, "Conceiving a DAD living multi-language glossary", and the last one "Contrasting naming systems used by teachers in describing their resources and documentation work, towards a deeper analysis of teachers' resource systems."

These reflections, among others, lead to the DAD-Multilingual project.

The DAD-Multilingual project, deepening a theoretical approach through its adaptation to a diversity of contexts

We describe here the origin of the project, the actors involved, and the translating processes. Any project is actually born from the convergence of a set of phenomena; and responds to a set of needs. It is indeed the case for the DAD-Multilingual project, being:

- The result of my personal experience, as a French native speaker having to go back and forth between English and French: introducing first DAD in French (Gueudet & Trouche, 2008), then in English (Gueudet & Trouche 2009); writing in English the entry DAD for the Encyclopedia of Mathematics Education (Trouche, Gueudet, & Pepin, 2020), then translating it in French (Trouche, Gueudet, Pepin, & Aldon, 2020). Doing so, I had in mind the Tuareg proverb: "travelling is going from oneself to oneself through others"…
- The result of my PhD supervisions experience: I realize, for example, that students from some countries had never the occasion to express themselves, in the frame of their studies, in their own language: "In spite of the numerous calls from education and language specialists, many countries still use the languages of wider communication instead of their native languages […] As a result, students are often required to learn subject material in the language of a former

1 The notion of "naming system" was inspired by the Lexicon project. "[This project involved] nine countries (Australia, Chile, China, Czech Republic, Finland, France, Germany, Japan, USA), and in each country a team of mathematics education researchers and experienced mathematics teachers. In this project, we consider that our experiences of the world and reflection on those experiences are mediated and shaped by available language, and that the use of English as *lingua franca* for international communication substantially limits what can be expressed and shared. The goal of the project is thus to document and compare the naming systems employed in mathematics teacher communities in the nine countries to describe the objects and events in their classrooms, in order to expand on the variety of constructs available for the purpose of theorizing about classroom practice and for identifying the characteristics of accomplished practice" (Artigue et al., 2019). But while the Lexicon project looked at the naming systems used to describe classroom activities, the young researchers workshop looked at the naming systems used to describe teachers' interactions with resources, before, during, and after class.

power; a language in which they may not be proficient "(Quigley et al., 2011). The DAD-Multilingual project appears then as a necessity for addressing equities issues;

- The result of interacting with researchers in various contexts (Algeria, Brazil, China, Japan, Lebanon, Mexico, Netherlands, Norway, or Senegal): these interactions have evidenced the fact that dealing with DAD in each new context (theoretical, cultural, curricular a well a linguistic) leads to new questions and potential enrichment. As it was said for didactics (Arcavi et al., 2016), DAD "goes travelling"…
- The result of developing an approach grounded on teachers' work with resources, involving naturally a diversity of supports and languages;
- The result of working over a long period with Ghislaine Gueudet and Birgit Pepin, committed both in international projects, Birgit having herself a long experience of crossing linguistic boarders…

These interactions evidenced also the need for enlightening the complex metaphoric structure developed by DAD, that could appear as a characteristic of the French community of mathematics education: "Despite the broad dispersion and wide-ranging accomplishments of didactique over the past decades, it has not had the influence outside the Francophone world that one might have expected […] part of the communication problem is that didactique carries some heavy baggage stemming largely from the language it employs and its metaphors in particular […] Didactique, in creating a precise vocabulary for its work, has made extensive use of the fundamental metaphoric structure identified by Pimm [1988, 2010], generating terms that need careful exegesis before they are used. Anglophones may find that English versions of those terms come laden with extra baggage that makes them difficult to interpret correctly." (Jimmy Kilpatrick in Arcavi et al., 2016)[2].

Finally, the presentation of DAD in the Encyclopedia of Mathematics Education gave us (Ghislaine, Birgit and me) the opportunity of a conceptual reversal. Each Encyclopedia, since Diderot and d'Alembert's work (1760), rests on a fundamental objective: making available all the knowledge of the world in a given place (a series of books) and a given language. Our project, reversing this objective, was to make available a small piece of knowledge in a diversity of languages, with the idea that this diversity will contribute to better understand the piece of knowledge at stake. Thus was born the DAD-Multilingual project (https://hal.inria.fr/DAD-MULTILINGUAL, aimed at adapting the Documentational Approach to Didactics entry into a diversity of languages.

The translating process involved 14 languages (in addition to English), actually the languages represented in the Re(s)source 2018 international conference (Gitirana *et al.* 2018): Arabic, Chinese, French, German, Greek, Hebrew, Hungarian, Italian, Japanese, Norwegian, Portuguese, Spanish, Turkish and Ukrainian. These 14 languages offer both elements of proximity (as for the roman languages: French, Italian, Portuguese and Spanish) and elements of distance, for example between European languages and Chinese one, leading to conceptualize differences of languages and of thought (Jullien, 2015).

The goals of the project, as announced on its website (https://hal.inria.fr/DAD-MULTILINGUAL), are the following ones:

- Making available a presentation of DAD in various languages, allowing the students and the researchers interested to refer to it in their own language;
- Deepening the DAD concepts themselves in thinking their possible instantiations in different languages;
- Questioning the translating process itself;

[2] Thanks to Tommy Dreyfus who, after his reading of a preliminary version of this paper, draws my attention on this Kilpatrick's contribution.

- Beyond the frame of DAD, questioning the notion of 'resource', resource for/from teaching;
- Designing (in a later step) a multilingual glossary of DAD.

The project involves a set of translators, reviewers (at least one translator and one reviewer for each language) and a scientific committee (see § 3). It would be excessive to say that they formed a community at the start of the project. Actually, there is not a DAD community, no regular event or specific journal allowing labelling such a scientific group. Since its beginning, in 2009, DAD develops as an « approach », with blurred boundaries, acting as a theoretical workshop for studying teachers-resources interactions, complementing or questioning already well established theoretical frameworks. During these 10 years, DAD has attracted PhD students, and researchers, in the frame of projects around mathematics teaching resources: e.g., in France, ReVEA (https://www.anr-revea.fr); in Europe, MC2 (http://www.mc2-project.eu); internationally, the French-Chinese joint project MaTRiTT (http://ife.ens-lyon.fr/ife/recherche/groupes-de-travail/matritt-joriss). The translators and reviewers have a diversity of links to DAD: interested as prospective users, or effective users, or co-designers (in particular, PhD students have enriched DAD with new concepts in their theses). They all are native speakers for the targeted language of a given translation; and they are sometimes go-between different languages, for historical reasons (e.g., Arabic-French in Lebanon) or PhD reasons (e.g. Chinese students having done their PhD in a frame of a co-supervision, using French and Chinese for collecting data; and English for writing their thesis), or a mix of these reasons (see Window 1).

Window 1 - The translator-reviewer pair in the Spanish language case

The translator was Ulises Salinas-Hernández, and the reviewer Ana Isabel Sacristán. Both have been members of the Department of Mathematics Education in Cinvestav-IPN (Mexico):

- Ulises obtained his PhD from Cinvestav; then has been doing a two-year post-doctorate at the ENS de Lyon with Luc Trouche, reflecting on theoretical networking, crossing DAD and the semiotic approach (Radford, 2008). His stays at ENS de Lyon gave him the opportunities to contrast the naming systems used by Mexican and Chinese teachers (Wang, Salinas & Trouche, 2019);
- Ana Isabel did her PhD at the University of London with Richard Noss and partially with Celia Hoyles. She has a long history of interacting with Luc Trouche, first at the ENS de Lyon during a three-month scientific stay in 2012, as well as in Cinvestav during a two-month scientific stay of Luc Trouche in 2017; these interactions gave rise to several papers (e.g., Trouche, Drijvers, Gueudet & Sacristan, 2013).

These close interactions have allowed flexible discussions on the translating process. This is a specific case for the translator-reviewer pair, and other cases can be found in the project, more or less close to DAD history.

The method of the translating process was as follows:

- English was the interface language (of course, this could constitute a bias for the on-going discussions);
- From the beginning, it was clear that the objective was not to produce a translation that was as close as possible to the original text, but instead to: make DAD understandable in a specific cultural, curricular and linguistic context (that means bridging it, if possible, with other frames well known of the targeted audience); enrich DAD in questioning its concepts when translating it;
- Issues that arose in the translating process were shared by each translator-reviewer pair, who had to fill, in English, a 'Translating issues report'. For such a report, a model was proposed (see Window 2), but such a model could be adapted according to the needs of the translator-reviewer pair.

Window 2 – The model (to be adapted) for the Translating issues report

Language: Translator: Reviewer:

Sources: English version and other linguistic versions?

- In a few lines, could you describe the main issues that emerged when translating the DAD entry or when interacting with the reviewer? Issues linked to the context (social, cultural, or curricular); issues linked to the concepts at stake; issues linked to the vocabulary
- Certain concepts raised difficulties, or discussions between the translator and the reviewer. We suggest that you explain these difficulties, and the choices you have made, for the notions of resource, document and for about three other notions, which seemed more particularly complex: Possible translations, and associated definitions (in English) - Final choice, and motivation - Scientific references using this word in the targeted language
- Other issues that you would like to share

Each translation was considered as an element of a collection, integrated in a French scientific Open Archive website (see Window 3).

Window 3 – The presentation of the Chinese translation on the Open Archive Website

文献纪录教学论

Luc Trouche, Ghislaine Gueudet, & Birgit Pepin1
中文翻译：王重洋（Chongyang Wang）
中文审阅：徐斌艳（Binyan Xu）

摘要

文献纪录教学论是数学教育百科全书的入门词条(Trouche, Gueudet & Pepin 2018)。词条于 2020 年更新(Trouche, Gueudet & Pepin 2020)。本文是中文译本，是 14 种语言译本之一（https://hal.archives-ouvertes.fr/DADMULTILINGUAL）。

文献纪录教学论是数学教育领域的一种教学理论。该理论创立的初衷是通过研究教师和资源之间的互动（包括使用和设计）来理解教师专业发展。本文主要阐述该理论的起源，理论背景，核心概念和相关的方法论。为达到理据结合的效果，我们将结合不同的研究项目作为案例诠释上述内容。本文面向的读者群是研究者，以及对入门文献纪录教学论有兴趣的非专家型读者（比如硕士研究生）。

Abstract

The 'Documentation Approach to Didactics' is an entry of the Encyclopedia of Mathematics Education (Trouche, Gueudet & Pepin 2018). This entry has been updated in 2020 (Trouche, Gueudet & Pepin 2020). This article is a Chinese adaptation of this updated version. It is part of a collection, gathering such adaptations in 14 languages (https://hal.archives-ouvertes.fr/DADMULTILINGUAL).

The documentational approach to didactics is a theory in mathematics education. Its first aim is to understand teachers' professional development by studying their interactions with the resources they use and design in/for their teaching. In this text we briefly describe the emergence of the approach, its theoretical sources, its main concepts and the associated methodology. We illustrate these aspects with examples from different research projects. This synthetic presentation is written for researchers, but also for non-specialists (e.g. master students) interested in a first discovery of the documentational approach.

关键词

课程材料；电子资源；文献起源；操作不变量（行动知识）；资源系统；教学资源；教师集体工作；教师专业发展。

Keywords

Curriculum materials; Digital resources; Documentational geneses; Operational Invariants; Resource systems; Resources for teaching; Teachers' collective work; Teacher professional development.

This website (https://hal.inria.fr/DAD-MULTILINGUAL) gives access to the presentation of the project and its actors; to the set of translations; to the set of Translating issues reports; and to different resources aimed to support the translating processes (most of them coming from the scientific committee); and to the analyses produced over the project (as this current lecture!).

What we have learnt until now from the feedback of the scientific committee

The scientific committee was at the beginning composed of 5 persons: Jill Adler, Nicolas Balacheff, Rongjin Huang, Janine Remillard and Kenneth Ruthven. They were called upon due to their knowledge in, and interest of: the international community of mathematics education; the resource approach to mathematic education; the semantic issues at stake in each translating process; or/and the interactions between different cultures and languages. From the beginning of the project, they were asked to comment on the way the project was organized (e.g., the model of 'Translating issues report', improved thanks to their comments), and to propose references that could support the reflections on the translating processes. Their full comments can be found on the project website (https://hal.inria.fr/DAD-MULTILINGUAL/page/translation-issues), where we underline what appear as their main contributions.

Kenneth Ruthven draws attention to the source language, with four fundamental questions:

- Why not retain key terms from the source language?
- Why not 'mark' key terms in some way to indicate the specialized usage intended (e.g., reSource)?
- Why not 'mark' key terms in some way to clarify the metaphor (e.g., resource-scheme-document, abbreviated, say, to res-sch-doc)?
- Does a concept (as 'resource system' see Ruthven 2019) need a sharper definition before it can become a key term of DAD? And would that sharper definition point to a more precise term (or phrase)?

In such cases, the support of dictionaries (e.g., the Cambridge Dictionary https://dictionary.cambridge.org/dictionary/ and the already accepted translated terms in the specialized domain (e.g., such as the case of 'scheme' in the field of psychology) should be followed.

Nicolas Balacheff recalls that "The issue of language is not just a question of words, as is too often stated, but of expression and the circulation of meaning" (Balacheff, 2018). The minimal condition for doing this work should be to complete the choice of translated terms with authority quotes attesting to their use, and allowing taking into account the "finesse" of the concepts.

Jill Adler considers that the focus on DAD concepts as isolated words is too narrow. She suggests to take into account the context in which these concepts are used (see also Arcavi et al., 2016; Pepin, 2002; Setati, 2003), and raised the issue of the link between teachers' discourse-resources (Adler, 2012), and the theoretical discourse analyzing them.

Janine Remillard, building on her experience in the Math3Cs project (Remillard, 2019), evokes Osborn's (2004) discussion of different types of equivalences in cross-cultural research and Clarke (2013)'s notions of validity when doing cross-cultural research. Like Jill Adler, she underlines the importance of the context, pointing to how the words themselves are "the tip of the iceberg". For facing these issues, the translation team needs to develop what Andrews (2007) calls *prerequisite intersubjectivity*, leading to a shared understanding of the core concepts (see also Pepin et al., 2019). In this perspective, the design of a multilingual glossary of key terms seems crucial – this is actually one of the objectives of the DAD-Multilingual project, already evoked at the Re(s)source 2018 conference (Trouche, 2019).

For taking into account the link between the cultural context, teachers' words, and the conceptualization of their interaction with resources, the coordinators of the project invited Michèle Artigue, involved in the Lexicon Project (see footnote 1), to join the scientific committee, and so it became composed of 6 persons. Until now, this scientific committee is only composed of members of the mathematics education community: other scientific fields could be, of course, considered (e.g., linguistics, computer science, anthropology, cultural studies…); perhaps to be discussed in a later stage of the project?

Some preliminary results

As I write the presentation of this lecture (August 20th 2020), the project is still on-going. 11 translations (of over 14) have been completed, and the three remaining translations will be achieved before the end of September. This is the first *productive* result of the project, corresponding to its first aim.

Regarding its *constructive* results – what do we learn from the translating process itself – the Translating Issues Reports, still under progress, as well as the interactions within the project, allow drawing some preliminary lessons.

First of all, each translation gave rise to very active processes, often mobilizing several sources, and more actors than the translator and the reviewer. For example, for the Turkish translation, the English and French versions were used, and the translation process was an opportunity to introduce, and to discuss the approach with doctoral students:

> About the translation, we checked the translation together but it seems it is not possible to finish it at the end of the April. Because the sentence type and the explanations are very different from English and French. Doctoral students also find the translation problematic and we are revising it according to their feedback (email from Burcu Nur Basturk, on April 13th 2020)

Second, each translation appeared as a complex process, involving several levels: vocabulary, scientific expressions, and structure of the sentences, as detailed in the Japanese report (Window 4).

Window 4 – Extract of the Japanese Translating issues report

Takeshi Miyakawa and Yusuke Shinno

After reading all through the translated text of the DAD entry, we found that the text was not really the one we usually write by ourselves in Japanese. One may find that this is the translation, not the original text. This would be due to the difficulties of translation at different levels.

First, at the level of vocabulary, there are many technical terms, which are not used in the ordinary language. We had to create an appropriate Japanese term for the English or French term. This difficulty is not only for the technical terms used in DAD, but also those used in the mathematics education, in the scientific papers in general, or in the ordinary language. For the technical term, we used sometimes the English phonetic expression, and other times the Japanese translated terms. The most difficult term we discussed a lot was the name of approach "Documentational approach to didactics". Even the usual term "approach" was not easy for us to translate.

The use of technical terms is also related to the context of scientific research. In the research on mathematics education in Japan, the scholars often try to use the terms which are comprehensible to others and actually use much less number of technical terms than in the didactics of mathematics in France. Japanese scholars therefore may be surprised with the use of technical terms in this text and sometimes might be uncomfortable with it.

There were also many difficulties of translation at the level of sentence. In Japanese language, the order of terms in a sentence is very different from French and English: for example, the verb is given at the end of the sentence; the subject is not sometimes given; and so forth. Due to this, we had to often split a sentence into several sentences. Further, we consider that the context in which the original English text was written would be a factor that makes our translation alien from the Japanese ordinary text. Some English sentences, which seem self-explanatory would not clearly explain the claim, and Japanese readers may feel the lack of sentences that complement the claim, since they are in the other context.

Third, translating DAD needed to think more globally on the theoretical background of this approach, and on the existing, or not, bridges towards the targeted language, requiring, sometimes, to go through a third language, such as Russian in the case of Ukrainian. In this perspective, the objects and methodologies have to be questioned (see Window 5).

Window 5 – Extract of the Ukrainian Translating issues report

Maryna Rafalska and Tetyana Pidhorna

There are no translations of the works of G. Brousseau, G. Vergnaud, and Y. Chevallard in Ukrainian. Thus, we faced the difficulties in translating the main notions used in their theories (e.g. “milieu”, “scheme”, “operational invariants”, “savoir à enseigner” and “savoir enseigné”, etc.). For the translation in Ukrainian of the notion of scheme we used Ukrainian articles [11, 12] that refer to the work of J. Piaget and the Russian translations of J. Piaget works [10]. We also used the Russian translation [12] of Rabardel's instrumental approach for translating the notions of instrumentation, instrumentalisation, and instrumental genesis. The Russian terms were translated in Ukrainian using the dictionaries that provide the meaning of these words and then we found the Ukrainian analogues (also using the dictionary to confirm that the meanings of found words are the same).

Translating the DAD showed us the differences in research objects and methodologies in different cultural contexts, e.g. French didactics of mathematics and Ukrainian method of teaching and learning mathematics (metodika navchannya matematiky). Thus, we noticed that in Ukrainian method of teaching and learning mathematics much less attention, compared to French didactics of mathematics, is given to the study of psychological constructs (e.g. scheme) that influence on teachers' choices as well as to transposition of mathematical knowledge in different institutions. The main accent in Ukrainian method of teaching and learning mathematics is given to the development of advanced methodical systems (systems of methods, forms and tools of teaching and learning of mathematics) and evaluation of their effectiveness via pedagogical experiments. Metodika has more practical objectives than didactics. For example, it aims to bring the answer to the following questions: what to teach (content), how to teach (what methods, 151organizational forms to use, tools), how to evaluate the teaching/learning results, etc.

10. Пиаже Ж. Избранные психологические труды. М., 1994.

11. Maksymenko S.D. Genetic epistemology of J. Piaget / S.D. Maksymenko // Problems of Modern Psychology : Collection of research papers of Kamianets-Podilskyi Ivan Ohienko National University, G.S. Kostiuk Institute of Psychology at the National Academy of Pedagogical Science of Ukraine / scientific editing by S.D. Maksymenko, L.A. Onufriieva. – Issue 32. – KamianetsPodilskyi : Aksioma, 2016. – P. 7–16.

12. Дубасенюк О.А. Підготовка майбутніх учителів до реалізації педагогічної дії. Матеріали Всеукраїнської науково-практичної конференції з міжнародною участю « Теорія і практика підготовки майбутніх учителів до педагогічної дії », 20-21 травня 2011 р., м. Житомир. – Житомир : Вид-во ЖДУ ім. Івана Франка, 2011. – С. 13 – 18.

Fourth, the evidence of central concept, such as ‘resource’, was called into question: “Understanding resource as something re-sourcing teachers' activity” can be transferred easily into French, or Italian, but is doesn't work in other languages, such as Portuguese:

> ‘Recurso’ (in Portuguese) is a word composed by the juxtaposition of the prefix «re» and the noun «curso», the first means repetition and the second a path already used, which is the meaning of the Latin recursus (NEGRI, 2007, p. 9). Therefore, recursar (verb in Portuguese) is unusual to give the same meaning of the verb re-source (in English). For that, we used the verb *reabastecer* ou *realimentar* with the idea of ‘source again’ (extract of the Portuguese Translation issues report, by Katiane Rocha, Cibelle Assis and Sonia Igliori)

It appeared, then, that there is a need to give a sharper definition of the critical concepts of DAD (see Ruthven's comment, § 3).

Fifth, thinking of *adaptation* instead of *translation* leads to establish links between DAD and frames already existing in a given culture, offering new opportunities to rethink the terrain of DAD; e.g.:

- The links between the instrumental and the documentational genesis through the lens of the mathematics laboratory for the Italian adaptation (Trouche, Gueudet, Pepin, Maschietto, & Panero, 2020);

- The potential links between the concept of instrumentalisation and the Guided Discovery Approach (Goztoniy, 2019), in the case of the Hungarian translation.

Finally, at this stage of the DAD-Multilingual project, these results appear quite promising regarding its aims: deepening the DAD concepts themselves in thinking their possible instantiations in different languages; questioning the translating process itself; questioning the notion of resource itself, resource for/from teaching. These results concern the 'resource' approach to mathematics education, beyond the community of mathematics education, and, beyond, the scientific fields interest in teacher education, cross-cultural studies, and translating processes.

Perspectives

I would like, as a conclusion, to imagine some perspectives, to be discussed with the actors of the project, perspectives internal to each language, crossing the languages or at a general level.

Each translation-adaptation could live its life in different natural ecosystems: being published in journals, discussed in scientific communities, being used in various research projects (for analyzing teachers interactions with resources, particularly their naming systems, in a variety of contexts), or teaching programs, and crossed with existing approaches. A given language could correspond to diverse cultural, national, or social contexts, e.g. the Spanish adaptation (in Mexico vs. in Spain), the Portuguese adaptation (in Brazil vs. in Portugal), or the Arabic one (in Lebanon vs. in Algeria or Morocco), English constituting a specific case (UK, USA, Australia or India). And, for a given language and a given country, the research context (at University level) and the teaching context (at schools levels) could provide different ecosystems where words and notions may follow their own trajectories. These appropriation processes, at a larger scale, would lead probably to updating each translation, and new issues to be addressed to/by the 'original' DAD frame.

The existing collection of translations opens also different perspectives; for example:

- Confronting the translating techniques, as detailed by Quigley et al. (2011) in the case of the English to Chinese translation:

  "*borrowing* (the source language word is transferred directly to the target language), *literal translation* (word-for-word translation), *transposition* (translating the words while paying attention to linguistic differences such as placement of adjectives before or after nouns), *modulation* (a technique often adopted when literal or transposition translation results in a utterance that, though grammatically correct, appears abnormal or awkward), and equivalence (a technique similar to modulation often used in idioms, proverbs, and phrases) […] in order to accurately translate documents, all these techniques must be used. Furthermore, translation through modulation and equivalence requires great attention to cultural, lexical, grammatical, and syntactic aspects of the text.

- Using the diversity of Translating issues reports for a mutual enrichment of each of them, leading towards an updated version of these reports and of the related translations;
- Organizing a new stage in interactions between pairs of languages using their proximity, or origin, for example Spanish-Portuguese, French-Italian, German-Norwegian, Chinese-Japanese, Greek-Ukrainian, Arabic- Hebrew…;
- Using the translated frame for analyzing data in the corresponding language;
- Proposing a special issue of a journal in mathematics education dedicated to the project, its results, issues and perspectives;
- Organizing working groups around concepts, leading towards a common glossary.

Finally, we could imagine a last stage of coming back to the original English presentation of the DAD entry, for updating and perhaps enlarging its scope in the perspective of the 'Resource Approach to Mathematics Education', as described by Gueudet, Pepin, & Trouche (2019).

These perspectives, of course, have to be questioned and enriched by the scientific committee, and discussed within the actors of the project, emerging as a community of (conceptual) enquiry (Jaworski, 2005). In this time of pandemic, sheltering at home, and cultivating a regular social distance, I would like to conclude with a personal statement: such a project, crossing the linguistic and cultural boarders, opens (at least for me) a breathing space, allowing to re-source, really, our conceptualization of teachers' work.

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RESEARCH COLLOQUIA

CONCEPTIONS AND CONSEQUENCES OF WHAT WE CALL ARGUMENTATION, JUSTIFICATION, AND PROOF

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Since PME-NA 2015, a working group has explored how argumentation, justification, and proof are conceptualized in the extant mathematics education literature and the consequences of these conceptualizations. We feature work from a forthcoming book in which scholars have critically examined the consequences of using particular conceptions of argumentation, justification, and proof as lenses for examining classroom practice from elementary through tertiary grade levels. Session participants will have opportunities to analyze data sets across grade bands and reflect on chapter authors' analyses of those data sets. Discussants will explore implications regarding how definitional choices impact both research and teaching practice. During the final session, we consider the next phase of the working group's focus for future PME-NA conferences.

Keywords: Reasoning and Proof; Advanced Mathematical Thinking; Teacher Knowledge

Theoretical Backgrounds

Although there is a large and growing body of research in mathematics education focused on argumentation, justification, and proof (Cirillo et al., 2015), the definitions and conceptualizations of these terms differ according to the perspective of the researcher, the focus of the research, and the particular data being analyzed (Reid & Knipping, 2010). Consequently, these differences cause inconsistencies in the literature. For example, there are debates about whether argumentation and proof are deeply intertwined or fundamentally separate activities, and there are inconsistencies related to students' success and difficulties with proof (Stylianides et al., 2017). Unpacking and making sense of these kinds of inconsistencies and debates has been a core focus of the Argumentation, Justification, and Proof working group.

History of the Working Group

The *Conceptions and Consequences of What We Call Argumentation, Justification, and Proof* Working Group (AJP-WG) met for the first time during the 37th Annual Meeting of the North American Chapter of the Psychology of Mathematics Education (PME-NA) in 2015. The group then met for three additional years in 2016, 2017, and 2018. The AJP-WG sessions were well-attended each year, and the group is active in between meetings. For example, following the 2015 and 2016 meetings, AJP-WG members published white papers to disseminate the group's work during the sessions (i.e., Cirillo et al., 2016; Staples et al., 2017). Following the 2018 meeting, AJP-WG members began work on an edited book. Across the authoring teams of the AJP-WG papers, the white papers, and the book, over 40 scholars have been involved in the group's work, including many graduate students. The goal of this colloquium is to share ideas from the forthcoming book (Bieda, Conner, Kosko, & Staples, forthcoming), which have developed during previous PME-NA AJP-WG sessions.

Areas of Discussion and Plan for the Research Colloquium

Each day will feature presentations from two chapter authors who analyzed the same grade level classroom transcript from different perspectives (e.g., justification and proof). Prior to author presentations, participants will have opportunities to review each transcript (i.e., the data set) and

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make their own observations. Participants will consider implications of the authors' definitional choices and conclusions with respect to research, working with teachers, or both.

Day 1: Opening Session and Focus on the Elementary Grades

The goals for the first session will be to (1) share the origins and evolution of the AJP-WG that motivated this research colloquium (2) introduce the aims and structure of the edited book that resulted from past efforts of the AJP-WG, and (3) have a set of book authors, whose chapters focused on justification and proof in the elementary grades, share reflections from their analyses. To achieve these goals, we anticipate the following organizational structure: M. Cirillo will lead Activities 1 and 2 [30 minutes] and K. Bieda will facilitate a review of the data set used by elementary section authors in the book [30 minutes]. Then, E. Thanheiser and C. Walkington will co-present major findings from their analyses of the elementary data set [20 minutes]. Finally, we will facilitate audience questions and discussion [10 minutes].

Day 2: Focus on the Middle Grades

The goals for the second day will be to explore the middle grades data and author analyses and to preview the data set for Day 3. K. Bieda will facilitate groups in reviewing the middle grades data set used by the book section authors [30 minutes]. Then, C. Gomez and K. Lesseig will share major findings from their analyses of the middle grades data set [20 minutes]. We will then facilitate audience questions and discussion [10 minutes]. Last, D. Plaxco will facilitate groups in reviewing the data set featured in the tertiary (post-secondary) book chapters [30 minutes].

Day 3: Focus on the High School and Tertiary Grades and Next Steps for Working Group

The goals for the third day will be to explore and reflect on the author analyses at the high school and tertiary levels, reflect on the implications of analyses across the presentations, and brainstorm about future working group foci as suggested by implications of the work presented across the three days. First, M. Cirillo will share findings from the Cirillo and Cox high school synthesis chapter which explores the findings of three authors who analyzed a common transcript through the lenses of argumentation, justification, and proof. Similarly, D. Plaxco will share major findings from analyses of the tertiary data set [30 minutes]. We will facilitate questions and discussion [10 minutes]. Then, A. Ellis will share reflections on the set of presentations featured during Day 1-3 of the colloquium [30 minutes]. We will then facilitate a conversation about potential topics for future PME-NA working groups based on implications from the analyses presented across the three days of colloquium sessions.

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RESEARCH TOOLS FOR COLLECTIVITY: TRACKING MATHEMATICS CLASSES

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Our research is guided by the question: 'How might we observe, document, display, and analyze data from a collective systems perspective?' In this colloquium, we will examine our current work involving the development of new methodological tools that address the research question.

Keywords: Research Methods

Introduction

Over the past 25 years, the research colloquium leaders have individually and in subgroups, been theorizing about, as well as collecting, analyzing, and reporting on data related to collective action in mathematics classrooms (e.g., Davis & Simmt, 2006; Martin, McGarvey & Towers, 2011; Martin & Towers, 2011; McGarvey & Thom, 2010; Proulx, Simmt & Towers, 2009; Thom & Glanfield, 2018). While our work has contributed to meaningful insights into mathematical understanding of learners and teachers, we realized that the methodological tools developed and used were limited due to the vast and intricate range of dynamic interactions (Martin, McGarvey & Towers, 2011; Simmt, 2011). This led us to working systemically on the mutual concern: How might we observe, document, display and analyze data from a collective learning systems approach? Building on our previous PME(NA) working group, NCTM research symposium, and PME research forum, in this colloquium we will present our work to date and provide opportunities for the participants to learn about, try out, and discuss the methodological tools we have developed.

Theoretical Background of the Research

We situate this research within a complex systems framework to inquire into how relationships between the parts of a system give rise to new and unanticipated collective behaviours of the larger system. We conceive mathematics classes as dynamic complex collectives; that is, mathematics classes emerge and evolve from the inextricable layering and entanglement of biological, social, societal, and environmental subsystems (Davis & Sumara, 2006; Davis & Simmt, 2003). Events within such systems may be unpredictable in foresight, but are potentially understandable in hindsight. As such, complex systems present "collective possibilities that are not represented in any of the individual agents" (Davis & Simmt, 2003, p. 140). In contrast to considering classroom interactions as strictly a series of distinct contributions by individuals, we recognize that teacher actions and decision making are often not based on the multitude of individual actions, but rather, on a teacher's sense of the class as a whole of which the teacher is a part (Burton, 1999; Towers, Martin & Heater, 2013).

Previous Engagement with Methodological Tools Work

At the 2015 PME(NA) Conference the research colloquium leaders offered a working group on "Collective learning: Conceptualizing the possibilities in the mathematics classroom" (McGarvey et. al., 2015). Participants discussed the theoretical and methodological concerns as well as practical implications related to mathematics class as a collective. Following the working group, we directed

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our attention on the metaphor of "vital signs" to guide the development of tools for observing the collectivity of mathematics classes. Similar to how health professionals monitor bodily systems, we explored how a mathematics class might also have *multiple vital signs*; that is, when monitored simultaneously, such signs afford robust insight into the systemic viability. Here, a suite of "classroom vital signs" were proposed to distinguish between different forms of collective classroom activity while pointing to key elements of dynamic engagement. At the 2017 NCTM Research Conference, we described the potential of vital signs as both a metaphor and tools to inquire into collective learning (McGarvey et. al., 2017) in a research symposium. In 2018, at the International PME conference, we facilitated a Research Forum on the "Vital signs of collective life in the classroom" where we shared our research team's efforts to develop methodological tools for observing the class as a collective (McGarvey et. al., 2018). To date, we have created six new tools with which to observe, track, and identify new patterns within the collective activity of mathematics classes. In this colloquium, three tools will be explicitly examined: Tool 1) ideational networks; Tool 2) dynamics of ideas; and Tool 3) classroom board activity.

Proposed Layout for Research Colloquium

<u>Session 1:</u> 15 min. introduction to the purpose and importance of the methodological research; 10 min. presentation about the development of Tool 1 to date; 20 min. for participants to observe the tool in action and dialogue about its use; 35 min. for participants to exchange ideas and collaboratively work with the tool; and 10 min. for closing remarks, discussion of future implications, and applications for mathematics education research.
<u>Session 2:</u> 20 min. overview of the Dynamical Model/Theory for the Growth of Mathematical Understanding (Pirie & Kieren, 1994) and presentation on the integration of it and the development of Tool 2 to date; 20 min. for participants to observe the tool in action and dialogue about its use; 40 min. for participants to exchange ideas and collaboratively work with the tool; 10 for closing remarks, discussion of future implications, and applications for mathematics education research.
<u>Session 3:</u> 10 min. presentation about the development of Tool 3 to date; 15 for participants to observe the tool in action and dialogue about its use; 25 min. for participants to exchange ideas and collaboratively work with the tool; 30 min. for participants to experiment with applying different combinations of the three tools and explicate the multiple perspectives they afford when observing them simultaneously "all at once"; 10 min. for closing remarks, discussion of future implications, and applications for mathematics education research.

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WORKING GROUPS

IMPLEMENTING AND RESEARCHING MATHEMATICS CONTENT-FOCUSED COACHING MODELS

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In this new working group, we will bring together those interested in sharing knowledge, information, insights, and current work related to implementing and researching content-focused coaching in mathematics. There are many variations of content-focused coaching currently referenced in the field which has led to a lack of clarity about how this coaching model is being implemented and researched. This working group is an opportunity for those engaged in the work of developing and/or researching coaching models to collectively explore (a) variations of content-focused coaching, (b) challenges in implementing content-focused coaching, including ways to support coaches, and (c) needs and future work for content-focused coaching in mathematics. We intend for this working group to continue into future PME-NA conferences as we build on this initial collaboration to impact our individual work and the field at large.

Keywords: Teacher Education-Inservice/Professional, Instructional Leadership, Research Methods

Content-focused Coaching in Mathematics

Coaching has become a widespread method of professional development for teachers (Campbell & Griffin, 2017). One particular form of coaching, content-focused coaching, has been shown to be a promising practice to impact teachers' instructional practices and student learning (e.g., Gibbons & Cobb, 2016; West & Staub, 2003). Content-focused coaching has two primary goals: (a) increasing the teacher's knowledge of a specific content area, such as mathematics, and (b) building the teacher's knowledge of effective instructional practices related to a specific content idea through a personalized, embedded program (Cobb & Jackson, 2011).

Researchers in multiple countries have highlighted the benefits of content-focused coaching (e.g. Becker, Waldis, & Staub, 2019; Gibbons & Cobb, 2016; Kreis, 2012; Kreis & Staub, 2011; Murawski, 2019). In Switzerland, for example, researchers found content-focused coaching to be beneficial for both prospective and practicing teachers, emphasizing the applicability of the model to other contexts (Becker et al., 2019, Kreis, 2012; Kreis & Staub, 2011). Similar coaching models have been used in Canada (e.g. Bengo, 2016), showing positive influences on teacher practice. The variability of contexts in which content-focused coaching is implemented in terms of demographics, teaching conditions, and teaching experience, highlights the need for collective exploration of content-focused coaching.

In addition, there is not a shared understanding in the field about how to define content-focused coaching, how to implement and support content-focused coaching programs, and how to study the actions of content-focused coaches and impact on teachers' practice. As a result, there are high-levels of variability in the ways mathematics coaches interact with teachers (e.g. Ellington, Whitenack, & Edwards, 2017) and inconsistent empirical findings when researching the effectiveness of coaching (e.g. Campbell & Griffin, 2017).

As a collective team, the proposal authors have extensive experience implementing content-focused coaching as a part of larger professional learning programs for teachers. The team has developed and supported content-focused coaches in both face-to-face and online contexts. We are currently researching the actions of content-focused coaches in an online environment and the impact of these actions on teachers' practices (see Author, 2019).

The intent of this working group is to build on and connect the knowledge and experiences of participants in order to establish a more robust understanding of content-focused coaching and to identify opportunities for future work. Participants will have an opportunity to collectively explore three themes: (a) variations of content-focused coaching, (b) challenges in implementing content-focused coaching, including ways to support coaches, and (c) needs and future work for content-focused coaching in mathematics.

Working Group Organization and Strategy

Each session of the working group will draw on the experiences of the authors and the participants related to content-focused coaching, with a different focus for each session based on the themes noted above.

The first session will invite discussion on two of the three themes: exploring variations of content-focused coaching to generate a common understanding, and exploring challenges in implementing content-focused coaching, including ways to support coaches. We will launch the session with the authors sharing their background and experiences to provide a foundation for discussion related to what constitutes content-focused coaching. We will then engage participants in small and full group discussions with the goal of generating a more cohesive understanding of the critical components of content-focused coaching. Attendees will then work in small groups to identify challenges in implementing content-focused coaching. We will share and discuss current work to overcome these challenges with the goal of networking around effective implementation of content-focused coaching.

Session Two will include brief presentations from several participants' work related to content-focused coaching. Participants will then engage in small group discussions on the third theme - what more needs to be known about content-focused coaching. Specifically, discussions will be guided by the following: (a) what has been/is being studied related to mathematics content-focused coaching, and, (b) what are areas of need for further contributions to the field. These discussions will culminate with groups creating questions relating to common research challenges and possible future research based on gaps in the literature. These questions will provide an extended opportunity for discussion with an *Open Space Protocol* structure. The Open Space Protocol provides time and space for participants to generate new knowledge about a particular question with which they find relevant. The process draws on the talent that will be in the room and positions participants to discuss questions important to their work within a supportive and structured environment.

In Session Three, we will use the Open Space Protocol to continue the conversations from the previous day in two more rounds of small group discussions. Attendees will be encouraged to move groups, if they desire, at the end of each round. To support follow-up and ongoing collaboration of participants, group notes and documents will be shared and distributed via a Google Folder that will be set up for this working group. This shared folder will provide a shared space for future collaborations and writing projects within the working group members.

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EMBODIED MATHEMATICAL IMAGINATION AND COGNITION (EMIC) WORKING GROUP

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The central aim of the EMIC Working Group is to connect, engage, and inspire colleagues in this growing community of discourse around theoretical, technological, and methodological developments for advancing the study of embodied cognition for mathematics education. This year, our fifth at PME-NA, we also will convene on Day 3 with the WG on Mathematical Play. Our community of scholars will use these sessions to continue to broaden the range of activities, practices, and emerging technologies that contribute to mathematics teaching and learning as well as to research on these phenomena.

Keywords: Cognition; Embodiment and Gesture; Informal Education; Learning Theory

Empirical, theoretical, and methodological developments in embodied cognition and gesture studies support the continuation of the regularly held Embodied Mathematical Imagination and Cognition (EMIC) Working Group for PME-NA. The central aim of EMIC is to attract, engage, and inspire colleagues in a growing community of discourse for advancing the study of embodied cognition for mathematics education, including mathematical reasoning, instruction, assessment, technology design, and learning in and outside of formal settings.

Views of learning as embodied experiences have grown from several developments in philosophy, psychology, anthropology, education, and the learning sciences that frame human communication as multimodal interaction, and human thinking as multimodal simulation of sensory-motor activity (e.g., Lave, 1988; Nathan, 2014; Wilson, 2002). Four ideas exemplify the plurality of ways EMIC is relevant for the study of mathematical understanding: (1) *Grounding abstractions in perceptuo-motor activity as an alternative to amodal symbol systems;* (2) *Cognition emerges from perceptually guided action*; (3) *Mathematics learning is always affective*, never detached from body-based feelings and interpretations; (4) *Mathematical ideas are conveyed via multimodal forms of communication, e.g., gestures, drawing, and objects.*

The interplay of multiple perspectives is vital for the study of embodied mathematical cognition to flourish. While there is significant convergence of theoretical, technological, and methodological developments in embodied cognition, there remain questions to be addressed through formulating and applying experimental design principles. We aim to: (1) synthesize the work of leading scholars into a theory of EMIC; (2) identify and negotiate focal ontologies and parameters that capture our theoretical, methodological, and technological variability; (3) curate and disseminate evidence-based design principles to enhance mathematics education and broaden participation in STEM fields; and (4) articulate a research agenda in embodied design.

Past Achievements, Current Organizers, and the Future of EMIC

This is the 5th year of the EMIC WG. Several activities and website have emerged to connect scholars and provide resources, https://www.embodiedmathematics.com. Two NSF workshops for researchers and instructors grew from this: "The Future of Embodied Design for Mathematical Imagination and Cognition" (May 20-22, 2019); and "EMIC: Professional Development for Undergraduate Mathematics Instructors" (June, 2021). An edited book is planned for the "Research in Mathematics Education" Series and an article is under review for *Frontiers in Education* Research Topic: "Futures of STEM Education."

As the WG matures, we are broadening the set of organizers to represent a range of institutions, perspectives, and applications. This enriches the WG experience and the long-term viability of the community. The organizers not included in the authorship list are listed here: Candace Walkington, Southern Methodist University; Carmen J. Petrick Smith, University of Vermont; Hortensia Soto, University of Northern Colorado; Ivon Arroyo, University of Massachusetts-Amherst; and Martha W. Alibali, University of Wisconsin-Madison.

EMIC 2020: Embodiment in Mathematics for Inclusion

Embodiment is an effective way to promote inclusive mathematics education research and practices. This year we will explore how embodied mathematics can bridge cultural divides and raise awareness of inclusion for those with different physical and perceptual abilities (Abrahamson et al., 2019). To demonstrate growth and relevance, we will also join the Mathematical Play WG on Day 3 to integrate EMIC and Play research and practice.

On Day 1 we will discuss the goals for PME-NA 2020 to bridge divides that arise across those with different physical and perceptual abilities. As is customary with EMIC, we will anchor this to mathematical activities (e.g., making human-scale polyhedra). For a portion of the time, participants will collaborate without sight, to identify how our bodies offer effective ways to engage. This will invite us to identify principles of inclusive pedagogy and activity design. This will be connected to the 4 EMIC themes: Grounding, emergence, affect, and multimodality.

On Day 2, we will explore mathematical collaboration when participants cannot rely on a shared language. In order to challenge the norms of mathematical activities (e.g., Bagh Chal from Nepal), we will include puzzles and games that are foreign to majority cultures.

On Day 3, EMIC will meet with the Math Play WG to consider overlapping interests and questions. The organizers of both WGs led a joint conference proposal that is currently under review, and Day 3 will be our first overt synthesis of the two bodies of work. The session will include small activity groups with Math Play participants who want to think about their work as embodied and EMIC participants who want to think of their work as math play opportunities. After, we will review Days 1 and 2, we consider how our themes overlap, ways to enhance future PMENA conferences, as well as the broader ways that embodiment, imaginative thinking, and play can be used to promote inclusivity. We will conclude by discussing continued engagement and dissemination opportunities available with both the EMIC and Math Play communities.

Acknowledgements

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Figure 1: A small selection of embodied activities created by EMIC organizers and experienced by EMIC participants. Clockwise from top left: experiencing geometric transformations, acting out geometry conjectures, constructing icosahedra first as small, then at human scale, and enacting topological relations.

COMPLEX CONNECTIONS: REIMAGINING UNITS CONSTRUCTION AND COORDINATION

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Students' construction, coordination, and abstraction of units underlie success across multiple mathematics domains. This working group aims to facilitate collaboration between researchers and educators with the particular aim of extending research on units coordination and construction across age groups, learning differences, and mathematical contexts.

Keywords: Cognition, Learning Theory, Number Concepts and Operations

Theoretical Background, Purpose, and History

Units coordination and construction refers to the number of levels and type of units children can construct and bring into a situation (Steffe & Olive, 2010). Children at young ages begin counting when first constructing *pre-numerical units* (relying on perceptual material and/or physical actions) with which to use as material for future activity (Steffe & Cobb, 1988). These units are first constructed through children's external activity before becoming *internalized* (imagined actions) and then *interiorized* (able to anticipate relationships between levels of units). In Steffe's 2017 plenary for PME-NA, he substantiated particular needs for investigating how children develop operations when constructing and coordinating units. The working group began at PME- NA 2018, with the aim of facilitating collaboration amongst researchers and educators sharing Steffe's concerns about (a) the need for supporting units construction and coordination for all learners and (b) the need for accompanying learning trajectories (curricula) appropriate for students' current level of units across grade levels.

Working Group Goals and Strategies, Past and Present

In the first year of the working group, goals included generation of related research topics of interest to PME-NA attendees, including the role of units coordination in early childhood education, special education, and secondary and post-secondary education (including teacher-education). Products included the creation of a website for organizing and collecting tasks used for assessing students' units coordination links to research papers addressing particular topics relating to units coordination: https://unitscoordination.wordpress.com/ At the second working group meeting at the 41st PME-NA (2019), novice and experienced researchers described inferences of students' units activity and shared perspectives of the affordances and constraints of assessing units coordination using particular tasks and settings. Our goals for the 2020 working group are to (1) continue to build on the productive discussions from the 2019 and 2018 working group meetings relating to issues of assessments of units coordination in different settings (within classrooms, via written instruments, via clinical interviews, and via individual or paired-student teaching experiments) as well as (2) to continue to bridge efforts to emerging research connecting units coordination across mathematical domains.

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Session 1: Shared Understanding of Units Coordination and Construction

GOAL: Participants (new and returning) will come to shared understandings of the main ideas of units construction and coordination as well understandings of differences in theoretical and conceptual perspectives. ENGAGEMENT: Prior to the working group, we will administer an entry survey of participants to determine interests and goals for collaboration. In the first day's meeting, participants will reflect on their own actions to solve tasks used to assess and support units construction and coordination and discuss in small groups how tasks from the literature afford students' units construction and coordination and associated mental operations. Subgroups will come together and discuss their understandings of relationships between student actions and task features when assessing and supporting students' units construction/coordination.

Session 2: Emerging Connections between Units Coordination and Subitizing, Units Coordination and Reasoning about Rates

GOAL: Explore and bring to focus the role of emerging research investigating connections between units construction and coordination across age groups and mathematical contexts (e.g., STEM fields, cognitive science fields). ENGAGEMENT: Participants will form three groups, one focused on units construction and coordination with young children (and their subitizing activity), one focused on units coordination with adult learners (and their reasoning about rate of change), and one focused on units coordination of elementary students with identified with learning (i.e., working memory) differences. Building on the discussions from the first session, participants will explore video-recordings of clinical interviews and teaching experiments and discuss theoretical and pragmatic connections between these constructs and describe their inferences and wonderings in these contexts by sharing their analyses in a shared google doc.

Session 3: Reflection and Taking Action

GOAL: Reflect on connections and embark on planning collaborations of interest to participants. ENGAGEMENT: During the first 30 minutes, we will discuss the results from the previous session as a whole group. Then, form small groups for each of these goals: (1) collaborations within participants' research projects (2) creation of content for the webpage and (3) identification of target journals and outlets or grants and funding sources. Administer exit survey of participants' interests and goals for collaboration.

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WORKING GROUP: DIDACTIC CONTRASTS BETWEEN CALCULUS AND ANALYSIS

GRUPO DE TRABAJO: CONTRASTES DIDÁCTICOS ENTRE CÁLCULO Y ANÁLISIS

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A new Working Group is proposed. In other forums, both PME and ICMI, working groups have worked and focused on the analysis of problems in teaching and learning Calculus and Analysis, both at the pre-university and university levels, but those did not start from a clear characteriza-tion and explicit statement of the differences that could exist between these two areas of mathematical knowledge and, consequently, without full awareness of the implications that this distinction could have for both educational research and teaching. This is precisely the theme that is intended to be the focus here. It is intended that the Working Group will be working until it reaches a certain maturity of the ideas that it plans to discuss.

Keywords: Precalculus, Calculus, Curriculum Analysis

Objective of the Working Group

Identify and characterize the conceptual, epistemological and didactic differences between Calculus and Mathematical Analysis, to guide both the specific problems of educational research, and the selection of content in pre-university or university Calculus courses for non-mathematicians.

Strategy

Promote the exchange of ideas and discussion among invited experts, as well as among the participants, around the issue raised and the stated goal. This work will be supported by the presentation of a base document for discussion, prepared by this new Working Group's proposers.

Invited researchers

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Coordinator of the Working Group

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Theoretical referents

From the theoretical referents provided by the Ontosemiotic Approach to Cognition and Mathematical Instruction (Godino and Batanero, 1994), researchers have been making different efforts to characterize both Mathematical Analysis and Differential and Integral Calculus, distinguishing the problem situations addressed by each of them, as well as the mathematical objects that are used and emerge in the historical development of each of these two areas of mathematical knowledge. In particular, emphasis has been placed on identifying and characterizing some of the mathematical practice systems and primary objects (problem situations, language, procedures, properties, arguments and concepts) in each of these two disciplines.

However, it is important to systematize and organize the partial results of these efforts in three directions. First, it is important to try to identify and characterize the greatest number of contrasts between Calculus and Analysis, so that the distinction is as complete as possible. Secondly, it is important to analyze these contrasts from the point of view of the cognitive abilities and professional needs of the students (pre-university and university) who enroll in a Calculus course. The second direction concerns directly the curricular design of the Calculus course. Thirdly, and this seems fundamental to us, it is important to identify, in light of these contrasts, the new psycho-pedagogical, epistemological and didactic questions that educational research will have to address but has not yet done it for lack of such clarification.

In a broader context, this panoramic distinction is also important in that it will allow us to clarify the vision with which we intend to introduce elementary ideas related to Calculus in early education, and particularly in secondary schools.

Participation dynamics

It is hoped that the participants of the Working Group (at least some of them), after analyzing the base document, will be able to contribute short essays related to the three aspects of the topic for analysis and discussion, as well as enrich the contributions of others.

Group work organization

The organization of the group work is planned in two stages. In the stage prior to PMENA 2020, researchers who have been working along the lines of the proposed theme will be contacted and invited to join the group, become familiar with the base document and prepare their essays or contributions, which will be made available to the group in the first week of October 2020. In the second stage, during the meeting, three 90-minute sessions will be structured to present the essays and discuss them. To conclude, a final presentation will be made by those in charge of the Working Group, who will summarize what has been achieved and outline the tasks for the future.

In order to make the Group's work as productive as possible, the discussion and analysis sessions will be structured based on the three lines mentioned above as follows:

First Session. Need for a clear and systematic distinction of didactic contrasts between Calculus and Mathematical Analysis. Epistemological, cognitive and didactic problems that emerge from this distinction.

Second Session. Curricular impact that this distinction would have on pre-university and university courses in Calculus for non-mathematicians. General outline of the main characteristics of these courses. Early Calculus curriculum.

Third session. Reorientation of educational research towards new psycho-pedagogical, epistemological and didactic problems that emerge as a result of the didactic reconceptualization of Calculus and Mathematical Analysis.

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GRUPO DE TRABAJO: CONTRASTES DIDÁCTICOS ENTRE CÁLCULO Y ANÁLISIS

WORKING GROUP: DIDACTIC CONTRASTS BETWEEN CALCULUS AND ANALYSIS

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Se propone un nuevo Grupo de Trabajo. En otros foros, tanto de PME como de ICMI, han funcionado grupos de trabajo centrados en el análisis de la problemática de la enseñanza y el aprendizaje del Cálculo y el Análisis, tanto en el nivel preuniversitario como universitario, pero sin partir de una caracterización clara y explícita de las diferencias que podrían existir entre estas dos áreas del conocimiento matemático y, en consecuencia, sin plena consciencia de las implicaciones que dicha distinción podría tener tanto para la investigación educativa como para la enseñanza. Esta es precisamente la temática que se pretende abordar. Se intentará sostener dicho Grupo de Trabajo hasta lograr la maduración de las ideas que en él se planea discutir.

Palabras clave: Precálculo, Cálculo, Análisis Matemático

Objetivo del Grupo de Trabajo

Identificar y caracterizar las diferencias conceptuales, epistemológicas y didácticas entre el Cálculo y el Análisis Matemático, para guiar tanto la problemática específica de investigación educativa, como la selección de contenidos de un curso de Cálculo preuniversitario o universitario para no matemáticos.

Estrategia

Promover el intercambio de ideas y la discusión entre expertos invitados, así como entre los participantes, en torno al tema planteado y su objetivo. Se apoyará dicho trabajo mediante la presentación de un documento base para la discusión, elaborado por quienes proponen este nuevo Grupo de Trabajo.

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Coordinador del Grupo de Trabajo

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Referentes teóricos

A partir de los referentes teóricos proporcionados por el Enfoque Ontosemiótico de la Cognición y la Instrucción Matemáticos EOS (Godino y Batanero, 1994), los investigadores han estado haciendo distintos esfuerzos para caracterizar tanto al Análisis Matemático como al Cálculo Diferencial e Integral, distinguiendo las *situaciones problema* que aborda cada uno de ellos, así como los *objetos matemáticos* que se emplean y emergen en el desarrollo histórico de cada una de estas dos áreas del conocimiento matemático. En particular, se ha puesto énfasis en identificar y caracterizar algunos de los *sistemas de prácticas matemáticas* y de los *objetos primarios* (situaciones problema, lenguaje, procedimientos, propiedades, argumentaciones y conceptos) en cada una de estas dos disciplinas.

Es importante, sin embargo, sistematizar y organizar los resultados parciales de estos esfuerzos en tres direcciones. En primer lugar, es importante tratar de identificar y caracterizar el may,or número de contrastes entre Cálculo y Análisis, a fin de que la distinción sea lo más completa posible. En segundo lugar, es importante analizar dichos contrastes desde el punto de vista de las posibilidades cognitivas y de las necesidades profesionales de los estudiantes (preuniversitarios y universitarios) que se inscriben en un curso de Cálculo. Esto último concierne de manera directa al diseño curricular del curso de Cálculo. En tercer lugar, y esto nos parece fundamental, es importante identificar, a la luz de dichos contrastes, las nuevas cuestiones psicopedagógicas, epistemológicas y didácticas que la investigación educativa deberá abordar y que no la hecho por falta de dicha clarificación.

En un contexto más amplio, esa distinción panorámica también es importante por cuanto permitirá precisar la visión con la que se pretenden introducir ideas elementales relacionadas con el Cálculo en la enseñanza temprana, y particularmente en la escuela secundaria.

Dinámica de participación

Se espera que los participantes del Grupo de Trabajo (al menos algunos de ellos), luego de analizar el documento base, puedan aportar ensayos breves relacionados con los tres aspectos del tema motivo de análisis y discusión, así como enriquecer las contribuciones de los demás.

Organización del trabajo del Grupo

Se planea organizar el trabajo del grupo en dos etapas. En la etapa previa a la realización de PMENA 2020, se contactará a investigadores que han estado trabajando en la línea de la temática propuesta y se los invitará a integrarse al grupo, familiarizarse con el documento base y preparar sus ensayos o aportes, que pondrán a disposición del grupo en la primera semana de octubre de 2020. En la segunda etapa, durante la celebración del encuentro, se estructurarán las tres sesiones de 90 minutos para la presentación de los ensayos y la discusión de los mismos. Para concluir se realizará una presentación final a cargo de los responsables del Grupo de Trabajo, quienes harán un balance de lo conseguido y esbozarán las tareas para el futuro encuentro.

A fin de que el trabajo del Grupo resulte lo más productivo posible, las sesiones de discusión y análisis se estructurarán con base en las tres líneas arriba mencionadas como sigue:

Primera Sesión. Necesidad de una distinción clara y sistemática de los contrastes didácticos entre el Cálculo y el Análisis Matemático. Los problemas epistemológicos, cognitivos y didácticos que emergen de dicha distinción.

Segunda Sesión. El impacto curricular que dicha distinción tendría sobre los cursos preuniversitarios y universitarios de Cálculo para no matemáticos. Esbozo general de las principales características de dichos cursos. El currículo de Cálculo temprano.

Tercera Sesión. La reorientación de la investigación educativa hacia los nuevos problemas psicopedagógicos, epistemológicos y didácticos que emergen como resultado de la reconceptualización didáctica del Cálculo y el Análisis Matemático.

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WORKING GROUP ON GENDER AND SEXUALITY IN MATHEMATICS EDUCATION: INFORMING METHODOLOGY WITH THEORY

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The Gender and Sexuality in Mathematics Education Working Group convened in 2018 and 2019. Over the past two working group sessions, working group members have (1) shared historical and contemporary research related to the topics of the working group; (2) clarified language related to gender and sexuality; (3) developed understandings related to language and its influence on methods, results, and interpretations; (4) explored how gender and sexuality are experienced by students and teachers, and studied by researchers, in international contexts; and (5) developed research relationships among participants to explore relevant ideas. Based on the discussions from past working groups, during the 2020 Working Group, we will strengthen our understanding of these topics by examining underlying theories of gender and sexuality and the affordances of these theories on both research and practice.

Keywords: Gender and Sexuality; Equity and Diversity; Affect, Emotion, Beliefs, and Attitudes

The Gender and Sexuality Working Group has met during the two previous PME-NA conferences. These meetings have resulted in a shared foundational knowledge of the research area and has helped us develop understandings related to how language choices in gender and sexuality influence research methods, results, and interpretations. The goal of the 2020 working group is to expand our communal knowledge on utilizing theories of gender and sexuality within our work in mathematics education. In order to reach this goal, this year's working group is organized to provide participants with opportunities to develop deeper understandings of theories from gender and sexuality studies—with a focus on conceptions of identities.

Theoretical Background

The previous Gender and Mathematics Working Group contributed significant understandings regarding girls' and women's experiences in mathematics (See Forgasz, Becker, Lee, & Steinthorsdottir, 2010). Early research in this area focused on biological sex-based disparities in mathematics achievement (Lubienski & Ganley, 2017). Subsequently, the field shifted to study gender, via the sociocultural factors that influence girls' achievement and participation in mathematics (Leyva, 2017). In response to calls for clarity in the way that mathematics education researchers define and operationalize gender (Damarin & Erchick, 2010), theories of gender as performative (Butler, 1993) are now being employed in mathematics education research (i.e., Chronaki, 2011; Darragh, 2015; Gholson & Martin, 2019). Conceptualizing gender as performative repositions gender as an aspect of identity that is interactionally (re)produced. Researchers who investigate identity in mathematics education have also tended to draw on a variety of epistemological traditions (Darragh, 2016; Langer-Osuna & Esmonde, 2017). While identity categories have been problematized in feminist theories, queer theory has deconstructed the concept of identity. Some researchers have proposed a gender-complex education in which the existence of queer students is reflected in curricula (Rands, 2013; Rubel, 2016). The 2020 working group will focus on bridging theories of identity and gender to more fully understand their affordances and limitations when applied to teaching and learning mathematics. We will also discuss newer methodologies for research in mathematics education from feminist and queer theories (Mendick, 2005b; Rands, 2009).

Organization and Structure of the Working Group

The organization and structure of the working group were created to maximize participation, while focusing on topics that prior participants have expressed interest in discussing further. In the working group sessions over the past two years, participants have implicitly explored notions of identity as narrative and the construction of mathematics as a masculine domain (Mendick, 2005a), and explicitly sought ways to collect stories avoiding a gender binary. By discussing the topics for the 2020 working group, participants will extend their understanding of gender, sexuality, and identities. On Day 1, summaries of feminism and queer theory will be provided. On Day 2, specific notions of identity and its relation to performativity will be discussed.

Day 1: Feminism and Queer Theory

The working group will begin with a 20-minute presentation by Ana Dias and Weverton Pinheiro about the history of feminism and queer theory. Based on their research and collaborative conversations, they will summarize the theoretical underpinnings from these two theories as well as include examples of how these ideas have been used in mathematics education research. After the presentations, Ana and Weverton will facilitate an activity (15 minutes) to review research from these traditions and lead a discussion (50 minutes) about feminism and queer theory in mathematics education research. The major focus of these activities will be based on the *Political Grammar of Feminist Theory*. In the activities, participants will explore feminist progress, loss, and return (Hemmings et al., 2011), and the perspective of queer theory. Day 1 will end with a preview of the topics and activities for Day 2.

Day 2: Identity and Performativity

The themes for Day 2 are identity and performativity. The impetus for these topics is the 2019 working group discussion in which participants began questioning whether these concepts are compatible. Brent Jackson will give a 20-minute discussion that extends the topic from Day 1 to include how the notions of identity and performativity have been used in feminism and queer theory. Brent will also address the implications of their use in mathematics education research. Brent will then facilitate an activity (15 minutes) regarding methods that employ varying constructs of identity and performativity in mathematics education research. To conclude the session, Brent will lead a discussion (50 minutes) on the topics from the two days and address how the ideas apply to working group members' current research or new research ideas that have been provoked from the past activities. Working group leaders will then elicit comments and recommendations for how to structure Day 3 to help participants achieve their own goals and work toward the goals of the working group.

Day 3: Working Group Plans

Based on the interests of participants, Day 3 will be organized as whole-group discussion, small break-out groups, or a combination. By the end of Day 3, the group as a whole will have generated a plan to continue working together toward the working group goals.

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MATHEMATICAL PLAY: ACROSS AGES, CONTEXT, AND CONTENT

JUEGO MATEMÁTICO: A TRAVÉS DE EDADES, CONTEXTO Y CONTENIDO

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In the proposed working group, we will build from the foundation of the past two years' working groups as well as our members' continuing collaborations with researchers outside of this group. Specifically, we propose three days of activity, each focused on different aspects of developing the body of mathematical play research. We have planned the three following foci: adapting existing mathematical tasks and curricula to increase opportunities for play (Day 1); the reverse, adapting voluntary play activity to support mathematical learning (Day 2); and developing synergistic dialogue with members of the EMIC research community through an intra-working-group discussion session (Day 3).

Keywords: Teaching tools and resources; Affect, emotion, beliefs and attitudes, Informal education

Understanding mathematical play at all ages is an important, yet under-investigated domain within mathematics education research (e.g., Holton et al., 2001; Wager & Parks, 2014). Over the past three years, members of this Mathematical Play working group have developed a community of colleagues focused on identifying and characterizing productive theoretical lenses and methodological approaches to investigate students' mathematical play. Central to this work has been the emergent characterization of mathematical play as (1) voluntary engagement in cycles of mathematical hypotheses with occurrences of failure (Authors), (2) often spontaneous and self-directed toward a player's emerging goals (e.g., Wager & Parks, 2014), and (3) supported or discouraged through physical or digital interactions (e.g., Authors; Sinclair & Guyevskey, 2018). In preparation for this year's working group proposal, our co-organizers have focused on situating our work based on the degree to which it might be characterized as pure play as well as the degree to which it can be characterized as structured mathematical instruction. This focus is consistent with what Wager and Parks (2014) discuss as two seemingly contrasting ideologies: groups advocating an increased focus on teacher-directed instruction and scholarship confirming that children learn best in play-based environments (p. 223). Wager and Parks (2014) also point to calls to identify practices that bridge the two ideologies, and this proposal is a direct response.

In order to address the existing theoretical divide between play and instruction, we will discuss and collaborate around theory and results from several projects that participating researchers might situate along the dimensions of play and instruction. These conversations will focus on how specific activities and instructional interventions might support shifts along those dimensions. That is, we will

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focus on two shifts: how might an education researcher who has traditionally situated their work around more traditional mathematical tasks alter their existing instructional approaches to afford greater opportunities for play? And the complement: how might educators shift their interactions with students in a play setting to better support meaningful mathematical development?

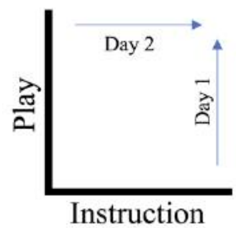

Figure 1: Graphic organization of two shifts we will explore – increasing the playfulness of high instructional tasks (Day 1) and increasing the instructional utility of high play tasks (Day 2)

Continuing the success of the last two years of the Mathematical Play PME-NA Working Group, we have developed the following overarching goals for this year's working group: (1) to engage participant researchers in conceptualizing the two shifts illustrated in Figure 1; (2) to share and discuss existing projects that are making or have made these shifts, specifically identifying frameworks and perspectives to support such shifts; and (3) to summarize these conversations and promote a synergistic dialogue with the EMIC working group.

Day 1 will center around examples of projects that originated as instructional activities but have shifted or are shifting toward more playful activities for students (Authors, Authors). Working group leaders will briefly introduce their projects and engage working group participants in tasks from their existing research projects. This will focus on identifying and discussing what play frameworks might be productive for supporting such transitions. The group will synthesize this discussion as a starting point to conceptualize how educators might incorporate playful activity within their existing instructional programs.

On Day 2, we will take a contrasting perspective as we explore design and facilitation practices that leverage mathematical play for learning. Leaders will engage group members in interactive play and board game activities with a focus on the mathematics that players draw on during their play. Leaders will then guide whole-group discussions to identify facilitation practices and pedagogical approaches to support meaningful learning in mathematical play.

On Day 3 the mathematical play working group will meet with the EMIC working group (Nathan et al., 2017) to explore areas of overlapping interest and potential convergence. Members of both groups will engage in intra-working-group conversations to highlight common theoretical and methodological approaches and identify opportunities for synergistic dialogue (i.e., mathematical play as an embodied way of learning, design considerations for embodied mathematical play, etc.).

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MATHEMATICS TEACHER EDUCATORS USING SELF-BASED METHODOLOGIES

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Narrative inquiry, self-study, and autoethnography (i.e., self-based methodologies) are becoming a more common choice of mathematics teacher educators (MTEs). This has opened new possibilities and challenges for early career MTEs as they try to disseminate their findings in mathematics education journals. Building from our working group at PME-NA 2018 and 2019, we respond to the need for creating a community where MTEs can feel supported in their study design, implementation, representation of findings, and publication using self-based methodologies. This year, we continue our focus on mentoring and scholarship on self-based methodologies. We invite English- and Spanish-speaking MTEs with research projects in any stage of preparation to join us in discussions meant to promote growth, sustainability, and continued insight into the use of self-based methodologies.

Keywords: Mathematics Teacher Educators, Research Methods, Narrative Inquiry, Self-study, Autoethnography

Context and Significance

Building from Hamilton, Smith, and Worthington (2008), we have adopted the language of *self-based methodologies* (Chapman et al., 2020) to refer to narrative inquiry (Clandinin & Connelly, 2000), self-study (LaBoskey, 2004), and autoethnography (Ellis & Bochner, 2000). These research methodologies focus on self-understanding based on personal professional experiences and are often used in teacher education (e.g., Grant & Butler, 2018; Grant, 2019; Kastberg et al., 2019; Ross, 2003; Sack, 2008; Samaras & Freese, 2009). In mathematics education, self-based methodologies are growing in use in journals (e.g., Chapman, 2011; Chapman & Heater, 2010; Cox et al., 2014; Grant & Butler, 2018; Kastberg et al., 2018; Kastberg et al., 2019) and conferences (Brand, & Jung, 2019; Clark et al., 2019; Cox & D'Ambrosio, 2015; Gallivan, & Rumsey, 2019; Kinser-Traut, 2018; Kosko, 2019; Lischka et al., 2018; Lischka et al., 2019; McGraw & Neihaus, 2018; Richardson & Zhou, 2019; Towers et al., 2019; Truxaw & Rojas, 2019). Yet, using self-based methodologies is still less widespread in the mathematics education field. Our goal is to keep growing our international community of MTEs, in which they feel sustained and empowered in their use of self-based methodologies.

History of the Working Group

Over the past two years, we have been building a community with the goal to support each other and to expand our network of MTEs who use self-based methodologies (Suazo-Flores et al., 2018; Suazo-Flores et al., 2019). After receiving positive responses from conference attendees in North America, our goal now is to diversify our membership by providing spaces for Spanish-speaking

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MTEs who want to learn, or are already using, self-based methodologies. We resonate with Whitcomb et al. (2009) call for creating spaces where MTEs feel energized. Over the last two years we have felt energized, which motivates us to continue providing spaces at conferences where MTEs who use self-based methodologies feel sustained and empowered in their professional practices (Jaworski & Wood, 2008). We think organizing such spaces contributes to the diversification in the use and acceptance of self-based methodologies (Bullock, 2012; Stinson & Walshaw, 2017).

Plan for the Working Group

We anticipate that many of the attendees at PME-NA 2020 will be Spanish speaking MTEs. Therefore, our working group will be facilitated in both Spanish and English. Given the personal nature of studies conducted under self-based methodologies, we will work on creating an atmosphere of trust and care. Day 1 will be a professional development day where the audience will learn about (1) communities of practice, (2) focus, (3) characteristics, (4) methods, and (5) professional growth in these methodologies. On Day 1 MTEs will also start drafting questions or topics they would like to explore. On Day 2, participants who have used self-based methodologies will be invited to share their work so that MTEs, who are new to such methodologies, can ask questions. We envision Day 2 also as an opportunity for the presenters to receive feedback on their ongoing work. On Day 3, MTEs will work in small groups where they can feel vulnerable and receive more personal feedback. MTEs who are new to these methodologies will continue planning inquiries to puzzling questions or topics of their interest using self-based methodologies. We will conclude our time together on Day 3 with the group planning for future meetings and projects, so that we keep nurturing each other in the use of self-based methodologies.

Conclusion

As MTEs increasingly use self-based methodologies, communities are needed to support their research practices including study design, implementation, representation of findings, and publications in mathematics education journals. This working group intends to be such a community, where over time MTEs feel supported and sustained in their use of self-based methodologies.

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NEW WORKING GROUP: TEACHING MATHEMATICS FOR SOCIAL JUSTICE IN THE CONTEXT OF UNIVERSITY MATHEMATICS CONTENT AND METHODS COURSES

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Goals:

There are three goals for this new working group: 1) To create a community of mathematics teacher educators (MTEs) who are (or are interested in) collaboratively teaching mathematics for social justice (TMfSJ) in their university content and/or methods classes. 2) To collaboratively select/develop/modify TMfSJ tasks and implement those in mathematics content/methods classes. 3) To research the implementation of TMfSJ tasks in content and methods classes.

Strategies to Reach Those Goals:

The organizers have all (to some level) incorporated TMfSJ into their teaching. At a recent workshop, many of the organizers collaborated on designing one task to implement in both content and methods courses focused on understanding gentrification across the United States and also locally in each collaborator's own city/area. This collaboration was highly beneficial and led us to envisioning this working group. Our goal is to create a community of MTEs who will collaboratively develop and implement TMfSJ tasks in their university courses and research the implementation for (in no particular order): (a) preservice teacher (PT) learning about the mathematics, (b) PT learning about the sociopolitical context, (c) impacts 'on PTs' view of mathematics and/or teaching mathematics, and (d) the potential for TMfSJ in university methods or content courses to ignite a call for action.

Background:

Children and youth in schools today are increasingly aware of and grapple daily with the social injustices that pervade our world. Mathematics educators face a moral and ethical imperative to support students in their struggles to make sense of and fight against these injustices (Stinson, 2014). Incorporating social issues into the mathematics curriculum offers one way to both deepen students' mathematics knowledge and encourage the application of mathematics to understand and potentially change their world (Frankenstein, 2009). The Teaching Mathematics for Social Justice framework (TMfSJ) includes two critical interrelated ideas. First, school mathematics can be used to teach and learn about issues of social and economic justice. Second, mathematics can be taught through the study of social justice issues - the development of mathematical literacy itself being an important social justice issue (Gutstein, 2003; Raygoza, 2016). A growing body of research shows how TMfSJ lessons can support PK-12 students to learn mathematics, interrogate social justice issues, and deep positive mathematics identities (e.g. Chao & Marlowe, 2019; Esmonde, 2014; Gutstein, 2003; Turner, Gutiérrez, Simic-Muller, & Díez-Palomar, 2009).

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Given the power of TMfSJ with PK-12 students, some MTEs seek to integrate TMfSJ tasks into their courses in order to give PTs, who have little or no experience with TMfSJ, opportunities to experience integrated learning of mathematics and social issues and to consider the relevance of TMfSJ to their local communities and instructional possibilities in their future classrooms. Research on TMfSJ in mathematics teacher education tends to focus on PT or teacher learning about the pedagogical conceptions and practices of TMfSJ (e.g. Bartell, 2013; Jong & Jackson, 2016). PTs, however, also need opportunities to develop mathematical knowledge (Ball, Thames, & Phelps, 2008) and political knowledge (Gutiérrez, 2017) for teaching mathematics generally and to enact TMfSJ, specifically. Little attention has been paid to the potential for TMfSJ in mathematics content and methods courses impact PTs' mathematics learning, understanding of social issues, and mathematics identities as well as their teaching practices. In other words, we seek to develop a research program that explores PTs learning not only *about* TMfSJ but also *through* TMfSJ.

Given the complexity of social issues and the challenge of using mathematics in authentic ways, TMfSJ proves more effective through multiple iterations over time (Harper, 2019). Accordingly, TMfSJ with PTs cannot happen in a single class. Instead, we seek to explore these ideas across content and methods courses at multiple spaces so that we can describe the complexity of these and other issues with our future teachers (and hopefully with their future students). This is especially true for content courses which allow elementary PTs to experience such tasks from a learner's perspective and to learn to read and write the world themselves. PTs can then build upon this in methods courses to explore the pedagogical practices for TMfSJ.

Many PTs enter their coursework believing that mathematics is neutral or universal (Greer, Verschaffel, & Mukhopadhyay, 2007; Keitel & Vithal, 2008). MTEs must address the fact that mathematics can never be neutral and no classroom is a neutral space (Frankenstein, 1983; Gutiérrez, 2013; Yeh & Otis, 2019).TMfSJ offers a means of engaging PTs in building their sociopolitical consciousness about the political implications of mathematics and how math can be leveraged to read and write the world (Gutstein & Peterson, 2005) from both a content and teaching methods perspective.

In some cases MTEs have met resistance from PTs when integrating social justice issues into the mathematics curriculum (Aguirre, 2009; Ensign, 2005; Felton-Koestler, Simic-Muller, & Menéndez, 2012; Rodríguez & Kitchen, 2004). However, MTEs have also found that they are able to broaden PTs' perspectives about mathematics and mathematics teaching (Bartell, 2013; Ensign, 2005; Felton & Koestler, 2015; Leonard & Moore, 2014; Mistele & Spielman, 2009) when PTs are given opportunities to engage in TMfSJ tasks during teacher preparation. This aligns with Gutstein's (2003) goal of supporting students in developing their sociopolitical consciousness, and possibly a stronger sense of agency and identity. Given the possibilities for TMfSJ to impact both PT and PK-12 student learning and mathematics identity in similar ways, PME-NA offers an ideal community for spearheading this work by bringing together experts in both student and teacher learning.

Participant Engagement

Session 1: Successes and struggles implementing TMfSJ tasks: 1) Organizers will present (30 minutes) on how they have used TMfSJ tasks in their classrooms. 2) Participants and organizers discuss the successes and struggle in implementing TMfSJ tasks. 3) Towards the end of the session, organizers will introduce one context to focus on for the next two sessions (e.g. gentrification) as well as an online media platform for continued participation with this group.

Session 2: Entry points for TMfSJ tasks: 1) We will discuss various entry points (focus on math and social issue) for TMfSJ tasks. 2) We will collaboratively engage in the use of one context in our classes and potential tasks that could go with that context. 3) Participants will share their own experiences and how they may envision using such a context in their class.

Session 3:1) We (in small groups) will collaboratively create/adapt TMfSJ task(s) to participants' localized contexts to use in their teaching. Participants will leave with a more nuanced understanding of TMfSJ tasks/implementation. 2) We will set up structures to follow up via online media after implementations. 3) The goal will be to meet at next year's PME-NA.

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STATISTICS EDUCATION ACROSS SOCIAL AND POLITICAL BOUNDARIES: SIMILARITIES, DIFFERENCES, AND POINTS FOR BUILDING COMMUNITY

EDUCACIÓN ESTADÍSTICA A TRAVÉS DE LAS FRONTERAS SOCIALES Y POLÍTICAS: SIMILITUDES, DIFERENCIAS Y ASUNTOS EN VÍAS DE LA CONSTRUCCIÓN DE UNA COMUNIDAD

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The concepts and practices of the discipline of statistics are crucial for engaging in government and society in the current information age. These concepts and practices are also a part of the school mathematics curriculum to help prepare students to be able to think and reason statistically in their daily lives. The relevance to learning statistics and the importance of its concepts and practices know no boundaries, as statistics is part of the human endeavor to make sense of the world we live in. In spite of this, due to sociopolitical forces, ideas, resources, and research often do not cross political and social boundaries. The goal of this group is to begin to break down some of those boundaries of isolation to create spaces for collaboration and leveraging our shared understandings for positive change. A particular focus given the location of the conference is to break down political and social barriers sharing ideas and resources.

Keywords: Data Analysis and Statistics, Cross-cultural Studies

Creating opportunities for students to engage in statistical investigations to learn statistical concepts and become attuned to statistical practices is crucial for mathematics educators (Franklin et al., 2007, 2015) making it relevant for PME-NA. The centrality of context to statistical inquiry makes its practices powerful for students to make sense of their world (Cobb & Moore, 1997; Wild & Pfannkuch, 1999). During the 2019 PME-NA Conference the goal of the statistics education working group was to create a space for those interested in researching issues around the teaching and learning of statistics to meet, discuss, synthesize past research, and begin to strategize ways of leveraging multiple perspectives and expertise to identify and address current challenges in statistics education. The goals of this year's group are still in a similar vein, but we want to take advantage of the location of the conference in Mexico to collaborate and discuss statistics education across political and social boundaries to share ideas and resources.

Education systems are very contextualized, which can lead scholars to focus on their specific context. One of our goals is to share challenges and insights that statistics educators have from the contexts that they work in to consider similarities and differences and to learn from one another. Such sharing is important because though North America includes Canada, Mexico, and the United States the political powers in those countries create obstructions physically, emotionally, psychologically, and discursively between their citizens and those of the other countries, which have isolating effects. However, people move back and forth across those made up boundaries spreading ideas and culture. Therefore, in education, a social science, we face some of the same challenges. Because of our contextual differences, we have also likely tackled challenges in different ways based on the resources we had available. Furthermore, based on differences in language and power structures we have likely faced different challenges that we may be able to help one another. Through sharing our experiences, we share insights on how to tackle common challenges, reflect on what we are missing in our work, consider important challenges for us to tackle together, reflect on what needs we have, and consider what help and support we can give. We plan to develop a mechanism for the communities to share and support one another with ideas, resources, data, and social and political experiences. We also hope to share ideas about ways we can talk about real-world controversial issues in our classrooms that are relevant to the lives of our students safely and critically – a very important aspect in all mathematics classrooms, but essential in statistics classrooms due to the nature of exploring and analyzing relevant data. Finally, we aim to find ways to share ideas across languages and cultural communities and find ways to break down barriers to the English dominant scholarship.

Theoretical Framing

To frame the work of the working group we draw from Communities of Practice (CoP; Lave & Wenger, 1991; Wenger, 1998). All of the authors are members of various CoP's relevant to the focus of the working group including those of statistics, statistics education, and mathematics education. We also come from different communities, particularly in the contexts we work within. We view this working group as a space were we come together to share challenges and lessons learned as well as consider our roles as boundary crossers and how we and the resources we produce might break down some of the boundaries of the communities we are situated and the communities we are interconnected by. This framing is consistent with past work of the authors in considering how CoP's can be used with teacher professional development (Gómez-Blancarte & Viramontes, 2014) and organizing researchers (Tauber et al., 2019).

Table 1: Plan for Active Engagement of Participants

Session	Activities
Session 1: Sharing	• Participant introductions. • Brief presentations of selected projects from authors' various contexts to highlight challenges in statistics education research to begin discussions. • Participants share their context for statistics education and challenges. • Identify similarities and differences in contexts and challenges. • Discuss how we can collaborate and what we hope to gain.
Session 2: Discussing	• Group discusses insights we have collectively on the similarities in contexts and challenges identified during the first session. • Group discusses resources they have to tackle the challenges identified. • Group will discuss what would be helpful and sustainable mechanisms for sharing ideas and resources beyond the conference.
Session 3: Planning for Action	• Participants brainstorm ways of tackling challenges different from their own identified during the first session to bring new ideas to bear. • Participants create a plan of collaboration for after the conference, research ideas, analyzing data together, writing together, etc.
After conference: Action	• Group continues to collaborate to implement the ideas and share resources based on connections made during the conference. • Group members begin longer term research collaborations.

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ALGEBRA, ALGEBRAIC THINKING AND NUMBER CONCEPTS

RESEARCH REPORTS

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SUPPORTING STUDENTS' MEANINGS FOR QUADRATICS: INTEGRATING RME, QUANTITATIVE REASONING AND DESIGNING FOR ABSTRACTION

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We describe how we supported middle-school students developing meanings for quadratic growth by bringing together three theoretical framings and designing a task sequence grounded in that framework. Specifically, this framework was assembled from the research on students' quantitative and covariational reasoning, Realistic Mathematics Education, and the theory of designing for mathematical abstraction. We present data from a teaching experiment to highlight how this task sequence supported students as they imagined constant increases in the amounts of change in one quantity, a kind of change they later understood to be a defining characteristic of quadratic growth. We conclude with a discussion of our findings and how our integration of the three frameworks proved useful to design a productive task sequence.

Keywords: Algebra and Algebraic Thinking; Middle School Education; Design Experiments

Quadratic relationships are an important topic in middle-school through college mathematics. In this study, we build on prior work (e.g., Ellis & Grinstead, 2008; Ellis 2011a, 2011b; Hohensee, 2016) that provides accounts of middle and high school students developing productive meanings for quadratic relationships via their quantitative and covariational reasoning. Whereas students in these studies leveraged numeric quantitative reasoning, we were interested in designing a task sequence that supported them in both non-numeric and numeric reasoning to develop meanings for quadratic growth. Hence, our goal is to address the research question: *How can middle-school students leverage non-numeric and numeric quantitative and covariational reasoning to develop meanings for quadratic growth?*

In what follows, we outline our operationalization of quadratic growth grounded in students' quantitative and covariational reasoning, and then describe a task sequence we developed with principles of Realistic Mathematics Education (RME) and designing for abstraction in mind. We highlight how this task sequence was productive in supporting students developing productive meanings for quadratic growth. We conclude by highlighting how the three theories interacted to inform the task design and development.

Quantitative Reasoning, Covariational Reasoning, and Quadratic Change

As conceptual entities, meanings for quantities can and do differ from individual to individual. Therefore, it is critical to attend to a *student's* meanings for quantities. Accordingly, Steffe, Thompson, and colleagues' stance on quantitative reasoning (Steffe, 1991; Smith III & Thompson, 2008; Thompson 2008) underscores the thesis that as students construct quantities in order to make sense of their experiential world (Glasersfeld, 1995), teachers and researchers cannot assume that students maintain understandings of quantities that are compatible with teachers' and researchers' intentions.

Quantitative reasoning can involve numerical and non-numerical reasoning (Johnson, 2012), but the essence of quantitative reasoning is non-numerical (Smith III & Thompson, 2008). Building on these

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prior descriptions of quantitative reasoning, Carlson et al. (2002) described covariational reasoning as entailing a student coordinating two quantities with attention to the ways the quantities change in tandem. They specified mental actions that allow for a fine-grained analysis of students' activity. The mental actions include coordinating *direction of change* (area increases as base length increases; MA2) and *amounts of change* (the *change* in area *increases* as base length *increases in equal successive amounts*; MA3). Whereas researchers (Johnson, 2012; Moore, 2016; Paoletti & Moore, 2017) have described productive ways high school and college students can engage in reasoning compatible with Carlson et al.'s mental actions, few studies have explored the possibility of middle-school students enacting these mental actions in productive ways (e.g., Ellis, 2011a). We pay particular attention to MA3 as critical to students developing meanings for quadratic relationships, and we characterize a productive meaning for quadratic change to entail a student understanding that as one quantity changes by equal amounts, the amounts of change of the second quantity increase (or decrease). Further, and consistent with meanings described by others (Ellis, 2011a, 2011b; Lobato et al., 2012), these amounts of change increase (or decrease) themselves by a constant amount. Hereafter, we refer to these constant amounts of change of the first amounts of change as a constant AoC of AoC.

Ellis (2011b) provides evidence that middle-school students are capable of engaging in such reasoning. In her examination of middle-school students' ways of reasoning about heights, lengths, and areas of a growing rectangle, she describes how the students numerically and pictorially represented the first and second differences to identify a constant second difference, which later supported them in developing meanings for quadratic change. Building on and extending this work, we are interested in exploring whether we can support students in first identifying constant AoC of AoC non-numerically prior to reasoning numerically to develop meanings for quadratic growth.

Designing for Mathematical Abstraction with RME Principles in Mind: An Example Task

We begin by outlining important aspects of RME and designing for mathematical abstraction that informed the task development. Then we introduce the *Growing Triangle Task*.

RME is an instructional theory that aims to find ways to connect what students already know to what they do not yet know (Gravemeijer, 2008). RME's emphasis is on opportunities for students to re-invent mathematics by organizing experientially real situations (Cobb et al., 2008; Gravemeijer, 2008). Tasks are experientially real to students if they can engage in personally meaningful mathematical activity; they need not refer to some 'real-world' situation or context.

By having students engage in an experientially real context, we intend to support their *horizontal mathematization*, a process that refers to a student's development of meanings for a specific context. A student's initial *model of* (Linchevski & Williams, 1999) the situation is specific to the context but should support her in developing informal strategies and representations that will be useful as she begins to generalize to other contexts. After beginning to mathematize the situation, a student can start the progressive process of *vertical mathematization,* which entails extending her informal mathematical representations or activity into more normative representations or activity. Thus, horizontal mathematization is preparation for vertical mathematization. Vertical mathematization may involve using conventional notations such as making a drawing, table, or graph (Cobb et al., 2008; Gravemeijer & Doorman, 1999). As students gather more experiences with similar problems, their attentions may shift towards mathematical relations and strategies which helps them develop further mathematical relations and a resulting shift from *models of* a context to *models for* a mathematical idea. This shift allows the students to use the model in a different manner (Gravemeijer, 2008) thereby becoming a *model for*. We describe in the Task Design section how we imagine this transition occurring in the context of quadratic growth.

Our operationalization of quadratic change emphasizes the importance of students constructing AoC of AoC as a quantity unto itself and then identifying constant AoC of AoC. We leveraged 3D-printing to design physical manipulatives we conjectured could support the students in both of these endeavors. As Greenstein (2018) noted, "The faithful mental representation of objects is critical, because conceptual thought proceeds from representational thought and representational thought proceeds from perception" (p. 3). Hence, leveraging principles of *designing for mathematical abstraction* (Pratt & Noss, 2010), we devised the physical manipulatives with the intention of making several quantities of interest, including AoC of AoC, available to students for abstraction through their sensorimotor engagement with those manipulatives (Piaget, 1970). We describe these manipulatives in the next section.

The Growing Triangle Task

We designed the *Growing Triangle Task* (https://bit.ly/2YTjwmj) with principles of RME, designing for abstraction, and theories of students' quantitative and covariational reasoning in mind. In this task sequence, students first interact with a dynamic GeoGebra applet showing an apparently smoothly growing scalene triangle (Figure 1a). Our intention is to provide students an experientially real context so that they could develop a *model of* the growing triangle situation. To support the students in attending to the triangle's area and base length (i.e., reasoning covariationally), we highlighted the base length of the triangle in pink and the area in green. The area and base length grow (apparently) smoothly as the longer slider increases (apparently) smoothly. With AoC in mind, we included a second smaller slider which allows students to increase the increment by which the pink length increases (e.g., to equal integer chunks versus apparently smoothly). We have the 'trace' option available so that students can visually identify the increasing AoC of area in the applet (i.e. the increasing size of the consecutive trapezoids shown in Figure 1b).

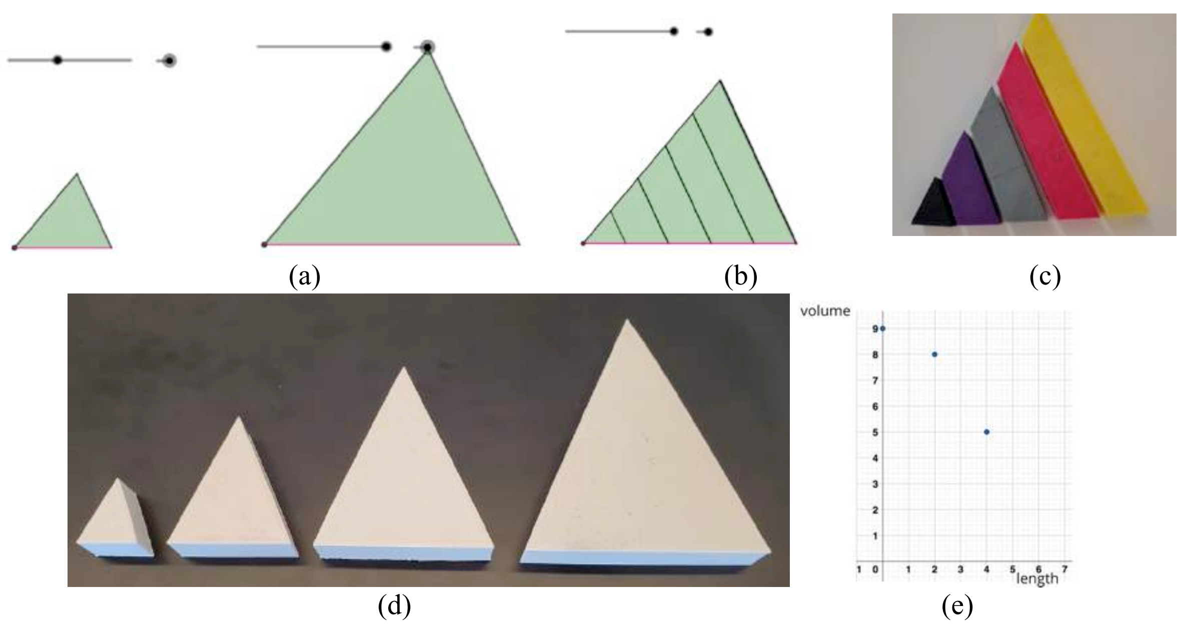


Figure 1: (a/b) Several screenshots of the *Growing Triangle Task* shown in applet, (c/d) images of manipulatives and (e) new task for vertical mathematization

We conjectured that although the visual representation provided by the applet may support students in identifying the increasing AoC (MA3), it was unlikely to support them in identifying the constant AoC of AoC. Hence, with designing for abstraction (Pratt & Noss, 2010) in mind, we 3D-printed a set of manipulatives that consisted of four consecutive gray triangles (Figure 1d) to represent the growing triangle at four equal integer increases of the base length and five AoC blocks (one triangle in black, and four trapezoids, Figure 1c). Thus, these designs make these representations of increase in the amount added to each triangle to get to the next consecutive triangle available to students through their mediated engagement with them. Figure 2b demonstrates this potential for abstraction: by stacking the physical representations of change on top of each other, students can construct non-numerical interpretations of the constant AoC of AoC. After doing so, we will ask them to create a table of values for the relationship to explore if (and if so, how) they identify the constant AoC of AoC in this representation.

After developing a *model of* the Growing Triangle situation, the next step is to support students in extending this model by developing a *model for* constant AoC of AoC (i.e., by continuing the process of vertical mathematization). To do this, we presented them with several graphs, each with three points, and the instructions, "For each of the following, we know the differences in the amounts of change of volume are constant with respect to the length of a side. Complete each graph". In the ensuing activity, after students determined additional points using the constant AoC of AoC, they will be introduced to a normative definition for quadratic growth to further support their vertical mathematization: "Whenever the *amounts of change* change constantly (i.e. we have a second constant difference), the relationship is *quadratic*." Additionally, we consider that this activity may support students in developing more sophisticated meanings for other polynomial relationships. In such cases, these relationships and their related properties will be discussed (e.g., cubic growth has a constant third difference, etc.).

Methods, Participants, and Analysis

To examine the potential of supporting students in developing meanings for quadratic change, we conducted a teaching experiment (Steffe & Thompson, 2000) situated in a whole-class setting. The teaching experiment occurred in a middle school that hosts a diverse student population (over 75% of the students are of color; 75% are entitled to free or reduced-price lunch), in the northeastern United States. The class was selected from a convenience sample. The teacher of the school's only accelerated 8th grade geometry course[1] invited the research team to explore new activities with her students. We taught five class sessions, with each session scheduled for 76 minutes. We covered a variety of topics in these sessions (see Table 1), with the focus of this paper being the fifth session. The class had eight students who worked in three groups on large whiteboards during class time. We video- and audio-recorded two of the groups (one group of three and one pair), capturing their utterances, motions, and written work. For brevity's sake, we focus this report on the group of Neil, Aaron, and Nigel (pseudonyms).

Adopting a radical constructivist perspective (Glasersfeld, 1995), we contend that a student's mathematics is inaccessible to us as researchers. Hence, to analyze the qualitative data, we performed conceptual analyses (Thompson, 2008) to generate and test models of each student's mathematics so that these models provided viable explanations of his observable words and actions. With the goal of building viable models in mind, we analyzed the records from the teaching episodes using open (generative) and axial (convergent) approaches (Strauss & Corbin, 1998). Specifically, we watched all videos to identify instances that offered insights into each student's meanings. Using these instances, we generated tentative models of each student's mathematics, which we compared to researcher notes taken during on-going analysis. We tested these models for viability by searching for supporting or contradicting instances in his other activities. When evidence challenged our

[1] Most students in the school take algebra in 8th grade. The terminology is used to be consistent with the school's name for the course.

models, we revised hypotheses to explain each student's meanings and returned to prior data with these new hypotheses in mind to modify previous hypotheses. This process resulted in viable models of each student's mathematics.

Table 1: Whole class teaching experiment sequence

Session	Task	Content topics for students to develop
1 & 2	Faucet Task (see Paoletti, 2018)	Coordinate Systems, graphs, and graphing
3	Triangle Task (Part I)	Non-linear change
4	Triangle/Rectangle Task (Paoletti, Vishnubhotla, & Mohamed, 2019)	Comparing non-linear and linear change, Systems of Relationships
5	Triangle Task (Part II)	Extending non-linear change to quadratic change
6	Post-test	

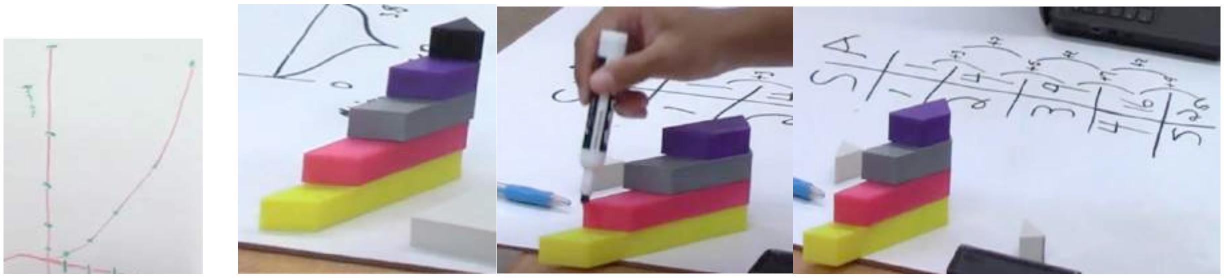
Results

In this section, we first highlight the students reasoning non-numerically about the quantities in the *Growing Triangle Task*. We then present the students' activity with manipulatives, which facilitated their identifying constant AoC of AoC (i.e., an instance of horizontal mathematization). We then characterize how the students develop a *model of* constant AoC of AoC when they are presented with a new task (shown in Figure 1: (a/b) Several screenshots of the *Growing Triangle Task* shown in applet, (c/d) images of manipulatives and (e) new task for vertical mathematization e), and how that *model of* transitions into a *model for* quadratic growth. We conclude with examples of student work on the post-test to characterize the extent to which students engaged in vertical mathematization.

The Growing Triangle Task: Horizontal Mathematization

Introducing the *Growing Triangle Task* on Day 3, the teacher-researcher (TR) distributed the manipulatives to the class intending to support the students in differentiating between the total area of the triangle and the changes in area for each increase in base length. To support the students developing meanings for area and change in area, the TR began with the first triangle (shown in Figure 1c) and added the first change. He asked the class, "When I add another unit of side length, is the amount of area I'm gonna add, is it the same amount, more, or less?" Neil responded, "More." The students were then asked to open up the applet on a laptop and begin making observations about the changes in the triangle. As they watched the applet play, the three students discussed how the changes were growing. When the TR asked, "So initially do we have a big jump, a small jump, medium jump?" Both Aaron and Neil simultaneously responded, "Small." As the TR asked what happened for the next two jumps, Neil responded, "It gets larger… even larger." In this activity, Neil's quantitative understanding of area was elicited as he described it growing by increasing amounts (MA3). After this interaction, Neil plotted points accurately representing this relationship between area and base length (Figure a). We infer Neil was reasoning covariationally as he leveraged non-numeric images of total area and AoC of area in the situation to graphically represent each quantity and the resulting relationship.

Whereas the focus of Day 3 was on supporting students in identifying increasing AoC, on Day 5 we returned to the triangle task with the intention of supporting students in imagining constant AoC of AoC in this context. Because the students had already identified increasing AoC of area with respect to side length, the TR asked the students to consider, "How do the amounts of change compare to one another?" The TR requested the students "play around with the manipulatives and explore." Neil and Nigel began to stack the trapezoidal manipulatives on top of one another in consecutive amounts (shown in Figure b). The following conversation ensued:



(a) (b) (c) (d)

Figure 2: (a) One group's graph; (b/c) students notice a constant second difference and use manipulatives to identify it; and (d) table displaying students' work

Neil: [*Places yellow trapezoid on bottom then places pink trapezoid on top*] Ooh! Look, look, it's a quadrilateral [*pointing to the piece at the end of yellow trapezoid not covered by the pink trapezoid, seen in Figure b*].

Aaron: Mmhmm. [*Agreeing*]

Neil: Right, look, if we do it again. [*Nigel places gray trapezoid on top of the pink trapezoid*] Same size [*pointing to quadrilaterals created by the difference in the yellow and pink trapezoids and pink and gray trapezoids to indicate these amounts are the same*]. Put that one on [*Nigel places the purple trapezoid on top of the stack*] Same size [*referring the quadrilateral formed by the difference in purple and grey trapezoids*].

TR: So what did you notice?

Neil: It's like, this quadrilateral [*pointing to quadrilateral shown by the differences Figure b*] keeps going I guess, it's added on to that [*pointing to each layer of the stack of trapezoids*].

TR: That, that piece we're adding on to the amounts of change is always the same? [*Nigel nods in agreement as TR speaks*]

Neil: [*as TR is completing his remark*] Yeah.

In this conversation, Neil characterized the amount being added to each trapezoidal AoC manipulative as equal (i.e. constant AoC of AoC). We conjectured from activity (e.g., adding consecutive trapezoids to the stack), agreement throughout the conversation, and their later activity, that Nigel and Aaron also understood that the relationship between area and side length exhibited constant AoC of AoC. We provide evidence for this shortly.

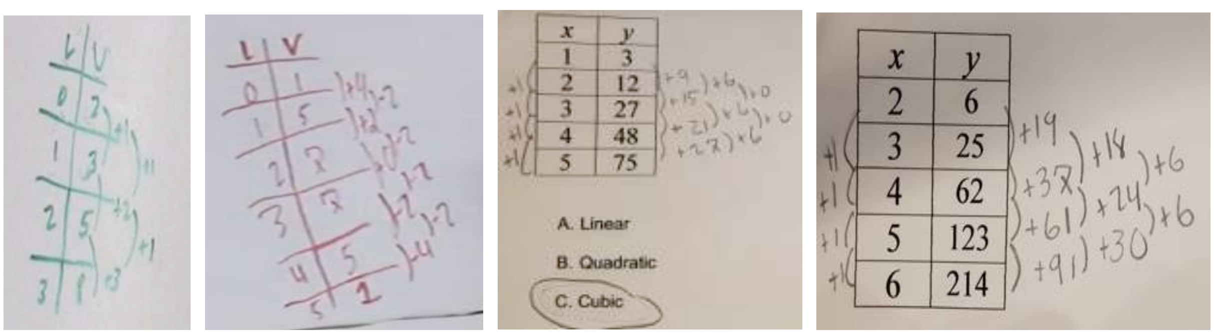
Beginning vertical mathematization for AoC of AoC

After identifying non-numeric constant AoC of AoC using informal representations, the TR prompted the students to create a table of values as a way to normatively represent such a relationship (i.e., an instance of vertical mathematization). His goal was for students to consider how the constant AoC of AoC would impact the numeric values in the table. The TR designated the smallest gray triangle as having an area value of one, and with this value of one, the students identified the purple trapezoid as having an area of three units and the total area of the resulting triangle as composed of four units. Similarly, when adding the pink trapezoid to the total area, Neil identified the area of the pink trapezoid as "seven," then immediately described the total area as "seven plus nine," with nine being the total area of the previous triangle. We infer Neil understood that in order to find a new total area he must first identify the AoC of area from the previous base length to the new base length and add that AoC-value to the previous total area. Using this reasoning, the three students completed their table (shown in Figure 2d).

To build on the students previously identifying non-numeric constant AoC of AoC, the TR asked the students how the values they had identified for the trapezoidal AoC manipulatives (i.e. first difference), and the differences in these (i.e. second difference), were represented in their table. As part of this process, the students identified the differences with Aaron notating with arrows and values (e.g., '+3', '+5') the AoC from one consecutive area to the next (seen in Figure d). As Aaron identified several AoC-values and notated them besides the corresponding total area values in the table, he spontaneously described how the constant AoC of AoC was represented to his peers. Pointing to the '+3' and '+5" written adjacent to the table, he said, "It's two" and then pointed to the quadrilateral made by differences of the AoC pieces (Figure c). We infer Aaron was connecting their informal identification of the constant AoC of AoC with the manipulatives to the changes in the AoC represented in their table.

Moving to a *model for* constant AoC of AoC

After each group identified the constant AoC of AoC in the table, we provided them with new tasks (example shown in Figure 1d) representing volume and length values for some new hypothetical situation. Indicative of the students engaging in extending their *model of* the *Growing Triangle Task* to a more general *model for* constant AoC of AoC, on the first task Aaron drew a table with the coordinate values, found the first AoC in the second quantity ('+1", "+2"), then identified an AoC of AoC value of '+1". Using the constant second difference, Aaron found the next first difference ('+3', shown in Figure 3a) and used this value to determine the next volume-value of eight. In the second task, Nigel engaged in compatible activity with a second difference value of '-2' (Figure 3b). We infer the students leveraged a meaning for constant AoC of AoC to calculate values of new points in a given relationship (i.e. a shift to a *model for* constant AoC of AoC). After engaging in this activity, we concluded the class episode by introducing the definition of quadratic (as well as cubic) growth as described above to extend students' understandings of a relationship that contains a constant second difference as quadratic.



(a) (b) (c) (d)

Figure 3: (a/b) Samples of students' vertical mathematization; post-test questions with students' work on (c) quadratic and (d) cubic relationships

Evidence of a *model for* quadratic growth

We present data from a post-test given on Day 6 as evidence of the three students extending their *models for* constant AoC of AoC to *models for* quadratic growth. The post-test included two tables (shown in Figure 3.c/d), with options linear, quadratic, cubic, or exponential. Given the first table (as shown with Nigel's work in Figure 3c), Neil and Aaron found a constant second difference and identified the relationship as quadratic. Nigel's work indicates he also successfully found a constant second difference, but then moved on to find a constant third difference of '+0' and concluded that the relationship was cubic. One possible explanation is that Nigel may not have noticed the constant second difference prior to finding the constant third difference; if this were the case, and Nigel had noticed the constant second differences, we conjecture he likely would have chosen quadratic. This conjecture is based in part on his response addressing the second table in which he and the other two students identified a constant third difference (shown in Figure 3d) to conclude the growth was cubic. Hence, we infer each student has at least begun to develop *models for* quadratic (and cubic) growth.

Discussion and Implications

To conclude, we highlight how we were able to draw from theories on students' quantitative and covariational reasoning (Carlson et al., 2002; Smith III & Thompson, 2008), designing for mathematical abstraction (Pratt & Noss, 2010), and RME instructional theory (Gravemeijer, 2008) to support students building *models for* quadratic growth (and potentially *models for* other polynomial growth as well). These findings extend previous examinations of middle-school students developing meanings for quadratic change (e.g., Ellis, 2011a) by describing how they leveraged their non-numeric quantitative and covariational reasoning to identify constant AoC of AoC before representing such relationships numerically.

We intend for our description of the task design and sequence to highlight ways in which these theories can work together to inform the design of a task sequence that is responsive to students' reasoning activity. For example, designing for mathematical abstraction supported us in developing manipulatives that we conjectured could support students constructing quantities, including total area, AoC of area, and AoC of AoC of area in ways compatible with our intentions. Further, these manipulatives supported students' non-numeric and numeric quantitative and covariational reasoning as well as their engagement in the mental actions described by Carlson et al. (2002). Connecting RME to the other theories, we note how the 3D-printed manipulatives were critical as the students transitioned to vertical mathematization as they identified constant AoC of AoC. Similarly, we highlight how the students leveraged their quantitative and covariational reasoning as they moved from *models of* the *Growing Triangle Task* to *models for* constant AoC of AoC, and further to *models for* quadratic growth.

We highlight the productive meanings for quadratic growth the three students described here developed as part of this study, and offer this research as both an existence proof and starting point for future researchers interested in exploring how to support larger populations of middle-school students in leveraging non-numeric and numeric quantitative and covariational reasoning to develop important mathematical ideas including quadratic and cubic growth. Integrating activities like these with activities described by Ellis (2011a), for example, may support students in developing more productive meanings for AoC and quadratic growth. Future researchers may be interested in exploring this possibility.

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UNIT TRANSFORMATION GRAPHS: A CASE STUDY

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We introduce a methodology for diagramming the ways students use sequences of mental actions to solve mathematical tasks. We studied 12 pre-service teachers as they solved a set of fractions tasks, ranked by cognitive demand. We present the unit transformation graphs for one of those pre-service teachers, to illustrate how she experienced and met cognitive demand across the fractions tasks. Specifically, the graphs illustrate how sequencing mental actions places demands on working memory and how units coordination structures can offload some of that demand.

Keywords: Cognition; Learning Theory; Number Concepts and Operations; Problem Solving.

Prior mathematics education research from a Piagetian perspective has identified students' construction and transformation of units as central to their development of number, extended to rational numbers and even algebraic reasoning as generalized arithmetic (Hackenberg, 2013; Steffe & Cobb, 2012; Steffe & Olive, 2010). The present study was motivated by a desire to explicitly identify mental actions that undergird the construction and transformation of units in the context of fractions. For example, students might construct 1/7 as a unit, which has a one-to-seven relationship with a whole unit. This relationship might be established by the mental action of partitioning a continuous whole into seven equal parts. Conversely, the relationship could be reversed by iterating a 1/7 part seven times to reproduce the whole. Thus, partitioning and iterating constitute reversible and composable mental actions that can be used to construct a 1/7 unit and transform it back into the whole unit (Wilkins & Norton, 2011).

The purpose of this paper is to introduce a methodology for modeling students' mathematics by explicitly identifying the sequences of actions and unit structures they use to solve mathematical tasks. Unit transformation graphs (UTGs) account for the constraints of working memory in sequencing actions, as well as the power of unit structures in offloading demands on working memory. We share the case study of a pre-service teacher (PST) with relatively high working memory and the ability to assimilate and operate on two-levels of units. Findings (and the UTGs themselves) illustrate how students might use their units coordinating structures to chunk sequences of actions into single units, thus reducing the cognitive demands of mathematical tasks.

Theoretical Framework

Mathematics educators have begun to explicitly account for students' actions in building models of their mathematical reasoning. Recent examples include activity-effect relations (Tzur & Simon, 2004) and the Learning Through Activity framework (Simon Placa, Avitzur, & Kara, 2018). Similar to those models, UTGs explicitly identify the mental actions students use to construct and transform units, but have two distinguishing features. First, they account for the role of working memory in sequencing actions. Second, they explicitly account for the role of unit coordinating structures in reducing demands on working memory.

Mental Actions for Constructing and Transforming Units

Following Piaget (e.g., Beth & Piaget, 1966), we characterize mathematical actions (operations) as mental actions that are potentially reversible and composable. We are particularly concerned with operations students use to construct and transform units. For example, a student might construct a unit by isolating a collection of items, treating them as identical, and taking them as a whole—a

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mental action called unitizing (Steffe, 1991). Students might also unitize a continuous span of attention, whether it be time, length, area, or volume. Once constructed, units can be transformed into other units: students might iterate a unit, making copies of it and integrating the copies within a new composite unit (a unit composed of units); or they might partition a unit into equal parts, forming smaller units.

Steffe (1992) originally defined a units coordination as a distribution of the units within one composite units across the units of another composite unit. For example, in determining the value of 7 times 4, a student might distribute seven 1s across the four 1s that comprise 4, making a sequence of four units of seven units of 1. This definition orients our thinking about how units might be transformed into other kinds of units, but we include additional transformations as units coordinations. The aforementioned operations of unitizing, partitioning, iterating, and distributing are all potentially reversible and composable, and all can be used to transform units into other units. In addition, disembedding enables a student to remove a unit, or collection of units, from a composite unit, without destroying the composite unit (Steffe, 1992). The student maintains the composite unit while considering some of its parts as units separate from that composite unit.

Sequencing Operations in Working Memory

Students might need to perform a long sequence of operations to solve a mathematical task. In our framework, the cognitive demand of the task would increase with the length of this sequence. This perspective aligns with Pascual-Leone's (1970) characterization of working memory as a mental-attentional operator (the M operator). "Working memory involves the process of holding information in an active state and manipulating it until a goal is reached" (Agostino, Johnson, & Pascual-Leone, 2010, p. 62). It is a limited resource used to implement mathematical problem solving strategies (Bull & Lee, 2014; Swanson & Beebe-Frankenberger, 2004) and one that predicts children's mathematical achievement (Blankenship et al, 2018; De Smedt et al, 2009).

In the context of mathematical problem solving, Pascual-Leone (1970) characterized this limited capacity (M-capacity) as "the number of separate schemes (i.e., separate chunks of information) on which the subject can operate simultaneously using his mental structures" (Pascual-Leone, 1970, p. 302). Commensurate with other measures of working memory, Pascual-Leone (1970) found that adults can typically hold in mind 5-7 schemes at once. In numerical contexts, such as solving fractions tasks, schemas might refer to the operations students use to construct and transform units. A student might hold in mind a sequence of seven such operations, but fractions tasks may involve multiple levels of units (e.g., the whole, unit fractions, measures of a unit fraction) with many transformations between them. A student might offload some of that demand on working memory through the use of figurative material, such as drawings or notations, or by assimilating some of the units and unit transformations into existing cognitive structures: units coordinating structures.

Units Coordinating Structures

In the absence of structures for assimilating multiple levels of units, each unit or unit transformation (e.g., partitioning a whole into n equal parts) would place separate demands on working memory. However, those units and the operations that transform them can be organized within units coordinating structures (Boyce & Norton, 2016; Hackenberg, 2007; Ulrich, 2016). The rectangle on the right side of Figure 1 represents a two-level structure for coordinating units—one that would organize the previously-described one-to-seven relationship between a whole unit and the unit fraction, 1/7. Note that the structure contains two units and a pair of reversible mental operations between them: the whole can be transformed into seven equal parts by via the operation of partitioning; and this mental action can be reversed by iterating one of those parts seven times, reproducing the whole. This unit coordinating structure acts as a single unit that can be used to assimilate two units and an action (in either direction), thus reducing cognitive demand from three to one.

Figure 1: Units coordinating structures

The mathematical power of units coordinating structures is well documented (Boyce & Norton, 2019; Hackenberg, 2007; Tillema, 2013). For example, Hackenberg and Tillema (2009) demonstrated that students who can assimilate three levels of units are able to reason through fractions multiplication problems in ways that other students cannot, including students who can assimilate two levels of units. This power can be explained, at least in part, by reduced demands on working memory. When units are assimilated into existing structures, working memory is freed to focus on ever more complex tasks. This structuring and offloading also explains the sense in which mathematics builds upon itself.

Methods

The data collected and analyzed for this paper is part of a larger project investigating the cognitive development of mathematics, behaviorally and neurologically. This paper reports on results from video recorded behavioral data.

Data Collection

Participants consisted of PSTs at a large university in the mid-Atlantic United States. All participants were enrolled in one of two sections of the same mathematics course—Mathematics for Elementary School Teachers—taught by the same instructor. PSTs comprise a special population of participants for the study because they practice metacognitive skills in the context of solving elementary school mathematics tasks. Specifically, they are encouraged to explain their reasoning when solving tasks. Twelve PSTs agreed to participate in the interviews.

Interviews lasted about 75 minutes and occurred in three parts: an assessment of their available structures for coordinating units (e.g., two-level structures like the one shown on the right side of Figure 1) using interview tasks from prior studies with middle school students (Norton et al, 2015); an assessment of working memory using backward digit span with digits read aloud (Morra, 1994); and the fractions tasks. Using the units coordination and working memory assessments from the first two parts of the interview, a subset of ranked fractions tasks was selected for each PST. Tasks were ranked by cognitive demand based on the number of unit constructions and transformations that might be required to solve the task, without reliance on units coordinating structures. We intended for initial tasks to impose low cognitive demand on PSTs and for later tasks to impose high cognitive demand, so generally, PSTs assessed with lower M-capacity began with simpler (lower ranked) tasks. Table 1 presents the four tasks on which we focus in this report.

Table 1: Fractions tasks (adapted from Hackenberg & Tillema, 2009)

Task #	Rank	Task Description
5	8	Imagine this [drawing a rectangle] is 5/9 of a whole candy bar. So, how could you make 1/9 of the whole candy bar from what you have?
6	10	Imagine a rectangular cake that is cut into 15 equal pieces. You decide to share your piece of cake fairly with one other person. So, how much of the whole cake would that person get?

8	12	Imagine you are at a party and a cake is cut into nine equal pieces. Two people show up to the party late and you decide to share your piece of cake with them. So, what fraction of the whole cake do the latecomers get together?
10	14	Imagine cutting off 1/4 of 5/6 of a cake. So, how much is that of the whole cake?

Data Analysis

Data analysis reported here consists of real-time and retrospective analysis of PSTs' responses to the fractions tasks. During the interview, we assessed PSTs' abilities to solve the tasks without using figurative material in order to determine whether to continue to more challenging (higher ranked) tasks. The interview continued with higher and higher ranked tasks until we inferred that the PST was unable to produce correct or confident solutions. In some cases, PSTs were explicit about their own perceived limitations; e.g., "I have no idea" or "my brain is confused now."

After all interviews were completed, the team began retrospective analysis of the behavioral video data. The videos were analyzed, PST by PST, moving from lowest ranked to highest ranked tasks. For each task, the video analysis consisted of two main parts: the first consisted of classifying the demand of the task for the students, and the second consisted of creating a UTG for the PST's actions in solving the task. A constant comparative analysis was used throughout both parts of analysis to promote consistency.

Analysis of the videos was done together by the research team with at least two of the three team members present. Each task was assigned a classification for the cognitive demand of the task, based on behavioral indicators during the PST's response to the task. Cognitive demand was coded as Low, High, or Over, depending on how challenging the task seemed to be for the PST in managing the units and unit transformations involved in solving the tasks. The Low code indicates that the PST's response was quick and confident. The High code was used when the PST struggled presumably operating near the limits of their M-capacity, as indicated by expressed doubt, rehearsing the task's solution, and requests for the task to be repeated. The Over code indicates that the task was beyond the students' ability to solve without figurative material or help from the interviewer.

Once cognitive demand codes were assigned for each task, the research team went back and watched the video again in order to build a UTG for each task's solution, illustrating the sequence of actions (operations) the PST used to reach a solution. The graphs serve as explanatory models for the PSTs' observed behavior in solving tasks by drawing on cognitive resources. Using the constant comparative method, we iteratively returned to prior graphs to ensure the models were consistent across tasks and PSTs. Adjustments were made to prior graphs as new features emerged in newer graphs.

Results

We chose to focus on PST 22 because she was one of two PSTs with the highest assessed M-capacity (7), and of those two, was the only PST operating at the lower stage of units coordination (constructing two-level unit structures but not three-level unit structures). Here, we analyze her responses to the four tasks shown in Table 1.

Task 5

PST 22 exhibited Low cognitive demand in solving Task 5. As soon as the task was posed, she responded, "well if you have five-ninths of it, taking away four ninths would give you one-ninth because five minus four is one." The UTG shown in Figure 2 illustrates the mental actions we inferred PST 22 used in solving the task.

PST 22 seemed to rely on a part-whole understanding of fractions. Rather than structuring 1/9 as a one-to-nine size relation between the 1/9 part and the whole, PST 22 seemed to partition the whole into nine parts (P_9) and, reversibly, take their unitized collection as the whole (U_9). The PST needed to disembed five of those parts from the whole (D_5) to establish 5/9 as five parts out of nine equal parts in the whole. Then, taking away four of those five parts, through a second use of disembedding (D_1), would leave 1/9 as one out of nine equal parts in the whole. As such, PST 22 would experience Task 5 as having an M-demand of 3, well below her M-capacity of 7.

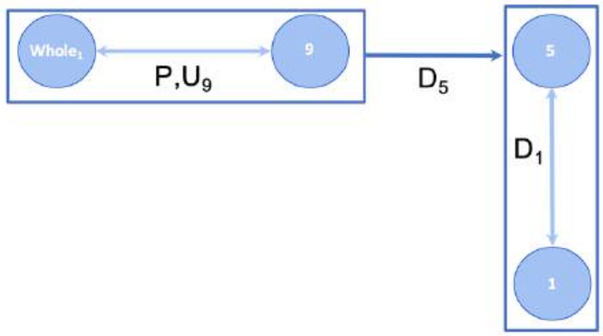

Figure 2: Making a unit fractional part from a non-unit fractional part (Task 5)

Tasks 6

Tasks 6 involved finding a fractional value (relative to the whole) when taking a unit fraction of a unit fraction. With the composition of two fractions, these tasks should impose additional cognitive demands, relative to Task 5. These increased demands are indicated in the PSTs' behavioral responses to the tasks, but behavioral indicators did not meet the threshold of High cognitive demand and, so, we categorized them as Low.

PST 22's response to Task 6 was immediate: "You get one fifteenth of the cake and split that in half. My first thought was one-thirtieth of the cake, because [makes splitting motion with hands in the air] splitting that in half, like if you were to split every piece of fifteen in half, then that would be like one thirtieth of the entire case." She seemed to imagine partitioning each one of the fifteen original parts into two parts to produce 30 parts in the whole. This mental action aligns with the distributing operation ($T_{2:15}$), but the production of 30 equal parts would be essential for PST 22's understanding of 1/30 as one out of 30 equal parts in the whole. We take such responses as indication PST 22 used her units coordinating structures to assimilate fractions as parts out of wholes. As indicated by the UTG (Figure 3) PST 22 experienced an M-demand of 4 for this task.

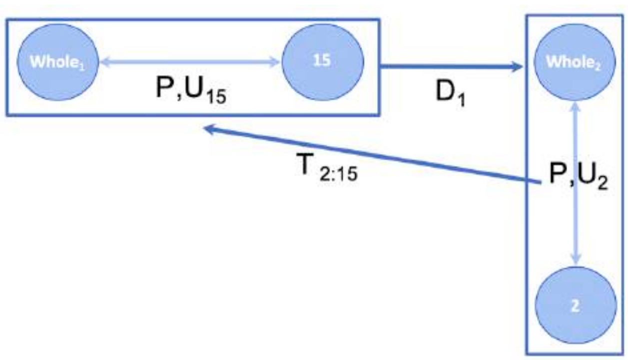

Figure 3: Finding a unit fraction of a unit fraction (Task 6)

Task 8

With the introduction of a non-unit fraction, Task 8 would also introduce one more unit and one more action to coordinate, increasing cognitive demand by 2 over Task 6. Indeed, the UTG for PST 22's response to Task 8 (not shown) would be structurally identical to the one for Task 6 (Figure 3), except it would include an additional action of disembedding two units from 27 (D_2) and the resulting unit of 2/27. Increased demand became evident in PST 22's response, which we took as indication of High demand.

PST 22: So, it's split up into nine equal pieces. So, then, you would split one ninth into…Two people come, but you still have a little bit? So, that… So, you would split that up into three. So, then I… Well, I guess you would do one ninth times two thirds to get how much they equal, like how much both their pieces would be. And then whatever that is, I guess it would be… two over… two eighteenths? Wait, that doesn't seem right. [pauses for five seconds] I feel like… I mean, I guess… You take those nine pieces, splitting that one ninth into thirds. But to find out how much two of those thirds are, you'd multiply one ninth by two thirds… Or no. You'd… you'd multiply the one ninth by one third, and then just do that twice? I don't know if that'd give you the same answer.

Researcher: Okay. Uh, let's… Maybe I can help you.

PST 22: Okay.

Researcher: If you want me to be your calculator again, I'll do it.

PST 22: [begins to draw on table with finger] So, you do one ninth, which divided by three, so you could times it by one third. So, then you'd have one over um… [pauses for five seconds.] Oh wait… [whispers to self] Three times nine, that's twenty-seven. Oh no, one over twenty-seven. And then you multiply that by two… to get two-thirds or to get two parts of the thing… So, then I guess… What's one over twenty-seven times two? Is that just two-twenty-sevenths? Okay.

Researcher: Nice, I like the way you reasoned through it. Yeah.

PST 22: Okay. I was like, because I was thinking one over twenty-seven times two over one and I was like I guess that's just two, twenty-sevenths.

This response indicates that PST 22 tried to rely on the standard algorithm for multiplying fractions but struggled to reconcile it with prior reasoning. She began as she had in Task 6, partitioning the whole, disembedding one of those parts, and then partitioning it into smaller parts. However, in contrast to Task 6, she then began referring to a fraction multiplication, 1/9 times 2/3. Fraction multiplication might have helped her keep track of the additional unit involved in this task (the 2 in 2/3), but she was not sure that the multiplication of fractions would generate the correct result. Her concerns were heightened when she mistakenly multiplied 9 times 2, instead of 9 times 3, to produce two eighteenths: "that doesn't seem right." So, she reverted to operating on the 9 units, partitioning them into thirds, which she was then able to reconcile with 1/9 times 1/3. Thinking of the task as the multiplication problem, 1/9 times 2/3, then did work for her by maintaining the 2 in two-thirds: "and then you multiply that by 2 to get two-thirds, or to get two parts of the thing."

We found PST 22's persistence in response to Task 8 impressive and indicative of her high M-capacity. Ultimately, she reasoned with parts out of the whole ("two parts of that thing"), as she had before, but was able to meet the cognitive demands of the task by organizing her operations around the fraction multiplication algorithm. So, while the task was highly demanding for her, we see evidence for how algorithms, when made meaningfully related to (or reconciled with) operations, can offload the demands of mathematical reasoning.

Task 10

For Task 8, the PSTs needed to distribute the new partitioning (thirds) across the nine units making up the whole, and then take two of the resulting parts (1/27ths). For Task 10, she needed to distribute fourths across both the five parts in five-sixths and the six parts in the whole. The increased demands caused PST 22 to lose track of the six parts making up the whole. Figure 5 presents the UTG for this response.

PST 22: You're cutting off one fourth of five sixths of a cake?
Researcher: Yes.
PST 22: [uses hands to show number of pieces on the desk and begins talking to self] So, you'd have, so you'd have six pieces…and out of those five…you want to cut off one fourth of that. Um…I guess you would…I mean I guess you could split those five pieces into four and get one of those, but I'm trying to think like numbers-wise what that would...I. well… [pauses for seven seconds] I guess of those five pieces you could…Split them into…Like you could get a…Split them into twenty pieces because five times four is twenty and then, um, you would take one fourth of that… I guess it would be five pieces. Yeah, it would be five pieces of that twenty to find the one fourth of the five sixth. Is that, do I need to explain it more?
Researcher: Okay, uh let's…
PST 22: Which would be, do you want me to draw it? [reaches towards paper]
Researcher: Well tell me the final answer and then we can draw it.
PST 22: Um, oh gosh it would be… [pauses for four seconds] Splitting twenty, it would be five…Well it would be five twentieths, which would equal one fourth, so like five of those, but then I don't know how to figure that out into sixths. I think that's my…
Researcher: Yeah that's cool, I like the way you're reasoning. Let's draw it, and I think you will figure it out.

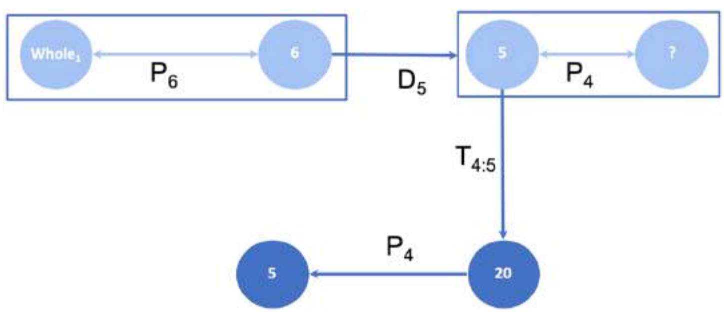

Figure 4: Finding a unit fraction of a non-unit fraction (Task 10)

Once again, PST 22 seemed to conceptualize the initial fraction (5/6) as a part-whole relation, partitioning the whole into six equal parts and disembedding five of them. This inference is supported by the PST's verbalization, "so you'd have six pieces…and out of those five." As PST 22 tried to find one-fourth of 5/6, she operated only on the five disembedded parts and lost track of the sixth part making up the whole. Thus, rather than fourthing 5/6, as originally intended, she ended up distributing four parts into each of the five parts, producing 20 parts, and taking one-fourth of those 20 parts instead.

Discussion

In line with the stated goals of PME-NA, we aimed to elucidate the psychological aspects of learning mathematics. Specifically, UTGs integrate the psychological construct of working memory with a construct from mathematics education research—units coordination—to explain how mathematics arises through the coordination of students' own mental actions (Beth & Piaget, 1966). The case study of PST 22's solutions to a ranked set of fractions tasks demonstrate the explanatory power of UTGs.

With an assessed working memory of 7, PST 22 would have experienced even Task 5 as highly demanding; her solution to the task involved four units and three transformations between them. However, PST 22 assimilated some of those units and transformations into two-level structures, thus chunking them into single two-level units so that she experienced Low cognitive demand. As a theoretical construct, chunking has its roots in cognitive psychology (e.g., Pascual-Leone, 1970), but UTGs illustrate how chunking can take a particular form in mathematics, as units coordinating structures (Steffe, 1992; Hackenberg, 2007).

Offloading cognitive demand with unit structures afforded PST 22 the ability to account for additional units and transformations in progressively more demanding tasks. In looking at the UTGs across the four tasks (as illustrated in Figures 2-4), we see how PST 22 introduced additional transformations (such as distributing) and units to solve Tasks 6 and 8, until she reached her threshold, with 7 units/transformations. Having reached that threshold, she was not able to proceed in solving Task 10, but we might imagine ways that she might further structure units into chunks to increase her mathematical power.

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LEARNING DIFFICULTIES TO BUILD ZERO AND ONE, BASED ON VON NEUMANN

DIFICULTADES DE APRENDIZAJE PARA CONSTRUIR EL CERO Y EL UNO, CON BASE EN VON NEUMANN

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This research project focuses on identifying the difficulties that children between 6 – 7 years-old have in learning natural numbers, when working a teaching model with a Von Neumann formal mathematical basis. In this study, we present the analysis of the experimentation with a first-grade class in the construction of zero and one numbers, contrasting the results with a case of the clinical interview. Our methodological theoretical framework is the Local Theoretical Model (LTM) and its four components: Formal, Cognitive, Communication and Teaching. With the theoretical contribution of each component, the categories of analysis are designed to observe and explain difficulties in the use of the Mathematical Sign System (MSS) involved in the construction of the first natural numbers, using iteration and the recursive process.

Keywords: difficulties, learning, natural numbers.

Research Problem

Learning numerical notions remains a concern in research and education policies. This project has been developed under the perspective of Educational Mathematics, stating that the learning difficulties that elementary school children have are not found in what the teacher does nor in the students, but in mathematics itself. Therefore, you have to know in depth the mathematical basis of what is taught.

In the curriculum of Mathematics primary education in Mexico (SEP, 2011) the development of numbers is focused on the use of cardinality in different contexts in contrast to that of ordinality. From the first degree, the number zero is used as a figure in the quantity representation and as an empty column, but there is no conceptual treatment. The lack of a mathematical conceptualization in teaching natural numbers as the basis of the arithmetic structure is accompanied by teaching practices that continue with the tradition of mechanization, memorization and exercise of the oral and written number sequence, and of algorithms of addition and multiplication.

From the formal point of view, Cantor and Peano contributed to the conceptualization of natural numbers. For Cantor, it is through the cardinality relations of the sets (Mosterín, 2000, pp. 105-108). While Peano proposes an axiomatic that involves a formal conceptualization of numbers (Op. Cit. pp. 54-55). However, this goes against how children cognitively process the first actions to order the world. It is considered essential to begin the learning of natural numbers with the construction of the number zero as it is a contribution to mathematical conceptualization and a contribution that goes beyond the numbering system, independent of the debates on its origin.

Therefore, the orientation of this work is to look at learning and observe learning difficulties from a formal mathematical proposal. According to Filloy, Puig & Rojano (2008), the analysis provided by the formal component indicates that difficulties must be sought in the most primitive mathematical actions. For the learning of natural numbers, we found these actions in the iteration. Hamilton & Landin (1961) introduce us to the construction of natural numbers based on the works of the logician

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mathematician Von Neumann, who uses the theory of sets and summarize the axiomatic of Peano in the principle of finite induction and iteration; each number is constructed from a finite number of iterations; order is implicit by the same construction.

The research questions are 1) What elements of the formal model in the terms referred to by Hamilton & Landin (1961) should we consider designing a Teaching Model that translates into specific activities for children aged 6 to 7? 2) What difficulties emerge when children work from the construction of the number zero and one, based on that teaching model?

To answer these questions, we propose as a general objective to identify students' learning difficulties in the construction of the numbers zero and one, in the framework of a teaching model based on Von Neumann.

The particular objectives are 1) Design and implement a teaching model by translating the Formal Model into sequences of specific activities, for the construction of the first natural numbers. 2) Identify and explain the difficulties that children 6 to 7-year old's have, when working with a Von Neumann's-based Teaching Model.

Theoretical Framework

The LTM (Filloy, et. al. 2008) is a theoretical and methodological framework for the experimental observation on research in Educational Mathematics. LTM relies on Peirce's semiotic approach (1987) to make sense of MSS, a theory for the interpretation of experimental observations. MSS focuses its attention on the production of intertexts through the reading/transformation of mathematical texts in relation to other texts. This allows users to produce meaning and mathematical meaning to communication processes that occur in classrooms, when activity sequences are implemented for a particular purpose. The sense of the local focuses the analysis on a specific phenomenon through the four components: formal, cognitive, communication and teaching.

The design of this LTM has been structured according to the four components

Formal. Von Neumann Model (Hamilton & Landin, 1961) proposes a logic of construction that requires MSS involved in natural numbers, starting with zero: "Zero is the empty set; i. e., $0 = \emptyset$", it continues with the number one as the set containing element zero: "$1 = \{0\} = \{\emptyset\}$". From this moment on, enter the successor definition:

> The set $x \cup \{x\}$ is the *successor* of the set x. If y is a set and if there is a set x such that y is the successor of x, then y is a successor. For each set x, the successor of x is x'. Thus, $1 = 0', 2 = 1'$, 3=2', etc. (Op. Cit. p. 77).

Each ordinal is the set of all the ordinals that precede it. Each of these sets is $\in -$ordered by the same construction, where for all x and for all y at least one of the following conditions is met: $x \in y, x = y, or\ y \in x$. Natural numbers are defined as "n is a $\in -ordered$ every nonempty subset of n has a leading element, if $x \in n$ then $x \subset n$, if n is not empty then *n* is a successor if $x \in n$ and x is not empty then x is a successor" (Op. Cit. 1961, p. 81). To count a set A is a one-to-one correspondence $\varphi: [1, n] \to A$ between $[1, n]$ y A, where $n \in N$." Cardinality is: If n is the result of a count of A, then A has n elements, or the number of elements in A is *n,* or the cardinality of A is *n*. We also denote the cardinality of A by $\#\ (A)$. (Op. Cit. pp. 99-101).

The sum is given, naturally when obtaining the successor, by iteration and starting from n, is $n + 1$. When *n* is a set, then the sum is the union of disjoint sets: $A \cup B$ and if m and n are their cardinals respectively, then $m + n$.

Cognitive. Based on the contributions of the theory of activity (Talizina, 2001) we expect to identify difficulties that obstruct (OB) students' competence in the use of MSS involved in the construction of natural numbers due to the ways in which children aged 6 to 7, have learned to use natural numbers in everyday life. These obstructions can hinder the transition from action to cognitive operation to foster the conceptual numerical development of zero, one and the notion of a successor.

Communication based on semiotics (Peirce, 1987), we analyzed induction, deduction and abduction arguments as processes of signification (ASP). The categories are built with significant relations to interpret what children do and say in the production of meaning and processes of the significance of the actions they perform in numerical activities. This is the Sense Endowment (SD). The logic of using MSS is related to iteration and recursion processes in the construction of natural numbers.

Teaching. It is understood as a collection of concrete texts that the students can understand, so that they gradually convert concrete texts into abstract texts, with a conventional mathematical meaning. Researchers designed a Teaching Model to translate Von Neumann's formal model into specific activities with the use of manipulative material.

Methodology Design and implementation of the Teaching Model. The design of the activities was carried out according to the mathematical definitions of the Von Neumann sequence according to Hamilton & Landin (1961, pp. 74 – 112), we have called them principles (Pi):

P_1: Beginning the construction with zero. The name of the empty set is zero.

P_2: Building the number one as a successor to zero number. Using the recursive process $0 = \emptyset$; $1 = \{\emptyset\}$; $2 = \{\emptyset, \{\emptyset\}\}$; $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$; and so on. Keys are replaced with bags, as shown in Figure 1.

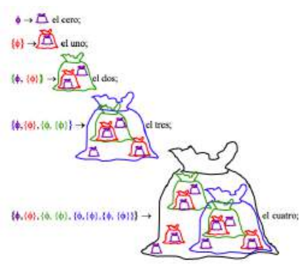

Figure 1: Recursive number processing, using sachets

P_3: One successor has been called *the following*.

P_4: Definition of Set n, as natural number.

P_5: Counting and cardinality.

P_6: Addition.

The use of a semi-straight has been included to represent the order of the numbers, from the interval definition "If $a, b \in N, [a, b] = \{x \mid x \in N \text{ and } a \leq x \text{ and } x \leq b\}. [a, b]$ is the interval of a to b." (Op. Cit. 1961, p. 97).

We selected two sequences of activities of the teaching model to present them in this work briefly described as follows:

Guess I am (P_1). -The empty set as the number zero.

- Teacher asks the students to look at an empty clear plastic bag and say what it contains (possible answers: nothing and empty).
- Teacher asks the students how to name this empty bag. The intention is to relate the notion of numbers to the bags to refer to bag/number.
- Students are asked by what number the empty bag can be named with.
- When students name the word zero, they are asked to identify the label of the number zero and paste it on the outside of the bag.
- Teacher asks students if they already know this number and what they think it is useful for.
- Teacher asks where students can paste it on the semi-straight (drawn on the board).

Can we build the next one? (P_2, P_3). - Construction of the successor.

- Once the empty set is related and named as number zero, the teacher asks: Can we build the next one? (The next is number one, it's the bag/number that contains item zero).
- Teacher takes another empty bag and asks students how they can build the next one. Teacher asks who the empty bag is, so that they name it as zero, they are asked to stick the zero label outside this empty bag/number and put it in the new bag named as number one; then stick the label for item one outside the bag.
- They are asked where to paste it in the semi-straight (it is expected to be after zero, building the sense of linear order).
- With this logic of construction, the successors are built, reflecting on each construction, who is before, who is after, who is in all numbers, who has no ancestor, who contains each bag/number.

This teaching model was used in the first phase when working with a first-grade class from a public school in Mexico City. Based on quantitative (written exercises) and quantitative (performance in each session) the students were classified into three strata High, Medium and Low. We selected one student from each stratum to participate in the clinical interview. The second phase was carried out in the next school year, with the application of the clinical interview.

The results of the analysis of model experimentation were published in (Rodríguez, et. al 2018) and (Rodríguez, et. al. 2019a, 2019b). Due to the space available in this report, we expose only a fragment of a group session and a snippet of Daniel's interview (low stratum).

Observation of the empirical experience with analysis categories. In the analysis of the experimentation of the Model, we identified recurring difficulties in the children's performances. These difficulties are grouped into three axes 1) The use of pragmatic/intuitive and spontaneous knowledge, that is, difficulties in identifying zero as a number; identify the successor and ancestor of any number. 2) Semantic use of numbers in rendering and counting actions, that is, difficulties in recognizing the number zero as an empty set, recognizing that zero is the only number belonging to any successor, identifying zero as the point of origin in the straight, and recognizing that every successor contains all his previous ones. 3) Syntactic use in operations, that is, difficulty using the form $a \cdot 10 + b$.

To understand and explain the difficulties and based on the theoretical framework, there are three categories of analysis designed, Cognitive Trends that constitute obstructions for learning (OB); Induction, Deduction and Abduction Arguments as Significance Processes (ASP) and, Significant Relations Indicators for Sense Endowment (SD).

Finally, to contrast whether the difficulties continue, or new ones appear, the clinical interview took place and it was interpreted based on the same categories of analysis.

The following is assigned as follows: Teacher (M), Children All (N_s), Nicole (N_e), N_1, N_2... refers to any child except those whose initial letter is the name of a specific child, Interviewer (E) and Daniel (D).

Fragment of dialogue of the activity sequence, "Guess I am":

M: What's with the bag? [Teacher shows the empty bag].
N1: Nothing [The teacher inserts some objects into the bag and then empties it in front of them].
M: How does the bag look?
N2: Empty.
M: How do we know that my bag is empty?
A: Because it has nothing in it.
F: If you don't put something in it, you have nothing.
E: If it is empty it is not heavy.

Ne: If you put something in it, then it's full.
E: Or by numbers.
M: What did you say?
Ns: By numbers.
M: And, what number do you think should be here?
Ns: One, two, three... [Labels shown from numbers 0 to 9].
N3: A zero.
M: Who's that number?
Ns: Nothing.
Ns: Zero.
M: Who wants to go to the front to look for number zero and stick it onto the bag?
K: Here it is [Karen chooses the label with number zero].
M: Where do I put it? [The teacher displays the bag/number zero, to place it on the semi-straight painted on the board].
F: At the end [Points to the left end of the straight but use the word end].
M: At the end?
F: Ah! At the beginning! [He corrects his answer].

Fragment of dialogue of the activity sequence, "Can we build the next one?"

M: Can we build the next one?
N_s: The one.
M: How are we going to build the next one?
Ne: Take another bag and put one on it.
M: What do we have here? [showing the empty bag].
N_s: Empty.
M: Empty, but I need...
N_s: The one.
M: How are we going to do it because this bag is empty?
N_s: Put the one on it! [The teacher inserts the bag of the zero that was previously built and asks them].
M: How many bags are inside? (...)
N_s: One. (...)
M: What number was formed here?
N_s: One.
M: Why is it one Emiliano?
E: Because the one comes first.
M: Nicole…
Ne: Because the one goes after the zero.
M: Where do I put it?
N_s: In first place.
M: Who's in the first one? [points to zero that is placed on the straight].
N_s: A zero number.
E: You pass it for the second [he refers to the right of the zero, on the straight].

Analysis of these fragments. "*Guess I am".* - In this fragment we observe the difficulties of identifying zero as an empty set and as the origin point on the straight. The zero as an empty set is observed in the action of inserting objects into the bag/number and removing them, children were able to relate the void to words: "nothing, empty, full, put in, take out, heavy" (A, F, E, Ne). This can be interpreted as SD-related actions, which allow an approach to the notion of zero as empty. (A) makes inductive reasoning (ASP) by using the word "nothing" to justify the vacuum, giving meaning (SD) to the absence of elements, which gave a pattern for his peers to follow the meaning of the notion of emptiness. When (F) verbally states, "Because if you don't put something in, you have nothing," he's making a deduction (ASP), allowing (E) to relate the weight to the vacuum, and (Ne) reasons inductively (ASP). Later, faced with the question of being able to name the empty bag, the answer to (E) allows us to observe that this is abductive reasoning (ASP), when proposing the use of numbers. However, for the rest of the group, it is a difficulty, which can be understood to be caused by a cognitive (OB) from their experiences with numbers, they have learned to repeat the number sequence starting with the one (Ns). While (N_3) manages to follow the idea of (E) and chooses the number zero, which can be understood as abductive reasoning (ASP).

The (F)'s difficulty in identifying zero as the origin point in the semi-straight, could be, because this student was sitting in front of the right end of the semi-straight. This cognitive (OB), of perception, made it difficult for him to establish a reversibility relation, to focus his attention on the semi-straight as an object and not only on visual perception. When the teacher questions it, he allows (F) to correct his answer.

Can we build the next one? In this fragment, we see difficulties to recognize that zero is the only number belonging to any successor and in recognizing that every successor contains his former. Children identify that the successor to zero is one, so we can say that they are making sense of the expression "the next one". However, the difficulty remains when children do not give meaning to the construction, they consider they need only the label of number one (N_2) "Take another bag and put the one on it", (N_s) "Put one on it!" This difficulty is constant in the first lines of the activity, because children do not make sense of the recursive process. They do not recognize that the bag/number one must contain at least one item. It is understood that this difficulty is due to a cognitive obstruction (OB) from how they have learned that numbers are only the oral and written repetitions of the counter sequence from one. The action of (M) by inserting the empty bag labeled with the number zero and asking them about the number that was formed, allowed some children to observe that it is number one, when it contains element zero (N_s): "One, one". But, for most children, they continue to associate it with label one: (E) "Because one is first," thus, observing that this difficulty is an (OB) with the use of reversibility relations to identify that the one's ancestor is zero; just as the zero's successor is the one. With the deductive argument (ASP) of (N_e): "One goes after zero", leads to (E) correcting his answer, using an inductive argument (ASP) when the teacher asks where to place it: "You move it to the second one".

Clinical interview. The objective was to compare whether the difficulties that occurred during the experimentation of the model continue or new ones appear.

Fragment of the dialogue in the sequence of activities: "Guess I am":

E: What do you have in the bag?
D: Nothing.
E: Nothing and how's the bag?
D: Empty.
E: How do you know it's empty?
D: Because it has nothing in it.
E: How can you say it has nothing?
D: It would be a little bit heavy.
E: How can we represent this bag that has nothing, that is empty? [Daniel keeps thinking for a few seconds, unanswered. So, the interviewer shows him the material he has on the table: a semi-straight, transparent rubber bags of different sizes, number labels in flexible plastic]. Can I use any of these?
D: This [Daniel points to the zero-number label heap].
E: What is this?
D: Zero. [Daniel sticks the number zero label on the front of the bag].

E: Now that we know that this bag is empty and that it is the number zero, where do we place it on the line?
D: Here [Daniel points to the left end of the straight].

Fragment of the dialogue in the sequence of activities: Can we build the next one?

E: What's next?
D: The one.
E: What do I need to make the number one?
D: A number inside.
E: Who's going to be that element inside?
D: The one?
E: Who was before the one?
D: Ah, zero!
E: Zero, so what do you have to do?
D: Grab a... [Daniel takes another empty bag, sticks the number zero label and inserts it into the new bag/number one].
E: What name am I going to give you?
D: The one. [Daniel points and takes a number one label and sticks it on the new bag/number one].
E: All right! Where are you going to put it in the straight?
D: Here [Daniel places it to the right of the bag/number zero].

Analysis of the fragments of the clinical interview. *Guess I am.* In this excerpt, we can observed that (D) relates the empty set to the number zero, by expressing, "it would be a little bit heavy", a deductive argument (ASP) that evoked the experience of the group session in the last school year, which, can be understood as a sense endowment (SD) to relate the notion of vacuum to the zero number, but it is not conventional yet. But D doubts when the teacher asks him how to name the bag/empty number, so (E) points out and asks, "Can I use any of these?", giving the guideline for (D) to choose any of the number labels. When he places it on the left end of the semi-fully, he is making sense (SD) of zero as the starting point of the construction and overcoming that difficulty.

Can we build the next one? - It is observed that there is an endowment of meaning (SD) when recognizing that the next number of the number zero is number one. The answer "A number that is inside" to the interviewer's question: "What do I need to make it number one?" can be understood as a deductive argument (ASP) to make sense (SD) to the notion of a successor. However, when the interviewer asks "Who is that element inside?", evidence of D's insecurity, he answers with another question "The one?", which can be interpreted as a difficulty to produce a sense of use of the recursive process to recognize that every successor contains all of the above. The question of E "Who was before", allows D to evoke the construction process, remembering that the number zero is the one that should be inside the bag/number one: immediately takes a smaller bag and labels it with the number zero, inserts it into the new bag/number one. With these actions, we understand that D makes sense of the use of MSS through recursion. He places the bag/number one in the semi-straight of the bag/number zero, consolidating the sense of order by the same construction.

Final discussion.

We close this space by emphasizing that the general objective is to identify learning difficulties when taught with von Neumann's formal mathematical model for the construction of natural numbers and the logics of using the MSS involved in that task.

During the experimentation, it was possible to check that the influence of the ways in which they have acquired numerical notions makes it difficult to understand and use recursion, but they do not constitute an obstacle. In the clinical interview, the recurring difficulties reappear, which are overcome in less time, making efficient use and making sense of MSS through iteration and recursion (elementary actions for the conceptualization of natural numbers).

For the results obtained, it seems valuable to recover the formal mathematical tradition in teaching from the first grades of elementary education, allowing children to participate in the construction of natural numbers, which gives them the possibility to consolidate generalization, as the basis for solid arithmetic thinking.

Finally, we consider that the conceptual work of numbers with children aged 6 to 7 is not trivial, memoristic or operational; but it can make it easier to develop the concept before symbolism. What this work seeks is to cultivate abstract thinking, which allows children to access higher levels of mathematical knowledge.

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DIFICULTADES DE APRENDIZAJE PARA CONSTRUIR EL CERO Y EL UNO, CON BASE EN VON NEUMANN

LEARNING DIFFICULTIES TO BUILD ZERO AND ONE, BASED ON VON NEUMANN

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Este Proyecto de investigación se ha centrado en identificar dificultades que tienen los niños (6 – 7 años) en el aprendizaje de los números naturales, cuando se trabaja un modelo de enseñanza con una base matemática formal de Von Neumann. En esta comunicación, se presentará el análisis de la experimentación con un grupo de primer grado, en la construcción de los números cero y uno; contrastando los resultados con un caso de la entrevista clínica. Nuestro marco teórico metodológico son los Modelos Teórico Locales (MTL) y sus cuatro componentes: Formal, Cognitivo, Comunicación y Enseñanza. Con el aporte teórico de cada componente se diseñon las categorías de análisis para observar y explicar dificultades de uso del Sistema Matemático de Signos (SMS) involucrado en la construcción de los primeros números naturales, usando la iteración y el proceso recursivo.

Palabras clave: dificultades, aprendizaje, números naturales.

Problema de investigación

El aprendizaje de las nociones numéricas, sigue siendo una preocupación en la investigación y en las políticas educativas. Este proyecto se ha desarrollado bajo la perspectiva de la Matemática Educativa, planteando que las dificultades de aprendizaje que tienen los niños de la escuela elemental, no está en lo que hace la maestra/o, ni en los alumnos; sino en la matemática misma, por lo que hay que conocer con profundidad la base matemática de lo que se enseña.

En la currícula de Matemáticas educación primaria en México (SEP, 2011) el desarrollo de los números está centrado en el uso de cardinalidad en diferentes contextos; en contraste con el de ordinalidad. Desde el primer grado, el número cero se usa como cifra en la representación de cantidades y como columna vacía, pero no hay un tratamiento conceptual. La carencia de conceptualización matemática en la enseñanza, de los números naturales como base de la estructura aritmética, se acompaña de prácticas docentes que continúan con la tradición de la mecanización, memorización y ejercitación de la secuencia numérica oral y escrita, y de algoritmos de adición y multiplicación.

Desde el punto de vista formal Cantor y Peano contribuyeron a la conceptualización de los números naturales. Para Cantor es a través de las relaciones de cardinalidad de los conjuntos (Mosterín, 2000, pp. 105-108). Mientras que Peano propone una axiomatica que involucra una conceptualización formal de los números (Op. Cit. pp. 54-55); pero va en contra de las maneras en que los niños procesan cognitivamente las primeras acciones para ordenar el mundo. Se considera fundamental iniciar el aprendizaje de los números naturales con la construcción del número cero, por ser una aportación que va más allá del sistema de numeración, independiente de los debates sobre su origen.

Por lo anterior, la orientación de este trabajo es mirar al aprendizaje y observar dificultades de aprender cuando los niños trabajan con una propuesta formal matemática. De acuerdo con Filloy, Puig & Rojano (2008), el análisis que nos brinda la componente formal nos indica que las dificultades deben buscarse en las más primitivas acciones matemáticas. Para el aprendizaje de los números naturales estas acciones las encontramos en la iteración. Hamilton & Landin (1961) nos introducen a la construcción de los números naturales con base en los trabajos del lógico – matemático Von Neumann, quien usa la teoría de conjuntos y encapsula la axiomatica de Peano en el principio de inducción finita y en la iteración; cada número es construído a partir de un número finito de iteraciones; el orden está implicito por la misma construcción; lo que permite observar las acciones matemáticas más simples al usar la *iteración*.

Las preguntas de investigación son:

¿Qué elementos del modelo formal en los términos señalados por Hamilton & Landin se deben considerar para diseñar un Modelo de Enseñanza que se traduzca en actividades concretas, dirigidas a niños de 6 a 7 años de edad?

¿Qué dificultades se pueden observar cuando los niños trabajan a partir de la construcción de del número cero y el uno, con base en ese modelo de enseñanza?

Para dar respuesta a estas preguntas, se propone como objetivo general:

Identificar dificultades de aprender la construcción del número cero y el uno, cuando se les propone un modelo de enseñanza basado en Von Neumann.

Objetivos particulares:

Diseñar e implementar un modelo de Enseñanza traduciendo el Modelo Formal a secuencias de actividades concretas, para la construcción de los primeros números naturales.

Identificar y explicar las dificultades, que tienen los niños de 6 a 7 años de edad, al trabajar con un modelo de Enseñanza con base en Von Neumann.

Marco Teórico

Los MTL (Filloy, et. al. 2008) son un marco teórico y metodológico para la observación experimental en la investigación en Matemática Educativa. Se apoyan en el enfoque semiótico de Peirce (1987) para dar sentido a los SMS, una teoría para la interpretación de observaciones experimentales. Los SMS centran la atención en la producción de intertextos a través de la lectura/transformación de textos matemáticos en relación con otros textos. Esto permite a los usuarios la producción de sentido y significado matemático a los procesos de comunicación que se producen en las aulas, cuando se implementan secuencias de actividades con un fin determinado. El sentido de lo *local*, focaliza su análisis en un fenómeno específico a través de los cuatro componentes: formal, cognitivo, de comunicación y de enseñanza.

El diseño de este MTL se ha estructurado de acuerdo con los cuatro componentes:

Formal: Modelo de Von Neumann (Hamilton & Landin, 1961) propone una lógica de construcción que precisa SMS involucrados en la construcción de los números naturales, comenzando por el cero: “El cero es el conjunto vacío; $0 = \emptyset$.”, el número uno es el conjunto que contiene al elemento cero: “$1 = \{0\} = \{\emptyset\}$”. A partir de este momento introduce la definición de *sucesor*:

> El conjunto $x \cup \{x\}$ es el sucesor del conjunto x . Si y es un conjunto y si hay un conjunto x tal que y es el sucesor de x, entonces y es un *sucesor*. Para cada conjunto x, el sucesor de x es x'. Por lo tanto, 1 = 0’, 2 = 1’, 3 = 2’, etc. (Op. Cit. p. 77).

Cada ordinal es el conjunto de todos los ordinales que le preceden. Cada uno de esos conjuntos es $\in -ordenado$ por la misma construcción, donde para todo x y para todo y se cumple al menos una de las siguientes condiciones: $x \in y$, ó $x = y$ ó $y \in x$. Se define a los números naturales como: “n es $\in -ordenado$; cada subconjunto no vacío de n pose un elemento principal; si $x \in n$, entonces $x \subset n$; si n no está vacío entonces n es un sucesor; si $x \in n$, y x no está vacío entonces x es un sucesor.” (Op. Cit. 1961, p. 81). Contar un conjunto *A* es una correspondencia uno-uno $\varphi: [1, n] \to A$ entre $[1, n]$ y A, donde $n \in N$.” La cardinalidad es Si *n* es el resultado de un conteo de *A*, entonces *A* tiene *n* elementos, o el número de elementos en *A* es *n*, o la cardinalidad de *A* es *n*. Esto también lo denotamos como la cardinalidad de *A* por *#(A)*. (Op. Cit. pp. 99-101).

La suma se da, de manera natural al obtener el sucesor, por iteración y partiendo de n, es $n + 1$. Cuando n es un conjunto, entonces la suma es la unión de conjuntos disjuntos: $A \cup B$ y si m y n son sus cardinales respectivamente, entonces $m + n$.

Cognitivo: Con base en los aportes de la teoría de la actividad (Talizina, 2001) se pretende identificar dificultades que obstruyen (OB) su competencia con el uso de los SMS involucrados en la construcción de los números naturales debido a las maneras en que los niños de 6 a 7 años, han aprendido a usar los números naturales en la cotidianeidad. Estos obstructores pueden dificultar el tránsito de la acción a la operación cognitiva para promover el desarrollo numérico conceptual del cero, el uno y la noción de sucesor.

Comunicación: Con base en la semiótica (Peirce, 1987), se analizan los argumentos de inducción, deducción y abducción como procesos de significación (APS). Con las relaciones significantes se construyen las categorías para interpretar lo que hacen y dicen los niños en la producción de sentido y procesos de significación de las acciones que realizan en las actividades numéricas, esto es la Dotación de Sentido (DS). La lógica de uso de los SMS está relacionada con los procesos de iteración y recursión en la construcción de los números naturales.

Enseñanza: Entendida como colección de textos concretos que entiendan los aprendices, con la finalidad de que gradualmente conviertan los textos concretos en abstractos, con un significado matemático convencional. Se ha diseñado un Modelo de Enseñanza traduciendo el modelo formal de Von Neumann a actividades concretas, con el uso de material manipulativo.

Metodología Diseño y ejecución del Modelo de Enseñanza. El diseño de las actividades se realizó siguiendo las definiciones matemáticas de la secuencia de Von Neumann de acuerdo con Hamilton & Landin (1961, pp. 74 – 112), las hemos llamado principios (P_i):

P_1: Inicio de la contrucción con el cero. El nombre del conjunto vacío es el cero.

P_2: Contrucción del número uno como sucesor del número cero. Uso del proceso recursivo $0 = \emptyset$; $1 = \{\emptyset\}$; $2 = \{\emptyset, \{\emptyset\}\}$; $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$; y así sucesivamente. Las llaves se sustituyen con bolsas, como se muestra en la figura 1.

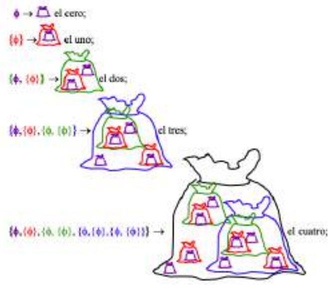

Figura 1: Proceso recursivo de los números, usando bolsitas

P_3: Un sucesor lo hemos llamado el siguiente.

P_4: Definición del Conjunto *n*, como número natural.

P_5: Conteo y cardinalidad.

P_6: Adición.

Se ha introducido el uso de una semirrecta para representar el orden de los números, a partir de la definición de intervalo "Si $a, b \in N, [a, b] = \{x\ x \in N\ y\ a \leq x\ y\ x \leq b\}. [a, b]$ es el intervalo de a hacia *b*." (Op. Cit. 1961, p. 97).

Para esta comunicación se han seleccionado dos secuencias de actividades del Modelo de Enseñanza, las cuales describiremos brevemente:

Adivina quien soy (P_1). El conjunto vacío como el número cero:

- Se les pide que observen una bolsa de plástico transparente vacía y digan que contiene, (posibles respuestas: nada y vacía).
- Se les pregunta cómo se puede nombrar esta bolsa vacía. La intención es relacionar la noción de números con las bolsas, para referirnos a bolsa/número.
- Se les pregunta con qué número se puede nombrar a la bolsa vacía.
- Al nombrar la palabra cero, se les pide identificar la etiqueta del número cero y la peguen en la parte externa de la bolsa.
- Se les pregunta si ya conocían ese número y para qué creen que sirva.
- Se les pregunta en qué parte de la semirrecta (dibujada en el pizarrón) pueden pegarla.

¿Podemos construir el siguiente? (P_2, P_3).- Construción del sucesor:

- Una vez relacionado y nombrado al conjunto vacío como número cero, se les pregunta: ¿Podemos construir el siguiente? (El *siguiente* es el número uno, es la bolsa/número que contiene al elemento cero).
- Se toma otra bolsa vacía y se les pregunta cómo se puede construir el siguiente. Se les pregunta quién es esa bolsa vacía, con la finalidad de que la nombren como *cero*, se les pide que peguen la etiqueta *cero* afuera de esta bolsa/número vacía, la introduzcan en la nueva bolsa nombrada como uno. Se les pide que peguen la etiqueta del *uno* por fuera de la bolsa.
- Se les pregunta dónde pegarla en la semirrecta (se espera que sea después del cero, construyendo el sentido de orden lineal).

- Con esta lógica de construcción, se van construyendo los sucesores, reflexionando en cada construcción, quién está antes de, quién está después de, quién está en todos los números, quién no tiene antecesor, a quién o quiénes contiene cada bolsa/número.

Este modelo de enseñanza se trabajó en la primera fase con un grupo de primer grado de una escuela pública en la CDMX. Con base en el análisis cuantitativo (ejercicios escritos) y cuantitativo (desempeño en cada sesión) los alumnos fueron clasificados en tres estratos (Alto, medio y bajo); se seleccionó un alumno representante de cada estrato para participar en la entrevista clínica. La segunda fase se realizó en el siguiente ciclo escolar, con la aplicación de la entrevista clínica.

Los resultados del análisis de la experimentación del Modelo, se publicaron en (autores 2018) y (autores 2019[a], 2019[b]). Por la falta de espacio en este reporte, sólo se presentará un fragmento de una sesión grupal y un fragmento de la entrevista de Daniel (estrato bajo).

Observación de la experiencia empírica con las categorías de análisis. En el análisis de la experimentación del Modelo, se identificaron *dificultades recurrentes* en las actuaciones de los niños. Las dificultades se agruparon en tres ejes: Uso de conocimientos pragmáticos/intuitivos y expontáneos: dificultades para identificar al cero como número; identificar al sucesor y antecesor de cualquier número. Uso semántico de los números en acciones de representación y conteo: dificultades para reconocer el número cero como conjunto vacío; reconocer que el cero es el único número que pertenece a cualquier sucesor, identificar al cero como punto origen en la semirrecta, reconocer que todo sucesor contiene a todos sus anteriores. Uso sintáctico en las operaciones: dificultades para usar la forma $a \cdot 10 + b$.

Para entender y explicar las dificultades y con base en el marco teórico, se diseñaron *tres categorías de análisis*: Tendencias Cognitivas que constituyen obstructores para el aprendizaje (OB); Argumentos de inducción, deducción y abducción como Procesos de Significación (APS) e; Indicadores de las Relaciones Significantes para la Dotación de Sentido (DS).

Por último, con el fin de confrontar a si las dificultades continúan, o aparecen otras nuevas, la entrevista clínica se aplicó e interpretó sobre la base de las mismas categorías de análisis.

Se usa la siguiente simbología: Maestra (M), Niños todos (N_s), Nicole (N_e), N_1, N_2,… cuando es un niño cualquiera y la letra inicial del nombre cuando es un niño en particular, Entrevistadora (E) y Daniel (D).

Fragmento del diálogo de la secuencia de actividades:"Adivina quien soy":

M: ¿Qué tiene la bolsa? [muestra la bolsa vacía].
N1: Nada [la maestra introduce algunos objetos a la bolsa y luego la vacía frente a ellos].
M: ¿Cómo quedó la bolsa?
N2: Vacía.
M: ¿Cómo podemos saber que mi bolsa está vacía?
A: Porque no tiene nada.
F: Por si no metes algo, no tienes nada.
E: Si está vacío no está pesado.
Ne: Si le metes algo, ya está lleno.
E: O por números.
M: ¿Cómo dijiste?
Ns: Por números.
M: ¿Y cuál creen que sea el número que debe estar aquí?
Ns: El uno, el dos, el tres, … [Se les muestran etiquetas de los números 0 al 9]
N3: Un cero.
M: ¿Quién es ese número?
Ns: Nada.
Ns: Cero.

M: ¿Quién pasa a buscar al número cero y pegarlo a la bolsa?
K: Aquí está [Karen, elige la etiqueta del número cero].
M: ¿En dónde lo coloco? [La maestra muestra la bolsa/número cero, para colocarla en la semirrecta pintada en el pizarrón].
F: Hasta el final [señala el extremo izquierdo de la semirrecta, pero usa la palabra final].
M: ¿Hasta el final?
F: ¡Ah! ¡En el primero! [Corrige su respuesta].

Fragmento del diálogo de la secuencia de actividades: "¿Podemos construir el siguiente?"

M: ¿Podemos construir el siguiente?
N_s: El uno.
M: ¿Cómo le haremos para construir el siguiente?
Ne: Tomar otra bolsa y ponerle el uno.
M: ¿Qué tenemos aquí? [Mostrando la bolsa vacía].
N_s: Vacía.
M: Vacía, pero necesito…
N_s: El uno.
M: ¿Cómo le haremos, porque esta bolsa está vacía?
N_s: ¡Póngale el uno! [La maestra introduce la bolsa del cero que se construyó previamente y les pregunta:]
M: ¿Cuántas bolsas hay adentro?
N_s: Una.
M: ¿Qué número se formó aquí?
N_s: Un uno.
M: ¿Por qué es uno Emiliano?
E: Porque el uno es primero.
M: Nicole...
Ne: Porque el uno va después que el cero.
M: ¿Dónde lo coloco?
N_s: En el primero.
M: ¿Quién está en el primero? [señala al cero que está colocado en la semirrecta].
N_s: Un número cero.
E: Lo pasas para el segundo [se refiere a la derecha del cero, en la semirrecta].

Análisis de estos fragmentos: "*Adivina quién soy*". En este fragmento se observan las dificultades para identificar el cero como conjunto vacío y como punto origen en la semirrecta.

El cero como conjunto vacío se observa con la acción de introducir objetos a la bolsa/número y luego sacarlos; pudieron relacionar el vacío con palabras: "nada, vacío, lleno, meter, sacar, pesado" (A, F, E, N_e). Esto se puede interpretar como acciones relacionadas con la (DS), los cuales permiten un acercamiento a la noción de cero como vacío. (A) hace un razonamiento inductivo (APS) al usar la palabra "nada" para justificar el vacío, dotando de sentido (DS) a la ausencia de elementos, lo que dio pauta para que sus compañeros pudieran seguir el sentido de la noción de vacío. Cuando (F) expresa verbalmente: "Porque si no metes algo, no tienes nada", está haciendo una deducción (APS) , lo que permite que (E) lo pueda relacionar el peso con el vacío, y (N_e), razona inductivamente (APS). Más adelante, ante la pregunta de poder nombrar a la bolsa vacía, la respuesta de (E) permite observar que se trata de un razonamiento abductivo (APS), al proponer el uso de los números. Sin embargo, para el resto del grupo, constituye una dificultad, que se puede entender es provocada por un (OB) cognitivo proveniente de sus experiencias con los números, han aprendido a repetir la secuencia numérica comenzando con el uno (N_s). Mientras que (N_3) logra seguir la idea de (E) y elige al número cero, lo que se puede entender como un razonamiento abductivo (APS).

La dificultad de (F) para identificar al cero como punto origen en la semirrecta, puede ser a que este alumno se encontraba sentado frente al extremo derecho de la semirrecta. Este (OB) cognitivo, de percepción, le dificultó establecer una relación de reversibilidad, para centrar su atención la semirrecta como objeto y no sólo en la percepción visual. Cuando la maestra lo cuestiona, permite que (F) corrija su respuesta.

¿Podemos construir el siguiente?.- En este fragmento se observan dificultades para reconocer que el cero es el único número que pertenece a cualquier sucesor; y que todo sucesor contiene a sus anteriores. Los niños identifican que el sucesor del cero es el uno, con lo que se puede decir que están dotando de sentido a la expresión “el siguiente”. Sin embargo, la dificultad se observa en que no le dan sentido a la construcción, consideran que sólo se necesita la etiqueta del número uno: (N_e) “Tomar otra bolsa y ponerle el uno”, (Ns) “¡Póngale el uno!” Esta dificultad es constante en las primeras líneas de la actividad, no le dan sentido al proceso recursivo. No reconocen que la bolsa/número *uno*, debe contener al menos un elemento. Se entiende que esta dificultad se debe a un obstructor cognitivo (OB) proveniente de las maneras en que han aprendido que los números es sólo la repetición oral y escrita de la secuencia contadora a partir del uno. La acción de (M) al introducir la bolsa vacía etiquetada con el número cero y preguntarles por el número que se formó, permitió que algunos niños observaran que es el número uno, cuando contiene al elemento cero (N_S): “Un uno”. Pero, para la mayoría de los niños lo siguen asociando con el primer elemento de la secuencia numérica: (E) “Porque el uno es primero”, con lo que se observa que esta dificultad es un (OB) con el uso de las relaciones de reversibilidad para identificar que el antecesor del uno es el cero; así como el sucesor del cero es el uno. Con el argumento deductivo (APS) de (N_e): “El uno va después del cero”, conlleva a que (E) corrija su respuesta, usando un argumento inductivo (APS) cuando la maestra pregunta en dónde colocarlo: “Lo pasas para el segundo”.

Entrevista clínica. Tuvo como objetivo comparar si las dificultades que se presentaron durante la experimentación del modelo continuan o aparecen nuevas.

Fragmento del diálogo en la secuencia de actividades: “Adivina quién soy”

E: ¿Qué tiene la bolsa?
D: Nada.
E: Nada y ¿Cómo está la bolsa?
D: Vacía.
E: ¿Cómo sabes que está vacía?
D: Porque no tiene nada.
E: ¿Cómo puedes decir que no tiene nada?
D: Estuviera un poco pesado.
E: ¿Cómo podemos representar esta bolsa que no tiene nada, que está vacía? [Daniel se queda pensando algunos segundos, sin contestar. Por lo que la entrevistadora le muestra el material que tiene sobre la mesa: una semirrecta, bolsas hule transparente de diferentes tamaños, etiquetas de números] ¿Alguno de estos puedo usar?
D: Este [Daniel señala el montón de las etiquetas del número cero].
E: ¿Cuál es este?
D: El cero. [Daniel pega la etiqueta del número cero en la parte frontal de la bolsa].
E: Ahora que sabemos que esta bolsa está vacía y que es el número cero ¿En dónde la colocamos en la recta?
D: Aquí [Daniel señala el extremo izquierdo de la semirrecta].

Fragmento del diálogo en la secuencia de actividades: ¿Podemos construir el siguiente?

E: ¿Cuál va a ser el siguiente?
D: El uno.
E: ¿Qué necesito para que sea el número uno?

D: Un número que esté adentro.
E: ¿Quién va a ser ese elemento que esté adentro?
D: ¿El uno?
E: ¿Quién era el que estaba antes?
D: ¡Ah, el cero!
E: El cero, entonces ¿qué tienes que hacer?
D: Agarrar una... [Daniel toma otra bolsa vacía, le pega la etiqueta del número cero y la introduce en la nueva bolsa/número uno].
E: ¿Qué nombre le voy a poner?
D: El uno. [Daniel señala y toma una etiqueta del número uno y la pega en la nueva bolsa/número uno].
E: ¡Muy bien! ¿dónde lo vas a colocar en la recta?
D: Aquí [Daniel lo coloca a la derecha de la bolsa/número cero].

Análisis de los fragmentos de la entrevista clínica. *Adivina quién soy.* - En este fragmento, se puede observar que (D) relaciona al conjunto vacío con el número cero, al expresar: "Estuviera un poco pesado", argumento deductivo (APS) que evocó la experiencia de la sesión grupal en el ciclo escolar próximo pasado. Lo que se puede entender como dotación de sentido (DS) para relacionar la noción de vacío con el número cero, pero aún no es convencional. Pero D duda cuando se le pregunta cómo nombrar a la bolsa/número vacía, se observa que duda, por lo que (E) le señala y pregunta: "¿Alguno de estos lo puedo usar?", dando la pauta para que (D) elija alguna de las etiquetas de los números. Al colocarlo en el extremo izquierdo de la semirrecta, está dotando de sentido (DS) al cero como punto origen de la construcción y superando esa dificultad.

¿Podemos construir el siguiente? - Se observa que hay dotación de sentido (DS) al reconocer que el número siguiente del número cero es el número uno. La respuesta de Daniel "Un número que esté adentro" a la pregunta de la Entrevistadora: "¿Qué necesito para que sea el número uno?", se puede entender como un argumento deductivo (APS) para dotar de sentido (DS) la noción de sucesor. Sin embargo, cuando la entrevistadora pregunta "¿Quién va a ser ese elemento que esté adentro?", se evidencia la inseguridad de D, al contestar con otra pregunta "¿El uno?", lo que se puede interpretar como dificultad para de producción de sentido de uso del proceso recursivo para reconocer que todo sucesor contiene a todos los anteriores. La pregunta de E "¿Quién estaba antes", permite que D evoque el proceso de construcción, recordando que el número cero es el que debe estar adentro de la bolsa/número uno, pues de inmediato toma una bolsa más pequeña y le pega la etiqueta del número cero y la introduce en la nueva bolsa/número uno. Estas acciones que realiza D nos permiten entender que le da sentido al uso de los SMS a través de la recursividad. A continuación, coloca la bolsa/número uno en la semirrecta a la derecha de la bolsa/número cero, consolidando el sentido de orden por la misma construcción.

Discusión final

Cerramos este espacio haciendo hincapié en que el objetivo general es identificar las dificultades de aprendizaje cuando se les enseña con el modelo formal matemático de von Neumann para la construcción de los números naturales y la lógica de uso de los SMS involucrados en dicha tarea.

Durante la experimentación se pudo cotejar que la influencia de las maneras en que han adquirido las nociones numéricas, dificulta la comprensión y uso pragmático de la recursividad, pero no constituyen un obstáculo. En la entrevista clínica vuelven a aparecer las dificultades recurrentes, mismas que son superadas en menor tiempo, haciendo un uso eficiente y dotando de sentido los SMS a través de la iteración y recursividad (acciones elementales para la conceptualización de los números naturales).

Por los resultados obtenidos parece ser valioso recuperar la tradición formal matemática en la enseñanza, desde los primeros grados de educación elemental y permitiría la participación de los niños en la construcción de los números naturales, lo que les brinda una posibilidad para consolidar la generalización, como base para un pensamiento aritmético sólido.

Finalmente, consideramos que el trabajo conceptual de los números con niños de 6 a 7 años de edad, no es una construcción trivial, memorística y operativa; pero puede facilitar el desarrollo del concepto antes del simbolismo. Lo que busca este trabajo es cultivar un pensamiento abstracto, que le permita a los niños acceder a niveles superiores de conocimiento matemático.

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INTRODUCING INVERSE FUNCTION TO HIGH SCHOOL STUDENTS: RELATING CONVENTION AND REASONING

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Researchers have identified students' difficulties reasoning about inverse functions. Through our review of this literature, three meanings stand out: a formal, 'undoing', and quantitative meaning. Using these meanings as a guide, we analyzed student work collected from a lesson on the topic of inverse functions taught by an experienced high school mathematics teacher, using a novel task. In analyzing the data, we noticed tensions between students' understanding of the context and of inverse function as treated in curricula. In this paper, we illustrate these tensions and describe potential implications for students' productive construction of the meanings of inverse function.

Keywords: Cognition, Algebra, High School Education, Representation and Visualization

Previous literature has identified that students in high school and beyond struggle with constructing productive inverse function meanings. We identify three different ways researchers discuss students' meanings for inverse relations: formal, "undoing", and quantitative. In this study, we characterize student work from a contextualized, problem-based lesson (Herbst, 2003) that our research team co-designed with an experienced high school mathematics teacher to support students in developing productive inverse function meanings in relation to the meanings characterized in the literature. In particular, we designed the lesson to support students in conceiving of and representing a quantitative relationship. The teacher who taught the lesson stated as part of their goal that students would understand that a function and its inverse function represent the same relationship and that the rule used to determine the function could be "undone" to determine the rule for the inverse function. Addressing the question "How do students reason when introduced to inverse function?", we use examples of student work and dialogue during their discussion to characterize the extent to which students exhibited these meanings of inverse function. We also highlight how the teacher's attempt to meet institutional obligations (Chazan, Herbst, & Clark, 2016) by introducing switching techniques (described shortly) during the lesson likely prompted students to move away from their initial reasoning.

Prior Literature on Students' Meanings for Inverse Relationships and Framework

We synthesize three meanings for inverse functions that are emphasized in the research literature examining the learning and teaching of inverse function: a formal meaning, an 'undoing' meaning, and a quantitative meaning. We use these meanings to categorize both the students' work from the classroom and the teachers' discussion of inverse.

Formal Meaning

Many researchers characterizing students' and teachers' meanings of inverse functions have emphasized aspects of the formal definition of inverse function: $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. This definition uses the notions of function composition (Brown & Reynolds, 2007; Even 1992; Vidakovic, 1996) and injectivity (Marmur & Zazkis, 2018; Wasserman, 2017). For example,

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Vidakovic (1996) provided a preliminary genetic decomposition of inverse function which closely resembled the formal definition. However, none of the students in her study developed inverse function meanings compatible with her genetic decomposition. Futhermore, none of the 26 pre-service teachers in Marmur and Zazkis' (2018) study noted the lack of injectivity of the function when asked to respond to a hypothetical student claiming the function $y = x^2 - 4x + 5$ had two inverse functions. The difficulties identified by Vidakovic (1996) and Marmur and Zazkis (2018) provide motivation for a continued need to explore ways to support students and teachers in developing meanings for inverse function.

Specifically, these aforementioned researchers and others (e.g., Paoletti et al., 2018) found that students' and teachers' meanings for inverse function are often constrained to engaging in specific actions in certain representations (e.g., switching-and-solving analytically, reflecting over a line graphically that may or may not result in equivalent inverse functions across these representations). For instance, Paoletti et al. (2018) noted a majority of the pre-service teachers in their study maintained disconnected meanings for inverse function that were constrained by such 'switching' techniques. Collectively, these disconnected meanings motivate a need to explore ways to support students and teachers in developing more coherent meanings for inverse function.

Undoing Meaning

Other researchers (Fowler, 2014; Martinez-Planell & Cruz Delgado, 2016; Oehrtman, Carlson, & Thompson, 2008; Teuscher, Palsky, & Palfreyman, 2018) have suggested having students develop meanings for an inverse function as "undoing" the original function process, often in lieu of focusing on formal mathematical properties of inverse function. Researchers who have adopted this stance have found that instruction emphasizing inverse functions as 'undoing' supports more students in addressing tasks relevant to decontextualized and contextualized inverse functions as compared to students who experienced instruction focused on formal definitions and switching techniques (e.g., analytically switching the x and y labels, reflecting over the line $y = x$). For example, across a sample of 3,858 college pre-calculus students, Teuscher, Palsky and Palfreyman (2018) reported that, in course sections with instruction emphasizing an undoing meaning, students accurately solved 48% of inverse function tasks compared to 32% of students whose instruction focused on switching techniques. We note that although emphasizing an 'undoing' meaning can be *more* productive when compared to emphasizing formal definitions or switching-techniques, over half of students in the former sections were still unsuccessful in addressing inverse function prompts.

Quantitative Meaning

Recently, Paoletti et al. (2018) and Paoletti (2020) have leveraged Thompson's (2011) theory of quantitative reasoning to characterize a quantitative meaning for inverse relations (and functions). A quantitative meaning for inverse relations entails a student understanding that a relation and its inverse relation represent an invariant relationship between quantities' values, regardless of how the relationship is represented. Thus, rather than foregrounding injectivity (cf. Marmur & Zazkis, 2018), a quantitative meaning entails the existence of an inverse relation regardless of whether the original or inverse represents a function. Students can determine if either relation is a function by examining if the univalence property (i.e., if for each value of one quantity there is exactly one value of the second quantity) holds for each relation.

Rather than focusing on a function and its inverse as processes that can be undone, a student with a quantitative meaning for inverse relations understands that a relation and its inverse are (or can be) represented by the same rule or graph (Paoletti, 2020). Paoletti (2020) provided an empirical example of one pre-service teacher reorganizing her unproductive inverse function meanings grounded in switching techniques, into a more productive, quantitative, meaning. By the end of the study the student, Arya, understood that a single graph or analytic rule represented a function and its inverse function and that switching techniques were used to maintain conventions commonly used in school mathematics (e.g., the independent quantity is represented by variable x on the horizontal axis). She particularly noted how confusing switching techniques were in contextualized situations as it was necessary to switch the quantitative referents of the variables when engaging in switching techniques (i.e., if in $F(C) = (9/5)C + 32$, F represents the temperature in Fahrenheit and C the temperature in Celsius, then in $F^{-1}(C) = (5/9)(C - 32)$, F represents the temperature in Celsius and C the temperature in Fahrenheit). In this paper, we present indications of other students naturally maintaining the quantitative referents of variables.

Methods

Our team worked closely with an experienced high school mathematics teacher to design a contextual problem-based lesson with the goal of introducing students to the concept of inverse function. The first step in the lesson design was for the teacher to create a problem that would provoke a need for this new idea, but that students could make progress on by drawing on knowledge and skills that they had developed previously. Next, the teacher created a detailed lesson plan that included anticipations of student work and potential scaffolds and responses to them. The teacher then implemented the lesson in which students would work with their peers in small groups and then with whole-class discussions. The teacher ended the lesson with a statement of the newly introduced idea. The final version of the problem that the teacher designed is presented in Figure 1.

Several of us that teach at [name of your school] are on a slow-pitch recreation softball team together. Your City Parks and Rec charges a “sponsor fee” of $350 to enter the league. This pays for umpire fees, softballs, grounds people, etc. In addition, individual players each have to pay a player fee of $17. Thus, **the total amount of money we need to pay the office depends on how many people we have on our team**.

1) Make a table of *Total Fee* vs. *Number of Team Members* for at least 6 points. We need at least eight people to play.
2) Write, in words, the calculation procedure you kept doing to get the total amount of money given the number of players.
3) Is this situation linear? How do you know?
4) What is the y-intercept? What does it represent in this situation?
5) Write a rule for this situation.
6) Graph this function on a piece of graph paper.

When I worked for the recreation department in My Town, near the end of the season I needed to be able to see which teams in each division still had a chance to win the league. This way, I could order enough “Champions” t-shirts for the team with the most players who had a chance to win. What I had was the inventory list that had the receipts for the amount of money each team turned in, and from that, I had to figure out the number of players they had.

Assuming this same scenario for Your City Parks and Rec, the function for the league supervisor is backwards: for him or her, ***the number of players on the team depends on the total fee.***

7) Make a table for the league supervisor that computes the number of players for teams that have paid $571, $622, $639, $673 and $724.
8) Explain in words the calculation procedure you did to compute the number of players from the total fee amount.
9) Write a rule that computes the number of players as a function of the total team fee.
10) Make a second graph on your graph paper that shows the relationship from this perspective (with the total paid as the independent variable and the number of players the dependent).

11) Compare the function from the front with that of the recreation supervisor from this side. What do you notice about how the tables, graphs, and rules are different?

Figure 1: The Softball Fees Problem

Relative to the meanings for inverse function, although the context included an injective relationship between quantities and could have led to a discussion involving composition, this introductory lesson to inverse function did not explicitly address composition. Rather, the lesson revolved around ideas of inverse function more closely related to the undoing and quantitative meanings of inverse function. In the discussion of the lesson, the teacher explicitly referred to the inverse equation as representing an "undoing" of the original function process. Moreover, in his lesson plan, the teacher described, as a mathematical goal, that "we can focus at this initial stage on simply what the relationship looks like if you want to change your perspective and have students recognize that the function and its inverse are related, but not the same" (which is consistent with an undoing meaning) and goes on to say "that the factual information [this many players equates to this much money] stays consistent regardless of what perspective you have" (which is consistent with a quantitative meaning) The teacher attended to constructing different representations for the function and its inverse and emphasized the importance of understanding that the two functions represent the same "factual information" for all representations. Thus, in this paper, we report on student work that stemmed from instruction that emphasized both an "undoing" notion of inverse function (particularly when representing the relationship between the quantities as rules) and maintaining the quantitative relationship that a function and its inverse represent.

The teacher worked at a large Midwestern public high school. He taught the lesson during three class periods to three different classes of students. The data collected from each implementation of the lesson included video recordings positioned strategically across the room to capture students' work in groups. Additionally, researchers took fieldnotes and created copies of written work from 60 students.

Our analysis focused primarily on the student work. Initially, we analyzed it using Balacheff and Gaudin's (2010) conception framework. Through this analysis, we noticed variation in the representations used—tables, rules, and graphs—by students as well as in how they operated on them in the process of finding the inverse function. We also noticed that there was an association between the representations/operations used and the students' control structure. Reflecting on these findings, literature emphasizing students' inconsistencies in representations of inverse relationships (Paoletti et al., 2018), and our own observations of students' attempts to update their work to fit the conventional ways of representing inverse relationships in each of the representations (i.e., tables with the independent variable on the left, rules in which the independent variable is represented as x, and Cartesian graphs whose horizontal axis represents the independent variable x), we shifted our attention to those pieces of student work that may present this tension. We then open-coded (Strauss & Corbin, 1994) the student work based on individual student's ways of representing inverse functions (e.g., column location of values, naming of quantities in expressions, orientations of graphs). We also analyzed the classroom video associated with the implementations of the lesson to learn what ideas about inverse functions the teacher emphasized and to confirm the use of the aforementioned conventional notation with the students. From there, attending to the idea that conventional student work would not be present in work *prior* to the class discussion, we identified key pieces of student work in which the student seemed to change the way they represented the inverse function. As we explain in the next section, these pieces of work provided insights into students' initial reasoning about representing inverse functions and the alterations they made to fit with the ways in which the teacher was asking the students to represent inverse functions. We connected the chosen pieces of student work to ways in which they did so to the aforementioned inverse function meanings. Collectively, this analysis allowed us to answer the question of how students reason when first introduced to inverse function and to identify some of the tensions that present in their different ways of representing their reasoning.

Results

We present samples of student work that provide evidence of reasoning emphasizing either an "undoing" or quantitative meaning for inverse function. Although each individual created tables, rules, and graphs, we only present specific responses relevant to specific representations of each student's work. Specifically, we describe, by type of representation, each piece of original student work and, if relevant, the updates that the students made to this work.

Reasoning with Tables

Addressing Question 7 (Figure 2), IS initially constructed a table for the supervisor that labeled the quantities "# of players" and "$" on the left and right, respectively. IS wrote the given fee values in the right-hand column, circling them. This table followed the same format as IS's first table from Question 1 (not pictured) in which IS had the number of players on the left side of the table and total fee amount on the right. To the right of that table in Figure 2 is an updated table in which the student constructed a table with switched columns (values and their associated quantitative referent). This example illustrates a student who considered their initial table as representing both a relationship and its inverse; using a single table to represent a function and its inverse aligns with the quantitative meaning for inverse function. Although students often chose different values or column labels in their tables, using the same table to address Question 1 and 7 was common. As a second example, MH preserved *x* and *y* labeling as well as the location of the quantities represented on the left and right in their table.

Figure 3 is another example of a student using the same table to address Questions 1 and 7. Moreover, KK's description of her original process for constructing her table across Questions 2 and 8 is consistent with an "undoing" meaning. Such activity may be indicative of the student understanding ways to connect her quantitative and undoing meanings for inverse function. She may understand that, while the function and its inverse represent the same relationship, in order to determine values of one quantity given a value of the second quantity, she must reverse the process by which she found values of the second quantity to determine the value of the first quantity. KK's table and descriptions also provide insight into the significant components of the table they were considering when updating their crossed-out table. Specifically, they drew several double-sided arrows on their (initial) crossed out table and, beside their redrawn table, they drew another blank table with the labels "ind" (independent) and "dep" (dependent). Comments from other students who drew new tables such as KK included a student writing "flipped around" above the new table and TK writing "should have put # of players right (y) on table." We conjecture such activity was spurned by the instructor who emphasized representing the independent variable on the left side of the table and the dependent variable on the right.

Lastly, consider the work in Figure 4 from DS. DS viewed a table as apt to represent both a relationship and its inverse. When asked to explain the calculation procedure for the supervisor, DS wrote, "You take the table from befor[e] and find the p[r]ice then writ[e] down what *x* is."

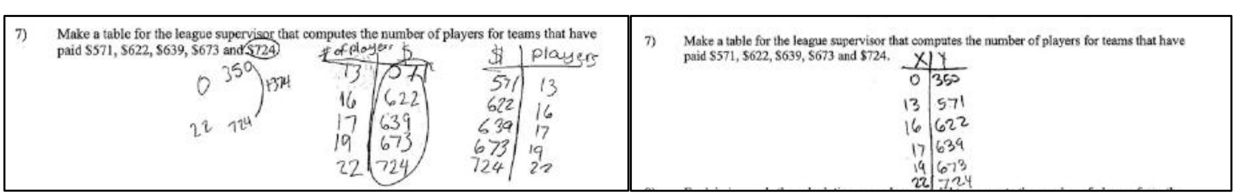

Figure 2: (left) IS's Two Tables for the League Supervisor and (right) MH's Table

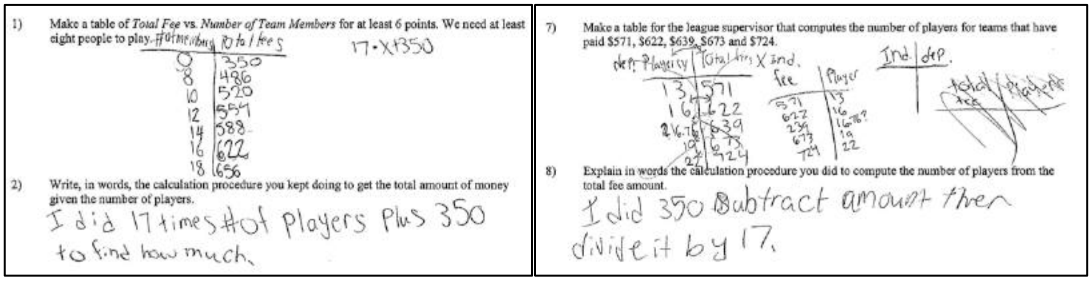

Figure 3: KK's Initial and Inverse Tables and "Undoing" Description

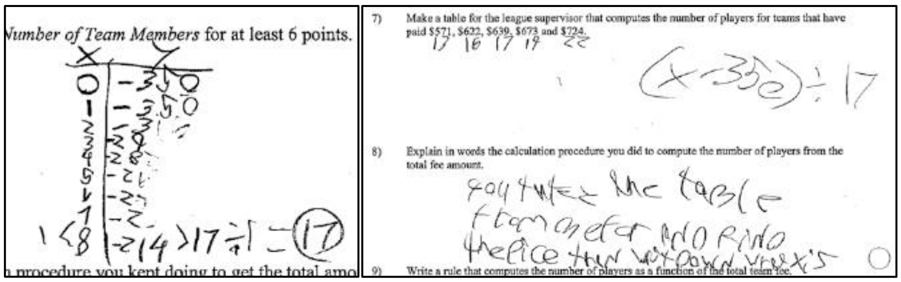

Figure 4: DS's Description of the Inverse Calculation Procedure

Reasoning with Rules

Figure 3 above provides an example of a student who provided a mathematical description of a process that undoes the original one. Perhaps due to current curricular treatments of inverse function, which emphasize the importance of representing the input quantity by the variable x on the horizontal axis, students engaging in writing a new rule to represent this undoing process may switch variables such that the independent quantity represents x and the dependent quantity represents y (or $f^{-1}(x)$). This point was raised by the teacher during the discussion. However, in students' initial work, there was variation among students' use of x, y, and their quantitative referents in their construction of a rule that computes the numbers of players as a function of the total team fee.

First—and indicative of maintaining a quantitative meaning for inverse functions—some students did not write a new rule for the inverse relationship and simply used their existing rule. For example, MH, who also did not construct a table with a different format for the inverse (Figure 2, right), used their rule "$y = 17x+350$" to substitute the given team fee values and solve for x (Figure 5). Thus, throughout their work on the task, x represents the "*number of team members*" and y represents the "*total fee*" consistently.

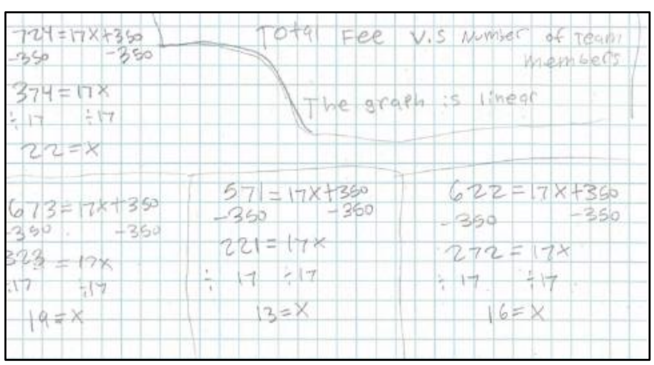

Figure 5: MH Substitution Strategy for Supervisor

Other students used their first rule and wrote a new rule in terms of the symbol representing the total team fee. For example, TK (Figure 6, left), who, like MH, did not construct a new table, had the rule "(y – 350) ÷ 17 = x". Here, y represented "Total fee" as labeled in the table. However, as seen in both MH's work and HV work, the x and y labels were not used consistently throughout individual students' work. For example, TK's initial rule, seen in Question 7 (i.e., $(y - 350)/17 = x$), maintains the quantitative referent of the variables in their table (i.e., "Total fee y"). However, when addressing Question 9, TK appears to have erased and then switched their original x and y labels in the rule used to create the table values. We conjecture this change may have been spurned by the classroom conversation based on TK's note that they "Should have put # of players right (y) on a table." Another student, HV, wrote the equation "(x – 350) ÷ 17 = y" but the labels on the table beside seem to indicate that x represented the number of players and y the total fee (Figure 6, right).

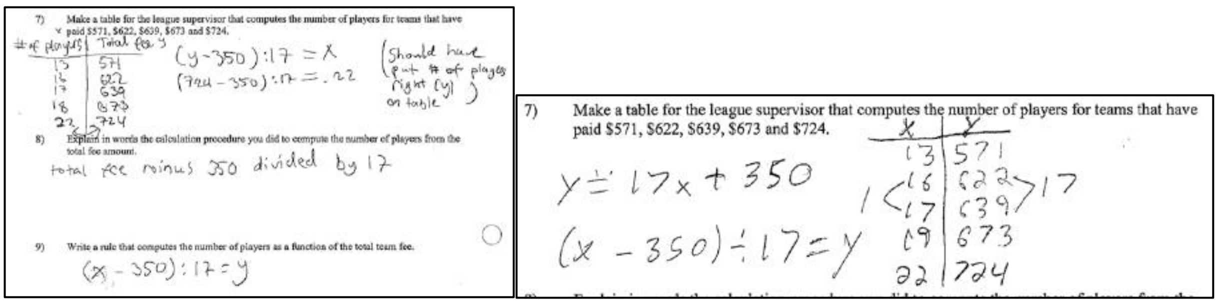

Figure 6: TK (left) and HV's (right) inconsistent use of *x* and *y*

Across the student work, we observed responses that were indicative of each of the meanings for inverse function described in the literature. Several students, like MH, exhibited a quantitative meaning for inverse function as they understood a single rule could be used to represent both a function and its inverse. Consistent with an undoing meaning for inverse function, other students, like TK, wrote a new rule that represented the opposite of the initial process that maintained the quantitative referents of the variables. Finally, several students created rules that inconsistently maintained the relationships between variables and quantitative referents, which may be indicative of their attempting to make sense of the classroom instruction that emphasized the importance of switching-and-solving.

Reasoning with Graphs

As a closing illustration, although most students constructed two perceptually different graphs with different axes labeled on the horizontal axis, nine students drew graphs (or at least labeled axes) to indicate that both requested graphs would have the same axes labels in the same locations. MX, for example, had "People" labeled on the horizontal axis for her first graph and seemed to intend for the number of people on the team to be represented on the horizontal axis for her graph for the supervisor, too (Figure 7). Like the students who only constructed one table (e.g., IS), these students seemed to indicate that a single graph orientation could represent both a relationship and its inverse.

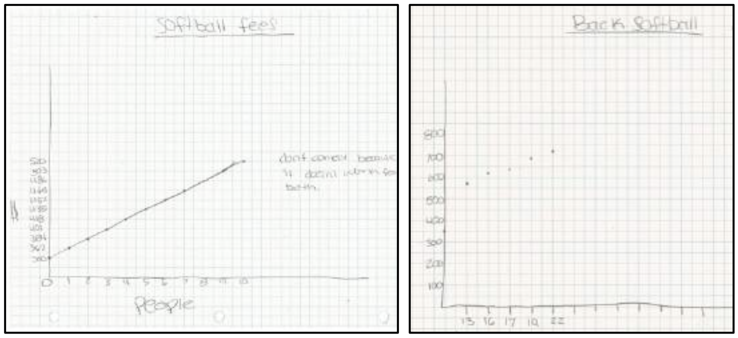

Figure 7: MX's Two Graphs with Same Axes Labels

Discussion

We use the aforementioned pieces of student work to highlight various ways in which students exhibited each of the three meanings for inverse relations present in the literature. The teacher designed the lesson purposefully to be contextualized and problem-based, and we argue reasoning with the context supported the students in understanding a relation and its inverse as representing the same quantitative relationships. The design of the task also supported an undoing meaning for inverse function. In particular, KK's work provides some evidence that these two meanings – quantitative and undoing meanings – can interplay with one another in possibly productive ways. We note that none of the student work contained the common struggles described in literature on students' inverse meanings (e.g., reflecting over a $y = x$ line on a graph, writing the multiplicative inverse of the function as the function's inverse, composing functions). We hypothesize this is because of the scaffolding of this introductory task and that the contextualization of the relationship they considered supported them in being able to reference the context to make sense of their results.

Despite the students providing several quantitative representations of their reasoning, the teacher likely felt an institutional obligation (Chazan, Herbst, & Clark, 2016) to carry out a classroom discussion in which students' reasoning, while perhaps quantitatively appropriate, needed to be amended to fit the conventional notations for inverse function. Although the teacher maintained consistent meanings for inverse function throughout this discussion, we note that students were faced with the tension of motivating changes in notation in their quantitatively appropriate work. We conjecture without having explicit conversations that allow students to reconcile the need for adjustments in their work (i.e. discussions regarding conventions), students may experience conflations (and perhaps a motivation to rely on memorizing techniques) when addressing inverse function tasks as is seen in the literature (e.g., Paoletti et al., 2018; Vidakovic, 1996). Future researchers may be interested in exploring how such conversations may be fruitful for teachers to have with students.

Acknowledgments

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FRAMEWORK FOR REPRESENTING A MULTIPLICATIVE OBJECT IN THE CONTEXT OF GRAPHING

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In this study, based on the analysis of a teaching experiment with middle school students, we propose a framework for describing meanings of a point represented on a plane in terms of multiplicative objects in the context of graphing. We classify those meanings as representing (i) non-multiplicative objects, (ii) quantitative multiplicative objects (Type-1 and Type 2), and (iii) spatial multiplicative objects. We then discuss implications of these meanings with respect to students' graphing activities.

Keywords: Cognition, Representations and Visualization, Modelling

Quantitative and covariational reasoning play a critical role in students' understanding of various ideas in mathematics (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Ellis, 2011; Johnson, 2015). There are numerous aspects and important constructs (e.g., quantitative structure, frames of reference, and quantification) that individuals could develop in order to engage in productive, powerful quantitative and covariational reasoning (Joshua, Musgrave, Hatfield, & Thompson, 2015; Moore, Liang, Tasova, Stevens, 2019; Thompson, 2011). One of these constructs is a *multiplicative object*. Thompson, Hatfield, Yoon, Joshua, and Byerley (2017) suggested that students "must construct a multiplicative object of quantities' attributes in order to reason about their values covarying smoothly and continuously" (p. 128). Constructing a multiplicative object is also important in the context of graphing as it imparts a productive meaning to a point on a graph (Frank, 2017; Saldanha & Thompson, 1998; Thompson & Carlson, 2017). However, much is left to understand about the extent and nature of students' meanings of a point as representing a multiplicative object in the context of graphing. Therefore, in this study and against the backdrop of empirical data, we provide a framework to classify students' meanings of a point on a plane in terms of representing a multiplicative object.

What is a Multiplicative Object?

A multiplicative object can be considered a conceptual object that is formed by uniting in the mind two or more quantities' magnitudes or values simultaneously (Saldanha & Thompson, 1998; Thompson, 2011; Thompson & Carlson, 2017). The mental operation of someone who constructs a multiplicative object is similar to the operation of someone who conceives a quarter coin as being, simultaneously, a circle and silver in color. In this operation, circle and silver, as two attributes of the object, have been considered, simultaneously, as one property of a quarter coin. For a dynamic example, imagine heating the quarter coin up to the melting point of silver. As a multiplicative object, coupling the two attributes, someone could track the variation of the coin's color with the immediate and persistent awareness that, at every moment, the temperature of the coin also varies.

In the context of co-variation, Thompson (2011) represented the multiplicative object formed by uniting two quantities' variations by using the following representation: $(x_e, y_e) = (x(t_e), y(t_e))$, where $x_e = x(t_e)$ represents a variation in the values of x, where t_e represents variation in t through conceptual time over the interval $[t, t+e)$. He explained that in order for students to reason covariationally, they must unite x_e and y_e by constructing (x_e, y_e), which simultaneously represents the two. Note that the corresponding representation of this conceptual object in graphical context would be a point in a coordinate plane. We next discuss the role of conceiving a point as a multiplicative object in developing meanings for graphs.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Multiplicative Objects in the Context of Graphing

Despite the notion that graphing is critical for understanding various ideas in STEM fields (Rodriguez, Bain, & Towns, 2019; Kaput, 2008; Leinhardt, Zaslavsky, & Stein, 1990), students face a number of challenges (e.g., graphs as pictorial objects) in interpreting and making sense of graphs (Clement, 1989; Leinhardt et al., 1990; Moore & Thompson, 2015). Thompson and Carlson (2017) conjectured that part of these students' difficulties were grounded in being unable to conceive points on a graph as multiplicative objects, and several researchers have provided evidence to this claim (e.g., Frank, 2016, 2017; Stalvey & Vidakovic, 2015; Stevens & Moore, 2017; Thompson et al., 2017). Given this evidence, conceiving points as multiplicative objects might be an integral part of constructing productive meanings for graphs, and thus a student's construction of points should not be taken for granted.

We note that students' meanings of points on a plane can be considered as a representation of multiplicative object if students conceive a point by engaging in multiplicative operation—the operation of uniting and holding in mind two attributes of an object (i.e., quantities) simultaneously—as defined by Inhelder and Piaget (1964). Inhelder and Piaget first introduced the role of a multiplicative operation to characterize children's thinking when classifying 2-atrribute objects (e.g., objects grouped according to shape and color, as described above). They reported two different ways of children's thinking, both of which lead to normative responses when identifying a missing element in a matrix arrangement (see Figure 1). One is based on twofold symmetries that involve relying on perceptual configuration of the matrix arrangement and treating it as an incomplete pattern. For example, squares are symmetric over the horizontal axis of the diagram, so the blank space should include a circle. Similarly, red objects are symmetric over the vertical axis, so the blank space should include a blue circle. The other way of thinking is based on a multiplicative operation on a logical structure with reasoning about objects and coordinating two classes. For example, classifying the given three objects simultaneously in terms of shape and color, then identifying two elements of squares already belong to the classification of red or blue, noting the given element of circle belongs to red, and then joining circle and blue to construct the missing element.

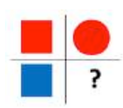


Figure 1. A matrix diagram designed based on the narratives of Inhelder & Piaget (1964).

We rely on these two types (e.g., perceptual features vs. reasoning about attributes) to classify students' meanings of points as representing non-multiplicative objects or representing multiplicative objects. In addition, we note that Inhelder and Piaget (1964) illustrated that an arrangement does not have to be in a matrix form for a child to think of objects in terms of two attributes. Inhelder and Piaget reported students could coordinate multiplicative classes without needing objects in a matrix form, and in our work, we illustrate that students could conceive points as representations of multiplicative objects without needing points represented in a Cartesian coordinate system (see spatial multiplicative object of the following framework). In other words, we considered that representing a multiplicative object is not restricted to plotting a point on a coordinate plane in the normative sense; it is about conceiving a point as a simultaneous representation of the two attributes of the same object.

Role of Multiplicative Objects in Emergent Shape Thinking

Moore and Thompson (2015) introduced the notion of *emergent shape thinking* to describe a person who envisions a graph "*simultaneously* as what is made (a trace) and how it is made (covariation)"

(p. 785). Constructing a graph from this perspective involves (1) representing two quantities' magnitudes and/or values varying on each axis of a coordinate system, (2) creating a point as a representation of a multiplicative object uniting those two quantities' magnitudes or values as a single object, and (3) generating a graph by conceiving the process of a multiplicative object leaving a trace when moving within the plane in ways invariant with the two covarying quantities. As we elaborate on different types of representing multiplicative objects, our framework informs the process in which a student could develop emergent shape thinking, which researchers (e.g., Frank, 2017; Moore, Stevens, Paoletti, Hobson, & Liang, 2019) have shown it is a productive way of thinking about graphs.

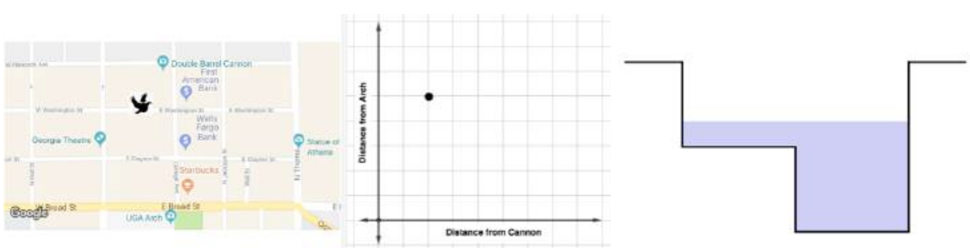
Method

Our work here stems from a semester-long teaching experiment (Steffe & Thompson, 2000) that occurred at a public middle school in the southeast United States. We recruited four seventh-grade students (age 12). Our goal was to investigate students' thinking involved in conceiving and representing various quantitative relationships. In this paper, we focus on two of the four students, Zane and Ella, since their meanings for points were consistent within their representational system and clearly described by them throughout the teaching experiment. We believe it is important to document these ways of thinking in order to add nuances to our models of students' conception of points in a graphing activity in terms of multiplicative objects.

The first author was the teacher-researcher (TR). We recorded all sessions using two video cameras to capture students' work and their gestures and a screen recorder to capture their activities on the tablet device. We transcribed the video and digitized students' written work for on-going and retrospective conceptual analyses (Thompson, 2008). Our analysis relied on generative and axial methods (Corbin & Strauss, 2008), and it was intended to develop working models of students' thinking based on their observable and audible behaviors.

Tasks

Before conducting the teaching experiment, the TR developed an initial sequence of tasks by considering particular design principles focused on graphing covarying quantities (e.g., Frank, 2017; Moore & Thompson, 2015; Stevens, Paoletti, Moore, Liang & Hardison, 2017; Thompson & Carlson, 2017). The TR revised and implemented those tasks based on on-going inferences and analysis of Zane and Ella's thinking. Each task was designed with a dynamic geometry software and displayed on a tablet device.



(a) (b) (c)

Figure 2. (a) The map of Downtown Athens, (b) Coordinate system with a point, and (c) A diagram of the pool

The Crow Task. The situation includes a map of Downtown Athens with a movable crow and fixed seven locations: UGA Arch (hereafter Arch), Double-Barreled Cannon (hereafter Cannon), First American Bank, Georgia Theater, Wells Cargo Bank, Statue of Athena, and Starbucks (see the map in Figure 2a). We also presented a Cartesian coordinate system whose horizontal axis is labeled as distance from Cannon and vertical axis is labeled as distance from Arch (Figure 2b). Students can control the crow freely by dragging it and see how the corresponding point in the coordinate plane changes (go to https://bit.ly/PMENA42 for the digital version of the tasks).

The Swimming Pool Task. This task was adapted from Swan (1985). We presented students a dynamic diagram of a pool (Figure 2c), where they could fill or drain the pool by dragging a point on a given slider. We designed the task to support students in reasoning with the inter-dependence relationship between two continuously co-varying quantities: amount of water (AoW) and depth of water (DoW) in the pool.

Framework for Representing a Multiplicative Object

In this framework, we describe students' meanings for a point in terms of multiplicative objects. We classify those meanings as representing (i) non-multiplicative objects, (ii) quantitative multiplicative objects (Type-1 and Type 2), and (iii) spatial multiplicative objects.

Representing a Non-Multiplicative Object

In this section, we illustrate a characterization of students' meanings for points on a coordinate system as contra-indication of representing a multiplicative object. Building off of limited number of studies (i.e., David, Roh, & Sellers, 2018; Frank, 2016, 2017; Thompson & Carlson, 2017; Thompson et al., 2017), this characterization emerged as we identified students correctly plotting points in the plane by carrying out a certain procedure (e.g., over and up), but the meaning of these points included solely an ordered pair of numbers and/or a location in the plane that did not symbolize or unite two quantities' magnitudes and measures.

Note that there are different students' meanings that could be classified as a non-multiplicative object. Aforementioned researchers have exemplified some of those meanings. For example, Thompson et al. (2017) argued that calculus students viewed the point (2, *f*(2)) in a coordinate plane as a value of the function, instead of the relationship between the value of the function (i.e., *f*(2)) and the value (i.e., 2) for which the function was evaluated. Similarly, David et al. (2018) reported that some students treated the output of the function as the location of the coordinate point in the plane, rather than on the vertical axis (i.e., *location-thinking*). Those students—and consistent with those in Thompson et al.'s study—did not think of 2 as a measure of a magnitude located on the horizontal axis and they did not think of *f*(2) as a measure of a magnitude located on the vertical axis in a canonical Cartesian plane.

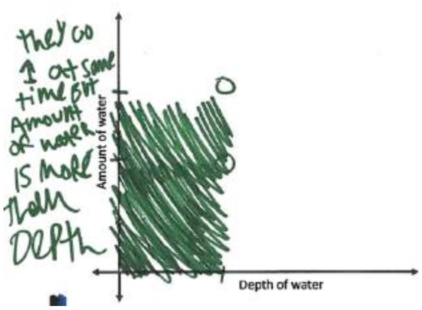


Figure 3. Ella's first draft.

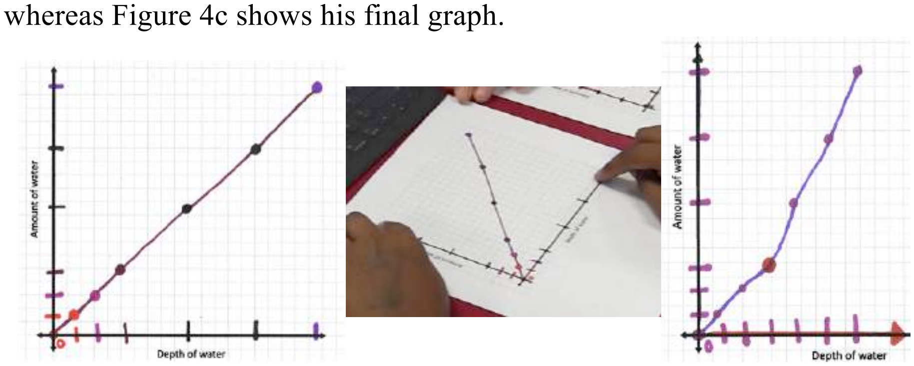
For an empirical example from our data set that falls under this category, we present Ella's graphing activity in Swimming Pool Task. We asked Ella to sketch a draft of a graph that represents the relationship between AoW and DoW as the pool fills up. She began by inserting two tick marks on each axis as an indication of AoW and DoW. As seen in Figure 3, she noted, as we fill the pool up, both tick marks go up along the axis at the same time and wanted to place the tick mark for AoW further along the vertical axis than the tick mark for DoW since she thought "amount of water is more than depth."

Then, Ella drew a small circle in the plane to show "where those two things [*tracing her finger horizontally from the tick mark on the vertical axis to the circle in the plane, then vertically down from the circle in the plane to the tick mark on the horizontal axis*] meet here." Then, she shaded the rectangular area in the plane, what she called "a box" (see Figure 3), to show "a bunch of water." When asked to explain what the small circle meant for her in terms of the pool situation, Ella said, "I don't know what it means" and further she explained "that is just like the dot between [*tracing her finger horizontally from the tick mark on the vertical axis to the circle in the plane*] here [*tracing her finger vertically down from the circle in the plane to the tick mark on the horizontal axis*] so I can just make this box [*pointing to the shaded area in the plane*]." We infer that Ella was able to plot a point in the plane respective of the tick marks that she placed on each axis. However, Ella conceived the point as a landmark to draw "the box," which was a contraindication of *representing* a multiplicative object. Although Ella reasoned about the attributes when placing the two tick marks on the axes and used those tick marks in order to generate the point (i.e., the circle in Figure 3), Ella's meanings of the point didn't include uniting the attributes of an object (i.e., AoW and DoW) in the plane; instead Ella conceived the point in terms of a mark as a part of a procedure to set the corner for the box.

Representing a Quantitative Multiplicative Object (QMO)

In this category, we describe meanings of students who construct and/or interpret a single point in the plane in relation to two quantities whose magnitudes or values represented on each axis. We illustrate this category by using Zane's graphing activity in the Swimming Pool Task.

We asked Zane to sketch a graph that shows the relationship between AoW and DoW as the pool fills up. Zane started with drawing tick marks on each axis. Zane referred to the quantity's magnitude by drawing a line segment from the origin to the tick mark on the axis to articulate his meanings of tick marks. Moreover, Zane simulated the quantities' variation by tracing his fingers along the axis as we played the animation to fill the empty pool (Figure 4b). After inserting tick marks, Zane plotted points for each related tick marks correspondingly (see his color-coded points and tick marks in Figure 4a), then he connected those points with line segments in the plane. Figure 4a shows Zane's earlier graph whereas Figure 4c shows his final graph.



(a) (b) (c)

Figure 4. (a) Zane's draft, (b) Zane moving his fingers on axes, and (d) Zane's final graph

To gain more insights into how he conceived his plotted points, we asked Zane to show the point on his graph representing when the pool is full. Zane first pointed to the far right and top purple tick marks on each axis (see Figure 4a, see also Figure 4b), and he then pointed to the corresponding purple point in the plane (see Figure 4a). Taken together with his description of a dot—"the dot represents both amount of water and depth of water"—his actions suggest that he could associate two tick marks (i.e., indication of quantities' magnitudes for Zane) on each axis to the corresponding point in the plane, which is an indication of representing a QMO.

So far, we demonstrated Zane's meanings of a single point in the plane. We, now, classify the instances of representing QMO in two ways in relation to conceiving a graph (e.g., a line drawn in the plane) when students represent a relationship as two quantities vary: (1) as a path or direction of movement of a dot in the plane, and (2) as a trace of the point consisting of infinitely many points, each of which showing the relationship of two varying quantities. We illustrate those types below.

Type-1 QMO. Type 1 includes students who envision points as a circular *dot* that represents two quantities' magnitudes or values simultaneously and envision that points on a graph (e.g., a line) do not exist until they are physically and visually plotted. Therefore, those students conceive the graph as representing a direction of movement of a dot on a coordinate plane. We illustrate by continuing to discuss Zane's graphing activity identified above.

We asked Zane whether his graph (see Figure 4c) showed every single moment of how the two quantities varied in the situation, Zane claimed no because one would need to stop the animation and plot an additional point in order to show the desired moment and state of the quantities. We infer that, for Zane, his line did not have points until they are visually plotted. He needed to physically plot additional points to represent moments in between two available points, even if there is a line connecting them. When questioned what the line segments that he drew in between dots meant to him, Zane responded that the line shows "where the dots go." By go, he meant a dot moving from one plotted point to the next plotted point, but not in a way that preserved an invariant relationship between those two points.

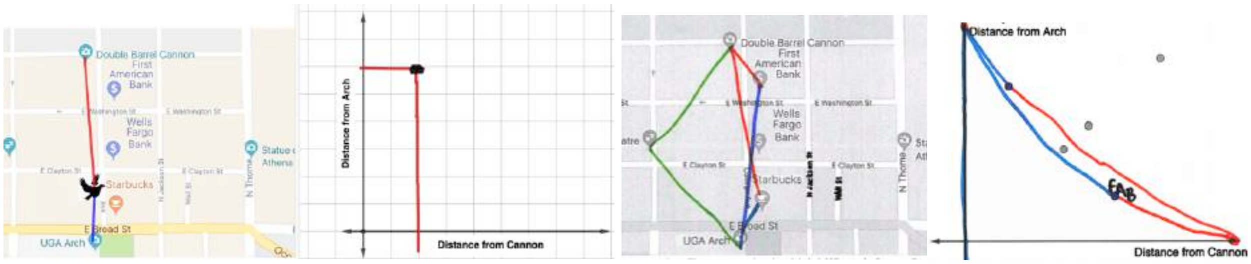
Therefore, despite his success in being able to conceive of a point as a multiplicative object, Zane assimilated his graphing activity as one dot moving along a line path instead of one dot generating infinitely many points by leaving a trace. We claim that his meaning for points and lines played a critical role in Zane's construction and constrained him from conceiving a graph as an emergent, in-progress trace (i.e., the third component of emergent shape thinking).

Type-2 QMO. Type 2 describes students who could envision a point as an abstract object that represents two quantities' magnitudes or values simultaneously, and envision a graph (e.g., a line) as composed of infinitely points, each of which represent two quantities' values or magnitudes, which is an indication of emergent shape thinking. We did not have data in our current study to show an empirical example of a student reasoning emergently (see Moore & Thompson, 2015, for an illustration).

Representing a Spatial Multiplicative Object (SMO)

This category emerged as we coded instances where the students assimilated a "point" in the plane as an object/location by focusing on the object's quantitative properties and engaging in quantitative reasoning (e.g., gross comparison of two quantities' magnitudes). Students who represented a SMO determined the object's *location* by coordinating and representing two (measurable) attributes of the object (e.g., the crow's distance from Arch and Cannon) *in* the plane, as opposed to representing those attributes on the axes of the plane. That is, they represented the two attributes considering the object's (i.e., the point in the plane) distance from each axis or from a certain location in each axis of the coordinate plane.

For an illustration, we present a moment from Zane's activity in a version of the Crow Task. In the previous version, Zane assimilated the given black point in the plane (see Figure 2b) as the crow. In this version, we hided the given point and asked Zane to plot a point that represents the crow's DfA and DfC when the crow is in a place on the map as seen in Figure 5a. Zane began drawing a horizontal line segment starting from the vertical axis to a certain place in the plane and drew a vertical line segment from that place to the horizontal axis (see Figure 5b). Making connection to the blue and red bars appearing on the map (Figure 5a), he referred to the horizontal line segment in the plane saying “the crow’s distance form Arch is shorter” and referring to the vertical line segment in the plane, he said “the crow’s distance from Cannon is longer.” Then, he plotted the black dot (seen in Figure 5b) where these line segments intersected.



(a) (b) (c) (d)

Figure 5. (a) and (b) Zane’s activity, (c) and (d) Ella’s activity

We infer that to *locate* the black dot (i.e., the crow for him) in the plane, Zane represented the crow’s distance from Arch as the distance from the vertical axis and the crow’s distance from Cannon as distance from the horizontal axis. Note that this activity is consistent with Zane’s earlier activity where he assimilated the axis of the coordinate system as Arch and Cannon. Although the position of the point he plotted is not normative in terms of a canonical quantitative coordinate system (Lee, Hardison, & Paoletti, 2018), this activity was valid for Zane as he was assimilating the black dot in the plane as the crow, and imagining Arch in place of the vertical axis and Cannon in place of the horizontal axis itself.

Note that our emphasis in this framework is not on students’ use of coordinate systems (cf. Lee, Hardison, & Paoletti, 2018; Paoletti, Lee, & Hardison, 2018); instead we categorize students’ meanings of points in terms of multiplicative objects represented in a space that may or may not be classified as any type of coordinate system that we, as researchers, know. For example, Zane’s graphing activity (Figure 5b) may suggest he *seemed* to be engaging in plotting a point as a multiplicative object in a *spatial coordinate system* (according to Paoletti et al., 2018), where Zane established a frame of reference considering the horizontal axis and the vertical axis as a reference point to represent the crow’s distance from Cannon and Arch, respectively. However, conceiving a point as SMO should not directly imply representing the relationship on a spatial coordinate system. Considering Ella’s activity in Downtown Athens Task (see Figure 5c and 5d), we infer that Ella essentially formed, from our perspective, a two-center bipolar coordinate system based on gross comparisons between the two quantities’ magnitudes. That is, she conceived Arch and Cannon as a location on the vertical and horizontal axis, respectively, implied by the labels (see orange dots on each axis in Figure 5d). Then, she made sense of the rest of the space by coordinating the radial distances between “places” in the plane and “Arch” and “Cannon” on each axis. For example, Ella labeled a point as FAB in the plane (see Figure 5d) indicating First American Bank. To justify why FAB, referring to the orange and blue line segments that she drew *in the plane* (Figure 5d), she said “the orange is shorter, and the blue is longer.” Referring to the line segments on the map (see Figure 5c), she added, “over here, like the same thing” showing FAB is closer to Cannon and farther from Arch in the map as well as in the plane.

Discussion

In this study, we illustrated different ways students’ graphing activity involved multiplicative objects when graphing quantitative relationships. We believe outlining such a framework is important as researchers can be more attentive to those meanings students hold for their representational activity. In this section, we discuss the different implications of these ways of representing multiplicative objects on students' graphing meanings.

QMO VS. SMO

We perceive that the goal for students who conceive a point as a SMO is locating the object in the space, which is why the primary attention is the plane rather than the axes. To locate the object on the space, students represent the quantities' magnitudes in the space by committing to a reference, such as the axis itself or a location on the axis, and where those magnitudes meet determines the location of the object (e.g., the crow, bike, or First American Bank). The meaning of this point on the space is different than students' meanings who conceive a point as QMO as they represent quantities' magnitudes on the axis and form the point by taking two orthogonal magnitudes along the axis and creating projections. In turn, the space is inconsequential beyond creating the point by joining the projections when engaging in representing QMO. For that reason, students who produce a graph by tracing a SMO will perform different actions (e.g., moving different directions on the space) than others who produce a graph by tracing a QMO. Interestingly, those two graphs produced by tracing a SMO and a QMO will be exactly the symmetry of each other over the line of $y = x$ in case of Zane's activity. Despite the fact that students construct non-normative graphs when tracing a SMO, their form of reasoning is productive in terms of completing the goal of the activity as they perceive it. Their activity should be considered as a different way of graphing relationships because it still requires students engage in quantitative coordination in a non-normative way. Calling this type as "spatial" should not imply dismissing the role of quantitative coordination in students' reasoning.

Type 1 and Type 2

To produce a graph via reasoning emergently, students trace a point and anticipate a graph including infinitely many points, each of which is a representation of a multiplicative object; a point as a multiplicative object is sustained throughout conceiving the emergence of the trace. Given the importance of reasoning emergently in developing productive meanings for graphs, by distinguishing two types of representing a QMO (i.e., Type 1 and Type 1), we note that conceiving a point as a multiplicative object is necessary, but not sufficient in envisioning "graphs as composed of points, each of which record the simultaneous state of two quantities that covary continuously" (Saldanha & Thompson, 1998, p. 298). As we illustrated above, students whom we classify engaging in representing Type 1 QMO can conceive a point in terms of multiplicative object, but they do not imagine a trace being produced as representative of that multiplicative object. Given there are numerous students (i.e., about 89% of secondary students [N=1798], as reported in Kerslake, 1981) not conceiving of infinitely many points on a line and believing there is no point on a line until they are plotted (Mansfield, 1985), it becomes important for us to be able to determine which type of multiplicative object the students forming and representing. In doing so, we can inform our instruction to foster and support students in developing productive meanings for graphs.

Acknowledgments

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A CONCEPTUAL ANALYSIS OF THE EQUAL SIGN AND EQUATION –THE TRANSFORMATIVE COMPONENT

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Mathematics education scholars have generally classified students' conception of the equal sign as either operational or relational. Adding to these conceptions, Jones (2008) introduced the idea of substitutional conception. Building off these scholars, I introduce a form of understanding the equal sign that includes a transformative equivalence component and extends the conceptions of the equal sign to conceptions of equations.

Keywords: Algebra and Algebraic Thinking, Cognition.

Introduction

Students' algebra achievement acts as a gatekeeper that affects their future academic success and employment (Rech & Harrington 2000; Ladson-Billings, 1998). As a result, students' understanding of algebra continues to attract attention from mathematics educators (Kieran, 1992; National Council of Teachers of Mathematics[NCTM] 2000; Wagner & Kieran, 1989). Researches have established that students' conception of the equal sign is fundamental to their learning of algebra (Knuth et al., 2006; McNeil & Alibali, 2005; Falkner et al., 1999). Consequently, aiding students in building a productive understanding of the equal sign may not only support students learning of algebraic concepts but also foster social equity. Building on such belief, this paper proposes one cognitive model for giving meaning to the equal sign.

The paper begins with a detailed summary of two important papers in which the authors separately addressed elementary school students' understanding of the equal sign and middle school students' understanding of the equal sign (Behr et al., 1980; Knuth et al., 2006). Both studies, although focused on students of different school ages, suggested a differentiation between an "operational" and a "relational" (or "equivalent") understanding of the equal sign, and such differentiation has been echoed by other researchers (Carpenter et al., 1999; Baroody & Ginsburg, 1983; McNeil et al., 2006; McNeil, 2008; McNeil et al.,2011). In general, an operational conception involves interpreting the equal sign as an announcement of the result of an arithmetic calculation, and the relational conception interprets the equal sign as indicating a mathematical equivalence (Knuth et al., 2006). After summarizing these perspectives, I introduce Jones's (2008) notion of the substitutive conception of the equal sign, which further divides the "relational understanding" into "substitutive relational" and "sameness relational." Following this cursory literature review, I explain the theoretical rationale for this paper and introduce a conceptual analysis—named *the transformative model*—along with a brief empirical result. The model extends the question from understanding students' conception of the equal sign to understanding their conception of equation, and it contributes a "*transformative equivalence*" component to previous discussions.

Background

The equal sign was not introduced until 1557 by Recorde, and it was universally applied around 1700. In the field of mathematics, it is not the only symbol that represents an equivalent relationship, and indeed it is a special symbol that only represents a certain category of mathematical equivalence (Molina et al., 2009). Therefore, one can reasonably assume that students' understanding of the equal

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sign might be as varied in both meaning and sophistication as its development across the history of mathematics.

One of the earliest works in studying students' understanding of equal sign can be found in Behr and his colleague's (1980) research, in which the authors studied elementary students' conception of the equal sign. One major finding is that students (around 6-7-year-old) hold a fixed belief that the equal sign has to appear after the operation symbol. For example, some students in their study read the expression "8=5+3" as "five plus three equals eight." Furthermore, students generally rejected a sentence in the form "$\square = 2 + 4$" but instead changed it to "$\square + 2 = 4$" or "$2 + 4 = \square$". One interpretation of them was that students had an inclination in using an "action" sentence rather than a non-action sentence. In such a case, the authors argued that students conceived the equal sign as a "do something signal," and one could only have an equal sign when there was an operation appears on the left (p.16). In other words, students did not conceive the equal sign as suggesting two equivalent expressions but an operation symbol. In the study, some students even changed expression "4=6+1" to "4=6+10" and saying 4 and 6 made 10. The authors further postulated that some students were merely treating equal symbols as symbols to connect numbers.

The aforementioned study provides evidence that many primary school students do not have a flexible way of using the equal sign and frequently see it as an operation signal, and McNeil and Alibali (2005) revealed similar patterns among high school and college students. Following this result, Knuth and his colleagues (2006) conducted a quantitative study on middle school students' conceptions of the equal sign. Based on their findings, they argued that middle school students lack "relational understanding" (or equivalence understanding) of the equal sign, and this influenced students' success in solving algebra problems.

Knuth et al. (2006) used two problems in their study. In the first problem, students were required to give an explicit description of the meaning of the equal sign. In the second problem, students were required to solve algebra problems such as "4m+10=70". The authors found that students who explained the equal sign with a relational description (i.e., equal sign means the same as) were more likely to solve the algebra problem correctly than students who explained the equal sign with a non-relational description (i.e., equal sign is a sign connecting the answer to the problem). The authors also illustrated that the students who gave relational descriptions were more likely to use algebraic methods in solving the later problem; the authors defined algebra methods as methods that involved algebraic manipulation (i.e., performing the same transformation on each side of the equation), and non-algebra methods are guessing and trying or direct arithmetic (i.e., 70-10=60, 60÷4=15). The authors further showed that both correlations hold when controlling for students' general mathematic abilities, that relatively few students hold a relational view, and that the percentage of students who hold such view did not increase significantly when students moved from grade 6 to grade 8. The authors suggested such a lack of progress might be related to the lack of explicit focus on the meaning of the equal sign in the curriculum.

The findings from these studies are consistent. In listed studies, we observe students may either understand equal sign as a "do something signal," which aligns with an operational meaning, or as "a sign of equivalent relation," which supports students to use algebraic method in solving it. In both studies, we also observe students with the operational conception have difficulties in doing algebra flexibility. Based on these studies, one can further conjecture that a student, if holds an operational conception, will likely avoid using algebraic methods since an algebraic method requires operations on each expression, but he/she may not want the right-side expression to involve an operation.

Adding to these researches, Jones and his colleague (Jones, 2008; Jones et al., 2012) studied the substitutive conception of the equal sign. They define the substitutive conception as realizing both sides of the equation can be used to substitute each other when needed. These studies are resonated with the study of "relational thinking" developed by Carpenter et al. (2005). In short, the key idea that concerns those researchers is that given "16+37=53", students should be able to evaluate "16+39" without doing a direct calculation but utilizing the possible connections between these two expressions. More importantly, Jones and his colleague (2012) argued there is a cognitive difference between "seeing two entities as equivalent" and "being able to interchange equivalent expressions when beneficial," and scholars frequently over-emphasized the sameness component and risks neglecting the substitute component. Jones (2008) found that while making substitutions, students were oblivious to the correctness of the equation. For example, when a student needed to replace 77, some picked 77=11+33. Jones et al. (2012) also observed cases where students realized that both sides of the equation are the same but did not substitute them in problem-solving. Though only a few numbers of researchers have studied this substitutive conception, studies in relational thinking have shown similar findings that it is challenging for students to relate both sides of the equal sign in solving problems (Molina & Ambrose, 2006, 2008). Therefore, the difference between recognizing the equivalence and being able to utilize such equivalence as a means for substitution should be marked.

Theoretical Rationale

Kaput (2000) proposed a movement to "algebrafying" the K-12 curriculum where he encouraged students to engage in algebraic reasoning. Carpenter and Levi (2000) answered with a proposal in re-conceptualization some mathematical topics taught in the primary grade. Following both proposals, I provide a re-conceptualization of the equal sign and equations such that students can reason with them more flexibly. The previous studies on students' conception of the equal sign offer a solid grounding for this re-conceptualization, but they share two potential limitations.

One limitation of previous research is their lack of clarity in classifying students' conception, or as Mirin (2019) argued, the differentiation between "operational" and "equivalent" is sometimes unclear. For instance, a primary school student who accepts the notion of 3=4-1 is considered as presenting a relational conception in Behr et al.'s study, but he/she can still have difficulties in solving equations with algebraic strategies when he/she moves into middle school. More importantly, there are inconsistencies between students' conception of the equal sign and their problem-solving strategies. For instance, in Knuth at al.'s (2006) study, 33 students in grade-eight used the algebra method in solving the problem, but only 31 students showed relational understandings. More surprisingly, 43 grade-seven students provided a relational definition to the equal sign, but only one student used the algebra method in problem-solving. These results suggest students' conception of the equal sign is not fully predictive of their performance in solving algebra problems. Especially in high school and above, it is likely that few students will not realize that equal sign represents an equivalence or do not believe in the legitimacy of 3=3, but their algebraic skills regarding equations can still be lacking. Therefore, to effectively extend the study of the equal sign to all k-16 education, especially for the high school and college students, it might be beneficial to reframe students' conception of the equal sign as students' conception of equation. In short, students' conceptions of equation are dependent on their conception of the equal sign, but student's conception of the equal sign is not fully predictive to their use of equation. Such expansion will bring issues, which we will address later.

The second limitation is that most studies on students' conception of the equal sign were conducted under the context of single equation solving but lacks research on modeling students' thinking within a system of equations. For instance, given the equation $x^2 + 2x = 1$ and ask students to evaluate $x + 1$. Besides using the standard procedure of solving for x and plugging, a student can also solve the question by re-writing the original equation as $x^2 + 2x + 1 = 2$ and taking square roots on both sides. The standard solve-and-plug method requires an equivalent understanding of the equal sign, but the later method further requires students to perform a transformation on both sides of the equation. Building off Jones's argument, I believe there is also a cognitive difference between "realizing the sameness or the substitutive nature of both sides of the equation" and "being able to transform both sides of equations flexibly in problem-solving." One could argue that those algebra skills are beyond the scope of understanding the equal sign. I contend that they are not beyond such a scope because there is an inherent kinship between students' making meaning for an equation and making meaning for the equal sign, a point we will revisit later.

The Transformative Model

In response to these potential limitations, I describe a first-order conceptual model (named the transformative model) on students' understanding of the equation as follows: In the first level, students conceive an equation as a call to execute an operation or calculation with an answer. In the second level A, students see the equation as representing the sameness of two expressions; in the second level B, students conceive equation as two parts that stay equivalent under some algebraic operation; In the second level C, students see equation as two parts that are interchangeable and used as means for substitution. In the third level, students see equation as a piece of information that only display a specific equivalent relationship explicitly but also includes a lot of implicit but inferable equivalent relationships that can be used in problem-solving. The third level is what I named the transformative equivalence conception (or transformative conception).

This model is of first-order as it is developed through analyzing my own thinking. It has two salient characteristics: Firstly, it incorporates the aforementioned research on students' conceptualization of the equal sign but extends such notion to the study of equation. Such extension is not a dramatic deviation from previous research on the equal sign, as many of them collected their data through observing students' solving equations (Alibali et al., 2007; Knuth et al., 2008). The other characteristic, which is also the central focus of this model, is the inclusion of the transformation component: conceiving equation as a piece of information that only display a specific equivalent relationship explicitly but also includes implicit but inferable equivalent relationships that can be used in problem-solving. I now use the following example to illustrate the above meanings:

Given $x^2 - 3x + 1 = 0$, find the value of $3x^3 - 8x^2 + x - 1 + \frac{3}{3x}$.

Notice to solve this problem, besides the common method of solving for x and then plugging the value, students will be benefitted from substituting "$x^2 - 3x + 1$" by 0. Students may also want to further conjecture the equations "$x^3 - 3x^2 + x = 0$" and "$x - 3 + \frac{1}{x} = 0$" from "$x^2 - 3x + 1 = 0$", and use these two new relationships to substitute x^3 and $\frac{3}{3x}$ in problem-solving. Here, I argue being able to recognize "$x^2 - 3x - 1$" as substitutivity equivalent to 0, and being able to conjecture a new equation and then recognize "$\frac{1}{x}$" $is\ substitutive\ equivalent\ to$ "$3 - x$" are cognitively different. I also postulate the second equivalence is more difficult to recognize since producing equation such as "$x - 3 + \frac{1}{x} = 0$" requires students to multiplicatively compare both expressions with respect to x, which is represented by dividing by x. Though dividing a variable to both sides is a common mathematical practice, it is often done to reduce the order or perform cancellation. However, since here dividing x will not accomplish either goal, students are less likely to use such a strategy as I illustrate in a subsequent section.

The model is also constructed with a partial hierarchical order, but it is not a linear progression where students gradually find new properties of equation. Instead, this model is an "emancipation process," where students gradually become less and less constrained to lower-level understandings. For instance, for students who solve "$x + 7 = 12$" with algebraic method in Knuth et al. (2006) study, when they subtracted 7 on both sides, they have performed a transformation, but their use of transformation can still be unnecessarily restricted and they might still be unable to perform other types of transformation in different questions. Therefore, though the model is hierarchical, the hierarchy is determined by the extent of how restricted students are using equations in problem-solving. Similarly, as Jones et al. (2012) suggested, students' development of different conceptions of equal sign does not follow a strict order, and one might develop substitutive conception before mastering an equivalent conception. Therefore, I put several conceptions as parallel, and the ordering in my model is certainly tentative rather than deterministic.

Methodology and Method

In considering my methodology, I follow radical constructivism and believe our knowledge is constrained by our experience in the sense that we do not have direct access to external realities or absolute truth (even assuming they exist). Consequently, each student constructs their own conceptualizations of mathematical ideas through their unique experiences. We do not have direct access to their understandings, but we can build hypothetical models of students' knowledge (Glaserfeld, 1995). Thompson (2013) reminded us of the importance of attending to students' meaning in mathematical activities and ensuring students understand mathematical objects productively. Conceptual analysis, as elaborated by Silverman and Thompson (2008), is then an approach in which researchers will model productive meanings of a concept such that those meanings are well connected with other mathematical ideas or students' life experiences. Educators, ideally, can then use those models to analyze students thinking and guide students to more productive understandings.

Following these beliefs, I conducted several semi-structurted clinical interviews with pre-service teachers and tried to identify possible ways that students are using or can use in conceptualizing the equal sign and equations. Research has reported the similarities between pre-service teacher and high school students in terms of their mathematics performance in non-college math topics (Moore & Carlson, 2012; Carlson, Oehrtman, & Engelke, 2010). In short, though the participants here represent a convenience sample, the results are applicable to a broader population. Here I will focus on my interview with Meki, who was a second-year undergraduate student registered in our math education pre-service teacher program. She has completed several college-level math courses but not high-level analysis courses. In the interview, I asked Meki to go through six algebra problems and explained her thinking, and most problems are algebraic questions that have multiple approaches but will be solved most efficiently by transforming the given equation and then using the substitution method. The interviews try to examine how i) interviewees' conceptualization of the equation is consistent/inconsistent with my conceptual model, especially regarding the transformative component; ii) what are the affordances and constraints of my conceptual model in modeling students' conceptualization of equation, especially regarding the transformative component.

Empirical Result

In this section, I report Meki's answer on two problems with detail. The first problem asks the student to evaluate $\frac{1}{x}+\frac{1}{y}$ when presented with the claims that $xy = 1; x + y = 1.$ Meki solved the problem by realizing $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{xy}$. However, when I asked about her thinking, she said that she did so as she felt there was no way to directly substitute the equations (pointing to equations $xy = 1; x + y = 1$) into the unknown expression, and she explained that there was no "xy" or "$x + y$" appeared in the equation. I then intervened directly and told her that there was a way of doing direct substitution; she, after some thought, realized the other solution which was substituting 1 by "xy" and change the expression $\frac{1}{x}+\frac{1}{y}$ to $\frac{xy}{x}+\frac{xy}{y}$ which equals $x + y = 1$. When I asked her why she didn't think of such substitution in the first place, she replied, "yeah, normally you don't put more variables into solve, cause you try to solve and get rid of the variable and find the numbers."

Meki displayed a substitutive conception of equation since she was comfortable in evaluating $\frac{x+y}{xy}$ by substitution, and also she mentioned that she started the problem by looking for some substitution. However, from the quote, I argue that her substitutive conception might be restricted in the sense that she normally does not substitute numbers by symbols. Therefore, I hypothesize that she lacks a transformative conception, and her use of equations is not flexible. Though this result does not directly substantiate the difference between the transformative conception and other conceptions, it at least suggests that a student with substitutive conception can still experience unnecessary restrictions in the potential ways of using the substitution method.

I then gave Meki the following problem, which was similar to the one I introduced earlier, and the difference is we had $\frac{3}{3x}$ in previous problem but $\frac{3}{a^2-1}$ here (which are equivalent)

Given $a^2 - 3a + 1 = 0$, find the value of $3a^3 - 8a^2 + a - 1 + \frac{3}{a^2-1}$.

In dealing with the expression $3a^3 - 8a^2 + a$, Meki started by extracting a common factor "a," and rewrote the expression into the form $a(a^2 - 3a + 1 + 2a^2 - 5a)$, which she then simplify as $a(a^2 - 5a)$. She then failed to make too much progress beyond. She mentioned she could do this trick again twice (referring to the trick of finding"$a^2 - 3a + 1$" in the expression and the substitute it by 0), but she tried some mental calculation and gave it up. In dealing with the term $\frac{3}{a^2+1}$, she rewrote it as $\frac{3}{a^2-3a+1+3a}$. She explained that since the expression "$a^2 + 1$" has two terms that were the same as the given equation's, she wanted to introduce some new terms to make it zero. It is important to mark here that she said she did not foresee the result would be $\frac{3}{3a}$, but she was trying to get rid of the two terms and had less term in the denominator.

Here I argue that Meki was using the substitutive conception, and she was working very hard to substitute the expression a^2-3a+1 by 0 in the exact form to the unknown expression, and she did not realize that one can also generate other ways of substituting (e.g. $a^3 = 3a^2 - a$) in solving the question. Though the problem can be solved by repetitively using direct substitution, the mental effort that is required in such a process was huge, and students are likely to give up. A student with a transformative conception may, however, have a very different approach to doing this problem. One approach that I observed from my cohorts was creating a "tool column" where he made a column of all equivalent forms such as $a^3 = 3a^2 - a$, and used these forms when he felt he needed it. Notice in such an approach, the student was still substituting one side of the equation with the other side, but such substitution requires a prior transformation of the original equation, and Meki seemed to be reluctant to perform it.

After I told Meki the answer and gave explicit hints on these potential equivalent equations, the students solved the problem without too many struggles. When reflecting on her thinking, she mentioned that she never did anything like that, and she said: "I was thinking a lot of it like taking things like this (circling the original equation) as it was instead of moving terms around." Her explained insistence in substituting the whole equation in its original form again suggested her use of substitution is restrictive, and a transformative conception of the equation is cognitively different from a substitutive conception.

When doing other problems, Meki presented similar thinking patterns where she is comfortable in making substitutions, but her use of the substitution method is unnecessarily restrictive. There were also some interesting findings that I noticed when I asked my cohort to experiment with some of the problems: For example, one of my cohorts realized $a^3 = 3a^2 - a$ and simplified most terms, but he struggled about $\frac{3}{3a}$. When I asked if he could conclude anything about $\frac{1}{a}$ from the given equation, he believed he could not. Certainly, by being able to simplify the higher-order terms, he displayed transformative conception, but his transformative conception does not fully support all forms of transformation (e.g., he did not notice here $\frac{1}{a} = 3 - a$).

Behr et al. (1980) summarized students' operational conception as "an extreme rigidity about written sentences, an insistence that statements be written in a particular form, and a tendency to perform actions (e.g., add) rather than to reflect, make judgments, and infer meaning." (p.16), and he named the flexible part as relational conception. Similarly, students can hold a relational conception but present the same rigidity in using algebraic methods in solving problems, which implies a lack of Jones's substitution conception. Furthermore, the student can also hold a substitution conception but present the same rigidity in using substitution, which implies a lack of the transformative conception. The case of Meki serves as a proof of such an argument where students show substitution conception but with unnecessary rigidity in the ways of substituting.

Conclusion and Compromise

Students' conception of the equal sign is important, but a differentiation between an "operational" and a "relational" is too broad to explain the wide spectrum of students' performance on operating with the equation. Consistent with Jone's finding that students can observe the equivalence between two sides of the equation while not substituting each side when helpful, I found some students can substitute each side but only perform substitutions in limited ways. In general, this study points out the variety of operations that exist in operating with equations, and students suffer from unnecessary restrictions in performing them as their conceptualization of how an equation could be operated in problem-solving is incomprehensive.

Here are two important questions to revisit: why include a transformative component of the equation into the study of the equal sign? How such an extension to the conception of equation may bring issues? For the first question, it is an observation that most studies of the equal sign are generalized through studying students' meaning of how an equation can be written/operated in problem-solving. Indeed, it is hard to imagine how one student can develop a conception of the equal sign without experiencing different ways of writing/manipulating equations. Besides, such a conception is tightly related to Jones' substitution conception, as both concern the potential ways of utilizing equations in problem-solving, and both believe there is a nonnegligible and important cognitive difference between understanding the equivalence and utilizing such equivalence in problem-solving. For the second question, since the conceptualization of equation goes beyond pure algebraic context (e.g., one may argue a complete conceptualization of equation has to contain the idea of function, as function is a special "equation" or a collection of an infinite number of "equations"), the provided conceptual analysis is incomplete. The model is created independently from concerning the students' conception of variables, the cognitive gap between arithmetic and algebra, and the rich real-life or mathematical contexts that an equation can be embedded. I concede all these factors are important points that a fully comprehensive study of equation should include. Certainly, this model is not claiming a complete analysis of students' conception of equation. Nevertheless, the primary aim of building such a model, which is to support emphasizing and studying the transformative property of equation in teaching and learning, should remain intact.

To contextualize this study into the broader field of education theory, I argue it echoes the general belief in fostering students' critical thinking and creativity as it relates to their mathematical learning. Adopting Ennis's (1996) definition of critical thinking with its emphasis on reflectiveness and making a choice from what to believe and what to do, the transformative conception may support creative reasoning and critical thinking since it invites students to think and consider all possible ways of writing an equivalent relation. Cobb(1998) raised an argument that viewing Mathematica learning as an "acculturation," and through providing more flexibility in problem thinking, the transformative conception may help students to conquer certain fixed norms in solving equations and experience the creative culture from the mathematics society (such as avoiding thinking substitution as only using numbers to replace symbols). Under such a perspective, mathematics equations and expressions become Lego blocks that students can play with and make creations accordingly, but if and only if there are multiple ways of playing, students have motivations to consider different possible mathematical operations, and what information can different equivalent expression produce. In such a way, students are potentially engaging with mathematics critically, creatively, and authentically.

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ALGEBRA, ALGEBRAIC THINKING AND NUMBER CONCEPTS:

BRIEF RESEARCH REPORTS

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MENTAL CONSTRUCTIONS OF INVARIANT SUBSPACE DEVELOPED BY AN APOS TEACHING DESIGN

CONSTRUCCIONES MENTALES SOBRE SUBESPACIO INVARIANTE DESARROLLADAS EN UNA PROPUESTA DE ENSEÑANZA DESDE LA TEORÍA APOE

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We present some of the preliminary results of a project about how the concept of eigenspace can be learned as special invariant subspace. The study is based on the theoretical and methodological elements of the APOS framework. Specifically, we present the part of the preliminary genetic decomposition, obtained from the theoretical analysis, corresponding to the concept of invariant subspace; as well as the criteria for classifying and contrasting the mental structures proposed in the genetic decomposition with those manifested during and after the instructional stage by eight students of Mathematics who participated in the study.

Keywords: Advanced Mathematical Thinking, Algebra and Algebraic Thinking, Technology, Design Experiments

The application of the concept of eigenspace and the link it has with various key concepts of Linear Algebra have guided various investigations regarding its learning. Wawro, Watson, and Zandieh (2019) point out that students find it difficult to argue when linear combinations of eigenvectors are eigenvectors, showing confusion to distinguish between a base for a given eigenspace and elements of such subspace; They conclude that: " A focus on eigenspaces as subspaces has the potential to mitigate these challenges and help students see connections across the linear algebra course" (p. 1122). Therefore, a study was carried out with the purpose of *analyzing the cognitive structures that a student might need for the learning of eigenspaces as invariant subspaces*, so that the cognitive demand of this task can be clarified. Some preliminary results about the difficulties in developing such mental constructs are presented below.

APOS theoretical framework

The APOS framework was chosen since it is a cognitive theory that allows a structural analysis on the learning of mathematical concepts and poses specific relationships between the cognitive analysis of mathematical learning and the design of teaching materials and experiments (Oktac, Trigueros & Romo, 2019). The model defines these stages as mental structures and classifies them as *Action, Process, Object* and *Schema* (hence the acronym APOS), which arise from the application of specific mental mechanisms: *internalization, encapsulation, coordination, reversal, de-encapsulation* and *thematization* (Arnon et al., 2014).

With an **Action** structure, an individual performs transformations to previously constructed *Objects* and its main characteristic is that each step of the transformation must be carried out explicitly, in a given order, by the individual and needs to be guided by external instructions. The **Process** structure comes from the *internalization* of an Action in a Process or by the *coordination* of different Processes into a new one. In the first case, the same operation is performed as the Action that is internalized, but in the individual's mind, achieving control over the transformation; in the second, the interaction of different processes leads to the development of a new one that in a certain way includes both. When it is possible to apply transformations onto the Process and such transformations can be constructed, the Process has been *encapsulated* in an **Object**. The **Scheme** structure has a

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different nature from the previous three, as it develops as a network of Action, Process and Object structures, as well as relationships between them; for example, a differential calculus scheme can be described as network centered on the concept of function (Arnon et al., 2014). The specific focus of this work leads us to study the first types of constructions and not include the development of Schemes.

The hypothetical model that explains the set Actions, Processes and Objects, as well as the mental mechanisms that a person could need for the construction of a specific mathematical concept is called *genetic decomposition*.

Methodology

Research in the APOS framework is commonly carried out following its *research cycle*, which consists of three components: *theoretical analysis, design and implementation of instruction,* and *data collection and analysis* (Arnon et al., 2014, p. 94). From the theoretical analysis a preliminary genetic decomposition is obtained, the teaching design generates sequences that seek to facilitate the mental structures and mechanisms proposed in the genetic decomposition; and from the collection and analysis of data, the *preliminary* genetic decomposition and the teaching design are obtained. Preliminary results of each methodological phase are presented below.

Theoretical analysis

Based on a review of various Linear Algebra books (Axler, 2015; Frieldberg, Insel & Spence, 1982) and research related to learning the concept of eigenspace (Sierpinska et al., 1999; Soto & García, 2002; Thomas & Stewart, 2011; Gol Tabaghi & Sinclair, 2013; Wawro et al., 2019), it was concluded that the need to compare a vector subspace with its image implied thinking that the image of an eigenspace satisfies that its image is the same subspace or the zero subspace. Likewise, that for the development of a conception of eigenspace as invariant, it was relevant that conceptions of both concepts were constructed first and that relations between eigenspaces and invariant subspaces were later determined.

It was determined that for the learning of *invariant-eigenspace*, students were required to have previous conceptions of *generated subspace*, *linear transformation*, and *eigenvalues and eigenvectors*.

In the preliminary genetic decomposition, the construction of eigenspace starts from invariant subspaces, followed by invariant one-dimensional and then *larger* subspaces (2 and 3 dimensions), eigenspaces, and then relationships between these structures are analyzed. The proposed concepts for invariant subspace are described below.

The **Action A_1** consists of manipulating the *spanning vector Object*, a single vector that spans a one-dimensional subspace, by applying a linear transformation (defined from $\mathbb{R}^2 \to \mathbb{R}^2$), identifying cases in which the image of the subspace is itself or the zero subspace. When Action A1 is repeated for different subspaces and different linear transformations, the reflecting on the results allows the internalization into **Process P_1.** This process allows the observing, assuming or arguing that the subspace generated by some vector is invariant without having to manipulate specific cases of subspaces or linear transformations.

Process P_1 is coordinated with the *spanned subspace Process* to obtain **Process P_3**, with this Process it is expected that students can identify that the images of the generators of an invariant subspace are not necessarily scalar multiples of themselves; to argue that the invariant subspaces are those subspaces that satisfy that the generators of the *image subspace* can be expressed as a linear combination of the spanning vectors of the subspace and recognize that the image of the generator set of the subspace does not generate the invariant subspace if the null subspace is different from the zero subspace.

Design and implementation of instruction

The ACE Teaching Cycles proposed from APOE were taken as a reference. These cycles are made up of three stages that give the cycle its name: *Activities* to initiate the development of mental structures; *Class discussions* to promote mental mechanisms; and *Exercises* to reinforce the mental constructions developed by the students (Arnon et al., 2014, p.58).

The design consists of four ACE type sequences. In sequence 1, students are familiarized with the graphic representations of the concepts and with the use of GeoGebra applets, additionally, previous conceptions about the concepts of generated subspace and linear transformations are evaluated. In sequence 2, the development of Action A1 and Process P1 is promoted; this paper refers to the second sequence. For the activity phase, an applet was designed available at: bit.ly/g-act1. The activity begins with the graphic exploration of the image of a one-dimensional subspace, line *l*, which can be modified by changing the spanning vector of the subspace. Each of the buttons T_1, T_2, T_3, T_4 and T_5, is associated with a different linear transformation and they are distributed by teams, while the group discussion is focused on the similarities and differences between the five cases.

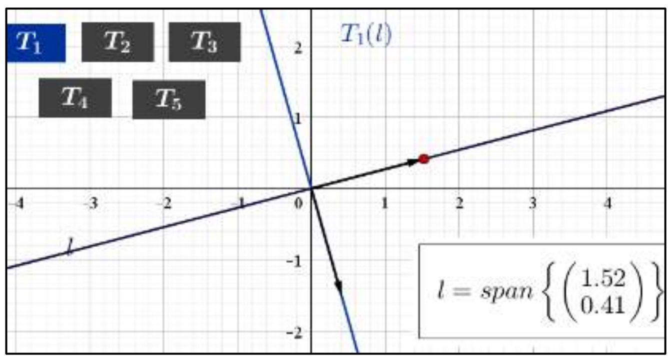

Figure 1 Sequence 2 GeoGebra applet

The implementation of the sequences here reported was carried out with eight students of the Bachelor of Mathematics enrolled in a second Linear Algebra course, the data and analysis shown below correspond to sequence 2, whose implementation was carried out in 3 sessions of 50 minutes.

Data collection and analysis

The student worksheets collected in the implementation stage complemented with notes and recordings of the class discussions constitute the data analyzed. A non-comparative case study was carried out, which sought to contrast the proposed structures from the theoretical analysis with those expressed by the students in their written productions. The following table shows a couple of student responses to a problem in the exercise section designed to assess Action A_1 conception.

Table 1 Differences in the development of conception Action A1.

Let $P_2(\mathbb{R})$ be the vector space of grade 2 or lower polynomials with real coefficients, and $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ a linear transformation such that $T(p) = p'$. Determine if a) $P_2(\mathbb{R})$ and b) $span\{x^2\}$, invariant subspaces under T	
Student **E1** set $p(x) = a + bx + cx^2$ and calculated the image of vector p, realizing that $T\big(p(x)\big) \in P_2(\mathbb{R})$ and concluding that the subspaces was invariant. In b), following a similar procedure, compared αx^2 with $span\{x^2\}$ and concluded that the subspace was not invariant because $T(\alpha x^2) = 2\alpha x \notin span\{x^2\}$.	Student **E3** set $p(x) = a_0 + a_1 x + a_2 x^2$, calculated its image and concluded that: "it is T-invariant because T(p) is a polynomial with grade equal or lower than 2, for each $p \in P_2(\mathbb{R})$". However, for b) he took an arbitrary αx^2, calculating that $T(\alpha x^2) = \alpha 2x \in P_2(\mathbb{R})$, finally concluding that, since for every $u \in span\{x^2\}$, $T(u)$ was a polynomial of grade 2 or lower.
Action	Pre-action

Student E1 can apply the definition for the specific subspace, determining whether or not it is met; this implies that she can determine for specific cases that the subspace is invariant if its image is the same subspace or the zero subspace. E1 resorts to a previous conception of set containment that involves thinking that given two sets A and B, $A \subseteq B$ if and only if for any $a \in A$ it is satisfied that $a \in B$, which is considered part of Action A_1 . In the case of E3, the initial procedure performed by the student is correct, however, the student does not recognize that in order to conclude that the subspace is or is not invariant, he must determine if $T(\alpha x^2) \in span\{x^2\}$. In his answer, he also does not write explicitly which is the subspace that is invariant under the transformation; the absence of this part of the action classifies it as an action in progress or pre-action.

Similarly, the responses for the other conceptions of the genetic decomposition are analyzed for its evaluation.

Discussion and preliminary conclusions

In general terms, the evaluation of sequence 2 allowed evaluating both the proposed mental structures from the theoretical analysis, as well as the design and implementation of teaching corresponding to Action A_1 and process P_1. The continuation of the analysis will allow evaluating the rest of the genetic decomposition and the teaching design.

The classification of the responses in terms of the concepts shown allowed us to identify the correctly developed mental structures and those that were emerging, as in the pre-action cases, in some cases showing they were not compatible with the preliminary genetic decomposition.

Regarding the Activities problems, the student responses were mostly classified as compatible with the described mental structure linked to them, although in several cases the structures are manifested in a developmental stage (Arnon et al., 2014, p. 139), this was primarily observed in responses in which the work was based on graphical representations, the students seemed to understand and be able to work the situation according to the required mental structure, however they found it difficult to give a sufficiently clear argument that would validate what they said; students showed better arguments when situation were stated in algebraic representations. This difficulty was associated with the fact that the students who participated in the study were unfamiliar with the graphic representations.

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CONSTRUCCIONES MENTALES SOBRE SUBESPACIO INVARIANTE DESARROLLADAS EN UNA PROPUESTA DE ENSEÑANZA DESDE LA TEORÍA APOE

MENTAL CONSTRUCTIONS OF INVARIANT SUBSPACE DEVELOPED BY AN APOS TEACHING DESIGN

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Se presentan algunos de los resultados preliminares de un proyecto en el que se analiza cómo se puede aprender el concepto de espacio propio como subespacio invariante especial. El estudio se sustenta en los elementos teóricos y metodológicos del marco APOE. Específicamente, se muestra la parte de la descomposición genética preliminar correspondiente al concepto de subespacio invariante, así como los criterios utilizados para el análisis de datos que permitieron clasificar y contrastar las estructuras mentales propuestas en la descomposición genética con las manifestadas durante y después de la etapa de instrucción por ocho estudiantes de la licenciatura en Matemáticas que participaron en el estudio.

Palabras clave: Pensamiento Matemático Avanzado, Álgebra y Pensamiento Algebraico, Tecnología, Experimentos de Diseño

Los campos de aplicaciones del concepto de espacio propio y el vínculo que tiene con diversos elementos del Álgebra Lineal han guiado diversas investigaciones sobre su aprendizaje. Wawro, Watson y Zandieh (2019) señalan que a los estudiantes se les dificulta argumentar cuándo combinaciones lineales de vectores propios son vectores propios, mostrando confusión para distinguir entre una base del espacio propio con los elementos del subespacio; concluyen que: "un enfoque en los espacios propios como subespacios tiene el potencial de mitigar estos desafíos y ayudar a los estudiantes a ver las conexiones a través del curso de álgebra lineal" (p. 1122). Por lo anterior, se realizó un estudio con el propósito *analizar las estructuras cognitivas que un estudiante podría necesitar en el aprendizaje de espacios propios como subespacios invariantes*, de manera que se pueda aclarar la demanda cognitiva de esta tarea. Se presentan a continuación algunos resultados preliminares acerca de las dificultares para desarrollar tales construcciones mentales.

Marco teórico APOE

Se eligió el marco APOE ya que es una teoría cognitiva que permite un *análisis estructural* sobre el aprendizaje de los conceptos matemáticos y plantea relaciones específicas entre el análisis cognitivo del aprendizaje matemático y el diseño de materiales y experimentos de enseñanza (Oktac, Trigueros & Romo, 2019). El modelo define estas etapas como estructuras mentales y las clasifica como *Acción, Proceso, Objeto* y *Esquema* (de aquí las siglas APOE), que surgen de la aplicación de mecanismos mentales específicos: *interiorización, encapsulación, coordinación, reversión, desencapsulación y tematización* (Arnon et al., 2014).

La estructura **Acción** el individuo realiza transformaciones sobre Objetos previamente construidos y se caracteriza porque cada paso de la transformación debe realizarse explícitamente por el individuo y necesita ser guiado por instrucciones externas. La estructura **Proceso** es desarrollada a partir de la *interiorización* de una Acción en un Proceso o por la *coordinación* de distintos Procesos. En el primer caso, se realiza la misma operación que la Acción que se está interiorizando, pero en la mente del individuo, logrando la toma de control sobre la transformación; en el segundo la interacción de Procesos distintos conlleva al desarrollo de uno nuevo que de cierta manera incluye a ambos. Cuando es posible aplicar transformaciones sobre el Proceso y se pueden construir tales transformaciones, el Proceso ha sido encapsulado en un **Objeto**. La estructura de **Esquema** tiene una naturaleza distinta a las tres anteriores, al desarrollarse como una red de estructuras Acción, Proceso y Objeto, así como relaciones entre éstas; por ejemplo, se puede describir un esquema de Cálculo diferencial como centrado en el concepto de función (Arnon et al., 2014). El enfoque puntual del presente trabajo nos lleva a estudiar los primeros tipos de construcciones y no incluir el desarrollo de esquemas.

El modelo hipotético que explica el conjunto Acciones, Procesos y Objetos, así como los mecanismos mentales que una persona podría necesitar para la construcción de un concepto matemático específico es llamado *descomposición genética*.

Metodología

Las investigaciones en el marco APOE se realizan comúnmente siguiendo su *ciclo de investigación*, el cual consta de tres componentes: *análisis teórico, diseño e implementación de enseñanza y recolección y análisis de dato*s (Arnon et al., 2014, p. 94). Del análisis teórico se obtiene una descomposición genética preliminar, el diseño de enseñanza genera secuencias que buscan favorecer las estructuras y mecanismos mentales propuestos en la descomposición; y de la recolección y análisis de datos se obtiene la valoración de la descomposición genética preliminar y el diseño de enseñanza. Se presenta a continuación resultados preliminares de cada etapa metodológica.

Análisis teórico

A partir de la revisión de diversos libros de Álgebra lineal (Axler, 2015; Frieldberg, Insel & Spence,1982) e investigaciones relacionadas con el aprendizaje del concepto de espacio propio (Sierpinska et al., 1999; Soto & García, 2002; Thomas & Stewart,2011; Gol Tabaghi & Sinclair, 2013; Wawro et al., 2019) se concluyó que la necesidad de comparar el subespacio con su imagen implicaba pensar en que la imagen del espacio propio satisface que su imagen es el mismo subespacio o el subespacio cero. Así mismo, que para el desarrollo de una concepción de espacio propio como invariante era relevante que se construyeran primero concepciones de ambos conceptos y que después se determinaran relaciones entre espacios propios e invariantes.

Se determinó que para el aprendizaje del concepto de *espacio propio invariante* se requería que los estudiantes tuvieran concepciones previas de subespacio generado, transformación lineal y valores y vectores propios.

En la descomposición genética preliminar se proponen concepciones para el aprendizaje de espacio propio partiendo de subespacios invariantes. Se construyen concepciones sobre subespacios invariantes unidimensionales y de dimensión mayor, espacios propios y luego se analizan relaciones entre estas estructuras. A continuación, se describen las concepciones propuestas para subespacio invariante.

La concepción **Acción $\mathbf{A_1}$** consiste en manipular el *Objeto de vector generador*, mediante la aplicación de una transformación lineal (definida de $\mathbb{R}^2$ en $\mathbb{R}^2$), identificando casos en los que la imagen del subespacio es él mismo o el subespacio cero. Cuando la Acción A_1 se repite para diferentes subespacios generados y diferentes transformaciones lineales, reflexionando sobre los resultados, se puede interiorizar en un **Proceso $\mathbf{P_1}$** que permite observar, suponer o argumentar que el subespacio generado por algún vector es invariante sin tener que manipular casos específicos de subespacios o transformaciones lineales.

El Proceso P_1 se coordina con el *Proceso de subespacio generado* para obtener el **Proceso $\mathbf{P_3}$**, con este Proceso se espera que los estudiantes puedan identificar que en general las imágenes de los generadores del subespacio invariante no son necesariamente múltiplos escalares de sí mismos; argumentar que los subespacios invariantes son aquellos subespacios que satisfacen que los generadores del subespacio imagen se pueden expresar como combinación lineal del conjunto generador del subespacio y reconocer que la imagen del conjunto generador del subespacio no genera al subespacio invariante si el subespacio nulo es diferente del subespacio cero.

Diseño e implementación de enseñanza

Se tomaron como referencia los *Ciclos de enseñanza ACE* propuestos desde APOE. Estos ciclos se componen de tres etapas que dan nombre al ciclo: ***A**ctividades* para iniciar el desarrollo de las estructuras mentales; *Discusiones de **C**lase* para promover los mecanismos mentales; y ***E**jercicios* para reforzar las construcciones mentales desarrolladas por los estudiantes (Arnon et al., 2014, p.58).

El diseño consiste en cuatro secuencias de tipo ACE. En la *secuencia 1* se familiariza a los estudiantes con las representaciones gráficas de los conceptos y con el uso de applets en GeoGebra y se valoran concepciones previas sobre los conceptos de subespacio generado y transformaciones lineales. En la *secuencia 2*, se promueve el desarrollo de la Acción A_1, Proceso P_1 y Proceso P_2; el documento se refiere a esta segunda secuencia. Para la actividad se diseñó un applet disponible en: bit.ly/g-act1. La actividad inicia con la exploración gráfica de la imagen de un subespacio unidimensional, la recta l, que puede modificarse al cambiar el vector generador del subespacio. Cada uno de los botones T_1, T_2, T_3, T_4 y T_5, está asociado a una transformación lineal distinta y se distribuyen por equipos para luego discutir grupalmente las semejanzas y diferencias entre los cinco casos.

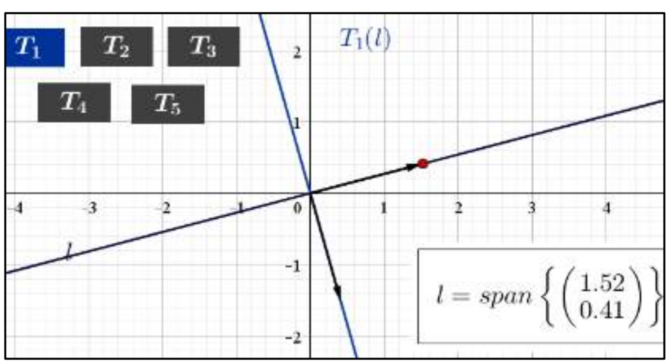

Figura 1 Applet de la Secuencia 2

La implementación de las secuencias que aquí se reporta se realizó con ocho estudiantes de la Licenciatura en Matemáticas inscritos a un curso de Álgebra Lineal II, los datos y el análisis que se muestra a continuación corresponden a la secuencia 2, cuya implementación se llevó a cabo en 3 sesiones de 50 minutos.

Recolección y análisis de datos

Se analizan las hojas de trabajo de las estudiantes, recolectadas en la etapa de implementación, complementando con notas y grabaciones de las discusiones de clase. Se realizó un estudio de caso no comparativo, con lo cual se buscó contrastar las estructuras propuestas a partir del análisis teórico con las manifestadas por los estudiantes en sus producciones escritas.

En la siguiente tabla se muestran un par de respuestas de estudiantes ante un problema de la sección de ejercicios diseñado para evaluar la concepción Acción A_1.

Tabla 1 Diferencias en el desarrollo de la concepción Acción A_1.

Sea $P_2(\mathbb{R})$ el espacio vectorial de los polinomios de grado menor o igual a dos, con coeficientes reales y $T: P_2(\mathbb{R}) \to P_2(\mathbb{R})$ una transformación lineal tal que $T(p) = p'$. Determine si a) $P_2(\mathbb{R})$ y b) $span\{x^2\}$, son subespacios invariantes bajo T	
La estudiante **E1**, tomó $p(x) = a + bx + cx^2$ y calculó la imagen del vector, llegando a que $T(p(x)) \in P_2(\mathbb{R})$ concluyendo que el subespacio era invariante. En b) realizó un procedimiento similar, comparando αx^2 con $span\{x^2\}$ y concluyó que el subespacio no era invariante porque $T(\alpha x^2) = 2\alpha x \notin span\{x^2\}$.	La Estudiante **E3**, en el caso a) tomó $p(x) = a_0 + a_1 x + a_2 x^2$, cálculo su imagen y concluyó: "Es T-invariante ya que $T(p)$ es un polinomio de grado menor a dos $\forall\, p \in P_2(\mathbb{R})$". Sin embargo, para el caso b) consideró un elemento arbitrario del subespacio αx^2, llegando a que $T(\alpha x^2) = \alpha 2x \in P_2(\mathbb{R})$ finalmente concluyó que se trata de un *T-invariante,* debido a que para todo $u \in span\{x^2\}$, $T(u)$ era "un polinomio de grado menor a 2".
Ejemplo de Acción	Ejemplo de Pre-acción

La estudiante E1 puede aplicar la definición para el subespacio específico, determinando si éste satisface o no la definición; esto implica que puede determinar para casos específicos que el subespacio es invariante si su imagen es el mismo subespacio o el subespacio cero. Se observa que E1 recurre a una concepción previa de *contención* que involucra pensar que dados dos conjuntos A y B, $A \subseteq B$ si y solo si para cualquier $a \in A$ se satisface que $a \in B$, lo cuál se considera parte de la Acción A_1. En el caso de E3, el procedimiento inicial realizado por la estudiante es correcto, sin embargo, la estudiante no reconoce que para poder concluir que el subespacio es o no invariante debe determinar si $T(ax^2) \in span\{x^2\}$, en su respuesta tampoco escribe explícitamente cual es el subespacio que es invariante bajo la transformación; la ausencia de esta parte de la acción lo clasifica como acción en desarrollo o pre-acción.

De manera similar se analizan las respuestas para las demás concepciones de la descomposición genética para su evaluación.

Discusión y conclusiones preliminares

En términos generales la valoración de la secuencia 2 permitió evaluar tanto las estructuras mentales propuestas a partir del análisis teórico, como el diseño e implementación de enseñanza correspondiente a la Acción A_1 y proceso P_1. La continuación del análisis permitirá evaluar el resto de la descomposición genética y el diseño de enseñanza.

La clasificación de las respuestas en términos de las concepciones mostradas permitió identificar las estructuras mentales correctamente desarrolladas y las que estaban emergiendo, como en los casos de pre-acciones, en algunos casos de forma no compatible con la descomposición genética preliminar.

En relación con los reactivos de las secuencias, las respuestas mayormente se clasificaron como compatibles con la estructura mental descrita, aunque en varios casos las estructuras se manifiestan en etapa de desarrollo (Arnon et al., 2014, p. 139), esto se observó primordialmente en respuestas en las que se trabajaba con las representaciones gráficas, los estudiantes parecían entender y poder trabajar la situación de acuerdo con la estructura mental requerida, sin embargo les costaba dar una argumentación suficientemente clara que validará lo que decían; los estudiantes mostraron mejores argumentaciones cuando el trabajo que se solicitaba involucraba representaciones algebraicas. Esta situación se asoció a que los estudiantes que participaron en el estudio se encontraban poco familiarizados con las representaciones gráficas.

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ASSESSING STUDENTS' UNDERSTANDING OF FRACTION MULTIPLICATION

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The purpose of this study is to investigate the range of strategies fifth graders used to solve a word problem involving fraction multiplication. We report a detailed qualitative analysis of elementary students' written work (N = 1472). The results demonstrate that students collectively use a wide range of strategies for fraction multiplication. Implications for teaching and learning are discussed.

Keywords: Number Concepts and Operations; Rational Numbers; Assessment and Evaluation

Perspectives

Kieren (1976) stated five subconstructs to define fractions: part-whole, ratio, quotient, operator, and measure (see also, Behr et al., 1992). A comprehensive understanding of the rational numbers demands students to be familiar with interpretations of various subconstructs as well as understand their interaction (Ball, 1993; Behr et al., 1983; Lamon, 2007, 2012; Ni, 2001). Teaching the algorithm for multiplying fractions seems easy (Johanning, 2019; Reys et al., 2007) but the conceptual underpinnings are complex (Tirosh, 2000; Tsankova & Pjanic, 2009). Usually, additive operations require dealing with one fractional unit, while multiplicative operations involve interaction between multiple units. A problem like 'Ben has 1/3 of a cup of sugar. He sprinkles 1/2 of the sugar onto brownies. How much sugar does Ben sprinkle?' requires – (i) coordination in the units involving '1/3 cup a sugar' with '1/2 of the 1/3 cup' and (ii) choice of a specific arithmetic operation.

This study explores students' conceptions on fraction multiplication for a contextual problem. The results might guide elementary teachers to design strategic problems to capture the implicit conceptions of their students' reasoning. This paper describes the patterns in students' responses to capture the reasons for selecting a specific option by examining their written work. The main question guiding our research is *What is the range and distribution of strategies that students use to approach and solve a fraction multiplication problem?*

Context

We designed a task to address the Grade 5 standard (CCSS.MATH.CONTENT.5.NF.B.4) focusing on students understanding of multiplication related to multiplying a fraction by a fraction (Figure 1). The question included two fractions with different meanings: the fraction 5/8 representing a certain length and functioning *as a measure* of the distance between Levi's home and his school and the fraction 2/3, functioning *as a ratio* between the distance he has walked and the school-home distance. The distractors for this question were purposefully designed to assess the participants choice of operation: (a) 1/24 is the result of subtracting; (b) 10/24 can be obtained by multiplying and is the correct response; (c) 16/24 can be obtained after finding equivalent fractions with a common denominator; and (d) 31/24 can be obtained by adding.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

7. Levi lives $\frac{5}{8}$ mile from school. After he walked $\frac{2}{3}$ of the way to school, he met Marta. How far, in miles, had Levi walked when he met Marta?

A. $\frac{1}{24}$ miles B. $\frac{10}{24}$ miles C. $\frac{16}{24}$ miles D. $\frac{31}{24}$ miles

Figure 1: The Problem

Data and Methods

The data is drawn from a larger study from a representative sample of fifth-grade students (*N* = 1427) in a Midwestern State. Participation was voluntary, and students were given 15 minutes to work on eight multiple-choice questions. For this paper, we have focused on one question involving fraction multiplication (Figure 1). The written work of the students was examined using qualitative software, MAXQDA version 18.1.1 (VERBI Software, 2016).

Qualitative Analysis of Students' Written Work

Research suggests that students' written work provides valuable evidence of their mathematical strategies, reasoning, and confusions (Brizuela, 2005; Kamii et al., 2001). We used thematic analysis (Braun & Clarke, 2006) to code students' written work. To ensure consistency, two coders coded one class (*n* = 57) during a training session to identify similarities in students' work and developed a list of themes. Certain pragmatic agreements were made, for example, the code of 'no written work' was used both for blank entries and if the student scratched out or erased their written work. As another example, the code 'unclear explanation' was used if something were written but a clear idea could not be deciphered. The coders used the initial codebook to code three classes individually and agreed on 100% of the cases after discussion. The final codebook had five themes each with several sub-themes (Table 1). Each code was defined in a code book along with prototypical examples to create consistent use.

Results

The unit of analysis for this part of the study is a *student's response to one specific item on fraction multiplication*. To address the research question, we first present the distribution of themes and then summarize the information captured from these themes.

Table 1: List of Themes with Distribution of The Students

	=1472	%
Code 10 Students who selected option 10/24 (**correct response**)	**435**	**9.55**
• 10A: Used multiplication as operator ('*' or '.' or 'of')	157	0.67
• 10A(a): Reduced the fraction to 5/12	36	2.45
• 10A(b): Used multiplication as second operator choice	7	0.48
• 10B: Used drawings	2	0.14
• 10C: Used "wrong" or "no" arithmetic operator	13	0.88
• 10D: Incorrectly written work	2	0.14
• 10E: No written work	145	9.85
• 10F: Unclear explanation	41	2.79
• 10F(a): Showed understanding of making the same denominators	14	0.95
• 10F(b): Subtract fractions and select the one with same multiples	6	0.41
• 10F(c): Added fractions as 7/11 or 7/24	11	0.75
• Guess	1	0.07

Code 1 Students who selected option 1/24	**414**	8.13
• 1A: Made the denominators same and subtracted both	125	8.49
• 1A(a): Subtracted fractions but did not write ‘-’ in their work	36	2.45
• 1A*: Made the denominators same, flipped the numbers and subtracted	119	8.08
• 1B: Unclear explanation	54	3.67
• 1B(a): Subtracted fractions and have written 3/5 as answer	6	0.41
• 1B(b): Used other/multiple operations	6	0.41
• 1C: No written work	68	4.62
Code 16 Students who selected option 16/24	**247**	6.78
• 16A: Made denominators same; selected the one with large numerator	60	4.08
• 16A(a): Compared fractions but used ‘-’ in their written work	10	0.68
• 16B: Unclear explanation	36	2.45
• 16B(a): Used multiple/other operators	11	0.75
• 16B(b): Transformed either 5/8 to 15/24 or 2/3 to 16/24	17	1.15
• 16C: No written work	113	7.68
Code 31 Students who selected option 31/24	**158**	0.73
• 31A: Made the denominators same and added both	97	6.59
• 31A(a): Added fractions but did not write ‘+’ OR wrote ‘-’	22	1.49
• 31B: Unclear explanation	9	0.61
• 31B(a): Added numbers as 7/11	2	0.14
• 31C: No written work	28	1.90
Code O Other Responses	**218**	4.81
• O(A): Students have written ‘what’ or ‘IDK’ or ‘guessed’ or ‘?’	2	0.14
• O(B): No response with unclear or clear written work	23	1.56
• O(C): No response and NO written work	193	3.11

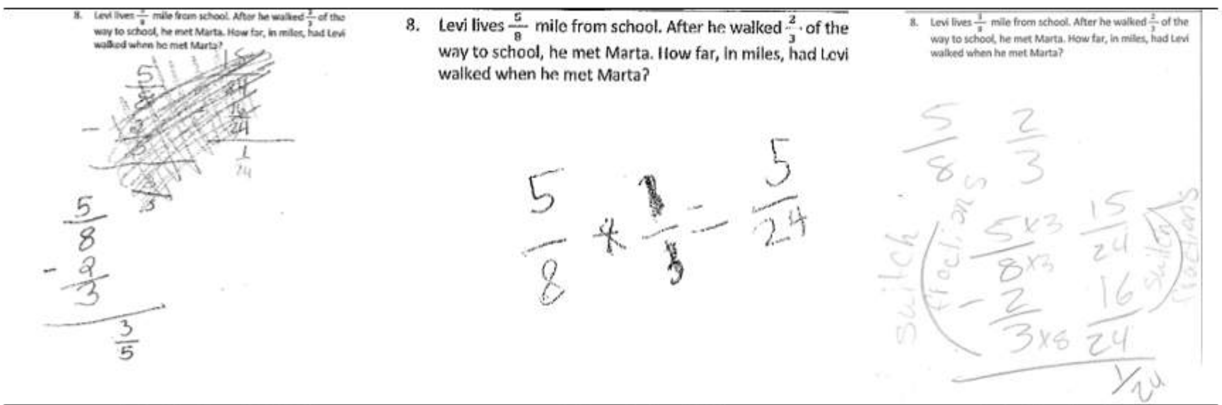
N: Number of Students; %: Percentage of Students

The students’ written work revealed their comprehension and conception of a fraction multiplication word problem. Around 435 students (29.55%) chose the correct option 10/24, but 62 (4.21%) of them had unclear explanations. Some students did not use the multiplicative operator as their first choice and employed a ‘guess and check’ strategy solving the task, e.g., making the denominator values the same, adding the fractions, etc. as their first attempt (**Code 10A(b)**, n = 7, 0.48%). **Code 10C** depicts the students (n = 13) who have either not used any or used ‘-’ as an arithmetic operator between 5/8 and 2/3. The reason of their selecting their operation is unknown to us but suggests avenues to be explored in the future using interviews.

Some students (**Code 10F(b),** n = 6, 0.41%) wrongly subtracted the fractions 5/8 and 2/3 as 3/5, and selected 10/24 (Figure 2(a)). A potential reason for this selection can be that the students might have realized that 3 and 5 are respective factors of 24 and 10. There is speculation in this inference but the best judgment we can make from their work. However, this code supported the idea of revisiting previously learned concepts because even if the subtracting fractions is a fourth-grade standard (CCSS.MATH.CONTENT.4.NF.B.3.A), misconceptions were visible in their present work. Similar reasoning has also been captured in **Code 1B(a)** (*n* = 6, 0.41%).

Some students demonstrated an advanced level of understanding by treating fractions as an operator. They considered 2/3 as 2*(1/3), multiplied 5/8 with 1/3 to get 5/24, and then doubled the output (Figure 2(b); **Code 10A**, n = 157, 10.67%). This shows a sophisticated level of reasoning as students changed the fraction into a unit fraction and then doubled it.

Many students chose the option 1/24 (**Code1;** *n* = 414, 28.13%) suggesting that they relied on a part-whole understanding of fractions. Students subtracted 5/8 and 2/3 after making the same denominator; this only makes sense if they were treating both as a distance. **Code1A*** (*n* = 119, 8.08%), an extension to **Code1A** (related to subtracting both fractions, *n* = 161, 10.94%), reflected a unique attribute of understanding where students made the denominators the same for both fractions and switched their order (Figure 2(c)). This code indicated students may have applied the rule of always subtracting the smaller number from the larger number. Similar reasoning was captured by **Code 31** (*n* = 158, 10.73%) where students interpreted this to be an addition problem.



(a) Code 10C(d) (b) Code 10A (c) Code 1A*

Figure 2: Examples of the Students' Work for Specific Codes

A few codes call for teachers' attention, for instance, **Code 16B(b)** (*n* = 17, 1.15%) depicts students using only one fraction to deduce the answer. One of the students mentioned, "I think it is 16/24 because 1/3 = 8/24 and he is 2/3, so you add and get 16/24". We also found that there were students who performed the correct calculation and obtained 5/12 but did not select any option, captured in the **Code O(B)**. Such students might have forgotten to mark an answer choice or potentially did not recognize equivalence between 5/12 and 10/24.

Discussion

The qualitative analysis of the students' written work helped in recognizing and identifying the strategies involved in answer choices. We find it plausible that students selected incorrect options because to answer correctly requires one to understand the different meanings of fractions and how to interpret the product (Wyberg et al., 2012). Instead, students often rely on their understanding of whole numbers (Tsankova & Pjanic, 2009; Wu, 2001). However, whole number strategies are not appropriate for finding the product of two fractions. The analysis shows that many students added or subtracted directly which implies they treated both fractions in the question (5/8 of a mile and 2/3 of the distance) as a measure. The students in this sample may lack understanding of a fraction as a ratio.

Previous researchers mentioned that instruction in the elementary classrooms is dominated by the part-whole interpretation of fractions (Ni & Zhou, 2005; Olanoff et al., 2014). Such instruction does not provide the conceptual understanding necessary to solve a problem involving fractions with a ratio meaning. We speculate that reinforcing the conceptual meaning behind all fraction sub-constructs might improve students' facility with fraction operations.

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STUDENTS' UNDERSTANDINGS OF THE TRANSFORMATIONS OF FUNCTIONS

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The goal of this study is to describe students' understandings of the transformations of functions in different representations based on an analysis of pilot interviews with two ninth and two twelfth grade students from the same urban public high school in Massachusetts, which serves a diverse community. Interview responses indicated that the students were unable to identify explicitly the type of transformation that described the relationship between two functions. Analyses of the interviews revealed that a student's flexibility in the use of representations of and approaches towards functions is an indicator of their understanding of functions and, therefore, that the ninth grade students interviewed have a less sophisticated understanding of functions than do the twelfth grade students interviewed.

Keywords: Algebra and Algebraic Thinking

Purpose of Study

Functions, as proposed by Schwartz and Yerushalmy (1992; Doorman & Drijvers, 2011; Schwartz, 1999; Zandieh et al., 2017), can be viewed as one of the fundamental objects of mathematics, and appears at all levels of the mathematics curriculum ranging from patterns in elementary school to real analysis in college mathematics. Landmark studies about the concept of function include, but are not limited to: (1) theoretical models on the development of the function concept; (2) teaching experiments that apply general theories to the specific concept of function; (3) students' and teachers' conceptions of functions; and (4) the use of technology in functions-based mathematics classes (e.g., Dubinsky & Harel, 1992b). The current study falls under the third category above, which includes more recent studies such as Ayalon and Wilkie (2019), Dubinsky & Wilson (2013), and Ronda (2015). It aims to contribute to this line of research by specifically analyzing and describing students' *understandings* of the *transformations* of functions in *different representations*. Thus far, researchers have identified that students experience difficulties with transformations of functions when asked to (a) visualize the transformations because of processing challenges with horizontal and vertical translations (Eisenberg & Dreyfus, 1994); (b) identify, graph and use transformations to solve problems because they have not interiorized the concept of function (Lage & Gaisman, 2006); and (c) translate functions because of cognitive and pedagogical obstacles (Zazkis et al., 2003). This study will describe some of the difficulties students encounter with transformations of functions, in particular, when asked to state the relationship, based on a transformation, between two given functions.

Theoretical Framework

Representations of Functions

There are various ways to represent a function. This study considers the algebraic, graphical, and tabular representations, which frequently occur in a high school math curriculum. The term algebraic representation refers to expressions or equations containing numbers and variables connected by mathematical operations. The term graphical representation refers to the Cartesian coordinate system, and the term tabular representation refers to a table of values displaying an input and an output.

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Conceptions of Functions

The most predominant distinction used for describing one's concept of a function is *process* versus *object*. A *process* conception of a mathematical concept is "a form of understanding of a concept that involves imagining a transformation of mental or physical objects that the subject perceives as relatively internal and totally under her or his control" (Dubinsky & Harel, 1992b, p. 20). An *object* conception of a mathematical concept is "a form of understanding of a concept that sees it as something to which actions and processes may be applied" (Dubinsky & Harel, 1992b, p. 19). One method for identifying one's conception of function is to consider one's approach towards functions, as is done in this study. One can have a pointwise or a global approach towards a function (Bell & Janvier, 1981; Janvier, 1978). For instance, if given the algebraic representation of a function and asked to create the graph, then a pointwise approach is to plot discrete points, and a global approach is to sketch the graph (Even, 1998).

Conceptions and Representations of Functions

A students' understanding of the concept of function can vary depending on the representation (Dubinsky & Harel, 1992a; Moschkovich et al., 1993). This is likely because the tabular representation is composed of discrete data points and requires a pointwise (process) approach, and the algebraic and graphical representations can be manipulated discretely (pointwise/process) or in their entirety (global/object). Further, to be able to translate between representations is associated with being able to transition between approaches (Even, 1998), and it is important to be able to move flexibly between representations and understandings (Moschkovich et al., 1993).

Methods

Participants

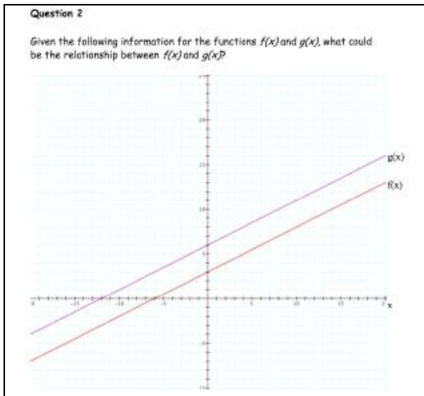
The participants included two high-performing ninth graders (Student 9-1 and 9-2) and two high-performing twelfth graders (Student 12-1 and 12-2), from the same public high school in an urban center in Massachusetts, which serves a diverse community. The ninth graders were learning basic algebra principles such as order of operations, as well as statistical concepts such as box plots; the twelfth graders were learning about polynomial functions and their characteristics. The students were selected by their mathematics teachers to participate in this pilot study based on their excellent grades and high skill level in their current mathematics class.

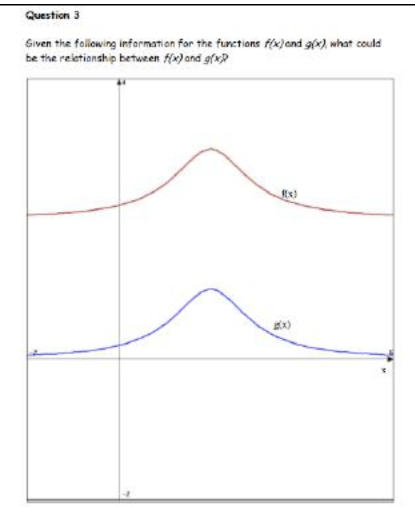
Individual Interviews

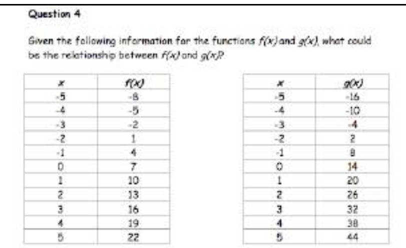
The individual interviews piloted seven questions pertaining to the definition of a function, and the transformations and comparisons of functions in different representations; however, this paper will focus only on the responses to the three interview questions regarding the transformations of functions (Questions 2, 3, and 4 – see Figure 1). In these questions, the participants were asked to state the relationship between the two given functions. The interviews were videotaped and lasted between 15 and 60 minutes, depending on the participant.

Analysis

The participants' responses to Questions 2, 3, and 4 were analyzed based on Figure 2, which was adapted from Moschkovich et al.'s (1993) study, and includes Even's (1998) approaches to functions – pointwise or global. If a participant's response was considered accurate, then it was coded with a 1, and if it was not, then it was coded with a 0. Also, the "most anticipated" response cells appear in boldface for each participant. The "most anticipated" responses were chosen based on two criteria: (1) it was in the representation in which the question was posed, and (2) it used an approach aligned with the representation. More specifically, for the second criterion, the graphical representation tends to evoke the object conception of function (Schwartz & Yerushalmy, 1992), while the tabular representation relies on discrete data points and, therefore, is more closely related to the process conception of function. Thus, each participant could have a "most anticipated" response, as well as accurate, non-anticipated responses.






Question 4

Given the following information for the functions f(x) and g(x), what could be the relationship between f(x) and g(x)?

x	f(x)
-5	-8
-4	-5
-3	-2
-2	1
-1	4
0	7
1	10
2	13
3	16
4	19
5	22

x	g(x)
-5	-16
-4	-10
-3	-4
-2	2
-1	8
0	14
1	20
2	26
3	32
4	38
5	44

Note: Functions can be expressed in other representations, and relationship is a translation.	Note: Functions cannot be expressed in other representations, and relationship is a translation.	Note: Functions can be expressed in other representations, and the relationship is a dilation.

Figure 1: Interview Questions

	Representation		
Approach	*Tabular (T)*	*Graphical (G)*	*Algebraic (A)*
Pointwise			
Global			

Figure 2: Framework for Examining Representations and Approaches to Functions

Results

Response Samples

Question 2. Student 9-2 was unsure as to what the question was asking and kept referring to the graphs as being representative of information usually seen in a table or as an equation. The student then pointed out that the two lines are parallel, and deduced the slope of each function from their respective graphs, which was found to be the same. The student continued to examine the two functions and pointed out that they both have different x- and y-intercepts, and finally concluded, "They're both the same, I guess. They're both the same lines…just in different positions … One's higher, and one's lower. They're placed…they're the same…the same two lines…just placed different on the axis."

Question 3. Student 12-1 started by examining the graphs and stating that if the functions were expressed in tabular form, then the x-values would be the same for each function, but the output values would be different by a power or a multiple. The student showed great difficulty in explaining the relationship but was able to say, "The difference between the y-axis on either graph is going to be the number that you are going to be ... adding or multiplying to [g(x)]."

Question 4. Student 12-2 displayed high confidence in responding to the question and stated that "There's…you can find the difference between the two if you have this, and you say, …g(x)…, …x and y table is, um…in order to get that, all you have to do is multiply by two. Then, you would…then you could easily find g(x). And, you could plot it out. And, you could discover the, um…the slope…and the y-intercept. And, you could find out the equation."

Response Summary

The analysis of the participants' responses to Questions 2, 3, and 4 is summarized in Table 1.

Table 1: Results of Questions 2, 3, and 4

		Question 2			*Question 3*			*Question 4*		
		Representation								
Participant	*Approach*	*T*	*G*	*A*	*T*	*G*	*A*	*T*	*G*	*A*
Student 9-1	*Pointwise*	*0*	*1*	*0*	*0*	*1*	*0*	*1*	*0*	*0*
	Global	*0*	*1*	*0*	*0*	*1*	*0*	*0*	*0*	*1*
Student 9-2	*Pointwise*	*0*	*0*	*0*	*0*	*0*	*0*	*1*	*0*	*0*
	Global	*0*	*1*	*1*	*0*	*1*	*0*	*0*	*0*	*0*
Student 12-1	*Pointwise*	*0*	*0*	*1*	*0*	*1*	*0*	*1*	*0*	*0*
	Global	*0*	*0*	*1*	*0*	*1*	*0*	*0*	*0*	*0*
Student 12-2	*Pointwise*	*0*	*0*	*1*	*0*	*0*	*0*	*1*	*1*	*1*
	Global	*0*	*0*	*1*	*0*	*1*	*0*	*0*	*0*	*1*

Discussion

Interview responses indicated one significant observation in terms of students' understandings of *transformations*: the students were unable to identify explicitly the type of transformation that described the relationship between any of the pairs of functions but were able to use other descriptive words for the same. More specifically, none of the students used the terms translation, dilation, or transformation in response to any of the questions. Instead, they used words such as "higher" or "lower" on the graph to describe a translation, and "multiplied" to describe a dilation, which indicates a lack of mathematical vocabulary. Also, Student 12-1 was unable to specifically determine if the functions in Question 3 represented a translation or a dilation, which indicates a lack of understanding of transformations. These observations regarding difficulties with transformations need to be substantiated with more research.

Analysis of the interviews highlighted two significant findings in terms of students' *understandings* of functions in different *representations*: (1) students were able to be flexible between representations for functions, moving from one to another even though the question they were answering was presented in a single type of representation; and (2) students were able to be flexible between approaches towards and conceptions of functions, providing evidence that they could approach single questions and functions embedded in them in both a pointwise (process) and global (object) way. Given our assumption that moving across representations of functions and adopting both pointwise and global approaches towards functions provides evidence for students' flexibility and therefore greater sophistication in their understandings about functions ((Even 1998; Moschkovich et al. 1993), these findings lead us to the preliminary conclusion that the ninth grade students interviewed have a less sophisticated understanding of the concept of function than do the twelfth grade students interviewed because neither ninth grade student exhibited any flexibility between representations and only one exhibited flexibility between approaches. This preliminary conclusion needs to be substantiated with further research.

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DIDACTIC SEQUENCE FOR THE LEARNING OF PLACE VALUE IN SECOND GRADE ELEMENTARY STUDENTS IN A MULTIGRADE SCHOOL

SECUENCIA DIDÁCTICA PARA EL APRENDIZAJE DEL VALOR POSICIONAL EN ALUMNOS DE SEGUNDO CICLO DE PRIMARIA DE UNA ESCUELA MULTIGRADO

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This study addresses the teaching of mathematics in a multigrade school through the design of a didactic sequence for the learning of place value. Based on the theory of didactical situations and didactic engineering, we designed a sequence of six didactic situations on place value that was implemented on a group of second grade elementary students (ages 8-9). The sequence encourages students to perform numerical decomposition, decimal grouping, and number ordering with different numeric ranges, to construct meanings about place value in a game-oriented way.

Keywords: teaching activities and practices, elementary education, rural education.

Background

In a multigrade elementary school, students from different grades are grouped in a classroom to be taught by the same teacher. These schools are in geographically inaccessible and sparsely populated communities, where the number of students is small and, therefore, it is not feasible to have a full-organization school. Teachers in these schools also perform management and administrative tasks (INEE, 2019).

Various studies reveal the difficulties faced by teachers and students in multigrade schools in the learning and teaching process; for example, the reduced time for teaching as teachers must perform management, administrative or janitorial functions. It is common for traditional teaching methods to predominate in these schools due to the lack of a suited curricular proposal for this modality (Reséndiz, Block and Carrillo, 2017).

Nonetheless, it has also been noted that the multigrade modality has favorable characteristics for learning, such as the possibility for students to learn from their peers as well as a greater flexibility to assign tasks to students based on their level of performance. However, for these features to become really favorable for learning, teachers must possess teaching and organizational strategies (Block, Ramirez and Reséndiz, 2015).

According to Santos (2011), the teaching events that occur within multigrade classrooms point to diversity; that is, the diversification of teaching activities, with criteria of concurrency and complementarity, abandoning unique, synchronized, and standardized didactic practices. At the same time, it is possible to observe the circulation of knowledge, which means going beyond the formalities of school grades for knowledge to flow.

This study focuses on the teaching of mathematics in primary education in multigrade classrooms, specifically on the learning of place value in the Decimal Numbering System (DNS). DNS is an instrument of measure of other mathematical learnings, so understanding the DNS is decisive in the subsequent school trajectory of students (Terigi and Wolman, 2007).

Understanding the DNS promotes the development of numerical sense, skill and reflection in arithmetic operations, mental calculation and estimates (Angulo, 2017; Galicia and Uzuriaga, 2015). The concept of place value is indispensable for the construction of the DNS. Place value is the value that a digit takes according to the position it occupies within the number (units, tens, hundreds).

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For the understanding of the DNS se should favor the understanding of all figures in a quantity to assign a name to the figures according to the order of location, that is, the value that each figure acquires within a number to represent the quantity described (Gallego and Uzuriaga, 2015).

In order for students to understand place value within a quantity, we proposed that a teaching sequence be designed to encourage students to discover the rules underlying the DNS. The question guiding this research is: How to promote the learning of place value in second cycle students (third or fourth grade) of a multigrade elementary school? Therefore, our goal is to design and implement a teaching sequence that favors this learning.

Theoretical Framework

The design of the didactic sequence is based on elements of the theory of didactic situations (Brousseau, 2007) and didactic engineering (Artigue,1995).

The design of didactic situations and the manipulation of teaching variables such as numerical range, allows us to approach a medium through which students interact with place value in all its complexity. According to Terigi and Vulture (2013) in the usual teaching of the DNS, the numbers are taught one by one, starting with the digits and respecting the order of the series. Ranges are established to sequence the teaching of numbers according to the years of schooling of the students: from 1 to 100 in first grade, up to 1000 in second, and so on. Since the beginning and along with the presentation of number 10, the notions of units and tens are incorporated. Therefore, one hypothesis that is incorporated into the design of didactic situations is that students can identify regularities and discover the recursion of the grouping, if an interaction with the SND through wide numerical ranges is allowed. Didactic situations, therefore, favor decimal grouping, numeric decompositions, number ordering and comparison, in different numeric ranges.

Some didactic sequences are designed and taught in a game-oriented environment, that is, games involving numbers. Fuenlabrada, Block, Balbuena and Carvajal (1992) state that "a good game allows you to play with little knowledge but, to start winning systematically, it requires the building of strategies

that imply the need of greater knowledge" (p.5). Therefore, didactic situations in a game-oriented context allow students to approach the situation with the knowledge they possess on place value and for meaning to be built through the construction of strategies.

Methodology

For the design of the didactic situations that make up the sequence, we used the phases of the didactic engineering (Artigue, 1995).

Situation 1 *Cashier 1* (Fuenlabrada, Block, Balbuena and Carvajal, 1992) consists of a set of dice and chips where students, organized into teams, take turns to throw the dice and ask a student playing the role as a Cashier for different color chips according to the number they obtain. Blue chips represent units, red ones represent tens, and yellow ones represent hundreds. Students must group the chips by tens and they trade their chips so as to obtain the yellow chip. The student who first gets the yellow chip wins.

In scenario 2, *Cashier 2*, students play with a board with blue, red and yellow chips. They add up 5 numbers that the *Cashier* shows them and record the sums on the board marking with the blue chip the position of the units, the tens with the red ones and the hundreds with the yellow ones. Whoever gets the sum of the five numbers correct wins.

Situation 3, *Math Bingo,* (Perez, 2016), is a game in which students have a card with several 5-digit numbers and they must place the value of one digit in one of those numbers. A number from 0 to 9 is drawn out of a tombola and, at the same time, place value: units, tens, hundreds, thousands, or tens of thousands is picked out of a deck of cards. The student will recognize the place and place a mark in one of the numbers to indicate the position.

Situation 4, *Broken Calculator 1*, (Galvez, Navarro, Riveros, Zanocco, 1994) is about students playing with a calculator, typing only the keys 1, 0, +, to create a number. In this situation students play with a number range from units to hundreds (1-999), so that the student can play without complications.

In situation 5 called *Decomposed Calculator 2,* a numerical range of hundreds to tens of thousands (100-99000) is worked to observe the procedures of students when facing the numbering system without restrictions.

Finally, in situation 6 *Clothes Line,* students compare numbers, previously selected, so that they can choose between the largest or smallest. Whoever has the highest or lowest number wins, depending on which one they chose at the beginning of the game. Students will then place the numbers on a rope sorted as they are directed, from highest to lowest or vice versa.

A multigrade rural primary school was chosen in the state of Guanajuato, Mexico, for the implement of the sequence. The school has four teachers who attend all students. The classroom is made up of third grade (8-year-old) and fourth grade (9-year-old) students

The community in which the school is located has a population of 311. The locals work in the fields or manufacturing factories.

A first approach to the group was made and an initial didactic situation of another topic was applied in order to identify the characteristics of the students and their level of participation with the researcher. The implementation of the teaching sequence is currently being carried out, recording each situation in audio and video for its subsequent analysis.

Discussion and results

The teaching sequence is intended to engage students in various contexts in which they can build meanings of about place value. As the research is in the implementation stage, here we identify some procedures of students that show how the sequence functions.

The use of teaching materials such as chips, calculators, bingo boards and dice, allows students to experience new ways of approaching mathematical activities. The situation that is created, in terms of the theory of didactic situations, stimulates the senses of the students and allows them to interact with their peers, building and validation their solution strategies.

The knowledge, procedures and skills developed by students is not determined by their age, which is associated with their school level. The curriculum indicates that the contents and learning in fourth grade (9-year-olds) are different because the level of complexity is higher, for example, the number ranges are wider than in third grade (8-year-olds). Nonetheless, the interaction between different grade students in the same classroom does not impede the construction of knowledge (Santos, 2011). On the contrary, students interact in a natural way and their arguments are validated collectively. Knowledge is generated in a didactic situation, designed from the curricular contents of the two levels.

The results of this study are located in the field of didactics of mathematics in multigrade schools and aim to contribute to improve teaching in this educational context.

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SECUENCIA DIDÁCTICA PARA EL APRENDIZAJE DEL VALOR POSICIONAL EN ALUMNOS DE SEGUNDO CICLO DE PRIMARIA DE UNA ESCUELA MULTIGRADO

DIDACTIC SEQUENCE FOR THE LEARNING OF PLACE VALUE IN SECOND GRADE ELEMENTARY STUDENTS IN A MULTIGRADE SCHOOL

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Este estudio aborda la problemática de la enseñanza de las matemáticas en la escuela multigrado a través del diseño de una secuencia didáctica para el aprendizaje del valor posicional. Con base en la teoría de situaciones didácticas y la ingeniería didáctica, se diseñó una secuencia de seis situaciones didácticas sobre el valor posicional, que se implementó con un grupo de alumnos de segundo ciclo de primaria (8-9 años). La secuencia promueve que los alumnos realicen actividades de descomposición numérica, agrupación decimal y orden de los números, con disntintos rangos numéricos, para construir significados sobre el valor posicional, en un contexo lúdico numérico.

Palabras clave: actividades y prácticas de enseñanza, educación primaria, educación rural.

Antecedentes

En la escuela primaria multigrado se agrupan dos o más grados escolares, en un mismo grupo, para ser atendidos por un mismo docente. Estas escuelas están presentes en comunidades geográficamente poco accesibles y con escasa población, en las que el número de alumnos es reducido y no viable contar con una escuela de organización completa. Los docentes de estas escuelas también realizan tareas directivas y administrativas que la gestión escolar implica (INEE, 2019).

Diversas investigaciones revelan las dificultades en los procesos de enseñanza y aprendizaje en estas escuelas, por ejemplo, la reducción de tiempos de enseñanza, pues los docentes realizan funciones directivas, administrativas o de aseo escolar. Es común que predominen los métodos tradicionales de enseñanza debido a la falta de una propuesta curricular para esta modalidad (Reséndiz, Block y Carrillo, 2017).

También se ha señalado que la modalidad multigrado tiene características favorables para el aprendizaje, como la posibilidad de que los alumnos aprendad unos de otros, o la mayor flexibilidad para asignar tareas a cada uno en función de su nivel de desempeño. Para que estas características se vuelvan realmente favorables para el aprendizaje, es necesario disponer de estrategias didácticas y de organización que lo hagan posible (Block, Ramírez y Reséndiz, 2015).

De acuerdo con Santos (2011), los acontecimientos didácticos que ocurren al interior de las aulas multigrado apuntan a la diversidad, esto es, la diversificación de actividades de enseñanza, con criterios de simultaneidad y complementariedad, abandonando las prácticas únicas, sincronizadas y uniformizadas en sus mecanismos. Al mismo tiempo, es posible observar la circulación de los saberes, lo que supone abrir las formalidades de los grados escolares para que los saberes fluyan.

La problemática de este estudio se centra en la enseñanza de las matemáticas en la educación primaria en aulas multigrado, específicamente en el tema de valor posicional del Sistema de Numeración Decimal (SND). El SND constituye el instrumento de mediación para otros aprendizajes matemáticos, por lo que el aprendizaje de este objeto matemático es decisivo en la trayectoria escolar posterior de los alumnos (Terigi y Wolman, 2007).

La comprensión del SND favorece el desarrollo del sentido numérico, la habilidad y reflexión en operaciones aritméticas, cálculo mental y estimaciones (Angulo, 2017; Gallego y Uzuriaga, 2015) y el concepto de valor posicional es indispensable para la construcción del SND. El valor posicional es el valor que toma un dígito de acuerdo con la posición que ocupa dentro del número (unidades, decenas, centena).

Para la comprensión del SND se debe favorecer la comprensión de todas las cifras en una cantidad para asignar un nombre a las cifras de acuerdo al orden de ubicación, es decir, el valor que adquiere cada cifra dentro de un número para representar la cantidad descrita (Gallego y Uzuriaga, 2015).

Para que los alumnos comprendan el valor posicional de las cifras dentro de una cantidad, se propone el diseño de una secuencia didáctica que favorezca que los alumnos descubran las reglas subyacentes al SND. Por lo tanto la pregunta de esta investigación es ¿cómo favorecer el aprendizaje del valor posicional en alumnos de segunco ciclo (tercero y cuarto grado) de una escuela primaria multigrado? De modo que nuestro objetivo es el diseño e implementación de una secuencia didáctica que favorezca este aprendizaje.

Marco Teórico

El diseño de la secuencia didáctica se fundamenta en elementos de la teoría de las situaciones didácticas (Brousseau, 2007) y en la ingeniería didáctica (Artigue,1995).

El diseño de las situaciones didácticas y la manipulación de las variables didácticas como el rango numérico, permite aproximarnos a un medio en el que los alumnos interactúen con el valor posicional en toda su complejidad. De acuerdo con Terigi y Buitron (2013) en la enseñanza usual del SND, se enseñan los números uno por uno, comenzando por los dígitos y respetando el orden de la serie. Se establecen cortes para secuenciar la enseñanza de los números según los años de escolaridad de los alumnos: de 1 a 100 en primer grado, hasta 1000 en segundo, y así sucesivamente. Desde el inicio y junto con la presentación del número 10, se incorporan las nociones de unidades y decenas. Por lo tanto, una hipótesis que se incorpora al diseño de las situaciones didácticas es que los alumnos pueden identificar regularidades y descubrir la recursividad del agrupamiento, si se permite una interacción con el SND, a través de rangos numéricos amplios. El medio de las situaciones didácticas favorece agrupaciones decimales, descomposiciones numéricas, establecimiento de orden y comparación de números, en diferentes rangos numéricos.

Algunas situaciones didácticas están contextualizadas en un entorno lúdico-numérico, es decir, juegos que involucran números. Fuenlabrada, Block, Balbuena y Carvajal (1992) afirman que "un buen juego permite que se pueda jugar con pocos conocimientos, pero, para empezar a ganar de manera sistemática, exige que se construyan estrategias que implican mayores conocimientos" (p.5). Por tanto, el medio de las situaciones didácticas, en el contexto lúdico-numérico, permite que los alumnos se aproximen a la situación con los conocimientos que tienen sobre el valor posicional y a través de la construcción de estrategias ganadoras, puedan construir significados.

Metodología

Para el diseño de las situaciones didácticas que conforman la secuencia, se utilizan las fases de la ingeniería didáctica (Artigue, 1995).

La situación 1 demoninada *Cajero 1* (Fuenlabrada, Block, Balbuena y Carvajal, 1992) consiste en un juego de dados y fichas en donde los alumnos, organizados en equipos, tiran los dados por turnos y de acuerdo al número que obtienen, piden fichas de colores a quien tiene el rol de cajero. Fichas azules representan unidades, fichas rojas representan decenas y fichas amarillas representa centenas. Los alumnos agruparán de diez en diez, realizarán equivalencias entre las fichas para cambiar fichas azules por fichas rojas y fichas rojas por amarillas. Gana el alumno que obtenga primero la ficha amarilla.

En la situación 2 denominada *Cajero 2*, los alumnos juegan con un tablero, con fichas de color azul, rojo y amarillo. Suman 5 números que les muestre quien tenga el rol de cajero y registrarán las sumas en el tablero, marcando con la ficha azul la posición de las unidades, con la roja las decenas y con la amarilla las centenas. Gana quien obtenga el resultado correcto de la suma de los cinco números mostrados.

La situación 3 denominada *Bingo matemático* (Pérez, 2016), es un juego en el que los alumnos tienen una tarjeta con varios números de 5 cifras y tienen que ubicar el valor de un dígito en uno de esos número. De una tómbola de números se toma uno del 0 al 9 y, al mismo tiempo, de un mazo de tarjetas se toma el nombre de una posición: unidades, decenas, centenas, unidades de millar, decenas de millar. El alumno reconocerá y pondrá una marca en la posición indicada, en alguno de los números de su trajeta.

La situación 4 denominada *Calculadora descompuesta 1* (Gálvez, Navarro, Riveros, Zanocco, 1994) consiste en jugar con una calculadora, tecleando sólo las teclas 1, 0, +, = para formar cualquier número. En esta situación se juega con un rango numérico que abarca de unidades a centenas (1-999), para que el alumno pueda acceder a la dinámica de juego sin complicaciones.

En la situación 5 denominada *Calculadora descompuesta 2*, se trabaja con un rango numérico de centenas a decenas de millar (100-99000) para observar los procedimientos de los alumnos al enfrentarse al sistema de numeración sin restricciones.

Finalmente, la situación 6 denominada *tendedero matemático* consiste en un juego en el que los alumnos comparan números, seleccionados previamente, con la intención de que puedan elegir el mayor o menor según sea el caso. Gana quien tenga el número mayor o menor, según se elija al inicio de la jugada. Luego, los alumnos colocarán los números en un listón o cuerda, ordenados según se les indique, de mayor a menor o viceversa.

Se eligió una una escuela primaria rural multigrado en el estado de Guanajuato, México, para la implemetación de la secuencia. La escuela cuenta con cuatro docentes que atienden todos los alumnos. El segundo ciclo, agrupa a los grados de tercero (8 años) y cuarto (9 años).

La comunidad en la que se ubica la escuela tiene 311 habitantes, que en su mayoría se dedican a labores del campo y al trabajo como obreros en fábricas manufactureras.

Se realizó un primer acercamiento al grupo y se aplicó una situación didáctica inicial de otro tema, para conocer las características de los alumnos y su nivel de participación con la investigadora. Actualmente se está realizando la implementación de la secuencia didáctica, registrando en audio y video cada situación didáctica, para su posterior análisis.

Discusión y resultados

La secuencia didáctica pretende favorecer el acercamiento de los alumnos con diversos contextos en los que puedan construir significados del valor posicional. Como la investigación se encuentra en fase de implementación, se identifican hasta el momento, algunos procedimientos de los alumnos que muestran el funcionamiento de la secuencia didáctica.

Por otra parte, el uso del material didáctico como las fichas, calculadora, tableros de bingo y dados, permite que los alumnos experimenten nuevas formas de acercarse a la actividad matemática. El medio que se construye, en términos de la teoría de situaciones didácticas, estimula los sentidos de los alumnos y les permite interactuar con sus compañeros, construir y validar sus estrategias de solución.

Los conocimientos, procedimientos y habilidades que desarrollan los alumnos no están determinados por sus edades, asociadas al grado escolar. El currículo indica que son diferentes los contenidos y el aprendizaje en cuarto grado (9 años), pues es mayor el nivel de complejidad, por ejemplo, los rangos numéricos más amplios que en tercer grado (8 años). Sin embargo, la interacción entre grados en el mismo grupo no fragmenta o imposibilita la circulación de saberes (Santos, 2011). Por el contrario, los alumnos interactúan de manera natural y sus argumentaciones se validan en colectivo. Se genera una circulación de saberes en una situación didáctica, diseñada desde los contenidos curriculares de los dos grados que se atienden en el mismo grupo.

Los resultados de este estudio se ubican en la línea de investigación de didáctica de las matemáticas en multigrado y pretenden contribuir a las propuestas para mejorar la enseñanza en esta modalidad educativa.

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INTERTEXTUALITY AND SEMIOSIS PROCESSES IN THE ALGEBRAIC RESOLUTION OF VERBAL PROBLEMS

INTERTEXTUALIDAD Y PROCESOS DE SEMIOSIS EN LA RESOLUCIÓN ALGEBRAICA DE PROBLEMAS VERBALES

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In this research, we used the Local Theoretical Models as a theoretical and methodological frame and Pierce's theoretical perspective on semiotics. At the end of the study carried out with secondary school students, authors identified strata of intertextuality derived from the processes of semiosis, based on acts of reading/transformation of texts or reading/translation/transformation of the textual spaces proposed in the teaching model through solving verbal problems using grade two equations with an unknown, adopting the Mixed Method (geometric and algebraic), and the Cartesian Methods that include algebraic procedures with paper and pencil and a symbolic calculator with a Computation Algebraic System.

Keywords: Intertextuality, semiosis processes, sense production, verbal problems.

There are various difficulties that secondary school students face during the process of learning algebra; for instance, they fail to identify the structures underlying algebraic expression, surface structure, and systemic structure as in Kieran (1989). Also, in Filloy & Córdoba (2013), Córdoba (2016), authors assert that it is usual when working with algebraic expression, that occur several types of syntax errors, either arithmetic or algebraic, or when solving equations of grade one (EG1) and of grade two (EG2), in such a way that students cannot be considered competent users of a Mathematical Sign System (MSS) yet. They are in the process of reading the mathematical text correctly and distinguishing the allowed transformations from those that are not.

Some researchers (Gallardo, 2002; Kieran, 2006; Solares, 2007; Rojano, 2010; Martínez, 2012; Filloy & Córdoba, 2013; Bonilla, 2014; Córdoba, 2016) consider that in the transition from arithmetic to algebra it is essential that secondary school teachers recognize as fundamental that students learn the properties and relationships of the arithmetic MSS (MSS1) and that the incomplete knowledge of these implies operational difficulties with MSS of algebra (MSS2).

Purpose of the study

The purpose of the qualitative experimental study carried out by Córdoba (2016) was to observe and analyze the semiosis processes of the students (study subjects), as a result of reading/transforming mathematical texts, for the production of meaning in the process of algebraic solving of verbal problems (VP) by EG2 with an unknown.

Theoretical framework

The theoretical and methodological framework proposed by Filloy (1999), for experimental observation in Educational Mathematics, called Local Theoretical Models (LTMs), allows us to account for the processes that are developed when certain specific mathematical contents are taught within the National Educational System to some students, which must be relevant to the phenomena under study; LTMs contemplate four interrelated components and their corresponding models: 1) Teaching Model, 2) Model of Cognitive Processes, 3) Model of Formal Competence, and 4) Model of Communication. Filloy (1999) defines a Teaching Model (TM), as a sequence of texts produced by both the teacher and the student, and these texts are the result of the work of both in problem teaching situations - which are taken as spaces textual. "In algebra, textual spaces are made up of Mathematical Sign Systems whose codes and traditions come from the meanings attributed to them for their social use". (Filloy, Rojano & Puig, 2011).

From Pierce's theoretical perspective on semiotics, what is a sign? For Peirce the fundamental thing in the sign is its function, when describing the semiotic process, rather than signs, it refers to sign functions. That sign, vital, constant, and meaningful process is the semiosis. Elizondo (2012). We find it relevant to differentiate text and textual space (TS), which corresponds to the distinction between meaning and sense. It is also relevant to understand the use that teachers and students assign to mathematical text. Filloy affirms that the notion of text is introduced to be used in the analysis of any practice of meaning production. (Filloy, Rojano, & Puig, 2008; Kieran & Filloy, 1989). A text is the result of the reading/transformation made on an TS, whose purpose is to produce sense and only extract meaning in that space (Filloy, Rojano, & Puig, 2008). Textual space is a system that imposes a semantic restriction on the person who reads it; the text is a new articulation of that space, individual and unrepeatable, made by a person as a result of an act of reading. Thus, an TM as a component of an LTM (Filloy, 1999) is a succession of texts that are taken as an TS to be read/transformed into another ET as the learner gives meaning in his readings (Rojano, 2010).

The MC process involves putting a problem into equations or a process of translating the given VP statement in natural language into algebraic language. On one hand, to accomplish an analytical reading that prepares the text of the problem by producing another text that in a certain way is in course to be translated into algebraic language. On the other hand, to work at the level of expression in the algebraic language that transforms the translated text (equation) into a text that can be solved, that is, the equation in its canonical form. (Puig, 2012).

Research Method

For this research, we designed and developed an LMT composed of the four models referred to. In the TM, we recovered elements for the didactic use of syntax errors in the development of algebraic thinking. From the study of Córdoba (2005); we observed with a clinical interview with teaching (ECT) situation, the performances of nine third graders of Secondary Education in the State of Mexico; but as a matter of space we will only refer to the case of Fer, when solving VP through EG2 using pencil and paper procedures and CAS as a symbolic manipulator. It was also used their modeling through the use of a didactic material called Algebraic Puzzle proposed by Larrubia (2005), which allowed analyzing the interaction of the learners with the MSS1, the MSS2 and the TM.

Results

In his doctoral thesis, Córdoba (2016) identified three types of intertextuality strata derived from the semiosis processes; it also described three levels of competence of MSS2 use (low, intermediate and high) of each; the above, from the ECT with students, in which the author asked them to solve VP using EG2 with an unknown, which are part of the TM. Below are the general characteristics of the intertextuality strata.

The first intertextuality stratum: *Reading/writing of texts in a network of textual relationships,* refers to the acts of reading/writing of algebraic text that a specific reader does base on a network of textual relationships derived from his/her prior mathematics and linguistics experiences. In particular, in this study, emerge three levels of MSS2 use competence of learners when asked them to solve EG2 with one unknown.

The second intertextuality stratum: *Reading/transformation of superficial structure of algebraic texts,* refers to the way the learners solve the given EG2 (complete and incomplete) from the recognition of shallow structure. Learner performed algebraic transformations between algebraic texts at the level of algebraic expression; this also require the learner to identify the shallow structure of the equation and its transformation from the factored form to the canonical form or vice versa.

The third intertextuality stratum: *Reading/translating/transformation of text,* refers to the algebraic resolutions of a VP through EG2 with one unknown; in this regards, it is common to use MC, defined as the process to pose a problem as an equation, which implies the process of translation and the algebraic transformations describe in second intertextuality stratum.

A brief description and analysis of a clinical interview episode with teaching, conducted by the Researcher (R) with Fer (F), is included to exemplify the third intertextuality stratum "Reading/translation/transformation of texts" in the resolution of a VP.

R: The square of the number plus the triple of the same number gives us fifty-four. What is that number and how do you state the equation?

F: Well, the first number is equis, and plus three times the same number, then it would be three equis! [writes with a certain separation x and 3x on the board, stays watching and mentions...] but as it says, the square of a number, we raise to the square equis, plus three equis equal to fifty-four [he writes $x^2 + 3x - 54 = 0$], the equation is square equis plus three equis equal to fifty-four, but if the fifty-four we pass it on the other side and match it with zero, then we would have... [he writes $x^2 + 3x - 54 = 0$ and continued the solving process by factoring the second-degree trinomial written on the first equation member until the two equis ($x_1 = -9$, $x_2 = 6$)].

R: Well, you obtained two values, one positive and one negative and the question of the problem is: What is the number? However, the VP refers: the square of a number plus the triple of the same number gives us fifty-four. What value do you consider adequate? [Fer stays thinking]

F: I will try first by substituting the value of six in the equation $(x)^2 + 3x = 54$, [he writes $(6)^2 + 3(6) = 54$, $36 + 18 = 54$ and obtains $54 \equiv 54$, then he doses the same with value -9, and concludes by saying that he value sought is six].

In this episode you can see significant progress towards algebraic syntax, his personal intertexts allow him to make an efficient reading of the proposed problem statement. He verbally evokes how he carries out his reasoning, which allow him to pose a EG2 with one unknown that represents the translation. The student's semiosis process shows his competence in the use of MSS2 at a high level; he also formulates the correct answer of the problem because he proved the equality of the equation by substituting the values of the unknown. The above fives sense to the algebraic procedures carried out.

Conclusions

The teaching model used in this study allowed to enhance the semiosis processes, the intertextuality and the production of meaning in the study subjects from the manipulation of Algebraic Puzzle pieces, that were useful for concrete modeling of the equations written in its canonical form $ax^2 \pm bx \pm c = 0$, where $a, b, c \in Z$, and thus provide meaning and senses to equivalent equations syntactically obtained from this solving process with pencil and paper, that the competent user of the MSS2 wrote in its factored form $(x \pm a)(x \pm b) = 0$, or by CAS.

The results presented are not conclusive. It is necessary to carry out studies that deepen the reading/transformation processes and the consequent production of meanings by the students, through the design of a teaching model that allow them to evolve towards the construction of the algebraic syntax around the MC of EG2 resolution, based on their intertexts during the semiosis processes that students use as competent users of the MSS2 to solve those VP posed.

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INTERTEXTUALIDAD Y PROCESOS DE SEMIOSIS EN LA RESOLUCIÓN ALGEBRAICA DE PROBLEMAS VERBALES

INTERTEXTUALITY AND SEMIOSIS PROCESSES IN THE ALGEBRAIC RESOLUTION OF VERBAL PROBLEMS

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En esta investigación se utilizó como marco teórico y metodológico los Modelos Teóricos Locales y la perspectiva teórica sobre semiótica de Pierce. Al final del estudio realizado con estudiantes de secundaria se identificaron estratos de intertextualidad derivados de los procesos de semiosis, con base en actos de lectura/transformación de textos o de lectura/traducción/transformación de los

espacios textuales propuestos en el Modelo de Enseñanza para la resolución de Problemas Verbales mediante Ecuaciones de Grado dos con una incógnita, empleando el Método Mixto (geométrico y algebraico) y el Método Cartesiano que incluye procedimientos algebraicos con lápiz y papel, y una calculadora simbólica con un Sistema Algebraico Computacional.

Palabras clave: Intertextualidad, procesos de semiosis, producción de sentido, problemas verbales.

Son diversas las dificultades que enfrentan los estudiantes de secundaria en su proceso de apredizaje del álgebra, por ejemplo, no identifican las estructuras subyacentes a las expresiones algebraicas: estructura superficial y estructura sistémica en términos de Kieran (1989), por otra parte en Filloy & Córdoba (2013), Córdoba (2016), se afirma que es común que se generen diversos tipos de errores de sintaxis al manipular expresiones matemáticas, ya sean aritméticas o algebraicas, o bien, en la resolución de ecuaciones de grado uno (EG1) y ecuaciones de grado dos (EG2), de tal forma que, aún no se les puede considerar usuarios competentes del Sistema Matemático de Signos del Álgebra (SMS2), están en proceso de leer los textos matemáticos de manera correcta y distinguir las transformaciones permitidas de las que no lo son.

Algunos investigadores (Gallardo, 2002; Kieran, 2006; Solares, 2007; Rojano, 2010; Martínez, 2012; Filloy & Córdoba, 2013; Bonilla, 2014; Córdoba, 2016) consideran que en la transición de la aritmética al álgebra es necesario que los profesores de secundaria contemplen como fundamental que sus estudiantes aprendan las propiedades y relaciones del SMS de la aritmética (SMS1), y que el insuficiente conocimiento de éstas se traduce en dificultades de operatividad con el SMS2 y con ello, la lectura/transformación de textos algebraicos

Propósito del estudio

El propósito del estudio experimental de corte cualitativo realizado por Córdoba (2016), fue observar y analizar procesos de semiosis de los estudiantes (sujetos de estudio), como resultado de la lectura/transformación de textos matemáticos, para la producción de sentido en el proceso de la resolución algebraica de problemas verbales (PV) mediante EG2 con una incognita

Marco Teórico

El marco teórico y metodológico propuesto por Filloy (1999), para la observación experimental en Matemática Educativa, denominado Modelos Teóricos Locales (MTL's), permite dar cuenta de los procesos que se desarrollan cuando se enseñan dentro del Sistema Educativo Nacional determinados contenidos matemáticos concretos a unos estudiantes dados, los cuales deben ser pertinentes para los fenómenos que son objeto de estudio; los MTL's contemplan cuatro componentes interrelacionados y sus correspondiente modelos:1) Modelo de Enseñanza 2) Modelo de los Procesos Cognitivos, 3) Modelo de Competencia Formal y 4) Modelo de Comunicación. Filloy (1999), define un Modelo de Enseñanza (ME), como una secuencia de textos producidos tanto por el profesor como por el alumno, y esos textos son el resultado del trabajo de ambos en situaciones de enseñanza problema —que se toman como espacios textuales—."En álgebra, los espacios textuales están constituidos por Sistemas Matemáticos de Signos cuyos códigos y tradiciones provienen de los significados atribuidos a ellos por su uso social". (Filloy, Rojano & Puig, 2011).

Desde la perspectiva teórica de Pierce en relación con la semiótica ¿qué es un signo? Para Peirce lo fundamental en el signo es su función, al describir el proceso semiósico, más que a signos, se refiere a funciones sígnicas. Ese proceso sígnico, vital, constante y significante es la semiosis. Elizondo (2012). Es pertinente hacer una distinción entre texto y espacio textual (ET), la cual se corresponde con la distinción entre significado y sentido, es importante comprender el uso que dan los profesores y los estudiantes a los textos matemáticos. Filloy afirma "… la noción de texto se introduce para ser utilizada en el análisis de cualquier práctica de producción de sentido." (Filloy, Rojano, & Puig, 2008; Kieran & Filloy, 1989). Un texto es el resultado de la lectura/transformación hecha sobre un ET, cuyo propósito es producir sentido y solamente extraer el significado inherente a dicho espacio. (Filloy, Rojano, & Puig, 2008). El espacio textual es un sistema que impone una restricción semántica a la persona que lo lee; el texto es una nueva articulación de ese espacio, individual e irrepetible, hecha por una persona como resultado de un acto de lectura. Así, un ME como componente de un MTL (Filloy, 1999), es una sucesión de textos que son tomados como un ET para ser leído/transformado en otro ET conforme el aprendiz da sentido en sus lecturas (Rojano, 2010).

El proceso del MC implica poner un problema en ecuaciones, o bien, un proceso de traducción del enunciado del PV dado en lenguaje natural al lenguaje del álgebra, implica: por un lado, la lectura analítica, que prepara el texto del problema elaborando otro texto que en cierta manera está preparado para ser traducido al lenguaje del álgebra, y por otro, un trabajo en el nivel de la expresión en el lenguaje del álgebra que transforma el texto traducido (la ecuación) a otro texto que se sabe resolver; es decir, la ecuación en forma canónica (Puig, 2012).

Método de investigación

En esta investigación se diseñó y desarrollo un MTL integrado por los cuatro modelos referidos, en el ME se retomaron elementos para el uso didáctico de los errores de sintaxis en el desarrollo del pensamiento algebraico, a partir del estudio de Córdoba (2005); se observaron en situación de entrevista clínica con enseñanza (ECE), las actuaciones de nueve estudiantes de tercer grado de Educación Secundaria en el Estado de México; pero por cuestión de espacio solo nos referiremos al caso de Fer, al resolver PV mediante EG2 utilizando procedimientos con lápiz y papel y CAS como manipulador simbólico, además se recurrió a la modelación de éstas, mediante el uso de un material didáctico denominado Puzzle Algebraico propuesto por Larrubia (2005), esto permitió analizar la interacción de los aprendices con el SMS1, el SMS2 y el ME.

Resultados

En la tesis doctoral de Córdoba (2016), se identificaron tres tipos de estratos de intertextualidad derivados de los procesos de semiosis, también se describen tres niveles de competencia de uso del SMS2 (bajo, intermedio y alto) de cada tipo, lo anterior, a partir de las ECE con estudiantes, en las que se les propuso resolver PV mediante EG2con una incógnita, los cuales forman parte del ME, a continuación se exponen características generales de los estratos de intertextualidad:

El primer estrato de intertextualidad: *Lectura/escritura de textos en una red de relaciones textuales*, se refiere a los actos de lectura/escritura de textos algebraicos que hace un lector concreto con base en una red de relaciones textuales derivadas de sus experiencias matemática y lingüística previas; se observaron tres niveles de competencia de uso del SMS2 de los aprendices cuando se le pide resolver EG2 con una incognita.

El segundo estrato de intertextualidad: *Lectura/transformación de la estructura superficial de textos algebraicos,* se refiere a la manera en que los aprendices resuelven las EG2 (completas e incompletas) dadas, a partir del reconocimiento de su estructura superficial. Las transformaciones algebraicas se realizan entre un textos algebraicos en el nivel de la expresión algebraica, esto requiere también que el aprendiz identifique la estructura superficial de la ecuación y su transformación de la forma factorizada a la forma canónica o viceversa.

El tercer estrato de intertextualidad: *Lectura/traducción/transformación de textos*, se refiere a la resolución algebraica de PV mediante EG2 con una incógnita, al respecto es común recurrir al uso del MC, definido como el proceso de poner un problema en ecuaciones, que implica el proceso de traducción y las transformaciones algebraicas definidas en el segundo estrato de intertextualidad.

Se incluye una breve descripción y análisis de un episodio de entrevista clínica con enseñanza, realizada por el Investigador (I) con Fer (F), para ejemplificar el tercer estrato de intertextualidad "Lectura/traducción/transformación de textos" en la resolución de un PV.

I: El cuadrado de un número más el triple del mismo número nos da cincuenta y cuatro. ¿Cuál es ese número? y ¿cómo plantearías la ecuación?

F: ¡Pues, el primer número es equis, y más el triple del mismo número, entonces sería tres equis! [escribe con cierta separación x y $3x$ en el pizarrón, se queda observando y menciona…] pero como dice, el cuadrado de un número, elevamos al cuadrado equis, más tres equis igual a cincuenta y cuatro [escribe $(x)^2 + 3x = 54$], la ecuación es equis cuadrada más tres equis igual con cincuenta y cuatro, pero si el cincuenta y cuatro lo pasamos del otro lado y lo igualamos con cero, entonces nos quedaría…[Escribe $x^2 + 3x - 54 = 0$ y continuó con el proceso de resolución factorizando el trinomio de segundo grado escrito en el primer miembro de ecuación hasta encontrar los dos valores de equis ($x_1 = -9$, $x_2 = 6$)].

I: Bueno obtuviste dos valores, uno positivo y otro negativo y la pregunta del problema dice: ¿cuál es el número? Sin embargo, el PV refiere: el cuadrado de un número más el triple del mismo número nos da cincuenta y cuatro. ¿Qué valor consideras adecuado? [Fer se queda pensando]

F: Probaré primero sustituyendo el valor de seis en la ecuación $(x)^2 + 3x = 54$ [escribe $(6)^2 + 3(6) = 54$, $36 + 18 = 54$, obteniendo $54 \equiv 54$, después hace lo propio con el valor de -9, concluye diciendo que el valor buscado es seis].

En este episodio se puede observar un progreso significativo hacia la sintaxis algebraica, sus intertextos personales le permitieron realizar una lectura eficiente del enunciado del problema propuesto, evoca de manera verbal cómo realiza su razonamiento, esto le permite plantear la EG2 con una incógnita que representa la traducción correspondiente; los procesos de semiosis del estudiante dan cuenta de su competencia de uso del SMS2 en un nivel alto, así mismo, enuncia la respuesta correcta del problema, con base en haber realizado la comprobación de la igualdad de la ecuación sustituyendo los valores obtenidos de la incógnita, lo anterior, dota de sentido a los procedimientos algebraicos efectuados.

Conclusiones

En particular el ME utilizado en este estudio, permitió potenciar los procesos de semiosis, la intertextualidad y la producción de sentido en los sujetos de estudio, a partir de la manipulación de las piezas del Puzzle Algebraico, siendo útil para modelar de manera concreta las ecuaciones escritas en su forma canónica: $ax^2 \pm bx \pm c = 0$ donde $a, b, c \in Z$ y con ello, dar significado y sentido a las ecuaciones equivalentes obtenidas en dicho proceso de resolución de manera sintáctica con lápiz y papel, y que son escritas por el usuario competente del SMS2 en su forma factorizada: $(x \pm a)(x \pm b) = 0$, o bien, por el propio CAS.

Los resultados expuestos no son concluyentes, se requiere realizar estudios que profundicen más en los procesos de lectura/transformación y la consecuente producción de sentido por parte de los estudiantes, diseñando un ET que les permita evolucionar hacia la construcción de la sintaxis algebraica alrededor del MC de resolución de EG2, teniendo como base el uso de sus intertextos personales durante los procesos de semiosis a los cuales recurren para resolver como usuarios competentes del SMS2, los problemas verbales que se les propongan.

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EXAMINING STUDENTS' REASONING ABOUT MULTIPLE QUANTITIES

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In this report, we discuss five forms of reasoning about multiple quantities that sixth-grade students exhibited as they examined mathematical relationships within the context of science. Specifically, students exhibited forms of sequential, transitive, dependent, and independent multivariational reasoning as well as relational reasoning. We use data from whole-class design experiments with students to illustrate examples of each of these forms of reasoning.

Keywords: Cognition, Algebra and Algebraic Thinking, Design Experiment, STEM/STEAM.

Variation, Covariation, and Multivariation

Reasoning about variation and covariation has been studied extensively in mathematics education as a way of supporting students' mathematics learning (e.g., Confrey & Smith, 1995; Carlson et al., 2002). More recently, we found that the use of variation and covariational reasoning also supported students' learning of science phenomena, such as the learning of gravity and the greenhouse effect (e.g. Author, 2019; Author, 2020). Science phenomena involve a complex interaction of variables and this provided a constructive space for students to reason about covariation in more complex ways. In these studies, we found that by manipulating the quantities involved in those phenomena using interactive simulations and studying what quantities are changing and how they are changing, sixth grade students exhibited some sophisticated forms of covariational reasoning. Specifically, students coordinated the direction of change of one quantity with the change in another quantity and also identified the bi-direction of change of some of those quantities. Students even discussed inverse relationships, such that as one quantity increases, the other quantity decreases, and predicted the change of one quantity if another is varied multiplicatively. While analyzing our data, we found that students also reasoned about more than two quantities changing simultaneously. Prior research on multivariational reasoning only focused on undergraduate mathematics education (Kuster & Jones, 2019). Therefore, this provided an opportunity to examine students' emerging forms of multivariational reasoning in earlier grades. This effort could eventually respond to Thompson and Carlson's (2017) call for more contributions on defining the covariation construct. Specifically, we aimed to explore: *How do sixth-grade students reason about multiple quantities as they explore complex quantitative relationships in scientific phenomena?*

Theoretical Framework

We use a *quantitative reasoning* lens (Thompson, 1994) to discuss students' forms of reasoning about multiple quantities in the context of science. We use the term *quantity* as one's conceived attribute of an object or phenomenon that is measurable, whether they have carried out that measurement or not (Thompson, 1993; 1994). In this manner, numeric or not, reasoning quantitatively involves analyzing a situation into "a network of quantities and quantitative relationships" (Thompson, 1993, p.1). Accordingly, Kuster and Jones (2019) defined multivariation as a situation with more than two quantities that change in relation to each other. They used this definition to discuss three forms of multivariational reasoning that students exhibited as they explored differential equations: dependent, nested, and independent multivariation. Specifically, they defined *dependent multivariation* as involving at least three quantities that are interdependent with each other, in which a variation in one quantity simultaneously influences the change in other interdependent quantities. They gave the example of reasoning that since P is a function of time, P' is

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also a function of time. They defined *nested multivariation* as involving a network of quantities, where the first quantity is embedded in the second quantity and the change in the second quantity influences the change in the third quantity. For instance, when the differential equation $P' = 2P+2t$ was presented, a student used nested multivariation to explain that a change in t influenced the change in $2t$, then variation in $2t$ changed P'. Finally, they defined *independent multivariation* as involving at least two quantities that are independent to each other and affect the change in another quantity. They gave the example of reasoning that the solution function $P(t)$ is dependent on t, but the rate of change, P', is not influenced by t. Although two independent quantities (t and P') are presented, we would argue that the example does not clearly show independent multivariation because the student does not clearly state that P' influences the function $P(t)$. However, we consider the types that Kuster and Jones presented to be foundational for initiating the discussion around the different forms of multivariational reasoning in the earlier grades.

Forms of Multivariational Reasoning

In this paper, we report on the data from whole-class design experiments (DEs) (Cobb et al., 2003) conducted in three different sixth-grade classrooms, each examining a specific scientific phenomenon: the sea level rise, the water cycle, and the rock cycle. We designed a simulation to dynamically model and study each scientific phenomenon. For example, in the rock cycle simulation students could manipulate a rock's depth and study the changes in its temperature and pressure. We accompanied the simulation exploration with questions that prompted them to reason about those quantitative relationships, such as "How would you describe the relationship between the quantities?" and "How does the change in one quantity affect other quantities?" In the following paragraphs, we discuss five forms of multivariational reasoning that students exhibited (Figure 1) by providing examples of students' episodes from all three DEs.

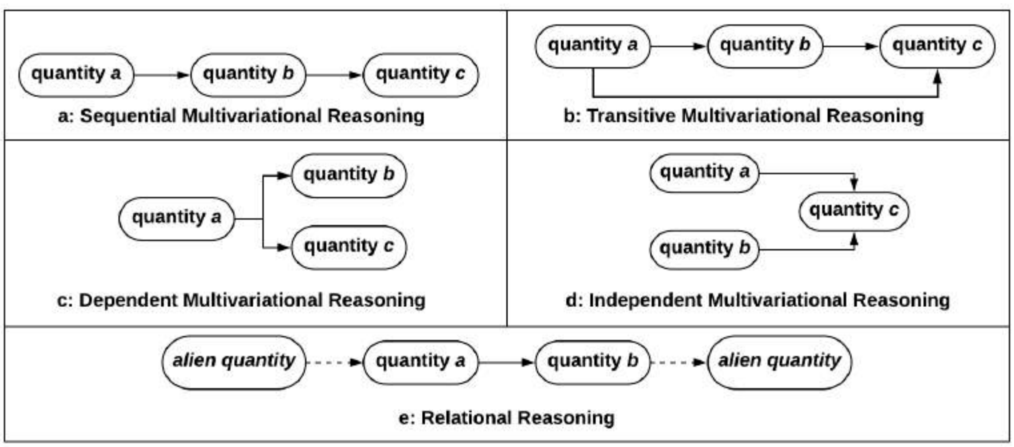

Figure 1: Forms of reasoning about multiple quantities.

Sequential Multivariational Reasoning

In students' articulations, we observed a form of multivariational reasoning that was not discussed in the Kuster and Jones' (2019) study. We refer to sequential multivariational reasoning (Figure 1a) as illustrating sequential changes in quantities, where a change in the first quantity (a) influences a change of the second quantity (b), and a change in the second quantity (b) affects a change in the third quantity (c). While exploring a simulation about sea level rise, students discussed the relationship between the global temperature rise, the height of future sea level, and the total land area. For instance, Myra explained that "The higher the global temperature, the higher the height of the future sea level, and the less the total land area." We interpret her reasoning to illustrate a sequential image of change: that the change in global temperature rise (quantity *a*) impacts the height of future sea level (quantity *b*), and that the change in height of sea level (quantity *b*) affects the change in total land area (quantity *c*).

Transitive Multivariational Reasoning

Our students also exhibited what we would define transitive multivariational reasoning (Figure 1b), a form of reasoning that supports that a change in the first quantity (*a*) leads to a change in the second quantity (*b*), and a change in the second quantity (*b*) in turn changes a third quantity (*c*), then a change in the first quantity (*a*) changes the third quantity (*c*). The difference between transitive reasoning and sequential reasoning is that the transitive reasoning involves the coordination of change in the first quantity (*a*) influencing a change in the third quantity (*c*), which is not illustrated in sequential reasoning. To illustrate this form of reasoning, we provide an example from the water cycle. The water cycle simulation presented a virtual ecosystem, in which students could manipulate the temperatures of air, mountain, land, and lake, and relative humidity and observe the change in the amount of water molecules in every phase of the water cycle. When asked to describe the relationship between evaporation and runoff, Ray stated, "If the rate of evaporation is higher, there could be higher rate of precipitation. If there's a higher rate of precipitation, there could be more runoff. So, the higher rate of evaporation, there can be more runoff." We consider Ray's coordination of the change in three quantities to illustrate transitive multivariational reasoning. In particular, Ray first explained how the change in evaporation (quantity *a*) influences precipitation (quantity *b*), and how the change in precipitation (quantity *b*) influences runoff (quantity *c*). Then he used those two relationships to reason about how a change in evaporation (quantity *a*) causes a change in runoff (quantity *c*).

Dependent Multivariational Reasoning

Our students also illustrated reasoning that we would characterize as a subset of Kuster and Jones' (2019) definition of dependent multivariational reasoning. In contrast to Kuster and Jones' definition in which all three quantities involved are interdependent, the students in our study coordinated a change in an independent quantity *a* which simultaneously affected changes in two dependent quantities *b* and *c*, while quantities *b* and *c* were not related to each other (Figure 1c). For example, when Michael was prompted to describe what he noticed as he explored the rock cycle simulation he stated, "I would say that, the deeper, the deeper you get, the higher the temperature is, and the higher the pressure is." We consider Michael's reasoning about the relationship of depth with the temperature and pressure to be dependent multivariational reasoning. Michael's language "the deeper" and "the higher" also shows an understanding of simultaneous change between the two dependent quantities (temperature and pressure) as influenced by one independent quantity (depth).

Independent Multivariational Reasoning

Our students exhibited independent multivariational reasoning, (Figure 1d), similar to Kuster and Jones' (2019) definition of coordinating a change in two independent quantities (quantities *a* and *b*) influencing the same dependent quantity (quantity *c*). For example, when Chloe and Justin were asked to use the water cycle simulation to release snow by manipulating only the air temperature and the land temperature, they reasoned that "We need both of them to be cold." Chloe explained that "if you just move for air temperature, it only snows a little bit, but if you put it with a land temperature, it starts to accumulate in the ground and it produces more." Chloe illustrated an example of independent multivariational reasoning as she coordinated the change of land temperature and air temperature as unrelated independent quantities with the change in snow as the dependent quantity.

Relational Reasoning

In addition to the above four types of multivariational reasoning, we also noticed instances where students related their explorations with quantities that were not part of the specific study. We refer to relational reasoning (Figure 1e) as the form of reasoning that connects the relationship of two quantities with a third quantity that students bring in from their prior experiences (what we refer to as an *alien* quantity). Relational reasoning can be expressed together with other forms, such as sequential multivariational reasoning. For instance, while exploring the water cycle simulation, we asked students to explain the model. Lorna connected the relationship between the amount of precipitation, runoff, and infiltration with the quantity of water that would go into the aquifers, which was not identified in the simulation or module. Lorna reasoned that "the more rain there is, there's more runoff. And the more runoff, the more water is going to go into the aquifers." Lorna first reasoned about the change in the quantity of rain with change the quantity of runoff. Then she coordinated the change in runoff with the amount of infiltrated water in the aquifer, an alien quantity to the simulation.

Conclusions

In 2017, Thompson and Carlson argued that while there are a wealth of studies employing variation and covariation as a framework for their investigations, these "do not contribute directly to defining the construct" (p. 427). Investigating how students may reason about more than two quantities makes a contribution to this call. The Kuster and Jones' (2019) study initiated a discussion about how we can define students' forms of multivariational reasoning. Our study built on their work to examine how students as young as sixth grade could reason about multiple quantities. By exploring the sequential and simultaneous variation of quantities involved in the water cycle, rock cycle, and sea level rise phenomena, students exhibited five different forms of reasoning about multiple quantities, namely sequential, transitive, dependent, and independent multivariational reasoning as well as relational reasoning.

The retrospective analysis showed that it was the students' interaction with the simulations and the probing questioning that provided a constructive space for them to study the variation in multiple quantities and reason multivariationally. Our initial goal in the study was to engineer opportunities for students to reason covariationally, therefore our tasks and questioning were restricted to only a few prompts to connect multiple quantities. In the next iteration of our design, we plan to engineer more opportunities of this type of reasoning. Through this process, we can examine the progression from covariational to multivariational reasoning and the tasks, tools, and questioning that assist students in exhibiting each specific form of reasoning about multiple quantities.

Acknowledgments

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INVESTIGATING THE LEARNING SEQUENCE OF DECIMAL MAGNITUDE AND DECIMAL OPERATIONS

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Research is mixed on whether understanding decimal magnitude supports operations with decimals or whether operations can be learned before and while students develop understanding of decimal magnitude. In the present study, we used a large scale, longitudinal design to investigate students' knowledge of decimal comparison and operation before and after decimal comparison alone was introduced in the curriculum. Student performance on a decimal comparison task did not increase, but there was an increase in performance on decimal subtraction and decimal multiplication tasks, topics which were not part of the mandated curriculum during the relevant period of instruction.

Keywords: Number Concepts and Operations, Rational Numbers, Elementary School Education, Cognition

Perspectives

Researchers examined various conceptual hurdles involved in meaningful interpretation and use of the notational system involving decimals (Resnick et al., 1989). Hiebert (1992) proposed three types of knowledge which are important to comprehend the decimal system: knowledge of the notation, knowledge of the symbol rules and knowledge of quantities and actions on quantities. The knowledge of notation comprises of "how the symbols are positioned on paper" (ibid, p. 290) rather than understanding of what '.' means or what quantities it represents. For instance, a student can compare two decimals correctly, but can have incorrect reasoning to explain their answers (see Resnick et al., 1989 for details on erroneous rules while comparing decimal numbers). The knowledge of the symbol rules prescribes on "how to manipulate the written symbols to produce correct answers" (Hiebert, 1992, p. 290). For instance, while adding and subtracting two or more decimals, the numbers need to be lined up systematically (Lai & Murray, 2015). This knowledge is analogous to Skemp's (1976) idea of instrumental understanding where an individual can manipulate mathematical syntactic symbols using appropriate rules, procedures, algorithms, etc. to produce the correct answer, even when without understanding the underlying reasons. Knowledge of quantities and actions includes the understanding of decimal numbers are representing quantities, i.e., measures of objects "…by units, tenths of units, hundredths of units, and so on" and comprehending the reasons that explain "what happens when the quantities are moved, partitioned, combined, or acted upon in other ways" (Hiebert, 1992, p. 291). Lai and Murray (2015) related the knowledge of quantities and actions on quantities with developing a comprehensive understanding of the decimal topics.

Decimal Comparison

Students build on whole number ideas when they engage with decimals, and this both helps and hinders learning. Lee and colleagues (2016) argued that due to the representational nature of decimal numbers, which is virtually indistinguishable from that of whole numbers, the students find decimal magnitude comparison tasks easier as compared to the fraction magnitude (see also, DeWolf et al., 2014; Iuculano & Butterworth, 2011). Researchers claim that students often perform well on the decimals comparison tasks by following syntactical rules (Lachance & Confrey, 2002), rather than developing a conceptual understanding of it. However, the common practice of teaching decimals as

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an extension of the whole numbers might convey an inadequate understanding of place-value system (Fuson, 1990; Martinie, 2014).

Sequence of Decimal Instruction

In the United States, decimal instruction begins in the fourth grade with comparisons of fractions written in decimal form with denominators of 10 and 100. A decimal is regarded as the one-dimensional magnitude of a fraction (a/b = c) expressed in the form of the standard base-10 metric system (Lee et al., 2016). This continues in fifth grade with decimal operations to the hundredths place (National Governors Association Center for Best Practices, 2010). Decimal instruction then continues through middle school (Rittle-Johnson et al., 2001).

Although curricula frequently sequence decimal comparison instruction before operations, there is little research available to support this sequence. The magnitude-before-computation sequence is supported for fractions and standards and curricula appear to follow it for decimals based on the idea that fractions and decimals are closely conceptually related, even though there has been very little research on this instructional order. Arguments for teaching decimal magnitude before tackling operations between decimals numbers are formed by research that shows that children who are less comfortable with fraction magnitudes are also not as good as their counterparts at computations involving fractions (Lortie-Forgues et al., 2015). Other researchers note that students can understand decimal magnitude without being able to understand the results of computations involving decimals, which implies that understanding decimal magnitude is a prerequisite for decimal operations (Siegler & Lortie-Forgues, 2015).

Decimal Comparison and Operations with Decimals

Even though decimal magnitudes are taught first and operations second, the concepts appear to be intertwined in the minds of students. Decimals are familiar to students before they reach fourth grade to some degree because they follow some of the whole number rules, even though students often misapply those rules for comparing and computing with decimals (Ren & Gunderson, 2019; Rittle-Johnson et al., 2001; Vamvakoussi & Vosniadou, 2004). As students make sense of decimals in school, they begin to apply what they understand to computation even if they have not been explicitly taught to do so. Hiebert et al. (1991) showed that children could learn about decimal concepts and structure and still show growth on decimal computation with symbols without detailed instruction on procedures. Other research has found that intermingling work on decimal place value with decimal addition and subtraction results in strong student performance, which is opposite of the assumptions surrounding magnitude-first instruction (Rittle□Johnson & Koedinger, 2009). Given the mixed findings from past research that has mostly relied on small-scale qualitative data, in the present study, we used a large scale, longitudinal design to investigate students' knowledge of decimal comparison and operation before and after it was introduced in the curriculum. In particular, we sought to answer two research questions. (1) *How does knowledge of decimal comparison and operations with decimal change during the year in which decimals are formally introduced in the curriculum?* (2) *Are the patterns that characterize students' responses at each time point more indicative of magnitude-before-operation or intermingled learning in the decimal domain?*

Methods

The data is drawn from a larger study that included a representative sample of Grade 4 elementary teachers in Indiana. These teachers administered 8-item tests to their Grade 4 students (N = 1467) in the Fall of 2017 and Spring of 2018, and the data we report comes from three items on that test. The participation in this survey was voluntary for the students and they were given 15 minutes to work on the test. McNemar's test was used to compare the pre-test and post-test results of the same students in grade four at two different points in the school year, so we had matched pairs of subjects with a dichotomous trait of correct or incorrect for each question. We used alluvial diagrams to search for patterns in pretest and posttest responses.

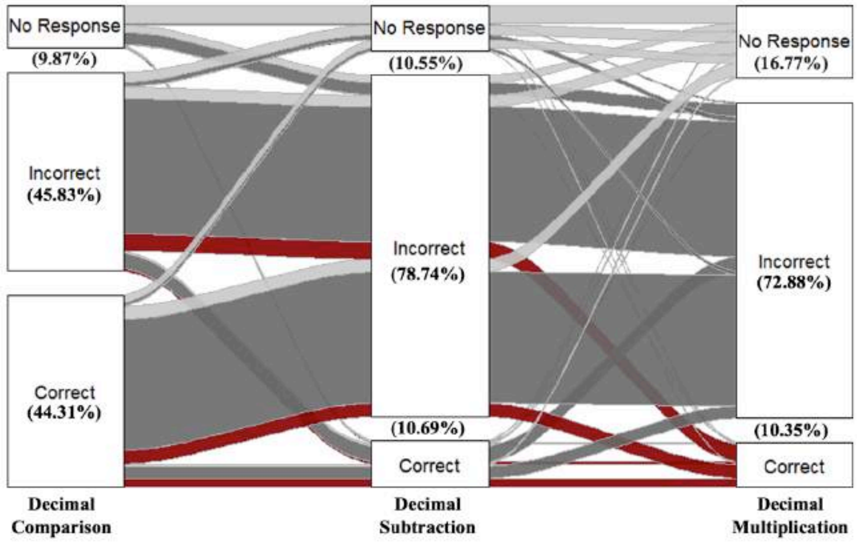

Figure 1. Pretest weighted flow of participants between response categories.

Results

An exact McNemar's test was used to compare the two conditions (correct and incorrect) on the pre- and post-test over fourth-grade fraction and decimal knowledge. The change in the number of students who correctly answered the comparison question from the pretest to the posttest was small, 44.31% to 45.07%. The analysis showed that there was not a statistically significant positive change between the pre- and post-test for ordering decimals from smallest to largest (p = 0.689). Of the 1,467 students who took the pre- and post-tests, 642 answered the ordering question correctly on the pre-test and 653 answered correctly on the post-test.

In contrast, there was a larger change in the number of students answering the decimal subtraction question correctly on the posttest from 10.67% to 28.57%, and this change was a statistically significant (p = 0.000). For decimal addition, 155 students answered correctly on the pre-test and 414 answered correctly on the post-test. For the decimal multiplication question, the increase was more modest, from 10.35% answering correctly on the pretest to 17.53% answering correctly on the posttest. Similar to the decimal subtraction problem, this increase statistically significant (p = 0.000). In the decimal multiplication question, 150 students answered correctly on the pre-test and 254 answered correctly on the post-test. The students' performance differences between the decimal magnitude question and the operation questions shows that as a group the students improved in their ability to operate on decimals without substantially increasing their understanding of decimal comparison.

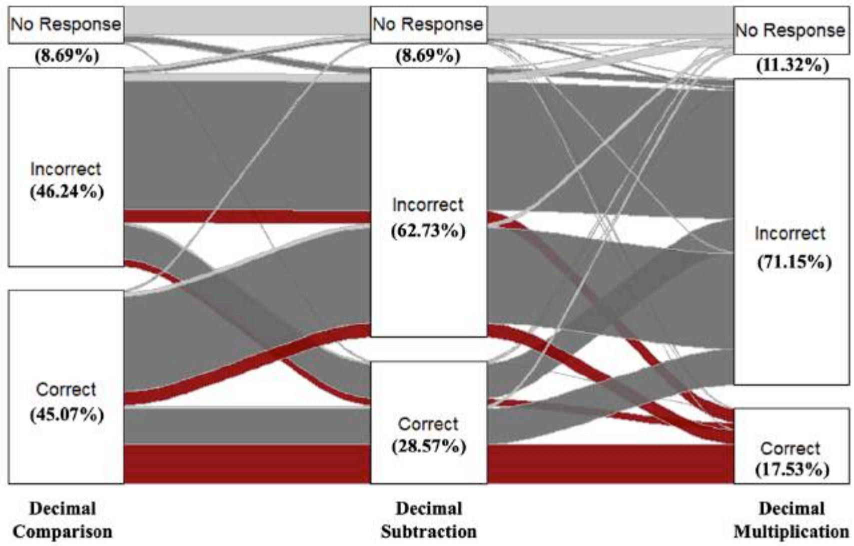

Figure 2. Posttest weighted flow of participants between response categories.

We examined the relationship between responses on these three items and compared these patterns of responses between the pretest and posttest. The alluvial diagrams below show weighted flows among the three decimal questions on the pre- and posttests from the first decimal question about comparing decimals, through the subtraction question, and then to the multiplication question. These diagrams illustrate how responses to decimal items early in the test were related to responses later in the test at each time point. In particular, although the number of students who answered the comparison question did not significantly increase at posttest, a much larger portion of those students went on to answer the two operation questions at posttest than at pretest (see large ribbon at the bottom of Figure 2).

Discussion and Implications

We expected that fourth grade students would show more growth on decimal comparisons than decimal operations in fourth grade because decimal comparison is a fourth-grade standard and decimal operations are a fifth-grade standard. Students were presumably receiving more instruction on magnitude comparisons than on decimal operations. What we saw instead was that growth in decimal comparisons was not statistically significant yet growth in decimal operations (subtraction and multiplication) were statistically significant. Furthermore, by comparing the patterns between items at each time point we noticed that a majority of the students who answered the multiplication question at posttest also answered the comparison and subtraction problems, suggesting most of the change from pre to post was driven by a cohort of students who solidified their understanding of operations during the year in which comparison was taught. These findings confirm at scale what other researchers have found in small, qualitative studies; namely, that the conceptual development of both comparisons of magnitude and operations happen concurrently rather than sequentially.

Acknowledgments

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LEVEL-ZERO COVARIATIONAL REASONING IN SECONDARY SCHOOL MATHEMATICS

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Based on the results from a research project currently in progress, we outline the hypothesis that exhaustive mathematical study of the behavior of a single variable quantity (E1VQ) constitutes, by itself, a whole cognitive structure (in the Piagetian sense) underlying the development of variational thinking, and that it can and should be fully addressed in secondary education, during and in parallel with the study of representation of numbers and quantities on the number line (a variable quantity – a number axis). This would allow the student to develop a series of conceptual (qualitative) images and quantitative mathematical tools that allow identifying, describing and naming the basic types of variational behavior of a variable quantity.

Keywords: Algebra and algebraic thinking, variable quantities, variational thinking.

Introduction

The ideas addressed in this work are the collateral product of a research project that aims to contribute arguments of a historical, epistemological and cognitive nature in efforts to reconceptualize the teaching of Calculus for non-mathematicians (Jiménez et al., 2020). These ideas point, on the one hand, to the need to promote early development of a variational way of thinking in secondary and high school students, and on the other, to rethink the vision under which the variational approach is embodied in the curriculum of these educational levels. These ideas are presented here as working hypotheses that require a great subsequent effort in educational research, either to corroborate, reject and modify them, or to develop a greater knowledge.

Variational Thinking Conception

The research literature has long documented the difficulties that many students experience in understanding and constructing Cartesian graphs and developing a variational way of thinking (Radford; 2009a, 2009b). In our opinion, one of the epistemological roots of such difficulties consists in the very interpretation of what *variational thinking* is. It is a term whose meaning has not yet been established in a clear and concise way, not to say precise and rigorous, as is typical of scientific work. Different research groups, affiliated with different theoretical approaches, assume their particular interpretation of the meaning of this notion (Vasco, 2010).

In this work, the term ***variational thinking*** refers to the type of mathematical thinking required to understand variation and *change in progress* (Thompson, Ashbrook, & Milner, 2016), and it develops and evolves as the study of such phenomenology occurs. This way of thinking is both *qualitative* (it implies the construction of dynamic images of variation and the reasoning about them, as well as the development of a language that reflects this dynamism) and *quantitative* (it has to do with numerical calculations, techniques and algebraic expressions).

By its nature, the way of thinking that a mathematical study of change in progress requires is a complex entity that we can conceive of as consisting of two components that Thompson and Carlson (2017) have respectively called *variational reasoning* and *covariational reasoning*. These are complementary aspects of the same type of mathematical thinking, which have to do with the mathematical conceptualization of variable quantities.

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Variational Reasoning

Variational reasoning consists of two critical moments. The first is to understand that variable quantities actually *vary*, that is, their numerical values change. Consequently, it is a *dynamic* way of thinking. Thus, this first moment, characterized by the fact of perceiving or conceiving, in a given situation of change, the intervention of one, two, three, four, etc., variable quantities (one of which may be time), and to inquire how they change in order to form dynamic mental images of their ways of changing, to create mathematical tools to represent and quantify such changes, to develop an appropriate language for describing those changes, and much more, is what constitutes the essence of variational reasoning.

There is an important aspect in conceptualizing a variable quantity that has an intuitive connotation. It is the fact that the numerical value a variable quantity takes at each moment is unique: it is not possible for a variable quantity to take two or more different numerical values at the same moment. This essential characteristic of the behavior of variable quantities is known as the *uniqueness principle*.

The second crucial moment in variational reasoning is in some sense an aesthetic appreciation: when a variable quantity changes, it does so smoothly. This "smoothness" is a quality of the processes of change in progress called *continuity*. Time runs smoothly from one instant to the next in an interval; an athlete running moves smoothly from one point to another in the path, and the height of the liquid changes smoothly as a container fills (or empties). The whole process is continuous, since it is continuous at every moment. Usually the process of change-in-progress is continuous, in the sense that it changes smoothly from one state to the next.

The *principle of continuity* is not exclusive to the movement of objects or filling/emptying of containers, but applies to all natural processes. As a process develops, it does not omit any state in its becoming. If the process is in one state at a certain time, and in another state at a different time, then it assumes all states between these two.

In summary, variational reasoning has to do with the mathematical conceptualization of continuous variation of a single variable quantity, this variation having a temporal background.

Covariational Reasoning

The second important stage in variational thinking is to make explicit the fact that in the analyzed situation there are at least two variable quantities present, whose numerical values change simultaneously, and are also related in some way. In this case, the ability to *coordinate* the joint change of these numerical values is crucial for the next level of analysis. Carlson et al. (2002) call this more complex way of thinking as *covariational reasoning*, and characterize it as the set of all "cognitive activities involved in the coordination of two variable quantities while considering how they change in relation to each other". In other words, covariational reasoning about quantities implies the consideration and/or construction of dependency relationships between numerical values of at least two variable quantities that change simultaneously and jointly.

The Early Development Of Covariational Reasoning: Level-Zero

It follows from the previous descriptions that covariational reasoning is cognitively and mathematically much more complex than variational reasoning and that, in order to develop in students the ability to reason covariationally, it is not only desirable but above all it is also ***necessary*** to previously develop their more elemental ability to reason variationally, that is, to reason mathematically about a single variable quantity. Despite this, educational research seems to assume as an unquestionable fact that variational reasoning is simple, unproblematic, natural and spontaneous, and that we should not be concerned with its development. In the overwhelming majority of research work on the subject, the student is involved from the very first moment with mathematical tasks typical of the level of covariational reasoning. The results of our own research point in another direction, suggesting that mastering mathematical work with variable quantities on the number line seems to be a fundamental prerequisite for subsequent work with two number lines in the Cartesian plane. In other words, variational reasoning is related to mathematical work with a single variable quantity on the number line, while elementary covariational reasoning is related to mathematical work with two variable quantities on the Cartesian plane.

Although we know that variable quantities never appear separately in the phenomena of the world in which we live in, isolating and studying them in this way seems to us a justified didactic decision, for several reasons. First, it is a relatively simpler cognitive task, since it does not require the explicit coordination of two variable quantities.

The second argument for first addressing the analysis of a single variable quantity, isolated from the others, is based on the historical development of scientific methodology itself. A methodological strategy to understand complexity is to consider simpler cases of it. The number line is a simpler object than the Cartesian plane.

Our third and main argument is that the basic mathematical ideas required for the coordination of two variable quantities can and should be developed in depth in this simplified, foregoing, hypothetical case (the study of a variable quantity in isolation from the others). Unfortunately, there is no place here to show that this is possible and that it does indeed contain great mathematical richness. In another work (Jiménez et al., work in progress) we argue that, in particular, it is possible to form and develop the mathematical images, tools and terminology necessary to describe and conceptualize the seven basic variational behaviors (uniform growth, accelerated growth, decelerated growth, uniform decrease, accelerated decrease, decelerated decrease, and zero growth or decrease), with a level of complexity similar to that of the case of covariation. Likewise, it is possible to approach some of the advanced ideas of Calculus, such as a first approach to derivatives of higher order and the Fundamental Theorem of Calculus. An outline of such approach is presented in Jiménez et al. (2020).

A cardinal statement of this work is the thesis that *exhaustive mathematical study of the behavior of a single variable quantity* (E1VQ) constitutes, by itself, a whole cognitive structure (in the Piagetian sense, 1968) on which the further development of covariational reasoning relies. A deep understanding of the behavior of a single variable quantity, its description in mathematical terms and its dynamic graphic representation on the number line are relevant and structural for the formation and development of variational thinking, both from a mathematical and a cognitive point of view. In particular, they are fundamental for understanding, interpreting, endowing with meaning and constructing graphs in the Cartesian plane. This is the initial step in the development of variational thought. Given that the process of development of covariational reasoning, according to the terminology proposed by Carlson et al. (2002), has been described in terms of five levels (called L1, L2, L3, L4 and L5), we have decided to name *level-zero* (L0) the corresponding stage of emergence and development of variational reasoning described above.

Related to this, it is sensible to assume that the development in students of a dynamic image of the variable, as well as of a dynamic meaning for it, is possible only relying on a specific mathematical work on the number line, provided that such mathematical work involves the representation of variable quantities and not only of numbers (Jiménez et al., 2020). However, as Thompson and Carlson (2017) rightly pointed out:

> ... there is relatively little research on students' meanings and understandings of number lines. Psychological research in this regard portrays number lines as nebulous objects on which researchers presume that people do informal arithmetic (...), the main interest being by what method people use it to determine sums, products, and so forth. Mathematics education research seems to see the target idea of a number line as being relatively unproblematic and

> focuses on using it as an instructional aid, helping students understand how to locate numbers on it, or using it as a tool in reasoning (...). In both cases, number lines seem to be taken by researchers as lines full of numbers. (Thompson and Carlson, 2017)

The study of the number line is certainly included in the school curriculum, but unfortunately not from the variational point of view. Students learn how to represent numbers by points on the number line, and to associate points with numbers (the coordinates), to identify line segments (intervals), etc. This mathematical work is essentially static in nature. But they never learn how to *represent variable quantities on the number line*, much less how to developn it in dynamic graphic images associated with the different basic variational behaviors, or how to develop and use mathematical tools that allow them to deepen the analysis of the behavior of variable quantities. The number line itself is not a constructed object; it is presented to students as a prefabricated object.

This does not imply that the mathematical work on the number line stipulated by the current curriculum is unnecessary and must be eliminated. On the contrary, it is adequate and necessary, although clearly ***insufficient*** to favor the formation and development of variational reasoning in students. For the latter, it will be necessary to incorporate another type of mathematical work on the number line, of a dynamic nature, related to graphic representation of variable quantities. This implies engaging in a deeper reflection on the construction of the number line itself.

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Stop Press Dedication from Carolyn Kieran: *I would like to dedicate the presentation of this paper to my young co-author, Cesar Martínez-Hernández, whose life was heartbreakingly taken away from him by COVID-19 on December 15, 2020. Cesar's role in the herein presented research study was crucial.*

DECOMPOSING, COMPOSING, AND RECOMPOSING NUMBERS IN NUMERICAL EQUALITIES: ALGEBRAIC THINKING BASED ON STRUCTURE SENSE

DESCOMPOSICIÓN, COMPOSICIÓN Y RECOMPOSICIÓN DE NÚMEROS EN IGUALDADES NUMÉRICAS: PENSAMIENTO ALGEBRAICO BASADO EN UN SENTIDO DE ESTRUCTURA

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This paper reports the manner in which primary school students from a Mexican public school, having just learned to validate the truth-value of numerical equalities (e.g., 8+2+16=10+12+4) based on two algebraic strategies, that is, transforming both sides of the equality to a third common form, and rewriting one side of the equality in the same form as the other side – both strategies being based on decomposing, composing, and recomposing the numbers – tended to favor the first strategy as their primary approach for validating numerical equalities. This tendency suggests that, although the students have developed an algebraic thinking based on structure sense in arithmetic, the second strategy would appear to be less consolidated than the first.

Keywords: Algebra and algebraic thinking; Structure sense in arithmetic; Decomposing, composing, and recomposing numerical expressions and equalities.

Background

Algebraic thinking in primary-grade students has been studied from different perspectives, although an emphasis on generalization has prevailed. Recently, it has been proposed that, in addition to generalizing, seeking and expressing structure is an important part of this kind of mathematical thinking (Kieran, 2018). Regarding the structural, some studies have focused on observing regularities in numerical equalities (e.g., Pang & Kim, 2018; Schifter, 2018). However, according to Mason, Stephens, and Watson (2009), structural thinking involves more than observing regularities. In this sense, Martínez-Hernández and Kieran (2018, 2019, in press), in their studies on structure sense in arithmetic, have reported on the strategies that primary school students use to validate numerical equalities. They found that these students tended at first to use computational strategies to validate numerical equalities of the form $a+b=c+d$ (Martínez-Hernández & Kieran, 2018); however, they were also able to use *ad hoc* strategies on the same kind of equalities, based on decomposing the given numbers (Martínez-Hernández & Kieran, 2019). According to Martínez-Hernández and Kieran (in press), the students were able to transition from an arithmetical thinking expressed by computational strategies to an algebraic thinking expressed by the decomposition, composition, and recomposition of the given numbers, using two strategies: (i) transforming both sides of a numerical equality to a third common form; and (ii) rewriting one side of a given equality in the form of the other side. Based on the emergence of these two strategies, the following question is posed: Which of these two strategies do the students tend to use in order to validate numerical equalities for which they previously used computational or ad hoc strategies?

Theoretical Framework

Early algebra has been characterized from different perspectives (e.g., Carraher & Schliemann, 2007; Kieran, 2018), but all of them recognize the algebraic character of arithmetic. According to Kieran (2004), algebraic thinking in the early grades involves the development of ways of thinking within activities for which the letter-symbolic could be used as a tool, or alternatively within activities that could be engaged in without using the letter-symbolic at all, for example, noticing structure. Recently, Kieran (2018) has emphasized and suggested that more attention be paid to the structural.

Structure in Numbers and Numerical Operations

According to Kieran (2018), the learning of high school algebra has included a focus on the structural aspect (e.g., Hoch & Dreyfus, 2004, Linchevski & Livneh, 1999) much more so than has been the case for early algebra. Kieran suggests that structure sense in arithmetic entails more than attending to the basic field properties. Structure sense in arithmetic involves looking through mathematical objects and decomposing and recomposing them in several structural ways. Based on Freudenthal (1983, 1991, cited in Kieran, 2018), Kieran proposes that the structure in numbers and in numerical operations is explained by the fact that the number system constitutes an order structure, in which, for each pair of numbers, a third number, for instance, its sum, can be assigned, thereby constituting an addition structure. Similarly, the multiplicative structure can be defined. Accordingly, expressing structure sense in arithmetic implies being aware of the different structural forms that numbers and numerical operations can take, for example, observing that the number 989 can be rewritten equivalently as 9x109+8 and as 9x110-1, also as $9x10^2+8x10^1+9x10^0$, among others.

In this way, Kieran (2018) suggests promoting in students, from the early grades, various experiences with equivalence of numerical expressions through the structuring processes of decomposing, composing, and recomposing. And, in line with Freudenthal who describes different means of structuring according to a variety of structures and properties, numbers and numerical operations can be decomposed and recomposed to show equivalence without calculation and involving properties related to, for example, the addition structure and the properties of equality. Thus, validating the truth-value of numerical equalities such as 67+86=68+85 by decomposing, composing, and recomposing the involved numbers implies re-expressing them in different forms, such as 67+86=60+7+1+85 and 68+85=60+7+1+85, and so it is true that 67+86=68+85.

Method

Participants

In the study, three students (S1, S2 and S3) from a Mexican public school participated, ages between 11 and 12 years. When data were collected, the students had just finished primary school. The three participants are the same students that were reported in Martínez-Hernández and Kieran (2018, 2019, in press).

Task Design and Data Collection

The study involved the design of four tasks. The focus in this paper is the fourth task, which asked students to indicate the truth-value of three numerical equalities (10+7=5+12, 530+200=300+430, and 8+2+16=10+12+4), without calculating the total of each side. The structure of the task for each of the equalities is the following:

- A true numerical equality is presented
- Students are asked to show, without calculating the total of both sides, that the equality is true.
- Students are asked for an explanation of their procedure.

The data were collected through a group interview, conducted by the first author of this paper. During the interview, each student was given the printed task sheet, which they first worked on individually for each given equality, followed by a group discussion of their answers. During the full interview, students had the opportunity to use the blackboard to explain their approaches.

Results and Discussion

When students had finished answering with respect to the veracity of the three equalities (10+7=5+12, 530+200=300+430, and 8+2+16=10+12+4) and after developing the two strategies based on decomposing, composing, and recomposing the given numbers [i.e., (i) looking for a third common form, observed when they validated 530+200=300+430 and 8+2+16=10+12+4; and (ii) transforming one side of the equality into the same form of the other side, which was observed when they validated 8+2+16=10+12+4 (see Martínez-Hernández & Kieran, in press)], they were asked to validate the equalities 10+7=5+12 and 530+200=300+430 once again. The reason for this was that the students had not used the above strategies to validate the initial equality 10+7=5+12 – they had first relied upon a computational strategy, which was followed by an *ad hoc* decomposition that had not involved an attempt to have the same numbers on both sides (see Martínez-Hernández & Kieran, 2019).

Episode 1

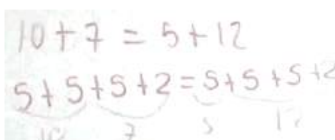
To investigate which of the two strategies based on structure sense would be used by the students, the Interviewer (I) asked them to show the veracity of the first given equality. See this episode in the following excerpt.

I: Let me go back to the first two equalities [Writes on the board 10+7=5+12]. How can it be re-expressed?

S1 and S3: Me, me, [Both want to go to the blackboard].

I: S1, go ahead, please. How could you tell this is true, without adding?

S1: [Writes on the board 5+5+5+2=5+5+5+2, see Fig. 1, left]. This [pointing to the 5+5 at the left side of the equality] I would take from the 10. This [pointing to 5+2 on the left side] from the 7. This [pointing to the 5 on the right side from the equality] from 5, and this [pointing to 5+5+2] from the 12.



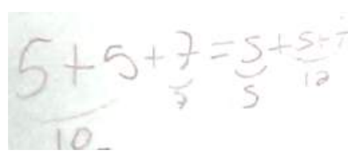


Figure 1: S1's decomposing strategy (left) and S3's (right)

As observed in Fig. 1 (left), S1 decomposes both sides of the equality into a third common form (5+5+5+2), and explains where each number comes from. In the same way, S3 proposes another decomposition (Fig.1, right) based on the same idea – also explaining where each number comes from in the rewritten equality.

Episode 2

In reaction to the students' behavior described in Episode 1, the interviewer asks them directly about the possibility of transforming the left side into the right side (or vice versa). The following excerpt presents the unfolding of the episode.

I: Finally, I am going to redo [Writes 530+200=300+430 on the board]. Can it be rewritten? You have already done this, but tell me how to do it in another way. This [Referring to the expression 530+200], for example, can it be rewritten in this way [pointing to the expression 300+430]? Or this [pointing to 300+430] in this way [pointing to 530+200]? How would you do it?

S3: Oh, yes, yes!

I: Go ahead S3

S1: Transforming

I: Tell us S1. Transforming how?

S1: By doing just as S2 and S3 just did [Referring to the strategy to validate 8+2+16=10+12+4; see Martínez-Hernández & Kieran, in press)]

S3: [Goes to the blackboard] I subtract 200 from this one [pointing to 530] to make it into 300, and add the 200 to this one [pointing to 200] to make it 430. Hold on, no! 230 [Finally writes the equality 300+430=300+430, see Fig. 2, left].

I: This [pointing to the left side 300+430 written by S3], where did it come from?

S3: From 530, I subtract 230 and add it to 200, so I get 430.

Figure 2: Rewritten equality by S3 (left); interpretation of his strategy (right)

As observed in the excerpt and in Fig. 2 (left), S3's strategy can be interpreted as follows: The left side is decomposed into 300+230+200 and then recomposed as 300+430; note that he does not have to transform the right side (see Fig. 2, right). The strategy in Episode 2 is based on a simultaneous relation that is discerned between both right and left sides of the given equality, which is not the case for the strategy in Episode 1. In other words, this strategy does not involve an arbitrary decomposing and recomposing of the left side; rather it is guided by the form of the right side.

Conclusions

According to the results, on the one hand, students tend to look for a third common form, by decomposing both sides of an equality (Episode 1) in order to validate the truth-value of numerical equalities. Hence, this strategy replaces the initial computational strategy that was spontaneously used by them at first (see Martínez-Hernández & Kieran, 2018). On the other hand, the Episode 2 strategy emerged from explicit interviewer intervention. As indicated by Martínez-Hernández & Kieran (in press), this second strategy would seem to be cognitively more demanding than the first one. In any case, both strategies go beyond that of simply observing regularities in equalities (e.g., Pang & Kim, 2018; Schifter, 2018) and therefore offer new findings with respect to the development of structure sense in numerical activity. As seen in the example of S3's work (Fig. 2), his way of decomposing, composing, and recomposing the numbers to transform one side of the equality into the form of the other side illustrates not only relational thinking based on structure sense but also the structural approach proposed by Kieran (2018) that involves looking through mathematical objects and expressing them in different structural forms. As a final comment, we envisage expanding the research so as to study the ways in which students understand the similarities and differences of their strategies.

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DESCOMPOSICIÓN, COMPOSICIÓN Y RECOMPOSICIÓN DE NÚMEROS EN IGUALDADES NUMÉRICAS: PENSAMIENTO ALGEBRAICO BASADO EN UN SENTIDO DE ESTRUCTURA

DECOMPOSING, COMPOSING, AND RECOMPOSING NUMBERS IN NUMERICAL EQUALITIES: ALGEBRAIC THINKING BASED ON STRUCTURE SENSE

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Este reporte da cuenta sobre cómo alumnos de primaria de una escuela pública Mexicana, una vez que son capaces de validar la veracidad de igualdades numéricas (e.g., 8+2+16=10+12+4) a partir de dos estrategias: transformar ambos lados de la igualdad en una tercera forma común, o reescribir un lado de la igualdad en la forma del otro lado–ambas relacionadas con la descomposición, composición y recomposición de los números– tienden a utilizar la primera de éstas como su opción inicial de validación de igualdades numéricas. Tal comportamiento de los alumnos indica que si bien muestran un razonamiento de tipo algebraico basado en un sentido de estructura en aritmética, su segunda estrategia no está suficientemente consolidada.

Palabras clave: Álgebra y pensamiento algebraico; Sentido de estructura en aritmética; Descomposición, composición y recomposición de expresiones e igualdades numéricas.

Antecedentes

El pensamiento algebraico en edades tempranas ha sido estudiado desde diferentes perspectivas, aunque ha imperado un énfasis en la generalización. Recientemente, ha sido propuesto que, además de lo general, el sentido de estructura es también parte importante de este tipo de pensamiento matemático (Kieran, 2018). Sobre lo estructural, algunos estudios se han enfocado en la observación de regularidades en igualdades numéricas (e.g., Pang & Kim, 2018; Schifter, 2018). Sin embargo, de acuerdo con Mason, Stephens y Watson (2009), un pensamiento estructural es mucho más que tal característica. En este sentido, Martínez-Hernández y Kieran (2018, 2019, en prensa), sobre el sentido de estructura en aritmética, han reportado estrategias de validación de igualdades numéricas que muestran alumnos de primaria. Los alumnos suelen utilizar estrategias de cómputo para validar igualdades numéricas de la forma $a+b=c+d$ (Martínez-Hernández & Kieran, 2018) y son también capaces de utilizar estrategias *ad hoc* para validar el mismo tipo de igualdades, basados en la descomposición de números (ver Martínez-Hernández & Kieran, 2019). En Martínez-Hernández y Kieran (en prensa) se muestran cómo los estudiantes transitan de un pensamiento aritmético manifestado en estrategias de cómputo hacia un pensamiento de tipo algebraico basado en un sentido de estructura; manifestado en la descomposición, composición y recomposición de los números, a través de dos estrategias: (i), transformar ambos lados de una igualdad numérica en una tercera forma común, y (ii) reescribir un lado de la igualdad en la forma del otro lado. A partir de estas, surge la siguiente pregunta: ¿Cuál de las dos estrategias aplican para volver a validar igualdades en las que previamente utilizaron estrategias de cómputo o bien aplicaron una estrategia *ad hoc*?

Marco Teórico

El algebra temprana ha sido caracterizado desde distintas perspectivas (e.g., Carraher & Schliemann, 2007; Kieran, 2018), en todas se reconoce el carácter algebraico de la aritmética. Sobre el pensamiento algebraico en edades tempranas, Kieran (2004) menciona que éste trata sobre el desarrollo de formas de pensamiento en actividades en las cuales el aspecto simbólico-literal puede usarse como herramienta, o alternativamente, en actividades que no requieren lo simbólico-literal, por ejemplo, observar la estructura. Recientemente, Kieran (2018) ha enfatizado y sugerido una mayor atención al aspecto estructural.

La Estructura en los Números y en las Operaciones Numéricas

De acuerdo con Kieran (2018) en el aprendizaje del álgebra, la importancia del aspecto estructural ha sido tomado en cuenta con amplitud (e.g., Hoch & Dreyfus, 2004; Linchevski & Livneh, 1999, Mason, Stephens & Watson, 2009), no así en el caso del álgebra temprana. En este sentido, Kieran propone que el sentido de estructura desde la aritmética implica mucho más que la estructura numérica basada en las propiedades de campo. El sentido de estructura en aritmética involucra observar a través de los objetos matemáticos, descomponerlos y recomponerlos en diferentes formas estructurales. Con base en planteamientos de Freudenthal (1983, 1991, citado en Kieran, 2018), Kieran propone que la estructura en los números y las operaciones se explica en el hecho de que el sistema de los enteros constituye una estructura de orden, en la cual, a cada par de enteros, un tercer número, por ejemplo su suma, les puede ser asignado, constituyendo así una estructura aditiva. De manera similar se define una estructura multiplicativa. Así, manifestar un sentido de estructura en aritmética implica ser consciente de diferentes formas estructurales que los números y las operaciones numéricas pueden tomar, por ejemplo, observar que el número 989 se puede reescribir como 9x109+8, 9x110-1, o bien como $9\text{x}10^2+8\text{x}10^1+9\text{x}10^0$, entre otras.

De esta manera, Kieran (2018) sugiere promover en los alumnos, desde los primeros grados escolares, la experiencia de la equivalencia de expresiones numéricas a través de la descomposición, recomposición y sustitución. En línea con Freudenthal, quien describe diferentes formas de estructurar, acorde a la variedad de estructuras y propiedades, los números y las operaciones numéricas pueden ser descompuestas y recompuestas para mostrar la equivalencia sin realizar cálculos y considerando propiedades relacionadas, por ejemplo, la estructura aditiva y las propiedades de la igualdad. De esta manera, para validar la veracidad de igualdades como 67+86=68+85 mediante la descomposición, composición y recomposición de los números involucrados implica re-expresarla en diferentes formas, por ejemplo: 67+86=67+1+85 y 68+85=60+7+1+85, por lo que 67+86=68+85 es verdadera.

Método

Participantes

Participaron en el estudio tres alumnos (S1, S2 y S3) de sexto grado de primaria de una escuela pública de México, de entre 11 y 12 años de edad; quienes al momento de la toma de datos estaban culminando su educación primaria. Los tres alumnos son los mismos participantes reportados en Martínez-Hernández y Kieran (2018, 2019, en prensa).

Diseño de la Tarea y acopio de datos

El estudio incluyó el diseño de cuatro Tareas. La aquí analizada corresponde cuarta, esta versa sobre mostrar la veracidad de tres igualdades sin calcular el total de cada lado (10+7=5+12, 530+200=300+430 y 8+2+16=10+12+4). La estructura de la Tarea para cada igualdad es:

- Una igualdad numérica es presentada
- Se solicita a los alumnos mostrar, sin calcular el total en ambos lados de la igualdad, que la igualdad es verdadera
- Se les solicita una explicación de su procedimiento

La recopilación de datos se llevó a cabo mediante una entrevista grupal, conducida por el primer autor (E, en adelante) de este reporte. Cada alumno contó con la tarea impresa, en la cual, los alumnos trabajaron primero de manera individual para cada igualdad dada, seguido por una discusión grupal de sus respuestas. En todo momento los alumnos tuvieron la oportunidad de pasar a un pizarrón para explicar sus procedimientos.

Resultados y Discusión

Al terminar los alumnos de responder sobre la veracidad de las tres igualdades (10+7=5+12, 530+200=300+430 y 8+2+16=10+12+4) y después de que desarrollaron dos estrategias basadas en la descomposición, composición y recomposición de los números de las igualdades dadas [i.e., (i) sobre la búsqueda de una tercera forma común, observada cuando validan 530+200=300+430 y 8+2+16=10+12+4, (ii) transformación de un lado de la igualdad en la misma forma del otro lado, observada cuando validan 8+2+16=10+12+4 (ver Martínez-Hernández & Kieran, en prensa)] les fue solicitado validar de nuevo las igualdades 10+7=5+12 y 530+200=300+430. Ello se debe a que no utilizaron tales estrategias para validar la igualdad inicial 10+7=5+12 – en esta, emplearon primero una estrategia de cómputo, y después, una estrategia de descomposición *ad hoc*, en la cual no buscan los mismos números en ambos lados (ver Martínez-Hernández & Kieran, 2019)

Episodio 1

Para indagar cuál de las dos estrategias, basadas en un sentido de estructura, podrían emplear los alumnos, el entrevistador (E) les solicita mostrar de nuevo la veracidad de la primera igualdad propuesta en la tarea, tal como se muestra en la siguiente transcripción.

E: Déjenme regresar a las primeras dos igualdades [Escribe en el pizarrón la igualdad 10+7=5+12]. ¿Cómo se puede re-expresar?

S1 y S3: Yo, yo, [A la vez manifiestan su interés por pasar al pizarrón].

E: S1, pasa por favor. ¿Cómo le harías para decir que sí es verdadera, sin hacer la suma?

S1: [Escribe en el pizarrón 5+5+5+2=5+5+5+2, ver Fig. 1, izquierda]. Éste [señala el 5+5 del lado izquierdo] lo agarraría del 10. Éste [señala el 5+2 del lado izquierdo] del 7. Éste [señala el 5 del lado derecho de la igualdad] del 5, y éste [señala 5+5+2] del 12.

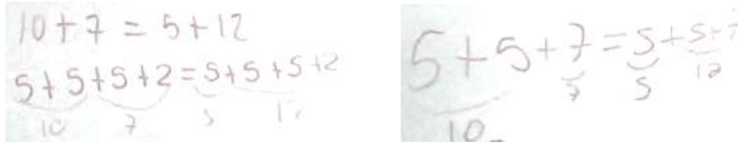

Figura 1: Estrategia de descomposición por S1 (izquierda) y S3 (derecha)

Como se observa en la Fig. 1 (izquierda), S1 descompone ambos lados de la igualdad en una tercera forma común (5+5+5+2), y explica de dónde surgen cada uno de los números de ésta. De la misma forma, S3 propone otra descomposición (Fig. 1, derecha), bajo la misma idea explica dónde surgen los números de la igualdad reescrita.

Episodio 2

Dado el comportamiento de los alumnos descrito en el episodio 1, el entrevistador les cuestiona directamente por la posibilidad de transformar el lado izquierdo en el derecho (o viceversa), esto se observa en la siguiente transcripción.

E: Finalmente, voy a escribirla [Escribe en el pizarrón la igualdad 530+200=300+430]. ¿Esa se puede reescribir? Ya lo hicieron ustedes, pero díganme otra forma más. ¿Esto [Refiriéndose a la expresión 530+200], por ejemplo, se puede reescribir de esta manera [señala la expresión 300+43] o esto [señala 300+430] de esta manera [señala 530+200]? ¿Cómo lo harían?
S3: ¡Ah, ya, ya! [Expresión coloquial de afirmación]
E: Pasa S3
S1: Transformando
E: Dinos S1, ¿transformando, cómo?
S1: Sería lo mismo que hizo S2 y S3, hace rato [Refiriéndose a la estrategia para validar la igualdad 8+2+16=10+12+4, ver Martínez-Hernández & Kieran, en prensa)]
S3: [Pasa al pizarrón] Le quito 200 a éste [señala el 530] para hacerlo 300 y se lo paso a este [señala el 200] para hacerlo para hacerlo 430. ¡Ah no! 230 [Finalmente escribe la igualdad 300+430=300+430, ver Figura 2 izquierda]
E: ¿Esto [señala es lado izquierdo 300+430 escrito por S3] de dónde salió?
S3: Del 530 le quito 230 y se los doy al 200, para que me resulte 430.

Figura 2: Igualdad reescrita por S3 (izquierda) interpretación de su estrategia (derecha)

Como se puede observar en la transcripción y en la Fig. 2 (izquierda), la estrategia desarrollada por S3 se puede interpretar de la siguiente manera: el lado izquierdo 530+200 lo descompone como 300+230+200 y después lo recompone como 300+430, mientras que el lado derecho no lo transforma (ver Fig. 2, derecha). La estrategia del Episodio 2, a diferencia de la del episodio 1, está sustentada en una relación simultánea entre los lados izquierdo y derecho de la igualdad dada. Es decir, no se trata de una descomposición y recomposición arbitraria del lado izquierdo, sino que está guiada por la forma del lado derecho.

Conclusiones

De acuerdo con los resultados, por un lado, los alumnos tienden a buscar una tercera forma común mediante la descomposición de ambos lados de una igualdad (Episodio 1) para validar el tipo de igualdades numéricas propuestas. Así, tal estrategia desplaza a la estrategia de cómputo inicialmente utilizada por ellos de forma espontánea (ver Martínez-Hernández & Kieran, 2018). Por otro, la estrategia del Episodio 2 emerge, de nuevo, a partir de la intervención explícita del entrevistador. Como se indica en Martínez-Hernández y Kieran (en prensa) esta segunda estrategia parece ser cognitivamente más demandante que la primera. En cualquier caso, ambas estrategias van más allá de observar regularidades en igualdades (e.g., Pang & Kim, 2018; Schifter, 2018), lo cual es muestran de nuevos resultados sobre el sentido de estructura en aritmética. Así, el trabajo de S3 (Fig. 2), la forma en que descompone, compone y recompone los números, para transformar un lado de la igualdad en la misma forma del otro, es un ejemplo de un pensamiento relacional basado en un sentido de estructura y de la aproximación estructural propuesta por Kieran (2018) respecto a observar a través de los objetos matemáticos y expresarlos en diferentes formas estructurales. Por ultimo, planteamos la necesidad de investigar sobre la forma en que los alumnos entienden las similitudes y diferencias de sus estrategias.

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INTUITION IN LINEAR TRANSFORMATION: SOME DIFFICULTIES

LA INTUICIÓN EN LA TRANSFORMACIÓN LINEAL: ALGUNAS DIFICULTADES

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An individual interview was conducted with five students who successfully completed a mathematics degree in Mexico. An instrument was applied that contained situations of both linear and non-linear transformations in the graphic and algebraic environment. The results were analyzed under the theoretical framework of Fischbein (1987) on intuition and intuitive models. It was obtained that the interviewed students have a universe of linear transformations that are known in the school context as a prototype. Students exclude the existence of a linear transformation in the geometric environment, when they fail to build said transformation under the composition of the range of prototype models.

Keywords: Advanced Mathematical Thinking, Algebra and Algebraic Thinking, Cognition, Representations and Visualization

Objective

The purpose of the research was to identify the intuitive models that some Bachelor of Mathematics students retain regarding Linear Transformations in R^2 (in the sense of Fischbein, 1987); It also sought to demonstrate the conceptions that students have regarding this concept. It was hypothesized that the universe that some students have regarding Linear Transformations in R^2 is reduced to rigid movements of the plane.

Theoretical framework

In the present work we are going to consider the term intuition as an instinctual knowledge, not as a method, not as a source of knowledge, but as a type of cognition. To make this approach clearer, Fischbein mentions the following examples:

> One intuitively admits that the shortest path between two points is the straight line, that each number has a successor, that the whole is greater than each of its parts, that a body must fall if it is not supported. (Fischbein, 1987)

These statements are accepted almost immediately, without the need to perform any formal test, that is; We can say that self-evidence is part of a characteristic of intuitive knowledge, but it is remarkable that there is a whole universe of statements or propositions that are not accepted so immediately, for example: if the product of the slopes of two lines is equal to one, then; these are perpendicular, or that $a^0 = 1, a \neq 0$ or the mathematical expression $\frac{n(n+1)}{2}$ generates the sum of the first n natural numbers.

> Intuition is not the primary source of truth, certainty and knowledge but this seems to be so, because this is exactly its role: to create the appearance of certainty, to attribute to various interpretations and representations a character of intrinsic certainty and unquestionable (Fischbein, 1987).

Students develop their intuition, because they resort to representations of mathematical objects. "Mental objects (concepts, operations and statements) must achieve a kind of intrinsic consistency and direct evidence, similar to the real one, external to material objects and events, if the reasoning process is a genuinely productive activity" (Fischbein, 1987) . In this way, mental representations are

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

not the product of memorization; but to the repeated experiences that the subject has had in the concept construction process: obstacles, conflicts, etc.

General characteristics of intuitive cognitions

Fischbein (1987) considers that at every level of mathematical reasoning, one must consider mainly three basic aspects.

1. *The formal aspect*: Which is essentially given by the structure logical-deductive mathematics, such as axioms, definitions, theorems and proofs.
2. *The algorithmic aspect*: Which refers to the procedures, the development of a approach until reaching the solution.
3. *The intuitive aspect*: That refers to the degree of acceptance of the concepts, mathematical propositions or statements, as something evident or true.

The research is particularly interested in the implicit models that students can develop in relation to the concept of linear transformation and the consequences of these models in learning the same concept.

Methodological elements

Five students (Saulo, Miguel, Max, David and Hugo) who graduated from the Bachelor of Mathematics at the Higher School of Physics and Mathematics of the National Polytechnic Institute in Mexico were interviewed. The interview was conducted individually on different dates over the course of a week. At the time of the interviews, each of the students belonged to a master's program in different institutions and specialties, with a common core in mathematics. It should be noted that the students were chosen for their good training and good performance in mathematics, since in this research we intend to identify those intuitive models that persist, and that are not necessarily strictly related to cognitive difficulties, etc.

The bibliography to which they referred when they were questioned about the textbook brought in during their training corresponds to the book on Linear Algebra published by the authors Hoffman & Kunze (1987), however they have consulted other authors such as Grossman (2012), Lang (1974) and Lay (2007).

The method

Once the instrument to be used was established, an a priori analysis was carried out, which hypothetically posed the possible arguments and situations to be presented during the interview. Once this analysis was carried out, we proceeded to the interview stage. The interviewer had the task of bringing, posing and explaining the situations that arose during the development of the interview, and one of his main tasks consisted of confronting the environments where the student had opposing or other people's arguments, which suggested a deeper analysis. Finally, the a priori analysis is contrasted with the a posteriori analysis, to obtain the following results.

Results

In one of the initial activities of the instrument, it requested the following:

a) Provide an example of a linear transformation.
b) Argue why the transformation you proposed is linear.

Both the student Saulo and the student Max provide examples very similar to those approached in a Linear Algebra course (see Figure 1 and Figure 2), since they are characteristic of those presented in textbooks.

Sea $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ / $T(x,y) = (2x, 2y)$

Figure 1. Saulo's production, following the example of TL.

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T(x,y) = (x+y, x-y)$

Figure 2. Max production, to the example of TL.

Fischbein (1987) considers this type of intuitive cognitions to be algorithmic in appearance; which refers to the procedures, the development of an approach until reaching the solution. The strategy used by the students (Saulo and Max) corresponds to first setting the definition of Linear Transformation and then applying it to their proposed examples until their demonstration is completed, with great skill in algebraic treatment. Attached to a priori analysis, this type of results are those expected by teachers, since they correspond to some exercises in textbooks or the first examples that are addressed to illustrate the concept of Linear Transformation. (Grossman, 2012, p.500).

Miguel provides the example of the Linear Identity Transformation (see Figure 3.), and applying his definition verifies the linearity of the transformation. It does not specify vector spaces, nor does it specify the membership of scalars to a field, as in the definition of Linear Transformation that you provided; highlights the notation used by the student presenting the definition with a “f ” referring to the function, thus also taking x & y as vectors, as a real analysis notation.

$$f(x) = x$$

$$\begin{aligned} f(\alpha x + \beta y) &= \alpha x + \beta y \\ &= \alpha f(x) + \beta f(y). \end{aligned}$$

Figure 3. Miguel's production, following the example of TL.

In the a priori analysis, this possible response by the student was warned, when conceptualizing Linear Transformations as functions in the context of their calculus courses.

For his part, the student David, argues that:

> David: There are several examples, we can start, to define the simplest transformation, which is the constant, in general the simplest functions are the constants, the ones that map all the space in a number here (indicating the counter-domain) but in this case , to be linear, it cannot be any constant. Given the structure we gave for the linear transformation, it can easily be verified that if the linear transformation is constant, then:

$$T(v) = 0 \quad \forall\, v \in V$$

Figure 4. David's production, following the example of TL.

David demonstrates his mathematical maturity by fluently providing the example of the Zero Linear Transformation. Very similar to the one shown in the textbook that he carried in his studies; "The zero transformation, defined by $0_\alpha = 0$, is a linear transformation from V to V" (Hoftman & Kunze, 1971, p.67).

It is evident that in his mathematical reasoning, David shows the formal aspect, in the sense of Fischbein (1987), which is essentially given by the logical-deductive structure of mathematics, such as axioms, definitions, theorems and demonstrations, this situation is reflected throughout the entire interview. David is very much akin to the implicit (or tacit) model:

A fundamental characteristic of a mental model is its structural entity. A model, like a theory, is not a simple isolated rule, rather a global, unitary, meaningful interpretation of a phenomenon or a concept. . . . of an implicit model is its concrete, practical nature, even if the model is an abstract construction. (Fischbein, 1987)

Of the students interviewed; the one that Fischbein (1987) classifies as an intuitive model was also observed when the following activity was requested.

a) Provide an example of a nonlinear transformation
b) Argue why the transformation you proposed is not linear.

The student Hugo does not provide any algebraic expression, as the rest of the interviewed students do; He takes Galileo's transformations, to illustrate the non-linearity of transformation

> Hugo: . . . well, here I'm going to use something like this. . . Galileo transformations, for example, are linear transformations, when. . . in fact they are valid for: when a reference system moves with respect to the other and in each of the two it is realized. . . for example the position of a particle or of a body in general, then this. . . and the Galileo transformations are linear transformations, but that only applies when the speed with which a system S 'moves with respect to another system S is constant (write), right now I don't remember what they are. . . how are the Galileo transformations established, but. . . if to this system S ', which moves with respect to the system S, we make this speed be different, not constant, that is, it has a acceleration, the resulting transformation is going to be a non-linear transformation, it is going to carry a term due to the acceleration, it is going to carry a square there. For that reason, it would be non-linear and that would also be the argument. . . Besides, if we check the calculations for this a bit, we would surely obtain a non-linear expression. . . and therefore it would not fulfill the two properties that a linear transformation must satisfy.

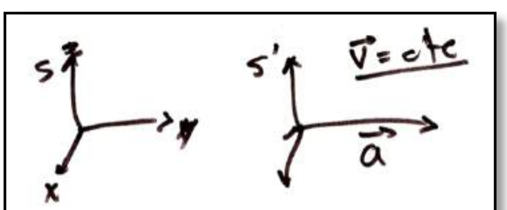

Figure 5. Hugo production, to the example of Nonlinear Transformation.

> An intuitive model is not necessarily a direct reflection of a certain reality, very often it is based on an abstract interpretation of that reality. The graph of a function is an intuitive model of that function and the function, for its part, is the abstract model of a true phenomenon […] Intuitive models that use conventional, graphical means are generally called diagrams. (Fischbein, 1987)

Conclusions

We observed that all the students showed a complete management of the concept of linear transformation, reaffirming their good training and performance in the area of mathematics, proof of this was that they all provided definitions of the concept of linear transformation presented in textbooks or that were acquired in their linear algebra courses, they also showed examples and counterexamples of linear transformations; It should be noted that all of these were different.

We verify that the students immediately identify a prototype linear transformation (expansions, contractions, rotations, reflections and the combination of them) in both environments. However, in the geometric part where the figures show a fixed vector, the students cannot identify the linear transformation, differing from the corresponding situation in the algebraic stage, which led to a confrontation of their arguments.

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LA INTUICIÓN EN LA TRANSFORMACIÓN LINEAL: ALGUNAS DIFICULTADES

INTUITION IN LINEAR TRANSFORMATION: SOME DIFFICULTIES

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Se realizó una entrevista individual a cinco estudiantes que cubrieron satisfactoriamente una licenciatura en matemáticas en México. En ella; se aplicó un instrumento que contenía situaciones de transformaciones tanto lineales, como no lineales en el ambiente gráfico y algebraico. Los resultados se analizaron bajo el marco teórico de Fischbein (1987) sobre la intuición y los modelos intuitivos. Se obtuvo que los estudiantes entrevistados, disponen de un universo de transformaciones lineales que son conocidas en el contexto escolar como prototipo. Los estudiantes excluyen la existencia de una transformación lineal en el ambiente geométrico, cuando no logran construir dicha transformación bajo la composición de la gama de modelos prototipos.

Objetivo

La investigación tuvo como propósito el identificar los modelos intuitivos que conservan algunos estudiantes Licenciatura en Matemáticas en cuanto a las Transformaciones Lineales en R^2 (en el sentido de Fischbein, 1987); asimismo se busco evidenciar las concepciones que poseen los estudiantes respecto a éste concepto. Se tuvo como hipótesis, que el universo que poseen algunos estudiantes respecto a las Transformaciones Lineales en R^2 se reduce a movimientos rígidos del plano.

Marco Teórico

En el presente trabajo el término intuición vamos a considerarlo como un conocimiento instintivo, no como un método, no como una fuente de conocimientos, sino como un tipo de cognición. Para hacer más claro este acercamiento, Fischbein menciona los siguientes ejemplos:

> Uno admite intuitivamente que el camino más corto entre dos puntos es la línea recta, que cada número tiene un sucesor, que el todo es más grande que cada una de sus partes, que un cuerpo debe caerse si no esta sostenido. (Fischbein, 1987)

Estas afirmaciones son aceptadas de forma casi inmediata, sin tener la necesidad de realizar alguna prueba formal, es decir; podemos decir que la autoevidencia forma parte de una característica del conocimiento intuitivo, pero es notable que existe todo un universo de afirmaciones o proposiciones que no son aceptadas de forma tan inmediata, por ejemplo: si el producto de las pendientes de dos rectas es igual a uno, entonces; estas son perpendiculares, ó que $a^0 = 1, a \neq 0$ ó la expresión matemática $\frac{n(n+1)}{2}$ genera la suma de los n primeros números naturales.

Una habilidad que desarrollan los estudiantes en una disciplina como las matemáticas corresponde a la intuición, la cual adquieren debido a la interacción ineludible al: conocer, comprender, construir y/o emplear objetos que no se puede acceder de manera directa, por su naturaleza abstracta. Una formación en matemáticas, no se reduce a un sistema deductivo de conocimientos, “la actividad creativa en matemáticas es un proceso constructivo en el cual los procedimientos inductivos, las analogías y las conjeturas plausibles, juegan un papel fundamental” (Gómez-Chacón, 2000, p.30). La intuición, es un término que no tiene un sentido universal en la comunidad de la matemática educativa; para ello recurrimos a la descripción propuesta por nuestro marco teórico.

> La intuición no es la fuente primaria de la verdad, certeza y conocimiento pero esto parece ser así, porque este es exactamente su papel: crear la apariencia de certeza, atribuir a diversas interpretaciones y representaciones un carácter de certeza intrínseca e incuestionable (Fischbein, 1987).

Los estudiantes desarrollan su intuición, porque recurren a representaciones de los objetos matemáticos. “Los objetos mentales (conceptos, operaciones y declaraciones) deben conseguir una especie de consistencia intrínseca y evidencia directa, similar a la real, externa a objetos materiales y eventos, si el proceso de razonamiento es una actividad genuinamente productiva” (Fischbein, 1987). De esta manera, las representaciones mentales no son el producto de la memorización; sino a las reiteradas experiencias que ha tenido el sujeto en el proceso de construcción del conceptos: obstáculos, conflictos, etc.

Características generales de cogniciones intuitivas

Fischbein (1987) considera que en todo nivel de razonamiento matemático, se deben de considerar principalmente tres aspectos básicos.

1. *El aspecto formal*: Que viene a estar dado esencialmente por la estructura lógico-deductiva de la matemática, como son los axiomas, las definiciones, teoremas y demostraciones.
2. *El aspecto algorítmico*: Que se refiere a los procedimientos, al desarrollo de un planteamiento hasta llegar a la solución.
3. *El aspecto intuitivo*: Que se refiere al grado de aceptación de los conceptos, proposiciones o afirmaciones matemáticas, como algo evidente o cierto.

La investigación se interesa particularmente los modelos implícitos que los estudiantes pueden desarrollar en relación con el concepto transformación lineal y las consecuencias de estos modelos en el aprendizaje del mismo concepto.

Elementos metodológicos

Se entrevistó a cinco estudiantes (Saulo, Miguel, Max, David y Hugo) egresados de la Licenciatura en Matemáticas en la Escuela Superior de Física y Matemáticas del Instituto Politécnico Nacional en México. La entrevista se realizó de manera individual en diferentes fechas en el transcurso de una semana. Al periodo de las entrevistas, cada uno de los estudiantes pertenecía a un programa de maestría en diferentes instituciones y especialidades, con un tronco común en matemáticas. Cabe aclarar que los estudiantes fueron elegidos por su buena formación y buen desempeño en matemáticas, ya que en esta investigación pretendemos identificar aquellos modelos intuitivos que persisten, y que no necesariamente guardan estricta relación con dificultades cognitivas, etc.,

La bibliografía a la que hicieron referencia cuando se les cuestionó sobre el libro de texto llevado en el transcurso de su formación, corresponde al libro de Álgebra Lineal publicada por los autores Hoffman & Kunze (1987), sin embargo han consultado a otros autores como Grossman (2012), Lang (1974) y Lay (2007).

El Método

Una vez establecido el instrumento a emplear, se realizó un análisis a priori, donde se planteaba de manera hipotética, los posibles argumentos y situaciones a presentarse durante la entrevista. Una vez realizado este análisis, se procedió a la etapa de entrevistas. El entrevistador tuvo la labor de llevar, plantear y explicar las situaciones que se presentaban durante el desarrollo de la entrevista, y uno de sus principales tareas consistió en confrontar los ambientes donde el estudiante tenía argumentos opuestos o ajenos, que sugerían un análisis más profundo. Finalmente se contrastan, el análisis a priori con el análisis a posteriori, para obtener los siguientes resultados.

Resultados

En una de las actividades iniciales del instrumento solicitaba lo siguiente:

a) Proporciona un ejemplo de una transformación lineal.
b) Argumenta por qué es lineal la transformación que propusiste.

Tanto el estudiante Saulo, como el estudiante Max proporcionan ejemplos muy similares a los abordados en un curso de Álgebra Lineal (ver Figura 1 y Figura 2), dado que son característicos a los presentados en los libros de texto.

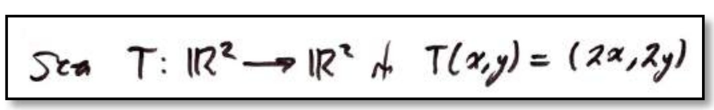

Figura 1. Producción de Saulo, al ejemplo de TL.

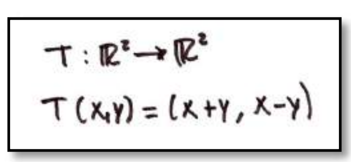

Figura 2. Producción de Max, al ejemplo de TL.

A este tipo de cogniciones intuitivas, Fischbein (1987) las considera como de aspecto algorítmico; las cuales se refiere a los procedimientos, al desarrollo de un planteamiento hasta llegar a la solución.

La estrategia que emplean los estudiantes (Saulo y Max), corresponde a situar en un primer momento a la definición de Transformación Lineal y posteriormente aplicar a sus ejemplos propuestos hasta cumplir su demostración, con gran habilidad en el tratamiento algebraico. Apegados al análisis a priori, este tipo de resultados son los esperados por los profesores, dado que corresponden a algunos ejercicios en los libros de textos o los primeros ejemplos que se abordan para ilustrar el concepto de Transformación Lineal. (Grossman, 2012, p.500).

Miguel roporciona el ejemplo de la Transformación Lineal Identidad (ver Figura 3.), y aplicando su definición verifica la linealidad de la transformación. No especifica los espacios vectoriales, como tampoco la pertenencia de los escalares a un campo, al igual que en la definición de Transformación Lineal que proporcionó; resalta la notación que maneja el estudiante presentando la definición con una "f "aludiendo a la función, así también tomando a x & y como vectores, como una notación de análisis real.

$$f(x) = x$$

$$\begin{aligned} f(\alpha x + \beta y) &= \alpha x + \beta y \\ &= \alpha f(x) + \beta f(y). \end{aligned}$$

Figura 3. Producción de Miguel, al ejemplo de TL.

En el análisis a priori se advirtió de esta posible respuesta por parte del estudiante, al conceptualizar a las Transformaciones Lineales como funciones en el contexto de sus cursos de cálculo.

Por su parte el estudiante David, argumenta que:

> David: hay varios ejemplos, podemos empezar, para definir la transformación más sencilla, que sea la constante, en general las funciones más sencillas son las constantes las que mapean todo el espacio en un número de acá (indicando el contradominio) pero en este caso, para que sea lineal, no puede ser cualquier constante. Dada la estructura que dimos para la transformación lineal, se puede verificar fácilmente, que si la transformación lineal es constante, entonces:

$$T(v) = 0 \quad \forall\, v \in V$$

Figura 4. Producción de David, al ejemplo de TL.

David evidencía su madurez matemática al proporcionar con soltura el ejemplo de la Transformación Lineal Cero. Muy semejante al que se muestra en el libro de texto que llevó en sus estudios; "la transformación cero 0, definida por $0_\alpha = 0$, es una transformación lineal de V en V" (Hoftman & Kunze, 1971, p.67).

Es evidente que en su razonamiento matemático, David evidencia el aspecto formal, en el sentido de Fischbein (1987), el cual viene a estar dado esencialmente por la estructura lógico-deductiva de la matemática, como son los axiomas, las definiciones, teoremas y demostraciones, esta situación se ve reflejada a lo largo de toda la entrevista. David está muy afín al modelo implícito (o tácito):

> Una característica fundamental de un modelo mental es su entidad estructural. Un modelo, como una teoría, no es una regla aislada simple, más bien una interpretación global, unitaria,

significativa de un fenómeno o un concepto. . . . de un modelo implícito es su naturaleza concreta, práctica, aun si el modelo es una construcción abstracta. (Fischbein,1987)

De los estudiantes entrevistados; también se observó aquel que Fischbein (1987) clasifica, como modelo intuitivo, cuando se solicitó la siguiente actividad.

a) Proporciona un ejemplo de una transformación no lineal
b) Argumenta por qué no es lineal la transformación que propusiste.

El estudiante Hugo, no proporciona expresión algebraica alguna, como lo hace el resto de los estudiantes entrevistados; él toma a las transformaciones de Galileo, para ilustrar la no linealidad de transformación.

Hugo: . . . bueno, aquí voy a usar algo que este . . . las transformaciones de Galileo, por ejemplo son transformaciones lineales, cuando . . . de hecho valen para: cuando un sistema de referencia se mueve respecto del otro y en cada uno de los dos se realiza . . . por ejemplo la posición de una partícula o de un cuerpo en general, entonces este . . . y las transformaciones de Galileo son transformaciones lineales, pero eso sólo vale cuando la velocidad con la que se mueve un sistema S' respecto de otro sistema S, es constante (escribe v = cte), ahorita de momento no recuerdo cuáles son las . . . cómo están establecidas las transformaciones de Galileo, pero . . . si a este sistema S', que se mueve con respecto al sistema S, hacemos que esta velocidad sea diferente, no sea constante, es decir, tenga una aceleración, la transformación resultante va a ser una transformación no lineal, va a llevar un término debido a la aceleración, va a llevar un cuadrado ahí. Por esa razón, sería no lineal y además ese sería el argumento . . . a parte que si nos metemos a comprobar un poco los cálculos de esto, seguramente obtendríamos una expresión no lineal . . . y por lo tanto no cumpliría con las dos propiedades que debe satisfacer una transformación lineal.

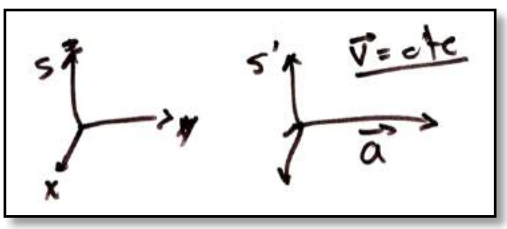

Figura 5. Producción de Hugo, al ejemplo de Transformación No Lineal.

Un modelo intuitivo no es necesariamente una reflexión directa de una cierta realidad, muy a menudo está basado en una interpretación abstracta de aquella realidad. El gráfico de una función es un modelo intuitivo de aquella función y la función, por su parte, es el modelo abstracto de un verdadero fenómeno[…] Los modelos intuitivos que usan medios convencionales, gráficos son generalmente llamados diagramas. (Fischbein, 1987)

Conclusiones

Observamos que todos los estudiantes mostraron un manejo íntegro del concepto de transformación lineal, reafirmando su buena formación y desempeño en el área de las matemáticas, prueba de ello fue que todos proporcionaron definiciones del concepto de transformación lineal que presentan los libros de texto o que fueron adquiridas en sus cursos de álgebra lineal, así también mostraron ejemplos y contraejemplos de transformaciones lineales; cabe señalar que todos estos fueron diferentes.

Comprobamos que los estudiantes identifican inmediatamente una transformación lineal prototipo (expansiones, contracciones, rotaciones, reflexiones y la combinación de ellas) en ambos ambientes.

Sin embargo, en la parte geométrica donde las figuras muestran un vector fijo los estudiantes no logran identificar la transformación lineal, difiriendo de la situación correspondiente de la etapa algebraica, lo cual desembocó en una confrontación de sus argumentos.

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A STUDY OF CONTRADICTIONS AS A LEVER FOR CONTINUING EDUCATION: A CASE INVOLVING ALGEBRAIC GENERALIZATION ACTIVITIES

ÉTUDE DE CONTRADICTIONS COMME LEVIER DE FORMATION CONTINUE: UN EXEMPLE AUTOUR D'ACTIVITÉS DE GÉNÉRALISATION ALGÉBRIQUE

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Transforming the teaching practice involves the adoption of new methods that must engage teachers in reflecting on their practice (CSE, 2004; Day, 1999). The roles of the researcher are multiplying. His quest to understand the teaching/learning activity is mirrored by a concern to examine the teaching profession from more than just a normative standpoint, thereby contributing to the reflection on actions and participating in the decisions that will guide their interventions (Bednarz, 2009). Influenced by historical and cultural theories (Radford, 2011; Engestrom, 1999) and more specifically by the concept of contradiction, our communication illustrates our way of "grasping" the awareness process and the coming into being of generality layers linked to the development of algebraic thinking occurring in the various classrooms of the teachers we are supporting and acting as trainers with them.

Keywords: Algebra and algebraic thinking, Teacher knowledge, Mathematical knowledge for teaching, Teaching activities and practices.

Background of the action research

Training mechanisms targeting the prescriptive training of teaching approaches are not obtaining the expected benefits (Bednarz & Proulx, 2010; Tardif, Lessard & Gauthier, 1998). Rather, the transformation of the teaching practice involves the appropriation and integration of new methods that must engage teaching professionals in reflecting on their practice (CSE, 2014; Day, 1999). The role of the researcher is thus changing. His quest to understand the teaching activity is mirrored by a concern to examine the teaching profession from more than just a normative standpoint, thereby contributing to the reflection on actions and participating in the decisions that will guide the interventions of the professionals (Bednarz, 2009).

From 2013 to 2017, we conducted an action research involving more than twenty high school teachers and academic advisors. The main objective: reflect together and improve the teaching practices aiming at the development of algebraic thinking among junior high school students (students aged 12 to 15). This research is the continuation of other studies (Kaput, 1998; Squalli, Mary & Marchand, 2011) suggesting to rethink classroom interventions fostering algebraic learning even before introducing literal language. To do this, different approaches (e.g. generalization to identify formulas and introduction to analytical reasoning through problem solving) promoting the introduction of algebra were considered and tested over the four years.

Reflecting on the development of algebraic thinking

Throughout this co-development work between the participants and the research team, the expression of a mathematical activity that we reformulated in terms of acting was the first step in inviting the participants to reconceptualize algebraic thinking as sensitive thinking as defined by Radford (2011). It is thereby considered a social process in the coming into being of historical and cultural methods and is mediated through gestures, the body, signs and artefacts provided to the student. These semiotic means are constituent parts of the thinking process. More specifically, the

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

three conditions linked to the development of algebraic thinking that we address like Radford (2014) were discussed regularly in such a way as to recognize their expression among students and to influence teaching to foster their emergence. These conditions are: 1) *Reason about the indeterminacy*, or the ability to exploit problems involving not-known numbers such as unknows, variables or parameters; 2) *Denotation*, which refers back to the use of signs (alphanumeric or non-conventional signs, gestures, natural language or a mix of those) to name, symbolize what is considered to be not-known; 3) *Reason analytically*, or treating indeterminate number(s) as though they were known and operate on them. The background of this project constantly brought us back to this desire to foster, among students, both the use of the letter without it being imposed by teachers and the development of algebraic expressions whose meaning could be made clear to their peers.

Following the presentation of the various roles that we, as researchers, played as part of the action research, this communication illustrates how the study and coming into being, among teachers, of certain contradictions inherent to the teaching of algebraic generalization help to enrich discussions with teachers on the way in which to introduce algebraic generalization.

Diversification of the roles of the researchers according to their objectives

Just like teachers and academic advisors, research assistants and researchers are here conceptualized as subjectivities expressing themselves in a common project whose goal is to develop algebraic thinking. The objectives of the researchers influence the intentionality of their activity and are:

- Support our participants and collectively reflect on the development of algebraic thinking among students
- Document algebra teaching/learning through the study of certain moments experienced in our participants' classrooms
- Document the support activity

These objectives thereby influence the different roles played as researchers. This text fosters the distinction of these roles although they were more often than not expressed in an overlapping way.

Table 1: Roles and tasks of the researchers

Roles	Tasks
Trainer	· Develop training activities · Host the training sessions · Share examples experienced in class by one of us while we were high school teachers · Plan and prepare the training days (methods, tools, etc.) · Anticipate / identify the learning obstacles and intervene if necessary · Manage the group in such a way as to facilitate professional learning · Evaluate the progress or achievement of the training objectives · Identify the needs and priorities
Classroom session investigator	· Observe and study the algebra teaching/learning activities as they take place in the classrooms · Produce and communicate research results that shed light on certain aspects of the activities studied
Educational resource investigator	· Study the educational kits provided to teachers and academic advisors

Investigator on the co-development activity between the participants and research team	· Observe and study the co-development activity between the participants, professional and research assistants as they evolve mainly on the discussion and training days and targeting the development of algebraic thinking among junior high school students · Produce and communicate research results that shed light on certain aspects of the activity studied
Support worker	· Develop and maintain collaboration relationships with the teachers, academic advisors, professional and research assistants · Offer (or make sure to offer when the research professional is involved) educational support to the teachers and academic advisors between the training days · Help find solutions to the problems perceived

Adaptation from Lessard (2008) and Gagnon (2010)

The activity theory as a tool for the researcher investigating the co-development activity between the participants and research team

As such, to report on the method we used to support and study the teaching/learning activity targeting the introduction of algebra in our teachers' classrooms and its transformation under the influence of the project, it seems useful to recall that the principles of the perspective used inevitably steer the production of data, its interpretation and more specifically, our way of conceptualizing the actions of the researcher in her different functions. The assumptions of the social, historical and cultural theories feed this text: the ways in which people do things, think and are are considered social practices that are mutually constitutive part of the culture in which they live and that must account for past similar activity forms (historical and genetic perspective). Phenomena must be studied holistically rather than isolating certain elements that would not account for the links they have between themselves (Langemeyer & Roth, 2006). As such, the activity is considered to be the smallest analysis unit allowing for researchers to give meaning to the participant coming into being process through the actions taken by the participants. To better grasp the activity concept, one must understand its dialectic (ideal, material) nature. According to Leontiev, a subject's (teacher, academic advisor or researcher) activity is always geared toward an object/a motive. The goal of this activity is reflected in the actions. In the special case of learning, Roth and Radford note that:

> "[…] students cannot know the object/motive of the learning activity: the object/motive itself has to be the outcome of the learning activities so that others – e.g. teachers – have to take on the regulative function that in other productive human activities exists in the known object/motive" (Roth & Radford, 2011, p.16).

As written by Radford (2015), student activity is the materialization of cultural archetypes seen in "actions, words, perceptions, gestures, symbols, reasoning (p.338)". Our role as trainers also calls us to insert methods linked to the historical and cultural development of algebraic thinking in the support activity of our participants and make the aspects of teaching/learning to consider when looking at it from an educational perspective come into being. As such, if for Roth and Radford (2011) the challenge of the learning activity for students is precisely that they recognize the object/motive through their own actions, we will see that the same goes for researchers who commit to contributing to the training of teachers and academic advisors.

Roth and Radford (2011) invite us to consider thought from an anthropological (its origins) and ontogenical (the conditions of its existence) perspective. On the one hand, thought is conceptualized as *praxis cogitans*. Thought is not static; it is activity and movement. Considered a potential, it reveals itself, is reflected in a singular and becomes/is a purpose/activity of consciousness. Radford (2011) then speaks of thought/activity as an awareness-building processes he called objectification. On the other hand, this awareness-building process depends on one's social existence. Researchers add that these individual and collective awareness-building processes are mutually constitutive.

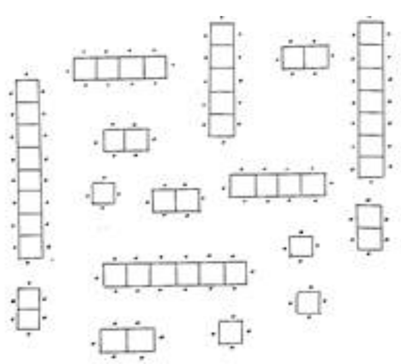
The concept of contradiction

Historical and cultural theories invite us to conceptualize activity as movement. Object/motive motivates an individual's activity; the object of the activity is materialized/reflected in his actions. This flow is considered to be the effect of the activity itself. Taking the work of linguists as an example, Roth (2012) illustrates the meaning to give to any activity. Our language changes in its uses. As written by Bakhtin (1981), a language dies as soon as it is no longer used. Every time a word is used, its meaning reifies itself, changes, much like the language itself is transformed by the use of a word. Thus, words, signs, in a dialectical approach, are not a single unique entity (Roth, 2014). In their uses, they carry and exhibit an internal contradiction that becomes apparent in the various ways in which people use signs. According to Roth and Radford (2011), these internal contradictions refer back to conflictual aspects that coexist dialectically in a phenomenon. Other work uses the concept of contradiction, but it is instead conceptualized differently. This is the case for the work of Engeström (2001) and Potari et al. (2018). The latter studies and distinguishes the activity of different communities (e.g. Department, teachers) considered as distinct systems. They then identify the contradictions in the contrary practices, the different choices between teachers or between a teacher and an external source (a researcher, manual or program). It is then said that the study of contradictions stemming from two systems can create learning opportunities and thereby transform the actions and goals of an activity. The transformation activity must then be looked at from a collective standpoint, all the while taking into account the different mediating factors (subject, tool, rules, community, division of work…) that influence this same activity.

Some elements of methodology

As part of this action research, the collaborative sessions with the teachers and academic advisors were all filmed. During these sessions, the project participants familiarized themselves with certain tasks, had analyzed videos of teachers facilitating these tasks and had attended the presentation of some theoretical content by the researchers, namely Radford's (2003) typology of generalization, which we will not refer to here. Also, the experiments conducted in the teachers' classrooms were also filmed. Transcriptions of these recordings were made. As part of this communication, we will discuss the experiments that followed the training session during which *Marcel's restaurant* situation was presented to the teachers (see Figure 1).

Marcel, a restaurant owner, has single tables in his restaurant. He places these tables one beside the other to be able to seat his clients when they arrive. He has tables of different sizes: large, small, medium…



Marcel would very much like not having to count the clients coming in every time to decide at which table to seat them. Can you help him find a way to quickly calculate the number of clients that can be seated at a table, regardless of the size of the table? Our owner lives far so he will be waiting for you to write to him about this.

Write him a **message with words** that would indicate a way to quickly find out how many people he can seat at a table, for any table.

Messages are long to read for Marcel so rewrite your message but this time as a text message so that he can read it quickly.

Figure 1. Marcel's restaurant situation

Before being able to discuss the potential and inherent intentions of the situation, the teachers had to solve it. So, for the first question, different word messages are possible. There are lots of good messages such as: 1) There are 2 people per table facing each other and one person at each end of the table; 2) 3 people can be seated at the end tables and 2 people facing each other for each table between these two tables; 3) We place one person per table and another person on one end of the table and we do this twice (we notice a symmetry), incorrect messages are also possible. These messages are formulated in context; here the motives presented are an essential support to the problem-solving process. In the second question, the purpose is to bring students to move on to symbolization, which helps to illustrate a size (here the number of tables) concisely and briefly.

Reflection on the contradictions recognized in the teachers' experiments

Through this situation, the goals of the researchers are to give meaning to symbolism through the emergence of symbolization that will be specific and spontaneous to each student. By not imposing the use of a sign (letter or other) in particular, we can expect a variety of symbolizations to represent the same variable. Other than this work on symbolization, the purpose is also to motivate the use of various algebraic expressions that will be equivalent and that will bring some flexibility in the way in which to perceive the pattern. This method opens the door to a work on equivalent expressions.

After the training session, numerous teachers experimented with this situation in their classroom to set out what they had observed in their students during the next meeting. Upon their return, the trainers observed diversity in the teachers' ways of doing things (see Table 1).

Table 2. The teachers' different ways of doing things for *Marcel's restaurant*

Stéphane	Annie	Alexandre
Stéphane considers this situation to be the first he uses to introduce algebra in his classroom. He referred back to it several times over the next few weeks. Especially, when he discussed the recognition of proportionality situations (words, value table and graph). During the first period, in teams of two, his students must find two different formulas expressed with algebraic symbols. The teacher validates the formulas obtained and invites a few teams to come explain them in front of the class by coordinating their explanations of the table visual. During the next class, the teacher focuses on the representation of the formulas taken from two representation	Annie mentions having dealt with the chapter on "geometric and numerical patterns" before. The students thus learned to extract formulas by studying the additive recurrence in the progression of different patterns. The teacher introduced the situation proposed with the intention of having the different messages she wishes to obtain in words or as a symbolic form extracted. During the problem solving, she also invites her students to build a value table to extract a formula. She is disappointed by the lack of variety of formulas.	This situation is the first that the teacher uses to introduce algebra according to him; however, he says that he started working on the translation of comparison relations only during games at the end of some periods. For this situation, the teacher promotes the emergence of different messages that are based on a study of the pattern. When returning to whole class, the focus is put on validating each message expressed in words by relying on the visual of the situation. He is therefore concerned with coordinating the visual with words. Following the first period, use of symbolism is decreased. Alexandre focuses on the potential of the formulas expressed in words to predict the maximum number of

registers: value table and graph. The teacher's purpose is to express the links existing between these three registers: table, graphic and symbolic.		people that can be seated according to the number of tables.

It can be noted that although the teachers all orchestrated *Marcel's restaurant* situation, the generalization activity or, more specifically, the coming into being process experienced by the students differs in each of the classrooms depending on what the teachers value in terms of the development of algebraic thinking. As trainers, we gave ourselves the role of documenting the teaching/learning activity lived in each of the participating teachers' classrooms. Here we can document the various facets of the same object (see Table 1). That said, the role of trainer is much more than that; it is about facilitating a group discussion on the different intentions driving the teachers when the experiment was set up and on the activities lived. While acknowledging the potential of these different methods, it is about going back to the initial intentions targeted (motive of the activity) by the trainers and set out during the previous meeting. It is also about making the teachers see the various facets (meaning of the letter, emergence of algebraic expressions whose meaning differs depending on the students' objectification process, generalization levels recognized…) of algebraic thinking that develops in each classroom.

In Alexandre's class, the trainers recognize the importance to giving meaning to algebraic expressions. They strive to make the teachers see all the work completed by the teacher to help his/her students coordinate the verbalization of the generalization messages extracted from the visual provided. They then make this contradiction (idealized/materialized in the classroom activity) be seen, which can be better explained by associating the generalization levels of Radford (2003). The whole class discussion in Alexandre's classroom is thus mainly in terms of contextual generalization. The latter still includes references to the specificities of the context objects and their characteristics in terms of the spatial and temporal situation. The generalization messages remain contextual in that their designation mode still depends on spatial properties. As specified by Radford (2003), the "arguments" or "variables" are no longer numbers, but generic objects that are designated by generic terms such as "the figure" or "the next figure". Other contradictions are felt by the trainers during the presentation of the approaches of Stéphane and Annie but they are somewhat different. Indeed, adding a value table in the situation (Annie's approach) limits the expressions extracted by the students, which the teacher observed. There are two levels of tension. On the one hand, it is no longer about working on the different possible symbolizations and a variety of expressions that mean something to the students; here, looking for a rule seems to be a priority. On the other hand, the generalization work undertaken by the students seems to be glossed over by the application of a procedure (for each table added, there are two more people so we write "times two" in the formula). This contradiction was a driving change for the teacher who became aware of it herself when she compared the activity in her classroom with that of her peers.

As for Stéphane's approach, the trainers did not intervene in the same way given that the intentions targeted are met. The focus is put on the work involving the representation registers; it is then about making aware the teachers that converting an expression in another register and coordinating the various significations is a daunting task for the students who are beginning to learn about algebraic generalization. It is not about telling the teachers that what they are doing is wrong or that there is a contradiction, in the sense of Engestrom (2001), according to "our trainer system", but rather about helping them look at the different facets that mesh together when it comes to reflecting on the work of algebraic generalization.

Conclusion

The study of the teaching/learning activities in the different classrooms of the teachers we are supporting illustrates the teachers' different methods that, although stemming from the same situation, have different motives. In our investigator role, we study the materialization of the expression of algebraic thinking in each class as contradictions between what the teachers aim and the materialized activity. In our role as trainers, it is by relying on these different ways of doing that, we make apparent different facets of algebraic thinking but which nevertheless shape it differently.

Studying certain contradictions that we are looking to make visible in our role as trainers allows the teachers to learn how to recognize the emergence of the different facets that influence the expression of algebraic thinking.

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ÉTUDE DE CONTRADICTIONS COMME LEVIER DE FORMATION CONTINUE: UN EXEMPLE AUTOUR D'ACTIVITÉS DE GÉNÉRALISATION ALGÉBRIQUE

A STUDY OF CONTRADICTIONS AS A LEVER FOR CONTINUING EDUCATION: A CASE INVOLVING ALGEBRAIC GENERALIZATION ACTIVITIES

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La transformation de la pratique enseignante implique un travail d'appropriation de nouvelles manières de faire qui doit engager les enseignants dans un travail réflexif sur leur pratique (CSE, 2004; Day, 1999). Les rôles du chercheur se multiplient. Sa recherche de compréhension de l'activité d'enseignement/apprentissage se voit doublée d'un souci de dépasser le regard normatif sur ce que font les enseignants pour ainsi contribuer à la réflexion sur les actions et prendre part aux décisions qui guideront leurs interventions (Bednarz, 2009). Sous l'influence de théories historico-culturelles (Radford, 2011; Engestrom, 1999) et plus particulièrement du concept de contradiction, notre communication illustre notre manière de «saisir» le processus de prise de conscience et de mise en apparence de couches de généralité associées au développement de la pensée algébrique qui a cours dans les différentes classes des enseignants que nous accompagnons et d'agir en tant que formatrices auprès d'eux.

Mots-clés: Algebra and algebraic thinking, Teacher Knowledge, Mathematical Knowledge for Teaching, Instructional activities and practices.

Mise en contexte de la recherche-action

Les dispositifs de formation qui visent une formation prescriptive d'approches de l'enseignement ne connaissent pas les bénéfices escomptés (Bednarz & Proulx, 2010; Tardif, Lessard & Gauthier, 1998). La transformation de la pratique enseignante implique plutôt un travail d'appropriation et d'intégration de nouvelles manières de faire qui doivent engager les professionnels de l'enseignement dans un travail réflexif sur leur pratique (CSE, 2014; Day, 1999). Le rôle du chercheur est alors en mutation. Sa recherche de compréhension de l'activité enseignante se voit doublée d'un souci de dépasser le regard normatif sur ce que font les enseignants pour ainsi contribuer à la réflexion sur les actions et prendre part aux décisions qui guideront les interventions des professionnels (Bednarz, 2009).

De 2013 à 2017, nous avons mené une recherche-action impliquant plus d'une vingtaine d'enseignants et des conseillers pédagogiques du secondaire. L'objectif premier : réfléchir ensemble et améliorer les pratiques d'enseignement visant le développement de la pensée algébrique chez les élèves du premier cycle du secondaire (élèves de 12 à 15 ans). Cette recherche est le prolongement d'autres études (Kaput, 1998; Squalli, Mary & Marchand, 2011) suggérant de repenser les interventions en classe qui favorisent l'apprentissage de l'algèbre avant même l'introduction au langage littéral. Pour ce faire, différentes approches (p.ex. généralisation en vue de dégager des formules et introduction au raisonnement analytique par la résolution de problèmes) favorisant l'introduction de l'algèbre ont été abordées et expérimentées au courant des quatre années.

Réfléchir le développement de la pensée algébrique

Au fil de ce travail de coélaboration entre les participants et l'équipe de recherche, l'expression d'une activité mathématique que nous avons reformulée en termes d'agir fut le prélude invitant les participants à reconceptualiser la pensée algébrique en tant que pensée sensible telle qu'elle est définie par Radford (2011). Elle est ainsi considérée en tant que processus social de mise en apparence de manières de faire historico-culturelles et est médiatisé par les gestes, le corps, les signes et les artefacts que l'on met à la disponibilité de l'élève. Ces moyens sémiotiques sont des parties constitutives de la pensée. Plus spécifiquement, les trois conditions associées au développement de la pensée algébrique que nous reprenons de Radford (2014) furent régulièrement discutées de manière à reconnaitre leur expression chez les élèves et à teinter l'enseignement de manière à favoriser leur émergence. Ces conditions sont : 1) *Raisonner sur l'indéterminé*, soit cette capacité à exploiter des problèmes qui implique des nombres qui ne sont pas connus; 2) *Dénoter* qui renvoie à l'usage de signes (signes alphanumériques ou non conventionnels, gestes, langage naturel ou mélange de ce qui précède) pour nommer, symboliser ce qui est considéré comme étant inconnu ; 3) *Raisonner analytiquement*, soit traiter les quantités indéterminées comme si elles étaient connues et opérer sur celles-ci. La trame de fond de ce projet nous renvoyait constamment à ce désir de favoriser chez les élèves, d'une part, le recours à la lettre sans qu'elle ne soit imposée d'emblée par les enseignants et, d'autre part, à l'élaboration d'expressions algébriques dont la signification saurait être explicitée à leurs pairs.

Suite à la présentation de la variété des rôles que les chercheures que nous sommes avons joués dans le cadre de la recherche-action, la présente communication illustre comment l'étude et la mise en apparence auprès des enseignants de certaines contradictions inhérentes à l'apprentissage de la généralisation algébrique contribuent à enrichir les discussions avec les enseignants sur des manières d'introduire la généralisation algébrique.

Diversification des rôles des chercheures selon leurs objectifs

Au même titre que les enseignants et les conseillers pédagogiques, les assistants de recherche et les chercheures sont ici conceptualisés en tant que subjectivités qui s'expriment dans un projet commun dont l'objet est le développement de la pensée algébrique. Les objectifs des chercheures colorent l'intentionnalité de leur activité et sont :

- Accompagner nos participants et réfléchir collectivement au développement de la pensée algébrique chez les élèves
- Documenter l'activité d'enseignement/apprentissage de l'algèbre par l'étude de certains moments vécus dans les classes de nos participants
- Documenter l'activité d'accompagnement

Ces objectifs teintent ainsi les différents rôles assumés en tant que chercheures dont ce texte favorise leur distinction bien qu'ils s'exprimaient plus souvent qu'autrement en superposition.

Tableau 1: Rôles et tâches des chercheures

Rôles	Tâches
Formateur	· Concevoir des activités de formation · Animer les formations · Partager des exemples vécus en classe par l'une de nous alors que nous étions enseignante au secondaire · Planifier et préparer les journées de formations (méthodes, outils, etc.) · Anticiper / identifier les obstacles à l'apprentissage et intervenir si nécessaire · Assurer une gestion de groupe facilitant un apprentissage professionnel · Évaluer la progression ou l'atteinte des objectifs de formation · Identifier les besoins et les priorités
Investigateur des séances de classe	· Observer et étudier les activités d'enseignement/apprentissage de l'algèbre telles qu'elles se déroulent dans les classes · Produire et communiquer des résultats de recherche qui éclairent certaines composantes des activités étudiées
Investigateur des ressources didactiques	· Étudier les ensembles didactiques mis à la disposition des enseignants et des conseillers pédagogiques
Investigateur de l'activité de coélaboration entre participants et équipe recherche	· Observer et étudier l'activité de coélaboration entre les participants, la professionnelle et les assistantes de recherche telle qu'elle évolue principalement dans les journées d'échange et de formation et visant le développement de la pensée algébrique chez les élèves du 1er cycle. · Produire et communiquer des résultats de recherche qui éclairent certaines composantes de l'activité étudiée.
Accompagnateur	· Développer et maintenir des relations de collaboration avec les enseignants, les conseillers pédagogiques, la professionnelle et les assistantes de recherche · Offrir (ou s'assurer d'offrir lorsque la professionnelle de recherche est mise à contribution) un soutien pédagogique aux enseignants et conseillers pédagogiques entre les journées de formation · Contribuer à la recherche de solutions aux problèmes perçus

Adaptation de Lessard (2008) et Gagnon (2010)

La théorie de l'activité un outil pour le chercheur investigateur de l'activité de coélaboration entre participants et équipe de recherche

Ainsi, pour rendre compte de notre manière d'accompagner et d'étudier l'activité d'enseignement/apprentissage visant l'introduction de l'algèbre dans les classes de nos enseignants et sa transformation sous l'influence du projet, il s'avère utile de rappeler que les principes de la perspective retenue orientent inéluctablement la production de données, leur interprétation et plus spécifiquement notre manière de conceptualiser l'agir du chercheur dans ses différentes fonctions. Les postulats des théories socio historico-culturelles alimentent ce texte: les manières de faire, de penser, d'être des individus sont considérées comme des pratiques sociales qui sont consubstantielles de la culture dans laquelle ils vivent et qui doivent rendre compte de formes d'activités semblables passées (perspective historico-génétique). Les phénomènes doivent être étudiés de façon holistique plutôt que d'en isoler certains éléments qui ne rendraient plus compte des liens qu'ils entretiennent entre eux (Langemeyer & Roth, 2006). L'activité est ainsi considérée comme la plus petite unité d'analyse permettant aux chercheurs de donner du sens au processus de mise en apparence des participants à travers leurs actions posées. Pour mieux saisir le concept d'activité, il est nécessaire de comprendre sa nature dialectique (idéale, matérielle). Au sens de Leontiev, l'activité d'un sujet (enseignant, conseiller pédagogique ou chercheur) est toujours dirigée vers un objet/motif. L'objet de cette activité se matérialise dans les actions. Dans le cas particulier de l'apprentissage, Roth et Radford notent que :

> «[…] les élèves ne peuvent pas connaître l'objet/motif de l'activité d'apprentissage : cet objet doit être le résultat de l'activité d'apprentissage dans laquelle les autres – p.ex. les enseignants – assurent une fonction de régulation permettant l'expression d'autres activités humaines dans l'objet de l'activité d'apprentissage [traduction libre]» (Roth & Radford, 2011, p.16).

Comme l'a écrit Radford (2015), l'activité des élèves est la matérialisation d'archétypes culturels qui se donnent à voir dans « l'agir, le parlé, le perçu, le gesticulé, le symbolisé, le raisonné (p.338) ». Notre rôle de formatrices convoque aussi une responsabilité d'insérer dans l'activité d'accompagnement de nos participants des manières de faire associées au développement historico-culturel de la pensée algébrique et de rendre apparentes des facettes de l'enseignement/apprentissage à considérer lorsqu'on porte un regard didactique sur celle-ci. Ainsi, si pour Roth et Radford (2011) le défi de l'activité d'apprentissage pour les élèves est précisément qu'ils reconnaissent l'objet/motif à travers leurs propres actions, on verra qu'il en va de même pour les chercheurs qui se donnent le mandat de contribuer à la formation d'enseignants et de conseillers pédagogiques.

Roth et Radford (2011) nous invitent à considérer la pensée sous ses dimensions anthrolopologique (ses origines) et ontogénique (ses conditions d'existence). D'une part, la pensée est conceptualisée comme *praxis cogitans*. La pensée n'est pas fixe, elle est activité, elle est mouvement. Considérée comme potentialité, elle se dévoile, se matérialise dans un singulier et devient/est objet/activité de conscience. Radford (2011) parle alors de pensée/activité en termes de processus de prise de conscience qu'il a appelé objectivation. D'autre part, ce processus de prise de conscience n'est possible qu'à condition d'exister socialement. Les chercheurs ajoutent que ces processus de prise de conscience individuel et collectif sont mutuellement constitutifs.

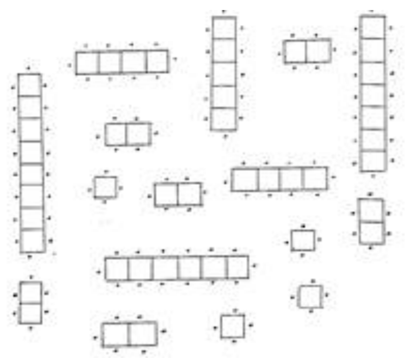
Le concept de contradiction

Les théories historico-culturelles nous invitent à conceptualiser l'activité en tant que mouvement. Le motif motive l'activité d'un individu, l'objet de l'activité se matérialise dans ses actions. Ce flux est considéré comme l'effet de l'activité elle-même. Prenant l'exemple des travaux de linguistes, Roth (2012) illustre le sens à donner à toute activité. Notre langue change dans ses usages. Comme l'écrit Bakhtin (1981) un langage meurt dès qu'il n'est plus utilisé. À chaque fois qu'un mot est utilisé, son sens se réifie, change tout comme la langue est elle-même transformée par l'usage d'un mot. Ainsi, les mots, les signes, sous une approche dialectique, ne sont pas une seule et unique entité (Roth, 2014). Ils portent et arborent, dans leurs usages, une contradiction interne qui se manifeste elle-même dans les différentes manières dont usent les individus des signes. Ces contradictions internes renvoient, selon Roth et Radford (2011) à des aspects conflictuels qui coexistent dialectiquement dans un phénomène. D'autres travaux recourent au concept de contradiction, mais il est plutôt conceptualisé différemment. C'est le cas des travaux d'Engeström (2001) et de Potari et al. (2018). Ces derniers étudient et distinguent l'activité de différentes communautés (p.ex. Ministère, enseignants) considérées comme des systèmes distincts. Ils identifient alors des contradictions dans les pratiques contraires, les choix différents entre enseignants ou entre un enseignant et une source externe (un chercheur, un manuel ou un programme). On avance alors que l'étude de contradictions émergeant entre deux systèmes peut créer des opportunités d'apprentissage et ainsi transformer les actions et buts d'une activité. Il s'agit alors de s'intéresser à l'activité de transformation d'un point de vue collectif tout en tenant compte de différents éléments médiateurs (sujet, outil, règles, communauté, division du travail…) qui influencent cette même activité.

Quelques éléments de méthodologie

Dans le cadre de la présente recherche-action, les séances de collaboration avec les enseignants et conseillers pédagogiques ont toutes été filmées. Dans le cadre de ces séances, les participants au projet s'étaient familiarisés avec certaines tâches, avaient analysé des vidéos d'enseignants qui animaient ces tâches et avaient assisté à la présentation de quelques contenus théoriques exposés par les chercheuses, notamment la typologie de la généralisation de Radford (2003) que nous n'exposerons pas ici. De même, les expérimentations qui ont eu lieu dans les classes des enseignants furent aussi filmées. Des transcriptions de ces enregistrements ont été effectuées. Dans le cadre de cette communication, nous discuterons des expérimentations qui ont succédé la séance de formation où la situation *Le Restaurant de Marcel* a été présentée aux enseignants (voir figure 1).

Marcel, le propriétaire du restaurant, dispose de tables simples dans son restaurant qu'il place l'une à côté de l'autre pour pouvoir placer ses clients lorsqu'ils arrivent. Il dispose ainsi de différentes tables de toutes sortes de grandeurs : des grandes, des petites, des moyennes…



Marcel aimerait bien ne pas avoir à compter à chaque fois les clients qui arrivent pour décider autour de quelle table il les place. Pourrais-tu l'aider à trouver une manière de calculer vite le nombre de clients qu'on peut asseoir autour d'une table, et ce, quelle que soit la grandeur de la table? Notre propriétaire habite loin alors il attend que tu lui écrives à ce sujet.

Écris-lui un **message en mots** qui lui indiquerait une manière de faire pour trouver vite combien de personnes il peut asseoir autour d'une table, et ce, pour n'importe quelle table.

Les messages sont longs à lire pour Marcel, réécris tes messages cette fois-ci sous forme de texto pour qu'il puisse les lire rapidement.

Figure 1. La situation du restaurant de Marcel

Avant de pouvoir discuter du potentiel et des intentions inhérentes à la situation, les enseignants ont eu à la résoudre. Ainsi, pour la première question, différents messages en mots sont possibles. On peut penser à de bons messages comme : 1) Il y a 2 personnes par table face à face et une personne à chaque extrémité de la table; 2) On peut asseoir 3 personnes sur les tables des extrémités et deux personnes face à face pour chacune des tables qui se situe entre ces deux tables; 3) On place une personne par table et une autre personne à une des extrémités de la table et on fait ceci deux fois (on voit une symétrie), des messages erronés sont également possibles. La formulation de ces messages se fait en contexte, les motifs présentés sont ici un support essentiel à la résolution. Dans la deuxième question, l'intention est d'amener les élèves à passer à la symbolisation, celle-ci permettant d'illustrer de façon concise et succincte une grandeur, ici le nombre de tables.

Réflexion sur des contradictions reconnues dans les expérimentations des enseignants

À travers cette situation, les intentions des chercheures sont de donner du sens au symbolisme en misant sur l'émergence d'une symbolisation qui sera propre et spontanée à chaque élève. En n'imposant pas l'usage d'un signe (lettre ou autre) en particulier, on peut s'attendre à une variété de symbolisations pour représenter la même grandeur. Outre ce travail autour de la symbolisation, l'intention est également de motiver le recours à différentes expressions algébriques qui seront équivalentes apportant une flexibilité dans la façon de voir des motifs. Cette façon de faire ouvre la porte sur un travail sur les expressions équivalentes.

Après la séance de formation, plusieurs enseignants sont allés expérimenter cette situation en classe en vue d'expliciter ce qu'ils avaient observé chez leurs élèves lors de la prochaine rencontre. Lors du retour, les formatrices ont observé une diversité dans les manières de faire des enseignants (voir tableau 1).

Tableau 2. Différentes manières de faire d'enseignants pour le *Restaurant de Marcel*

Stéphane	Annie	Alexandre
Cette situation est considérée par Stéphane comme étant la première qu'il utilise en classe pour introduire l'algèbre. Il reviendra sur celle-ci à plusieurs reprises durant quelques semaines. Il la réinvestit lorsque vient le temps d'aborder la reconnaissance de situations de proportionnalité (mots, table de valeurs et graphique). Lors de la première période, en équipes de deux, ses élèves doivent trouver deux formules différentes qui sont exprimées à l'aide de symboles algébriques. L'enseignant valide les formules obtenues et invite quelques équipes à venir les expliquer à l'avant en coordonnant leur explication du visuel des tables. Au cours suivant, l'enseignant axe sur la représentation des formules dégagées dans deux registres de représentation : table de valeurs et graphique. L'intention de l'enseignant est d'exprimer les liens existants entre ces trois registres : table, graphique et symbolique.	Annie mentionne avoir traité le chapitre «des suites» auparavant. Les élèves ont donc appris à dégager des formules en étudiant la récurrence additive dans la progression de différents motifs. L'enseignante introduit la situation proposée en ayant comme intention de faire dégager différents messages qu'elle souhaite obtenir en mots ou sur forme symbolique. Lors de la résolution, elle invite aussi ses élèves à construire une table de valeurs pour dégager une formule. Avec déception, elle partage ne pas avoir eu une diversité de formules.	Cette situation est la première qu'utilise l'enseignant pour introduire l'algèbre selon lui. Il dit avoir toutefois avoir amorcé un travail sur la traduction de relations de comparaison uniquement dans des jeux de fin de période. Pour la présente situation, l'enseignant favorise l'émergence de différents messages qui s'appuient sur une étude du motif. Lors du retour en grand groupe, l'attention est portée sur la validation de chacun des messages exprimés en mots en s'appuyant sur le visuel des motifs. Il a ainsi le souci de coordonner le visuel avec les mots. Suite à la première période, le recours au symbolisme est minoré. Alexandre met l'accent sur le potentiel des formules exprimées en mots pour prédire le nombre maximal de personnes qui peuvent s'asseoir selon le nombre de tables.

On peut remarquer que bien que les enseignants aient tous orchestré la situation du *Restaurant de Marcel*, l'activité de généralisation ou plus spécifiquement le processus de mise en apparence vécu par les élèves diffère dans chacune des classes selon ce que les enseignants valorisent au sujet du développement de la pensée algébrique. Comme formatrices, nous nous sommes données comme rôle de documenter l'activité d'enseignement/apprentissage vécue dans chacune des classes des enseignants participants. On peut ici documenter les différentes facettes d'un même objet (voir tableau 1). Mais le rôle de formateur va plus loin, il s'agit d'animer une discussion de groupe sur les différentes intentions qui ont animées les enseignants lors de la mise sur pied de l'expérimentation et sur les activités vécues. Tout en reconnaissant le potentiel de ces différentes manières de faire, il s'agit de revenir aux intentions ciblées au départ (motif de l'activité) par les formatrices et explicitées lors de la rencontre précédente. Il s'agit aussi de faire voir aux enseignants les différentes facettes (sens de la lettre, émergence d'expressions algébriques dont le sens diffère selon le processus d'objectivation des élèves, niveaux de généralisation reconnus…) d'une pensée algébrique qui se développe dans chacune des classes.

Ainsi dans la classe d'Alexandre, les formatrices reconnaissent l'importance accordée à donner du sens aux expressions algébriques. Elles s'attardent à faire voir aux enseignants tout le travail réalisé par l'enseignant pour aider ses élèves à coordonner la verbalisation des messages de généralisation dégagés au visuel fourni. Elles font alors voir cette contradiction (idéale/matérialisée dans l'activité de la classe) qui peut mieux s'expliquer dans l'association des niveaux de généralisation de Radford (2003). Le retour en grand groupe animé dans la classe d'Alexandre reste donc principalement au niveau de généralisation contextuelle. Celle-ci comporte encore des références aux particularités des objets du contexte et de leurs caractéristiques en référence à la situation spatio-temporelle. Les messages de généralisation restent contextuels au sens où leur mode de désignation dépend encore des propriétés spatiales. Comme le spécifie Radford (2003), les «arguments» ou «variables» ne sont plus des nombres, mais des objets génériques qui sont désignés par des termes génériques tels que «la figure», «la prochaine figure». D'autres contradictions sont ressenties par les formatrices lors de la présentation des approches de Stéphane et d'Annie mais elles ne sont pas du même ordre, de la même nature. En effet, l'ajout d'une table de valeurs dans la situation (approche d'Annie) limite les expressions dégagées par les élèves, ce qu'a pu observer l'enseignante. Des tensions sont de deux ordres, d'une part, il ne s'agit plus ici de travailler sur les différentes symbolisations possibles et sur une diversité d'expressions qui sont porteuses de sens pour les élèves, la recherche d'une règle semble prendre le pas. D'autre part, le travail de généralisation engagé par les élèves semble escamoté par l'application d'une procédure (à chaque table qui s'ajoute, il y a deux personnes de plus donc on inscrit «fois deux» dans la formule). Cette contradiction a été moteur de changement chez l'enseignante qui en a elle-même pris conscience en comparant l'activité de sa classe avec celle de ses pairs.

Dans le cas de l'approche de Stéphane, les interventions des formatrices ne sont pas du même ordre puisque les intentions visées sont rencontrées, mais l'accent est mis sur le travail autour des registres de représentation. Il s'agit alors de faire voir aux enseignants que l'exercice de convertir une expression dans une autre registre et d'y coordonner les différentes significations est une tâche exigeante pour les élèves qui amorcent leurs apprentissages au sujet de la généralisation algébrique. Il ne s'agit pas de dire aux enseignants que ce qu'ils font est incorrect ou qu'il y a contradiction, au sens d'Engestrom (2001), selon «notre système formatrice», mais plutôt de les aider à percevoir les différentes facettes qui s'enchevêtrent lorsqu'il s'agit de réfléchir au travail de généralisation algébrique.

Conclusion

L'étude des activités d'enseignement/apprentissage qui ont cours dans les différentes classes des enseignants que nous accompagnons illustre différentes manières de faire d'enseignants qui, bien que partant d'une même situation, ont des motifs différents. Dans notre rôle de chercheures investigatrices des séances de classe, nous étudions la matérialisation de l'expression de la pensée algébrique dans chaque classe soit cette expression de contradictions entre ce que les enseignants visent et ce qui se matérialise. Dans notre rôle de formatrice, c'est en nous appuyant sur ces différentes manières de faire que l'on rend apparentes différentes facettes qui colorent toutes l'expression d'une pensée algébrique mais qui la façonnent toutefois de façon différente.

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THE ALLOCATION OF SENSE TO THE QUOTATIVE DIVISION OF FRACTIONS

LA ASIGNACIÓN DE SENTIDO A LA DIVISIÓN CUOTATIVA DE FRACCIONES

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This research paper is part of a doctoral study under development. In this research report, we analyze the cognitive and operative processes that first graders in secondary school perform in assignments involving division of fractions. The study includes some contributions from researchers who have adopted the problem solving about division of fractions tasks through the quotative model, as well as the use of different levels for coordination of units proposed by Hackenberg (2007). Tasks here presented include problems solving through canonical and pictorial algorithms, and form part of a questionnaire that was thoroughly analyzed, by contrasting the arithmetic-symbolic notations and the expressions of the language.

Keywords: Division of fractions, canonical and pictorial algorithm.

Introduction

In the semantic analysis carried out in this research report focused on measuring how much a part fits in the given hole, that Fischbein, Deri, Sainati, Sciolis (1985) called a quotative model of division and recognized the pictorial algorithm by Kieren (1985), we intend to make sense of dividing fractional numbers using a didactic support that makes feasible its comprehension and achieves good results in the applied tasks; otherwise, if only the canonical algorithm is applied the operation of division of fractions will be mechanic. This situation is illustrated below with an example.

Operations that students carry out can not be restricted to operational-algorithm use of fractional number; on the contrary, we intend that they help to achieve a semantic interpretation through a pictorial algorithm proposed in each task. We consider that the use of the pictorial algorithm is substantial in the present research work because through it we can observe the steps students follow when facing diverse problematic situations posed and there is no doubt about first graders' competency in Telesecondary schools (an educational modality in Mexico that is taught to the most remote places through television images) to solve tasks in which division of fraction is involved within the quotative model.

Research question and objectives

Mexican students' assessments in secondary school are applied through an official test called Early Warning System (Sistema de Alerta Temprana-Sisat, 2018). For now, results are not quite positive for first graders in Telesecondary modality, which take us to investigate which are the performance and strategies students use to face problematic situations that involve the quotative division of fractions?

General aim:

Identify through the pictorial algorithm, the different strategies used by that first graders in Telesecondary to give sense to the quotative division of fractions to understand why it is divided, by recognizing how much a fraction fits in another fraction.

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Theoretical framework

Rational numbers have been a fairly broad research topic around the world, just to mention some of the most outstanding authors such as Kieren (1983), who define several subconstructs to understand the meaning of fractional numbers. Streefland (1991) highlights realistic models where operational and conceptual operation of rational numbers is necessary. Vergnaud (1988) came up with theorems in action, which will help us to analyze the students' conceptual difficulties when solving the proposed tasks. Fischbein et al. (1985) defines the quotative division that is essential in the present research work. Tirosh (2000) emphasizes the semantic and operative elements that required to solve a division of fraction task, but much students lack them. Valdemoros (2004) points out how semantic contents are vital to achieve the transition from natural language to an arithmetic language, all which is necessary to define with considerable accuracy regarding quotative division of fractions.

Students' limitations regarding the division of fractions have been documented over the last decades, as well as those limitations that the in-service teachers also show, as Izsák (2012) reports, who express how most of in-service teachers can solve tasks involving division of fractions through a canonical algorithm, but they have serious difficulties to solve such tasks when this type of number appears in problematic situations in the division of fractions. This situation also occurs in secondary students. Izsák (2012) shows a task from Armstrong & Bezuk (1995), in which teachers accepted that the required situation to solve it was a multiplication of fractions, but when asked to explain their thought or draw diagrams to unify the algorithmic-canonical result to that of pictorial one (by using geometric representations) and understand what kind of unit were using in each case, teachers showed many difficulties to recognize their cognitive conflicts, this situation that also arises in secondary students and it is one of the fundamental reasons that lead us to investigate about such problematic.

To construct a quotative model for dividing fractional numbers, the different levels of unit coordinations proposed by Hackenber (2007) are taken into account; although, the focus of the paper is the construction of improper fractions, its original contributions of how students carry out the coordination of different levels of units for the construction of proper fractions is fundamental to this research work.

The correct identification of appropriate reference units (Schwartz, 1988) for each fraction is fundamental for solving tasks that involve the division of fractions; and the coordination of unit structures in the three levels mentioned in the previous paragraph, are essential for understanding the mathematical knowledge, all of this through drawing geometric models, diagrams and other pictorial elements that students use to appropriate the arithmetic of rational numbers.

Method

Participants

There were 28 first graders belonging to a scholar group which we applied a questionnaire in a Telesecondary school located on the boundary of the state capital of Puebla. The school was selected because the teacher of the group showed great interest in the didactic treatment about knowledge of fractional numbers.

Socio-Cultural Situation

The school has 198 students divided into six groups (2 groups for each grade). The school has the Service and Support Unit for Regular Education, what is in charge to assist students with low school achievement; it also has institutional support resources such as the Internet, food service, psychological attention, library, civic and sports activities.

Methodological instruments

We designed a questionnaire for the first graders in Telesecundaria. It consisted of eight tasks that involved mathematical problems about the division of fractions. For each task, we provided two ways of solving it, canonical and pictorial algorithms (tasks should be solved in both ways). The following is one of the tasks: How many parts of ½ meter of a wooden board can I obtain in 2 wooden boards if each board measures 2 meters? Toño achieved the first result through the canonical algorithm. He operated $2 \div \frac{1}{2} = 4$ parts in each wooden board. The second solution came through the pictorial algorithm; Pepe represented his operation with two rectangular figures divided into four parts each and said ½ meter of wood fits 4 times in each wooden board. Students solved another task equal to the previous explained, but the fraction unit was 1/3.

Analysis of collected data

In this research paper, we consider those answers that used both models of solving because from these models we can build appropriated meanings for the operations the students carried out, in particular the quotative meanings. Andrea's answer was right. In her first answer, she operated $2/1 \div 1/3 = 6/1 = 6$. In her second answer, she drew two rectangular figures divided into 3 parts each one, so her answer to the task was: 6 parts.

As we can observe in Andrea's answer, she formed a structure of two levels of unit coordination, where a two-unit range is divided into three equal parts. This answer allowed us to determine that Andrea understood the quotative meaning of the division of fractions by using two levels of unit coordination.

Results

The students could solve tasks in the questionnaire by using those strategies they considered appropriate; however, when the solution involves the use of the pictorial algorithm this allows us to determine in which moment and circumstances students constructed the different levels of unit coordination, which in turn allows us to know if the reasoning used is correct or there are some deficiencies in their development.

The students in first grade gave only one answer through the canonical algorithm. They had difficulties to explain each of their answers, though they were correct, their interpretation of results was unreliable and their oral explanations were inconsistent. So, the didactic interview with feedback (Valdemoros, 2004, Valdemoros, Ramirez and Lamadrid, 2015) and the group interview (Werscht, 1993), which will be carried out soon, will be of great help in clarifying the students' conceptualizations that participate in this research.

Conclusions

The first graders' performance in the Telesecondary modality can be observed in the tasks included in the questionnaire; unfortunately, results are a little favorable because most of the students tried to solve tasks by changing the fractional numbers to decimal numbers. Others seemed not to have previous knowledge regarding the pictorial algorithm, which at the end was inconvenient and in some cases caused serious conflicts of interpretation of the proposed tasks. To solve these inconveniences, we propose to carry out a series of didactic interviews with feedback (also described as a didactic interview) applied to some of the students in first grade as well as to apply a group interview in the following phase of the research. Currently, the theoretical-methodological and empirical design of the collective interview is in process.

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LA ASIGNACIÓN DE SENTIDO A LA DIVISIÓN CUOTATIVA DE FRACCIONES

THE ALLOCATION OF SENSE TO THE QUOTATIVE DIVISION OF FRACTIONS

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El presente trabajo de investigación forma parte de un estudio doctoral en desarrollo; en este reporte de investigación analizamos los procesos cognitivos y operatorios que presentan los alumnos de primero de secundaria para realizar tareas en las que se involucra la división de fracciones. En el estudio se incluyen aportaciones de investigadores que adoptan la resolución de tareas de división de fracciones por medio del modelo cuotativo, así como la utilización de los distintos niveles de coordinación de unidades propuestos por Hackenberg (2007). Las tareas que se presentan en este estudio incluyen la resolución de problemas por medio de los algoritmos canónico y pictórico. Dichas tareas fueron integradas en un cuestionario, el cual se analizó de forma exhaustiva, por medio del contraste entre las notaciones aritmético-simbólicas y las expresiones de la lengua que acompañaron a las primeras.

Palabras clave: División de fracciones, algoritmos canónico y pictórico.

Introducción

El análisis semántico realizado en el presente documento de investigación se centra en la medición del cuánto cabe una parte en el todo dado, denominado por Fischbein, Deri, Sainati, Sciolis (1985) como modelo cuotativo, a través del algoritmo pictórico (Kieren, 1985), con el cual pretendemos otorgar sentido a la división de números fraccionarios, dando un tratamiento didáctico que facilite su comprensión y la obtención de buenos resultados en las tareas a resolver; ya que si sólo se aplica el algoritmo canónico estaremos haciendo mecánica la operación de división de fracciones, más adelante se ilustra con un ejemplo esta situación.

Las operaciones realizadas por los alumnos no pueden quedar restringidas al uso operacional-algorítmico de los números fraccionarios, sino por el contrario, se pretende que puedan facilitar una interpretación semántica por medio del algoritmo pictórico propuesto en cada una de las tareas a realizar. Consideramos que el manejo del algoritmo pictórico es substancial en el presente trabajo de investigación, ya que a través de él podemos observar los pasos que sigue cada estudiante ante las diversas situaciones problemáticas planteadas, y no dudamos de la capacidad de los alumnos de primero de telesecundaria (modalidad educativa en México que se imparte hasta los lugares más alejados, a través de imágenes televisivas) para resolver las tareas en las que se involucra la división de fracciones, a través del modelo cuotativo.

Pregunta de investigación y objetivos

La evaluación que se elabora en México para los alumnos de secundaria se realiza a través de la prueba oficial denominada Sistema de Alerta Temprana (Sisat, 2018), los resultados obtenidos hasta ahora son poco favorables en los alumnos de primero de telesecundaria, lo que nos permite preguntar, ¿cuál es el desempeño y las estrategias que utilizan los alumnos de primero de telesecundaria frente a situaciones problemáticas en las que se involucra la división cuotativa de fracciones?

Objetivo general:

Identificar, por medio del algoritmo pictórico, las distintas estrategias que utilizan los alumnos de primero de telesecundaria para otorgar sentido a la división cuotativa de fracciones para comprender por qué se divide, a través de reconocer cuánto cabe una fracción en otra fracción.

Marco teórico

Los números racionales han tenido un tratamiento de investigación bastante amplio alrededor del mundo, con autores muy reconocidos de distintos países, sólo por mencionar algunos de los más destacados como Kieren (1983), define a lo que llama subconstructos para entender las significaciones de los números fraccionarios. Streefland (1991) destaca los modelos realistas en los que es necesario el manejo operativo y conceptual de los números racionales. Vergnaud (1988) propone los teoremas en acción, los cuales nos servirán para analizar las dificultades conceptuales de los alumnos al resolver las tareas propuestas. Fischbein et al. (1985) define la división cuotativa que es central en este trabajo de investigación. Tirosh (2000) hace énfasis en los elementos conceptuales y operatorios que necesitan los estudiantes para resolver una tarea de división de fracciones, pero que en muchas ocasiones carecen de ellos. Valdemoros (2004) señala cómo los contenidos semánticos son de vital importancia para realizar la transición del lenguaje natural al lenguaje aritmético, lo cual en nuestro estudio es necesario definir con la mayor claridad respecto a la división cuotativa de fracciones.

Las limitaciones de los alumnos de secundaria con respecto a la división de fracciones se han documentado durante las últimas décadas, así como las limitaciones que tienen también los futuros profesores como lo reporta Izsák (2012), quien identifica cómo la mayoría de los maestros en servicio pueden resolver tareas en las que se involucra la división de fracciones a través del algoritmo canónico y, sin embargo, tienen serias dificultades para resolver dichas tareas cuando este tipo de números se presentan en situaciones problemáticas de división de fracciones, esta misma situación se presenta en alumnos de secundaria. Izsák (2012) propone una tarea extraída de Armstrong y Bezuk (1995), en la cual los maestros reconocieron que la situación requerida para resolverla era una multiplicación de fracciones, pero cuando se les solicitó que explicaran sus pensamientos o dibujaran diagramas para unificar la solución algorítmica-canónica con la pictórica (mediante el uso de representaciones geométricas) y comprender qué tipo de unidad se estaba utilizando en cada caso de la situación problemática, presentaron muchas dificultades para reconocer sus conflictos cognitivos, circunstancia que se presenta también en los alumnos de secundaria y es uno de los motivos fundamentales que nos llevaron a indagar sobre esta difícil problemática.

El modelo cuotativo de división de números fraccionarios se realiza aquí tomando en consideración los distintos niveles de coordinación de unidades propuestos por Hackenberg (2007), aunque el foco de atención del artículo es la construcción de fracciones impropias, es central en este trabajo de investigación su aportación original de cómo los estudiantes realizan la coordinación de los distintos niveles de unidades para la construcción de fracciones propias.

La correcta identificación de unidades de referencia (Schwartz, 1988) apropiadas para cada fracción es fundamental para la resolución de las tareas que implican división de fracciones, y la coordinación de estructuras de unidades en los tres niveles mencionados en el párrafo anterior, son esenciales para la comprensión del conocimiento matemático, todo esto a través de modelos geométricos dibujados, esquemas, y demás elementos pictóricos que utilizan los alumnos para apropiarse de la aritmética de los números racionales.

Método

Participantes

Se aplicó un cuestionario a 28 alumnos de primero de telesecundaria ubicada en la periferia de la capital del Estado de Puebla, la escuela fue seleccionada porque la maestra titular del grupo muestra gran interés por la apropiación de conocimientos de los números fraccionarios.

Entorno socio-cultural

La escuela seleccionada tiene un total de 198 alumnos repartidos en seis grupos (2 grupos de cada grado), cuenta con una Unidad de Servicio y Apoyo a la Educación Regular, la cual se encarga de atender a los alumnos que presentan bajo aprovechamiento escolar. La institución dispone además de recursos de apoyo institucional como: servicio de internet, comedor, asistencia psicológica, biblioteca escolar y de grupo, actividades cívicas y deportivas.

Instrumentos metodológicos

Se aplicó un cuestionario a los alumnos de primero de telesecundaria, con ocho tareas en las que se involucraban situaciones problemáticas de división de fracciones; en cada una de las tareas se proporcionaron dos formas de solución, los algoritmos canónico y pictórico (ambas debían ser resueltas). La tarea propuesta fue la siguiente: ¿cuántas partes de 1/2 de metro de tabla de madera, puedo obtener en 2 de tablas de madera, si cada tabla mide 2 metros? La primera solución se presentó por medio del algoritmo canónico, Toño realizó la operación de $2 \div 1/2 = 4$ partes en cada una de las tablas. La segunda solución se expuso a través del algoritmo pictórico, Pepe representó su operación con dos figuras rectangulares divididas en 4 partes cada una y dijo 1/2 metro de madera cabe 4 veces en cada una de las tablas. La tarea a realizar por los alumnos fue la misma que se expuso al principio del párrafo, solo se modificó la fracción unitaria a 1/3.

Análisis de datos recopilados

La postura del presente trabajo de investigación es tomar en cuenta aquellas respuestas que utilizaron los dos modelos de solución propuestos en el ejemplo, ya que a partir de estos modelos podemos construir significados apropiados para las operaciones realizadas por los alumnos, en especial los significados cuotativos. La respuesta de la alumna Andrea fue correcta, ya que su primera respuesta fue realizar la operación de $2/1 \div 1/3 = 6/1 = 6$. La segunda respuesta de Andrea fue dibujar dos figuras rectangulares divididas en 3 partes cada una, por lo que su respuesta a la tarea fue: 6 partes.

Como se observa en la respuesta de Andrea, forma una estructura de dos niveles de coordinación de unidades, en la que un intervalo de dos unidades se divide en tres partes iguales; esta respuesta nos permite determinar que Andrea entendió el significado cuotativo de la división de fracciones, utilizando dos niveles de coordinación de unidades.

Resultados

Los alumnos pueden resolver las tareas propuestas en el cuestionario aplicado utilizando las estrategias que consideren más apropiadas, sin embargo, cuando la solución implica la utilización del algoritmo pictórico nos permite clarificar en qué momento y circunstancias se apropian de los diferentes niveles de coordinación de unidades, lo que a su vez nos permite conocer si el razonamiento utilizado es correcto o si tienen algunas deficiencias en su apropiación.

Los alumnos que solamente dan una respuesta a través del algoritmo canónico, tienen serias dificultades para explicar cada una de sus respuestas, ya que aunque tengan una solución correcta, su interpretación del resultado es poco confiable y su explicación verbal tiene muchas inconsistencias, por lo que la entrevista didáctica y con retroalimentación (Valdemoros, 2004, Valdemoros, Ramirez y Lamadrid, 2015) y la entrevista colectiva (Werscht, 1993), a realizar próximamente, serán de gran ayuda en el esclarecimiento de las conceptualizaciones de los alumnos seleccionados.

Conclusiones

El desempeño de los alumnos de primero de telesecundaria se ve reflejado en las tareas propuestas en el cuestionario aplicado, desafortunadamente los resultados son poco favorables, debido a que la mayoría de los alumnos intentó resolver las tareas cambiando los números fraccionarios a números decimales y otros, al parecer, no tenían conocimientos preliminares concernientes al algoritmo pictórico, lo que a la postre resultó inconveniente y en algunos casos causó graves conflictos de interpretación de las tareas propuestas. Para resolver este tipo de inconvenientes, se propone una serie de entrevistas didácticas y con retroalimentación (también descrita como entrevista didáctica) para llevar a cabo en la siguiente fase de la investigación con algunos alumnos seleccionados, así como la aplicación de una entrevista colectiva. El diseño teórico-metodológico y empírico de la entrevista colectiva se está elaborando actualmente.

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EXPRESSIONS OF MATHEMATICAL GENERALIZATION AMONG CHILDREN IN GRADE FIVE OF PRIMARY SCHOOL

EXPRESIONES DE GENERALIZACIÓN MATEMÁTICA EN NIÑOS DE QUINTO GRADO DE PRIMARIA

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We report progress from a longitudinal study focused on identifying expressions of mathematical generalization articulated by students at early ages through the design of 17 tasks that are organized into structured blocks, comprising numerical sequences and everyday situations. We present the results obtained with respect to solving numerical sequences.

Keywords: Algebraic thinking, Primary education.

Purpose of the Research

We present preliminary results of a study focused on the analysis of expressions of mathematical generalization that arise in early ages, particularly among students ranging between 10 and 12 years of age when solving generalization mathematical tasks. It is expected that expressions of generalization shown by students in a natural and incipient way will be refined as they develop their capacity for generalization, which could contribute to development of algebraic thinking at later levels.

Reference Framework

Generalization is a topic of interest to various researchers and is analyzed from different perspectives (e.g. Radford, 2000, 2002; Rivera, 2006, 2018; Schliemann, Carraher y Brizuela, 2012). The approach in this study is based on the contributions of John Mason regarding the ability possessed by students to generalize, and how that ability can be developed through mathematical tasks that lead them to articulate generality.

Expressing the regularities that are observed in a generalization task is a reference to algebraic work, albeit incipient, that arises in students. Mason, Graham, Pimm and Gowar (1999) are of the opinion that expressing generality is a very important process, because it contributes to the acquisition of algebraic language. This process can be considered a continuous spiral of actions (Mason, 1996), which is summarized as the *manipulation* (first action) of particular examples to *obtain meaning* (second action) of what is happening in them, in order to *articulate* generalities (third action), and express them in some useful mathematical form.

Manipulation (of physical, mental or symbolic objects) provides the basis for detecting patterns, relationships, generalities, etc. Discovering what is happening allows one *getting a sense of* some characteristic or property of the objects that are being manipulated, which allows one to *articulate* and manifest the expression of generality. When this occurs, such an expression becomes a new entity that can be manipulated and used to find other properties, allowing it to continue ascending the spiral and starting a new cycle of actions. But when resolution is difficult, and the conjectures are wrong, the sensible thing is to return to the corresponding cycle of actions and *manipulate* more examples in order to be able to ascend.

Likewise, Mason et al. (1999) consider three stages in the generalization process: seeing, saying and recording. “Seeing” refers to the mental identification of a schema, structure or relationship. Seeing generalities means that students can identify key factors and combine them to produce a rule that

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works; this can happen after a certain time, working with a number of particular examples until the identification of something common is achieved. "Saying", either to oneself or to someone in particular, is to articulate in words what has been recognized. Before saying the observed generalities, there is an expression of what occurs in particular cases. "Recording" a pattern or relationship leads to symbolization and written communication, which is not easy to do. The register can involve a variety of formats: drawings, word drawings, mostly words and some symbols, or mostly symbols with some words (Mason et al., 1999).

Method

The study is qualitative and descriptive in nature (Cohen, Manion y Morrison, 2007). It is longitudinal and done in cohort as the same students are followed through two school cycles, beginning in grade five of primary school. The study plan for both cycles corresponds to 2011. Identifying regularities in numerical sequences in order to find nearby missing terms or to continue the sequence (Secretaría de Educación Pública, 2011) is one of the expected learning outcomes for grade five, but generalization is not promoted. Textbooks for grades four through six include one or two activities involving these types of number sequences. Participants were deemed to have no experience with generalization tasks at the time of the study.

Design of Mathematical Tasks

17 tasks were designed for the two school cycles, which were organized in four blocks with a structured order involving a gradual transition from number sequences to everyday situations. Blocks I and II correspond to numerical sequences, Blocks III and IV to everyday situations. The analysis of responses corresponding to Blocks I and II is reported here.

Block I (numerical sequences 1-8). The mathematical structure underlying these sequences corresponds to the functions $f(x) = x$, $f(x) = 2x$, $f(x) = 2x - 1$, $f(x) = x^2$, $f(x) = 3x$, $f(x) = x + 3$, $f(x) = 4x$, $f(x) = x + 4$. Each sequence is presented through a dotted figure and the corresponding figure number. Inductive reasoning (*manipulation* of particular cases) is fostered with the intention of having students identify the correspondence relationship between the number of points and the figure number (*get a sense of*) to *articulate* the correspondence rule (generalization). When students, using the first elements of the numerical sequence, can find nearby elements, it can be said in terms of Mason (1999), that they perceive generality. By requesting considerably distant elements, the articulation of the generalization rule is promoted. As students manipulate particular cases, they are expected to "see" the corresponding structure or relationship, identifying what remains and what varies so as to articulate the generalization rule that characterizes each task. In each subsection, a figurative, numerical or explanatory record is requested. Figure 3 is an example of the numerical sequence in Block I, they all have a similar format, the variant lies in the function involved.

Block II (numerical sequences 9-10). The mathematical structure of these sequences corresponds to the functions $f(x) = 2x$, $f(x) = x + 2$. Each sequence is presented in a table that includes numerical terms along with the number that corresponds to the term, omitting the figurative (Figure 1). An inductive reasoning is also fostered, and the registry stage is highlighted during the resolution of the task. The tasks in both blocks are aimed at analyzing the articulation of the (written) ideas that lead to the generalization rule, but in this Block II, the request for a rule is made explicit. Subsection e) encourages students to suggest a number (it could be near or far); according to Mason (1999) these assignments tend to stimulate in students the scope of the generality they are expressing.

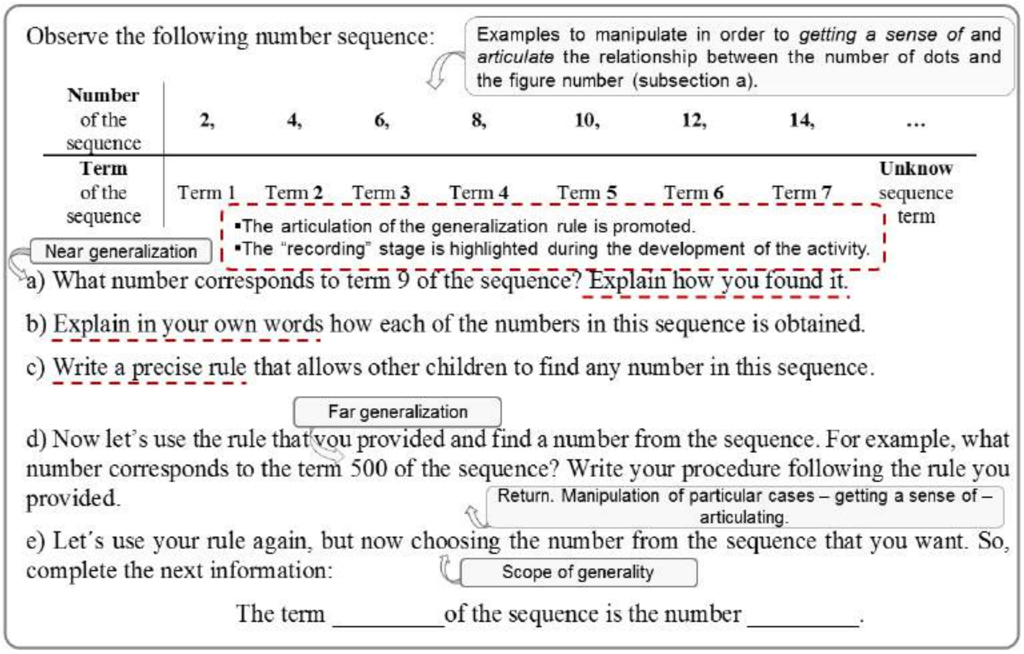

Figure 1: Task 9. Block II

Participants, Application and Analysis of Responses

The participants were 25 students from a public school located in Tekit, Yucatán, Mexico. A printout of each task was provided, students worked individually and there was no intervention by the researcher. The analysis is carried out individually to identify the expressions of generalization by student and by block, given that the tasks in each block have common characteristics. Figure 2 provides the scheme designed for the analysis of Block I responses under the Mason (1996) spiral of actions.

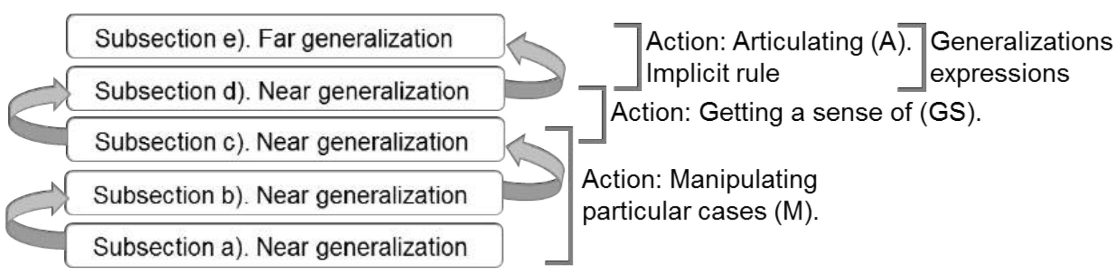

Figure 2: Maison (1996) spiral of actions in the tasks of Block I

Figure 3 provides an example of the analysis of the responses of Student 10 (S10) in Task 2 of Block I. Table 1 shows the summary of the analysis of actions that were recognized in this task by 25 students. We can see that S10 correctly constructs figures 5, 6 and 7 (part a), demonstrating the adequate manipulation of the particular cases, both figuratively and numerically (part b). The manipulation of increasingly distant examples (parts c and d) can be interpreted as getting a sense of generality, which leads to articulation and concise expression of the rule of correspondence between the number of points and the number in the figure (part e). The rule is recorded in natural language, using words (without drawings or symbols). In an analogous way, the responses of the tasks in Block II are analyzed with the corresponding scheme.

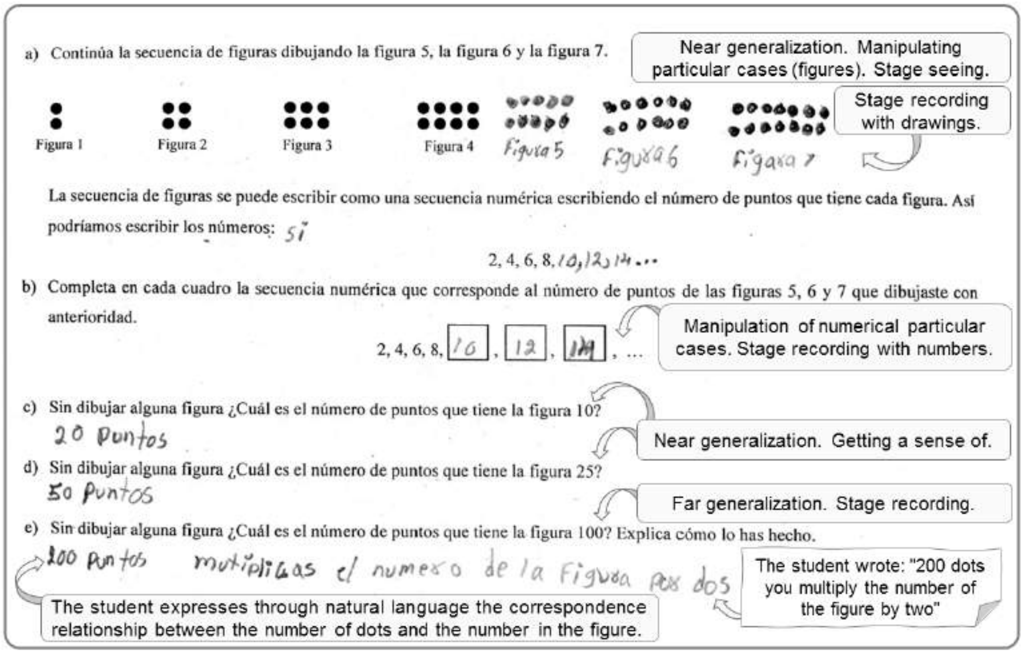

Figure 3: Analysis S10 responses in Task 2 of Block I

Table 1: Summary of actions in Task 2 of Block I

S	a)	b)	c)	d)	e)	S	a)	b)	c)	d)	e)	S	a)	b)	c)	d)	e)
S1	M	M	M	GS	A	S9	M	M	M	GS	A	S18	M	M	M	GS	A
S2	M					S10	M	M	M	GS	A	S19	M	M			
S3	M	M	M	GS	A	S11	M	M	M	GS	A	S20	M	M			
S4						S12	M	M	M	GS	A	S21	M	M	M	GS	A
S5	M	M	M	GS	A	S13	M	M	M	GS	A	S22	M	M	M	GS	A
S6	M	M	M	GS	A	S14	M	M	M	GS	A	S23	M	M	M	GS	A
S7	M	M	M	GS	A	S15	M	M	M	M	NA	S24	M	M			
S8	M	M	M	GS	A	S16	M	M	M	GS	A	S25	M	M			
						S17	M	M	M	GS	A						

Note: S=Student; M=Manipulating; GS= Getting a sense of; A= Articulating; NA=No answer; Blank space=Student fail any of the M-GS-A actions.

Results and Preliminary Conclusions

Students provide evidence of their ability to generalize by identifying, in numerical sequence tasks, those details that remain unchanged and those that change. Also, they can identify regularities and patterns, read and interpret tabular registers and above all articulate expressions of generalization. These expressions were communicated through numerical answers, words or words with some arithmetic symbols, which resulted in 5 categories: 1) expressions that state only the regularity of the pattern, that is, the variation between the terms; 2) numeric expressions that arise from using previous results, being correct only for the functions of the form $f(x) = ax$; 3) generalized number, that is, the generalization is expressed only for the requested number; 4) expressions that incipiently state what happens with the relationship between the two variables involved and 5) expressions denoting generalization considering any term (such as the one reported in Figure 3). In the next phase of the study, we will analyze the stage "saying" using the expressions of generalization done by interviewing the 25 students when they solve everyday context tasks. Finally, in our opinion the work provides references on the development of algebraic thinking and language that can contribute to the teaching of basic education.

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EXPRESIONES DE GENERALIZACIÓN MATEMÁTICA EN NIÑOS DE QUINTO GRADO DE PRIMARIA

EXPRESSIONS OF MATHEMATICAL GENERALIZATION AMONG CHILDREN IN GRADE FIVE OF PRIMARY SCHOOL

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Se reportan avances de una investigación longitudinal cuyo interés se centra en identificar las expresiones de generalización matemática que articulan los estudiantes en edades tempranas a través del diseño de 17 tareas organizadas en bloques estructurados que comprenden secuencias numéricas y situaciones cotidianas. Se presentan los resultados obtenidos en la resolución de las secuencias numéricas.

Palabras clave: pensamiento algebraico, educación primaria.

Propósito de la Investigación

Se presentan resultados preliminares de un estudio cuyo propósito es analizar las expresiones de generalización matemática que se manifiestan en edades tempranas, particularmente en estudiantes de 10 a 12 años cuando resuelven tareas matemáticas de generalización. Se considera que las expresiones de generalización que manifiestan los estudiantes de manera natural e incipiente se irán refinando conforme vayan desarrollando su capacidad de generalización, lo que podría contribuir al desarrollo de un pensamiento algebraico en niveles posteriores.

Marco de Referencia

La generalización es de interés para diversos investigadores y es analizada desde diferentes perspectivas (p.ej. Radford, 2000, 2002; Rivera, 2006, 2018; Schliemann, Carraher y Brizuela, 2012). En esta investigación el enfoque se apoya en las contribuciones de John Mason respecto a la capacidad de generalizar que poseen los estudiantes, y cómo ésta puede ser desarrollada mediante tareas matemáticas que los conduzcan a articular la generalidad.

Expresar las regularidades que se observan en una tarea de generalización es referencia del trabajo algebraico, aún incipiente, que surge en los estudiantes. Mason, Graham, Pimm y Gowar (1999) consideran que expresar la generalidad es un proceso importante porque contribuye a la adquisición del lenguaje algebraico. Este proceso puede considerarse como una espiral continua de acciones (Mason, 1996), la cual se resume en la *manipulación* (primera acción) de ejemplos particulares para *obtener sentido* (segunda acción) de lo que está ocurriendo en ellos, con el fin de *articular* generalidades (tercera acción) y expresarlas en alguna forma matemática útil.

La *manipulación* (de objetos físicos, mentales o simbólicos) proporciona las bases para detectar patrones, relaciones, generalidades, etc. Descubrir lo que está sucediendo permite *obtener sentido de* alguna característica o propiedad de los objetos que se están manipulando, lo que da lugar a *articular* y manifestar la expresión de generalidad. Cuando esto ocurre, tal expresión se convierte en una nueva entidad que puede ser manipulada y usarse para encontrar otras propiedades, lo que permite continuar ascendiendo en la espiral empezando un nuevo ciclo de acciones. Pero cuando la resolución resulta difícil, y las conjeturas son erróneas, es sensato regresar al ciclo de acciones correspondiente y *manipular* más ejemplos para poder ascender.

Asimismo, Mason et al. (1999) consideran tres etapas en el proceso de generalización: ver, decir y registrar. “Ver” se refiere a la identificación mental de un esquema, estructura o relación. Ver generalidades implica que los estudiantes puedan identificar factores clave y combinarlos para producir una regla que funcione; esto puede ocurrir después de cierto tiempo, trabajando con un número de ejemplos particulares hasta que se logra la identificación de algo común. El “decir”, ya sea a uno mismo o a alguien en particular, es articular en palabras aquello que se ha reconocido. Antes de decir las generalidades observadas se suele decir qué ocurre en casos particulares. “Registrar” un patrón o relación conduce a la simbolización y la comunicación escrita, lo cual no es fácil de realizar. El registro puede involucrar una variedad de formatos: dibujos, dibujos con palabras, la mayor parte palabras y algunos símbolos o la mayor parte de símbolos con algunas palabras (Mason et al., 1999).

Método

El estudio es de naturaleza cualitativa y de corte descriptivo; es longitudinal y de cohorte (Cohen, Manion y Morrison, 2007) en virtud de que se realiza un seguimiento a los mismos estudiantes durante dos ciclos escolares, empezando en quinto grado de primaria. El plan de estudios de ambos ciclos corresponde al año 2011, entre los aprendizajes esperados en quinto grado se encuentra la identificación de regularidades en sucesiones numéricas para encontrar términos faltantes cercanos o continuar la sucesión (Secretaría de Educación Pública, 2011) pero no se promueve la generalización. Los libros de texto de cuarto a sexto grado incluyen una o dos actividades para trabajar con ese tipo de secuencias numéricas. Se considera que los participantes no contaban con experiencia sobre tareas de generalización al momento del estudio.

Diseño de las tareas matemáticas

Se diseñaron 17 tareas para los dos ciclos escolares, organizadas en cuatro bloques con un orden estructurado que conlleva ir gradualmente de secuencias numéricas a situaciones cotidianas. Los

Bloques I y II corresponden a secuencias numéricas, los Bloques III y IV a situaciones cotidianas. Se reporta el análisis de respuestas correspondientes a los Bloques I y II.

Bloque I (secuencias numéricas 1-8). La estructura matemática que subyace a estas secuencias corresponde a las funciones $f(x) = x$, $f(x) = 2x$, $f(x) = 2x - 1$, $f(x) = x^2$, $f(x) = 3x$, $f(x) = x + 3$, $f(x) = 4x$, $f(x) = x + 4$. Cada secuencia se presenta mediante una figura con puntos y el número de la figura correspondiente. Se promueve un razonamiento inductivo (*manipulación* de casos particulares) con la intención de que los estudiantes identifiquen la relación de correspondencia entre el número de puntos y el número de figura (*obtener sentido de*) para *articular* la regla de correspondencia (generalización). Cuando los estudiantes, apoyándose en los primeros elementos de la secuencia numérica puedan encontrar elementos cercanos se puede decir en términos de Mason (1999) que ellos perciben generalidad. Al solicitar elementos considerablemente distantes se promueve la articulación de la regla de generalización. Se espera que a medida que manipulen los casos particulares "vean" la estructura o relación correspondiente, identificando aquello que permanece y lo que varía para articular la regla de generalización que caracteriza a cada tarea. En cada inciso se solicita hacer un registro de tipo figurativo, numérico o explicativo. La Figura 3 es un ejemplo de secuencia numérica del Bloque I, todas poseen un formato similar, la variante radica en la función involucrada.

Bloque II (secuencias numéricas 9-10). La estructura matemática de estas secuencias corresponde a las funciones $f(x) = 2x$, $f(x) = x + 2$. Cada secuencia es presentada mediante una tabla que incluye términos numéricos y el número que le corresponde al término, omitiendo lo figurativo (Figura 1). Nuevamente se promueve un razonamiento inductivo y se destaca la etapa registrar durante la resolución de la tarea. En las tareas de ambos bloques se pretende analizar la articulación de las ideas (escritas) que llevan a la regla de generalización, pero en este Bloque II, se hace explícita la solicitud de una regla. El inciso e) promueve que los estudiantes propongan un número (éste puede ser cercano o lejano), Mason (1999) expresa que estas encomiendas tienden a estimular en ellos el alcance de la generalidad que están expresando.

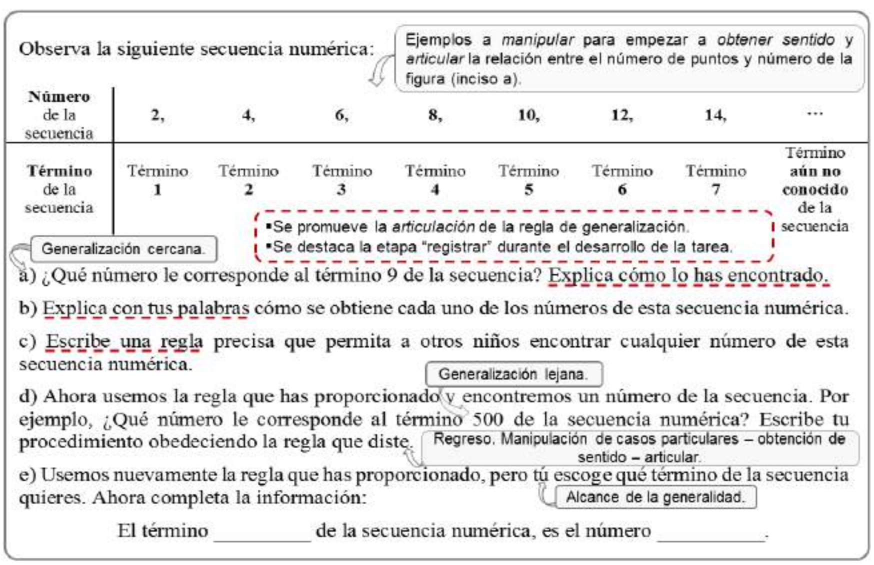

Figura 1: Tarea 9. Bloque II

Participantes, aplicación y análisis de las respuestas

Participaron 25 estudiantes pertenecientes a una escuela pública ubicada en Tekit, Yucatán, México. Cada tarea fue entregada impresa, los estudiantes trabajaron de manera individual y no hubo intervención por parte del investigador. El análisis se realiza individual para identificar las manifestaciones de generalización en cada estudiante y por bloque, en virtud de que las tareas de cada bloque presentan características comunes. En la Figura 2 se presenta el esquema diseñado para el análisis de las respuestas del Bloque I bajo la espiral de acciones de Mason (1996).

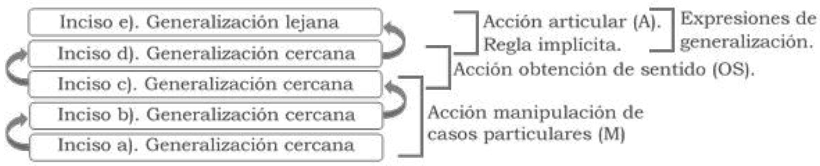

Figura 2: Espiral de acciones de Maison (1996) en las tareas del Bloque I

La Figura 3 presenta un ejemplo del análisis de las respuestas del Estudiante 10 (E10) en la Tarea 2 del Bloque I. La Tabla 1 muestra el resumen del análisis de acciones que se reconocieron en los 25 estudiantes en dicha tarea. Se puede identificar que E10 construye correctamente las figuras 5, 6 y 7 (inciso a) evidenciando la manipulación adecuada de los casos particulares tanto figurativa como numéricamente (inciso b). A medida que manipula ejemplos cada vez más distantes (incisos c y d) se puede interpretar que va obteniendo sentido de la generalidad, lo que le conlleva articular y expresar de manera concisa la regla de correspondencia entre el número de puntos y el número de la figura (inciso e). El registro de la regla lo realiza en lenguaje natural, mediante palabras (sin dibujos, ni símbolos). De manera análoga se analizan las respuestas de las tareas del Bloque II con el esquema correspondiente.

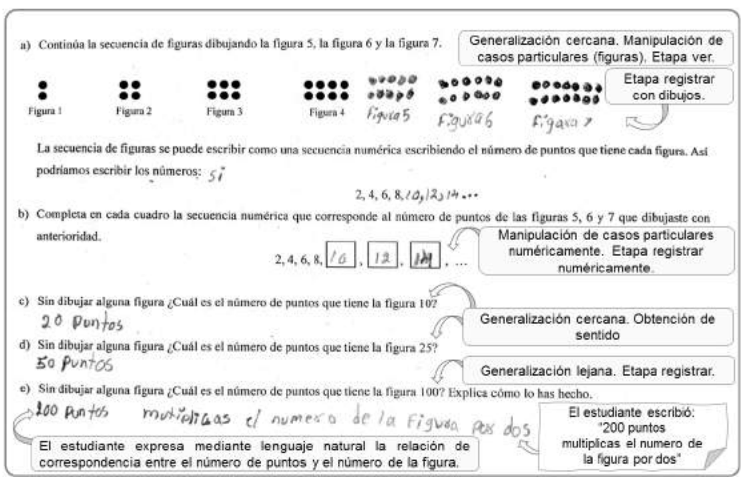

Figura 3: Análisis de las respuestas del Estudiante 10 en la Tarea 2 del Bloque I

Tabla 1: Resumen de acciones en la Tarea 2 del Bloque I

S	a)	b)	c)	d)	e)	S	a)	b)	c)	d)	e)	S	a)	b)	c)	d)	e)
S1	M	M	M	GS	A	S9	M	M	M	GS	A	S18	M	M	M	GS	A
S2	M					S10	M	M	M	GS	A	S19	M	M			
S3	M	M	M	GS	A	S11	M	M	M	GS	A	S20	M	M			
S4						S12	M	M	M	GS	A	S21	M	M	M	GS	A
S5	M	M	M	GS	A	S13	M	M	M	GS	A	S22	M	M	M	GS	A
S6	M	M	M	GS	A	S14	M	M	M	GS	A	S23	M	M	M	GS	A
S7	M	M	M	GS	A	S15	M	M	M	M	NA	S24	M	M			
S8	M	M	M	GS	A	S16	M	M	M	GS	A	S25	M	M			
						S17	M	M	M	GS	A						

Nota: E=Estudiante; M=Acción Manipular; OS=Acción Obtención de Sentido; A=Acción Articular; NC=No contestó; Espacio en blanco=El estudiante no logra alguna de las acciones M-OS-A.

Resultados y Conclusiones Preliminares

Los estudiantes proporcionan evidencia de su capacidad para generalizar al identificar en las tareas con secuencias numéricas aquellos detalles que permanecen invariantes y los que cambian. También, pueden identificar regularidades y patrones, leer e interpretar registros tabulares y sobre todo articular expresiones de generalización. Dichas expresiones fueron comunicadas mediante respuestas numéricas, con palabras, o palabras con algunos símbolos aritméticos, encontrándose 5 categorías: 1) expresiones que enuncian solo la regularidad del patrón, es decir, la variación entre los términos; 2) expresiones numéricas que surgen de usar resultados anteriores, siendo correctas solo para las funciones de la forma $f(x) = ax$; 3) número generalizado, es decir, se expresa la generalización sólo para el número solicitado; 4) expresiones que enuncian de manera incipiente lo que ocurre con la relación entre las dos variables involucradas y 5) expresiones que denotan generalización considerando un término cualquiera (como la que se reporta en la Figura 3). En la siguiente fase de la investigación se analizarán las expresiones de generalización obtenidas en la etapa "decir" al entrevistar a los 25 estudiantes durante la resolución de las tareas de situaciones cotidianas. Finalmente, se considera que el trabajo proporciona referentes acerca del desarrollo del pensamiento y lenguaje algebraico que pueden contribuir en la enseñanza de la educación básica.

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CHILDREN'S INTEGER DIVISION: EXTENDING ANALOGIES AND DIRECT MODELING

DIVISIÓN INTEGRAL DE NIÑOS: EXTENDIENDO ANALOGÍAS Y MODELADO DIRECTO

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Even famous mathematicians, such as Pascal and Diophantus, rejected the possibility of negative integer solutions to algebraic equations (Bishop et al., 2011, 2014; Dehaene, 1997; Gallardo, 2002). Historical struggles make sense when we look at the limitations of physical models and instructional models for integers (Martínez, 2006; Peled & Carraher, 2008; Schwarz et al., 1993–1994; Wessman-Enzinger, 2019). Consider modelling 2 – -5 with discrete chips. Although this is possible with two-colored chips or tiles (Flores, 2008) and using concepts of zero pairs (e.g., Dickman & Bofferding, 2017), it is not necessarily intuitive and has constraints (Murray, 2018; Vig et al., 2014). These challenges in the physical embodiment of negatives integers extend to thinking and learning about multiplicative situations (e.g., -4 × -5). It is important to understand the ways that children use and extend their previous whole-number to negative integers in order to best support their thinking in our classrooms.

Theoretical Framing: Division with Integers

Cognitively guided instruction illustrates problem types and strategies for thinking about division (Carpenter et al., 2015). We can describe *partitive division* as having a total amount, dividing that amount by a certain number of groups, and determining the amount per group (total ÷ number of groups = amount per group). We can describe *measurement division* as having a total amount, dividing that total by the amount per group, and determining the number of groups (total ÷ amount per group = number of groups). These two-problem types, partitive and measurement division, are well-accepted in the field (e.g., Jansen & Hohensee, 2016; Nueman, 1999). Children's *direct modeling strategies,* as they solve division problems, often align with these problem types (e.g., Carpenter et al., 2015; Mulligan & Mitchelmore, 1997). Despite constraints with the physical embodiment of the negative integers (Martínez, 2006), direct modeling can be productive for students to extend their whole number reasoning to multiplication with negatives (Carpenter & Wessman-Enzinger, 2018).

When children engage in solving addition and subtraction problems with negative integers, they construct a variety of productive strategies, including drawing on analogies (Bishop et al., 2014, 2016, 2018; Bofferding & Wessman-Enzinger, 2018; Wessman-Enzinger, 2019; Whitacre et al., 2017). For example, they productively compare -2 + -3 to 2 + 3 with analogical reasoning (Bishop et al., 2016, 2018; Bofferding, 2010, 2011; Whitacre et al., 2017). Yet, sometimes these analogies break down for multiplication (Carpenter & Wessman-Enzinger, 2018)—a student incorrectly determined -4×-2 is -8 with a logical analogy to $4 \times 2 = 8$.

The work reported in this research brief extends the discussion on children's thinking about negative integers by highlighting the ways two Grade 5 children reasoned with analogies and direct modeling with integer division.

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Methods

Alice and Kim, from a rural Midwest school in the United States, participated in a 12-week teaching experiment (Steffe & Thompson, 2000) centered on integer addition and subtraction. The children in this study became accustomed to exploring mathematical ideas, struggling, and sharing their invented thinking. The session described here departed from the study on addition and subtraction and is an exploratory session on division. I wondered how children would *construct* integer division. In this session, two of the fifth-graders, Alice and Kim, worked together on three different division open number sentences with negative integers for fifty minutes. The three division problems they discussed in this session included: -24 ÷ ☐ = -2, ☐ ÷ -3 = -10, and -21 ÷ ☐ = 7. Teaching experiment methodology draws on using conceptual analysis, which requires examining the thinking of individuals iteratively (Thompson, 2008).

Results and Discussion

Alice and Kim used analogical thinking and direct modeling strategies that drew mostly on partitive thinking as they solved division problems with negative integers. Their ways of thinking, as well as affordances and constraints, are addressed in this section.

Analogical Thinking with Integer Division

Alice and Kim solved 24 ÷ ☐ = -2 first in the session. Both Alice and Kim’s inaugural approach of this open number sentence included constructing analogies to 24 ÷ 12 = 2 and reasoning that -24 ÷ -12 = -2. Alice shared, “I think it’s -12 because twenty-four divide by twelve is two. And, these two are negative (pointing at -24 and -2), so I thought that this would have to be negative.” And, Kim shared, “Yeah. There’s both negatives (shrugs). I just compared it to something easier. I just did 24 divide by twelve.”

For ☐ ÷ -3 = -10, Alice and Kim made analogies again. They compared ☐ ÷ -3 = -10 to 30 ÷ 3 = 10. Kim reasoned, “You can multiply these two (points at -10 and -3 on the paper), but in a negative way, in the negative side. So, three times ten is thirty, so just add the negative symbol onto it.” Alice and Kim both reasoned that -30 is the solution to ☐ ÷ -3 = -10 to because 30 ÷ 3 = 10.

There are similarities in analogical reasoning across these integer division number sentences; Alice and Kim compare both to fully positive integer number sentences. Children often draw on analogies productively for both addition and subtraction with negative integers. In fact, problem types like -2 + -5 and -6 – -3 are readily solved by children who see negative integers for the first time by comparing -2 + -5 to 2 + 5 and -6 – -3 to 6 – 3 (e.g., Bofferding, 2011; Whitacre et al., 2017). Note the structural similarities between these analogies: addition and subtraction with negatives are compared to fully positive numbers sentences, but result in correct solutions. Therefore, it is not surprising that Alice and Kim applied an analogy that worked well for integer addition and subtraction to these open number sentences with division, -24 ÷ ☐ = -2 and ☐ ÷ -3 = -10. However, this structurally similar analogy (comparing 24 ÷ 12 = 2 to -24 ÷ ☐ = -2) does not result in the correct solution.

Although the analogies described above did not support correct solutions, Alice solved -21 ÷ ☐ = 7 correctly using an analogy. Alice thought the open number sentence -21 ÷ ☐ = 7 was challenging division open number sentence type; Alice wrote “nope” and the solution -3 (-21 ÷ -3 = 7) and Kim wrote “wung it!” and the solution 3. Kim reasoned that the solution to -21 ÷ ☐ = 7 is 3. She made an analogy similar to what she did previously, comparing -21 ÷ 3 to 21 ÷ 3. Kim noticed something different in the structure, as she wrote “wung it” and shared uncertainty.

Alice suggested the correct solution of -3 when solving -21 ÷ ☐ = 7. Alice was unsure of her solution and verbalized her struggle. She, for example, wrote “nope” on her paper conjecturing, “it’s wrong.” She shared the challenge with the structure of this integer division problem: “I mean it doesn’t make sense that a negative divided by a negative would equal a positive.” Alice compared -

21 ÷ ☐ = 7 to -21 ÷ 3 = -7 and 21 ÷ 3 = 7, conjecturing -21 ÷ -3 = 7. Although unsure about -3 a solution, she maintained a strong confidence the answer is not 3, stating how numbers like -21 and 21 are different.

One challenge analogical reasoning that analogies that work well for integer addition and subtraction (comparing -2 + -3 to 2 + 3), to not readily extend to integer division (comparing -24 ÷ -12 = -2 to 24 ÷ 12 = 2). An affordance of analogical reasoning is that Alice was able to comparing -21 ÷ ☐ = 7 to 21 ÷ 3 = 7 and know that solution could be negative.

Direct Modeling: Partitive and Measurement Thinking with Integer Division. Alice used a direct modelling strategy for -24 ÷ ☐ = -2 when prompted to explain her analogical reasoning above. Both Kim and Alice used direct modeling strategies for justifying their analogies as they solved ☐ ÷ -3 = -10. For the last open number sentence in the session, -21 ÷ ☐ = 7, both Kim and Alice used analogical reasoning only.

Alice drew out tallies and groups, a component of direct modeling, because she wanted to justify the analogy she made when first solving -24 ÷ ☐ = -2. The transcript below illustrates the Alice's discussion about her tallies and group in Figure 1:

> I did two rows (pointing at tallies). They both had… they are negatives. And, there's two (draws two circles around groups of tallies). … Because if it was positive twelve, then this would be (points at the -2) … I don't know. … I did twenty-four (shows drawing with 24 tallies) and what I did is twelve and twelve and it makes two (points at two groups of 12 tallies in the drawing).

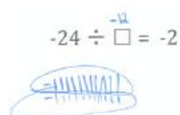
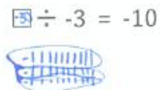


Figure 1: Alice's use of direct modeling when solving -24 ÷ ☐ = -2

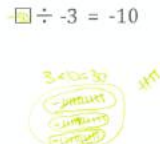
Alice created a total of 24 tallies, stating "they are negatives." As she made these 24 tallies, she placed them in two groups—she drew on a partitive division reasoning. Each of her groups has 12 tallies (see Figure 1), which represents -12. Alice actually illustrated -24 ÷ -12 = +2 with her direct modeling, and even refers to "two groups" (+2) instead of -2. She used partitive division because she doles out the tallies into two groups using how many tallies are in each group (12 tallies, or -12) as the unknown. For Alice, the number of groups was known, using +2 instead of -2, from the open number sentence. She stated that the amount in each group was unknown in -24 ÷ ☐ = -2, putting tallies in two groups and counting the tallies after.

One challenge that Alice faced is that her drawing is well-suited for -24 ÷ -12 = +2, instead of -24 ÷ 12 = -2. A second challenge that Alice encountered is making sense of what -2 groups with partitive division entails; indeed, -2 groups used in this way has physical has constraints. The physical constraints could potentially be why Alice states +2 groups instead of -2 groups. Suppose Alice used partitive reasoning and direct modeling in a way that resulted in the correct solution for -24 ÷ ☐ = -2. How could Alice physically take the existing 24 tallies that represent -24 and put the +12 tallies into -2 groups? Having -2 groups seems a bit ridiculous. However, in this case Alice could have used a direct modeling strategy with measurement division and it could have worked. For example, if Alice, instead, took her -24 tallies and looked for an unknown number of groups (12) that have -2 each in them, this could have worked. Using Alice's strategy with tallies, she could have started making groups of negative two or two tallies. If she did this, there would be 12 groups.

Both Alice and Kim drew on direct modeling strategies when justifying their analogies as they solved ☐ ÷ -3 = -10 (see Figure 2a and 2b, respectively). Both of their drawings produced for this open number sentenced used a direct modelling strategy similar to what Alice created (see Figure 1).



(a)



(b)

Figure 2: Alice and Kim’s use of direct modeling when solving ☐ ÷ -3 = -10

The following excerpt of transcript highlights how Kim discussed her drawing in Figure 2b:

> Um, I did three groups. I did three groups. The negatives is, is the negatives symbol which it tells us that the answer is going to be a negative. These numbers are negatives. I did three groups of ten tally marks in each. Like how you do tally marks regularly, like three, four, and equal five. I did that. I screwed up, but it doesn’t matter though. …These circles represent three groups. … Negative thirty is all of these combined together (motions around drawing). … Um, once you add all these up it equals negative thirty. And, you divide by negative three. …If you add all these up, it equals thirty, you can just divide it by three because of the three groups….The negative ten is in each of these (points at the tallies in the groups). I screwed up there’s six in each of these, but there’s supposed to be five.

Alice and Kim both used partitive reasoning for division, where they interpreted dividing by -3 as indication that there are +3 groups (not -3). They put ten tallies in each group, which represents -10 (albeit Kim has groups of 6 tallies instead of 5, but she counts them as five and recognizes the extra tallies). Both Alice and Kim determine that the solution is -30 to ☐ ÷ -3 = -10 by counting the total amount of “negative” tallies they have.

Kim interpreted the “÷ -3” in ☐ ÷ -3 = -10 as the number of groups positive three groups (+3), which is a challenge whether you use partitive division or measurement division. Number sentences like -24 ÷ ☐ = -2, can be supported by the students’ partitive reasoning if they twelve groups of -2. But, is partitive or measurement division intuitive with ☐ ÷ -3 = -10? The open number sentence ☐ ÷ -3 = -10 differs from -24 ÷ ☐ = -2 in the sense that the positive number (i.e., ☐ or 30) is the dividend. Analogical reasoning may be more productive for ☐ ÷ -3 = -10.

Concluding Remarks

This qualitative description of thinking about integer division highlights affordances and constraints of both analogies and direct modeling with integer division. Analogies for whole number addition and subtraction do not extend readily, but can be used productively. Although direct modeling had limitations with partitive division for -24 ÷ ☐ = -2, Alice’s direct modeling highlighted the potential of measurement thinking with this particular open number sentence. Yet, both partitive and measurement definitions of division have limitations for number sentences like ☐ ÷ -3 = -10—highlighting potential for analogical reasoning.

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FOUNDATIONAL ALGEBRAIC REASONING IN THE SCHEMES OF MIDDLE SCHOOL STUDENTS WITH LEARNING DISABILITIES

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During the course of a three-month teaching experiment, two middle school students with learning disabilities were found to form operations foundational to algebraic reasoning as they constructed mathematical equivalence schemes. In particular, the students' schemes were considered to be algebraic because they contained the cognitive roots of the distributive property, quantitative unit conservation, and solving linear equations of the form ax=bc with whole number solutions (where a, b, and c are whole number constants). The algebraic character of the students' operating was based on the extent to which the students operated on the structure of their additive and/or multiplicative schemes as they solved tasks that required them to create equivalence between two multiplicative compilations.

Keywords: Algebra and Algebraic Thinking, Number Concepts and Operations, Special Education

Introduction

The purpose of this paper is to demonstrate how the equivalence schemes of students with learning disabilities can be considered algebraic. Additionally, I suggest the algebraic character of such schemes is rooted in how students create and coordinate composite units. This study is grounded in the work of early algebra researchers who have focused on the algebraic character of student's ways of operating prior to a formal algebra course (cf. Blanton & Kaput, 2005; Carraher, Schliemann, Brizuela, & Earnest, 2006; Hackenberg & Lee, 2015). Research in the area of early algebra has become crucial for schools as the number of students who encounter algebra in middle school has increased substantially in the last twenty years.

At the same time, opportunities to be included in grade-level general education have grown for middle school students with learning disabilities due to education policy (Every Student Succeeds, 2015; Individuals with Disabilities Education Act, 2004) (Hord et al, 2019). But access is not enough. Hord et al. (2019) explains that for students with learning disabilities to be successful in middle school and as they progress to formal algebra in high school, they will need to make sense of complex algebraic concepts (Bouck, 2017; National Council of Teacher of Mathematics (NCTM), 2000). This report addresses how the whole number operations of students with learning disabilities can form the foundation for reasoning algebraically and learning complex algebraic concepts. Specifically, I focus on how the cognitive roots for the distributive property, quantitative conservation, and solving linear equations of the form ax=bc (where a,b, and c are whole number constants) can be found in relational equivalence schemes.

Relational Equivalence Schemes and Algebraic Reasoning

The schemes and reasoning I investigated were not formal algebraic schemes, but rather, the multiplicative and equivalence schemes of students that form the underpinnings for formal algebraic concepts. From an ontogenetic perspective, this means schemes are described as algebraic if they can later be reorganized into algebraic schemes (Hackenberg, 2006; Steffe, 2001). In this section, I first describe schemes of mathematical equivalence (Woodward, 2016). I then discuss formal algebraic ideas related to the equivalence schemes: the distributive property, quantitative unit conservation, and solving linear equations.

Mathematical Equivalence Schemes

Students have the opportunity to construct equivalence schemes when they are asked to produce equality between related multiplicative compilations. For example, a student may be told that person A has 3 baskets of apples with 4 apples in each basket, while person B has 8 baskets of apples with 4 apples in each basket. They are then asked to make it so that each person has the same amount of apples. In this case the two multiplicative compilations differ by the number of composite units (CU) (3CU of 4 singletons and 7CU of 4 singletons), but they also could differ by the size of the CUs (3CU of 4 singletons and 3CU of 5 singletons). There are many solution methods to such a task, including some that indicate a student is engaged in unidirectional thinking as they focus on transforming only one quantity. Others, though, suggest a student is considering relationships between quantities across the compilations as well as notions of balance between them.

I have identified two schemes that students construct while solving tasks like those mentioned in the previous paragraph, a Relational Equivalence (RE) scheme and a Quantitative Relational (QRE) scheme (Woodward, 2016). RE and QRE both incorporate additive balancing operations that support creating equivalence between the two multiplicative compilations. When operating with an RE scheme, a student first multiplicatively produces the totals of 1s from each compilation (3x4=12; 7x4=28 in the example above). They then find the difference in 1s between the totals (28-12=16) and create equivalence by operating on the totals with some or all of the 1s in the difference. For example, they may halve the difference between the two compilations and then re-distribute the 1s (16/2=8; 12+8=28-8).

When a student constructs a QRE, they can operate in the same way, but they can also do more. Instead of operating on 1s, they may choose to focus on the CUs. They can lift their operations on 1s to operations on CUs that require anticipation of multiplicative structures. A student operating with a QRE may first produce a difference in CUs between the two compilations (7CU-3CU=4CU). They then can create equivalence via additive operations on the CU. For example, they may re-distribute the CUs in the difference (4CU/2=2CU, 3+2=7-2) or transform the CU in one of the original compilations to create equivalence (3CU+4CU=7CU).

In this paper I argue that these two schemes are of particular importance because they can provide valuable insight into how students can engage with the distributive property, quantitative unit conservation, and solving linear equation prior to a formal introduction. Moreover, they may be able to provide a path to generate these ideas from students' whole number operating.

Distributive Operations

Researchers have described how students exhibit operations needed for the formal distributive property in algebra, $a(b+c)=ab+ac$ where a, b, and c are real numbers, in whole number (e.g., McClintock et al., 2011; Tzur et al.; 2009) and fractional situations (e.g., Hackenberg & Lee, 2015; Hackenberg & Tillema, 2009). In the case of whole numbers, McClintock et al. (2011) demonstrated a student's use of a distributive operation as the student found the difference between two multiplicative compilations. For example, when presented with two multiplicative compilations such as 19 boxes of candy with 6 pieces of candy in each box and 15 boxes of candy with 6 pieces of candy in each box, a student could anticipate the multiplicative structure of each compilation and identify 4 boxes of candy as the difference. Implicitly included in their mental operations is the underlying quantity of 6 pieces of candy in each box. It is distributed over each set of composite units. The student could mentally anticipate the units-coordination in each of the three sets of composite units without enacting them. For such a student, 19 groups of 6 minus 15 groups of six is the same as 4 groups of 6.

Distributive operations have also been described in fractional reasoning. Hackenberg & Tillema (2009) provided evidence of a student's distributive operation as they multiplied two fractions. The student produced a fractional amount of each unit and then added them together to produce a fractional amount of the sum of the units. For example, when asked to find 1/5 of 4/7 of a mile, a student may first partition 4/7 into 4 units of 1/7 and then partition each 1/7 mile into 5 units. In doing so, they distribute the 1/5 across the 4 units of 1/7 miles. They can then solve the problem by adding 4 copies of 1/5 of 1/7 [1/5(4/7) = 1/5 (1/7+1/7+1/7+1/7)]. Whether the situation is multiplicative or fractional, the key is that the student anticipates the result of distributing one unit across another. Such an anticipatory scheme is considered to be algebraic because it can be reorganized into the formal algebraic idea of the distributive property (Hackenberg & Tillema, 2009).

Quantitative Unit Conservation

Olive and Cağlayan (2008) described *Quantitative unit conservation* as a "coordination of coordinated quantities" (p.271). Quantitative unit conservation emphasizes how quantities found in numerical or algebraic tasks often not only need to be coordinated locally within an equation, but also need to be coordinated across the whole equation. Writing and solving systems of equations, for example, requires schemes constituted by complex relationships that also focus on relating the underlying quantities in the task.

Suppose a student is asked to deposit \$1,000 between two bank accounts from banks A and B, where Bank A provides an interest rate of 4% and bank B has a rate of 6%. If a goal is set to earn 5% interest, two equations could be written and then solved via methods for systems of equations. A possible equation relating the interest earned from each account to the total interest could be $.04x+.06y=.05\text{x}1{,}000$ (x is the amount invested at bank A and y is the amount invested at bank B). However, it is not uncommon for a student to leave off the \$1,000 and only write $.04x+.06y=.05$. The cause of this can be linked to quantitative unit conservation. A student may focus on equating the percentages instead of considering the global coordination of the quantities and the need to equate quantities of money.

Solving Linear Equations

Hackenberg describes how the equation $ax=b$, where a and b are real numbers and x is an unknown, can be thought of as a statement of division (2006). Similarly, the equation $ax=bc$, where a, b, and c are real numbers, can also be thought of a statement of division (once b and c are coordinated multiplicatively). This is not the only way, though, to conceptualize the equation $ax=bc$. For the purposes of my study, the equation $ax=bc$, where a, b, and c are whole numbers, can be thought of as a statement of equivalence. Many researchers have espoused the need for children to view the equal sign as a relational qualifier (e.g., Baroody & Ginsburg, 1983; Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006) when solving linear equations in formal algebra. When a QRE scheme is formed, a scheme of balance is constructed that can be reorganized into a relational understanding of the equal sign.

Furthermore, in the special case where a, b, and c are whole numbers and x is also a whole number, students who do not have fractional reasoning available can still enlist their multiplicative schemes to find a solution if they view the equal sign as a relational qualifier. For example, when solving the equation 4x7=__x14, a student who sees the equal sign as a unidirectional symbol may write 28 or 392 in the blank provided because they multiply the 4 and 7 or all three of the numbers. A student with a relational scheme of equivalence, in contrast, can reason the left side is equal to 28, which is equal to 2 times 14. Moreover, a student with a QRE scheme can reason 4 times 7 is like having 4 groups with 7 in each group, and doubling the group size from 7 in each group to 14 in each group means the number of groups must be halved from 4 groups to 2 groups if equivalence is to be maintained. Whether the unknown is symbolized as a blank space or a letter, a QRE scheme provides a foundation to create meaning for the operations needed to solve the equation.

Conceptual Framework

Examination of students' operations and schemes in my study was grounded in the reflection on activity-effect relationship (*Ref*AER*) framework (Simon et al., 2004; Tzur & Simon, 2004). *Ref*AER* is a refinement of constructivist scheme theory (Piaget, 1985; von Glasersfeld, 1995) that provides a lens for interpreting how a student forms a new conception through two types of reflections on their mental activity (Simon, Tzur, Heinz, & Kinzel, 2004). The definitions of schemes and operations are those described by von Glasersfeld (1995). Schemes are comprised of mental actions, or operations, and consist of three parts. The first part requires an experiential situation; with the caveat that the experiential situation is nothing more than the student's perception of what is offered. As they perceive the situation, the student sets a *goal* as to what activity to engage in. The second and third parts of a scheme are, respectively, the specific mental activity that is called up by the experience (and attached goal) and the student's expected result.

In addition to equivalence schemes, constructs integral to this study were children's multiplicative reasoning and number schemes. The Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988) and the Generalized Number Sequence (GNS) (Steffe, 1994) provide a distinction for the students' operations on CUs. In each, the student can make sense of the nested relationship between compilations with like units. For example, 7 CUs of 4 (7 baskets of 4 apples) and 6 CUs of 4 (6 baskets of 4 apples) are embedded within 13 CUs of 4 (13 baskets of 4 apples). Additionally, a student with GNS can anticipate a multiplicative structure made up of abstract, iterable units (four 1s distributed over each of the 7 CU) prior to operating with it (Steffe, 1994).

Methodology

Over two and a half months, a pair of 8th graders, Joe and Javier, participated as a pair in 14 teaching episodes taught by the author as part of a teaching experiment. The videotaped teaching sessions occurred twice per week and lasted 30 to 45 minutes. A witness-researcher was present for each of the sessions. Data analysis was conducted during the planning and evaluations of teaching sessions and also retrospectively. During the ongoing analysis, critical events were identified, discussed, and used to build second-order models (Steffe & Thompson, 2000) of the students' ways of operating. These models informed decisions as to which tasks and prompts to select for subsequent episodes. The tasks and prompts had a two-fold purpose: to facilitate the construction of new conceptions or to test the anticipatory nature of current conceptions. In the retrospective analysis, video segments and students' written work were used to make inferences about the ways of operating of the students and the algebraic character of their operating.

Joe was selected for the study for two reasons. First, he was attending twice-weekly pull-out sessions for math with a special education teacher. This was because he was identified by school personnel to be a student with a learning disability in reading who needed additional support with areas of math such as solving word problems. The school psychologist identified the Wechsler Intelligence Scale for Children (WISC) and the Wechsler Individual Achievement Test Second Edition (WIAT-II; Wechsler, 2005) as the two main assessments used by the school to identify students with learning disabilities. The second reason Joe was selected was that a base-line assessment revealed he was operating with at least an Explicitly Nested Number Sequence (ENS) (Steffe & Cobb, 1988).

Javier was also purposefully selected for two reasons. The first was his identification by school personnel to be a student with a mild cognitive disability. He received mathematics instruction in a self-contained classroom with a special education teacher. Secondly, Javier was considered to be operating with a Tacitly Nested Number Sequence (TNS) (Steffe & Cobb, 1988). Javier did not operate at the same level as Joe, but he was paired with Joe and part of the study because he was able to multiplicatively coordinate quantities fluently in multiplicative and divisional contexts.

Data Excerpts

In this section I describe how both Joe and Javier enlisted their equivalence schemes to solve a particular task. Javier operated on the structure of his additive schemes as he enlisted an RE scheme (Woodward, 2016). Joe went further and operated on the structure of his multiplicative schemes as he created equivalence using his QRE scheme. During their 10^{th} session on May 15th, Joe and Javier were given the task of creating equality between the following two multiplicative compilations: Javier buys 19 bags of candy with 6 pieces of candy in each bag. Joe buys 15 bags of candy with 6 pieces of candy in each bag. An additional constraint of *no written work allowed until after they had a solution* was imposed by the researcher in an attempt to provoke Javier and Joe to move past initially coordinating the two quantities in each compilation to produce a total of 1s.

To solve the task, Joe thought for only a couple of seconds and then wrote:

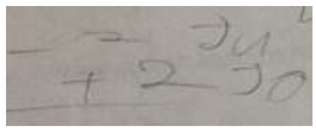

Figure 1: Joe Operates on Composite Units to Create Equivalence.

Unifix cubes were provided to the students in the form of 19 groups comprised of 6 cubes and 15 groups also comprised of 6 cubes. It was common for the students to be asked to demonstrate their solution with cubes after solving the task. At this point I asked Joe about his solution in Figure 1. Joe's response was, "Yeah, I was thinking about bags." Joe then proceeded to demonstrate his solution by moving 2 groups of 6 from the 19 groups to the 15 groups so that each compilation became 17 groups of 6. From Joe's explanation, I inferred he operated on the composite units *between* the compilations. Joe anticipated the multiplicative structures that would be formed if he coordinated the two quantities in each compilation. Such anticipation took the form of a re-presentation to himself of a quantity of composite units with identical sizes from each compilation. Joe was operating with iterable composite units (Steffe, 1994), and so the structures of these two compilations were essentially the same except that there were more composite units in the larger compilation. This understanding enabled Joe to reflect on their relative sizes and to conceptualize the smaller compilation as nested in the larger compilation (15 boxes of 6 nested within 19 boxes of 6). For Joe, the difference in composite units was a way to describe the difference between the two compilations.

Next Joe enlisted his additive schemes as he found the difference in composite units between the two compilations (19-15=4 boxes). His operating to find the difference was predicated on his anticipation that the difference in composite units also described the difference between the total 1s. This anticipation was available to him because the "4 boxes" signified 4 composite units with 6 single units in each that could be coordinated to produce a total of 1s if he chose to enact the coordination. To create equality, Joe increased the smaller compilation (15 boxes with 6 candies per box) by half the difference (4 boxes with 6 candies per box) and decreased the larger total (19 boxes with 6 candies per box) by half the difference. Joe had previously enlisted similar operating, but it was on 1s and not composite units. He re-distributed half the *composite units* in the difference (2 boxes) to the larger compilation (19 boxes) and the other half (2 boxes) to the smaller compilation (15 boxes).

At the end of the task, the students were also asked to write an equation representing their solution. Joe's equation (see Figure 2 below) again provided evidence that the single units (6 cubes per box) were still available to him even though he had operated solely on composite units throughout the process.

Equation: $19 \cdot 6 - 12 = 15 \cdot 6 + 12$

Figure 2: Joe Represents His Solution with an Equation.

It was a requirement for the equations to contain the original quantities from the task. This meant the 6 pieces of candy in each bag needed to be present. When Joe wrote 19 times 6 and 15 times 6, he recognized they represented a quantity of 1s. For the equation to make sense, the two bags from the difference that were re-distributed also then had to be a quantity of 1s. Joe accomplished this by converting the 2 bags of 6 into 12 pieces of candy. Doing so enabled the equation to make sense globally with each representing a quantity of 1s that were equivalent.

Javier also created equality between the two compilations, but he relied on his prowess with multiplying two numbers rather than operations on the multiplicative structures. He first mentally produced the totals (19 boxes of candy with 6 pieces of candy per box=114 pieces and 15 boxes of candy with 6 pieces of candy per box=90 pieces). Next, he added 10 to 90 and subtracted 10 from 114. He produced two new totals of 100 and 104. Javier purposely added a quantity of 1s to the smaller total (90+10) and subtracted a quantity of 1s from the larger total (104-10). I inferred he performed this operation because he anticipated that enacting his additive schemes would bring the two totals closer together by increasing the smaller total and decreasing the larger total. Moreover, when his transformations did not immediately produce equality, Javier continued operating on the smaller quantity to increase it and bring it into balance with the larger quantity. He finished by adding 4 more to 100 so that both totals were now equal to 104. In other sessions, Javier demonstrated that he could continue with more successive operations on both quantities if necessary.

Javier operated on two levels of units, the composite units and the units they contained. Moreover, the composite units were not abstract in nature. This meant he could not *anticipate the results of coordinating the units or the multiplicative structures* they would produce. Thus, he had no opportunity to operate on or with the multiplicative structures. Even when given compilations with large numbers and constrained to mental calculations, Javier computed the totals first and then operated on 1s.

Discussion

In this section I describe how I considered Joe's schemes to be algebraic, while Javier's schemes were algebraic to a lesser degree.

Distributive Operations

Joe operated on CUs as he subtracted 19 boxes of candy minus 15 boxes of candy to yield 4 boxes of candy. Joe mentally anticipated the units-coordination in each of the three sets of CUs involved in the computation as the unit rate of 6 pieces of candy in each box was distributed over each set of CUs. Hence, Joe's mental operations implicitly included the underlying quantity of 6 pieces of candy in each box. He reasoned that 19 groups of 6 minus 15 groups of 6 is the same as 4 groups of 6. I would symbolize this distributive reasoning as 19x6-15x6=(19-15)x6=4x6. Additionally, to produce equality he added two boxes to the smaller compilation and subtracted two boxes from the larger compilation. His reasoning was that 19 groups of 6 minus 2 groups of 6 is the same as 15 groups of 6 plus 2 groups of 6. This reasoning can be symbolized as 19x6-2x6=(19-2)x6=15x6+2x6=(15+2)x6. Again, this representation illustrates the distributive property as 6 is distributed across the quantities.

Joe operated on and with the multiplicative structures of the two compilations provided. His reasoning took on an algebraic character as he enlisted his additive schemes to operate *on* the anticipated multiplicative structure as he found the difference in composite units between the two compilations (19-15=4 boxes). To create equality, Joe also inserted the multiplicative structure of the compilations and the difference into the structure of his equality scheme. The multiplicative structure was readily available to him prior to operating because of his GNS. Joe's reasoning was algebraic because he operated on the structure of his equality scheme *with* the multiplicative structures of the difference (as quantities of composite units) and the existing compilations.

Quantitative Unit Conservation

Joe also exhibited quantitative unit conservation. He demonstrated such reasoning with his equation (see figure 1) and when he described his solution to the ask. After Joe grabbed two groups of cubes from the larger compilation and moved them to the smaller one, he stated, "I take these 2, and then I give them to me. And then they'd be equal. But I just can't say 2. I have to label. So I just put 12." Joe's explanation indicated that he coordinated quantities *within the equation* and *across the whole equation* when generating relationships between the quantities. The key coordinating quantities within an equation and across the whole equation was anticipating the multiplicative structure and operating on the composite units. Joe operated on the composite units (19-2 and 15+2) within the equation. At the same time, he considered the relationship of the multiplicatively coordinated quantities that also contained the unit rate (6 cubes per tower) across the equation. As he demonstrated his solution with the concrete objects (Unifix cubes), the 1s that comprised the composite units were present and available when he needed to include them in his equation. Joe's explanation of his use of the word "two" indicated that he was aware of the need to coordinate the quantities across the equation. Joe reasoned that 19·6-2 was not equal to 15·6+2 because the underlying total of 1s was not equal. As he equated the composite units, he also recognized the totals in 1s needed to be equal. His reasoning was algebraic as it formed the cognitive root of quantitative unit conservation.

Solving Linear Equations

The relational equivalence schemes constructed by both Joe and Javier can serve as a basis for developing a relational understanding of the equal sign. In the discussion above, I illustrated how they both enlisted relational equivalence schemes as they operated on either 1s or composite units simultaneously. This can be contrasted with a student who only operates on one quantity at a time and creates equality by transforming one quantity into the other, a unidirectional scheme of equivalence.

When Javier initially operated on both quantities simultaneously, he purposely added a quantity of 1s to the smaller total (90+10) and subtracted a quantity of 1s from the larger total (104-10). This operation was performed because he anticipated that enacting his additive schemes would bring the two totals closer together by increasing the smaller total and decreasing the larger total. Moreover, when his transformations did not immediately produce equality, Javier continued operating on the smaller quantity to increase it and bring it into balance with the larger quantity. Javier also demonstrated that he was able to do successive operations on both quantities if necessary.

Javier's reasoning was algebraic because he operated with the structure of his additive scheme on the structure of his equality scheme as he operated on both quantities simultaneously. Moreover, he anticipated that the results of his operations could be used for further operating. As Javier created equality via successively smaller transformations, he anticipated a relationship between these transformations and a single, larger transformation. He provided evidence for this anticipation by describing adding 14 to 90 rather than adding 10 and 4. Adding 14 was a single transformation that was a sum of the two smaller transformations of adding 10 and adding 4.

Furthermore, I suggest their RE and QRE schemes form the cognitive roots for symbolizing and solving linear equations of the form $ab=xc$ (where a, b, and c are constants and x is an unknown). Javier could use his relational scheme in conjunction with his operations on tacit composite units to solve *ax*=*bc*. He could multiplicatively produce the total from *ab*, next guess a value for x, then multiplicatively produce the total from *xc* to check to see if matches the total from *ab*. If the totals do not match, he could repeat the process. Without operating with the multiplicative structure of *ab* and *xc*, this would be the primary method available to Javier.

I also suggest Joe could operate with the multiplicative structure of *ab* and *xc* to solve the equation. He could anticipate the multiplicative coordination of the quantities *a* and *b*. He could then reflect on the relationship of this coordination to the multiplicative coordination of quantities *x* and *c*, even though *x* was yet to be determined. He could then solve for *x* via reasoning about *multiplicative relationships* between the multiplicative structures of the compilations *ab* and *xc*. For example, if *c* was twice as large as *b*, Joe could reason that *x* must be half as large as *a*. The key differences between his reasoning and Javier's would be his operations with abstract composite units and his anticipation of the multiplicative structure.

Conclusion

This research demonstrates how the algebraic reasoning of middle school students with learning disabilities is afforded and constrained by their whole number operations. It also provides examples of how students with learning disabilities can operate on and with the structure of their schemes while engaging in complex algebraic reasoning. Finally, this research also supports Hackenberg's reorganization hypothesis (2016) for algebraic reasoning within the context of whole number operations.

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“H IS NOT A NUMBER!” EXAMINING HOW NUMBER INFLUENCES VARIABLE

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Students' conceptions of variable do not always support normative interpretations of equations, and research links limited conceptions of variable to operations on composite units (i.e., units of units). This study examines how one ninth grade Algebra 1 student, Alex's, concept of number is related to his conceptions of variables when writing and interpreting linear equations and expressions. Alex had constructed an advanced tacitly nested number sequence (aTNS). An aTNS is the third stage out of five in the number sequence hierarchy, and indicates that he operates on composite units but does not reason multiplicatively. Analysis links the cognitive structures that define Alex's aTNS to his applied conceptions of variable, and finds that non-normative conceptions of variable manifest due to limitations operating on composite units. Being constrained to additive reasoning limited Alex's use of variables in multiplicative situations.

Keywords: Algebra and Algebraic Thinking; Number Concepts and Operations

Understanding how symbols are used in mathematics is critical to students' success, but algebra curriculum tends to focus more on manipulating symbols than it does on the meaning of symbols (Sherman, Walkington, & Howell, 2016). Letters are particularly difficult for students (Bush & Karp, 2013) because they can represent one number (e.g., e); one unknown quantity (as when solving for x); a “pattern generalizer” (e.g., representing odd numbers as $2n + 1$; Usiskin, 1988, p. 9); or a varying quantity (e.g., $A = l \cdot w$; Baroody, 1998). Students must discern these four uses of letters in algebra in order to make sense of algebraic expressions and equations.

With these four meanings in mind, it is unsurprising that variable misconceptions have been well documented. Knuth, Alibali, McNeil, Weinberg, and Stephens (2005) categorized the ways in which students in grades six through eight conceived of variables. The categories included conceiving of variables as representing multiple values, a specific value, or an object. In this study, the percentage of students who responded that n represents multiple values in an expression like $3n$ increased from less than 50% in sixth grade to more than 75% in eighth grade.

MacGregor and Stacey (1997) similarly documented higher instances of variable misconceptions in later years of schooling (years 8-10) compared to earlier (year 7). Stacey and MacGregor's (1997) research also generated categories for students' misconceptions, including: abbreviated word, alphabetical value, numerical value, use of different letters, letter ignored, labels, variable equals one, and general referent. They found that interference from new learning and poor instruction may account for increased variable misconceptions in older students.

Hackenberg, Jones, Eker, and Creager (2017) have studied the mental structures that support students' conception of variable. They found that operating on composite units (i.e., units of units) supports students' conception of variable as an unknown because a quantitative unknown consists of a composite unit containing an unknown number of units of one. This directly ties students' conceptions of variable to their operations on composite units. Zwanch (2019, 2020) also found that operations on composite units and multiplicative reasoning are related to students' interpretations of linear equations. The present study examines how an Algebra 1 student's conceptions of variable are related to his concept of number.

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Theoretical Framework

Students’ units construction and coordination (Ulrich, 2015, 2016) specifies that the levels of complexity with which they interpret and operate on number is related to their ability to create different levels of units. These levels of units inform students’ number sequences (Steffe & Olive, 2010; Ulrich, 2016b), which are the “recognition template of a numerical counting scheme. … [C]hildren use their number sequences to provide meaning for number words” (Steffe & Olive, 2010, p. 27).

Tacitly Nested Number Sequence (TNS)

The TNS is the second of five in the number sequence hierarchy (Steffe & Olive, 2010). TNS students construct composite units in mental activity. To construct composite units in activity means that TNS students interpret a number word, such as “seven,” as seven individual units, or counting acts (Ulrich, 2015). Thus, “seven” is thought of as the seven numbers from one through seven, or from 34 through 40, for instance. TNS students can engage in mental activity to construct a composite unit of seven, but can neither operate nor reflect on the composite unit.

Advanced Tacitly Nested Number Sequence (aTNS)

In the number sequences, an aTNS is the third stage out of five, and it is characterized by assimilatory composite units (Ulrich, 2016b). An *assimilatory composite unit* allows students to interpret a number word, such as “seven,” as a single unit containing seven individual units of one (i.e., seven is one unit of seven units), and supports operations on composite units (Ulrich, 2016a). This means that aTNS students can construct a third level of units in mental activity (e.g., 21 as a unit containing three units of seven). Following activity, the third level of units decays leaving aTNS students to reflect on 21 as a composite unit containing 21 units of 1. An assimilatory composite unit advantages aTNS students’ numerical reasoning over TNS students, but aTNS students remain limited to additive reasoning.

Explicitly Nested Number Sequence (ENS)

An ENS is the fourth stage in the number sequence framework (Ulrich, 2016b; c.f. Steffe & Olive, 2010). Like aTNS students, ENS students also assimilate with composite units and can construct three levels of units in mental activity. However, ENS students have also constructed multiplicative reasoning that supports them in thinking of seven, for instance, as a unit that is seven times the size of a unit of one.

This research study examines an aTNS student’s conception of variable. Specifically, this research asks: (1) What concepts of variable does the student apply when writing and interpreting linear equations and expressions? and (2) In what ways does operating on composite units but not reasoning multiplicatively support or limit his concept of variable?

Methods

This study was conducted in a middle and high school in the rural southeastern United States. 326 students across grades six through nine were given a survey. The purpose of the survey was to attribute a number sequence to each student (Ulrich & Wilkins, 2017). Based on the number sequences attributed by survey analysis, 18 students participated in semi-structured clinical interviews. Each student was interviewed on two days, for approximately 45 minutes each day. The first portion of the interviews confirmed the students’ number sequence. Questions were taken from the methods of Ulrich and Wilkins (2017). The next portion of the interviews characterized student’s algebraic reasoning. Tasks will be described in the results.

The participant reported here was a ninth grade, Algebra 1 Part 2 student named Alex (a pseudonym). Algebra 1 Parts 1 and 2 was a two-semester course that covered the content of high school algebra. Alex was a tenacious problem solver and was quick to explain his thinking. Alex was identified as an aTNS student by the survey.

Results & Analysis

Alex reasoned about a variable as an object on four of the 11 algebra tasks. One such task asked him to write an equation to represent the following situation: “This week the soccer team scored three fewer points than they did last week.” In this situation, he was told that last week’s and this week’s scores were unknown. Alex wrote $LW - 3$, and this explanation followed:

> Alex: Last week minus three. … Last week minus three equals fewer? I don’t know. Fewer points? … [If] last week they scored 10 and this week they scored less than three, that would be, like, seven.
> Interviewer: … Can we put a variable on that side of the equation?
> Alex: Yeah. $7p$ for points. (Writes $LW - 3 = 7p$)
> I: So does that work, then, if they scored, let’s say five points last week?
> Alex: … Five minus three equals two. Ok, think. It’d be two points this week. (Writes $LW - 3 = 2p$ on the next line.)

Alex was able to conceive of LW, which stood for last week’s score, as an unknown quantity in the context of the expression $LW - 3$. Consistent with Hackenberg et al.’s (2017) conclusion that an unknown comprises a composite unit, Alex’s aTNS supported his conception of LW as a composite unit containing an unknown number of units of one. His aTNS also supported additive operations on LW, evidenced by the expression $LW - 3$.

Alex could not, however, think of $LW - 3$ as an entity equal to a second unknown. He said that it equaled fewer points. To conceive of $LW - 3$ in relation to this week’s score, Alex needed to construct a three-level unit structure. This was supported by Alex’s additive operations on composites, however, following the mental operations, the third level of units decayed leaving Alex to reflect only on $LW - 3$ as representing “fewer points.” Then, Alex initiated a numerical example in which $LW = 10$. Substituting specific numbers for unknowns decreased the complexity of the unit structure, allowing Alex to think about $LW - 3$ as $10 - 3$, which he could equate to seven. When asked to incorporate a second variable, however, he said, “$7p$ for points.” This is evidence of his conception of p as a label on 7, rather than an unknown.

Finally, the interviewer attempted to perturb Alex’s thinking by asking if the equation $LW - 3 = 7p$ would work if the team scored five points last week. Alex was not perturbed, however, and wrote a second equation rather than recognizing the limitation of the equation he had written. This is evidence that conceiving of the additive relationship between two unknowns is beyond the limits of Alex’s algebraic reasoning. His aTNS supported conceiving of one variable as an unknown, and supported additive operations on the unknown; it did not support his normative inclusion of a second unknown into the equation because an aTNS does not support reflection on a third level of units.

At another point during the interview, Alex was asked to represent the weight of a $\frac{1}{24}$ share of a candy bar, given that the candy bar weighed h ounces (adapted from Hackenberg & Lee, 2015). Alex determined two numerical examples (24:1 and 48:2), but he could not think about h as the unknown weight of the candy bar.

> Interviewer: [Can you] represent the weight of just your piece if the whole thing weighs h?
> Alex: h is not a number!
> Interviewer: (Surprised) You said, “h is not a number?”
> Alex: Yeah! That’s not a number!

Later, Alex redirected the discussion back to h. When asked if he understood the task, he said, “Yes and no. The ‘no’ is how you got a number from a letter. … I don’t think you can say that. You can’t say that a candy bar equals h ounces, cause it’s not a number.”

The candy bar task is different because it required Alex to conceive of the multiplicative relationship between two unknowns. In a multiplicative context he could not conceive of h as an unknown because he could not construct and reflect on the three-level unit structure representing the relationship between the whole candy bar and his piece. This is a limitation of his aTNS that manifested differently on the multiplicative task than on the additive tasks. Rather than Alex conceiving of the variable as an object, he insisted that the variable not be present at all, and that the relationship can only be represented numerically. Reasoning about numerical examples instead of unknowns reduced the complexity of the unit structure, thereby allowing Alex to construct and reflect on 24 units of 1, rather than h as 24 units of an unknown number of ones.

Discussion

Alex is a ninth grade aTNS student in the second semester of an algebra class. Regardless, he could not write one-step equations to represent additive or multiplicative relationships. The first research question asked what concepts of variable Alex demonstrated. When writing additive expressions, Alex reasoned about a variable as an unknown, but when asked to reason about the relationship between the expression $LW - 3$ and this week’s score, Alex reverted to numerical examples and interpreted variables as objects.

The inability to establish the relationship between two variables when writing an equation is a persistent difficulty for high school algebra students (Bush & Karp, 2013), and the results of the second research question provide a theoretical lens to understand the complexity that contributes to this difficulty for aTNS students. aTNS students assimilate with composite units and can perform additive mental operations on composite units. Such operations supported Alex’s expression writing. However, the results of the mental operations decay following activity. This manifested behaviorally in Alex’s inability to reflect on the relationship between the expression and the second unknown. To compensate for this limitation, Alex reverted to numerical examples, and interpreted the variables in his equations as labels on numbers.

On the candy bar task, Alex did not introduce a variable at all. Hackenberg and Lee (2015) found that students who assimilate with composite units and reason multiplicatively may represent the unknown weight of the smaller piece of the candy bar as $\frac{h}{24}$. Alex was not able to write $\frac{h}{24}$, and concluded that “h is not a number!” This illustrates the manner by which aTNS students, who assimilate with composite units and are constrained to additive reasoning, may be more limited in comparison to their peers who assimilate with composite units and reason multiplicatively (i.e., ENS students).

On the tasks presented here, Alex reasoned with non-normative concepts of variables or did not include variables. These results bring to light the importance of studying not only students’ operations on composite units as prerequisite cognitive structures to their algebraic reasoning, but also their construction of additive versus multiplicative reasoning. Furthermore, Alex’s solutions demonstrate that he is capable of reasoning algebraically to some extent. Thus, research should continue to examine the ways in which algebra instruction can productively support aTNS students’ concept of variable and their additive and multiplicative operations on variables.

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ALGEBRA, ALGEBRAIC THINKING AND NUMBER CONCEPTS:

POSTER PRESENTATIONS

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EPISTEMIC ANALYSIS OF A LESSON ON LINEAR EQUATIONS OF A MEXICAN TEXTBOOK

ANÁLISIS EPISTÉMICO DE UNA LECCIÓN SOBRE ECUACIONES LINEALES EN UN LIBRO DE TEXTO MEXICANO

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Keywords: Algebra and Algebra Thinking, Teaching tools and resources

Mathematics teachers use many resources to prepare and teach their classes. Official syllabi or academic texts might be a part of them, but as said by Kajander & Lovric (2009) "most teachers still use the textbooks as their primary resource" (p. 173). In this regard, analysing textbooks is important not only for identifying the author's meaning on the different mathematical objects but also, to be aware of the possible conflicts that may araise while using them. This study identifies the promoted epistemic and teaching paths, and possible semiotic conflicts, of a lesson on linear equations in the textbook *Matemáticas 1* by Block, D., García, S., & Balbuena (2018).

The Ontosemiotic Approach (OSA) (Godino, Batanero, & Font, 2007) proposes types of primary mathematical objects: language, situations, concepts, propositions, procedures, and arguments. In this regard, the *epistemic path* is defined as the distribution of these six components along the didactic episode. From this perspective, the *teaching path* describes the actions that the teacher (or, in this case, the textbook) is doing: motivating, assigning, regulating, or evaluating. Also, a *semiotic conflict* is "any disparity or difference of interpretation between the meaning ascribed to an expression by two subjects" (p. 133).

This study analyzes lessons where linear equations are studied, which correspond to sequences 5 and 12. Each sequence is divided into 4 and 3 lessons, respectivly, and, at the end of each sequence, there is a section titled "math lab", where review activities are proposed.

In every unit of analysis, we determine the primary objects that are studied and the order in which they appear, giving as a result the *epistemic path* promoted by the textbook. Also, we identify possible *semiotic conflicts* and the *teaching path* promoted in the study of linear equations.

In this poster we will share findings adressing the following objectives:

a) Identify the primary objects studied in each unit of analysis.
b) Describe the epistemic path promoted by the textbook.
c) Identify possible semiotic conflicts in each unit of analysis.
d) Describe the teaching path promoted by the textbook.

All of them, in the context of the study of linear equations in the aforementioned textbook.

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A PROPOSAL FOR TEACHING MATHEMATICS TO HIGH SCHOOL STUDENTS FROM MODELING APPROACH

UNA PROPUESTA DE ENSEÑANZA DE LAS MATEMÁTICAS PARA ALUMNOS DE PREPARATORIA DESDE LA MODELIZACIÓN

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Keywords: Modeling, High School Education, Algebra and Algebraic Thinking.

The teaching proposal resulting from the study is based on Sadovsky´s (2005) modeling perspective. According to this, mathematical modeling refers to a process that integrates knowledge of a diverse nature and encompasses mathematical tasks, which is divided into several parts: recognizing a problematic, choosing a theory to deal with it and producing new knowledge about it; perspective in which the role of context in modeling problems is highlighted. Likewise, the use of digital technology (GeoGebra) was incorporated into the proposal from the standpoint that it allows observing and establishing, in a simpler way, relationships between the various elements of a problem, compared to working only with pencil and paper.

The research question was as follows: What role does the context of problems, the use of technology and teacher interventions play in promoting students' mathematical modeling work? To answer this, a didactic sequence was designed on problems using algebraic language in mathematical modeling supported by digital technology, which is based on Segal and Giuliani´s (2008) proposal. A previous analysis was made, then a pilot survey was implemented, based on this, the problems were adjusted.

The sequence was implemented to a group of 22 senior students in area I (physical-mathematics) of the National Preparatory School No. 8 in Mexico City. The sequence consisted of three problems, distributed in three sessions of 50 minutes (distributed in two days). Video recordings were used to record group work, teacher interventions and their interactions with students; also, a pair of students was recorded each day. Students were given sheets with the problems and white sheets so they can reflect their work. In general, the activities were designed to be work in pairs and with support of technology: two applets designed in GeoGebra for problems 2 and 3.

In general, the roles played by the context, the use of technology and the teacher's interventions to promote the mathematical work of the students were appreciated. The context favored the evocation/establishment of relationships between the elements of the problems and the appropriation of the context of the problems. The use of technology allowed them to analyze the problems (observe relevant characteristics for the resolution) and obtain approximations of the answers. The teacher's interventions helped students evoke certain knowledge related to the problems, which allowed them to solve them. In this way, it was observed how these elements (context, use of technology and interventions) were intertwined and favored the mathematical work of the students.

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UNA PROPUESTA DE ENSEÑANZA DE LAS MATEMÁTICAS PARA ALUMNOS DE PREPARATORIA DESDE LA MODELIZACIÓN

A PROPOSAL FOR TEACHING MATHEMATICS TO HIGH SCHOOL STUDENTS FROM MODELING APPROACH

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Palabras clave: Modelación, Educación Media Superior, Álgebra y Pensamiento Algebraico.

La propuesta de enseñanza, producto del estudio realizado, está sustentada en la perspectiva de modelación de Sadovsky (2005). De acuerdo con ésta la modelación/modelización matemática se refiere a un proceso que integra conocimientos de diversa naturaleza y que abarca el quehacer matemático, el cual se encuentra dividido en varias partes: reconocer una problemática, elegir una teoría para tratarla y producir conocimiento nuevo sobre dicha problemática; perspectiva en la que se resalta el papel del contexto en los problemas de modelación. Asimismo, se incorporó en la propuesta el uso de tecnología digital (GeoGebra) bajo la postura de que ésta permite observar y establecer, de manera más sencilla, relaciones entre los diversos elementos de un problema, a comparación de trabajar únicamente con lápiz y papel.

La pregunta de investigación fue la siguiente: ¿qué papel juegan el contexto de los problemas, el uso de la tecnología y las intervenciones del docente para promover el trabajo de modelización matemática de los estudiantes? Para contestarla se diseñó una secuencia didáctica de problemas sobre el uso del lenguaje algebraico en la modelización matemática apoyada en tecnología digital, la cual se basó en la propuesta de Segal y Giuliani (2008). Asimismo, se realizaron un análisis previo y, luego, un levantamiento piloto, a partir de los cuales se ajustaron los problemas.

La secuencia se implementó en un grupo de 22 estudiantes de sexto año de área I (físico-matemáticas) de la Escuela Nacional Preparatoria No 8 en la Ciudad de México. Constó de tres problemas distribuidos en tres sesiones de 50 minutos (distribuidas a su vez en dos días). Se usó video para registrar el trabajo grupal, las intervenciones del docente y sus interacciones con los estudiantes; asimismo se grabó a una pareja de estudiantes cada día. Se dieron hojas con los problemas y hojas blancas para que los estudiantes reflejaran su trabajo de manera escrita. Las actividades fueron diseñadas para trabajarse en parejas y con apoyo de tecnología: dos applets diseñados en GeoGebra para los problemas 2 y 3.

En general, se apreciaron los papeles que jugaron el contexto, el uso de la tecnología y las intervenciones del docente para promover el trabajo matemático de los estudiantes. El contexto favoreció la evocación/establecimiento de relaciones entre los elementos de los problemas y la apropiación del contexto de los problemas. El uso de la tecnología les permitió analizar los problemas (observar características relevantes para la resolución) y obtener aproximaciones de las respuestas. Las intervenciones del docente ayudaron a los estudiantes en la evocación de ciertos conocimientos relativos a los problemas, los cuales les permitieron resolverlos. De esta forma se observó cómo dichos elementos (contexto, uso de tecnología e intervenciones) se compenetraron y favorecieron el trabajo matemático de los estudiantes.

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PARAMETERS AND SYSTEM OF LINEAR EQUATIONS

PARÁMETROS EN SISTEMAS DE ECUACIONES LINEALES

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After a long process of learning algebraic procedures, solving systems of linear equations (SLE) is often confused with obtaining a numerical solution, which hinders the idea of solutions of a parametric nature. Therefore, it is important to address the idea of solution in a SLE.

Proper interpretation of infinite solutions creates problems. Even when the solution exists and is unique, it may not be valid. In these cases, the use of parameters allow us to obtain valid solutions, depending on the conditions of the problem (Liern, 2018). However, this possibility is not attended to in classrooms, even though it is relatively simple to come about this situation;or example, an SLE with two equations and three unknowns.

This ongoing research suggests that once we have the resources to solve a square SLE (2x2 or 3x3) we can use a SLE that requires the use of parameters to calculate the solutions. In addition, we can compare the effect that these have on the original SLE and the graphs of the solution set.

Considering parameters as emerging variables associated with the constrains of associated problems, allows high school students to explore situations where, even though they have the solution, theycan modify it depending on the constrains, which helps students realize that parameters are a certain type of variable that modifies other variables and are manageable.

To carry out this research, we will take the point of view of the Theory of Objectivation (Radford, 2020) using computer software as a symbolic tool to solve SLE with two equations and three unknowns. This would promote the idea of a parameter as a variable, that in particular, makes others vary; allowing its interpretation both graphically and algebraically in its respective domain of solution to take the most appropriate decision making of the problems posed.

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PARÁMETROS EN SISTEMAS DE ECUACIONES LINEALES

PARAMETERS AND SYSTEM OF LINEAR EQUATIONS

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Keywords: Álgebra y algebraico, Pensamiento de la educación preparatoria, Sistema de ecuaciones lineales

Tras un largo proceso de aprendizaje de los procedimientos algebraicos, resolver sistemas de ecuaciones lineales (SEL) se confunde con frecuencia con la obtención de una solución numérica, lo que obstaculiza la idea de soluciones generales de naturaleza paramétrica, por lo que es de importancia en abordar la idea de solución en un SEL.

La interpretación adecuada de las soluciones infinitas genera problemas, pero incluso cuando la solución existe y es única, ésta puede no ser válida. En estos casos el recurso del uso de parámetros permite obtener soluciones válidas dependiendo de las condiciones del problema, Liern (2018). Sin embargo, esta posibilidad no es atendida en los salones de clase, pese a que es relativamente simple encontrar esta situación, por ejemplo, cuando tenemos un SEL con dos ecuaciones y tres incógnitas.

Esta investigación en curso plantea que una vez que se sabe cómo resolver un SEL cuadrado (2x2 o 3x3) de manera tradicional, podemos usar un SEL no cuadrados que requieren del uso de parámetros para calcular las soluciones y resolverlos, además de la ventaja de analizar el efecto que tienen éstos en el SEL original, así como el análisis de las gráficas del conjunto solución.

El considerar los parámetros como variables emergentes asociadas a las restricciones de los problemas asociados, permitiría a los estudiantes de preparatoria explorar situaciones en las que, si bien se tiene la solución, ésta puede modificarse dependiendo de las restricciones y lo utilizado; esto contribuye a que el estudiante se dé cuenta de la idea de que los parámetros son cierto tipo de variables que modifican otras variables y que podemos manejar.

Para llevar a cabo esta investigación tomaremos el punto de vista de la Teoría de la Objetivación (Radford, 2020) usando un software de computadora como herramienta simbólica que ponga en juego los saberes necesarios para resolver SEL de dos ecuaciones y tres incógnitas. Esto motivaría el uso de parámetros, lo que propicia el encuentro con la idea de parámetro como una variable que, en particular, hace variar a otras; también permite su interpretación, tanto gráfica como algebraicamente, en un espacio solución para la toma de decisiones más adecuadas de los problemas planteados.

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INTRODUCING VARIABLES TO GRADE 4 AND 5 STUDENTS AND THE MISCONCEPTIONS THAT EMERGED

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Algebra's role as gatekeeper is well documented in helping students build a solid foundation early is important (Stephens, 2005). For this research study, we explored: What misconceptions do elementary students display when generalizing patterns using variables to justify a hypothesis?

We used communities of practice (Wenger, 1998) as an overarching theoretical framework. We sought to engage the students in a community that would be similar to how mathematicians practice mathematics. We used this theory as a way to engage the students in joint enterprise and mutual engagement as a way to hold the participants accountable for their goals of generalizing a conjecture through the use of variables.

Two students, one in Grade 4 and one in Grade 5, participated in three semi-structured, task-based interviews (Goldin, 2000). Each interview lasted approximately sixty minutes. During the interviews, students worked on the unsolved mathematical task called The Graceful Tree Conjecture. They examined graceful labelings of four different classes of tree graphs including: Stars, Paths, Caterpillars, and Comets. We encouraged the students to develop a justification or generalize a pattern for each of the classes to document that all graphs in the given class could be labeled gracefully. All of the students' work was collected and the interviews were video recorded. All the interviews were transcribed and analyzed. For analysis, we documented each instance where students attempted to use a variable to create a generalized pattern or discussed the use or meaning of a variable.

When attempting to create a generalization the students displayed several misconceptions about variables. First, when discussing where the biggest number for would be, they were able to label it as BN. Later, one student said that if it was the biggest number it would have to be infinity rather than the largest number in the set. Second, when attempting to label one less than the biggest number they wanted to label it as SB for second biggest. We pushed them to further their thinking and were hoping for a label of b-1. One student said they could label it as A or negative A because that is one less than B. The other student said it would be E because it is like half of the biggest and if you cut the humps off B it would become E, so E is half of B. Third, when discussing if the smallest number was one (which we always used one as the smallest number in our set), one student said the smallest number is not one, but should be -9999…. These misconceptions need to be addressed for students to later be successful in algebra courses.

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ELEMENTARY ALGEBRAIC THINKING WITH PATTERNS IN TWO VARIABLES

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Usiskin (1999) described four conceptions of algebra: Algebra as Generalized Arithmetic, Algebra as a Study of Procedures for Solving Certain Kinds of Problems, Algebra as the Study of Relationships among Quantities, and Algebra as the Study of Structures. The Algebra as the Study of Relationships among Quantities conception relates to the NCTM (2000) Algebra Standard expectation that students "understand patterns, relations, and functions" (p. 37). Algebraic thinking "includes being able to think about functions and how they work, and to think about the impact that a system's structure has on calculations" (Driscoll, 1999, p. 1). Analyzing students' algebraic thinking with patterning tasks in two variables allows researchers to understand how students think about functions, how they work, and how the representation provided in the question impacts student thinking about the structure of the problem. In this study, one elementary student solved patterning problems in two variables with different representations during a task-based interview (Goldin, 2000).

Preliminary findings suggest that this student used different reasoning strategies when given pattern problems in two different representations. On a task consisting of a visual pattern of figures growing in an arithmetic sequence, the student visualized how the growth occurred in each successive figure. The student used the rate of growth to compute the size of the figure at future iterations. In the context of this task, the evidence suggests that the student was thinking covariationally (Confrey & Smith, 1994) about the relationship between the increase in figure size and increase in figure number.

When presented with a task showing a linear relationship between values in an input-output table of numbers, the student was asked to determine the output value when the input value was 38. Upon receiving this question, the student intensely looked at the problem before stating:

> Oh, I see it now. Okay, so I see if you multiply this by – each number [points at all the numbers in the left input column] by two and add 1, that's the number on this side [points at all the numbers in the right output column]. So take 15 for example. 15 times 2 is 30, plus 1 is 31 and that is in the out. [15 and 31 correspond to each other in the table. 15 being in the input column and 31 being in the output column].

The student used this mapping between the numbers in the input column and the output column to determine 38 corresponds to 77. In this context, the student used a correspondence approach (Confrey & Smith, 1994) to determine the output when the input was 38.

In conclusion, both tasks contained the same structure as linear functions. However, the student thought differently about how the functions "worked" when given a visual pattern of growth as opposed to when given an input-output table. This student showed the capacity to reason through covariation and correspondence while the context of the problem may have influenced the approach. The poster presentation will provide evidence and vignettes from the task-based interview.

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STUDENT'S STRATEGIES TO SOLVE RATIO COMPARISON PROBLEMS IN ELEMENTARY SCHOOL

ESTRATEGIAS DE LOS ESTUDIANTES PARA RESOLVER PROBLEMAS DE COMPARACIÓN DE RAZONES EN PRIMARIA

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Fischbein (1987) describes intuition as a type of cognition that allows attributing intrinsic certainty to knowledge in order to develop a reasoning endeavor. An intuitive knowledge is not necessarily grounded on a logical reasoning. It was desired to observe if third grade students have intuitive notions about proportion even if they hadn't received formal instruction on this subject, and if these ideas prevailed on sixth grade students that had already received this lesson. To address this purpose, we suggested the following research question: which strategies are used by third and sixth grade students to solve ratio comparation problems?

Grounded theory, that was used to elaborate the current research, is a qualitative methodology that proposes to build theory from data. One of the analytic tools proposed by this theory is constant comparisons which consist in coding, grouping and categorizing data (Corbin & Strauss, 2014). The work was developed with 33 third grade and 25 sixth grade students that went to the same elementary school. Four ratio comparison problems like the next one composed the instrument: Pedro blows up 16 balloons in 8 minutes, Juan blows up 20 balloons in 40 minutes, which child blows up more balloons in less time?

Table 1: Approaches and Strategies Used by Students

Approach	Description	Strategies	%3rd	%6th
1. Meaningless	Use of sums and products with no sense	Algorithmic	1	19
2. Absolute	Comparison of only one of two variables (e.g. balloons with balloons)	Univocal comparison	79	6
3. Comparative	Comparison of all the variables by different methods without achieving the correct result	Equivalences chart	0	8
		Incipient notion of proportionality	16	0
4. Proportional	Students get correct results by conventional methods	Constant of proportionality	0	17
		Equivalent ratio	4	28
		Unitary value	0	22

In order to classify the results, it was considered that each element of the proportional relation was a variable: in this case balloons and minutes were two different variables. The students strategies were categorized into four different mutual exclusive bottom-up approaches. The first approach was considered the furthest from proportional reasoning (see Table 1).

Some questions arise from this research. Why was there a major proportion of sixth grade students that answered using algorithms if third grade students already presented intuitive responses with incipient notions of proportionality? How can teaching canalize and make the students intuitions contribute to achieve proportional reasoning?

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ESTRATEGIAS DE LOS ESTUDIANTES PARA RESOLVER PROBLEMAS DE COMPARACIÓN DE RAZONES EN PRIMARIA

STUDENT'S STRATEGIES TO SOLVE RATIO COMPARISON PROBLEMS IN ELEMENTARY SCHOOL

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Palabras clave: Conceptos de Números y Operaciones, Resolución de Problemas

Fischbein (1987) describe la intuición como un tipo de cognición que permite atribuir certeza intrínseca a un conocimiento, y a partir de éste, desarrollar un trabajo de razonamiento. Un conomiento intuitivo no necesariamente se fundamenta en un razonamiento lógico. Se pretendía observar si los estudiantes de tercero tenían nociones intuitivas acerca de la proporcionalidad incluso sin haber recibido instrucción formal en esta materia, y si estas ideas prevalecían en los estudiantes de sexto quienes ya habían recibido esta instrucción. Con este objetivo se propone la siguiente pregunta de investigación: ¿qué estrategias utilizan los estudiantes de tercer y sexto grado de primaria para resolver problemas de comparación de razones?

La teoría fundamentada, que se utilizó para elaborar la presente investigación, es una metodología cualitativa que propone construir teoría a partir de los datos. Una de las herramientas analíticas propuestas por esta teoría es la comparación constante que consiste en codificar, agrupar y categorizar datos (Corbin & Strauss, 2014). El trabajo se desarrolló con 33 estudiantes de tercero y 25 estudiantes de sexto que pertenecían a la misma escuela primaria. Cuatro problemas de comparación de razones como el siguiente conformaron el instrumento: Pedro infla 16 globos en 8 minutos, Juan infla 20 globos en 40 minutos, ¿cuál niño infla más globos en menos tiempo? Para clasificar los resultados se consideró a cada uno de los elementos de la relación de prorcionalidad como una variable: en este caso los globos y los minutos fueron dos distintas variables. Las estrategias de los alumnos se categorizaron en cuatro enfoques distintos mutuamente excluyentes de manera ascendente. El primer enfoque se consideró el más lejano al pensamiento proporcional (ver Tabla 1).

Tabla 1: Enfoques y Estrategias Usadas por los Estudiantes

Enfoque	Descripción	Estrategias	%3ro	%6to
1. Sin sentido	Uso de sumas y productos sin sentido.	Algorítmico	1	19
2. Absoluto	Comparación de una variable (e.g. globos con globos)	Comparación unívoca	79	6
3. Comparativo	Comparación de todas las variables usando diferentes métodos sin lograr el resultado correcto	Tabla de equivalencias	0	8
		Noción incipiente de la proporcionalidad	16	0
4. Proporcional	Los estudiantes obtienen el resultado correcto por métodos conveniconales	Constante de proporcionalidad	0	17
		Razón equivalente	4	28
		Valor unitario	0	22

A partir de la investigación surgieron algunas preguntas. ¿Por qué hubo una proporción mayor de estudiantes de sexto que respondió usando algoritmos si los estudiantes de tercero ya presentaban respuestas intuitivas con nociones incipientes de proporcionalidad? ¿Cómo la enseñanza puede canalizar y hacer que las intuiciones de los alumnos contribuyan a lograr un razonamiento proporcional?

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THE ADVANCED TACITLY NESTED NUMBER SEQUENCE: WHY DOES IT MATTER IN THE MIDDLE GRADES?

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Keywords: Middle School Education; Number Concepts and Operations

Steffe and Cobb (1988) defined four stages of elementary students' understanding of number, which are based on the levels of units that students construct and coordinate (Ulrich, 2015, 2016a). First is the initial number sequence, marked by counting on; next is the tacitly nested number sequence (TNS), marked by double counting and constructing composite units (i.e., units of unit) in activity; third is the explicitly nested number sequence, which is defined by multiplicative reasoning; and finally, the generalized number sequence. In her work with sixth-grade students, however, Ulrich (2016b) defined an additional stage, the advanced tacitly nested number sequence (aTNS). The name is literal in its meaning: aTNS students can apply the operations of their TNS in advanced ways. Research has since identified substantial numbers of aTNS students in the middle grades (Ulrich & Wilkins, 2017; Zwanch & Wilkins, in review). This study builds on Ulrich's (2016b) definition of an aTNS by characterizing the ways in which aTNS students' numerical and algebraic reasoning typify an *advanced* TNS.

The data for this study were collected from students in grades six through nine at a small school district in the southeastern United States. 18 students (2 TNS, 8 aTNS, 6 ENS, 2 GNS) participated in two, 45-minute, semi-structured clinical interviews. Each student's number sequence attribution was confirmed, and the students' algebraic reasoning was characterized.

One characteristic use of number that distinguished aTNS students was their use of skip counting to solve multiplicative problems. TNS students construct composite units in activity (Ulrich, 2015), thus when asked to solve the bar task (Figure 1), they tended to partition the larger bar into four pieces and add 8 four times. Constructing composites in activity supported double counting, but not multiplicative reasoning. aTNS students operate on composite units (Ulrich, 2016b) making it possible for them to think about repeating composite units of eight. Accordingly, aTNS students tended to partition the larger bar into four pieces, then skip count by 8 four times. Most aTNS students stated that they had multiplied eight times four, but their behavior was inconsistent with multiplication. This shows a retrospective awareness of the multiplicative relation, but even after solving several similar tasks, aTNS students did not generalize the multiplicative nature of one bar task to the others. Algebraically, similar behaviors manifested in students' solutions to solving systems of equations. aTNS students used guess and check methods wherein the checking was heavily dependent on students' skip counting. Thus, aTNS students' operations on composite units supported their solutions to algebraic tasks in a manner that confirms their categorization of having constructed an *advanced* TNS. This also demonstrates how aTNS students' numerical and algebraic reasoning is distinct from their peers, making it necessary to consider how we support their learning in the middle grades.

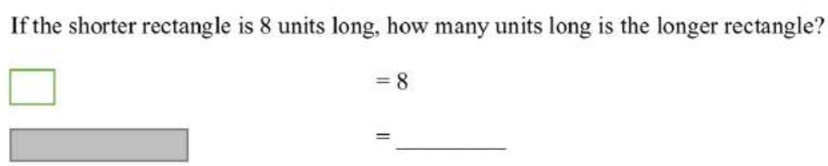

Figure 1. Bar Task from Ulrich and Wilkins (2017)

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CURRICULUM, ASSESSMENT & RELATED TOPICS

RESEARCH REPORTS

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WHAT MAKES A MATHEMATICS LESSON INTERESTING TO STUDENTS?

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How can we design mathematical lessons that spark student interest? To answer this, we analyzed teacher-designed and enacted lessons that students described as interesting for how the content unfolded. When compared to those the same students describe as uninteresting, multiple distinguishing characteristics are evident, such as the presence of misdirection, mathematical questions that remain unanswered for extended time, and a greater number of questions that are unanswered at each point of the lesson. Low-interest lessons did not contain many special narrative features and mostly had questions that were answered immediately. Our findings offer guidance for the design of lessons that can shift student mathematical dispositions.

Keywords: mathematics curriculum, narrative, aesthetic, mathematical story

What if mathematics lessons could be designed and enacted so they were as stimulating as *Harry Potter*, where students eagerly await the next installment? Literature is purposefully designed to capture and hold readers' attention; why not mathematical sequences? The ability to craft mathematical sequences that catch student attention and nurture a desire to continue to learn math would arguably have a positive impact on math education. That is, when a student becomes curious, they are more likely to engage with the content and increase attention, thereby increasing the potential for learning and deepening understanding (Csikszentmihalyi, 1990; Dewey, 1913; Guthrie, Hoa, Wigfield, Tonks, & Perencevich, 2005; Wong, 2007).

Despite major investment to improve mathematics curriculum, the content in most classrooms often sends the meta-message, *study this content because you need it to study related content later that you also will likely have little interest in or realize is even coming* (Chazan, 2000). Rather than provoking student imagination and curiosity in mathematics through sequencing curricular material, textbook authors rely on worldly contexts (conveying, *you should be interested in this because someday you might own a business and need to maximize profit*). Sinclair (2001) argues that this practice of relying on sources outside of mathematics to make mathematics interesting "endorses the belief that mathematics itself is an aesthetically sterile domain, or at least one whose potentialities are only realized through engagement with external domains of interest" (2001, p. 25). Drawing on Dewey's (1934) notion of *aesthetic* as a felt response to an experience rather than an attribute of an object, Sinclair asks, "Could we reverse the direction of the aesthetic flow, so that it originates in the mathematics?" (2001, p. 25).

Unfortunately, little is known about how decontextualized secondary school mathematics can be designed to be interesting and engaging. In this paper, we explore the characteristics of decontextualized secondary mathematics lessons that spur students' curiosity, captivate students with complex mathematical content, and compel students to engage and persevere, which we refer to as "mathematically captivating learning experiences" (MCLEs). The purpose of this paper is to address: *What characteristics, if any, distinguish high school mathematics lessons that students identify as interesting from those they describe as not interesting?*

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Theoretical Framework

In order to identify differences in how the mathematical ideas emerge and change as the lesson unfolds, this study interprets mathematical sequences that connect a beginning with an ending as a *mathematical story* (Dietiker, 2013, 2015b). Built from Bal's (2009) narratological framework, this interpretation foregrounds how mathematical characters are acted upon through mathematical action in mathematical settings. For example, mathematical characters are the mathematical objects brought into existence (objectified) through reference in the story, such as a function. Mathematical action describes the work of an actor (such as a student or teacher) in changing the mathematical ideas or objects of study, such as composing two functions to create a new function. Mathematical characters and actions are brought into being in a constructed "space" such as a white board or a coordinate plane, referred to as the mathematical setting.

For this study, a particularly important quality of a mathematical story is its *mathematical plot*; that is, the way it captivates and holds the interest of its audience. When a mathematical story hints of a future revelation, it may spur the formulation and pursuit of questions ("Why did the composition just end up with *x*?"), similar to how a reader of a literary story might wonder how the story will progress and continue reading. Thus, the mathematical plot describes the dynamically changing tension between what is already known and desired to be known by the participants as the story progresses (Dietiker, 2015b). It enables the description of how a mathematical sequence can generate suspense (by setting up anticipation for a result) and surprise (by revealing a different result than the one anticipated). Questions may span the entire story or may represent brief puzzles or mysteries. The progress made on each question, from when it is asked, to how students' understanding of it changes, to how it is abandoned or answered, constitutes a *story arc*. Since a mathematical story may involve answering multiple questions at any point along a sequence, multiple story arcs may arise over the course of the lesson and overlap at different intervals. The changing number of questions under pursuit by students throughout the lesson can be referred to as its *density* of inquiry.

Methods

This study identifies the distinguishing characteristics of interesting lessons that were designed and taught in the first of three design research cycles (Cobb, Stephan, McClain, & Gravemeijer, 2001; Edelson, 2002). The larger research project is an exploration of whether and how designing high school lessons as mathematical stories impact the aesthetic experiences of students. Six high school teachers, each with at least 4 years of experience, from three high schools with different curriculum and diverse demographic settings in the Northeastern region of the USA were recruited to participate in this study. The teachers worked in pairs, along with researchers, to design MCLEs for one or more of their classes. The participating courses, selected by teachers, spanned entry-level (e.g., Integrated Math 1) to advanced-level (e.g., calculus) and included both honors and non-honors. To support the design of MCLEs, the teachers attended a two-week professional development during the summer of 2018 where they learned about the mathematical story framework (Dietiker, 2015a, 2015b, 2016; Ryan & Dietiker, 2018) and participated in analyzing the mathematical plot of one lesson enactment.

An analysis of the complete set of MCLEs from the 2018-2019 school year, when compared with non-MCLEs from the same teacher and classes, revealed that the MCLEs did impact student interest measures positively (Dietiker et al., 2019). Yet, if these positive student reports are connected with non-mathematical factors (i.e., mood of the teacher, point in the semester), then the mathematical stories of the interesting lessons should not be significantly distinguishable from those of less interesting lessons. Thus, we designed the present study to learn whether the unfolding content of those lessons that students describe as interesting have characteristics are qualitatively different from those that students indicate are not interesting.

All 32 lessons generated in the 2018-2019 school year were observed by multiple researchers using the same protocol so that students would not be able to infer whether some lessons were special or not. The lessons were filmed using three video-cameras placed strategically to capture the teacher, students' facial expressions in the whole class, and the progress of a focus group of students. In addition to a central microphone, audio recorders were placed around the classroom to capture student discourse and the teacher wore a lapel mic. Immediately following each observed lesson, all participating students took a Lesson Experience Survey (LES) on their digital devices. In this survey, students were asked to rate their overall interest in the lesson on a scale of 1 to 4 and select three terms to describe their view of the lesson from 16 given descriptors, including negative, neutral, and positive options. More information about the design and testing of the LES can be found in Riling et al. (2019).

To recognize characteristics that distinguish high-interest lessons from low-interest lessons, the research team composed two groups of lessons by identifying each teacher's highest-interest and lowest-interest lessons based on students' LES responses. Pairing a low- and a high-interest lesson per class allowed both sets to include reports from the same students, rather than two groups of students who might have differing dispositions to mathematics to begin with. Only lessons with surveys from at least 10 students were included (this eliminated one lesson). We selected the lesson with the highest average interest measure for each teacher, using students' selection of positive descriptor to break ties. After selecting a teacher's high-interest lesson, the lesson for that class with the lowest average student interest level (with negative descriptors used to break ties) was selected for the low-interest group. Although the lessons were selected based on the interest level and student-selected descriptors from the LES, all lessons in the resulting "high interest group" were MCLEs and all lessons in the "low-interest group" were non-MCLEs.

To analyze the lessons in the high- and low-interest groups for their mathematical story characteristics, we first coded each for its mathematical plot. Then, we compared the mathematical plots of the two groups and identified characteristics that distinguished them. The plots relied on detailed transcripts for each lesson that included the discourse of a focal group, allowing us to note the progress those students made on the mathematical questions raised during groupwork and to include any questions the students asked while collaborating.

The transcripts were analyzed on three separate coding passes. On each pass, the research team coded separately in groups and then met to resolve differences. On the first pass, the team identified *acts* by tracking what mathematical characters, actions, and settings were in focus throughout the transcript and noting when these changed. On the second pass, the research team identified all mathematical questions that were raised, considered, and addressed throughout the lesson by a teacher, student, or some type of curriculum materials. Whereas some questions were recognized explicitly through verbal or written statements (e.g., "Find the root"), others were raised implicitly by images or situations experienced by students. Questions that were not mathematical (e.g., "Can I present my solution?") were not included.

On the final coding pass, the research team coded how what was known about each question changed across the acts of the lesson. For each question, the researchers used codes adapted from Barthes (1974) narrative theory to code contributions by teachers and students. These codes included foreshadowing of a question ("*proposal,*" marked with a 0 in the plot diagrams), when a question was raised ("*question,*" 1 when raised by a teacher and 2 when raised by a student), any explicit messages that the question would be answered ("*promise,*" 3), any progress made on answering the question ("*progress,*" 4 if made by a teacher, 5 if made by a student), and when, if ever, it was answered and thus closed ("*disclosure,*" marked with "D"). In addition to these codes, we also coded for interruptions to progress when the topic shifted so far from the question that it is no longer reasonable to assume the question may be addressed ("*suspension,*" marked with 9), or when there is a threat to progress toward an answer ("*jamming,"* marked with 8*)*. Finally, we coded any evidence of misdirection, in which lesson participants are misled in a consequential way. There are two types of misdirection: a *snare* (marked with 7) is an explicit error or lie, while an *equivocation* (marked with 6) is an encouragement to make a faulty assumption. To separately track who or what was responsible for a contribution, we also identified for each code whether the contribution came from the teacher, student, or environment (e.g., a worksheet).

These coding passes result in a comprehensive mapping of how participants within each lesson are moved to raise and answer questions, representing the mathematical plot of the lesson. For each question, the acts during which it is open (i.e., it is unanswered and it is reasonable to think that there is still a possibility of further progress on the question), along with all the codes for that question, form a story arc. The set of all story arcs describe how all of the mathematical content emerges and changes throughout the acts of the story.

Next, we qualitatively compared the mathematical plots and identified characteristics that appeared to distinguish the high- and low-interest lessons. We then compared quantitative dimensions of their mathematical plots such as number of acts, number of questions opened throughout the lesson, and story arc length. We compared the average number of coded questions per act, the average number of questions open and in progress per act ("mean density per act"), the proportion of story covered by these open questions ("mean arc length as proportion of story"), and the percentage of questions open for more than one act ("proportion of extended story arcs per total arcs"). A paired samples *t*-test was conducted to compare the mean differences between the two means of these measures for the two groups of lessons for $\alpha < 0.05$. A box plot diagram was made to compare the high- and low-interest lessons for measures.

Findings

In this section, we illustrate the differences between high-interest and low-interest lessons by describing a pair of high- and low-interest mathematical stories taught to one group of students by the same teacher. These mathematical stories are presented in present tense in the sequence in which events unfolded to highlight how what was known changed through the lesson. Next, we introduce contrasting characteristics of the stories' mathematical plots. Finally, we describe general patterns of high- and low-interest lessons, identifying characteristics of lessons that students find interesting.

This pair of lessons was selected because each contains many characteristics common to the other high- or low-interest lessons. These lessons were taught in an Algebra 2 Honors course with sophomores and juniors by Ms. Elm (pseudonym). The class had 28 students, 25 of whom participated in the study.

Low-Interest Mathematical Story

The class begins (Acts 1 through 3) with students working in partners on a "Do Now," which asks students to make sense of a newly defined operation ($a\&b = 3a - b$), including determining whether it is commutative and what its domain is. After Ms. Elm collects their work and verifies one of the answers, she reviews general principles of commutativity and domain.

In Act 4, Ms. Elm distributes a handout with questions about a range of topics, including domain, range, and percentages. The questions are multiple choice practice tasks for standardized tests. The focal group spends the rest of Act 4 working on a question about domain. In Acts 5 and 6, the group discusses a question about the range of a different function. In Act 7, they briefly complete more tasks about algebraic equations and operations.

In Acts 8 through 11, the focal group shifts to work on questions about percents. Each question is about a different aspect of percents and based in a different context: the reduced price of clothing item, what percentage one number is of another, and how much a company's profit increased.

Students engage with each other on each problem, although there is little evidence that they are challenged. The teacher concludes class by stating “we’re gonna stop there for today.”

High-Interest Mathematical Story

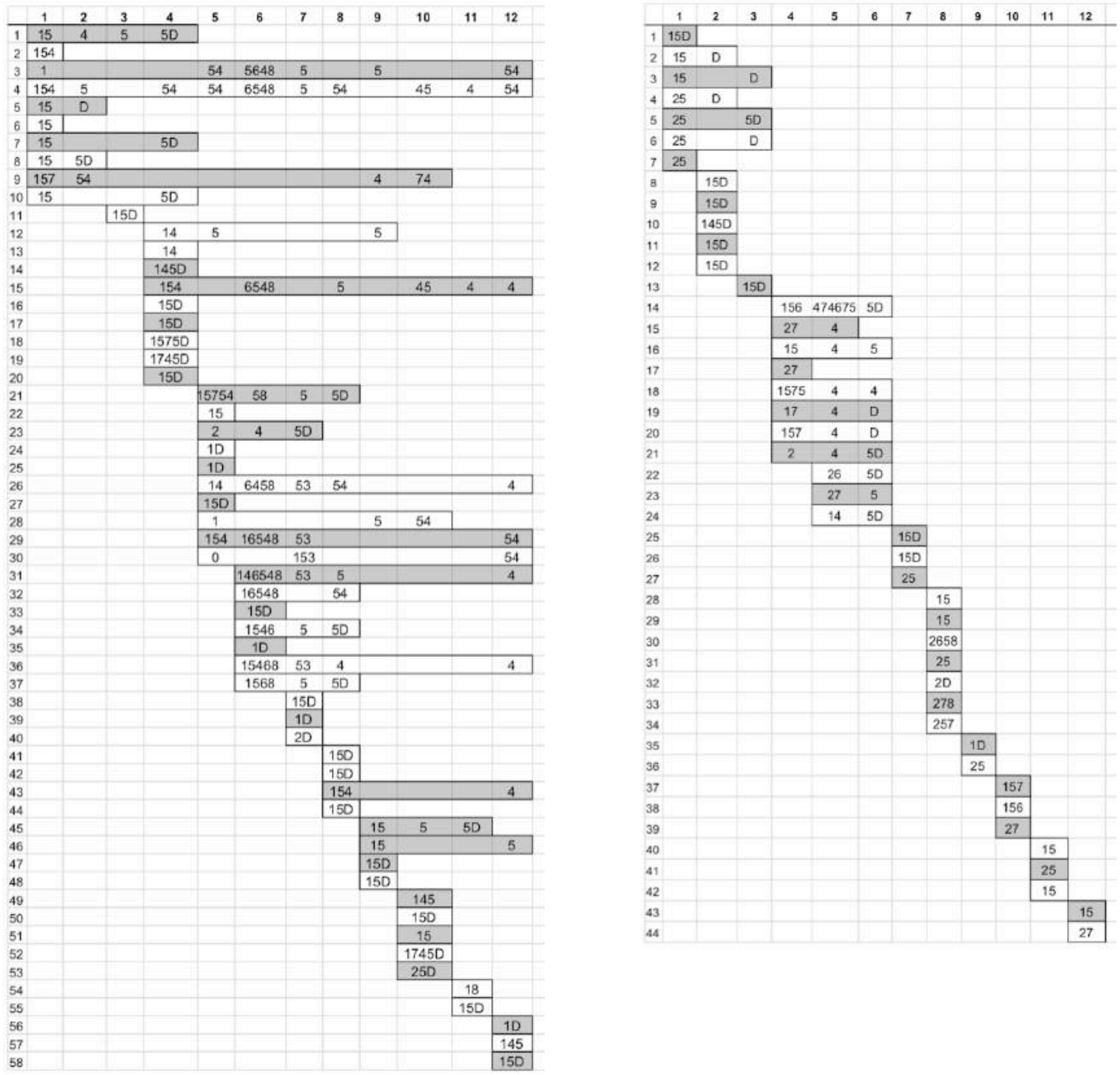
In Act 1, Ms. Elm explains that the lesson objective is to develop a strategy to identify the roots of polynomial functions. After displaying $f(x) = x^3 - 5x^2 - x + 5$ on the board and assigns each pair of students a value between –9 and 9, she challenges students to figure out whether they are “guilty as a root.” First, she asks students to predict whether their value is a root and brainstorm ways to verify their guess. The focal group, which has the value 7, predicts that they cannot be a root because “if you do factoring by grouping, you wouldn't get 7 at all.” In Act 2, Ms. Elm reviews synthetic division. In Acts 3 and 4, each group checks whether they are a root using synthetic division. The group whose value is 5 thinks they might be a root, which is confirmed. The focal group celebrates with a little dance. Next, the class finds the remaining roots (i.e. 1 and -1) by factoring the polynomial.

Students next test whether their group’s value is a root of $f(x) = 2x^3 + x^2 - 16x - 15$. In Act 5, a student says that 8 is likely a root, explaining, “this is gonna sound kinda weird, but because of the two and the sixteen.” This is the first time a student connects the polynomials’ coefficients and roots. When it turns out that 8 is not a root, another group shares that 3 is a root. In Act 6, a student suggests that since 3 is a root, then -5 might be, because the y-intercept is -15. Ms. Elm responds by saying, “that's weird right, ‘cause I actually really kind of agree here. That we need to do something to get ourselves to fifteen.” She asks groups to use synthetic division to find the remaining roots. As students begin to do so, in Act 7, a student spontaneously claims that 2.5, not 5, is another root “because the leading coefficient, you need to divide that. It's two x minus five.” Ms. Elm asks him to “hold that thought” and he reacts with contained excitement. In Act 8, the whole class finds the remaining quadratic once the cubic is divided by $x - 3$. Then, once the constant term is determined to be positive, students recognize that one of the remaining roots is not 2.5, but rather –2.5.

In Act 9, Ms. Elm elicits enthusiasm when she proposes a challenge: The class has four chances to identify the roots of $f(x) = 6x^4 + 35x^3 - 49x^2 + x + 7$. The first two choices, 6 and 7, are found to not be roots. A group proposes -7 as a root, and a student confirms it using synthetic division. In Acts 10 and 11, the class fully factors the polynomial and finds the remaining roots: 1, -⅓, and ½. In the last act, there is a growing sense during a class discussion that there is a multiplicative relationship between the coefficients and the roots. For homework, Ms. Elm asks students to reflect on how to identify polynomial roots.

The Comparison of the Mathematical Plots

These mathematical plots have several distinguishing characteristics, as can be observed in the mathematical plots of the high-interest lesson (Figure 1a) and low-interest lesson (Figure 1b). There were considerably more questions opened throughout Ms. Elm’s high-interest lesson (75) as compared with her low-interest lesson (44). Interestingly, the percentage of the story arcs that remained open for more than one act was similar in both the high-interest lesson (27 out of 75, or 36%) and the low-interest lesson (15 out of 44, or 34%). However, the average arc length of the high-interest lesson is 2.6 acts long, which constitutes 22% of the lesson and is one act longer than the average length questions remained unanswered in the low-interest lesson (1.6 acts, or 13% of the lesson).



(a) (b)

Figure 1: The Mathematical Plots for Ms. Elm's (a) High-interest Lesson and (b) Low-interest Lesson. Each row represents a story arc. See methods section for code references.

In addition, most story arcs in the high-interest lesson contain acts in which no change in what is known about the question occurs. In almost a quarter of the story arcs (18 of 75), there is at least one act with no codes at all, providing the opportunity for students to build curiosity. This also provides a sense that not all questions that are raised will be answered immediately. In contrast, in the low-interest lesson, only three questions (#3, 5, and 6) have an act during which the question is open and yet no codes appear. For all other questions (41 of 44, or 93%), every act in which the question is open contains some change.

Stark differences are also evident in the density of these lessons (see Figure 3). Since the two lessons have a different number of acts, density is graphed across the percentage of the lesson that has passed. In the high-interest lesson, the density generally increases and then remains high, providing a lasting sense of mystery. Act 1 has 10 open questions, and by Act 7, almost twice as many questions are open (19). The number of open questions remains relatively high through the end of the lesson, as students continue to pursue their ideas about how to identify the roots of a given polynomial function. There is variation in the density of both lessons, as the tension alternately increases and decreases. Yet in the high-interest lesson, this variation does not return to the initial lower level, whereas the density in the low-interest lesson remains low with temporary dips. Overall, the density in the high-interest lesson was an average of 14.25 questions per act. In comparison, the low-interest lesson had a maximum density of 10 questions and an average density of 5.7 questions per act.

Figure 2: The Density of Ms. Elm's High-interest (red) and Low-interest (blue) Lessons.

Lastly, we found differences in the occurrence of special mathematical plot codes that represent interruptions and misdirection. The mathematical plot of the high-interest lesson has more instances of *jamming* in comparison with the low-interest lesson (11 vs. 2, respectively). Additionally, the high-interest lesson has a *proposal*, offering a sense of mystery, and five instances of *promise*, offering anticipation for an answer to come, whereas low-interest lesson has neither. In addition, the high-interest lesson has twice as many *equivocations* than low-interest lesson (10 vs. 5), while the latter has twice more (12 vs. 6) *snares*. Interestingly, the high-interest lesson had more misdirection from the teacher, both in terms of *equivocations* and *snares*, than the low interest lesson (3 vs. 1 and 1 vs. 0, respectively).

Characteristics that Distinguish High-interest and Low-interest Lessons

Across all 12 lessons, we identified multiple characteristics that distinguished high- and low-interest lessons significantly. Figure 4 shows the comparative measures for high- and low-interest lessons after the measures were standardized (i.e., mean = 0, vertical axis indicates the number of standard deviations from the mean). The questions in high-interest lessons remained unanswered for significantly more acts, as shown by a higher proportion of story arcs that lasted for more than one act (what we refer to as "extended questions") ($t(5) = 3.16$, $p < 0.05$), a longer mean arc length ($t(5)= 2.85$, $p < 0.05$), and the average arc length spanning a longer portion of the lesson ("mean arc length as % of story"; $t(5)= 4.21$, $p < 0.01$). In addition, the average number of open questions per act ("mean density per act") for the high-interest lessons is significantly greater than that of the low-interest lessons ($t(5)= 3.93$, $p < 0.05$). Similarly, the average number of changes to what is known (i.e., codes) per question in high-interest lessons is significantly greater than that of the low-interest lessons ("mean total codes per question"; $t(5)= 2.96$, $p < 0.05$). We found that high- and low-interest lessons have a similar percentage of acts with codes and disclosed formulated questions. Additionally, although higher interest lessons have more formulated questions and acts than low interest lessons, this difference is not distinguishable.

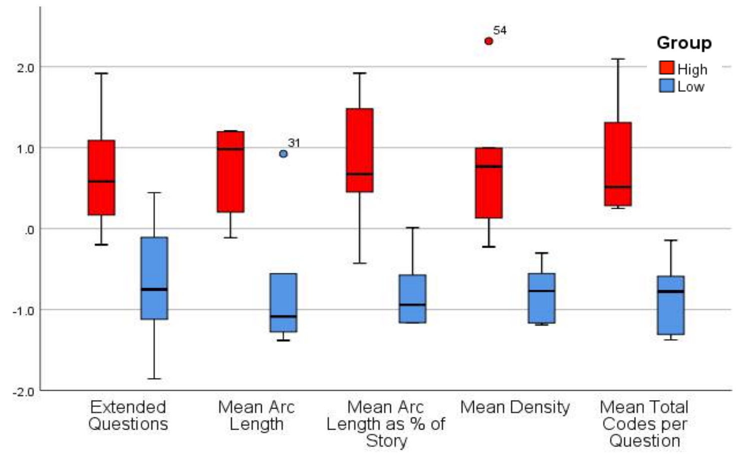


Figure 3: Comparison of Mathematical Plot Measures.

When comparing the frequency of the special mathematical plot codes by teachers (jamming, snare, equivocations, and promise), the high-interest lessons had a greater average frequency than the low-interest lessons. However, these differences were not statistically significant.

Discussion

We started this paper wondering how we can design mathematics lessons that compel students to become curious or excited. The characteristics described in this paper begin to answer this question. For example, the higher percentage of questions that are open for larger proportions of a lesson provide students extended opportunities to mathematically wonder, build understanding, and thus enjoy underlying mathematical concepts. In contrast, when questions of low-interest lessons are answered almost immediately, we are concerned that students may not have enough time to think deeply about mathematical concepts or relationships. Being confused, without the benefit of curiosity or anticipation, may hinder joy.

Knowledge of the characteristics of high-interest lessons can support educators and curriculum developers who wish to design mathematically captivating lessons that can positively impact students' experiences. For example, teachers might decide to mindfully include equivocations, which were not present in any of the low-interest lessons in our data, in order to enable surprise during their lessons. Or curriculum designers may decide to encourage teachers to delay giving answers to students, in order to permit students to wonder and anticipate for longer periods of time. Further research will hopefully uncover additional features that can be used to design reliably engaging mathematical learning experiences.

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IDENTIFYING OPPORTUNITIES TO ENGAGE IN LITERACY PRACTICES: A FRAMEWORK FOR ANALYZING CURRICULUM MATERIALS

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In this report I have created an exploratory framework to identify opportunities to engage in literacy practices within mathematics curriculum materials. This framework describes "unstructured literacy opportunities" and "structured literacy opportunities" for each of the language modalities of reading, writing, speaking, and listening. Different structures within each modality are also detailed along with connections between the modalities. The framework is then applied to Illustrative Mathematics (IM) curriculum materials to reveal patterns of how different types of literacy opportunities are addressed and connected.

Keywords: Curriculum Analysis, Communication, Classroom Discourse, Instructional activities and practices

Written and spoken language serve important functions for communicating, receiving, and retaining information. Language also manifests in complex ways. Aguirre and Bunch (2012) describe how students must navigate a myriad of language demands in mathematics classrooms, differentiating such demands into modalities of reading, writing, speaking, and listening (along with a "representing" demand critical to mathematics in particular). The authors also describe how these demands are connected to one another: writing and speaking are *productive* activities while reading and listening are *receptive*. Additionally, reading and writing are "linked" to "written mathematical conventions," while listening and speaking encompass "oral language" (Aguirre & Bunch, 2012, p. 185). This multifaceted description of language aligns well with Moschkovich's (2015) definition of academic literacy, which positions the learner as an active participant in mathematical discourse. Moschkovich explains this phrasing by stating her desire to "shift from a focus on language as words to a broader sense of *literacy* as participation in practices and discourses" (p. 45).

When literacy is seen in the Vygotskian sense as "the understanding and communication of meaning" (Moll, 1992, p. 8), its relevance to mathematics education becomes even more apparent. After all, achieving learning with understanding is considered the "Holy Grail" of mathematics education (Hiebert & Carpenter, 1992, p. 65). If literacy within written and oral language is a core conduit through which mathematical ideas are communicated, then knowing how best to engage students in literacy practices is of upmost importance.

This report extends Aguirre and Bunch's and Moschkovich's exploration of language and literacy in mathematics by focusing on a curricular rather than instructional lens. The research questions considered are: (1) In what ways do mathematics curriculum materials present opportunities to address literacy demands of reading, writing, speaking, or listening? (2) In what ways do such curriculum materials connect opportunities for reading, writing, speaking, or listening?

Theoretical Framework

A contribution of this report is a framework which describes structured versus unstructured opportunities for reading, writing, speaking, and listening in mathematics. Wiggins (2001) notes how interdisciplinary curricular materials often position one discipline as subservient to the other, which results in an "approach [that] has little to do with teaching the concepts of [either discipline]" (p. 42). Genuine integration should instead promote shared concepts and processes present across the relevant disciplines, thus leading to better understanding of both disciplines (Wiggins, 2001).

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Shanahan and Shanahan (2008) discuss in particular how generalized literacy skills are insufficient within contents such as mathematics as students advance into secondary grades. Instead, teaching in these grades should emphasize disciplinary literacy skills unique to the specialized nature of the individual content (Shanahan & Shanahan, 2008). Curricular tasks which address literacy skills in generalized or subservient ways are considered *unstructured literacy opportunities* in this framework, while tasks which address mathematics-specific literacy skills (i.e. disciplinary literacy) are considered *structured literacy opportunities*. This framework expands beyond Shanahan and Shanahan's reading focus to consider Aguirre and Bunch's (2012) language demands of reading, writing, speaking, and listening. The disciplinary focus of each type of structured literacy opportunity is also described for each modality, as the mathematical purpose of curricular activities may vary. Such modality-specific descriptions of these opportunities follow.

Unstructured reading opportunities include any written mathematical text which students may read, while structured reading opportunities are tasks which specifically address elements of reading comprehension. This includes referencing surface comprehension of syntax and text features for students (Hoffer, 2012) or metacognitive modeling of deeper comprehension (Hoffer, 2012; Kenney, 2005). With such structures in mind, the role of vocabulary within these texts is also of interest. Certain words (called academic vocabulary) have more utility across domains while also being critical to comprehending mathematical text (Hoffer, 2012; Bay-Williams & Livers, 2009), so their inclusion in written tasks could better afford such opportunities. Thus, it is worth considering whether academic text is part of a structured reading opportunity or whether it is absent (which this framework describes as *simple text*).

Structured writing opportunities are found in tasks which require written explanations or justification, as well as tasks which promote refinement of mathematical ideas through written language (Hoffer, 2012; Kenney, 2005). Although many mathematical tasks use words/phrases that imply writing (e.g. "explain", "show how you know," etc.), this alone is unstructured. Since explicit guidance around when and how to write is critical to building writing skill in mathematics (Thompson, 2008), evidence of such opportunities is a prerequisite for structured writing. Additionally, such mathematical writing can serve different purposes. Writing can act as a *summative* assessment which provides better insight into student understanding (Miller, 1991), but it can also be used *formatively* throughout the problem-solving process to better develop students' metacognitive skills than oral communication alone (Pugalee, 2001).

Unstructured listening is a ubiquitous expectation of many mathematics classrooms (Aguirre & Bunch, 2012). Hintz and Tyson (2015) refer to "complex listening" as a structure to support this demand, where mathematics sense-making is encouraged in part by "*Directing* students towards what to listen for and whom to listen to" (p. 315, emphasis in original). While listening opportunities can serve to help students internalize information for themselves (e.g. understanding what the teacher is saying), they also can be used to help facilitate responses to others' ideas within discourse (Aguirre & Bunch, 2012).

Structure can be given to speaking opportunities by directing students toward cooperative speaking (Thompson, 2008) or exploratory talk (Mercer, Wegerif, & Dawes, 1999), where students are explicitly supported in understanding how to share time and space with their peers and collaborate respectfully. While such structures drive speaking in dialogically oriented math classrooms, classrooms which adhere to direct instruction emphasize students sharing their work individually in order to receive immediate, corrective feedback from the teacher (Munter, Stein, & Smith, 2015). This shows how speaking opportunities might be structured in collaborative *or* individualized ways depending on the preferred orientation to mathematical instruction.

Aguirre and Bunch (2012) detail how language modalities of reading, writing, speaking, and listening also connect with one another. The authors note that speaking and writing share productive connections (ideas are produced) while reading and listening share receptive connections (ideas are received). Reading and writing are also both expressions of written language while speaking and listening are forms of oral language (Aguirre and Bunch, 2012). Thus, when multiple literacy connections appear in the same task, they can share connections related to student action (productive and receptive) or format (written and oral language).

Altogether these distinctions between unstructured and structured literacy opportunities, the varying types of structured opportunities, and the types of connections across language demands form the foundation of this report's framework. These are summarized in Figure 1.

Unstructured Literacy Opportunities (ULOs)

- Unstructured Reading Opportunities (UR)
- Unstructured Writing Opportunities (UW)
- Unstructured Listening Opportunities (UL)
- Unstructured Speaking Opportunities (US)

Structured Literacy Opportunities (SLOs)

- Structured Reading Opportunities (SR)
  - Academic Text (SR-A)
  - Simple Text (SR-S)
- Structured Writing Opportunities (SW)
  - Formative Writing (SW-F)
  - Summative Writing (SW-S)
- Structured Speaking Opportunities (SS)
  - Cooperative Focus (SS-C)
  - Individualized Focus (SS-I)
- Structured Listening Opportunities (SL)
  - Listening to Respond (SL-R)
  - Listening to Internalize (SL-I)

Literacy Connections

- Written Language Connections (WLC) – reading and writing practices are both present in the task.
- Oral Language Connections (OLC) – speaking and listening practices are both present in the task.
- Receptive Connections (RC) – reading and listening practices are both present in the task.
- Productive Connections (PC) – writing and speaking practices are both present in the task.

Figure 1: Opportunities to Engage in Literacy Practices

Modes of Inquiry

A curricular analysis has been conducted to test the exploratory framework of this report. The Illustrative Mathematics (IM) curriculum has been chosen for this analysis because it is well-suited for the focus of this investigation. The IM curricular overview explicitly cites Aguirre and Bunch's 2012 chapter and claims that "embedded within the curriculum are instructional supports and practices to help teachers address the specialized academic language demands in math when planning and delivering lessons, including the demands of reading, writing, speaking, listening, conversing, and representing in math" (Illustrative Mathematics, 2019b, para. 1). This suggests that the "intended curriculum" (Herbel-Eisenmann, 2007, p. 344) of IM should at the very least align with Aguirre and Bunch's model and could also reveal how these ideas play out in the transition towards the written curriculum.

Specifically, this investigation will analyze the first 6 lessons of *Unit 2: Introducing Proportional Relationship* from the 7th grade IM curriculum materials. These lessons are chosen because they emphasize multiple representations of proportional relationships, and representation is seen as a conduit which supports enactment of literacy demands and drives holistic connections across literacy opportunities (Aguirre & Bunch, 2012). Since this unit is students' first introduction to the terminology of "proportional relationship" there are also opportunities for addressing academic vocabulary in the selected lessons.

Taken together, these different aspects of the unit afford ample opportunity for a preliminary analysis of literacy opportunities. The IM lessons are broken into distinct activities, so a level of analysis smaller than the lessons themselves is appropriate: different literacy demands can be addressed in the same lesson without necessarily being connected to one another if they arise in discrete activities. Because IM activities generally break down into a "Launch" stage where a task is introduced and worked on and a "Synthesis" stage where student work or results are discussed, this study uses such task stages as the unit of analysis.

This approach created 40 task stages which were analyzed across the 6 lessons. A codebook was created which differentiated each type of opportunity and connection as summarized in the theoretical framework of this report. Reliability was established by enlisting a colleague to apply this codebook to the data set and checking for alignment with the author's results. Any initial disagreements in the data were discussed and resolved by the author and colleague. Each task stage was examined with the framework to determine whether any structured or unstructured literacy opportunities were described. When structured literacy opportunities (SLOs) were identified, they were also coded according to their appropriate sub-code (see Figure 1). The SLO results were then examined at the task level to determine if any tasks involved multiple SLOs. These tasks were coded as literacy connections as represented in Figure 1, because they provide students with opportunities to experience multiple related language modalities.

Results

The results of this analysis are presented in four main sections. The first two sections speak to the first research question regarding the ways that the Illustrative Mathematics (IM) curriculum materials address literacy opportunities of reading, writing, speaking, and listening. Results concerning structured literacy opportunities (SLOs) are examined in the first section, and these show that the curriculum materials employ multiple SLOs in every examined lesson but that productive opportunities for writing and speaking are more commonly found than receptive opportunities for reading and listening. The second section explores the unstructured literacy opportunities in these materials, and findings show that unstructured speaking and listening opportunities often appear together. The third section addresses the other research question regarding literacy connections and shows that the curriculum materials emphasize productive over receptive connections. The final section considers how the literacy practices were presented throughout the materials, noting how the way that such practices are phrased might promote an idea that SLOs should be reserved only for certain student populations.

Structured Literacy Opportunities

One overarching result of this analysis is that these curriculum materials employ structured literacy opportunities (SLOs) throughout their lessons, although with more focus on writing and speaking. Not only does every examined lesson include SLOs, but there is not a single lesson which does not employ at least 2 different SLOs. Additionally, all eight sub-codes for types of literacy opportunities were found, suggesting that these materials intentionally provide a range of opportunities to engage in literacy practices. As shown in Figure 2, 53 SLOs were coded across the tasks in these lessons. However, the different types of SLOs did not appear with equal frequency.

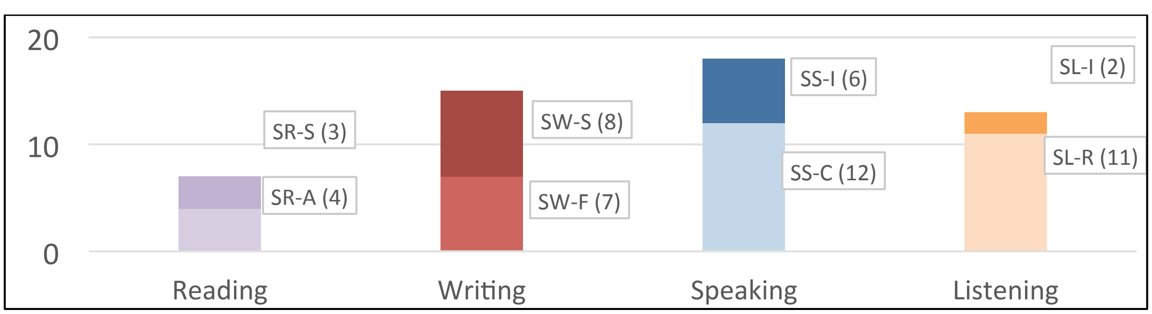


Figure 2: The frequency of each type of structured literacy opportunity with sub-codes

Structured opportunities for reading appear 7 times, writing 15 times, speaking 18 times, and listening 13 times. Despite every task being presented in the written language, reading comprehension is the least frequent structure seen in these tasks. Interestingly, listening SLOs (the other receptive modality besides reading) is the next least frequently referenced practice in these lessons. Given that most tasks in the IM curriculum materials consist of a “Launch” section with problems to complete (often with writing elements) and a “Synthesis” section grounded in discussion, it is not surprising to see high numbers of writing and speaking SLOs. What is surprising is that the receptive complements to these modalities are less frequently addressed. This becomes especially noted when looking at listening: Although there are 13 occurrences of such opportunities, 11 of these are *Listening to Respond* and 7 of those are paired with speaking SLOs. Figure 3 shows one such example from activity 5.2: The four sentence stems are structured speaking opportunities (cooperative focus) since they all orient students towards how to share time and space with their peers and collaborate respectfully (Thompson, 2008; Mercer, Wegerif, & Dawes, 1999), while “Why did you…?” and “I agree/disagree because…” are also structured listening opportunities (listening to respond) since they elevate complex listening skills (Hintz & Tyson, 2015) by directing students how to respond to specific arguments made by their partner. Given that these 7 paired routines blur the line between speaking and listening (since they emphasize discourse), this limits an explicit focus on listening. Such findings suggest that additional structures could be provided for reading and listening.

Support for Students with Disabilities

Engagement: Develop Effort and Persistence. Encourage and support opportunities for peer interactions. Prior to the whole-class discussion, invite students to share their work with a partner. Display sentence frames to support student conversation such as: “First, I _____ because ….,” “I noticed _____ so I ….,” “Why did you…?,” “I agree/disagree because….”
Supports accessibility for: Language; Social-emotional skills

Figure 3: A paired speaking and listening opportunity (Illustrative Mathematics, 2019a)

Unstructured Literacy Opportunities

This analysis also shows that unstructured literacy opportunities (ULOs) exist within in the IM curriculum materials. Such ULOs appear throughout the analyzed lessons, with unstructured speaking and listening opportunities often being paired together. As seen in Figure, there are 40 unstructured literacy opportunities coded throughout these lessons, including 4 opportunities for reading, 2 for writing, 19 for speaking, and 15 for listening . These 40 ULOs number fewer than the 53 structured practices but still represent significant numbers.

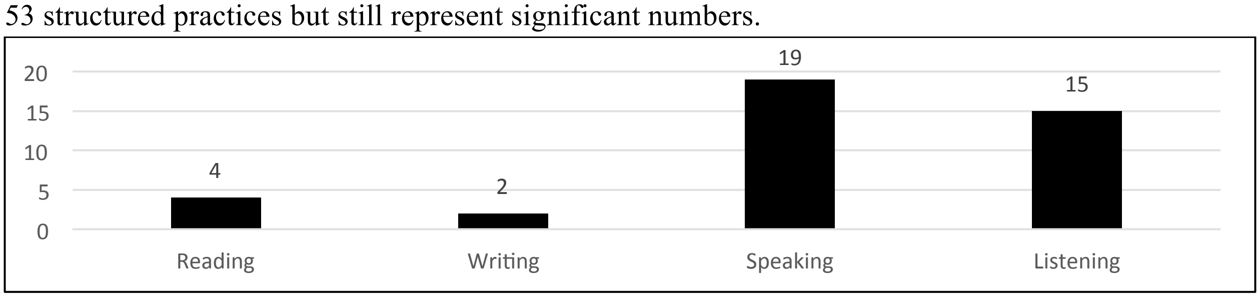


Figure 4: Frequency of unstructured literacy opportunities

24 of these instances arise from 12 pairs of listening and speaking ULOs. These unstructured pairs result from students being asked to compare or discuss with their peers (typically one partner) but without any oral language support structures (see Figure 5). Given that structured speaking and

listening opportunities *are* present throughout the IM curriculum, it is interesting that they are not consistently used when students are asked to engage in discourse. A more thorough analysis of the data could shed light on when structured speaking and listening opportunities are used versus when students are simply given unstructured speaking and listening opportunities.

Ask students to compare answers with their partner and discuss their reasoning until they reach an agreement.

Then, invite students to share how they used their equation from question 2 to answer question 3 with the whole class.

From task 4.4

Arrange students in groups of 2. Give students 6 minutes of partner work time followed by whole-class discussion.

From task 5.4

Figure 5: Unstructured speaking and listening opportunities (Illustrative Mathematics, 2019a)

Literacy Connections

The second research question for this report relates to the connections between literacy opportunities. Out of the 20 tasks explored, 14 of them had at least one connection (70% of all tasks) while 6 tasks showed two connections. Overall, 20 literacy connections were found. However, one finding of this analysis (shown in Figure 6) is that the types of literacy connections in the IM curriculum are not equally distributed. Productive connections and oral language connections both occur in 8 tasks, while written language connections only appear in 3 tasks and receptive connections do not appear at all. Additionally, 5 of the tasks include both productive and oral language connections, with each of these tasks having students write a response and then use that for discussion (see Figure 7 for such an example from activity 5.3). While this overall inconsistency is of note, the complete absence of receptive connections is especially a surprise. None of the analyzed tasks gave students structured opportunities to both read *and* listen. The limited number of written language connections (between reading and writing) further indicates that reading overall is an underutilized dimension of literacy in these materials.

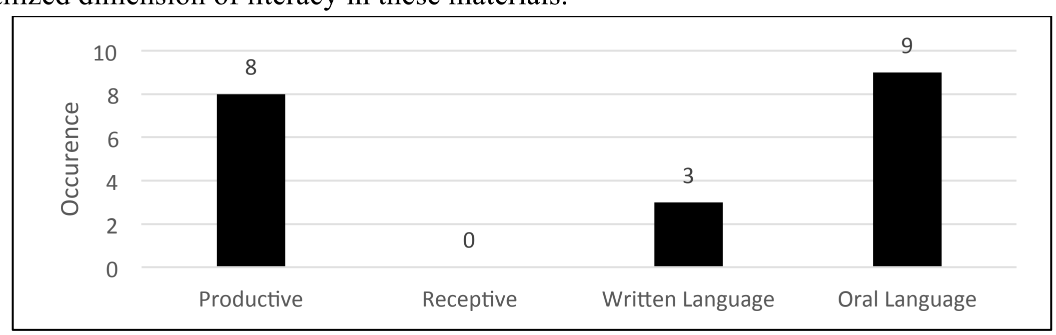
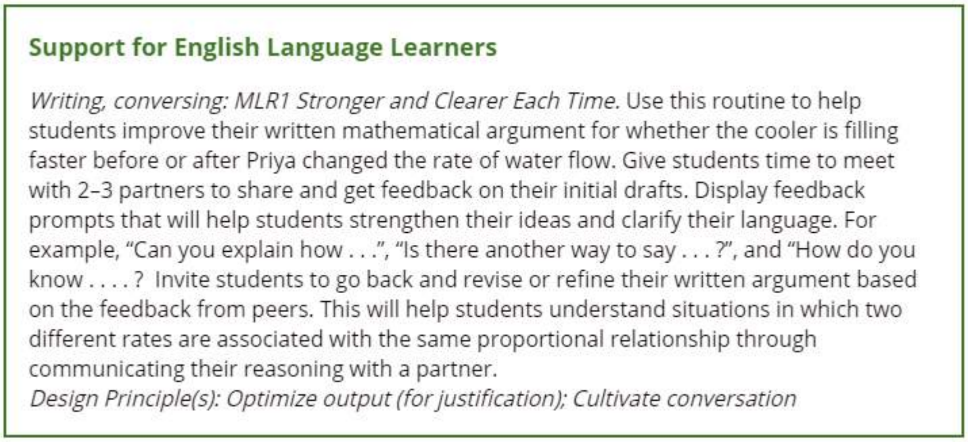


Figure 6: The number of each literacy connection that occurred in the selected IM lessons


Support for English Language Learners

Writing, conversing: MLR1 Stronger and Clearer Each Time. Use this routine to help students improve their written mathematical argument for whether the cooler is filling faster before or after Priya changed the rate of water flow. Give students time to meet with 2–3 partners to share and get feedback on their initial drafts. Display feedback prompts that will help students strengthen their ideas and clarify their language. For example, "Can you explain how . . .", "Is there another way to say . . . ?", and "How do you know . . . . ? Invite students to go back and revise or refine their written argument based on the feedback from peers. This will help students understand situations in which two different rates are associated with the same proportional relationship through communicating their reasoning with a partner.
Design Principle(s): Optimize output (for justification); Cultivate conversation

Figure 7: Productive and oral language connections within a task (Illustrative Mathematics, 2019a)

Framing of Literacy Practices in the IM Curriculum

Finally, an interesting trend that arose from the analysis is the way in which the IM curricular materials address literacy demands across different student populations. Illustrative Mathematics (2019) claims to integrate support for English Language Learners (ELLs) into their curriculum through what they call "Mathematical Language Routines" or MLRs (para. 22). As shown in Figure 7, this ELL priority is clearly established by bounding every MLR in a green border and titling it "Support for English Language Learners." Additionally, some structured literacy opportunities are included in bounded boxes titled "Support for Students with Disabilities," as shown in Figure 3. The MLRs ultimately account for 27 of the 53 structured literacy opportunities coded in this analysis, representing just over half of all such findings. The "Support for Students with Disabilities" directions account for another 4 opportunities. This means that such "bounded" curricular components account for a sizeable share of the total literacy opportunities found. All other opportunities, such as that shown in Figure 8, are embedded (unbounded) within the task launch or synthesis teacher guidance. This reveals two underlying conclusions: First, the MLRs that IL states are a core part of their curricular materials *do* appear in the analyzed tasks. Second, these routines are presented with a caveat – they are for ELLs and students with disabilities. While it is certainly true that language structures support ELL students (Aguirre and Bunch, 2012) and organizational structures aid low-performing students (Kenney, 2005, p. 45), the benefits of literacy opportunities are not limited to such groups. IM themselves admit as much when they state that "these instructional supports and practices (MLRs) can and should be used to support all students learning mathematics" (Illustrative Mathematics, 2019, para. 1). Despite this, the bounded nature of such supports and their frequent placement *after* all other teacher guidance within the curriculum materials could indicate exclusivity. Exploring how practitioners interpret structured literacy practices when they are built into the overall teacher-facing task instructions versus when they are separated is a consideration for future study.

Activity Synthesis	Select students to discuss and share their solutions. Ask them to identify difficulties which might include: getting started, noticing the pattern, dividing with decimals, completing the values in the table, creating the equation. This problem increases the level of difficulty by having so much missing information, and by using decimals in the table. It is important to identify if there are parts that are confusing for students to move them forward.

Figure 8: An embedded structured speaking opportunity from task 4.3 (Illustrative Mathematics, 2019a)

Discussion

This report provides numerous avenues for discussion for educators, curriculum designers, and researchers. First, this framework's descriptive language and distinction of different types of structured literacy opportunities can act as a roadmap for addressing such structures in mathematics curriculum materials. These definitions can also serve as a tool for strengthening otherwise unstructured literacy opportunities within curriculum materials or recognizing opportunities for connecting different modalities of literacy more consistently.

This framework can also better illuminate how curriculum materials are or are not considering multiple ways in which language relates to the learning of mathematics. Because each structured literacy opportunity has two distinct sub-codes, these allow for more nuanced discussion about the curriculum. For instance, the focus on speaking in the IM curriculum materials was usually cooperative rather than individual, and the focus on listening was largely to respond to peers' thinking rather than to internalize ideas for oneself. While this fits within the dialogic orientation to teaching, it limits opportunities aligned with the direct instruction model. Such an imbalance between the two sub-codes of speaking and two sub-codes of listening (but not an *absence* of any sub-code) gives rise to worthwhile questions about curriculum design and enactment: Are only the dialogically aligned opportunities desired? Should some curricular balance exist between cooperative versus individualized speaking, or listening to respond versus listening to internalize? This is an avenue for future research to consider.

Together, this framework and its application to IM curriculum materials provides insights into the role of language and literacy within mathematics. Applied only to a small sampling of materials, the framework illustrates the ways in which literacy opportunities are being structured and connected within tasks and demonstrates just how many literacy opportunities remain unstructured in these materials. Such results give clarity to the complex manifestations of literacy in the curriculum while also pointing towards further considerations which could advance our understanding of this critical aspect of learning mathematics.

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DESIGNING MATHEMATICS LEARNING ENVIRONMENTS FOR MULTILINGUAL STUDENTS: RESULTS OF A REDESIGN EFFORT IN INTRODUCTORY ALGEBRA

DISEÑANDO AMBIENTES DE APRENDIZAJE EN MATEMÁTICAS PARA ESTUDIANTES MULTILINGÜES: RESULTADOS DE UNA INVESTIGACIÓN DE DISEÑO EN ALGEBRA INTRODUCTORIA

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We describe results of a long-term design experiment focused on promoting mathematics learning among multilingual ninth graders classified as English Learners. The intervention at a linguistically diverse public high school in the US focused on a unit introducing concepts related to linear rates of change. We analyze results from a curriculum-aligned pre- and post-unit assessment used to document student learning across each design cycle. The main result is that students in both the Pre-Intervention and Redesigned classrooms made gains on the pre- and post-unit assessments. However, student gains on the assessments were higher in the Redesigned classrooms than in the Pre-Intervention classrooms. Additionally, on the assessment in the Redesigned classes, students classified as English Learners made larger gains than their non-EL peers, and the majority of the gains occurred on conceptually-focused items.

Keywords: Algebra and Algebraic Thinking, Design Experiments, Curriculum, Equity and Diversity

Objective

Research on the mathematics education of emerging multilingual learners[1] has shown that such students often experience procedurally-focused instruction (Callahan, 2005), are provided low cognitive demand tasks (de Araujo, 2017), and may have limited opportunities to engage in disciplinary and discourse practices (Zahner, 2015; Moschkovich & Zahner 2018). Recently, mathematics educators have engaged in design research to study how to transform the learning environment in linguistically diverse classrooms with the goal of promoting more robust forms of student learning (e.g., Chval et al., 2014; Prediger & Zindel, 2017). These design efforts have yielded promising results and frameworks for integrating mathematics and language learning. In this report, we describe results from a four-year design research effort that took place in a linguistically diverse ninth grade mathematics classes in an urban secondary school in the US. We analyze trends observed in the student responses to a curriculum-aligned assessment used to evaluate the efficacy of the design effort.

Our overarching research question is: To what extent did the design effort meet its goal of promoting student learning? In particular, we address the three specific questions

1. What was the effect of the redesign effort as measured by student response patterns on curriculum-aligned assessments?
2. How did the assessment results differ for students classified as ELs and those not classified as ELs (non-ELs)?

[1] In our research context, multilingual students who are learning the language of instruction are classified as "English Learners" (ELs). We use ELs when describing students as they are classified by the school. However, when referring to this group of students more generally, we will use "multilingual students" to highlight the assets of students learning the language of instruction.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

3. On what kinds of problems, conceptual or procedural, in the assessments of the Redesigned lessons did the students make the largest gains?

Framework

This design research (Cobb et al., 2003) project utilized the Academic Literacy in Mathematics (ALM) framework (Moschkovich, 2015, Moschkovich & Zahner, 2018), a sociocultural framework for analyzing and designing mathematics learning environments for emerging multilingual students. The ALM framework highlights that developing academic literacy entails developing mathematical proficiencies (in particular, procedural fluency and conceptual understanding), engaging in disciplinary practices, and participating in mathematical discourse. In our design efforts, we focused on developing students' conceptual understanding of the slope of a linear function as representing a rate of change (Lobato & Thanheiser, 2002). Based on prior research, we know that many students develop procedure-bound understandings of linear rates of change. For example, students can calculate a rate of change given a well-ordered table, but then they may struggle to interpret the rate in a given problem context (Lobato & Siebert, 2002). Therefore, one of our goals was to develop students' conceptual understanding of a linear rate of change as a multiplicative relationship between quantities.

Building on insights from prior design efforts in linguistically diverse mathematics classrooms (Chval et al., 2014; Prediger & Zindel, 2017) and the ALM framework, our redesign efforts were based on three guiding principles: (a) aligning a conceptual focus and problem contexts across the unit to minimize linguistic complexity, (b) integrating mathematical language goals linked to the conceptual focus, and (c) incorporating language supports in daily lesson activities to allow all members of a linguistically diverse classroom to engage in classroom discussions. In support of this design research effort we collected a wide array of data and we are conducting both quantitative and qualitative analyses of these data. In this report, we report on quantitative evidence of student learning measured in pre-unit and post-unit assessments. These assessments were designed by the participating teachers and researchers to align with the content of the unit, and to target both conceptual and procedural knowledge. In this sense, the assessment provides data related to the first component of the ALM framework (Moschkovich, 2015).

Data and Methods

Design Cycles

The classroom-based design research entailed three main phases and two design cycles, shown in Figure 1. In Phase I, Pre-Intervention, the researchers observed ninth grade mathematics classes at City High across a unit on linear rates of change. Students in the observed classes completed a pre-unit assessment at the start of the unit and a post-unit assessment at the conclusion of the unit. Additionally, during Phase I, the researchers conducted clinical interviews with selected students from the observed classes. Due to space restrictions, the interviews are not discussed further here.

At the conclusion of Phase I, the teachers and researchers analyzed the Phase I data with the goal of redesigning the unit on linear rates of change. The Redesigned lessons were then pilot tested in Phase II during Teaching Experiment (TE; Prediger, Gravemeijer, & Confrey, 2015). The teaching experiment lessons were taught in an after school setting, which allowed the researchers and teachers to make adjustments in the lesson designs from week to week. The TE lessons were designed using the project design principles outlined above. The teaching of the lessons in the TE was the conclusion of the first iteration of the design cycle and the start of the second iteration.

Finally, in Phase III, the team of teachers and researchers analyzed the data from the Phase II TE lessons and redesigned the unit lessons from Phase II for use during the regular school day. The participating teachers implemented the Redesigned lessons during their typical school day in Phase

III. During this second iteration of the design cycle, the classroom lessons were observed and the students in the Phase III classes completed a pre-unit and post-unit assessment (paralleling the Pre-Intervention observations from Phase I). In the results reported here, we focus on the assessments that were administered at the start and conclusion of the classroom observations in the Pre-Intervention (Phase I) and Redesigned (Phase III) units.

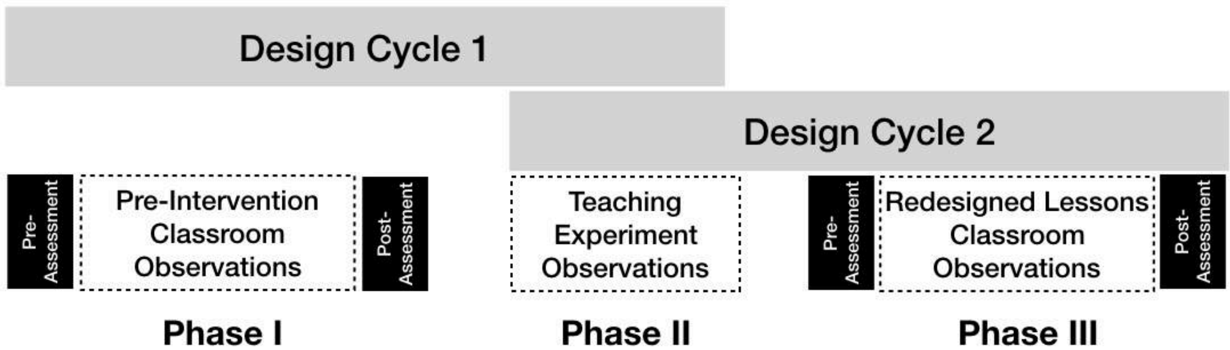

Figure 1: Overall project design and phases of design cycles.

Setting and Participants

Data collection took place in ninth grade mathematics classes at City High, a large, urban high school enrolling a linguistically diverse group of students. City High is located near the US-Mexico border, and it serves a relatively large number of recent immigrant and transborder students. Throughout the study, approximately 30% of City High students were classified as ELs, and an additional 50% of the students were formerly classified as ELs. Over 75% of the students were identified as Latinx in school demographic data, and most students classified as ELs spoke Spanish as their primary language. Additionally, the majority of City High students were from households with low socioeconomic status.

Six teachers participated in the design effort across the four-year project. In this analysis, we focus on the assessment results for the students of Mr. S because he was the only teacher who participated in all phases of the design effort. Mr. S had six years of experience at the start of the study, holds a teaching credential in mathematics, is bilingual in Spanish and English, and has taken some courses on teaching English learners in the content areas as part of his mathematics teaching certification program. He participated in the study to further develop his expertise teaching mathematics to multilingual students.

We included all students in Mr. S's focal classes who took the pre-unit and post-unit assessments during the Pre-Intervention (n=18) and Redesigned (n=28) phases of the research. We note that the assessments were designed to align with the respective units. Therefore, we were not able to use the same items on the assessments for the Pre-Intervention and Redesigned lessons. To account for these differences in the assessment design, we converted the total scores on each assessment to the proportion of correct answers.

Assessment

The pre- and post-assessments were designed by the researchers in collaboration with the teachers. In both the Pre-Intervention and Redesigned units, the assessments included a mixture of procedurally-focused items that did not require explanation, as well as conceptually-focused items characterized by prompts for explanation. Aligning with the design guidelines for the ALM framework (Moschkovich & Zahner, 2018), the focus of the intervention was developing students' conceptual understanding of rate of change. The conceptual items reflected this focus and assessment goal. Figure 2 shows an example of a procedural item (top) and conceptual item (bottom) from the Phase III assessment. In the Results section, we compare patterns of student responses to the conceptually-focused and procedurally-focused items from Phase III.

1) Deondra is saving to buy a new bike. She starts off with $10 in her account and saves $20 a month babysitting. She writes an equation that represents the total amount of money (y) she has saved in any given month (x):

$$y = 20x + 10$$

How much money did Deondra save after 6 months?

- ❑ $20
- ❑ $100
- ❑ $120
- ❑ $130
- ❑ $150

6) Connie ran 45 meters in 10 seconds. Jen ran the same speed as Connie, but Jen ran a different amount of time and distance.

a) Which of the following are possible for the time and distance that Jen ran? **Select all that apply.**

- ❑ 90 meters in 5 seconds
- ❑ 54 meters in 12 seconds
- ❑ 22.5 meters in 5 seconds
- ❑ 90 meters in 20 seconds
- ❑ 50 meters in 15 seconds
- ❑ 4.5 meters in 1 second

b) Explain: How can you check if Jen's distance and time makes the same speed as Connie?

Figure 2: A procedural item (top) and a conceptual item (bottom) from the Redesigned assessments.

The assessments were scored using a rubric by the research team. In general, students were awarded points for correct responses as well as valid explanations (on the conceptual items). For the analysis of assessment gains, we used comparisons of means and paired effect sizes (Cohen, 2013) to identify the direction and magnitude of change from the pre-assessment to the post-assessment in each phase. For the comparison of student performance on the conceptually-focused versus procedurally-focused items, we also compared means and measured paired effect sizes. We used paired effect sizes because we had both pre-assessment and post-assessment data by student. The details of these analyses are in the Results.

Results

We note a caveat at the outset of this section: we had relatively small sample sizes. These small sample sizes are a result of engaging in intensive work with a small group of teachers across multiple years. Our sample sizes were reduced by teacher turnover across study years and student absences/mobility within each design cycle. These realities, while not ideal for statistical research, are byproducts of working in our local school setting. In light of the small samples, we used Hedges' correction whenever presenting claims about growth as measured by effect sizes.

Research Question 1: What was the effect of the redesign effort as measured by student responses on the pre- and post-assessments?

To answer RQ1, we compared results of the pre- and post-assessment gains in Pre-Intervention and the Redesigned units. Table 1 shows the pre- and post-assessment mean scores along with the paired effect size to measure the magnitude of the gain.

Table 1: Pre- and post-assessment means for the Pre-Intervention and Redesigned units.

	Pre-assessment	Post-assessment	Gain	Paired effect size (Magnitude)[1]
Pre-Intervention (n=18)	0.42 (0.17)	0.57 (0.18)	+0.15	0.80 (L)
Redesign (n=28)	0.24 (0.16)	0.44 (0.19)	+0.20	1.12 (L)

Note: [1]S: |d|<0.20, M: 0.20 < |d| < 0.50, L: |d| > 0.50 (Cohen, 2013)

In summary, there was significant growth in both the Pre-Intervention and Redesigned units as measured by the pre- and post-assessments. The effect size comparing the magnitude of the pre-post gain was larger in the Redesigned unit. While the direction of the difference in effect sizes was favorable for our design effort, given the small sample size, we were not able to use a statistical test such as an ANOVA to compare the difference in effect sizes.

Research Question 2: How did the assessment results differ for students classified as ELs and those not classified as ELs (non-ELs)?

Table 2 shows the pre- and post-assessment mean scores broken out by Pre-Intervention and Redesigned Lessons and student EL classification. As with Table 1, we included a paired effect size to illustrate the magnitude of the differences (Cohen, 2013). The data in Table 2 indicate that while there was significant growth from the pre- to post-assessment scores in the Pre-Intervention unit, the effect size was larger for non-ELs (0.81) than for Els (0.58). That is, during the Pre-Intervention unit, the students who were not classified as ELs benefitted more from the business as usual teaching. During the Redesigned unit, however, the pattern in effect size was reversed. That is the effect size was larger for ELs (1.44) than for non-ELs (0.94). In terms of the mean gain scores, ELs in both the Pre-Intervention and the Redesigned lessons had larger gains than their non-EL classified peers. One interpretation of this result is that the intervention led to larger gains for ELs than non-ELs. However, this result is moderated by the fact that ELs in the Redesigned unit pre-assessment started relatively low compared to the non-ELs.

Table 2: Pre- and post-assessment mean scores by EL classification for the Pre-Intervention and Redesigned units.

		Pre-assessment	Post-assessment	Gain	Paired effect size (Magnitude)
Pre-Intervention					
	Non-EL (n=14)	0.45 (0.16)	0.59 (0.18)	+0.14	0.81 (L)
	EL (n=4)	0.33 (0.21)	0.49 (0.21)	+0.16	0.58 (L)
Redesign					
	Non-EL (n=20)	0.27 (0.17)	0.45 (0.20)	+0.18	0.94 (L)
	EL (n=8)	0.17 (0.11)	0.43 (0.19)	+0.26	1.44 (L)

RQ3: On what kinds of problems, conceptual or procedural, in the assessments of the Redesigned unit did the students make the largest gains?

Our final research question examines patterns in student responses to the pre- and post-assessments from the Redesigned unit. The assessment for the Redesigned unit included five conceptually-focused and three procedurally-focused items. Conceptually-focused items included a prompt for explanation while the procedurally-focused items were multiple-choice questions with no prompt for further explanation. To conduct the analysis of student results based on item type, we calculated a subscore for each type of item (e.g., a subscore for the three procedural items, and a subscore for the five conceptual items). Table 3 summarizes the results by type of question (Conceptual versus Procedural).

Table 3: Pre- and post-assessment mean scores by type of item for the Redesigned unit.

Type of Item	Pre mean (SD)	Post mean (SD)	Growth	Paired Effect Size (Magnitude)
Procedural	0.47 (0.22)	0.44 (0.26)	-0.03	0.12 (S)
Conceptual	0.20 (0.16)	0.44 (0.21)	0.25	1.27 (L)

Table 3 shows there was little change in the mean for the procedural items from the pre-assessment to the post-assessment. However, there were significant gains in the conceptual items. This result aligns with the focus of the redesign efforts on fostering student development of conceptual understanding related to linear rates of change.

Discussion & Conclusion

In this report we have described quantitative findings based on pre- and post- unit assessments that were used as part of a design research effort. The goal of the design research effort was to create a classroom learning environment in a linguistically diverse secondary mathematics classroom where students learn to reason about linear rates of change. The main findings presented in this report are:

1. Students in the Redesigned unit showed larger gains as measured by paired effect size with Hedges correction,

2. ELs in the Redesigned unit had larger gains when compared to their non-EL counterparts. The gains by Els in the Redesigned unit were also relatively larger than their gains in the Pre-Intervention lessons, and
3. In the assessment of the Redesigned unit, the students had more growth on the conceptually-aligned items than on the procedural items.

While we have shown that the paired effect size was larger in re-designed unit, we are not able, at this point, to make a causal claim about learning linked to our design efforts. However, we can hypothesize that some of the growth observed in our analyses can be traced back to the design principles and the instructional activities that were developed across the two design cycles. For example, in the Pre-Intervention lessons, we observed that most students had limited opportunities to solve conceptually demanding tasks during their regular mathematics classes. Therefore, in the Redesigned unit, we intentionally included conceptually demanding tasks, along with targeted linguistic supports for students classified as ELs, to allow all students in the Redesigned classroom an opportunity to solve more conceptually-focused problems. For example, during one of the Redesigned lessons students were challenged to make up a story related to a graph showing distance and time. Then they were tasked with solving non-routine problems about rates of change in other graphs. These instructional tasks, which consistently included prompts for students to explain their reasoning and justify responses, were similar to the conceptually-focused item in Figure 2.

One question that remains for our design efforts is why the students in the Redesigned unit did not show gains on the procedural items. When we ran the analysis for RQ3, we expected to see growth on both the conceptual and the procedural items. However, looking at the results in Table 3, there was little change on the procedural tasks from the pre-assessment to the post-assessment. One possible explanation for this lack of gain is that the learning opportunities within the Redesigned unit did not include many tasks focused on developing students' procedural fluency. Reflecting on the data in Table 3 prompts us to consider whether the learning tasks in the Redesigned lessons overemphasized a conceptual focus and did not include enough opportunities for students to develop procedural fluency.

Despite the limitations induced by the small sample size of this study, the results presented here highlight potential avenues for further development and research. This is of particular interest since there are relatively few studies, like the one presented here, that have engaged in design research focused on meeting the needs of linguistically diverse students. One potential follow-up study might be to use the materials developed in this project in a study with a larger number of teachers and students. Such a study, particularly one using a more controlled design, could test of the robustness of the results presented here.

Another avenue for follow-up on this research, which we are undertaking presently, is qualitative analyses of the design effort. In these analyses we are examining the forms of student reasoning that were developed during the Redesigned lessons and tracing those forms of reasoning back to the classroom learning environment. For example, we are examining the quality of student explanations on the conceptually-focused items, and then connecting the student reasoning observed on the assessments to the classroom observations (paralleling the analysis of Zahner, 2015). This analysis will allow us to understand *how* a linguistically diverse group of ninth grade students developed particular forms of reasoning about linear rates of change.

Acknowledgments

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CURRICULUM, ASSESSMENT & RELATED TOPICS:

BRIEF RESEARCH REPORTS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

CHARACTERIZING COHERENCE WITHIN ENACTED MATHEMATICS LESSONS

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Curricular coherence has been emphasized by leaders in mathematics education as it enhances deeper understanding by enabling students to see connections between mathematical ideas. Although there are different forms of curricular coherence, the coherence of lesson has received considerably less attention. Little is known about what constitutes coherent lessons or how to measure the degree of coherence. Using the data from a larger study in which lessons are intentionally designed for coherence, we propose a tool for examining lesson coherence and describe characteristics of the lessons with different levels of coherence.

Keywords: Curriculum, Curriculum Enactment, Instructional Activities and Practices

What explains poor U.S.A. performance in mathematics according to international comparison studies? According to Stigler and Hiebert (1999), who compared enacted lessons from seven countries including the U.S.A., one of the factors that distinguishes mathematics lessons in the U.S.A. from high-performing countries is their degree of coherence. They explain:

> Imagine the lesson as a story. Well-formed stories consist of a sequence of events that fit together to reach the final conclusion. Ill-formed stories are scattered sets of events that don't seem to connect. As readers know, well-formed stories are easier to comprehend than ill-formed stories. And well-formed stories are like coherent lessons. They offer the students greater opportunities to make sense of what is going on. (p. 61)

Much of the prior research has focused more on *curricular coherence*, which refers to how mathematics topics are connected across grade levels (Schmidt, Wang, & McKnight, 2005). Coherence *within* a lesson (what we refer to as *lesson coherence*) has received considerably less attention. Our goal is to learn what constitutes a coherent lesson and to what extent one lesson's coherence differs from that of another. Using lesson data from a larger study in which lessons were intentionally designed for coherence, the present study aims to answer the following questions: When enacted lessons are analyzed for how mathematical ideas build within each lesson and how its parts are interconnected, to what extent are the lessons distinguishable? What are the characteristics of lessons in each type of lesson coherence?

Theoretical Framework

As Stigler and Hiebert (1999) suggest, mathematics curricula (i.e., lessons, units, entire courses, and so on) can be thought of as *mathematical stories* (Dietiker, 2015). Mathematical stories foreground how the mathematical content unfolds across a lesson, connecting a beginning with an ending. The sequential parts of a mathematical story form its *acts* during which students' understanding of *mathematical characters* (e.g., numbers or geometric objects), *mathematical actions* (e.g., procedures), and/or *mathematical settings* (e.g., representations) changes.

A story's coherence is the extent to which parts of stories fit with one another and come together as a whole (Richman, Dietiker, & Riling, 2019). Incoherent mathematical stories make it harder for students to see connections between lesson parts (i.e., acts) and prevent students from fully comprehending a mathematics lesson. However, it does not necessarily follow that more coherent lessons are always better; students may feel boredom during a predictable lesson.

Methods

In order to learn about types of lesson coherence, we analyzed recordings of secondary mathematics lessons expected to represent a range of coherence. The lessons were taught by six teachers from three high schools in Northeastern USA. About half of the lessons were teachers' typical lessons and the rest were designed as mathematical stories, a process that we predicted would increase coherence. Data includes video- and audio-recordings of full lessons. At the end of each lesson, consenting students completed a survey describing their experience. In order to achieve maximal variation in coherence, we identified each teacher's lessons with the most positive and negative student aesthetic reactions. We also included a lesson for which the teacher had participated in analyzing a previous enactment as a story, which we thought might result in a unique form of coherence.

Members of the research team coded independently and met for consensus throughout the coding process. The team first identified *acts* by noting changes in mathematical characters, actions, or settings (e.g., when students shift to a new task). Within each act, the team identified questions that arose. For each question, researchers marked changes in what was revealed publicly, such as when a teacher asks clarifying a question when or students shift to a new task.

In order to examine the connectivity of each lesson, we created a *coherence map* using graph theory. Nodes represent acts and edges reflect that two acts contain progress on the same question(s). Lessons were then grouped based on the connectedness of their graphs. For each level, at least two research team members examined the transcripts of the lessons to demonstrate the characteristics of each level.

Findings

We identified three levels of coherence within our data set: incoherence, partial coherence, and strong coherence. Here, we present coherence maps of lessons selected to represent each coherence level and articulate features and characteristics of each level.

Level 1: Incoherence

Two lessons analyzed contain discontinuities between topics and a lack of overarching themes across tasks. One of the lessons (see Figure 1) begins with a warm-up in which students describe properties of an operation (i.e., a&b = 3a – b). Yet, these features are not relevant to the next task, about the range of a function of x. After that, the lesson has another disconnect when the focus in Act 7 shifts to an unrelated topic (percent) without explanation. Although all tasks in Acts 7-12 are about percentages, they jump from calculating percentages of numbers to calculating prices as percent discounts, and so on. No work that students do to complete one task supports them to complete the rest. It is unlikely students will connect these tasks beyond recognizing that each is about percentages. Because these tasks are so independent, there exists no obvious sequence that would support students in building an understanding of percentages.

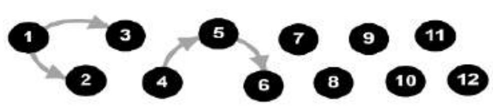

Figure 1: A Visual Model of an Incoherent Lesson

Level 2: Partial Coherence

Two of the analyzed lessons show some extent of coherence within substantial portions of the lessons. The coherence maps of such lessons contain sections with some internal connections, but no connections across sections. One of these lessons (see Figure 2) beings with students observing four graphs of systems of linear inequalities (Acts 1-6). During this activity, students work on questions like, *why are the different parts shaded?* After that, however, this question is not pursued by the teacher or students as the students work on a worksheet with other types of inequalities (e.g., one variable inequality) in Acts 8-14. In Act 15, the teacher briefly review the answers to the worksheet problems and ends the lesson. The lessons in this level include more connections between acts than the incoherent lessons do. However, these connections rely heavily on a couple of acts (e.g., Acts 6 and 15), which build some extent of coherence but not a strong amount.

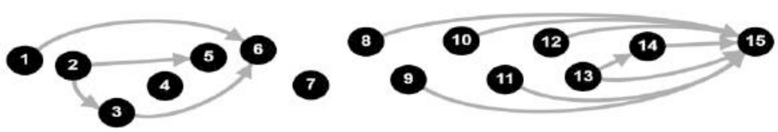

Figure 2: A Visual Model of a Partially Coherent Lesson

Level 3: Strong Coherence

Some lessons in our data set were strongly coherent. That is, a student would likely understand why they were engaged in a given activity and know how parts of the lesson connected to one another. Three sub-types of strong coherence—retroactive coherence, coherence with brief diversions, and strong coherence—are described below.

Retroactive coherence. In some highly coherent lessons, there are portions of the lesson that appear disconnected, but are later shown to be connected. These lessons are similar to partially coherent lessons in which teachers review answers at the end of the lesson, but retroactive coherence is richer because students have opportunities to make conceptual connections across ideas from the lesson. Consider a lesson with this type of coherence about repeated roots of polynomials (see Figure 3).

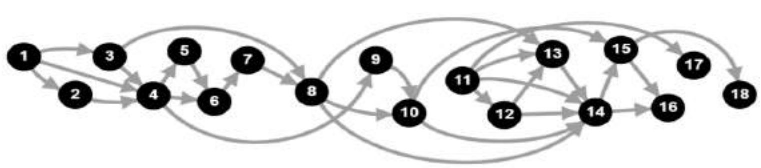

Figure 3: A Visual Model of a Retroactively Coherent Lesson

In Acts 1 through 10, students are shown a graph of a polynomial with a repeated root and work collaboratively to find its equation. In Act 11, the teacher explains that they will no longer be making progress on questions regarding the graph and distributes a new worksheet with equations to graph. In the final acts, the teacher enables students to see connections between the two seemingly distinct portions of the lesson by turning their attention to broad concepts that apply to both. Several early questions that did not refer to specific equations or graphs, but do broadly apply to them, become relevant again in Act 14. Retroactive coherence is possible because the teacher does not disclose many questions from the first part of the lesson before beginning the second part, so students may still wonder about them later on.

Coherence with brief diversions. Sometimes, most acts were connected to each other, with a few brief diversions consisting of acts that were somewhat, or not at all, connected. We found four such lessons. Two have brief diversions that appear as tails at the beginning of a lesson, typically due to teachers reviewing prior work. The others have tails later in a lesson, when teachers introduced alternate solution strategies that are not addressed as the lesson continues. An example of this second type is a lesson about exponential equations (Figure 4). In Act 15, the teacher introduces a new, efficient way of solving for x in the equation $1.04^x = 2$. The teacher asks students to think of way to find x that would be more efficient than the predominant solution strategy (i.e., guessing and checking) used in the lesson. He then solves the problem using the new method (i.e., taking the logarithm of both sides and using the power property to solve for x).

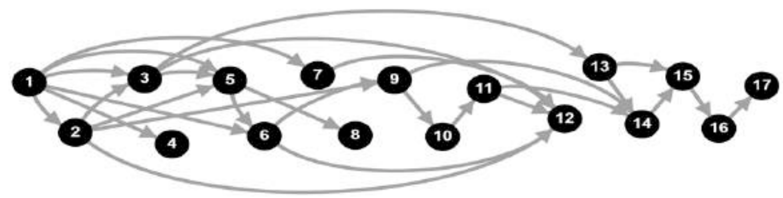

Figure 4: A Visual Model of Coherence Lesson with Brief Diversions

Strong coherence. Several lessons designed as mathematical stories displayed an incredibly high level of connection across acts with no diversions or temporary coherence gaps. The coherence map of one lesson with strong coherence is represented in Figure 5. In this lesson, students built understanding of the Rational Root Theorem by investigating potential roots. The lesson's high degree of coherence is evident in the multiple complete subgraphs (e.g., Acts 5-7). The strong coherence is due to both a set of questions from Act 1 and new questions introduced in subsequent acts that remain open for most of the lesson. The teacher permits the students to gradually explore and refine their ideas as they consider new challenges, prompting them to explore their initial questions during each subsequent task. The progressively complex nature of each new task (e.g., checking provided roots versus selecting their own potential roots) likely makes it so that students do not grow bored of their investigation.

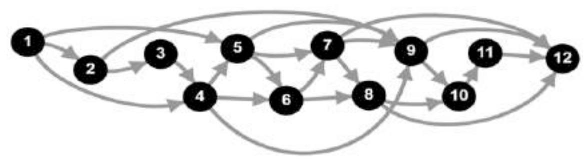

Figure 5: A Visual Model of Strong Coherence

Discussion

We do not claim that these three types of coherence are discrete. There might be additional intermediate levels, or even a continuous coherence spectrum. The presented levels are only samples of this possible spectrum; our goal is to present a way to describe the coherence of a mathematics lesson. Lesson coherence is not only a quality indicator of a mathematics curriculum but also a useful dimension for making a lesson more captivating in terms of student engagement. Increasing coherence requires purposeful design and management of mathematical inquiry. Through coherence mapping, lesson coherence that is often implicit can be visualized so that teachers will be able to see how they can make stronger connections between parts of a lesson.

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AN ANALYSIS OF COORDINATE SYSTEMS PRESENTED IN GRADE 6-8 TEXTBOOKS

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We analyzed a total of 1129 tasks in grade 6–8 textbooks to examine the type of coordinate system presented and the associated graphing activity required in each task. We share our findings and discuss educational implications of such findings.

Keywords: Representations and Visualization, Coordinate System, Graph, Textbook Analysis

Middle school is a critical time for students to develop robust understandings of coordinate systems and graphs. In the Common Core State Standards for Mathematics (CCSSM; NGA Center & CCSSO, 2010), for example, students are first introduced to the Cartesian plane and learn to plot or interpret points in the first quadrant in 5th grade; thereafter, students are expected to use the Cartesian plane for exploring and representing other mathematical ideas including geometrical shapes, proportional relationships, number systems (6-7th grade), and graphs of linear relationships (8th grade). In this report, we share results from an analysis on the types of coordinate systems presented and associated graphing activity required in grade 6–8 textbooks and discuss educational implications of such findings.

Conceptual Framework

By coordinate system, we mean a representational space in which an individual systematically coordinates quantities (Thompson, 2011) to organize some phenomenon. We have previously distinguished between two types of coordinate systems depending on the goal they serve: spatial and quantitative coordinate systems (Lee, Hardison, & Paoletti, 2018; 2020). A spatial coordinate system is used to quantitatively organize a space in which a phenomenon is situated (Figure 1). Constructing a spatial coordinate system involves an individual organizing a space by (mentally) overlaying a coordinate system onto some physical or imagined space being represented where objects within that space are tagged with coordinates. For example, in Figure 1b a coordinate system is overlaid onto a region of a city from a bird's eye view where the *x*- and *y*-axes coincide with roads in the city.

On the other hand, a quantitative coordinate system is used to coordinate sets of quantities by constructing a geometrical representation of the product of measure spaces (Figure 2). Constructing a quantitative coordinate system involves an individual extracting quantities from the space in which a phenomenon occurs and projecting them onto a new space, different from the space in which the quantities were originally conceived. For example, in Figure 2b the coordinate system is showing the relationship between time (in minutes) and the number of boxes a machine packages over time where both quantities were taken from a space separate from the presented coordinate system.

Relatedly, graphs represented in each of these coordinate systems are fundamentally different (Lee, Hardison, & Paoletti, 2018; 2020). Graphs created on spatial coordinate systems can be viewed as projections or traces of physical objects or phenomena onto an analogous space containing the original objects or phenomena. Whereas, in a quantitative coordinate system, graphs are not projections of physical objects or phenomena from the same space containing the original objects or phenomena. Due to this distinction, different ways of reasoning could be productive when creating and interpreting spatial and quantitative coordinate systems and their associated graphs (c.f., Lee, Hardison, Paoletti, 2020).

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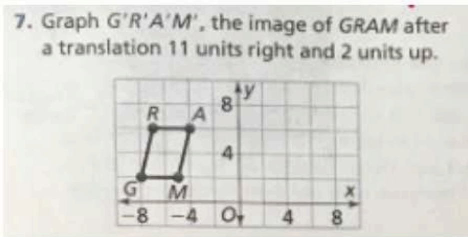
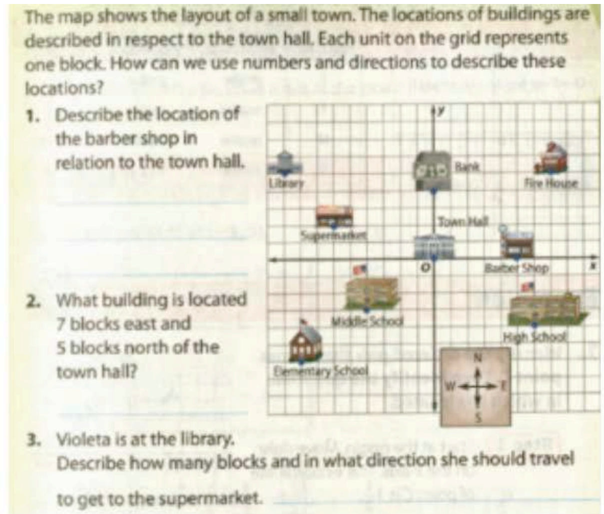
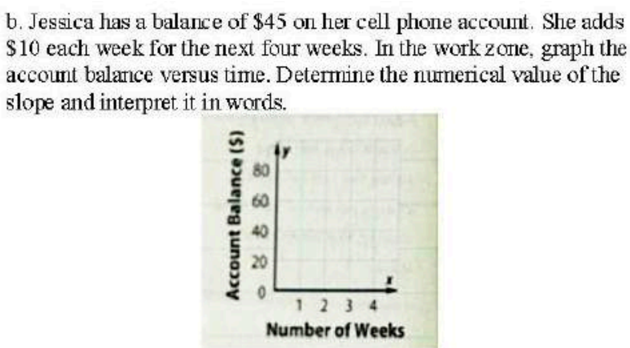


(a) Task from Berry III et al. (2017f), p. 301.

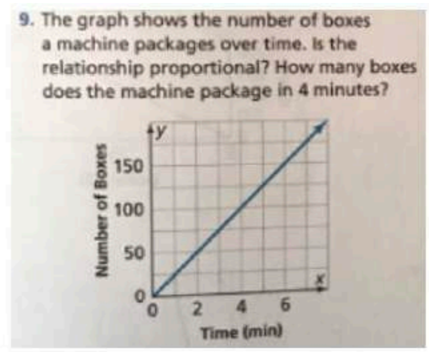


(b) Part of a task from McGraw-Hill Education. (2015a), p. 85.

Figure 1. Examples of tasks using a spatial coordinate system.



(a) Task from McGraw-Hill Education. (2015d), p. 464.



(b) Task from Berry III et al. (2017c), p. 119.

Figure 2. Examples of tasks using a quantitative coordinate system.

Using this framework, our goal in this study was to investigate the different types of coordinate systems presented in grade 6–8 textbooks. Specifically, we examined (a) what type of coordinate systems textbooks present, (b) what type of graphing activities problem solvers are prompted to engage in, and (c) how frequently these coordinate systems and graphing activities appear in textbooks. We emphasize that the textbook analysis involved *our* interpretations of the textbook author's intended use of coordinate systems, which do not necessarily coincide with how students might perceive of the coordinate system.

Methods

We extracted and coded a total of 1129 tasks from three major textbook series (*Mathematics in Context*, *enVision Math 2.0 Common Core,* and *Texas Math TEKS*; see references) for grades 6–8 to date. The three series were selected to represent a variety of curricula. The first step in our analysis involved selecting and extracting tasks. The criterion for inclusion was that the task presented a pre-constructed two-dimensional coordinate system, either left blank or containing a graph. Relatedly, we also included tasks referring to previous tasks containing a pre-constructed coordinate system. We excluded tasks that had coordinate system-like grids but did not explicitly involve problem solvers to attend to the coordination of quantities (e.g., a grid superimposed onto a shape to find its area, box-and-whisker plots, and bar graphs with categorical data). Our unit of analysis, what we call a task, is a sequence of explanations or questions surrounding a single context or a single coordinate system. This means that explanations/questions about a single context with several coordinate systems (e.g., comparing several graphs) and explanations/questions with multiple contexts around a single coordinate system (e.g., graph several things on the same coordinate system) all counted as a single task.

After extracting tasks, we coded each task along two dimensions. The first dimension is the type of coordinate system. A task received the code *spatial* if the coordinate system is spatial; *quantitative* if the coordinate system is quantitative; *both* if the task involves both types of coordinate systems in a single task (e.g., comparison tasks with both types); and *neither* if it was difficult to discern as spatial or quantitative due to lack of context (e.g., *x-y* graph without specification of what *x* and *y* represent). When necessary, we referred back to previous or subsequent tasks in the textbook to determine the context of the task.

The second dimension was the type of graphing activity required in the task. A task received a *create* code if it requires problem solvers to create a graph by plotting a point or collection of points (e.g., tasks in Figures 1a and 2a). A task received an *interpret* code if it requires problem solvers to make sense of a pre-constructed graph, such as describing the relationship between two variables or constructing an algebraic equation that describes the graph (e.g., tasks in Figures 1b and 2b). Our distinction between create and interpret is similar to Leinhardt et al.'s (1991) distinction between construction and interpretation; however, different from Leinhardt et al., we consider building algebraic functions for a graph as interpretation and not construction. A task received a *both* code if it required the problem solver to both create and interpret a graph; a *neither* code if there were no requirement for the problem solver to create or interpret a graph.

We reiterate that codes were attributed to each task based on our interpretations of the authors' intention of the task. Once all tasks were coded, we compared our codes and when there was a disagreement, we discussed them to come to a consensus. Finally, we recorded the mathematical topic covered in each task using the topic names used in the textbook.

Findings and Discussion

The results are summarized in Table 1. Because our purpose was not to compare textbooks, we report only on the total frequencies for each code across all textbooks within each grade. As shown in Table 1, the coordinate systems were predominantly *quantitative* and most tasks required problems solvers to *interpret* a graph. There were consistently and predominantly more tasks that required students to interpret a graph rather than create a graph across all three grades. However, the trend in coordinate system type changed across grade levels. In grade 6, both types appeared close in frequency. Many grade 6 textbooks introduced the Cartesian plane, asked students to plot points or enact operations on coordinates (i.e., coordinate geometry) and then used the Cartesian plane to represent graphs of functions. As such, coordinate systems were often first introduced spatially and students were expected to transition from a spatial to quantitative coordinate system unproblematically. In grade 7, there was a stark difference in the number of quantitative coordinate systems (n=128) in comparison to those that were spatial (n=8). Relatedly, in grade 7 textbooks we coded, statistical graphs (e.g., bar graphs, histograms) and linear relationships were the main focus of content. Finally, in grade 8, there was a more balanced use of quantitative and spatial coordinate systems, with more quantitative (58%) than spatial (42%). In grade 8, textbooks covered linear relationships (functions, systems of equations) and statistical graphs in which quantitative coordinate systems prevailed; however, they also covered topics such as transformations of shapes and the distance between points, in which spatial coordinate systems were used.

Table 1. Number of Tasks by Problem Type, Coordinate System Type, and grade level.

Grade	Problem Type	Coordinate System Type				Total
		Quantitative	Spatial	Both	Neither	
6	Create	10	0	0	7	17
	Interpret	42	51	0	15	108
	Both	18	12	0	3	33
	Neither	0	0	0	1	1
	Grade Total	70	63	0	26	159
7	Create	9	0	0	15	24
	Interpret	83	3	0	19	105
	Both	35	5	0	4	44
	Neither	1	0	0	5	6
	Grade Total	128	8	0	43	179
8	Create	27	22	0	34	83
	Interpret	225	151	0	124	501
	Both	84	71	0	44	199
	Neither	0	0	0	8	8
	Grade Total	336	244	0	210	791
Grand Total		534	315	0	279	1129

Note there were no tasks that involved both types of coordinate systems but a total of 279 tasks coded neither for coordinate system type. Previously, Paoletti et al. (2016) analyzed graphs in STEM textbooks and practitioner journals at the undergraduate level and found that the majority of graphs either mathematized a spatial situation or represented two (contextual) quantities. On the other hand, they found that most graphs in commonly used precalculus and calculus mathematics textbooks represented two decontextualized quantities, finding a discrepancy between graphs students experience in their math classes and those used in other STEM fields. Looking across our *interpret* graph tasks, 22.13% of those tasks used decontextualized coordinate systems, thus received a *neither* coordinate system type code. Although not as dramatic as Paoletti et al.'s findings, the mathematics textbooks we analyzed also presented decontextualized graphs for students to interpret, which increased over time (15, 19, 124 tasks in grades 6, 7, 8, respectively).

Based on our findings, we propose the following changes in curricula (and relatedly, in teaching) on coordinate systems and associated graphs to support students' mathematical development as well as potential future STEM courses and careers: (a) tasks drawing from contexts that afford opportunities to develop a balanced understanding of both coordinate system types; (b) a more balanced graphing activity associated with coordinate systems, and hence more opportunities for students to create graphs; and (c) better support in curricula materials assisting students' transitions from spatial to quantitative coordinate systems.

In this study, we specifically focused on tasks that presented pre-constructed coordinate systems. Future research directions include identifying the different types of activities required for coordinate systems (e.g., create or interpret a coordinate system) in conjunction with the extant three dimensions (coordinate system type, graphing activity type, and mathematical topic).

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TYPES OF PROBLEMS LINKED TO THE DEVELOPMENT OF ALGEBRAIC REASONING IN CHILEAN PRIMARY EDUCATION

TIPOS DE PROBLEMAS VINCULADOS AL DESARROLLO DEL RAZONAMIENTO ALGEBRAICO EN LA EDUCACIÓN BÁSICA CHILENA

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The Chilean Curricular Bases of primary education (students from 6 to 12 years old), establish in the Patterns-and-algebra axis that algebraic thinking will be developed through a problem-solving approach. However, international research shows that working with the algebraic objects involved in these types of problems is not the only way of enhancing the development of such thinking; there is another type of arithmetic-nature-problem which also enhances it at proto-algebraic level. This paper aims to identify and characterize this type of problems, associated with the Numbers-and-operations axis in the official Chilean textbooks. For this, the model of Algebrization Levels is considered as a theoretical reference, which is a theoretical-methodological tool proposed by the Onto-Semiotic Approach of mathematical knowledge and instruction.

Background

There are approximately three decades in which educational authorities and researchers in the field of Mathematics Education have shown a great interest in introducing algebraic ideas to the primary education curriculum. The introduction of these ideas aims to develop algebraic reasoning in students from 6 to 12 years old. This approach is currently known as Early Algebra. There are studies at a global level (Cai, 2004; Fong, 2004; Lew, 2004; Watanabe, 2008) developed under this approach, which have identified the characteristics with which these ideas have been introduced into the curriculum.

In the Mexican context, Aké and Godino (2018) state that, although the tasks proposed in the first-grade textbook are not aimed at promoting algebraic thinking in children, since numerical register is prioritized, some implicit algebraic objects are identified in some mathematical tasks and, from the authors' perspective, those tasks guide towards proto-algebraic ways of thinking. In the case of Colombia, Castro, Martínez-Escolar, and Pino-Fan (2017) report the promotion of algebraic reasoning through mathematical tasks from first to sixth grade of primary education which, for example, at the first grade include expressing numerical sentences in an equivalent way ($42 - 9 = 43 - \square$), and solving numeral sentences ($\square + 4 = 4$). Those tasks require, for instance, solving inequalities and then graphing the solution, verifying numerical conjectures, etc. Finally, in the American context, four standards of algebraic context are established: understanding of patterns, relations and functions; representation and analysis of mathematical situations and structures using algebraic symbols; use of mathematical models to represent and understand quantitative relations; and the analysis of change in various contexts (National Council of Teachers of Mathematics, 2000).

These studies, and others, have shown that currently there is no consensus in the field on the characteristics of algebraic reasoning and, even more, on what is meant by this concept. Thus, in Blanton and Kaput (2011), they refer to it as an activity of generalization of mathematical ideas. In addition, the work with undetermined quantities, the use of variables, algebraic symbolization, relations between quantities, unknowns, equations, patterns, and the study of change, are considered as other notions related to the development of algebraic reasoning. In addition, other studies relate the development of this type of reasoning based on the work with some notions of arithmetic. These

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results somehow support and justify the existence of the Early Algebra approach, which emerged as a counterweight to the curricular separation created between arithmetic and algebra. Molina (2009) points out that some authors suggest that this separation accentuates and prolongs the difficulties of students. This is why the work with activities that enable the transition and integration of algebra and arithmetic is proposed, with a different approach to the computational one (which predominates in the first grade) and that benefits the development of algebraic and arithmetic modes of thinking.

In the Chilean context, the few studies developed to characterize algebraic reasoning, focus on how algebraic ideas are introduced to the primary education curriculum. Mejías (2019) published a study on the presence of algebra in Chilean textbooks from first to sixth grade of primary education, and its incorporation into the national curriculum, focusing the attention on the Patterns-and-algebra axis. This axis, together with Numbers-and-operations, Geometry, Measurement, and Data-and-probability axes, are part of the curricular organization proposed in the Curricular Bases (Ministry of Education of Chile [MINEDUC], 2018) from first to sixth grade of primary education (students aged 6 to 12 years old). Particularly, with the study of concepts such as pattern, in this axis it is expected that the bases for the development of more abstract mathematical thinking at higher levels, especially algebraic reasoning, will be established (MINEDUC, 2018). However, in the Numbers-and-operations axis it is expected that some concepts that promote this type of reasoning can be identified. As a consequence of the above, in this work it is proposed to characterize a typology of problems, related to the Numbers-and-operations axis in the Chilean context, which promote algebraic reasoning.

Theoretical And Methodological Aspects

This work is based on the theoretical notions developed by the Onto-Semiotic Approach to mathematical knowledge and instruction (Godino, Batanero, & Font, 2007, 2019), specifically on the Algebrization Levels model (Godino, Aké, Gonzato, & Wilhelmi, 2014). This model is considered as a theoretical-methodological tool, which allows characterizing algebraic reasoning in terms of the representations used, the generalization processes, involved, as well as the analytical calculation that is put into play during the (personal or institutional) mathematical practice related to a type of algebraic task. It should be noted that this model has been implemented in other studies, and it has been evident that it is a good predictor of (intended or assigned) algebrization levels for mathematical tasks in textbooks collections (e.g., Aké & Godino, 2018; Castro, Martínez-Escobar, & Pino-Fan, 2017).

The levels related to Primary Education are listed below, which are of particular interest to this work.

- Level 0 (*absence of algebraic reasoning*): mathematical objects are expressed through natural, numerical, iconic, or gestural language; if symbols that refer to an unknown value intervene, the result of such value is obtained by operating on particular objects.
- Level 1 (*incipient level of algebrization*): properties, numerical equivalences, and relations from particular tasks are identified; languages are natural, numerical, iconic, or gestural.
- Level 2 (*intermediate level of algebrization*): mainly, undetermined quantities or variables expressed in symbolic or symbolic-literal language intervene, in order to refer to recognized intensives, although linked to information from the spatial-temporal context.
- Level 3 (*consolidated level of algebrization*): intensive objects, represented in a symbolic-literal way, are generated and operated on; transformations of algebraic (symbolic) expressions are performed.

It is pertinent to clarify that these levels are not established by the problem itself, but by the practice (or practices) developed to solve the problem. A mathematical practice is defined as “any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems" (Godino & Batanero, 1994, p. 334).

This work has a qualitative nature (León & Montero, 2003). The research technique is content analysis (Gil-Flores, 1994), specifically focused on mathematical practices (proposed in official textbooks). The analysis is carried out following these phases: 1) preliminary study and selection of problems related to the Numbers-and-operations axis, 2) resolution of the selected problems and detailed analysis of mathematical practices, and finally 3) the categories of problems are generated. Namely, mathematics textbooks distributed by the Chilean Ministry of Education for primary education levels are analyzed. In this work, we particularly report the results obtained from the first and second grade mathematics textbooks.

Preliminary Results

The analysis evidenced that a series of institutional mathematical practices proposed in the context of the Numbers-and-operations axis, would enable the development of algebraic reasoning. Arithmetic is seen by several researchers as a key access to algebra (Warren, 2003). For example, Carpenter, Frankie and Levy (2003) state that, when studying algebra, students do not understand that the procedures they use to solve equations and simplify expressions are based on the properties of numbers. Meanwhile, Molina (2009) argues that "being able to count requires working algebraically since it is necessary to have a structured and organized way of counting" (p. 137, personal translation from Spanish). Therefore, it is considered that, despite the gap between arithmetic and algebra in the context of Chilean Primary Education, the problem situations included in the textbooks and their respective practices, would serve as promoters of algebraic reasoning at incipient levels (specifically, levels 0 and 1).

19 types of problems have been identified in the first-grade textbook (16 related to the Numbers-and-operations axis; 3 related to the Patterns-and-algebra axis), and in the second-grade textbook, 21 types of problems were characterized (18 related to the Numbers-and-operations axis; 3 related to the Patterns-and-algebra axis). The problems belonging to the Patterns-and-algebra axis are not considered in this study, because the current curricular document already established that these kinds of problems develop algebraic thinking (MINEDUC, 2018). In the case of the Numbers-and-operations axis, the entire typology of problems has been quantified, however, attention is only focused on those which promote algebraic reasoning through their mathematical practices. Among the typology of problems, related to the first and second grade of primary, the following can be found:

- Apply the algorithm of composing and decomposing numerical quantities to solve problems.
- Compare quantities (greater than, less than or equals to, it is more than, it is less than, <, >, =).
- Identify the condition to know when an element belongs to a set.
- Identify and represent the cardinality of a set by means of a number.
- Operate numerical quantities (addition or subtraction).

The practices promoted by the typologies of previous problems are related to a level 0 of algebrization (absence of algebraic reasoning), that is, mathematical objects are mainly represented in the numerical, iconic or mother language. Numerical quantities (extensive objects) are decomposed as the sum of two (or more) numerical quantities. In addition, tabular representations are used to organize the information, mainly when using algorithms for addition and subtraction. The concept-definitions are: addition, subtraction, compose, decompose, abacus, place value, additive decomposition, etc. The procedures are related to the use of algorithms, and they are also supported by the use of materials such as the abacus. Propositions and arguments are primarily related to how numbers can be decomposed or composed in accordance with the place value of their digits.

On the other hand, level 1 (*incipient level of algebrization*) has only been identified in the practices promoted in the second-grade textbook, in which the typology of problems related to them is as follows:

- Identify the additive properties of natural numbers (commutative, associative) (5 problems),
- Identify the properties of multiplication of natural numbers (existence of multiplicative neutral, zero property of multiplication) (10 problems).

Among the algebraic objects identified the following can be found: notion of an unknown value, structure properties, notion of a numerical equivalence. Practices at this level allow us to identify that extensive objects (examples) intervene that explicitly promote generalization (a rule) through mother language. Properties and relations are identified from the structural tasks.

Projections

Analogously to the analysis developed for the first and second grade textbooks, it is intended to characterize typologies of problems related with the development of algebraic reasoning for the remaining texts, from third to sixth grade of primary education, but not only for the Numbers-and-operations axis, but in Patters-and-algebra axis too. It is considered that a characterization of this nature will allow identifying the didactic-mathematical knowledge that primary teachers should have in order to develop instructional processes in accordance with curricular approaches.

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TIPOS DE PROBLEMAS VINCULADOS AL DESARROLLO DEL RAZONAMIENTO ALGEBRAICO EN LA EDUCACIÓN BÁSICA CHILENA

TYPES OF PROBLEMS LINKED TO THE DEVELOPMENT OF ALGEBRAIC REASONING IN CHILEAN PRIMARY EDUCATION

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En las Bases Curriculares de primaria (6 a 12 años) de Chile, se establece que con el estudio de problemas del eje de Patrones y álgebra se desarrollará el pensamiento algebraico. Sin embargo, investigaciones internacionales han demostrado que no sólo el trabajo con los objetos algebraicos involucrados en ese tipo de problemas potencia el desarrollo de dicho pensamiento, sino que existe otra tipología de problemas, de índole aritmética, que también lo promueve en niveles conocidos como proto-algebraicos. En este trabajo se pretende identificar y caracterizar esa tipología de problemas, asociados al eje de Números y operaciones, en los textos escolares oficiales de Chile. Para ello, se considera como referente teórico el modelo de Niveles de Algebrización, herramienta teórico-metodológica propuesta en el Enfoque Onto-Semiótico del conocimiento y la instrucción matemáticos.

Palabras clave: álgebra y pensamiento algebraico, análisis del currículo, educación primaria

Antecedentes

Aproximadamente son tres décadas en las que autoridades educativas e investigadores del campo de la Educación Matemática han mostrado un gran interés por introducir ideas algebraicas al currículo de educación básica. La introducción de estas ideas al currículo tiene por objetivo el desarrollo del razonamiento algebraico en estudiantes entre los 6 y 12 años. A este enfoque se le conoce actualmente como Early Algebra. Existen investigaciones a nivel mundial (Cai, 2004; Fong, 2004; Lew, 2004; Watanabe, 2008), elaboradas dentro de este enfoque, que han dado cuenta de cuáles son las características con las que dichas ideas han sido introducidas al currículo.

En el contexto mexicano, Aké y Godino (2018) declaran que, si bien las tareas propuestas en el texto de estudio de primer año de primaria no están dirigidas a promover un pensamiento algebraico en los niños, puesto que se prioriza el registro numérico, sí se identifican algunos objetos algebraicos implícitos en algunas tareas matemáticas y que, desde la perspectiva de los autores, orienta hacia formas de pensamiento proto-algebraico. En el caso de Colombia, Castro, Martínez-Escobar y Pino-Fan (2017), reportan la promoción del razonamiento algebraico a través de tareas matemáticas desde primero a sexto grado de educación primaria, las que, en el primer grado, por ejemplo, contemplan expresar sentencias numéricas de manera equivalente ($42 - 9 = 43 - \square$), resolver sentencias numéricas ($\square + 4 = 4$), que requieren, por ejemplo, resolver inecuaciones y luego graficar la solución, verificación de conjeturas numéricas, etc. Finalmente, en el contexto estadounidense, se establecen cuatro estándares de contenido algebraico: comprender patrones, relaciones y funciones; representar y analizar situaciones matemáticas y estructuras usando símbolos algebraicos; usar modelos matemáticos para representar y comprender relaciones cuantitativas y analizar el cambio en diversos contextos (National Council of Teachers of Mathematics, 2000).

Estas investigaciones, y otras más, han demostrado que actualmente no existe un consenso en el campo sobre las características del razonamiento algebraico y, más aún, qué se entiende por este concepto. Así, en Blanton y Kaput (2011), se refieren a éste como una actividad de generalización de ideas matemáticas. Además, se considera al trabajo con cantidades indeterminadas, el uso de variables, la simbolización algebraica, las relaciones entre cantidades, las incógnitas, las ecuaciones, los patrones, el estudio del cambio, como otras nociones asociadas al desarrollo del razonamiento algebraico. Otros estudios, además, relacionan el desarrollo de este tipo de razonamiento a partir del trabajo con algunas nociones de la aritmética. Estos resultados de alguna manera abonan y justifican la existencia del enfoque Early Algebra, el cual surgió como contrapeso a la separación curricular creada entre la aritmética y el álgebra. Molina (2009), señala que algunos autores sugieren que esta separación acentúa y prolonga las dificultades de los alumnos y es por ello que se propone trabajar actividades que posibiliten la transición e integración de ambas, con un enfoque diferente al computacional (el cual predomina en los primeros cursos escolares) y que beneficie el desarrollo de modos de pensamiento algebraico y aritmético.

En el contexto chileno, los pocos trabajos desarrollados para caracterizar el razonamiento algebraico se centran en cómo son introducidas las ideas algebraicas al currículo de educación básica. Mejías (2019), elabora un estudio de la presencia del álgebra en los libros de texto chilenos de primero a sexto año de enseñanza básica y su incorporación en el currículo nacional, centrando su atención en el eje Patrones y álgebra. Este eje, en conjunto con Números y operaciones, Geometría, Medición y Datos y probabilidades, forman parte de la organización curricular propuesta en las Bases Curriculares (Ministerio de Educación de Chile [MINEDUC], 2018) de primero a sexto básico (niños de 6 a 12 años). En este eje, en particular se espera que, con el estudio de conceptos como el de patrón, se establezcan las bases para el desarrollo de un pensamiento matemático más abstracto en niveles superiores, en especial el razonamiento algebraico (MINEDUC, 2018). No obstante, se considera que, en el eje Números y operaciones, además, se pueden identificar algunos conceptos que fomenten este tipo de razonamiento. Como consecuencia de lo anterior, en este trabajo se propone caracterizar una tipología de problemas, asociados al eje Números y operaciones en el contexto chileno, que fomenten el razonamiento algebraico.

Consideraciones Teóricas Y Metodológicas

Este trabajo se sustenta en nociones teóricas desarrolladas en el Enfoque Onto-Semiótico del conocimiento y la instrucción matemáticos (Godino, Batanero y Font, 2007, 2019), específicamente en el modelo de Niveles de Algebrización (Godino, Aké, Gonzato y Wilhelmi, 2014). Este modelo es considerado como herramienta teórico-metodológica, la cual permite caracterizar el razonamiento algebraico en términos de las representaciones utilizadas, los procesos de generalización implicados, así como el cálculo analítico que se pone en juego durante la práctica matemática (personal o institucional) asociada a un tipo de tareas algebraicas. Cabe señalar que este modelo ha sido implementado en otros estudios, y se ha evidenciado que es un buen predictor de los niveles de algebrización (pretendidos o asignados) para las tareas matemáticas en colecciones de libros de texto (e.g., Aké y Godino, 2018; Castro, Martínez-Escobar y Pino-Fan, 2017).

A continuación, se enuncian los niveles asociados a la Educación básica, y que son de particular interés para este trabajo.

- Nivel 0 (*ausencia de razonamiento algebraico*): los objetos matemáticos son expresados mediante lenguaje natural, numérico, icónico, o gestual; de intervenir símbolos que refieren a un valor desconocido, el resultado de dicho valor se obtiene operando sobre objetos particulares.
- Nivel 1 (*nivel incipiente de algebrización*): se identifican propiedades, equivalencias numéricas, y relaciones a partir de tareas estructurales; los lenguajes son de tipo natural, numérico, icónico, o gestual.
- Nivel 2 (*nivel intermedio de algebrización*): intervienen, principalmente, cantidades indeterminadas o variables expresadas con lenguaje simbólico, o simbólico-literal, para referir a los intensivos reconocidos, aunque ligados a la información del contexto espacial-temporal.
- Nivel 3 (*nivel consolidado de algebrización*): se generan objetos intensivos, representados de manera simbólico-literal, y se opera con ellos; se realizan transformaciones de las expresiones algebraicas (simbólicas).

Es pertinente aclarar que el nivel no se establece al problema per se, sino a la práctica (o a las prácticas) desarrollada para dar solución al problema. Se define como práctica matemática a "toda actuación o manifestación (lingüística o no) realizada por alguien para resolver problemas matemáticos, comunicar a otros la solución, validar la solución y generalizarla a otros contextos y problemas" (Godino y Batanero, 1994, p. 334).

El trabajo es de carácter cualitativo (León y Montero, 2003). La técnica de investigación es el análisis de contenido (Gil Flores, 1994), específicamente de las prácticas matemáticas institucionales (propuestas en textos oficiales). El análisis de desarrolla en las siguientes fases: 1) estudio preliminar y selección de los problemas asociados al eje Números y operaciones, 2) resolución de los problemas seleccionados y análisis pormenorizado de las prácticas matemáticas, y finalmente 3) se generan las categorías de problemas. A saber, se analizan textos de matemática, distribuidos por el Ministerio de Educación de Chile para los niveles de educación básica. En este trabajo, en particular, se reportan los resultados obtenidos del libro de matemática de primero y segundo básico.

Resultados Preliminares

El análisis desarrollado ha evidenciado que una serie de prácticas matemáticas institucionales propuestas en el contexto del eje Números y operaciones, posibilitarían el desarrollo del razonamiento algebraico. Diversos investigadores consideran a la aritmética como acceso clave al álgebra (Warren, 2003). Carpenter, Frankie y Levi (2003), por ejemplo, manifiestan que, al estudiar álgebra, los estudiantes no entienden que los procedimientos que usan para resolver ecuaciones y simplificar expresiones están basados en las propiedades de los números. Por su parte, Molina (2009), argumenta que "ser capaz de contar requiere trabajar algebraicamente ya que es necesario tener una forma, estructurada y organizada de contar" (p. 137). Por lo tanto, se considera que, a pesar de que existe una brecha entre la aritmética y el álgebra en el contexto educativo básico de Chile, los problemas y sus respectivas prácticas, fungirían como promotores del razonamiento algebraico en niveles incipientes (0 y 1 específicamente).

Se han identificado 19 tipos de problemas en el libro de texto de primero básico (16 asociados al eje de Números y operaciones, 3 a Patrones y álgebra), mientras que en el libro de texto de segundo grado se han caracterizado 21 tipos de problemas (18 en Números y operaciones, 3 en Patrones y álgebra). En este estudio no se consideran los problemas pertenecientes al eje Patrones y álgebra, ello debido a que el documento curricular vigente ya establece que éstos desarrollen el pensamiento algebraico (MINEDUC, 2018). Para el caso del eje Números y operaciones, se ha cuantificado toda la tipología de problemas, no obstante, sólo se enfoca la atención en aquellos que fomentan, a través de sus prácticas matemáticas, el razonamiento algebraico. Entre la tipología de problemas, asociada a ambos grados, se encuentran:

- Aplicar el algoritmo de componer y descomponer cantidades numéricas para resolver problemas.
- Comparar cantidades (mayor que, menor que o igual que, es más que, es menos que, <, >, =).
- Identificar la condición para saber cuándo un elemento pertenece a un conjunto.
- Identificar y representar mediante un número la cardinalidad de un conjunto.
- Operar cantidades numéricas (suma o resta).

Las prácticas promovidas por las tipologías de problemas anteriores se asocian al nivel 0 de algebrización (ausencia de razonamiento algebraico), es decir, los objetos matemáticos se representan, principalmente, en lengua materna, lenguaje numérico o icónico. Las cantidades numéricas (objetos extensivos) se descomponen como la suma de dos (o más) cantidades numéricas. Además, se hace uso de representaciones de tipo tabular para organizar la información, principalmente cuando se usan algoritmos para la suma y resta. Los conceptos-definiciones son: suma, resta, componer, descomponer, ábaco, valor posicional, descomposición aditiva, etc. Los procedimientos se asocian al uso de algoritmos, y, además, se apoyan en el uso de materiales como el ábaco. Las proposiciones y argumentos se asocian principalmente, en cómo los números se pueden descomponer o componer según el valor posicional de sus dígitos.

Por otra parte, el nivel 1 (*nivel incipiente de algebrización*) sólo se ha identificado en las prácticas promovidas en el libro de texto de segundo grado, en el cual la tipología de problemas asociada es la siguiente:

- Identificar las propiedades aditivas de los números naturales (conmutativa, asociativa) (5 problemas),
- Identificar las propiedades de la multiplicación de los números naturales (existencia del neutro multiplicativo, propiedad cero de la multiplicación) (10 problemas).

Entre los objetos algebraicos identificados se encuentra: noción de valor desconocido, propiedades de estructura, noción de equivalencia numérica. Las prácticas en este nivel permiten identificar que intervienen objetos extensivos (ejemplos) que promueven la generalización (una regla) de manera explícita mediante lengua materna. Se identifican propiedades y relaciones a partir de las tareas estructurales.

Proyecciones

Análogamente al análisis desarrollado para los libros de texto de primero y segundo básico, se pretende caracterizar tipologías de problemas asociados al desarrollo del razonamiento algebraico para los textos restantes, de tercero a sexto básico, pero no sólo en el eje de Números y operaciones, sino que además en el de Patrones y álgebra. Se considera que una caracterización de esta naturaleza permitirá identificar los conocimientos didáctico-matemáticos que los profesores de educación básica deberían poseer para desarrollar procesos instruccionales acordes a los planteamientos curriculares.

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REVIEW OF SLOPE IN CALCULUS TEXTBOOKS

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In this study, we report on how slope is reviewed in a convenience sample of 28 common calculus textbooks published in English. While most calculus textbooks studied included reviews of slope, findings suggest that the reviews are written for students who already have a fairly solid understanding of slope. Slope as a ratio, whether approached visually or nonvisually, serves as a foundational notion for derivative and was the most common conceptualization used in the textbook reviews studied. However, the lack of alternative conceptualizations and connections between various conceptualizations of slope may hinder students deeply understanding other calculus topics. Future study should look at each of these in depth to determine how slope is needed and leveraged when particular calculus concepts are introduced.

Keywords: Calculus, Post-Secondary Education, University Mathematics, Curriculum Analysis

Dietiker (2013) argues that mathematics textbooks can be interpreted as narratives that present mathematical ideas in a purposeful, influential order. Mathematics textbooks create a link between natural language and symbolic mathematical language (Fang & Schleppegrell, 2010), where both languages work using nonvisual elements, such as equations, and visual elements, such as graphs (O'Halloran, 2015). Textbooks play an important role in the way professors shape and sequence their instruction (Davis, 2009) and in how students choose strategies to consider and solve problems (Massey & Riley, 2013). Love and Pimm (1996) have suggested that textbooks are primarily geared toward students. So, textbooks often impact how students learn, aiding students as they form ideas on how to solve problems (Massey & Riley, 2013). Previous calculus textbook research has focused on a) how students consider and solve textbook problems (Lithner, 2003; Lithner, 2004), b) how instructors use textbooks in their teaching (Mesa & Griffiths, 2012), c) how textbooks present and structure examples (Mesa, 2010), d) what is required of students in examples (Özgeldi & Aydın, in press), and e) how representations are coordinated in particular reform textbooks (Chang, Cromley & Tran, 2016).

Slope is a secondary mathematics topic that becomes foundational in post-secondary (i.e., university) mathematics. It plays a key role when contrasting the covariational behavior of linear and nonlinear functions in algebra (Carlson, Jacobs, Coe, Larsen & Hsu, 2002; Teuscher & Reys, 2010) and in the development of derivative in calculus (Zandieh & Knapp, 2006). Research on slope in post-secondary mathematics has increased in recent years even extending into how slope plays a role in multivariable calculus (e.g., McGee & Moore-Russo, 2015). However, examining slope and how it is presented in single variable calculus (henceforth, simply referred to as "calculus") textbooks, has not received attention. Ideally, students should follow and use relations between conceptualizations of slope at will, demonstrating a flexible, integrated understanding of this notion. However, little is known about how, or even if, calculus textbooks review slope. This study considers both calculus for science, technology, engineering, and math (STEM) majors (deemed "STEM textbooks") and applied calculus textbooks often used in classes for business majors as well as life and social science majors (deemed "non-STEM textbooks) to see which conceptualizations of slope are included and if the textbooks are capitalizing on visual approaches in addition to linguistic resources (Moore-Russo & Shanahan, 2014). More specifically, we seek to answer the following three research questions:

1. Is slope reviewed in calculus textbooks? If so, where are slope reviews located?

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2. Which conceptualizations of slope are used in textbook reviews? Are visual or nonvisual approaches to slope taken in textbook reviews?
3. What common links, if any, exist between the conceptualizations of slope reviewed in calculus textbooks?

Literature Review

Importance of Calculus

While a calculus course is required of STEM majors (Bressoud, 2015), applied calculus, without any trigonometry, is often required of business majors as well as social and life sciences majors. Failing, or only marginally passing calculus, is one of the main reasons post-secondary students change their majors (Hensel, Sigler & Lowery, 2008; Kaabouch, Worley, Neubert & Khavanin, 2012; Bressoud, 2015). Many STEM degree programs require a grade of C or higher in calculus to count for credit, with calculus being prerequisite to other courses required in the major, and it is often recommended that students pass calculus at a high level before moving on to further courses (Koch & Herrin, 2006). Engineering students who fail calculus lack the foundation needed for required courses in their majors (Koch & Herrin, 2006; Veenstray, Dey & Herrin, 2008). Student struggles in STEM calculus are not limited to engineering students; studies have shown that calculus attrition rates (i.e., receiving a grade of D or F or withdrawing) for students in physical science or math may be as high as 40% to 50% (Pilgrim, 2010; Fayowski & MacMillan, 2008).

Slope as a Foundational Topic for Calculus

Some mathematics courses follow a vertical path in which certain concepts rely on previous concepts (Treisman, 1992). Many key concepts in calculus build on topics introduced in algebra and precalculus (Habre & Abboud, 2006). In order to develop a robust understanding of foundational ideas in calculus, such as instantaneous rates of change and derivatives, students must first understand average rates of change and the difference between linear and nonlinear functions. Yet, individuals often a) have isolated notions of slope (Dolores Flores, Rivera López & García García, 2019); b) have trouble interpreting different representations of slope (Glen, 2017; Tanışlı & Bike Kalkan, 2018); c) are only able apply slope in particular problem contexts (Byerley & Thompson, 2017); and d) have a limited understanding of linear functions in general, even when able to transition between different representations of linear functions (Adu-Gyamfi & Bossé, 2014).

As students enter post-secondary institutions, the ways they think of slope may be quite limited and different from the ways their professors think of and communicate slope (Nagle, Moore-Russo, Viglietti, & Martin, 2013). Two reasons for this may be related to the limited understanding of slope held by some high school teachers (Coe, 2007; Moore-Russo, Conner & Rugg, 2011; Nagle & Moore-Russo, 2014a; Stump, 1999) and the differences in the way state standards and textbooks address slope (Nagle & Moore-Russo, 2014b; Stanton & Moore-Russo, 2012). No matter why, a lack of prerequisite knowledge often leads to difficulty in understanding later topics, which corresponds with poor performance (Pyzdrowski et al., 2013).

Slope Understanding

Previous studies have considered how slope is characterized in the U.S. and Mexican curriculum (Stanton & Moore-Russo, 2012; Dolores Flores, Rivera López & Moore-Russo, 2020) and conceptualized by a variety of individuals (Moore-Russo, Conner & Rugg, 2011; Nagle, Martínez-Planell & Moore-Russo, 2019; Stump 1999, 2001b). The meaning that an individual makes related to slope, or any other mathematical notion, often depends on what the task at hand evokes (Tall & Vinner, 1981), the representations used to communicate ideas (De Bock, Van Dooren & Verschaffel, 2015) and the individual's prior knowledge or experiences (Vinner, 1992). In short, slope can be conceptualized in many ways, but previous research suggests that both students and teachers often fail to make connections between the various conceptualizations of slope (Coe, 2007; Hattikudur et al., 2011; Hoban, in press; Lobato & Siebert, 2002; Planinic, Milin-Sipus, Kati, Susac & Ivanjek, 2012; Styers, Nagle, & Moore-Russo, in press).

Stump's (1997, 2001a) research findings suggest that teachers rely primarily on ratios as the dominant representations of slope. Even though secondary teachers express concern for students' understanding of slope, they often reduce slope to procedural computations and neglect to regard the importance of developing a conceptual understanding of slope (Stump, 1999). Slope is often reduced to mnemonics that hinder students' understanding of slope as a rate of change (Walter & Gerson, 2007). As a result students enter calculus with isolated notions of slope and are not able to connect slope as a ratio to other ways of conceptualizing slope, such as a measure of steepness (Nagle & Moore-Russo, 2013a; Stump, 2001b). Students are not always able to work with slope in a conceptual way in application tasks (Lingefjärd & Farahani, 2017) nor are they always able to interpret slope in non-standard settings, such as when nonhomogenous coordinate systems are used (Zaslavsky, Sela & Leron, 2002).

Teuscher and Reys (2010) found that while the majority of calculus students could determine over what interval a variable changed by a certain rate, which involved slope, only half of the students were able to determine the interval with the greatest rate of change. They suggested that part of the reason for the difficulty was the vocabulary used in textbooks. Concepts such as steepness, slope, and rate of change are described in different ways among different textbooks, leading to misunderstandings of the questions for some students (Teuscher & Reys, 2010), those who lacked a deep, connected understanding of slope.

Theoretical Framing: Reader-Oriented Theory

Weinberg and Wiesner (2011) wrote that most academic research on textbooks has framed them as static collections of ideas, simply describing students' reading of the texts as extracting information. They sought to characterize the ways in which students interpret textbooks using reader-oriented theory. Reader-oriented theory centers on the idea that meaning of a text is constructed by the reader, not by the text itself. Three ideas about the readers of textbooks emerge from reader-oriented theory: the intended reader, the implied reader, and the empirical reader. The *intended reader* is "the idea of the reader that forms in the author's mind" (Wolff, 1971, p.166, as cited in Weinberg & Wiesner, 2011). The *empirical reader* is the person who actually reads the textbook. The *implied reader* is a concept used to describe the understandings an empirical reader must possess in order to make sense of a mathematics textbook (Weinberg, 2010). Authors should ensure the intended reader and implied reader coincide. This study will use these notions of different readers as a lens to interpret findings.

Methods

The data set consisted of a convenience sample 28 introductory calculus textbooks published in English between 2011 and 2019 that the research team recognized as including the calculus textbooks most commonly used in the United States. These 28 were used since they were available to the lead researcher as sample copies on an electronic book platform through her academic institution. Of the 28, 14 were non-STEM calculus textbooks and 14 were STEM calculus textbooks. In each textbook, the researchers first reviewed the index for any occurrence of the word "slope" to find all instances where slope was reviewed without any calculus content (e.g., limits, derivatives, etc.). All such review instances were included in the study.

Coding

For the current study, Nagle & Moore-Russo's (2013b) slope coding scheme was revised slightly. Five categories were used for the conceptualization of slope. Each of the five categories was divided further in two subcategories to see if the textbook relied on a *visual* or *nonvisual* approach. Table 1 displays descriptions for all of the conceptualization category-approach pairs.

Table 1: Slope Conceptualization and Approach Coding

Conceptualization	Approach	Description
Slope as a *Ratio* (RA)	Visual	rise/run or vertical change divided by the horizontal change
	Nonvisual	$(y_2 - y_1)/(x_2 - x_1)$ or change in y over change in x
Slope as a *Behavior Indicator* of a line (BI)	Visual	line increases, decreases, is horizontal, is vertical (looks like /, \, -, |) for positive, negative, zero, undefined slope respectively
	Nonvisual	line increases, decreases, is constant, or is not a function in other words (i.e., $y_2 > y_1$ for $x_2 > x_1$) for positive slope, (i.e., $y_2 < y_1$ for $x_2 > x_1$) for negative slope, (i.e., $y_2 = y_1$ for $x_2 > x_1$) for zero slope, or (i.e., $x_2 = x_1$ for $y_2 > y_1$) for undefined slope respectively
Slope as denoting *Steepness* of line's angle of inclination with respect to horizontal (ST)	Visual	relates to how inclined, tilted, slanted, or pitched a line is <u>seen</u> as being; greater value of |slope|, line is more steep (i.e., closer to vertical); closer to zero value of |slope|, line is less steep (closer to horizontal); since horizontal lines have no tilt, they have zero slope
	Nonvisual	relates to how extreme a line is <u>calculated</u> as being; the greater the value of |slope|, the more steep the line over an interval (e.g., the closer to infinity the value of $\|y_2 - y_1\|$ is); the closer to zero the value of |slope|, the less steep the line over an interval (e.g., the closer to zero the value of $\|y_2 - y_1\|$ is); horizontal lines have $\|y_2 - y_1\|= 0$ for all y values; so, slope is zero
Slope as a *Constant Parameter* (CP)	Visual	emphasis on the uniform "straightness" of the line's entire graph; no matter which segment of the line is considered, the straight slope remains the same between any two points due to similar triangles
	Nonvisual	emphasis that a single constant holds a property for the line's equation/table (not dependent on input); for any interval of a line, slope calculations remain the same between any two points
Slope as *Determining Relationships* between lines (DR)	Visual	two unique lines have the same slope if and only if they never intersect in two-dimensions (i.e., are parallel); two unique lines have different slopes if and only if they intersect at a common point; two unique, nonvertical lines have negative reciprocal slopes if and only if their intersection is at a right angle
	Nonvisual	two unique lines have the same slope if and only if a system of these two lines has no solution; two unique lines have different slopes if and only if the system of these two lines has one solution; two unique, nonvertical lines have negative reciprocal slopes if and only the product of their slopes is -1

A textbook was used as a unit of analysis and coded as an entity, meaning that if a textbook had more than one instance of one of the 10 categories (i.e., *visual* and *nonvisual* approaches to a *Ratio*, *Behavior Indicator*, *Steepness*, *Constant Parameter*, and *Determining Relationships* conceptualization), it was only marked once. To explain coding, consider this example involving two different approaches to the same conceptualization category. For example, consider the numerical computation of slope between two points (coded RA_n, for *Ratio-nonvisual*) being accompanied by a graph with labeling of Δy for the vertical displacement and Δx for the horizontal displacement (coded RA_v, for *Ratio-visual*). This graph of a line with Δy and Δx labeled and accompanied by $m = \frac{\Delta y}{\Delta x}$ would be coded as the link RA_v-RA_n. The implied reader, in this case, is meant to have an understanding of the slope computation and how it correlates with the visual markings indicated on the graph. Similar coding was used if two different conceptualization approaches were linked.

Findings and Discussion

Research Question 1: Slope Reviews in Calculus Textbooks

Since the concept of slope is an important building block for students taking calculus (Noble, Nemirovsky, Wright & Tierney, 2001), it was not surprising that all but one of the STEM textbooks in this study contained at least some review of slope. Most (22 of the 28) textbooks had the slope review at the beginning of the textbook only, while 3 had some slope review at the beginning and at the end of the textbook. This suggests that most calculus textbook authors feel that a review of slope should be available to students (or covered through instruction) prior to the introduction of derivatives and other calculus concepts. Calculus textbook authors appear to envision intended readers as students who need to have a solid base of prerequisite knowledge that includes an understanding of slope.

Research Question 2: Slope Conceptualizations in Calculus Textbooks

We now consider which conceptualizations of and approaches to slope were used in the sample of textbooks used in this study focusing on the implied reader to consider how concepts emphasized in different textbooks' slope reviews involve different understandings readers must possess in order to make sense of the calculus concepts presented in the textbooks. Table 2 displays the findings from the textbooks. All five of the conceptualizations of slope were used in at least one of the textbooks.

Almost all (25 of 28) textbooks used the *Ratio* conceptualization of slope. Postsecondary instructors and calculus students have been found to respond to open-ended questions about slope with responses that included a *visual* or *nonvisual* approach to *Ratio* (Nagle et al., 2013), as have high school teachers (Stump, 1999). So, this finding suggests that the slope reviews in the textbooks might be trying to connect with how students often think of slope. Understanding slope as a *Ratio* is important in calculus, especially in the understanding of derivative, when the limit of a difference quotient ties together the concepts of slope, limit, and derivative. So, it is not surprising that *Ratio* is the most prevalent conceptualization in calculus textbooks.

For calculus, students need a solid understanding of the idea of ratio (not just in the sense of slope, but in general) that goes beyond chanting "rise over run" or plugging and chugging into a formula. This is important since research (e.g., Carlson, Madison, & West, 2015) has shown that students often do not consider slope as representing the ratio of two covarying quantities in complex problems, such as those found in calculus. Textbook authors who do not consider this may lead to a disconnection between the intended and implied readers. In calculus, readers need to understand that the visually-oriented, rise-to-run graphical comparisons of a linear segment and the corresponding algebraic formulas that represent the slopes of secant lines connecting two points on a curve approach the value of the slope of a tangent line to the point on a curve in order to understand how a derivative is defined. However, in their reviews of slope, most textbook authors did not mention that slope is a foundational topic for understanding calculus concepts, and focused solely on finding slope as a numerical value (*nonvisual*) or as a property associated with the image of a line (*visual*).

Table 2: Slope Conceptualizations and Approaches in Textbooks

Conceptualization	Approach	Textbook Type		Total
		non-STEM	STEM	
Ratio (RA)	Visual Only	0	0	0
	Nonvisual Only	0	0	0
	Both	14	11	25
Behavior Indicator (BI)	Visual Only	3	7	10
	Nonvisual Only	1	0	1

	Both	10	4	14
Determining Property (DP)	Visual Only	1	0	1
	Nonvisual Only	1	2	3
	Both	9	7	16
Constant Parameter (CP)	Visual Only	2	2	4
	Nonvisual Only	1	1	2
	Both	6	7	13
Steepness (ST)	Visual Only	2	3	5
	Nonvisual Only	0	0	0
	Both	6	2	8

Behavior Indicator tied as the most used conceptualization of slope in the reviews, present in 25 textbooks (as was the *Ratio* conceptualization). However, all *Ratio* coded textbooks used both *visual* and *nonvisual* approaches, while *Behavior Indicator* textbooks did not. STEM textbooks more often used *visual* approaches only. Slope is often associated by students visually as the way a line is displayed on a graph (Moore et al., 2013). In Nagle and colleagues' 2013 study, this was a conceptualization commonly reported by post-secondary calculus students but not by their instructors. In the cases of textbooks emphasizing *Behavior Indicator*, the implied reader should be able to build an understanding the ideas of "increasing" and "decreasing" as well as related conventions of graphing to slope. In order to understand the idea of derivative and related topics (e.g., role that a slope of zero has in identifying potential relative extrema in the first derivative test), students will need to know more than just how the slope of a tangent line behaves visually; they will need to be able to work with intervals of functions using formulas.

Determining Property was the conceptualization used third most by textbook authors. In the textbooks where this conceptualization was noted, implied readers should be able to build on an understanding of concepts such as parallel, perpendicular, reciprocal, and so on. In the case of textbooks using this conceptualization, readers are typically asked to interpret two or more lines that are being compared (either graphically or using formulas). This should help prepare readers for tasks involving identification of the equation of a normal line, which is perpendicular to the line tangent to a curve at a point. Students also need to know that parallel lines have the same slope in order to understand the Mean Value Theorem and Rolle's Theorem.

The second least used slope conceptualization was *Constant Parameter*. Textbooks which include this conceptualization require their students to understand what the word "constant" means in a mathematics context for linear functions where slope acts as a parameter that results in a constant numerical change seen in tables or in the graphical straightness noted in a *visual* display of a line. Implied readers need to leverage an understanding of the straightness of a line for approximations over sufficiently small intervals when using linearization.

Steepness was the least used conceptualization of slope used in the calculus textbooks. Students should be able to construct ideas of "steepness" in a physical sense that relates the higher the absolute value of the slope of a line is, the steeper that line is. This understanding is often needed for related rates problems involving angles of inclination and right triangles, such as tasks involving where the vertical rate of ascension for a rising balloon is given.

Research Question 3: Links between Slope Conceptualizations

To answer the third research question, we now consider common links between the slope conceptualizations in the textbooks. In order to be coded as a link, the textbook had to indicate that two conceptualization-approach pairs related to the same idea. Table 3 displays the type of links present in the textbooks, and the total number of textbooks per link type. The RA_n-RA_v link was most frequent (22), which supports that the implied reader must to be able to relate the *visual* and *nonvisual* approaches of the *Ratio* conceptualization in order to understand slope. This was frequently shown as the graph of a line with the rise and run indicated, accompanied by the corresponding numerical calculation. Table 3 displays most frequent links, those that occurred in at least 10 textbooks.

Table 3: Links between Conceptualization-Approach Pairs in Textbooks

Link	Textbook Type		Total
	non-STEM	STEM	
RA_n-RA_v	12	10	22
CP_n-RA_n	9	6	15
DP_n-DP_v	7	7	14
CP_v-RA_n	7	6	13
BI_n-BI_v	8	4	12
CP_n-CP_v	6	6	12
CP_n-RA_v	6	6	12
CP_n-RA_v	5	6	11

Links containing either the *Ratio* or *Constant Parameter* conceptualization (with either a *visual* and *nonvisual* approach) were the most frequent. Given the prevalence of the *Ratio* conceptualization in textbooks, it is not surprising that its *nonvisual* and *visual* approaches would be linked most often.

Research (e.g., Nagle & Moore-Russo, 2013b) has referred to slope as a constant ratio, whereas *Constant Parameter* and *Ratio* were defined separately in this study. As such, the links containing either the *Ratio* or *Constant Parameter* being the most common is not surprising. In other words, understanding that slope is a constant rate of change between two covarying quantities, an equivalence class of ratios appears to be considered by authors as pivotal when first learning the concept of derivative. This suggests that the implied readers typically will need to make the link that slope can be considered as both a *Ratio* and a *Constant Parameter*.

The *Behavior Indicator* conceptualization occurred just as often as the *Ratio* conceptualization in 25 of the 28 textbooks. However it was not linked to other conceptualizations of slope very frequently. It does not seem that textbook authors deemed this way of thinking about slope to need to be connected to other ways of thinking of slope. This lack of connection could lead readers to concentrate on procedures without a connected, conceptual understanding of why first derivative tests are used to determine functional behavior in calculus.

Conclusions, Instructional Implications, Further Study

Slope is not heavily reviewed in calculus textbooks. Given the importance of understanding slope for calculus and how textbooks review slope, textbook writers seem to be assuming that the intended readers of these texts have a healthy understanding of slope upon entering calculus. Some conceptualizations of slope are sparsely represented in textbooks, and many textbooks do not provide a well-rounded, connected review of slope. Instructors must be aware that they may need to provide additional review over what is offered in textbooks to ensure that students are making connections between different conceptualizations of slope so that students have the robust understanding of slope that is needed as a foundation for the topics they encounter throughout calculus. Instructors should read the slope reviews, encourage students to do the same, and then be aware of other conceptualizations of and approaches to slope that they may need to provide to their students that are not present in textbooks.

The emphasis on slope as a *Ratio* and linking this idea to slope as a *Constant Parameter* should prepare students for the limit definition of a derivative; however, the lack of connections between the various conceptualizations of slope may result in students' failure to deeply understand other calculus topics that require alternative notions of slope. The role of slope in introducing derivatives is documented, but it is important that instructors also consider how textbooks leverage slope to introduce other calculus topics, such as how students come to think about average rates of change, why a derivative of zero may yield relative extrema, how parallel lines play a role in the Mean Value Theorem, how the "straightness" of a line is leveraged in linearization, etc. Future study should look at each of these in depth to determine how slope is needed and leveraged when particular calculus concepts are introduced.

One limitation of this study is that it only considered the stand-alone reviews of slope. It is possible that textbook authors are using a just-in-time review approach and connecting different conceptualizations of slope while introducing the calculus concepts themselves. Future research should consider this possibility.

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COMPUTATIONAL THINKING PRACTICES AS A FRAME FOR TEACHER ENGAGEMENT WITH MATHEMATICS CURRICULUM MATERIALS

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Teachers routinely make adaptations to their mathematics curriculum materials as they plan and enact lessons. In this paper, I explore how encouraging two elementary teachers to examine their mathematics curriculum materials through the lens of computational thinking practices – abstraction, debugging, and decomposition – supported them in adapting tasks from their curriculum materials in ways that raised the cognitive demand.

Keywords: Computational Thinking; Curriculum; Elementary School Education

Background and Purpose of the Study

Mathematics curriculum materials (CMs) can serve as supports for teachers in creating high-quality mathematics instruction (Stein & Kaufman, 2010; McGee, Wang, & Polly, 2013). One way CMs can act as a support is by providing tasks with high cognitive demand (Stein, Smith, Henningsen, & Silver, 2000) or starting points for such tasks. For the potential of this support to be realized in practice, teachers must learn ways of interacting with CMs that allow them to thoughtfully choose among tasks and adapt them in ways that support students' engagement with productive mathematics (Brown, 2009). Existing research has revealed teachers differ in the specific strategies they use to approach CMs (Remillard, 2012; Sherin & Drake, 2009), and not all such strategies result in instruction that maintains high cognitive demand for students (e.g., Amador, 2016). On the other hand, a number of studies have supported the notion that teachers can create instruction that maintains high cognitive demand when they engage with CMs through lenses related to big mathematical ideas (Stein & Kaufman, 2010), student thinking and knowledge (Choppin, 2011; Grant et al., 2009), and the connections between these two elements (Drake et al., 2015). In this paper, I present a post-hoc analysis of how two teachers adapted tasks when reviewing a lesson in their CMs through the lens of three computational thinking (CT) practices: decomposition, debugging, and abstraction. In particular, I focus on how these three CT practices supported teachers in transforming low cognitive demand tasks presented in their curriculum materials into tasks of higher cognitive demand by helping to focus teachers' attention on the big mathematical ideas of the lessons and students' potential strategies.

Conceptual Framework

This study utilized the mathematical task framework, which has two parts. First, Smith and Stein (1998) developed four cognitive demand categories for mathematics tasks. Two categories —Doing Mathematics and Procedures with Connections—are high cognitive demand because they engage students in thinking about mathematical concepts and relationships. The others—Procedures without Connections and Memorization—are low cognitive demand because they focus on use of procedures and correct answers. Second, Smith, Grover, and Henningsen (1996) argued any task passes through three phases when used in instruction: (1) the task as it appears in instructional resources, (2) the task as set up by a teacher, and (3) the task as implemented by students. In this study, I categorized the tasks in participants' CMs and the tasks they planned according to the cognitive demand categories. I focused on the transition from the first task phase to the second: the changes teachers made to CM tasks to the way they planned to set up the tasks.

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Methods

Study Context and Participants

The purpose of the CT4EDU project is to support elementary teachers to incorporate CT into their mathematics and science teaching. CT is a broad set of thinking practices used by computer scientists (Yadav, Stephenson, & Hong, 2017). The CT4EDU project is focused on big ideas in CT, including *abstracting* important information from situations, *decomposition* of complex problems into simpler parts, and *debugging,* or finding and fixing errors. In a professional development workshop, participating teachers worked in groups to plan a math lesson, starting from CMs, that incorporated at least one of the CT practices ideas mentioned above. Two teachers from the project were chosen for inclusion in this study. Alice and Cindy (both pseudonyms) were using *Math Expressions* (Fuson, 2012), the CMs mandated by their district. Alice was a fourth-grade teacher with 15 years of experience. Cindy was a fifth-grade teacher with five years of experience.

Data and Analysis

I examined the *Math Expressions* lessons referenced by Alice and Cindy, as well as the tasks that were the focus of their conversations. I classified these tasks according to level of cognitive demand. Next, I used transcripts of Alice and Cindy's planning conversations to create descriptions of the tasks these teachers planned to pose to students. I classified these tasks according to cognitive demand. To understand how CT played a role in how teachers adapted the tasks, I read the transcripts of the planning conversations to identify decisions related to changes to the tasks. Next, I examined the explanations the teachers articulated for these decisions. I considered an explicit mention of a CT practice or reference to a CT handout as potential evidence of influence of CT on the teachers' reasoning. I coded the decision as influenced by CT when the teacher (1) related a decision to a CT practice's description, (2) described how a proposed change would provide opportunities for students engage in a CT practice, or (3) connected a decision to a CT practice as she reflected on the lesson-planning process.

Results

Alice

Alice was working from a *Math Expressions* lesson on estimation and mental math. Table 1 shows the initial tasks posed in the CMs and the task Alice set up in the classroom. Both tasks on the left can be classified as Procedures without Connections. Students are asked to produce an estimate and an exact total in the first task, but not to explain their reasoning. Students are asked to provide a solution method and a yes-or-no answer for the second task. The task on the right, by contrast, can be classified as Procedures with Connections. To engage with this task, students must think about the impact of estimating via rounding to the nearest hundred on the real-world context of the problem rather than merely do the rounding for themselves. Thus, the planned version of the task has a higher level of cognitive demand than the tasks as posed in the CMs.

Three of Alice's decisions were influenced by CT. First, Alice chose a task from the CMs to use as the main task in her lesson. She primarily attended to the two tasks at the left in Table 1, and decided to start with the latter because she wanted to give students an opportunity to *decompose* a problem. She felt the two-part format of the first task did the decomposition for students: "I feel like now, looking at this, this wouldn't be good because they're giving it to them. They're telling them how to break it down." Second, Alice changed the statement of the problem to prompt a discussion about different possible estimates and how those estimates differ from the exact total. Third and relatedly, Alice changed the numbers in the task. According to Alice, students always rounded to the highest place value—two-digit numbers to the nearest 10, three-digit numbers to the nearest 100, and so on. She expected students to use this rounding technique as they made estimates, and felt that changing the numbers to be in the hundreds would lead to estimates farther away from the exact total: “These numbers aren’t gonna have them overestimate. So maybe change them so that the numbers are higher?” Estimates further away from the total, reasoned Alice, could lead to a discussion of *debugging*.

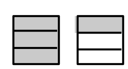
Table 1: Alice’s Starting Tasks and Task as Set Up in the Classroom

Starting Tasks from *Math Expressions*	Task as Set Up by Alice
The best selling fruits at Joy’s Fruit Shack are peaches and bananas. During one month Joy sold 397 peaches and 412 bananas. a) About how many peaches and bananas did she sell in all? b) Exactly how many did she sell? Tomas has $100. He wants to buy a $38 camera, a $49 CD player, and 2 CDs that are on sale 2 for $8. How can Tomas figure out if he has enough money for all four items? Does he have enough?	My friend gave me $930 to purchase items for a trip. The exact costs are $651 for his plane ticket, $112 for clothes, and $156 for meal gift cards. I rounded the amounts and added them to get an estimate of $1000. I told my friend he did not give me enough money, but he said I was wrong. I rounded the costs to the nearest hundred and added: 700 + 100 + 200 = $1000. Can you help me figure out what I did wrong? Did he give me enough? Did I round incorrectly?

Cindy

Cindy was working from a lesson on fractions greater than 1. Table 2 shows the tasks from the CMs and the task Cindy set up. The CM tasks can be classified as Procedures without Connections. Students can complete them by following the procedures given in the examples. The task on the right can be classified as Procedures with Connections. Students must think about the whole and provide two other representations of a fraction greater than 1, given one representation. Thus the cognitive demand of this task is higher than the tasks given in the CMs.

Table 2: Cindy’s Starting Tasks and Task as Set Up in the Classroom

Starting Tasks from *Math Expressions*	Task as Set Up by Cindy
Change each mixed number to a fraction. **Example:** 2½ = 2 + ½ = 1 + 1 + ½ = 2/2 + 2/2 + ½ = 5/2 3 2/5 = ____ 2 3/8 = ____ (etc.) Change each fraction to a mixed number. **Example:** 13/4 = 4/4 + 4/4 + 4/4 + ¼ = 1 + 1 + 1 + ¼ = 3¼ 10/7 = ____ 12/5 = ____ (etc.)	Fill in the missing parts. In the unit fraction column, draw a ring around the whole. **Picture** **Sum of Unit Fractions** **Fraction** ¼ + ¼ + ¼ + ¼ + ¼  12/5 (additional rows were given)

Cindy made four decisions influenced by her attention to CT. First, Cindy decided to teach the *Math Expressions* lesson over two days to allow her to spend more time on representing fractions greater than 1. Cindy credited this decision to thinking about *decomposition*:

> The CT is helpful to me as the teacher, in a sense that I’m now looking through a finer lens at the lesson itself and thinking, gosh, the workbook does go in this order, this fast. But really breaking it down and trying to think like the students are, and really think about what challenges they have. And how I can decompose the lesson itself into smaller pieces.

Second, Cindy decided to launch the lesson by showing students one representation at a time (picture, or sum of unit fractions) and discussing how students could change one representation into the other. Third, Cindy incorporated pictures and sums of unit fractions into the student page so students' independent work would more closely mirror the class discussion. Fourth, she limited the examples to numbers less than 3, so drawing models and writing sums of unit fractions remained a viable strategy. Cindy related these three decisions to supporting students in realizing that symbolic representations of fractions are an *abstraction*:

> Yeah, that abstraction is heavy. Even having them consciously aware of what that abstraction feels like and looks like here. To talk and have that discussion when you go from the visual to the sum of unit fractions or to the mixed number and really highlighting that idea.

Discussion

Alice's attention to CT supported her in thinking deeply about how students would approach tasks. Thinking about decomposition led her to consider the impact of the CMs breaking problems into subparts for students—which is one way of lowering the cognitive demand of a task by changing a challenge into a nonproblem (Stein et al., 2000). Thinking about debugging led Alice to consider how she expected her students to approach rounding and the impact that approach may have in a real-world context. This suggests that thinking about CT practices supported Alice in making curriculum adaptations based on student thinking, which other studies have suggested leads to productive use of CMs (Choppin, 2011; Grant et al., 2009).

Cindy's attention to CT supported her in thinking about big mathematical ideas in her lesson. As she considered symbolic fractions as an abstraction, she began to consider the multiple ideas encapsulated in those representations (e.g., 7/5 is an abstraction intended to show that wholes are divided into 5 equal parts, we are considering 7 of parts, and so on). Cindy realized she did not think students would be able to "see" all this information in a symbolic fraction without more experience with other representations. This led her to decompose the lesson. Ergo, examining her CMs through the lens of CT helped Cindy unpack big mathematical ideas—another strategy research suggests leads to productive CM use (Stein & Kaufman, 2010).

This data does not allow me to empirically examine *why* the lens of CT practices led teachers to consider and adapt their CMs in this manner, but considering the conference theme suggests one possibility. While the CT practices highlighted here resemble disciplinary practices in mathematics, the import of these ideas from another disciplinary culture, computer science, may have aided teachers in engaging with them in ways that supported new kinds of pedagogical thinking. Decomposition, for example, is discussed in the Common Core State Standards (CCSSI, 2010), but only in reference to decomposing mathematical objects such as numbers or geometric shapes. Computer scientists tend to discuss decomposition of *problems* (Yadav et al., 2017). This broader nature of the object being decomposed seemed to support Alice in thinking about decomposing the steps of a problem (rather than a number) and to support Cindy in thinking about decomposing the multiple mathematical ideas in her lesson. As computer science education emerges as a unique research area, math education researchers may benefit from cross-disciplinary conversations that offer new perspectives on existing ideas.

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BROADER INTENTIONS: EXPLORING THE ROLE OF AIMS FOR SCHOOL MATHEMATICS IN TEACHER CURRICULAR DECISION MAKING

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The current study develops theories about why the system of mathematics education in the United States is struggling to meet many of its beyond-the-classroom aims by exploring to what extent and how aims permeate curriculum planning and enactment systems and the role that aims play in the decision making of teachers in these systems. It examines the written, planned, and enacted curriculum of three high school algebra lessons. It finds that aims influence the decision making of all three teachers, but permeate the lessons differently in ways that are potentially explained by teacher aims, the topic taught, the types of evident aims, the number of years the teacher has been teaching, and how long they have been using their textbook.

Keywords: Curriculum, Curriculum Analysis, Curriculum Enactment

Secondary mathematics students often lament "Why do I have to learn this stuff?" There is good reason to take this question seriously. There are a number of broader goals that school mathematics is intended to support and the system of mathematics education in the United States is struggling to meet many of these aims. For example, US schools have had limited success developing students' ability to use quantitative information to make day-to-day decisions (Kastberg et al., 2016), participate in the labor market (Carnevale & Desrochers, 2003; Deloitte, 2015) and succeed in college STEM majors (Ganter & Barker, 2004).

I refer to the rationales for teaching and learning school mathematics, such as developing students' abilities to use mathematics to make day-to-day decisions or preparing students for the labor market, as *aims for school mathematics.* These are the beyond-the-classroom benefits that are attributed to the teaching and learning of mathematics in K-12 schools. The system's failure to achieve many of these aims raises an important question: Are aims considered in the curricular work of mathematics education decision makers and if so, how?

The current exploratory study adds to what is known about curricular decision-making systems by examining the curricular stages and changes that occur in three different high school algebra lessons as the teachers transform them from written textbook lessons to plans to an enacted lesson perceived by students. This examination is guided by the following research questions: 1) *To what extent and how do aims for school mathematics permeate these curriculum planning and enactment systems? 2) What role do aims for school mathematics play in the decision making of these teachers in these lessons?*

Theoretical Framework

I describe any desired ends of school mathematics as *curricular intentions*. I refer to classroom-based curricular intentions that are oriented toward improving students' mathematical proficiency (National Research Council, 2001) as *mathematical goals* and beyond-the-classroom benefits that mathematical proficiency provides as *aims for school mathematics*.

In order to understand the role that aims play in curricular decision making, different aims need to be identified and categorized because it is likely that the role of aims will differ depending on the type of aim invoked. I have compiled and categorized a list of aims mentioned in a variety of policy and research literatures and categorized them based on common characteristics (e.g., Geiger et al., 2015; González & Herbst, 2006; Gutiérrez, 2017; NCTM, 2000; Sinclair, 2001; Steen, 2001; Usiskin, 1980; Williams, 2012) (see Figure 1).

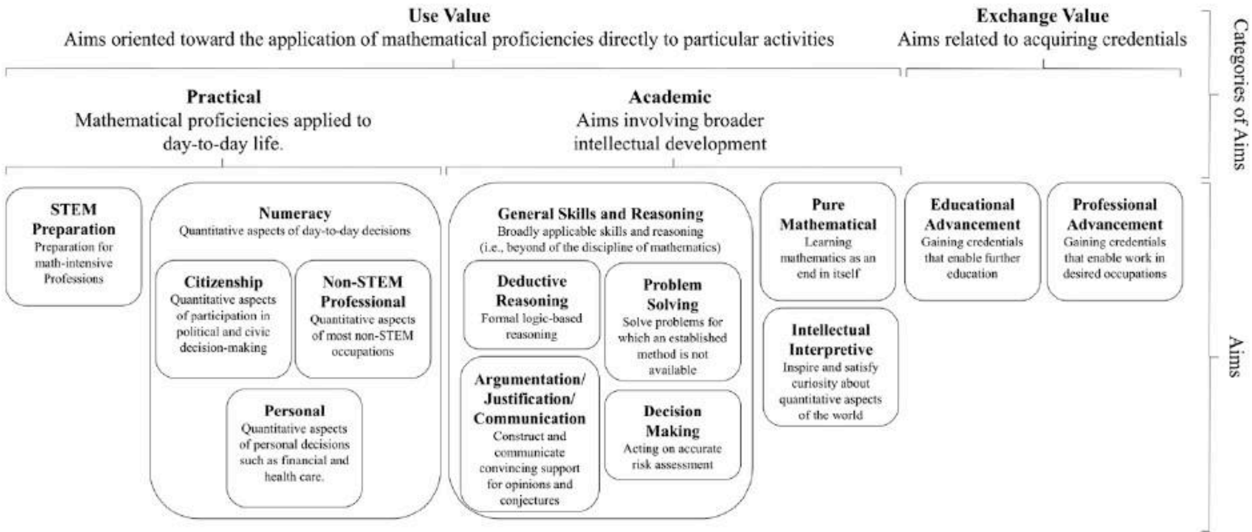

Figure 1: A Conceptual Framework for Aims for School Mathematics

Curricular decision making takes place within *curriculum policy, design, and enactment systems*. These systems include three stages: 1) the curriculum formulated before instruction (*intended)* which includes system-level expectations for student learning, textbooks, and teacher plans; 2) the curriculum that emerges as students and teachers interact (*enacted)*, and 3) the curriculum learned by students (*student learning)* (Remillard & Heck, 2014).

In planning and enacting curriculum, teachers vary widely in the extent to which they modify written materials (Remillard, 2005; Sherin & Drake, 2009). This variation can be described as a continuum in which some offload their design decisions to text, some adapt the text, and other improvise (Brown, 2009). Teachers' skill in making these decisions in order to achieve their intentions can be described as their *pedagogical design capacity* (Brown, 2009). One important element of this capacity is a teacher's developing knowledge of how their curriculum materials function in their particular context, their *curriculum context knowledge* (Choppin, 2009).

A key issue in investigating the role of aims in curriculum decision making is the extent to which aims permeate the system. At one extreme is the *low permeation* model whereby a particular set of mathematical goals, both process and content (e.g., NCTM, 2000; NGACBP/CCSSO, 2010), are established as supportive of the range of aims set for the system and teachers focus on mathematical goals without explicitly considering aims. At the other extreme is the *high permeation* model adopted by teachers who place aims at the center of their day-to-day decision making.

In investigating aims permeation, it is important to determine the types of curricular activities (what I will call *curricular structures*) in which aims are evident. This study investigates three kinds of structures, *tasks*, *discussions*, and *connectors*. I define tasks as anything students do that involves more than conversation. I define discussions as the verbal substance of the lesson. Connectors are verbal or written exposition that come before a task or discussion to frame it, after a task or discussion to summarize it, or between lesson elements as a transition. Tasks can be further categorized by their contextuality as not contextual (*mathematical)*, containing all of the complexity of a real-life problem (*authentic contextual)* somewhat simplified but still might reasonably occur outside of the classroom *(practical contextual),* or contextual but unrealistic (*prototypical)* (Csikos & Verschaffel, 2011; Palm, 2009; 2018).

Methods

The current study examines the curricular systems of three high school algebra teachers in economically and racially diverse high schools. Tobin[1] is in her seventh year of teaching, Megan in her fourth, and Rose in her third. Tobin and Megan are in their second full year using their text while Rose is in her first full year using hers (although she previously used elements of it).

For each lesson, data was collected from teacher interviews, classroom observations, student interviews, and textbooks. General interviews were conducted with teachers to learn their perspective on aims for school mathematics, the outside forces that impact their decisions, and how they use the written curriculum provided to them. Teachers were also interviewed before observed lessons, and teachers and up to four students per classroom were interviewed after each lesson. All interviews were audio recorded and transcribed. Lessons were observed and audio recorded. Additionally, Tobin and Megan's first days of school were observed and audio recorded and Rose described her first day of school in an interview. Introductory textbook materials were collected along with observed textbook lessons, relevant teacher-created materials, and pictures were taken of the classroom environment during the lesson.

Textbook overviews and general interviews for each teacher were analyzed for evident aims. A thematic analysis (Braun & Clarke, 2006) was conducted, using the previously described conceptual framework as initial codes, to create a coherent description of the textbook or teacher's perspective on aims for school mathematics. The lesson-based data was first analyzed to determine the structure of the lesson. Then the written, planned, and enacted stages were coded for stated intentions of the lesson, any other evident aims and contextual intentions, and any other mathematical goals that were connected to aims or the stated intentions of the lesson. Furthermore, any curricular decision described by the teacher was coded for any intentions cited by the teacher as justification for that decision.

Findings

The practical aim of effective financial decision making is evident in all curricular stages of Tobin's lesson as well as in her decision making. The stated intention of Tobin's textbook lesson is to use what students know about linear and exponential functions to help them understand the difference between simple and compound interest. In the lesson itself, there is considerable attention paid to developing a continuous model for calculating compounding interest. In her planning and enactment, Tobin makes significant adaptations that she describes as focusing the lesson on the value of interest in general and the power of compounding. This shift focuses the lesson more explicitly on financial decision making and less on the underlying mathematics.

In the textbook lesson, the aim of financial decision making is evident in an opening discussion, an opening written passage, and four practical contextual tasks. There is also a practical contextual task and a lesson summary in which the aim is not evident. Tobin's adaptations in her planning all relate to the aim. They include changing the framing and summaries of tasks, adding an authentic contextual task, a prototypical contextual task, and some teacher exposition, modifying tasks, and eliminating tasks. In enactment, Tobin makes further aims-related changes. She adds task framings and summaries, more teacher exposition, two personal asides, and gets four unplanned student-initiated conversations. The two students interviewed from Tobin's class cite financial decision making as evident in the lesson.

Aims are evident in all curricular stages of Rose's lesson as well as in her decision making but less so than in Tobin's lesson. Furthermore, Rose's perspective on aims differs from her text book so her adaptations change the nature of evident aims. The stated intention of Rose's textbook lesson is

[1] All names are psudonyms.

solving real world problems using systems of linear equations. In the written lesson, this practical aim is evident in two suggested class discussions and four practical contextual tasks, three of which are in a business context. Rose, however, is more focused on her students' general problem solving skills and in having them collaborate so they will enjoy the lesson and thus be more likely to consider STEM careers. As a result, her adaptations from planning to enactment end up eliminating the practical tasks. She uses prototypical tasks and modifies them to incorporate more problem solving and collaboration. Interestingly, despite these adaptations, two of the four students interviewed after this lesson identified the practical use of systems of equations as an aim for the lesson.

Unlike Tobin and Rose's lessons, aims are not evident in Megan's lesson, yet the aim of communication drives some of Megan's decisions and multiple aims are perceived by students. The stated intention of Megan's written lesson is for students to be able to add and subtract rational expressions. Megan's aim-related adaptation is to ask all of the groups to present their solutions to the first task in the lesson, a change that she explicitly ties to the aim in interviews, but not in the class. Despite this lack of evident aims in the written and enacted lesson, two the four students interviewed identify the aim of communication as evident in the lesson and one of four identifies collaboration and problem-solving. This is consistent with passages in the textbook introduction and teacher exposition on the first days of school that link mathematical goals such as communication, collaboration and problem-solving to broader aims.

Conclusions and Discussion

Thus, there are a variety of curricular structures in which aims can be evident, including a range of tasks, discussions, and connecting activities. Most notably, Rose and Tobin's textbook lessons and planning demonstrate how authentic contextual tasks can make practical aims evident in a lesson. Tobin's enactment suggests that aims-related personal asides and summaries may make aims evident in ways that register with students as they seem to inspire both student-initiated conversation and student-perceived aims. In contrast, Rose's enactment suggests that a lack of these structures may lessen the impact of evident aims. It also demonstrates the power of the teacher to eliminate aims to which she is not attending. Megan's lesson shows that intending to support mental discipline aims is not the same as making them evident in the lesson. However, it also suggests that explicit connection of mathematical practices to mental discipline aims in overview materials and general teacher exposition may have an impact on student perception of aims in later lessons even if the aims are not explicitly evident in the lessons themselves.

The differences in evident aims between these three lessons may be due, in part, to the topics. It is unsurprising that lessons on exponential functions and systems of equations would be more clearly connected to practical aims than one on simplifying rational functions. However, the finding that Tobin more effectively adapts her lessons suggests that more experience with her curriculum may have helped her develop more curriculum context knowledge (Choppin, 2009) and more years in the classroom may have allowed her to develop more pedagogical design capacity (Brown, 2009). Megan's use of overviews to link goals to aims may suggest another element of curriculum context knowledge and pedagogical design capacity.

Overall, this analysis suggests that aims can, indeed, permeate curricular processes, and provides some initial ideas for how this permeation may be indicative of teacher skill in using curriculum and how it might influence the achievement of aims. It lays the groundwork for future research to explore whether and how this kind of curricular work can, in fact, support the achievement of aims for school mathematics.

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CURRICULUM, ASSESSMENT & RELATED TOPICS:

POSTER PRESENTATIONS

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SECONDARY MATH METHODS SYLLABI ACROSS CULTURES

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Keywords: Teacher Education – Preservice; Cross-cultural Studies

Introduction

This poster presentation focuses on the comparison of syllabi for secondary mathematics methods courses across cultures. The authors uncover similarities and differences between these documents and their relevance. According to Wikle and Fagin (2014): "Within most college or university courses the syllabus serves as the principal tool for course planning" But is this true across cultures? Similarly to Kaki (2000), the authors believe culture does play a role influencing syllabus production.

Research Question and Design

The research questions guiding this comparison study are: a) how are secondary mathematics methods courses similar and/or different across cultures? b) How are the objectives of those courses similar and/or different according to their syllabi? And c) How are the evaluation methods similar and/or different according to their syllabi?

To conduct the comparison, the authors collected secondary mathematics methods syllabi from five institutions: two United States (US) colleges of education, one public teacher preparation program in Uruguay, one college of Engineering in Colombia, and one college of education in Korea. The syllabi were review first structurally, to see what sections they included, and then the content of those sections, specially objectives and evaluation methods, similarly to what Parkes, Fix and Harris (2003) and DuBois, Burkemper (2002) did.

Summary of Findings

Structurally, the US syllabus contained the greatest number of sections, and were in general the longest documents. They had the most coincidence with Korean syllabi, even though these were the shortest along the Uruguayan one. This one also was the one with the least number of sections, notoriously not including any information about how the students would be evaluated for the course, which was also the case for Colombia. Korea and US included outlines of the content by week. Standards and accreditors information was only included in the US syllabi.

When looking at the objectives all of the syllabus mentioned planning of instruction, and classroom assessment. All of them but the Colombian one also mentioned societal issues like diversity and equity through mathematics education. And the two south American syllabi also mentioned the importance of considering mathematics methods as a scientific discipline.

Last, looking at the course evaluation methods, the authors of this poster found that only the US cases have sections explaining assignments. There were some coincidence on the topics for evaluation: readings, planning, and assessment. Korea had a very broad explanation of how students would be evaluated, mentioning lecture, discussion/presentations, xperiment/practicum, and field study.

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MATHEMATICS INSTRUCTION DURING PROSPECTIVE TEACHERS' PERFORMANCE-BASED ASSESSMENT IN SPECIAL EDUCATION CLASSROOMS

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Keywords: Assessment and Evaluation, Special Education, Teacher Education – Preservice

This research project examined the perceived tensions of prospective teachers' (PTs) between their university coursework centered on student-led mathematics instruction and their internship placements' scripted mathematics program for students with disabilities in special education settings. The scripted program was a mandated district initiative featuring a mathematics curriculum with set pacing, required daily lesson materials, and a teacher script. A yearlong case study followed five PTs enrolled in a dual major Elementary Education and Special Education program. All participants were completing student teaching internship requirements in K-5 special education classrooms: four traditional resource classrooms (children with disabilities received instruction in a pull-out setting), one autism spectrum disorder classroom. Participants were also completing their performance-based teaching assessment called the edTPA, which focused on the instructional needs of one student with a disability in the area of mathematics.

The edTPA is a performance-based assessment intended to determine if beginning teachers are prepared to enter the classroom. This performance-based assessment was developed by Stanford University and the Stanford Center for Assessment, Learning, and Equity (SCALE). The Special Education edTPA is evaluated across 15 scored rubrics in areas of planning, instructing, and assessing. Currently, 920 Teacher Preparation Programs in the United States complete the edTPA (SCALE, 2019). The edTPA faces critique (Behizadeh & Neely, 2018; Gitomer et al., 2019), however, specific to this case study, it required that the PTs justify how their instruction integrated the personal, cultural, and community assets of their students. This focus on students' assets was emphasized in PT coursework that urged them to follow their students' mathematical contributions rather than follow a lockstep curriculum (Carpenter et al., 2014).

The study's qualitative PT data sources included: (a) focus groups pre, during, and post completion of the edTPA, (b) interviews, (c) written responses to the edTPA prompts, (d) written reflections about the process, and (e) open-ended survey responses from the teacher preparation program's feedback survey about the edTPA. Grounded in theory and literature around teacher preparation assessments (Darling-Hammond, 2020), qualitative data was analyzed to note emergent codes which resulted in persistent themes of PTs' perceptions of math teaching and perceptions of teaching self, contextualized in the tension of their setting and the edTPA. edTPA scores were included as quantitative data sources whose analysis produced descriptive statistics of the overall case and revealed edTPA subsets of challenge or above-average performance that provided insight into why certain tensions may have been magnified. Findings indicated that the PTs expressed tension between following the scripted program versus following their learner was exacerbated by the edTPA. Although the edTPA could have been used as a PT's catalyst for rejecting the scripted program, the PTs instead settled with a disjointed compromise among their perceived demands of the edTPA, their Clinical Educators, the scripted program, and their own expressed beliefs about teaching and learning.

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LOCATING A PRODUCTIVE CLIMATE: MEASURING STUDENT COMPUTATIONAL FLUENCY IN A TIER 2 SYSTEM

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Instead of interventions that focus on direct instruction, number strings provide opportunities for students to engage in mathematical discourse, both in describing their strategies and connecting with the mathematical strategy of others. Research on number strings has found that students participating in number string routines can adopt new strategies (O'Loughlin, 2007) and make connections between conceptual understanding and procedures. Studies on number strings have not previously focused on students with disabilities or students who are significantly underperforming in mathematics, investigating this problem can provide information that could lead to solutions that improve computational fluency in multiplication and division. In particular, interventions for students with disabilities (Lambert, 2018; Lambert & Tan, 2020) within the Multi-Tiered System of Support (MTSS) in mathematics.

To explore student computational fluency, we designed a standard aligned number string intervention for students (N= 35) with disabilities ages 8 through 11 significantly underperforming in multiplication and division (Lambert, Mendoza, & Nguyen, 2020). A number string is a short (15–20-minute) daily instructional routine in which a teacher presents a carefully designed sequence of problems one at a time for children to solve mentally and modeling student thinking with a representation (Lambert, Imm, & Williams, 2017). Research question: What are the effects of a Tier 2 number strings intervention in multiplication and division for students needing additional support in these areas?

Findings

In order to assess whether student computational fluency improved after eight number string interventions and three iterations of the Multiplication and Division CCSS CBM Math Assessment. . Using a dependent samples t-test, students improved their scores after the 8 session intervention, $t(32) = 2.30$, p=0.028. (We used a dependent sample t-test since repeated measures were taken from the same sample). After the pre-intervention test scores (first iteration of the assessment) of ~29.10%, the post-test scores (3rd iteration of the assessment) increased to ~36.93%. This is a 26.9% multiplicative increase for all students.

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TEACHER'S COLLABORATION WITH FRESHMEN UNDERGRADUATES TO IMPROVE FEEDBACK PRACTICES THROUGH COGENERATIVE DIALOGUES

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Keywords: Classroom discourse; Assessment and Evaluation; Instructional practices

Providing students with written feedback on their homework yields many benefits on their learning (Black, Harrison, Lee, Marshall & William, 2003; Fyfe, 2016; Landers & Reinholz, 2015). These benefits include providing students with information on their current levels of understanding, and in providing possible next steps for student improvement. However, in many cases, teachers take the position of "presenter of knowledge" (Black et al., 2003, p. 89), providing students with feedback that they assume the students need, and which may or may not match the actual students' needs. Students, on the other hand, rarely read and implement change on their work based on the written feedback (Black et. al., 2003), making the feedback lose its effectiveness. To improve the effectiveness of written feedback, researchers suggest that teachers seek different ways of using written feedback in mathematics classrooms (Frey & Fisher, 2011). We conjectured that using cogenerative dialogues (cogens) (Emdin, 2016; Tobin, 2006) to invite students into deciding on the nature of written feedback they would be receiving, and ways of using it efficiently, may support more student-centered forms of feedback.

We carried out a practitioner-inquiry study (Samaras & Freese, 2009) in a first year undergraduate Probability and Statistics class taught by Wambua. The class consisted of ten students at a private university in the Northeastern region of the United States. We hypothesized that we could improve the effectiveness of written feedback in classroom tasks by leveraging student autonomy to shape the nature of the feedback they received and in deciding how they use the feedback to produce new work. For six weeks, we held 15-minutes long cogens (structured teacher-student dialogues aimed at co-constructing classroom practices) after each class with all participants. For data analysis, we used constant comparison method (Savin-Baden, & Major, 2013). We coded all cogen-transcripts, students' worksheets and teacher's feedback to look for evidence of improvement in the teacher's feedback based on students' comments.

Working closely with students through weekly cogens focused on improving written feedback as a formative assessment technique generated improvements in three areas: (a) helping the teacher transition from giving general feedback to give specific feedback, (b) providing students opportunities to state how immediate feedback supported their learning and (c) in transforming the classroom culture. By listening to and learning from the students, the teacher improved her written feedback from using general statements like "Good job" and "explain more" to providing more elaborate feedback that articulated what was right or wrong in the students' work, and gave suggestions on how to correct the mistakes. The students highlighted that the cogens provided opportunities for prompt and focused feedback that helped them curtail practicing the same mistake in future. Finally, the cogens boosted students' motivation in their mathematical abilities and served as an evidence that their perspectives and experiences are valuable in co-constructing classroom practices. In future, we hope to explore how cogens could leverage feedback provision in larger class sizes and in education levels beyond undergraduate.

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EQUITY AND JUSTICE

RESEARCH REPORTS

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CRITICALLY ANALYZING AND SUPPORTING DIFFICULT SITUATIONS (CARDS): A TOOL TO SUPPORT EQUITY COMMITMENTS

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Mathematics educators sometimes have trouble enacting equity-based pedagogy. Part of this is due to the lack of authentic opportunities to engage in analyzing difficult scenarios involving power, identity, and access. Additionally, mathematics educators sometimes have trouble finding, sharing, and collaborating on activities and materials that interrogate these spaces. In this research report, we present how a collective of equity-oriented mathematics educators created, enacted, and studied the use of scenarios presenting difficult situations to pre- and in-service mathematics teachers. More importantly, this report shows how we, as a field of mathematics educators, can enact large-scale collaboration that disrupts the capitalistic norms of knowledge ownership and neoliberal approaches to teacher preparation. The CARDS tool was not created by an individual, but by an amorphous group aligned and committed to equity.

Keywords: Social Justice, Teacher Education – Preservice, Teacher Education – Inservice / Professional Development, Teaching Tools and Resources

The 2020 PME-NA conference theme, *Across Cultures,* promotes the exchange of ideas and collaborations across cultures in addition to thinking beyond traditional forms of educational research. This research report introduces a curricular tool, whose design, implementation and subsequent research of, directly aligns with this *Across Cultures* theme. The Critically Analyzing and Responding to Difficult Situations (CARDS) tool promotes dialogue across the many cultures and stakeholders involved in dismantling inequities embedded within mathematics education. The CARDS, intended to support each of us in rehearsing and preparing for difficult conversations associated with equity issues, aligns with the perspectives of the PME-NA Equity Statement (2019). Additionally, this research report is not just about presenting the CARDS tool, but about sharing and analyzing the process in which the CARDS were refined over several years by a collective of critical mathematics educators across institutions, generations, and geographies.

The MathEdCollective is a loosely organized and open membership group organized in the Fall of 2017 in response to attacks on mathematics educator Dr. Rochelle Gutiérrez by white supremist media (Gutiérrez 2017b, 2018). The CARDS were designed to help pre-service teachers, in-service teachers, and teacher educators develop and promote critical perspectives of mathematics as sociopolitical (Gutiérrez, 2010/2013). The CARDS ask users to image or enact difficult scenarios and practice potential responses, largely based upon Gutiérrez's (2015) "In My Shoes" activities and Crockett's (2008) case studies for mathematics educators to contemplate the intersection of culture and mathematics teaching.

Enterprise

The MathEdCollective

When mathematics education scholar Dr. Rochelle Gutiérrez (2017b, 2018) was attacked for her scholarship, white supremacist attacks on equity and justice work in mathematics education in the

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U.S. rose to a previously unseen level of vitriol. Professional and personal attacks crossed every imaginable boundary, causing colleagues to maintain open lines of communication via email and weekly video calls to organize, provide solidarity, and support to Gutiérrez and other scholars who might be the next to come under attack. As a result, the MathEdCollective was formed to organize and exhibit solidarity with U.S. mathematics educators and organizations under attack. Quickly, the MathEdCollective defined its collective principles based on used various sources of inspiration, including U.S. Civil Rights Era activism, works by Paulo Freire and bell hooks, and the hacker-activist group Anonymous (The MathEdCollective, 2019). As the group grew in number and increased its activities, a series of implicit organizing principles began to evolve.

The MathEdCollective (2019) practices (1) shared ownership of ideas, which creates a community that can shield individuals from further harm by "anonymizing" their ideas through the MathEdCollective's voice, (2) heterarchical and open membership, which means that it is without hierarchy, "leaders", "representatives", defined membership, or email list, (3) collective action, meaning no decisions or action by the MathEdCollective reflect individuals but reflect the consensus of the moment with whoever happens to be participating at the time, and (4) taking the high road, transforming negative hate and energy into something positive and productive.

The CARDS Emerge

The previously mentioned attacks on mathematics educators and researchers underscore an ideological war present within our field, making necessary the need for preparing teachers to engage in these difficult and critical conversations. During one call with several mathematics educators and graduate students, the MathEdCollective discussed the importance of the reactive work of the collective which provided support and trauma aftercare to those attacked. But we also recognized that being reactive was not enough, we needed to move toward a more proactive, educative approach. One member of the collective suggested using a set of playing cards that could be used to help a person rehearse for difficult situations. Essentially, the idea involved supporting a broad range of stakeholders in mathematics education through helping them be ready to respond to difficult conversations by practicing in advance using real-life scenarios and simulations of politically dangerous interactions that might emerge within the work of teaching (Crockett, 2008; Kazemi et al., 2016).

Teaching rehearsals are a simulation of conversations, interactions, situations, and/or relationships that might emerge during the work of teaching (Kazemi et al., 2016). By engaging in and practicing these rehearsals, teachers can improve "routine and improvisational decisions in practice" (p. 18). Therefore, the CARDS describe scenarios rooted in the social, historical, and institutional contexts related to the work of mathematics teaching. These scenarios, similar to and inspired by "In My Shoes" activities (Gutiérrez, 2015), provide opportunities to rehearse such conversations as a means for developing preservice teachers' sociopolitical toolbox. Like the "In My Shoes" activities, the CARDS aim to develop "nuanced perspectives on situations" and to consider multiple options or a "repertoire of moves that can be used and the kind of language that would accompany those moves in challenging situations" (pp. 22–23) for mathematics educators of all levels of experience.

The CARDS reflect not only a commitment to collective enterprise and shared intellectual ownership, but also to emergent design research (Tom, 1996). All persons who were involved with the ideation, refinement, and conversation around the CARDS are recognized as equal participants in the design, feedback, piloting, and intellectual development of this work. The mathematics teacher educators, the teachers we worked with, the various members of professional organizations who attended conference sessions or engaged in conversations related to the CARDS, and the MathEdCollective are collectively always "gathering" and "generating" data and knowledge in the form of anecdotes, videos, and feedback. This collective participatory design research method (Bang & Vossoughi, 2016) means that all participants helped guide the research questions, the research design, and even the interpretation of the data. Additionally, the development of the CARDS reflects the anti-hierarchical organizing principles of the MathEdCollective that disrupts colonial ways of defining people, land, and ideas as things to be owned, taken, and sold (MathEdCollective, 2019; Patel, 2016).

Development of the CARDS

The development of the CARDS started with solicitation and creation of scenarios based upon conversation within the MathEdCollective calls. This led to the idea of engaging in mathematics teacher education that was proactive in helping mathematics teachers be prepared for difficult conversations in their careers. We briefly describe the three cycles of development of the CARDS here.

This first cycle of the CARDS development consisted of soliciting and creating scenarios and possible responses to the scenarios, sharing those scenarios and responses with the MathEdCollective, and then creating a public, online space in which various individuals could add comments or questions about the scenarios and responses. The scenarios and responses were formatted to be able to be printed as physical cards that are approximately four inches by four inches each. The CARDS and pilot data based on small enactments within our mathematics methods courses were shared at the Association of Mathematics Teacher Educators (AMTE) conference in 2019. After that presentation, several members of the MathEdCollective immediately used these cards in their own teaching and reported back about issues they encountered through emails to the CARDS development team or directly on the CARDS google document itself.

Once a set of pilot cards had been developed, using the process and resulting in the materials as described above, we set forth on a second cycle to implement and, this time more intentionally, study the sorts of learning that the CARDS seemed to support. We decided to each use the cards in ways that made sense in our particular mathematics teacher education contexts: some of us work with pre-service teachers, some with in-service teachers; some of us with work with elementary teachers, some with middle or high school mathematics teachers. Some of us were most interested in using the CARDS as an in-class activity, others assigned them as outside-of-class work to be completed individually; some incorporated them with reflective writing, others used them as prompts for discussion, and others had teachers interact with the cards over innovative, virtual spaces for communication. Some of us were additionally interested in having a graduate student or a "more junior" mathematics teacher educator lead others in the use of the cards, so that we could study how the cards were taken-up and implemented by a mathematics teacher educator who was NOT involved in their development.

The second cycle involved refining and printing out eight physical cards for distribution to the MathEdCollective for use in their practice. The authors used these cards in their own mathematics methods courses in various ways. The authors then came together to engage in constant comparative analysis (Dye et al., 2000) through reading transcripts, listening to audio interviews, and watching video responses that the pre-service teachers generated when encountering the scenarios to create a list of themes. We then engaged in conversation to collapse these themes to those that seemed most important, and then re-analyzed specific points of data with this collapsed list of themes in mind.

A third cycle followed, involving the creation and solicitation of eight additional scenarios and then sharing them in three more situations: the 2020 AMTE conference, an elementary mathematics methods course with 25 pre-service teachers, and a professional development course with 20 practicing elementary teachers and assistant teachers in a public elementary school.

Through these cycles of analysis, multiple questions arose, such as: How are people using these cards? How do people respond to the cards? How are the enactments mitigating the historical violence that might be triggered through some of these scenarios? What are potentially dangerous assumptions that we and the MathEdCollective have made in the creation and implementation of these scenarios? While we collected a large amount of data that can be analyzed in multiple ways, we focus this paper on unpacking the critical conversations that the authorship team engaged in after these three cycles. We feel this analysis best serves the goal of the PMENA conference, engaging in our own difficult conversations about and across cultures.

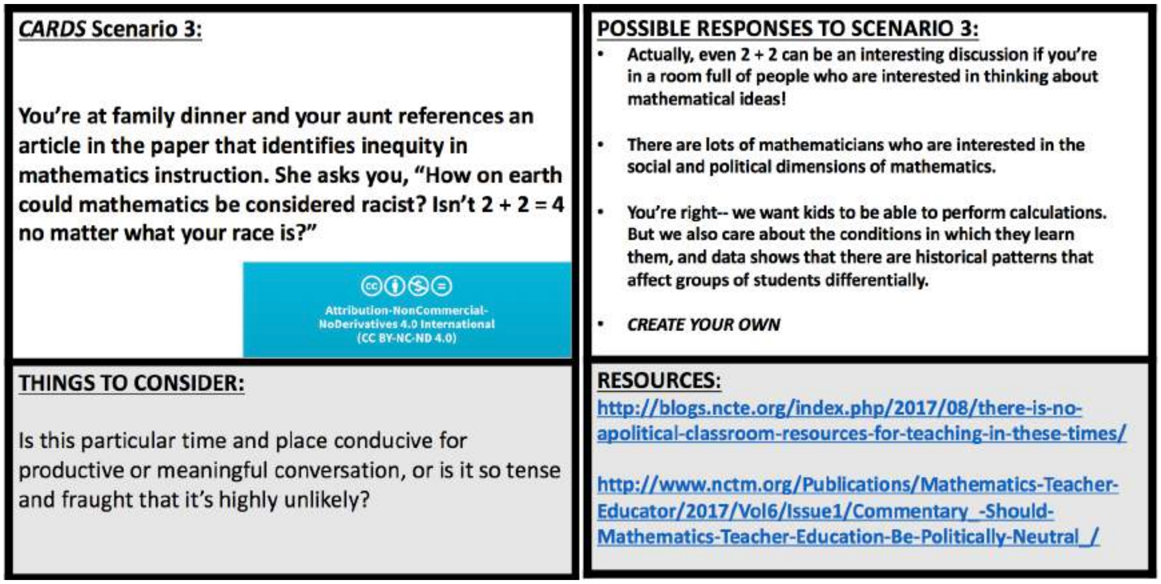
These various solicitation and collaborative cycles informed the design and content of the cards in several ways. We regard everyone who has been engaged with the CARDS as co-authors/designers. These individuals provided feedback on current scenarios and responses and made suggestions for new scenarios, things to consider, example responses, and resources. In addition, the co-authors/designers offered suggestions for a user guide. These suggestions were often a result of their own local contexts and connected to their own experiences, allowing the CARDS to involve scenarios that moved across cultures. These interactions and reflections provided valuable data on how the CARDS were taken up, both in their purpose and structure. Co-authors/designers also provided key reminders of how critical tools such as the CARDS could potentially cause harm, discomfort, or trauma for people using the cards. The feedback also included ways to think carefully about various audiences, how to scaffold the CARDS, and the necessary pre-work in building a community of trust.

Design of the CARDS

The CARDS are the result of an on-going collaborative effort involving a broad community of mathematics teacher educators, teachers, coaches, pre-service students, etc. Many voices contributed to the content and design considerations of the evolving tool. The CARDS (see Figure 1) were designed to be a tool intended to both 1) serve as a catalyst to open up dialogue between us about a variety of situations related to equity in mathematics teaching/education; and 2) support the development of our preparedness to engage in difficult conversations with others.

The front of each CARD includes a short scenario related to a topic in mathematics education. The scenarios are deliberately short, often with a variety of details omitted. This is purposeful. The vagueness of some of the scenarios provide an opportunity for further conversations and things to consider. Below the scenario, a list of “Things to Consider” are included. These are meant to be used to help the user think about external issues that might impact one’s response to the scenario.

The back of the CARD includes a range of possible responses as well as a list of resources that may support the topic or issue raised in the scenario. The possible responses are NOT intended to be used as actual statements to be used as response to but rather to further open up conversations among us as we engage with the CARD. The range of possible responses also aims to further promote dialogue about the topic and how the response contributes to a productive conversation or not. For example, some provided responses could be taken up as inviting others into a conversation versus some that intentionally shut people out.



CARDS Scenario 3:

You're at family dinner and your aunt references an article in the paper that identifies inequity in mathematics instruction. She asks you, "How on earth could mathematics be considered racist? Isn't 2 + 2 = 4 no matter what your race is?"

POSSIBLE RESPONSES TO SCENARIO 3:

- **Actually, even 2 + 2 can be an interesting discussion if you're in a room full of people who are interested in thinking about mathematical ideas!**
- **There are lots of mathematicians who are interested in the social and political dimensions of mathematics.**
- **You're right-- we want kids to be able to perform calculations. But we also care about the conditions in which they learn them, and data shows that there are historical patterns that affect groups of students differentially.**
- ***CREATE YOUR OWN***

THINGS TO CONSIDER:

Is this particular time and place conducive for productive or meaningful conversation, or is it so tense and fraught that it's highly unlikely?

RESOURCES:

http://blogs.ncte.org/index.php/2017/08/there-is-no-apolitical-classroom-resources-for-teaching-in-these-times/

http://www.nctm.org/Publications/Mathematics-Teacher-Educator/2017/Vol6/Issue1/Commentary_-Should-Mathematics-Teacher-Education-Be-Politically-Neutral_/

Figure 1: Front and Back of Scenario 3 CARD

Findings: Impact of the CARDS

We present three of the main impacts that we found as a result of our initial implementations with the cards. We refer to these as impacts in the sense that these are three ways in which the cards seem to provoke moments of learning for us mathematics teacher educators: about the cards themselves as curricular materials, about our preservice teachers' experiences in teacher preparation programs; and about our own orientations toward our work as mathematics teacher educators working with teachers. Below we briefly present three initial findings about these impacts, sharing illustrative examples from the interview transcriptions; the video responses, and written assignments. As a reminder of our data analysis process: we reviewed selections the data together (we watched videos, listened to interviews, read transcripts and submitted assignments) and engaged in consensus-building discussion about what themes seemed to emerge in multiple instances (i.e., in more than one preservice teachers' response) and what themes seemed most compelling to us and most immediately informative to our work as mathematics teacher educators.

The Importance of Place and Space

As preservice teachers reacted to the scenarios on the cards, they considered both the individual scenarios and the possible responses. In doing this, we noticed that the specificity of place (i.e., the grocery store or a family dinner) was a significant detail in how several participants responded to scenarios, particularly as they thought about the ideas of the "appropriateness" of having "sensitive" conversations in those places. Several preservice teachers said they would want to talk about things in a different place or at a different time. For example, as she was interacting with the cards, Marta was also influenced by the setting of the scenario. She mentioned that if she were at the grocery store, the scene of Scenario 1, she would probably be "in a rush" and "not be in the right mindset to have that kind of serious conversation." Other preservice teachers told us that they would want additional information about the topic before they would feel comfortable responding if they were in similar situations. We reflected on the ways that the preservice teachers' responses may have been affected. That is, we see the consideration of place as very understandable and relatable, but realize that the content on the cards may have influenced responses of this nature. As retreating is a frame of whiteness, in how white-identified peoples hold the privilege to be able to retreat from difficult situations (Picower, 2009), we realize the need to have preservice teachers, and ourselves, explore the different possibilities for responding to issues of power, identity, and access given the different settings, times, and people involved in our interactions.

Considerations of Our Teachers

Students of color or from marginalized identities bear some emotional weight related to personal experiences with the content of the cards. In Kat's interview, she expressed how a recent experience in her university courses influenced whether or not she was comfortable with certain responses to the Columbus Day scenario (Scenario 2). For example, she recalled the racial tension in her class after speaking up about an issue. She said: I was in a class, and me being the only, well, one out of two black students that were in the class. I spoke up and said something about, something relating to social studies and Columbus Day and things like that, where I felt like people in the class weren't taking seriously… And I received a lot of backlash...it was hard for weeks...now things are getting a lot better. But it was just hard because you can feel the racial tension and just the divide. The weight of that experience influenced which possible responses to the scenarios she saw as acceptable and which felt like "an attack to somebody".

As Abdulah worked with the cards, he specifically connected with Scenario 5, where the scenario is related to a teacher who is unable and unwilling to learn the correct pronunciation of his students' names. Abdulah shared personal experience of how his name was very often mispronounced. In sharing his experience, Abdulah discusses the need for teachers to learn the correct pronunciations of students' names from the start, but he also mentioned that his name is never pronounced correctly due to differences in language. Although Abdulah said that he took no offense when people mispronounced his name, another preservice teacher reacted differently this saying, "My name has always been mispronounced which makes me feel uneasy." The situations on the cards brought out the emotional weight these preservice teachers have experienced throughout their schooling in a range of different experiences, and we want to acknowledge concerns we have heard from MTEs about using these cards framed as worry for upsetting or triggering their students of color.

Moving from Deficit to Asset Perspectives of our Teachers

The data were a reminder that all of our students have backgrounds, knowledge, and experiences that they can draw on in meaningful ways during activities with the cards. As the preservice teachers worked with the cards, they referenced what they knew based on their experiences in schools. For example, Norah mentioned experience from her field placement school where students were making predictions based on the cover of a book. On the cover, a group of children were standing around another child holding a cap. As the students in the field placement classroom made predictions, one of the students predicted that the two Black children on the cover were brother and sister because they had the same skin tone. Norah used this as an opportunity to begin a conversation in the classroom. Several other preservice teachers related the scenarios on the cards to concepts they had discussed and explored in other coursework, such as their social studies methods course, educational policy courses, literacy course, among others. They weren't required to make connections to what they knew or had learned in their teacher preparation courses but many of them did.

Discussion and Next Steps

Working Groups and professional conference sessions extended our methodology, in which participants learned about, received, and were challenged to use an early iteration of the CARDS. These opportunities to work across cultures (mathematics educators, teachers, parents, other stakeholders) served as an impetus for multiple MTEs to implement the cards in their own institutions. The feedback received has helped make improvements related to content, context, form, and delivery. Content and context details have been added along with the addition of new scenarios. Variations of the cards are being developed to be used with different audiences (teacher candidates, teachers, administrators, etc.) The form of the cards is being modified to become more user-friendly and incorporate additional resources and technology. Different delivery options (online, game formats, etc.) are also being explored and will be included in a user guide. We did not "construct" the cards and then evaluate them, rather they were a result of emergent design research (Tom, 1996). In other words, and an ongoing effort of collectively collaborating to create/design tools that come out from the larger community.

As a next step, we reiterate our invitation and our plea that our colleagues (and by that we mean anyone who plays a role in mathematics teacher education) continue to join in the shared creation and improvement of the cards. We will continue to share the cards, user guide, and other relevant materials freely- not only to download, but to add to and to suggest revisions. These materials are licensed under creative commons copyright law, so as long as users are doing so with attribution and for educational, non-commercial purposes, they are free to do so. This is what we mean by collective ownership.

In fact, we consider the collective ownership of the cards to be one of, if not the, most significant findings of this project thus far. Our field is dominated by scholars who work in higher education institutions, which are organized by a capitalist framing of knowledge creation and ownership. We recognize that we ourselves are a part of this problematic structure which inhibits true collaboration or idea sharing, in which individual knowledge creation and selfish ways of thinking are rewarded and incentivized. We seek to enact a decolonizing stance to educational research (Patel, 2015) to think about ways to create true collaboration in our field and to enact research designs that honor all voices. In this project, we have offered a glimpse of what this might look like, through the ideation, creation, enactment, and study of a tool for mathematics teacher education that has been completely collaborative from start to finish. Even now, we do not present the CARDS project as a finished project, but as an evolving tool that will grow through collective action.

Turning our attention to use of the cards: there are many concepts and frameworks regarding equitable mathematics teaching practices from which mathematics teachers can base their work with these CARDS. Paradigms and frameworks such as Guitierrez's *Four dimensions of equity,* Tuner, Drake, McDuffie, Aguirre, Bartell, and Foote's (2012) *learning trajectory for building on children's multiple mathematical knowledge bases* (Project TEACH MATH), and the *antiracist tool for mathematics teacher educators* (A3IMS Project) all offer foundations from which MTEs might regard the cards as resources. We do not intend to homogenize differences between these approaches, thereby trivializing the significance of theoretical framework and the intellectual work involved in articulating the affordances of any one perspective. We ourselves have been intrigued by Love's description of abolitionist teaching (2019), and the statements in our data expressed by Abdullah and Kat echo Love's call that students of color--in our case: our preservice and inservice teachers of color--need to know that they *matter*. They need to be repeatedly and resolutely assured that their communities, their families, their neighborhoods, their stories, their bodies, their hopes and their dreams *matter.* The Cards provide one mechanism to support our practice of inviting, eliciting, and honoring these multiple dimensions of our students' and teachers' lives into our interactions with them.

For our colleagues in mathematics teacher education who share our interest in developing deep, theoretical understanding alongside a skillful, practical usage of the cards, we invite researchers to develop and apply their various conceptual and analytic frameworks. As we have analyzed the data together, we have begun to wonder in what ways interactions with the cards reveal whiteness within our teachers, ourselves, and our systems and institutions, our practices and policies. Battey and Leyva's (2016) *framework for understanding whiteness in mathematics educatio*n offers one tool for this.

Interestingly, and perhaps counterintuitively: as a result of the efforts described here, we are learning more and more about what might be the most appropriate intentions for using these cards. When we began developing these resources, we admittedly did not have a clear objective. We were guided by curiosity, a sense that these would be useful, and a lot of encouragement and expressions of interest from colleagues. In some ways, these cards were developed antithetically to the "design" approach so ubiquitous in education research wherein problems of practice are identified and then answers are proposed, studied, and refined. Our process was more akin to an "undesign" approach (Leander & Boldt, 2018), wherein we were motivated by a "What can we do with these?" wondering about a curricular resource that we found ourselves compelled by. The "What can we do with this?" curricular approach has been adopted by mathematics teachers in productive and interesting ways for posing rich mathematical tasks and we are intrigued by what this stance could bring to mathematics teacher education.

Fundamentally, we do not view equity, diversity, and inclusion as "problems of practice" that need to be "solved:" rather we view them as the existential commitment most relevant and animating to our work as mathematics teacher educators--individually and collectively. We will never be finished, we will never get to cease remaining vigilant in our commitment to remaining watchful and attentive to harm that arises in mathematics teaching and learning as a result of abuses of power. Contributing toward more equitable mathematics classrooms is a practice: we will need to hold ourselves accountable and reaffirm our commitments again and again. These cards-and this approach to collaboratively developing and owning them--are one way to do this. They do not "hold" or "convey" the right answers because there are no such things, but a commitment to return to them again and again, always adding to our collective understanding, is a stance worth taking.

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BETWEEN THE BOUNDARIES OF KNOWLEDGE: THEORIZING AN ETIC-EMIC APPROACH TO MATHEMATICS EDUCATION

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Researchers and reformers across multiple areas of scholarship have challenged the idea of mathematics as fixed, politically neutral, and value-free. Ethnomathematics has brought attention to the mathematical practices of particular cultural groups that differ from Western ways of mathematical understanding. These practices raise the following question for mathematics education, especially within Indigenous communities: whose knowledge should be taught and from whose perspective? One response to this question is to teach both dominant and non-dominant perspectives on mathematics, which can be considered an "etic-emic" approach to mathematics education. Drawing on the literature on decolonizing studies in education, I offer a theorization of this etic-emic approach in terms of re-mythologizing mathematics, pursuing recognition and reconciliation, and refusing colonization.

Keywords: Ethnomathematics

The sociopolitical turn in mathematics education calls for an examination of the ways mathematics is framed, conceptualized, and presented in the curriculum (Gutiérrez, 2013a). For instance, ethnomathematics researchers have brought attention to the mathematical practices of particular cultural groups that differ from Western ways of understanding mathematics (D'Ambrosio, 1985; Barton, 1996). These mathematical practices, even when they are not explicitly labeled as ethnomathematical, can serve as important resources for mathematics educators seeking to draw connections between dominant and non-dominant forms of knowledge and challenge the notion that there is only one way to learn, understand, and do mathematics.

Ethnomathematics seeks to promote expanded views of what counts as mathematical activity. This raises the question for mathematics education, especially within Indigenous communities, of how to balance the perspectives created by and enacted through dominant and non-dominant mathematical practices. One response is for teachers to teach both dominant and non-dominant perspectives on mathematics, a practice that I will refer to as an etic-emic approach to mathematics education. In this paper, I seek to theorize this etic-emic approach and, in doing so, to highlight how historicized and ongoing effects of colonization make it difficult, if not impossible, to reconcile dominant and non-dominant ways of knowing. I begin with existing conceptions of and approaches to ethnomathematics in order to provide background on the various ways in which researchers have brought attention to and grappled with multiple systems of mathematical knowledge, particularly as the existence of these multiple systems implicate approaches to mathematics education. I proceed by reviewing the work of scholars who have proposed an etic-emic approach to mathematics education. Drawing on the literature on decolonizing studies in education, I conclude by theorizing this etic-emic approach in terms of re-mythologizing mathematics, pursuing recognition and reconciliation by Indigenous communities, and refusing colonization.

Ethnomathematics

Ethnomathematics began as an endeavor to identify and elaborate on the practices of cultural groups, particularly from the point of view of one immersed in scholarly mathematics (D'Ambrosio, 1985). Over the past four decades, ethnomathematics has branched out in several directions. Barton (1996) identifies a few of these directions, including an interest in the ways mathematics is culturally

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based, the nature of mathematical thought and activities across cultures, the evolution of mathematics from a socio-anthropological perspective, the politics of mathematics, the use of culturally specific contexts in schools, and the relationship between mathematics education and society. He proposes the following definition for ethnomathematics:

> “a research program of the way in which cultural groups understand, articulate and use the concepts and practices which we describe as mathematical, whether or not the cultural group has a concept of mathematics” (Barton, 1996, p. 214).

The term *mathematics* refers to the work of school and scholarly mathematics (e.g. algebra) and the term *mathematical* refers to concepts and practices identified as somehow related to mathematics (e.g. kinship systems that can be interpreted in terms of algebraic structures) (Barton, 1996). Barton (1996) points out that this definition of ethnomathematics is not absolute or definitive. The meanings of the terms are culturally situated and depend on the person or group using them. Ethnomathematics is a culturally specific practice performed by one cultural group seeking to make sense of another, often by reference to a specific conceptualization of mathematics (Barton, 1996).

Barton’s approach to define and frame ethnomathematics is joined by other perspectives. D’Ambrosio (2006) argues that ethnomathematics concerns the history and philosophy of mathematics with pedagogical implications, the goal of which is to develop a broader vision of knowledge by making cross-cultural comparisons of the ethnomathematics of different groups. Borba (1990), Gerdes (2005), and Powell and Frankenstein (1994) emphasize the importance of ethnomathematics for education, pointing out the ways mathematical practices of cultural groups can be brought into the classroom. Pais (2013) suggests a path for ethnomathematics that critiques its own directions and purposes, particularly those that would render ethnomathematics a mere pedagogical tool for importing cultural contexts into schools.

Pais (2011, 2013) and Vithal and Skovsmose (1997) raise concerns regarding ethnomathematics. When a mathematical lens is applied to a cultural practice, there is a risk this lens becomes a “gaze” that suggests a group’s cultural activity is valuable only because one can see mathematics in it (Pais, 2013, p. 3). This gaze also highlights the unidirectional nature of ethnomathematics, the alternative being that ethnomathematics can and should be directed back toward dominant mathematics through a critical examination of how mathematics has taken its current form and how it powerfully formats reality in ways that are often unjust (Pais, 2013).

In the context of mathematics education, a mathematics gaze focuses on bringing local knowledge into mathematics classrooms in the name of promoting diversity and highlighting that mathematics appears everywhere in the world (Pais, 2013). Although this is often accompanied by good intentions and the promises of multicultural education, there is a risk that an essentialist view of culture will be promoted that positions communities and peoples as foreign Others and ignores the tensions inherent in cultural approaches to education in multicultural contexts (Pais, 2011, 2013; Vithal & Skovsmose, 1997). For instance, in South Africa the meaning of *ethno* in *ethnomathematics* was used by policymakers to separate individuals into supposed cultural groups organized by race, wherein white students were provided a higher quality mathematics education (Vithal & Skovsmose, 1997). Ethnomathematics thus became associated with the racism of apartheid. Vithal and Skovsmose (1997) point out that in using ethnomathematics to structure the learning experiences of students in South Africa, there was a failure to specify the relationship between culture and power and a failure to recognize the formatting power of dominant mathematics and to teach toward cultural competence and self-empowerment.

Etic and Emic Perspectives

Albanese, Adamuz-Povedano, and Bracho-López (2017) propose two distinct approaches to incorporating ethnomathematics into mathematics education. Under the first approach, the mathematics of cultural groups are studied, understood, and taught from the point of view of dominant mathematics. The researcher, teacher educator, or teacher identifies and chooses cultural material to translate into the formal language of mathematics, even if this formal language does not exist within the studied community (Albanese et al., 2017). For example, Ascher and Ascher (1986) analyzed kin relationships found among the Aranda of Australia using diagrams and group theoretical terms. Borrowing terminology from anthropology (Rosa & Orey, 2012, 2013), Albanese et al. (2017) call this an *etic* perspective. In short, the etic approach is "the *recognition* of *mathematics in cultural practices*" (Albanese et al., 2017, p. 324). The goal of the etic approach is to build bridges between dominant mathematics and the mathematical practices of other cultures and to establish communication between them (Albanese et al., 2017). It suggests that dominant mathematics is a universal system that can be found everywhere, including within the cultural practices of communities that would not necessarily characterize these practices as mathematical (Pais, 2013). The pedagogical implication is that teachers ought to bring cultural contexts into the classroom under the assumption that students have experiences with or interests in out-of-school mathematical practices and that relating school mathematics to students' life experiences will lead to better learning (Pais, 2013).

The second approach is *emic*, which takes into account the categories and schemes of thinking of the community or cultural group of interest (Albanese et al., 2017). This leads to "the *discovery* of different *ways of thinking*" (Albanese et al., 2017, p. 324). For instance, bricklayers in some rural areas of Mozambique build houses with rectangular floors but do not have tools for designing right angles (Albanese et al., 2017). They use sticks and ropes of equal measure to find the vertices of a rectangle. This practice may be identified as deploying the property of rectangles that diagonals are equal and bisect each other (i.e. an etic perspective) or may be identified as a practice-embedded, operational definition for these bricklayers (i.e. an emic perspective). Barton (1999) explains that this approach to ethnomathematics rejects the idea of a universal mathematical or logical system to which both scholarly and cultural mathematics are a mere approximation. Mathematics in its most general form is instead a system for dealing with quantitative, relational, and spatial aspects of human experience, which Barton (1999) calls a "QRS system" (p. 56). Certain cultural groups have their own QRS systems. Dominant mathematics is one QRS system, and the purpose of ethnomathematics is to explore how different QRS systems relate to one another. Rather than being used to locate cultural contexts to import into classrooms, the emic approach suggests that ethnomathematics should be incorporated into a larger project of critiquing schooling and the curriculum (Knijnik, 2012; Pais, 2011, 2013).

Albanese et al. (2017) argue that both etic and emic approaches should be considered in every ethnomathematics project, including the use of ethnomathematics for teaching and learning mathematics. For instance, artisan-architects on an Indonesian island use a stick and pencil to find the midpoint of a segment based on a sequence of moves that yield a better approximation with each iteration (Albanese et al., 2017). An emic perspective would acknowledge the situatedness of this practice while an etic perspective would view the practice in mathematical terms, observing the ways the practice resembles error reducing algorithms (Albanese et al., 2017). Problems arise when both perspectives are not brought into dialogue. Merely contextualizing mathematical tasks without reflecting on the nuances between dominant and non-dominant forms of mathematics misses out on the opportunity to think about different ways of knowing (Albanese et al., 2017). Focusing exclusively on cultural ways of knowing misses out on the opportunity to seek correspondences between dominant mathematics and the cultural practices of other communities (Albanese et al., 2017).

Theorizing an Etic-Emic Approach to Mathematics Education

The previous discussion highlights the diversity in how one might approach the study, understanding, framing, and teaching of mathematics, particularly in light of the findings of ethnomathematics researchers that multiple mathematical systems exist. This motivates a framework for making sense of how an understanding of multiple mathematics might impact ones approach to mathematics education. For Albanese et al. (2017), a reasonable response is to seek dialogue between etic and emic perspectives on mathematics knowledge. In doing so, they acknowledge the significant formatting power of dominant mathematics and its role in modern society while still embracing different ways of knowing mathematics. However, if we are to take this suggestion seriously, there is a need to theorize what exactly such an etic-emic approach would entail. The next section proposes three dimensions to this theorization: re-mythologizing mathematics, pursuing recognition and reconciliation for Indigenous communities, and refusing colonization. Through these dimensions, a theorization would propose to do three things—deepen our understandings of ethnomathematics and its role in mathematics education, emphasize the relevance of decolonizing studies to mathematics education research, and speak to broader conversations about balancing dominant and non-dominant perspectives on knowledge within school curricula and teachers' instructional practices.

Re-Mythologizing Mathematics

Scholars in fields as diverse as anthropology, sociology, and education have shown the many ways in which mathematics is neither universal nor politically neutral (Appelbaum, 1995; Borba, 1990; D'Ambrosio, 2006; Eglash, 1997; Ernest, 1998; Gerdes, 1998; Gutiérrez, 2013a; Hersh, 1999; Iseke-Barnes, 2000; Knijnik, 2012; Skovsmose, 2011). Modern conceptions of mathematics have been shown to be rooted in mathematics' alleged purity and close connections to technology and the natural sciences (Skovsmose, 2011). The theorems and objects of mathematics have been shown to be cultural products created through human activity (Ernest, 1998; Hersh, 1999). Bishop (1990) challenges the idea that dominant mathematics is value-free, pointing out that such mathematics is grounded in four values: rationalism, objectism, power and control, and progress and change. Related to this conception of mathematics is a Western-based hierarchy of rationality that privileges abstract thought as the highest form of intellect (Gutiérrez, 2013b). Drawing on Foucault and Wittgenstein, Knijnik (2012) argues that dominant mathematics "expels 'out of its margins'" different kinds of mathematics by constraining the circulation of divergent mathematical discourses (p. 97). In each case, a critique is made that not only seeks to challenge mathematics education but also seeks to challenge the status of mathematics itself.

An etic-emic approach to mathematics education can be seen as part of this larger project to re-right views of mathematics that perpetuate myths about its universality and political neutrality. Wagner and Herbel-Eisenmann (2009) call on scholars to "re-mythologize" mathematics by reconceptualizing it with human stories that are not traditionally part of dominant mathematics discourses. The purpose is not to discredit dominant mathematics nor is it to "de-mythologize" dominant mathematics in an attempt to render it powerless. It must be acknowledged that the "myth of mathematics" continues to powerfully position students, teachers, and practitioners (Wagner & Herbel-Eisenmann, 2009) and that dominant conceptions of mathematics inevitably impose themselves on interactions among doers of mathematics. Ethnomathematics has and can continue to be used as a counter-narrative and engine for re-storying a plural understanding of mathematics. By balancing etic and emic perspectives on mathematics knowledge, reformers can continue to dismantle notions that Western mathematics is the only legitimate mathematical system. The ability to shift from one mathematical system to another promotes the view that Western mathematics is simply one of a multitude of culturally based mathematical systems, where each system is grounded in human activity and a particular set of values. In teacher education contexts, this approach can be used to support mathematics teachers in undergoing the epistemological shifts that Knijnik (2012) and Pais (2011, 2013) say are necessary for widespread mathematics education reform.

Pursuing Recognition and Reconciliation

An etic-emic approach to mathematics education can also be seen in relation to broader politics of recognition and reconciliation for Indigenous peoples. The dominant status of Western mathematics means that throughout history, alternative conceptions of mathematics among Indigenous peoples have been, and continue to be, marginalized or subject to erasure. Bishop (1990), for instance, shows how through dominant conceptions of mathematics, Western explorers sought to replace Indigenous mathematics through regimes of trade, administration, and education, which mediated a process of cultural invasion by dominant methods of measurement and numerical procedures and by a value system grounded in rationalism. Takeuchi (2018) finds that hierarchies created by dominant conceptions of mathematics led Filipina mothers to undervalue their mathematics knowledge—particularly with respect to calculating international currency conversions—and involvement in school education for their children. As Takeuchi (2018) explains, "[P]arents' funds of knowledge...can be masked through school practices if only certain ways of knowing are treated as legitimate and valued" (p. 139).

This process of erasure of Indigenous ways of knowing is not a mere accident of history but rather one of historicized and ongoing colonization (Bernales & Powell, 2018; Iseke-Barnes, 2000; Stathopoulou & Appelbaum, 2016). Bernales & Powell (2018) point to the Programme for International Student Assessment (PISA) and the tendency to "unreflectively copy the developed countries curricula, reinforcing power structures in the societies" (p. 566) as well as the "hominization of curricula that OECD's PISA causes on national curricula" (p. 566). An etic-emic approach can be seen as serving the project of reversing erasure and pursuing recognition and dignity in the face of dispossession of land and knowledge (Stathopoulou & Appelbaum, 2016). In schools, this would mean modifying curricula and pedagogical practices so that instruction begins with students' out-of-school knowledge and worldview of local culture while still giving students access to dominant mathematical discourses. Doing so provides recognition and dignity to students whose contemporary and heritage practices are delegitimated and displaced by Western mathematical values and practices (Stathopoulou & Appelbaum, 2016), and it can promote social justice through the fundamental values of ethnomathematics, which include respect, solidarity, and cooperation with other cultural groups (D'Ambrosio, 2007). Developing fluency around both dominant and non-dominant mathematical forms confronts what Battiste (1998) calls the educational model of "cognitive imperialism" (p. 17), which comprises of "Eurocentric strategies that maintain their knowledge is universal, that it derives from standards of good that are universally appropriate, that the idea and ideals are so familiar they need not be questioned, and that all questions can be posed and resolved within it" (Stathopoulou & Appelbaum, 2016, p. 38). This approach not only gives students access to mathematical practices necessary for social mobility but it also aids toward a process of "reconciling [the] dignity of each person" in light of "legacies of centuries of privilege and power, cultural authority and school-based deligitimation practices" (Stathopoulou & Appelbaum, 2016, pp. 39-40).

This approach of balancing "insider" and "outsider" views of mathematics must be seen as part of a larger movement in education to provide marginalized and minoritized youth with both dominant and non-dominant knowledge. Ladson-Billings (1995), for instance, puts forth a framework of culturally relevant pedagogy (CRP), which calls for academic excellence, cultural competence, and sociopolitical consciousness for students. CRP entails the development of "literacy, numeracy, and technological, social, and political skills in order to be active participants in a democracy" (p. 160). Paris and Alim (2006) extend CRP by proposing culturally *sustaining* pedagogy as a way to emphasize the preservation of students' heritage and contemporary practices and to foreground the way students often enact their cultural identities in novel ways. Stathopoulou & Appelbaum (2016) call for a similarly expansive view of culture within ethnomathematics, which have historically been based on static colonialist categories of culture. In contrast to these static colonialist categories, an etic-emic approach emphasizes a *fluidity* not just in how mathematics can be viewed but also how people and knowledge can be granted dignity and recognition.

Refusing Colonization

In contrast to the politics of recognition, an etic-emic approach can be theorized in terms of a *politic of refusal* (Coulthard, 2014; Grande, 2018; McGranahan, 2016; Mignolo, 2011; Simpson, 2007; Tuck, 2009; Tuck & Yang, 2014). Grande (2018) describes this politic of refusal in terms of Indigenous sovereignty, noting that refusal is not about attaining recognition but rather about reconstructing culture and tradition in a way that "positively asserts Indigenous sovereignty and peoplehood" (p. 59). Drawing on scholars such as Glen Coulthard, Audra Simpson, Walter D. Mignolo, and Anibal Quijano, Grande (2018) theorizes refusal as "a stance or space for Indigenous subjects to limit access to what is knowable and to being known" (p. 59) and a form of "epistemic disobedience" (p. 59) that severs the link between Indigenous and Western understandings of knowledge. Two important points must be made about refusal. First, refusal is an alternative to recognition, which seeks reconciliation with the state—an idea that several critical Indigenous scholars criticize "as a technology of the state by which it maintains its power (as sole arbiter of recognition) and thus settler colonial relations" (Grande, 2018, pp. 49-50). Second, refusal is connected to settler colonialism, which refers to colonialism premised on the removal of Indigenous peoples from land followed by the creation of labor and knowledge systems and infrastructures to make the land productive for settlers (Bonds & Inwood, 2016; Grande, 2018). In this light, refusal is premised on the idea that decolonization "is a political project that begins and ends with land and its return", and thus "the very nature of settler colonialism precludes reconciliation" (Grande, 2018, p. 53).

A politics of refusal troubles the possibility of taking an etic-emic approach to mathematics education. By seeking to recognize and reconcile both insider and outsider perspectives of Indigenous mathematical practices, one continues to legitimate the Western "gaze" as the arbiter of recognition. That is, attempts to reconcile dominant and non-dominant perspectives on mathematics reproduce configurations of colonial power that have and continue to deprive Indigenous people of knowledge. Raising the issue of *psycho-affective attachment* to colonialist forms of recognition, Grande (2018) discusses the "unequal exchange of institutionalized and interpersonal patterns of recognition between the colonial society and the marginalized" (p. 54). By seeking recognition and reconciliation, one may not be able to avoid the feelings of attachment to dominant knowledge forms felt among the colonized, as such feelings often stem from "inducements" (Wolfe, 2013), which manifest as the material and psychological rewards often associated with success in dominant mathematics.

This is not to say that Indigenous communities should not be taught dominant forms of mathematics knowledge, which can be instrumental for social and economic mobility. Rather, a politics of refusal highlights the competing impulses that can arise when one attempts to take an etic-emic approach to mathematics education—on one hand, a yearning to reconcile dominant and non-dominant ways of knowing but, on the other hand, a refusal of Western mathematics premised on Indigenous sovereignty and the delegitimization of the settler colonial state. For instance, Stanton (1994) describes a "both ways" mathematics curriculum for aboriginal teacher education, pointing out the extent to which dominant conceptions of mathematics were entrenched within the beliefs and attitudes of participants. He further describes the tensions created when teachers expressed the need for Aboriginal children to become prepared for key positions within their community through the mastery of dominant mathematical techniques. Stanton (1994) is ultimately optimistic about cross-cultural attempts to bridge dominant and non-dominant forms of mathematics, though his concerns highlight the difficulty that historicized and ongoing effects of colonization create with respect to reconciling etic and emic perspectives on mathematics.

Concluding Remarks

There has been a significant increase in attention toward concerns for equity and social justice within mathematics education, and yet ethnomathematics and decolonizing studies in mathematics education remain niche areas of research. This is despite the fact that dominant mathematical activity is a form of ethnomathematics. I discuss the project of balancing etic and emic perspectives on mathematics not only because it represents a key tension within ethnomathematics and decolonizing studies in mathematics education but also because it highlights how these research areas can speak to larger conversations about the role of dominant and non-dominant ways of knowing in curriculum and instruction. In this paper, I have raised the question of whose knowledge should be taught and from whose perspective. Although a reasonable response to this question might be to say, "everyone's knowledge and everybody's perspective", I have sought to nuance and problematize such an attempt at an etic-emic approach to education. The historicized context of math-making cannot be separated from the mathematical practices we seek to teach in classrooms. We cannot avoid the fact that much of the mathematics knowledge we seek to teach youth is laden with histories of settler colonialism, racial violence, and white supremacy. How, then, do we move forward? This paper's theorization does not offer a definitive resolution. At best, I offer this theorization to highlight the relevance of ethnomathematics and decolonizing studies for mathematics education research and to urge researchers to continue to critique dominant and oppressive forms of knowledge as part of a larger project of individual and collective empowerment.

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MEASURING HIGH SCHOOL STUDENTS' FUNDS OF KNOWLEDGE FOR LEARNING MATHEMATICS

MEDICIÓN DE LOS FONDOS DE CONOCIMIENTO DE LOS ESTUDIANTES DE SECUNDARIA AL APRENDER MATEMÁTICAS

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Mathematics experienced by students can be derived from the contextually situated "real world" experiences of the educator, which is typically White and middle class and not a reflection of the demographics of many classrooms in the United States. Activities where students find connections to their lives and interests have shown promise in enhancing student performance and experiences in mathematics classrooms. In this study, mathematics funds of knowledge are assessed in a novel survey instrument, reinforcing the salience of relating math experiences to students' lives and acknowledging skills and knowledge originating from experiences outside of the math classroom.

Keywords: Culturally Relevant Pedagogy, Equity and Diversity, High School Education

Many believe that there is no culture in mathematics. This would mean that arithmetic is the same no matter who a person is or where a person is physically in the world. However, the "real" in "real world" problems being solved is different for each student. Developers pull from their personal world but may not consider the perspective of their audiences. As such, many math word problems tend to be written from a white, middle class perspective (e.g., see Gerofsky, 2009; Frankenstein, 2009).

Expectations and decision-making of teachers are grounded in their cultural systems that may not align with those of their students. As teacher cultural expression is displayed and enacted through classroom practices, the lack of acknowledgement of those expressions and how they affect students can be problematic. The knowledge and awareness of how one's beliefs, experiences, values, and expectations are linked to their cultural identity is a major tenet of culturally responsive teaching (Griner & Stewart, 2013).

The racial gap between teachers and students in the United States continues to widen as the rate of diversity in teachers is declining more slowly than the rate of enrollment of students of color (NCES, 2017). According to NCES, White teachers make up 80% of teachers while over half of US students are Black, Latinx, Asian, Pacific Islander, Native American/Alaska Native or other races/ethnicities. This disparity in teacher-student ethnic demographics, structural barriers in advanced mathematics courses (U.S. Department of Education, 2018), lower expectations for African-American students (Gershenson & Papageorge, 2016), and racist policies for students being recommended for gifted courses (Ford, 1998), perpetuates the idea that there is no place for Students of Color in these classrooms.

Because of the opportunity gap that exists for students of color, math performance and preparation for college pathways in STEM for these students have become a matter of high interest and importance to teachers, schools and leaders in education policy and reform (Contreras, 2011; Gullatt, 2003; Noble, 2013; Marzocchi, 2016; Balfanz, 2006; Kotok, 2017). In this paper, we describe a research study conducted with high school students who were predominantly Latinx and African-American and were poised to become first-generation college students. Using a survey instrument we developed, we assessed students' everyday funds of knowledge for learning mathematics. The purpose of developing this assessment was to drive further instruction and enrichment in mathematics for these students. Here we report on the properties of the survey instrument, in the

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hopes that it could be used by teachers and researchers to assess their students' funds of knowledge and enact culturally responsive teaching.

Theoretical Framework

Culturally Responsive Teaching

According to Ladson-Billings (1995), Culturally Responsive Teaching (CRT) includes (a) students having the opportunity for collective success, (b) students developing or maintaining cultural competence, and (c) students developing a critical consciousness through which they challenge the status quo of the current social order. These things are not typically found in everyday lesson plans and many teachers require assistance to include these characteristics of CRT into their pedagogy. CRT involves teachers maintaining a high level of "cultural competence" in their pedagogy (Lindsey, 2009). These teachers are able to gather from the cultural experiences brought into their classroom via their students in their learning environment. CRT not only requires acknowledgement of student culture, but also the use of student culture as a learning tool in the classroom. The culture of the classroom transcends ethnic backgrounds, but also includes sexual orientation, disabilities, religion and language (Rogers, 2016), among other identifications and groups. Culture also includes the community inside and outside of school.

Funds of Knowledge

Funds of Knowledge (FoK) are the everyday knowledge bases from which students experience and learn within their homes and communities (Gonzalez, et al., 2005). Posited from the works of Velez-Ibanez and Greenberg (1992), the concept of FoK conveyed the notion of the existence of skills, talents, aptitudes, and inter-cultural exchange within Mexican-American homes. Moll et all. (1990b) focused on FoK originating solely from the household, but updated this opinion later (2005) crediting that FoK must incorporate information accumulated outside of the household in varied settings and activities. Moje et al. (2004) found that students' FoK derived from a combination of the home, peers and other networks. Andrews and Yee (2006) concurred that FoK stems from students' lived lives including their personal interests and influences. These personal interests serve as a source of knowledge not only useful for the wellbeing of the household, but of the student as well. Barton and Tan (2009) actually describe FoK as deriving from students' interests and talents.

Although it has been shown that Latinx households demonstrate strength in the complexities associated with sharing recourses and social networks within their community (Velez-Ibanez, 1988), educators have historically not used these FoK as a resource inside the classroom. Taking educators out of the classroom and into the homes and communities of their students has been highlighted in the works of Gonzalez, Moll, and Amanti (2005). In this study, teachers became researchers seeking and discovering the household knowledge of students developing rich relationships which set the stage for transactions of knowledge between teacher and student targeting student interest. FoK were found to focus on activities and tactical knowledge deriving out of culture (i.e., social, economic, political) essential to household functioning and progression. The involvement of student background into daily lessons and teacher pedagogy requires teachers to take an authentic inventory of their students and those students' cultural experiences to truly direct their pedagogy with a culturally responsive lens.

FoK emphasizes the salience of both academic and personal background knowledge of students. A culturally responsive educator focuses and utilizes this accumulated lived experience and knowledge to build upon it, increasing student learning. The FoK used to navigate social contexts are affirmed in culturally responsive lessons when properly addressed and utilized. As educators facilitate culturally responsive opportunities for students to broaden their FoK including their world views shaped by cultural, historical and political events, students' sociopolitical and critical consciousness are influenced to critique the inequities in their own educational and societal establishments.

Funds of Knowledge for Mathematics

Funds of Knowledge in mathematics and connections between home and community involvement has been the subject of considerable research (Civil, 2007; Gonzalez, Andrade, Civil & Moll, 2001; Civil & Andrade, 2002; Nasir, 2002; Nasir, et al., 2008). The work of Civil (2007) on mathematical FoK focused on everyday life activities and how they connect to mathematics. Civil's work highlighted success in using FoK in the development of mathematics learning objectives through an educator's authentic desire to learn about their students' community, and to understand and leverage the community's resources and the knowledge originating from students' households. As opposed to making cultural generalizations about the community, teachers took invested interest in learning about their students' home and community lived experiences. Certain dilemmas are addressed in Civil's work, such as the tension between authentic problem-solving opportunities that relate to home and community experiences and dealing with socio-mathematical norms in the classroom where students are conditioned that mathematics "work" involves worksheets inside the classroom.

Work on FoK in mathematics has also been connected to research on personalizing instruction. Personalizing students' mathematics learning by drawing upon their FoK can affect student interest and performance in mathematics, as demonstrated in Walkington and Bernacki (2015). In their article, Walkington and Bernacki had students pose mathematical problems based upon their out-of-school interests in areas like sports or video games, harnessing their FoK to develop more meaningful connections to mathematical concepts. Walkington and Hayata (2017) also describe a series of teaching experiments where students posed, solved, and shared algebra problems related to their out-of-school interests.

Algebraic story problems can be presented to students as a method of contextualizing life experience to confront inequalities. In Turner, et. al., (2016), using students' FoK, an educator gave students opportunities to discuss mathematical situations where injustices they have experienced in their lives that could be represented as inequalities. Prompting students within the context of injustices with inequalities, students gained deeper understanding of constructing these types of equations and contrasting them against others.

While there is a large body of work that support the use of FoK within an educational context, there are also critiques of this practice. Zipin (2009) notes the absence of what he calls "dark" or negative pedagogies including abuse of others and substances, mental health issues and alcoholism. Here, the idea is presented that educators wish to only focus on "light" or positive FoK as opposed to the whole lived experience of students, light and dark, which may seem troubling for educators to consider in the classroom (Zipin, 2009). Although dark FoK may challenge traditional approaches by educators, it provides a rich and authentic source of knowledge from which students could draw. In addition, knowledge as metaphorical capital has been criticized for being incomparable to financial capital and has been framed as an inappropriate connection to the negative economic and political dominance of capitalism (Hinton, 2015). Oughton (2010) also points out the theory of funds of knowledge literature has morphed from deriving from household knowledge into various sources of knowledge which may be influenced by what the researcher has determined to be FOK. However, the power of individual, parental and educator agency as potential propelling forces behind the acquisition of knowledge and increases of performance in students is highlighted in by Rodriguez (2013).

Research Purpose

Very few prior studies have focused on quantifying FoK. In one such study (Rios-Aguilar, 2010), 212 K-12 Latinx students took the Latino/Hispanic Household Survey where connections to FOK were made to both academic and non-academic outcomes focused in reading and literacy. Rios-Aguilar (2010) found that activities and experiences in Latinx households contributed to both academic and non-academic outcomes for these students.

Here, our purpose was to expand this work and consider specifically what knowledge bases students might have that are relevant to learning mathematics. Our first research questions is: (1) How are different areas of FOK in mathematics related to each other? Our remaining research questions are: How do measures of students' FOK for mathematics (both overall and in individual areas) predict: (2) Student interest in mathematics? (3) Math grades? (4) AP math course-taking? (5) Dual Credit math course-taking? (6) Desire to pursue a career in STEM?

Method

Participants

This study included students participating in an educational program outside of school hours. These students are from various high schools in a large, urban, US Southwestern school district. Students' ages ranged from 14 to 17 years old. All students are low socioeconomic status and most are Black and Latinx. Similar to other US urban school districts, the one the students from this study come from serves over 150,000 students where 87% of the community is economically disadvantaged and 44% are limited in English proficiency. Students in the district were 69.6% Latinx, 22.5% African-American, 5.4% White, 1.4% Asian-American, 0.3% Native American, and 0.7% two or more races.

There were 72 students participating in the study with 49 (68%) female students and 23 (32%) male students. Students' self-reported grades in their prior math class were 24 (33%) with As, 41 (57%) with Bs, 6 (8%) with Cs, and 1 (1%) Ds. Seniors made up 14% (10) of total students while 44% (32) were Juniors, 24% (17) were Sophomores and 18% (13) were Freshman. Students' current high school math class included 10 (14%) in Algebra 1, 22 (30%) in Algebra 2, 3 (4%) in AP Calculus, 4 (6%) in College Algebra, 4 (6%) in AP Statistics, 12 (17%) in Geometry, 1 (1%) in Math Models, 13 (18%) in Precalculus, 1 (1%) in Trigonometry and 2 (3%) students that did not answer. Sixty-five percent of students indicated interest in pursuing STEM major in college and 68% of students were interested in STEM careers.

Students completed the survey on their phones. There were no incentives for the students to complete the survey. All included participants gave assent along with having parental consent.

Measures

This study used the researcher-created Mathematics Funds of Knowledge Survey (MFoKS) to assess students' level of FOK. The MFoKS is used to quantify students' level of Funds of Knowledge in 9 "bins" including: Money, Travel, Sports/Fitness, Social Media, Video Games, Cooking, Health, Art, and Directions. The survey is a 70 item mixed-model of qualitative (9) and quantitative questions (61). Each bin had 3-11 items. The survey includes a 5-point Likert scale including 1 for "almost never," 2 for "at least once a year," 3 for "at least once a month," 4 for "at least once a week," and 5 for "almost daily." The survey calculates FoK as a frequency of usage in each of the bin which will generate a score for each bin. Some examples of items from the MFoKS include: "How often do you check how many retweets or shares a post [on social media] has gotten?" and "How often do you pay attention to how many plays, spins or streams a song has?" and "How often do you consider different shipping rates when shopping online?"

The overarching survey that students in the educational program took also included questions about students' interest in mathematics using a scale from Renninger and Schofield (2014) with 24 Likert questions ranging from 1 to 5. Students had an average score of 2.89 in math interest.

Data Analysis

To answer the first research question, a correlation matrix was generated to determine which categories were highly correlated or not correlated. The remaining research questions were answered using regression models. The *lm()* command was used in RStudio. The outcomes were average math interest (rated on a 1-5 scale), grade in their last math course (4=A, 3=B, 2=C, 1=D), AP Track (i.e., a 0/1 variable denoting whether the student was in AP math classes), Dual Credit (i.e., a 0/1 variable denoting whether the student was in Dual Credit classes) and STEM Major interest (i.e., a 0/1 variable denoting whether the student was interested in majoring in STEM). Math interest and last math grade were fit as linear outcomes. The *glm()* command was used for AP Track, Dual Credit, and STEM Major Interest, as these were 0/1 variables - we used the binomial family and logistic regression. Predictors in the models were average 1 – 5 Likert scale ratings of the 9 areas in funds of knowledge. Control variables (not shown in table for brevity) were what math course the student was currently enrolled in and their gender. The Dual Credit model also included an additional control for what year in high school the student was currently enrolled, as dual credit opportunities are usually for older students.

Results

Students rated their FOK for mathematics highest for cooking, with an average rating of 3.56. The lowest FOK rating across student was for Money with an average rating of 2.46.

With respect to our first research question, when examining how students' ratings of the 9 different areas of FOK were related (Table 1), none of the correlations were over 0.6, which suggests that all 9 of the areas of funds of knowledge surveyed may be distinct from one another. It was also found that there were higher correlations between money and travel, health and art, and cooking with travel, sports, health, and art. FOK areas found to have low correlations with each other were money and art, and video games with travel, sports, social media, health, cooking and distance, with the lowest correlation found between video games and travel.

For the second research question, we found that in the regression models (Table 2) math interest was significantly positively predicted by certain areas of FOK. The FOK areas showing significance include money (p = .004), travel, (p < .0001), cooking (p = .019), distance, (p = .034), and, overall mathematics FOK (i.e., a composite average from the 9 areas; p = .005). For the third research question, it was found in the regression models that travel showed significance in positively predicting math grade (p = .04). The fourth research question addressed predicting the AP Track of students, and it was found that travel (e.g., recognizing the use of numbers and distances while using various modes of transportation) showed significance as a negative predictor (p = .023). For the fifth research question, it was found in the regression models that social media showed significance in negatively predicting Dual Credit enrollment (p = 0.035). The sixth research question addressed STEM Career interest which was significantly and positively predicted by travel (p = .008), social media (p = .046), and overall mathematics funds of knowledge (p = .034).

Table 1: Means, standard deviations, and correlation matrix

FOK Variable	M	SD	1	2	3	4	5	6	7	8	9
1. Money	2.46	.726	--								
2. Travel	2.55	1.080	.45	--							
3. Sports	3.16	1.216	.28	.35	--						
4. Social Media	2.92	1.212	.39	.26	.40	--					
5. Video Games	2.86	1.557	.30	.00	.02	.15	--				
6. Health	3.1	1.219	.23	.29	.43	.37	.18	--			
7. Art	3.08	1.215	.11	.22	.38	.39	.33	.50	--		

8. Cooking	3.56	1.260	.28	.54	.58	.46	.04	.56	.47	--	
9. Distance	3.38	1.367	.35	.39	.38	.39	.14	.42	.32	.34	--

Table 2: Regression Results

	Math Interest	Last Math Grade	AP Track	Dual Credit	STEM Career
Overall Math FOK	.37(.13)**	.05(.10)	-.25(.36)	-.10(-.37)	.81(.39)*
Money	.40(.14)**	.02(.11)	-.61(.40)	-.10(.40)	.67(.43)
Travel	.49(.08)***	.16(.07)*	-.64(.24)*	-.37(.30)	1.03(.39)**
Sports	.15(.09)	.01(.07)	-.21(.24)	-.20(.24)	.40(.24)
Social Media	.08(.09)	-.09(.07)	-.04(.23)	-.59(.28)*	.51(.25)*
Video Games	.07(.07)**	.01(.05)	-.16(.19)	.17(.19)	.23(.19)
Health	.14(.90)	-.02(.07)	-.04(.23)	-.19(.25)	.16(.24)
Art	.15(.09)	.07(.07)	.12(.24)	.12(.25)	.35(.24)
Cooking	.19(.08)*	.10(.06)	-.20(.24)	-.17(.23)	.19(.22)
Distances	.21(.08)*	.07(.06)	-.11(.22)	.28(.24)	.31(.21)

Discussion and Conclusion

In the present study, we developed a quantitative measure of students' FOK for learning mathematics, and discussed the results of administering this measure to a sample of high school students. Our measure covered 9 facets of everyday funds of knowledge, which correlational results suggest were distinct areas – with some facets of FOK being more highly related to each other than others. Our next step will be to conduct a factor analysis and calculate internal consistency values for the survey instrument. We also found that students' FOK for mathematics were predictors of mathematics outcomes that we would care about for students – particularly interest in learning mathematics and their interest in careers in STEM related fields. This suggests potential directions for future work – leveraging students FOK in the classroom could enhance students' interest in learning mathematics, as could asking students to increasingly apply a mathematical lens to their everyday activity. Cultivating students' mathematics FOK could also influence career interest in STEM fields, an area underrepresented by Latinx and African-Americans (Funk & Parker, 2018). A well-designed survey instrument could help teachers assess their students' FOK to be used in targeting instruction; this paper represents the first step towards creating such an instrument.

The purpose of the MFoKS is to collect and quantify students' levels of FoK focused in mathematics for daily living. When teachers and administrators have access to student math FOK, they are able to utilize student interests in personalized lesson plans and identify strengths of their students for participation and opportunities. Students' FoK data counters the narrative about deficits students have with experience with mathematics and numbers outside of the classroom. Lastly, the MFoKS can provide administrators and community leaders with the blueprints for establishing programs to build math FoK in the community.

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PREDICTING THE MATHEMATICS PATHWAYS OF ENGLISH LANGUAGE LEARNERS: A MULTILEVEL ANALYSIS

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This study employed hierarchical linear modeling to investigate the student- and school-level factors associated with the secondary mathematics achievement of English language learners (ELLs) and non-ELL students among a nationally representative sample of ninth graders in the United States. While certain characteristics, such as socioeconomic status, attitudes and interest in mathematics, and school engagement and belonging were predictive of access to and achievement in mathematics for both student groups, the direction and relative magnitude of the predictors differed. School-level variables, such as whether the school was public or private and administrator perceptions of school climate, were only predictive of mathematics grade point average (GPA) for non-ELLs. Implications of the findings are discussed.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Equity and Diversity; High School Education

English language learners (ELLs) are among one of the fastest growing groups of students in the United States. This group of students vary considerably in terms of English language proficiency, educational experiences, and many other factors (National Center for Education Statistics, 2004; Ryan, 2013). Adapting to these shifting demographics has proven challenging for educators, who must contend with difficulties in the fair identification of ELLs (Carlson & Knowles, 2016), simultaneous attention to language and content (Janzen, 2008), and development of appropriate assessments (Abedi et al., 2005; Bailey & Carroll, 2015).

These issues become increasingly urgent in the face of evidence that ELLs continue to face limitations in access to multiple educational outcomes. ELLs are less likely to graduate high school (National Center for Education Statistics, 2004) and some researchers have found that ELLs encountered more restricted access to college preparatory courses and postsecondary planning (Callahan & Shifrer, 2016; Kanno & Cromley, 2013, 2015). The lack of access to resources may lead ELL students to conclude that academic success is not be for them (Kanno & Kangas, 2014; Menken & Kleyn, 2010). Such beliefs are not reflective of the possibilities of public education, and as educators it remains our duty to address these discrepancies in opportunities to learn.

Note that the term "English language learner" or "ELL student" is used throughout the text to align with the phrasing used by the federal data analyzed in the study, and the language that continues to be used across many policy documents. The author more strongly recommends use of the term "emergent bilingual" to refer to linguistically diverse students, as it demonstrates a greater respect for the student, their existing knowledge of a home language, and their emergent language skills in other languages.

Purpose of the Study

Much past research has focused on general educational outcomes of ELL students, with more attention paid recently to ELL students' progress through secondary mathematics course-taking (e.g. Thompson, 2017). Special attention to ELL students progression through secondary mathematics is key for several reasons. Mathematics courses are required for graduation, and almost always serve as gatekeepers for postsecondary access (Adelman, 2006). While there remains a significant language factor involved in the study of mathematics (Schleppegrell, 2007), these courses retain an important

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place in the majority of students' access to postsecondary access, secondary graduation, and future access to a wide range of STEM fields.

Given the disadvantages facing ELL students, as well as the importance of mathematics for future academic success and attainment, it is critical to understand how the mathematical progress of ELLs in high school differs from that of their English-proficient peers, and whether factors predictive of success differ between the two. The primary research question was the following: What student- and school-level factors are significantly related to the secondary mathematics attainment of ELL and non-ELL students, and to what extent do key factors differ between the two groups?

Theoretical Framework

The present study was framed using Bronfenbrenner's ecological model of human development (Bronfenbrenner, 1976, 1977). Bronfenbrenner proposed that human development evolved through increasingly complex interactions between the organism and its ecological environment. The environment was conceptualized as multiple nested levels of influence, where levels ranged from the environment surrounding the individual (microsystem), up to relationships with others inside and outside the environment (mesosystem, exosystem), and expanded to include how society impacts the individual (macrosystem) and how influences change over time (chronosystem). In education, this model posits that students' learning is closely connected to the learning environment (classroom,) and shaped by interactions with actors in the environment, such as students, teachers, and administrators (Bronfenbrenner, 1976).

The microsystem was of primary interest to the present research, defined as "the complex of relations between the developing person and environment in an immediate setting containing that person" (Bronfenbrenner, 1977, p. 514). This level reflects the initial point of interaction between the individual and the environment. Variables at this level include perceptions of the environment and roles adopted within that environment, as well as relationships between student, teachers, and peers. Bronfenbrenner's model was used to select variables likely to affect mathematics outcomes in secondary school. Students' development in the mathematics classroom may be impacted by factors at multiple levels, ranging from psychological or cognitive factors to social or cultural. The present research drew on the ecological model to select variables that may impact students' mathematics attainment primarily in the microsystem.

Method

This study used two-level hierarchical linear modeling (HLM) to analyze restricted-use from the High School Longitudinal Study of 2009 (HSLS: 09). The HSLS: 09 consisted of data from a nationally representative group of approximately 24,000 students from over 900 schools. The study collected data beginning in 2009, with follow-ups in 2011, 2012, 2013, and remains ongoing. Analyses in this study were restricted to students with data regarding their status as an ELL student prior to 9^{th} grade, and weighted using NCES-provided weights to account for non-response bias in the sample from 2009 through 2013. This restriction resulted in an analytic weighted sample of 3,220,965 students, with 124,042 ELLs prior to 9^{th} grade nested within 91 schools and 2,930,349 non-ELLs nested within 920 schools.

Variables

A total of 13 student-level variables and 9 school-level variables were selected for study. Student-level variables included factors such as socioeconomic status (SES), race/ethnicity, affective characteristics, and school engagement and belonging. School-level variables included factors such as whether the school was public or private, school climate, and percentage of ELL students enrolled. The outcomes of interest were the number of mathematics credits earned and mathematics grade point average (GPA). These two measures were chosen as a proxy for the number of mathematics courses completed, as well as an average success in those courses.

Data Analysis

Analysis began with the unconditional model containing only one outcome variable and no independent variables. The unconditional model allows for examination of whether the school grouping variable has a significant impact on student-level scores. Next, full models were developed that introduced both student- and school-level variables to the unconditional model. First, to examine the absolute effects of the independent variables, each student- and school-level variable was tested individually. Statistically significant variables ($p < 0.01$) were then introduced together to examine relative effects. To achieve parsimony, variables that were no longer statistically significant were removed one by one, beginning with the variable with the largest p value and proceeding until all remaining variables were statistically significant. The resulting models reduced the overall complexity of the final model, and allowed us to focus only on those variables that had both absolute and relative effects on the outcomes of interest. The same procedures were carried out separately for both ELL and non-ELL students.

Results

Descriptive Statistics

Examination of the descriptive statistics in Table 1 indicated that students who had been previously classified as ELL demonstrated lower number of mathematics credits earned and lower mathematics GPA. There was a significant difference in scores for both mathematics credits earned and mathematics GPA ($p<0.001$). Previously ELL students also had significantly lower SES and standardized mathematics assessment scores.

Table 1: Descriptive Statistics for the HSLS Student Sample

	ELL Previously		Never ELL	
	M	SD	M	SD
Mathematics credits earned	3.50	1.26	3.64	1.16
Mathematics GPA	2.12	0.92	2.38	0.94
Socioeconomic status	-0.58	0.73	0.03	0.79
9^{th} grade standardized score	-0.23	1.33	0.06	1.08
Gender				
Male	0.48	--	0.51	--
Female	0.52	--	0.49	--
Race/Ethnicity [a]				
Asian	0.06	--	0.03	--
Black/African-American	0.03	--	0.13	--
Hispanic	0.82	--	0.17	--
White	0.09	--	0.58	--
School-Level Variables				
Public	0.97	--	0.92	--
Catholic or Other Private	0.03	--	0.80	--
City	0.46	--	0.30	--
Suburb	0.33	--	0.34	--

Town	0.06	--	0.12	--
Rural	0.15	--	0.24	--

[a] Native Hawaiian/Pacific Islander, American Indian/Alaska Native, and More than one race categories are excluded due to low proportions relative to the entire sample size (<0.01)

Unconditional Models

Table 2 presents the unconditional models for both mathematics credits and mathematics GPA models for ELL and non-ELL students. Calculation of the ICC of each model proceeded by dividing the between-school variance by the total variance. The ICC ranged between 28% and 55%, suggesting that a significant proportion of variance of the outcome measure was at the school-level. This implied a multilevel nature of the data and justified further use of HLM.

Table 2: Comparison of Unconditional Models

	ELLs Credits		Non-ELLs Credits		ELLs GPA		Non-ELLs GPA	
	Estimate	SE	Estimate	SE	Estimate	SE	Estimate	SE
Fixed Effects								
Intercept	3.485**	0.102	3.667**	0.023	2.147**	0.067	2.420**	0.016
Random Effects								
Intercept variance	0.950**	0.142	0.477**	0.022	0.403**	0.060	0.249**	0.012
Level-1 variance	0.778**	0.003	0.937**	0.001	0.421**	0.002	0.654**	0.001
ICC	0.550		0.337		0.489		0.276	
Deviance	321,455.480		7,909,070.128		245,291.578		6,847,178.614	
# Parameters	3		3		3		3	

* p < 0.01, ** p < 0.001

Predictors for ELL Students

Table 3 presents the final models for ELL students. At the student-level, 10 out of the 12 variables were significant. SES, mathematics utility, and school engagement were all strong predictors of increases in credits earned. Mathematics assessment score and school belonging were also positively related, although smaller in magnitude. Alternatively, both self-efficacy and interest in 2009 mathematics course were negatively related to credits earned.

Regarding mathematics GPA, 10 out of the 12 student-level variables were significant. Mathematics assessment score, SES, mathematics identity, self-efficacy, and school engagement were all positively related, while interest and school belonging were negatively related.

Of the 7 school-level variables examined, none were statistically significant predictors of either credits earned or GPA for ELL students.

Table 3: Final Model Predicting Outcomes of Interest for ELL Students

	Credits		GPA	
	Estimate	SE	Estimate	SE
Fixed Effects				
Intercept	3.043**	0.097	2.169**	0.059
Student Level				
Gender (0 = Male)	--	--	0.058**	0.009

Hispanic (0 = Yes)	0.545**	0.012	--	--
Black (0 = Yes)	1.088**	0.022	0.132**	0.013
Asian (0 = Yes)	0.912**	0.016	0.584**	0.009
Socioeconomic status	0.214**	0.005	0.152**	0.003
Mathematics assessment score	0.080**	0.004	0.219**	0.003
Mathematics identity	--	--	0.149**	0.003
Mathematics self-efficacy	-0.050**	0.004	0.162**	0.003
Mathematics utility	0.161**	0.004	--	--
Interest in 2009 math course	-0.047**	0.004	-0.051**	0.003
School engagement	0.184**	0.003	0.168**	0.002
School belonging	0.043**	0.004	-0.108**	0.003
Random Effects				
Intercept variance	0.824**	0.125	0.307**	0.047
Level-1 variance	0.589**	0.003	0.242**	0.001
Intraclass correlation	0.583		0.559	
Deviance	230,281.809		143,170.007	

* $p < 0.01$, ** $p < 0.001$

Predictors for Non-ELL Students

Table 4 presents the final models for non-ELL students. At the student-level, 10 out of the 12 variables were statistically significant. Specifically, increases in SES, assessment score, identity, self-efficacy, belonging, engagement, and interest were all significantly and positively related to mathematics credits earned.

Regarding mathematics GPA, 11 out of the 12 student-level variables were significant. Increases in SES, assessment score, identity, self-efficacy, engagement, interest, and belonging were positively related to GPA, while mathematical utility was negatively related.

Of the 7 school-level variables examined, none were significant predictors of credits earned for non-ELL students. However, 2 of the 7 school-level variables were significant for non-ELL students, with private schools and positive school climates related to increases in GPA.

Table 4: Final Model Predicting Outcomes of Interest for Non-ELL Students

	Credits		GPA	
	Estimate	*SE*	*Estimate*	*SE*
Fixed Effects				
Intercept	*3.598***	*0.023*	*2.232***	*0.018*
Student Level				
Gender (0 = Male)	*0.079***	*0.001*	*0.278***	*0.001*
Hispanic (0 = Yes)	*-0.071***	*0.002*	*-0.095***	*0.002*
Black (0 = Yes)	*0.130***	*0.001*	*-0.117***	*0.002*
Socioeconomic status	*0.162***	*0.001*	*0.155***	*0.001*
Mathematics assessment score	*0.138***	*0.001*	*0.371***	*0.001*
Mathematics identity	*0.043***	*0.001*	*0.105***	*0.001*
Mathematics self-efficacy	*0.058***	*0.001*	*0.108***	*0.001*
Mathematics utility	--	--	*-0.054***	*0.001*
Interest in 2009 math course	*0.016***	*0.001*	*0.021***	*0.001*
School engagement	*0.089***	*0.001*	*0.107***	*0.001*
School belonging	*0.052***	*0.001*	*0.032***	*0.001*
School Level				

School type (0 = Public)	--	--	*0.120**	*0.041*
School climate	--	--	*0.057***	*0.016*
Random Effects				
Intercept variance	*0.483***	*0.023*	*0.135***	*0.007*
Level-1 variance	*0.813***	*0.001*	*0.381***	*0.000*
Intraclass correlation	0.373		0.381	
Deviance	6,239,817.966		3,414,698.368	

* $p < 0.01$, ** $p < 0.001$

Discussion and Significance of the Study

The present study explored the associations between mathematics credits earned in high school, mathematics GPA, and student- and school-level factors for ELL and non-ELL students. While many variables were predictive of both outcomes for ELL and non-ELL students, effects differed between the two groups. Such differences highlight the need for educators to approach students who are or have been classified as ELL in ways that specifically target these differences, acknowledging that assistance which may be beneficial to non-ELL students may not be as effective or necessary for ELLs. Key differences are discussed below.

Perhaps one of the most concerning findings is that of the statistically significant negative relationship between interest in 9th grade mathematics and both access and achievement in mathematics outcomes for ELL students. “Interest” in this survey was operationalized as a composite of students’ responses to six survey questions, all of which addressed the extent to which the student enjoyed mathematics and spoke of it as a preferred subject. In both final models, interest was related to decreases in the outcome measure, indicating that ELLs who began with higher levels of interest were more likely to experience worse access and achievement. Such findings indicate the need for investigation into the experiences of ELL students who exhibit mathematics interest early on, but perhaps then struggle to pursue that interest. Early interest in mathematics is vital in encouraging future growth, and efforts must be made to turn ELLs’ interest in 9th grade mathematics into a positive predictor of future success.

While many of the relationships between attitudes, school perceptions, and the outcomes were significant, the effects of variables such mathematics identity, efficacy, and engagement were inconsistent between the two groups. For example, belonging was positively related to GPA for non-ELLs, but negatively related for ELLs. Mathematics identity was a significant positive predictor for non-ELLs’ credits earned, but insignificant for ELLs. One potential reason for these contradictory findings is that while mathematics identity and sense of belonging in the classroom are representative of the role the student adopts in their learning environment, this role may not necessarily be recognized by others in that environment (e.g. teachers, administrators, counselors). ELL students may find themselves fighting inaccurate academic placements or reduced opportunities to learn, despite their own beliefs and self-perceptions. These findings indicate also that the relationships between student attitudes and feelings of belonging and engagement differ for ELLs compared to their English proficient peers, and these details are difficult to identify. As educators, it remains critical to attend to all students’ feelings of belonging and engagement, self-efficacy, and views of mathematics. However, it is necessary to keep in mind that such efforts may not have the same effects on all students, and students who were previously classified as ELLs may require more positive focus on mathematics utility, self-efficacy, and feelings of school belonging.

Finally, the final models for both ELL and non-ELL students included few school-level variables, with many significant predictors entered at the student-level and much variance unaccounted for at the school-level. Such findings indicate that there are other unaccounted for school-level variables which contribute to the school-level variance, likely related to aspects of institutional climate or environmental issues beyond the control of the student. Further investigations should include other more school-level variables to account for these differences.

The mathematics pathways of ELLs and non-ELLs differed significantly. Students previously classified as ELL were significantly more likely to earn fewer mathematics credits and a lower mathematics GPA. Some factors, such as SES and mathematics assessment score, exhibited similar positive relationships with the outcomes for both groups. In these cases, educators may continue to address both student groups in similar ways. For other factors, such as a student's interest in their 9th grade mathematics course, school belonging and engagement, and other attitudes, it is necessary to proceed with caution. A student's linguistic background and prior classification may have long-lasting effects on their mathematics success, and it is critical to acknowledge the ways their school experiences differ. Careful attention should be paid to fostering and maintaining early interest in mathematics, as well as developing mathematics self-efficacy and utility for ELL students. While school-level factors did not appear as often as student-level factors, the two are often intertwined, and educators should seek to positively impact ELL students' experiences throughout their school experience.

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WHY THEORIZING AFFECT MATTERS FOR MATHEMATICS EDUCATION RESEARCH

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This paper briefly examines theories of affect and some of its possibilities and limits for mathematics education research. First, psychological, socio-cultural, embodied, and new materialist perspectives are considered. The paper juxtaposes emerging and older theories of affect in mathematics education with alternative approaches in the humanities and social sciences. Then, the paper briefly historicizes some of the changing and enduring economies for affect in mathematics education across three historicized "moments" of U.S. mathematics education reform circa the 1830s, 1890s, and 1930s. This section aims to consider some of the ways 'bodies' have become differently legible for theorizing affect in problematic ways with potential implications for future research.

Keywords: Affect, Equity, Body Studies, Emotions, Inclusive Materialism, Reform, History

What is affect and why might its theorizations matter for mathematics education research? To start, affect, however conceptualized, is widely considered to be important to learning school mathematics. Most commonly, affect has been defined as a complex psychologized construct located in individual minds, distinct from cognition, and bundled with some amalgam of emotions, attitudes, moods, feelings, beliefs, and/or values (de Freitas & Sinclair, 2019; Hannula, 2012; McLeod, 1994). Somewhat less frequently, affect in mathematics education has also been theorized to include physiological effects, such as changes in neuronal firing patterns, pulse rates, skin sensations, and other changes – sometimes at levels considered outside the range of conscious awareness (de Freitas & Sinclair, 2019; cf. Dowker, Sarkar, & Looi, 2016). More recently, socio-cultural and -historical perspectives have begun to (re)consider affect as something that may also exceed analysis at the level of the individual, such as by theorizing its emergence through practices, activity, and/or norms (Hannula, 2012; Roth, 2012).

The latter focus on practices, norms, and activity have also intersected with a recent proliferation of "body studies" (de Freitas & Sincliar, 2019; Roth, 2012). Here, perspectives tend to eschew long-standing dualisms familiar to mathematics education research (e.g., cognition/emotion, mind/body, abstract/concrete). For example, a "mathematical concept" conceptualized with a Deleuzean-Spinozan perspective, may be approached as a kind of 'body' that affects (and is affected by) classroom atmospheres, teacher-student conversations, and corporeal body movements (de Freitas & Sinclair, 2014). This notion of affect need not privilege human agency or center the human as 'in control' of the 'bodies' that are made through shifting material-discursive practices and/or "affective networks" (de Freitas, Ferrara, & Ferrari, 2019; see also, Barad, 2007).

At the same time, despite considerable social and political shifts occurring through the COVID-19 pandemic, a 'mathematical concept' may at once circulate as a 'new' amalgam of messy and dynamic 'bodies' that, without alternative possibilities for thinking-doing, tend nonetheless to stabilize what (school) mathematics may be, become, and/or do. Put another way, while 'bodies' may produce complex and emergent amalgams of *affectivity*, much not only necessarily escapes capture by the research apparatus - (school) mathematics is also stabilized by grids of capture that treat affect as a knowable 'object' or 'objects' that take school mathematics as a more or less neutral site for producing affect based on various pedagogies (see also, Popkewitz, 2008). Indeed, mathematics education research has little theory to engage research questions that attend to such (always partial and incomplete) complexity, messiness, and multiplicity. Yet, theories of affect in the humanities and

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social sciences have for several decades offered possible entry points that do not seek to foreclose (school) mathematics or the 'bodies' with potential for affectivity as stable or necessary objects of inquiry (see also, Sinclair & de Freitas, 2019).

Of course, opening to new possibilities may also invite old dangers. For example, and as discussed briefly below, the emergence of efforts to theorize the mind as something interdependent with the corporeal body and, at times, as intimately linked with emotionality and materiality have historical antecedents that were also problematic. For example, Herbartian efforts to mathematize and study 'correct' sensation as the basis of truth claims were carried forward through social control projects and scientific racisms in the late nineteenth century in ways that continue to haunt contemporary education (see, e.g., Crary, 1988).

This theoretical paper, then, aims to provide a brief introduction to theories of affect in mathematics education research and consider both possibilities and limits for mathematics education research. In the first section, the paper provides a brief overview of research on how affect has been conceptualized in mathematics education research. In the second section, examples of three well-circulated perspectives on affect theory from the humanities and social sciences that are largely absent in mathematics education are considered. Finally, the focus shifts to three sketches of "moments" in U.S. discourses taken from a larger study investigating how objects of inquiry (or, 'bodies') in psychologized and socio-cultural approaches to research have shifted with respect to changing economies for affect in U.S. mathematics education. Because an in-depth discussion is not possible in this space, the "moments" selected are not intended as comprehensive 'histories.' Rather, they are intended as entry points for further discussion about how affect has become differently available with shifting notions of 'bodies' and their presumed relations. Finally, the approach to the three "moments" also seeks to draw attention to how emphasis on 'affect' may inadvertently stabilize the 'bodies' it purports to investigate in ways that obscure their historical traces in making the present appear thinkable and actionable (see Popkewitz, 2008; Yolcu & Popkewitz, 2019).

Theorizing Affect

Affect in Mathematics Education Research. There has been wide agreement across many fields in the social and mind sciences that affect is messy at best. As noted above, affect has been operationalized and defined in mathematics education research as something primarily bio-psychological, and, with less frequency, as something socio-culturally and historically contingent. As a psychological construct, affect has typically been considered something available to self-report in the form of emotions, beliefs, attitudes, values, and moods; where each category is presumed to differ primarily with respect to duration and/or intensity, and, at times, with respect to relation with (but distinct from) cognition and sometimes behavior (Hannula, 2012, 2019; McLeod, 1994). For example, a belief may seem more durable over time when compared with an emotion, mood, or feeling and have different 'effects' on cognition (McLeod, 1994). In studies prior to those considered explicitly as part of what became considered domain-specific studies of "affect" in the late 1980s, attention to 'affect' tended to center on constructs of anxiety and attitudes (see, e.g., McLeod, 1989; Zan, Brown, Evans, & Hannula, 2006). More recently, psychologizing approaches have also begun to include notions of identity and motivation among the collection of psychologized constructs (Hannula, 2012).

Socio-cultural perspectives on identity formation have also circulated widely in 'equity'-oriented literature. However, such scholarship rarely attend explicitly to theories of affect. This is particularly of note given the prominent role given to theories of affect once had on formulating theories of identity formation (e.g., that school mathematics appears to *affect* identity formation differently across racialized, gendered, and abilized categoriess) (see, e.g., d' Ambrosio, 1987; Fennema, 1979). More recently, theories of affect in 'equity'-oriented scholarship may be implicit in concerns with identity formation in multiple ways. For example, calls for improving "engagement" with school mathematics often require assumptions that something beside 'cognition' matters for learning, such as with what appears to constitute a "racialized narrative", grids of "positionality", an activity, or an artifact (e.g., Nasir & Shah, 2011).

In more recent socio-cultural approaches dealing explicitly with theories of affect, some scholars have claimed that intellect and thought are fundamentally inseparable from emotion (and/or affect) in ways that are culturally and historically specific (e.g., Radford, 2015; Roth & Walshaw, 2019). Additionally, Hannula (2012) has argued that affect understood through "situatedness" or "enactivism" may be understood as a social as well as mental process. Related scholarship has argued that recruiting the corporeal body and/or social groups to make generalizations based on movements and/or practices as adaptation to situations and environments also work to erode long-standing emotion/thought and mind/body binaries (Hannula, 2012). However, the majority of scholarship from socio-cultural and -historical perspectives has not engaged in substantive historicizing or moved much away from emphasizing that 'emotions' and 'bodies' are cultural and historical productions (see, for examples, Radford, 2015; Roth, 2012; cf. Yolcu & Popkewitz, 2019).

Biological and/or physiological theories of "affect" are often located somewhere between the confluence of second-order cybernetics and new materialisms, neuroscience, and physiology. At one pole, for example, transcranial electrical stimulation (tES) has been offered as a potential hope for treating "mathematics anxiety" (Dowker et al., 2016), reducing affect to something solely in the brain. At another pole, theories of embodiment that avoid brain-based reducationisms abound. Examples include perspectives linking gestures to semiotic constructions (e.g., Abrahamson, 2009), embodiment as generative of metaphorical worlds-forming (Lakoff & Nuñez, 2000), or relational embodiment as immanent to what it means to be and become a mathematical 'body' (de Freitas & Sinclair, 2014). While the former two perspectives on embodiment have been discussed at length (see, e.g., Hannula, 2012; de Freitas & Sinclair, 2013; Radford, 2009), inclusive materialisms are more recent arrivals to the literature. In brief, and often drawing from feminist and post-structural perspectives, inclusive materialisms assume relationality as an *ontological* commitment, where matter and mathematical concepts, diagrams, or other objects, much like corporeal bodies and feelings, are not bracketed out as something independent of language or thought. Matter, like subjectivity, from this perspective, is thus necessarily ongoing, immanent, unfinished, agential, and perspectival (de Freitas & Sinclair, 2013). In other words, 'bodies' are *made* mathematical by a "dance of agency" that does not start or finish as a property of or in people or things (p. 454). Rather, "agency" is understood as emerging through the complex and ephemeral ways human-nonhuman assemblages become differently intelligible for thought and action, such as through tools, symbols, pedagogies, curricular texts, corporeal body gestures, and research.

In some ways, new materialisms also invite comparisons with second-order cybernetics and enactivisms that themselves recall lines of research drawing from American cybernetics-inspired radical and social constructivisms (de Freitas et al., 2019; Hannula, 2012; see also, Eisenhart, 1988; von Glasersfeld, 1995). For instance, relational and systems-oriented ways of knowing do not presume or center an observer that is independent from complex social and material fluxes that are always partial and emergent. However, inclusive materialisms differ in important ways. For one, material (or the 'environment') is not a neutral space devoid of its own agency. Rather, matter is both produced through discourse *and* produces discourse. Put another way, 'maps' of minds, groups, flows, and/or practices are only possible because the various objects bracketed through apparatuses of observation also 'map back' in ways that 'cut' and make 'bodies' differently legible, invisible, and/or 'able' (Barad, 2007; Yolcu & Popkewitz, 2019). For instance, 'mapping' the purported cognition of a child also participates in *making* the child's cognition as something available for new strategies of intervention through pedagogy, policy, and research (see also, Popkewitz, 2008). In the next subsection, additional alternative approaches to theorizing affect in the social sciences and humanities are considered as potential supplements to existing scholarship in mathematics education.

Affect Theory. In one reading of affect theory, Sara Ahmed (2010) defines affect in part with the term "affective economy" as "what sticks, or what sustains or preserves the connection between ideas, values, and objects" (p. 29). Here, affect is not understood as an emotion or feeling *per se* (though whatever may be understood as an emotion or feeling are not excluded from participating in affective economies). Rather, affect is considered to be *circulated* as a kind of *economy* in the sense that objects become affective and "sticky" through their circulations. In other words, affective economies circulate in ways that may recruit notions of the cultural, psychological, and the political, but they also exceed capture as something *primarily* psychological and cultural that can be located as something 'in' individuals or 'in' social groups. Rather, affective economies gesture to how different notions of interiority and exteriority are produced through material-discursive practices. For one example, by thinking of "growth mindset" as something that may produce neuronal growth (and thus rebiologized notions of "intelligence") in the brain (Boaler, 2015), the brain (and neuronal growth) also becomes exteriorized and mappable as a site appearing to justify new forms of intervention (e.g., those that may promote "mindset" changes) and create new notions of interiors that can be divided and sorted in ways that perpetuate exclusionary practices (e.g., those that 'have' growth mindset and those that do not). Yet, whether or not "growth mindset" is considered in this way or otherwise, the notion of "mindset" has also become "sticky" and circulated in 'other' economies of affect concerned with enhancing and optimizing bodies and selves assigned various degrees of risk. A possible critique of perspectives of affect for Ahmed include the potential 're-centering' of affect as something involving emotions, feelings, and/or something of the psyche, however deconstructed.

From another perspective, affect theory is concerned with how bodies are 'affected' and 'affect' each other *prior* to and/or between 'capture' and labeling as emotions, feelings, etc. (Massumi, 1995; de Freitas et al., 2019). Here, affect is *not* emotion, feeling, attitude, belief, etc. – it is potentiality. This analytic shift involves theorizing both the messiness and the incompleteness of the assemblage of 'bodies' that appear to make emotions and feelings possible for capture and 'self' through processes that exceed human control and/or agency. By 'decentering' the human, some analyses have emphasized studying physiological and neuronal changes that seem to anticipate *and* exceed conscious 'capture' in the messiness of the everyday (Massumi, 1995). If, for example, mathematics anxiety (or joy) is approached not as a state or dynamic construct but as something necessarily partial, emergent, and messy, what assemblages of 'bodies' (e.g., tools, symbols, spaces, temperatures, political 'moods' and 'atmospheres', texts, etc.) may be virtually affective *before* it it put into feelings labeled as 'anxiety'? Or, how do not-quite-yet sensations and changes in 'bodies' become differently available for capture in research apparatuses when viewed as potential pathologies (e.g., as anxiety) through cultural theses about which "mathematical bodies" are desirable (and having 'health') and which are not (and thus needing 'intervention')? Despite the promise of new lines of inquiry that take seriously health discourses in relation to school mathematics in such ways, it is also of note that critics have argued that this approach may invite new and problematic universalisms and reductionisms, especially when affect appears to be something relocated in neurons and/or physiological responses in ways that recall various forms of humanism (see, e.g., Rutherford, 2016).

Additional perspectives in affect theory have built on the work of Sedgwick and Frank (1995) and suggested that even more familiar constructs such as 'emotions' like *shame* can be quickly denaturalized by attending to its complex, contingent, and multifaceted messiness that exceeds capture as a clear construct. For instance, shame as an amalgam of "interest-excitement" and "surprise-startle" and "contempt-disgust" *at the same time* invite a kind of messiness and incompleteness less familiar to more orthodox psychological renderings that have since been circulated as entry points into rich description in gender and sexuality studies. Turning to mathematics education, the always already excess of categories may provide entry points into studies that question either/or paradigms that often assign affect constructs along continua of duration, intensity, and absence/presence. Further, they may invite new renderings that open discussions about identity that exceed efforts to generalize and reduce identity to debates between essence and/or environment. Such perspectives may also open new lines of inquiry into 'old' problems, in part by noting how questions of knowledge-power in mathematics education-related spaces are not simply matters that can be hashed out on an 'empowerment-oppression' continuum. Critiques may question the extent to which complicating existing constructs move away from centering the 'self' and experience as the primary focus of theories of change.

Finally, affect theory also may have something to say about how research is presented, as the *approach* to writing also matters in much contemporary affect theory (Massumi, 2015; Seigworth & Gregg, 2010). It is not enough, for example, to address racializations/racisms in mathematics education through counter-narratives, histories, quantitative studies, or meta-analyses without *also* attending to the (political) aesthetics of presentation and the circulations of affective 'bodies' that are differently "sticky", such as via terms like equity, urban, or diverse. Additionally, if the human is not centered as the primary agent or subject-object of analysis, how might affect theory at once contribute to analyses of a broad field of possible considerations involved in the makings of what produces possibilities for feeling, action, and/or thought while not collapsing the potential for 'strategic essentialisms' that may offer new points of resisting oppression? This may not be an either/or – perhaps by explicitly attending to the messiness of how geographies of social categories (e.g., race, gender, class, ability) become configured through the messy and ongoing emergence of new 'body' assemblages that exceed the possibilities of capture, research may open to generative spaces that do not require school mathematics as a kind of "slow emergency" that doubles as a 'necessary' condition for mattering in the world (Anderson, Groves, Rickard, & Kearns, 2020; Sinclair & de Freitas, 2019).

In short, affect theory, at least via some perspectives, may offer one set of possibilities for attending to the messiness and spillage of the 'everyday' that may trouble (while not necessarily jettisoning) assumed categories, boundaries, representations, states, rules, modes of capture, etc.; if research is less concerned with making definitive statements or claims about what 'happened' and more concerned with how different 'bodies' come to matter in ways such that their incompleteness and messiness are no longer pathologized but offer springboards into the necessarily unknowable (de Freitas & Sinclair, 2019). In this sense, rather than seeking to define "affect" *as* 'this' and *not* 'that', many approaches seek to explore affect as something *necessarily* processual, messy, as multiplicity, 'not yet' and thick (Seigworth & Gregg, 2010).

In the next and final section, I pivot to attend to some of the limits of affect theory as a strategy for research that do not attend meaningfully to how 'bodies' are themselves historically and culturally contingent. To do so, I shift to provide a brief sketch of the additional need to historicize how some of the objects of inquiry of mathematics education research and affect theory (minds, bodies, emotions, and their presumed relations) have become differently circulated and made intelligible through different 'economies' for producing affect (see also, Baker, 2013). Brief engagements from 3 "moments" of considerable ontological and epistemological change in ways of thinking about 'bodies' and school mathematics reform discourses were selected that continue to resonate with some of the social, cultural, and political themes that continue to move with mathematics education (research).

3 "Moments"

Circa 1830s. Schooling and society in the post-revolutionary United States saw many changes that directly affected how school mathematics was to be learned and taught. In addition to the emergence of publicly-funded school systems, educational journals, and a proliferation of organizations and institutions, the first quarter of the nineteenth century also saw a marked increase in circulation of mathematics textbooks and materials intended specifically for children (P. Cohen, 1999; Monroe, 1917). On the one hand, such changes were not surprising, given the perceived stability of the Republic and the increasing efforts to link mathematics education (mainly as arithmetic and cyphering) with managing commercial and industrial affairs and promoting mental cultivation for the presumed rigors of democratic (and 'white male') citizehnship (P. Cohen, 1999).

Within this milieu, the concept of "mental discipline" – a term widely circulated as marking aspects of nineteenth century theories of mathematics – emerged as a central hope for what school mathematics could provide beyond applications to practical affairs (e.g., Stanic, 1986). Briefly, mental discipline has typically been described as a doctrine suggesting the mind was like a muscle composed of separate but interdependent faculties, where 'exercise' of any of the faculties (such as via arithmetic) offered routes to strengthening the mind and, by extention, the intellect (e.g., Clason, 1970; Stanic, 1986). However, such a perspective may obscure the nuances and traces of how mental discipline and faculty psychology were also circulated in their historical present through a wide array of new theories of minds and bodies. For example, faculty psychology accompanied new theories such as phrenology that began to consider capacities for mathematics as something at once 'in' the brain, correlated with head shape and character, and, at times, differently modifiable through physical exercise (Tomlinson, 2005). [1]

Further, mental discipline was also understood in part through new medical discourses. For example, from the perspective of the influential U.S. physician Benjamin Rush, bodies could now be conceptualized as systems of "oscillatory matter", where mental "laws" were equated with physical "laws" (Altschuler, 2012). Further, for Rush, mind was understood as influenced by diverse and distributed social and material systems, including blood circulation, political affiliation, occupation, commercial trade, and perceived racialized/racist effects of institutional slavery on health and physiognomy (Herschthal, 2017). And, with phrenology, the faculties of mind that appeared to be exercised through school mathematics also circulated with new theories of associations between physiognomy and intellectual capacity. The corporeal body and the emotions/passions were considered interdependent with 'healthy' mental cultivation and theories linking heredity to dispositions toward mind-body-spirit im/balance and the future of the Republic (Ziols, 2019).

Finally, new theories of childhood also emerged that located children as ontologically distinct from immature adults. As such, children were considered particularly vulnerable to too early or too intense exposure to mathematics (though they were also felt to be especially sensitive to Lockean sense impressions and capacity for cultivation) (Ziols, 2019). Within this milieu, it is perhaps not surprising that school mathematics became a subject that was increasingly to be designed specifically for children in ways that included new hopes and fears about the potential effects of school mathematics in new economies for affect. For example, Samuel Goodrich (1818) was among the first arithmetic textbook authors who argued that arithmetic should be "attractive" to children and "divest[ed]" of "all that is not necessarily difficult or disagreeable" (p. iii). Goodrich's hopes also included making arithmetic more "inviting", satisfying, and pleasurable. Although Goodrich's textbook is among the more explicit in this way, subsequent textbooks across a range of "systems" for teachers also began

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[1] Tomlinson (2005) has argued that phrenology discourses *a la* George Combe and Horace Mann were woven into the inception of U.S. public schooling as a "moral technology" linking the "exercise" of minds-bodies with racist, sexist, and abilist efforts to eliminate the "abnormal" as routes to 'improving' humanity.

to include appeals to securing interest and preventing injury. For example, in addition to advocating arithmetic as a site for “mental discipline”, Colburn felt that using fingers and objects had made arithmetic learning more appealing for younger children in “both sexes and among all classes” (Colburn, 1830, as quoted in Monroe, 1912, p. 465). At the same time, with an arithmetic textbook intended for children as would-be missionaries, school mathematics was less about learning to ‘reason’ per se and more about converting ‘heathens’ through “Christian benevolence” and by “training a rising generation to esteem the privilege, and the practice of doing good” (Weeks, 1822, p. iv).

In short, school arithmetic had now become available through changing notions of children, mental cultivation, schooling, and their presumed relations with new theories of minds, bodies, and groups (e.g., by sex, race, age, nation, language, profession, brain size, blood circulation, physiognomy, ability, class) (Ziols, 2019). In this economy for ‘affect’, the arrival of mental discipline was not simply about ‘beliefs’ or ‘ideas’ that mathematics could cultivate the mind and/or affect the emotions – it also required complex changes and new amalgams that included colonizing logics, racialized and newly ‘bodied’ notions of mathematics learning, and theories of reasoning that recruited the corporeal body in different ways (Ziols, 2019). It is perhaps no surprise, then, that school mathematics also began to emerge as a school subject that could potentially injure children by ‘unbalancing’ mind-body-spirit amalgams in ways that reinforced exclusionary discourses (Ziols, 2019; see also, Jenkins, 2010).

Circa 1890s. By the 1890s, what might have seemed “difficult or disagreeable” in Goodrich’s historical present shifted to new ways of thinking. Some texts lamented old fears of arithmetic study in new ways. For example, an article in *The Journal of Education* claimed a “juggernaut Arithmetic” was “grinding” children into “physical and nervous wrecks” and making “a sound body, steady nerves, and a clear brain impossible” (Arent, 1896, p. 77). (Such “grinding” pain was also described as the mental equivalent of a man being hung by his thumbs as punishment for committing a crime). Those at risk for becoming “wrecks” retained earlier nineteenth century fears that pathologized the ‘precocious’ child as those considered to be “ambitious for high scholarship” (Arent, p. 77; Ziols, 2019). However, the risks of too much study were now put in the language of experimental physiology and the “new” psychology (Popkewitz, 2008; Stanic, 1986). In short, new amalgams of ‘danger’ accompanied a (partial) erosion of mental discipline logics, as mathematics education became increasingly visible as a topic of concern in a growing number of “attacks” on U.S. school mathematics (Stanic, 1986). Importantly, responses to such “attacks” also recruited new assemblages of ‘bodies’ for reform-oriented justifications for the ‘why’ of school mathematics.

For instance, the “new” psychology sought to map the child’s mind as a scientific object for study and planning the future through the mind and social sciences in response to fears of (im)migrant populations moving to the cities (Popkewitz, 2008; Ziols, 2019). Also, as learning school mathematics became largely domain specific and less tethered from its value for ‘mental discipline,’ it paradoxically became increasingly relied upon as a standard for mental testing, partly linking scientific racism and eugenics through new bio-psychologized notions of intelligence, character, and individual difference (Danziger, 1997; see for an example, Thorndike, 1922).

Importantly, though, mapping the child’s mind was not simply a project related to studying the child’s mind as a psychologized object. It also appeared to require the study of children *in situ*, a perspective informed in part by the study of ‘othered’ cultures such as through ethnology, folklore studies, and history (see, e.g., Dewey, 1884, 1886). In brief, by providing a sense of teleological progress as the inevitable directionality necessary for a desired moral order, a technoscientific future could be secured through application of scientific ‘expertise’ to pedagogy (Popkewitz, 2008; Yolcu & Popkewtiz, 2019). Progress had also become entangled with Spencerian notions of human agency and cultural ‘development’, secured in part through efforts to predict and control behavior by studying the duration and intensity of various senses (Crary, 1988). In short, as the mind-body-spirit became set in linear notions of technoscientific progress, it also became increasingly available (along with disciplinary mathematics) as a social and cultural construct that both required and exceeded notions of human agency to secure 'advancement' along rank-ordered developmental continua on a 'great chain of being' (Baker, 2013; Crary, 1988).

Within this milieu, James McClellan and John Dewey (1895) in the *Psychology of Number* argued that scientific principles would provide a plan for the "natural" alignment of children's mathematical activity with "civilization." Culturally-mediated activity would foster discernment and reason in ways that would move 'inward' to the corporeal body and 'outward' to secure the mind (and civilization) from ethical and psychological "destruction" (McClellan & Dewey, 1895, see pp. 4-5). The avoidance of 'destruction' also entailed efforts to tie activity to the production of quantity through accurate measurement, discernment, and relation as strategies to predict, control, and direct the future. For example, McClellan and Dewey argued: "The child and the savage have very imperfect ideas of number, because they are taken up with the things of the present moment. There is no imperative demand for the economical adjustment of means to end; living only in and for the present, they have no plans and no distant end requiring such an adjustment" (p. 38). By moving 'out' of the present moment, the 'child' and the 'savage' required developing an "idea of quantity" through "arrang[ing]… acts in a certain order, to prescribe for himself a certain course of conduct so as to accomplish something remote" (p. 38). In one section, the racializing/racist psychologizing of number was also one that could be summarized by "embodying the idea that number is to be traced to measurement, and measurement back to adjustment of activity" (p. 52).

Also, with this 'new' onto-epistemological framing, the psychologizing of number accompanied new claims that *all* humans and some animals had mathematical 'capacities.' Cultural 'activity' then, was, what explained purported differences in the power and rigor of 'culturally'-specific mathematical practices and distinctions made between humans and animals (Ziols, 2019; see for different examples, Dewey & McClellan, 1895; McGee, 1898). Further, 'access' into 'civilization' was explained not only by converting and/or assimilating the activities of the child-as-savage into 'civilization' but also as a political project that located 'access' based on theories of cultural tool use. For example, ethnologists, working in part to establish anthropology as the pinnacle of the sciences, argued that cultural tools (e.g., the hand, objects, and written signs) provided the levers that would 'liberate' the 'primitive' mind from its 'mystical' past (see, e.g., Conant, 1896, McGee, 1898). From one perspective, by directing one's goals toward an imagined future through targeting cultural group "leaders" (as the 'strong'), educators-as-scientists could offer the most direct routes for 'racial' uplift and desired social change (Haller, 1971; Ziols, 2019). Such notions not only exacerbated racist and ethno-centric discourses centered around 'Western' concepts of mathematics – they linked static notions of the environment and climate with the 'extranatural' (or socio-cultural), unconscious, kinetic, and linguistic as sites for securing imperial notions of Spencerian 'progress' in social groups defined as on separate developmental trajectories (see, e.g., Haller, 1971).

Finally, although the modern psychologized notion of the personality (and the person) as a set of discrete constructs was not yet thinkable (Danziger, 2012), character-building and habit-forming discourses also permeated journals and books related to mathematics education. For an example distinct from *The Psychology of Number*, an article in the *Journal of Education* suggested that "reasoning about things" in early school mathematics was subsidiary to learning to "use the signs and symbols" of arithmetic and "by every ingenious contrivance" to "cultivate habits, habits, HABITS, of accuracy, rapidity, and neatness, both in mental and manual activity, and of speech as well" (emphasis original, Allyn, 1892, p. 281). However, it was not 'enough' to simply cultivate 'mathematical' practices believed to align the mental and the manual through activity and speech. A teacher was also to cultivate "above all… truthfulness and honest sincerity" in ways that avoided "making dunces who hate math" (p. 281).

However partial the above account, the intention here has been to note that although the mind may have been conceptualized as distinct from the body, mind-body amalgams were not eschewed with the arrival of new theories about how 'mental' processes were produced through social and cultural practices, the corporeal body, the senses, language, and notions of character and conduct. Similarly, though reworkings of a mind/body split were certainly present in such discourses, new notions of materialisms and efforts to mathematize sensation also undergirded them (Crary, 1988). In the next subsection, a third shift in economies for affect are considered, where new notions of character and individual difference were increasingly scientized through hopes and fears about securing democracy, promoting the 'adjustment' of the modern personality, and impressing the 'cultural value' of mathematics.

Circa 1930s. Around the 1930s, the institution of mandatory secondary schooling in the United States accompanied efforts to address new fears, particularly those involving the inclusion of the "other 50%" now required to attend secondary school (see Lagemann, 2000). One primary site for addressing 'new' hopes and fears of a mathematics education under "attack" was via the insertion of mental hygiene and cultural value-creation into policy documents and pedagogies (Ziols, 2019). On the one hand, reform discourses sought to reconceptualize the "un-emotional subject *par excellence*" of mathematics education as one intimately requiring both emotion *and* intellect as essential and interdependent for meangingful learning (Progressive Education Association [PEA], 1940). By "understanding the student", mathematics educators would avoid the potential dangers of psychological, social, and physiological 'maladjustment' (PEA, 1940). Or, from the joint Yearbook published by the Mathematics Association of America and the National Council of Teachers of Mathematics (1940), educators were to focus on the "problem of the dull normal" who differed in "degree" from the "gifted" (p. 133). And, though the "[t]he data" appeared to show the "fact that the slow group grows in the same proportion as the fast group though on lower levels of development" (MAA & NCTM, 1940, p. 134), the 'problem' was considered in part to be one of "implanting the cultural value of mathematics" such that students would "comprehend certain essential elements of the civilization they are to share" (NCTM & MAA, 1940, p. 48).

To achieve a sense of cultural value, the 'why' of mathematics education also appeared to need justification. On the one hand, justification was now expressed in new (eugenic) theories of health that included not only concerns with 'intelligence' but also with attention to emotion and conduct, attitudes, and "traits" (MAA & NCTM, 1940; PEA, 1940; Zachry & Lighty, 1940). On another hand, approaches such as those in the *Mathematics Teacher* included efforts to link the "mathematic of a culture" to that culture's purported "soul" or "spirit" (Schaff, 1930). Such a perspective was not unique. Schaff drew in large part from Oswald Spengler's (1965) widely circulated book, *Decline of the West,* that argued every culture was defined by its "mathematic", where developments in mathematics marked a 'culture' as either ascendant or in decline. Schaff, however, argued for a more optimistic interpretation: Mathematics as a human and cultural product suggested that humans were the "the law givers of the universe" and that it was "possible… that the greatest of our material creations is the material universe itself" (Sullivan, as quoted in Schaff, pp. 502-503).

Additionally, creating a shared sense of a cultural value for regulating emotion and conduct involved another object of inquiry: the "modern" personality (Danziger, 1997). As an amalgam of "traits", the "modern" personality emerged through efforts to measure a person's character – a perspective drawing heavily from Galtonian-inspired eugenics (see Danziger, 2012). In related U.S. mathematics education journal articles and policy documents, some authors explicitly argued for "reform" as a "concern with the effect of arithmetic on personality" that required "a major reorganization of subject matter and methods" (Buswell, 1941, p. 10). Further, in the same chapter, such "reorganization" involved engineering "an organized body of number experiences from which both mathematical insight and social significance may be derived" (p. 10), where a "number experience" was what might offer "positive contributions to the development of desirable personality traits" (p. 10).

Lastly, shifts in the 1930s could also be summarized in part by the PEA's (1940) companion report to *Mathematics in General Education* titled *Emotion and Conduct in the Adolescent* (Zachry & Lighty, 1940), where what constituted the "un-emotional subject" of school mathematics was now juxtaposed with perspectives on how emotion and conduct were to be reconceptualized by bucking the purported status quo. Namely, it was argued that the "Puritan tradition" believed "responsible [for]… the tendency in all Anglo-Saxon cultures paradoxically both to discount emotion and to counsel its mastery" was to be challenged (Zachry & Lighty, 1940, p. 5). "Emotion thus broadly conceived" was to be "fused with thinking - for the most part harmoniously - in the healthy, competent individual" (Zachry & Lighty, 1940, p. 5).

In short, school mathematics had become in part a translation device for addressing fears of "maladjustment" in adolescents and children through reforms centered on "understanding the student" to regulate emotion and conduct, establish and secure social cohesion, cultural unity, and "democratic order," and strengthen and/or muting un/desirable personality and character "traits" during times of perceived crisis in school and society (see also, Yolcu & Popkewitz, 2019).

Concluding Remarks

This paper has had two aims. First, it has made an argument that mathematics education research may broaden its scope by engaging with theories of affect that eschew analysis of the individual or group. Second, it has argued that affect theory is also limited with respect to what it may take as assumptions of ahistorical continuity across different material-discursive assemblages and space-times. While neither argument is entirely new to mathematics education research (see, e.g., de Freitas & Sinclair, 2019; Popkewitz, 2008), little research has addressed how and why mathematics education research continues to locate desirable 'affect' as something messy and seeming to be a problem of largely ahistorical approaches to methodology *despite* rather dramatic onto-epistemological shifts in how affect (and mathematics education) has become intelligible. Engaging with affect as historically and culturally contingent assemblages of 'bodies' requiring further scrutiny and historicizing may thus provide important new entry points for future research.

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EQUITY AND JUSTICE:

BRIEF RESEARCH REPORTS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

STUDENT ENGAGEMENT AND GENDER IDENTITY IN UNDERGRADUATE INTRODUCTION TO PROOF

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I examined how gender identity shapes engagement experiences in undergraduate mathematics classrooms through a mixed methods study. Data collected from 12 classroom observations, stimulated recall interviews, and student-reported data on engagement were used to answer the question, "How does gender identity shape students' engagement experiences in undergraduate mathematics classroom?" The findings indicate that students' in-the-moment engagement is characterized by classroom environments that foster collective mathematical discussions and group work.

Keywords: Equity and Diversity, Gender and Sexuality, Inclusive education

Introductions

Student engagement is a strong predictor of achievement and behavior in school irrespective of students' socioeconomic situation (Klem & Connell, 2004), making it a powerful factor in education. Additionally, in the social sciences, women earned a majority of bachelor's degrees (55%) and master's degrees (57%) from 1993 to 2015 (NSF, NCSES 2019). On the other hand, of all STEM degrees awarded in 2016, women earned about half of bachelor's degrees, 44% of master's degrees, and 41% of doctorate degrees, about the same as in 2006 (NSF, NCSES 2019). Based on these data, it is possible that gender identity plays a role in the field(s) one chooses to pursue. At the classroom level, gender identity and participation in mathematics are related in that students' gender identity can influence their decision to continue studying mathematics (Boaler, 2002a; Boaler & Greeno, 2000). To better understand how students learn mathematics, there is potential in better studying connections between students' engagement and gender identity in mathematics learning environments.

Student Engagement in Mathematics Education

Engagement manifests itself in activity, including observable behavior and mental activity involving attention, effort, cognition, and emotion (Middleton et al., 2017). Engagement is thus a complex meta-construct consisting of cognitive, affective, and behavioral dimensions (Fredricks et al., 2004). For students to learn mathematics, they must be engaged. For instance, Bodovski & Farkas, (2007) found that student engagement is a significant contributor to students' mathematical growth in early elementary school. At the middle and high school level, researchers found that higher cognitive, behavioral, and emotional engagement predicted students' academic achievement (Wang et al., 2016). However, the literature lacks studies of student engagement in mathematics classrooms at the undergraduate level (cf. Williams, 2017).

Gender Identity

In the early 1900s, researchers used sex hormones to explain masculinity and femininity (Bell, 1916; Lillie, 1939), replacing religious justifications with scientific ones for restricting women's roles (Risman et al., 2017). In the latter part of the century, social scientists began to push back against using scientific justification to explain gender and viewed masculinity and femininity through the lens of roles that were assigned to men and women by society. In the 21st century, sociologists including gender research expert Risman (2017), argue that gender identity is a socially constructed

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component of one's identity built as a result of internal and societal interactions. For the purpose of this study, gender identity is comprised of personal identity made up of psychological characteristics and a social identity encompassing salient group classifications which differs across cultures and can change.

Gender Identity and Student Engagement

At the college level, most studies regarding student engagement and gender identity strive to either explain the growing disparity between degrees awarded to men and women or differences in undergraduate enrollment at baccalaureate-granting institutions between women and men. Thus, at the college level, most studies on student engagement and gender focus solely on how many males and females either graduate or drop out from degree programs. However, such conceptualization of student engagement is different from student engagement as defined in this study nor are the labels "male" and "female" sufficiently capturing what is meant by gender identity (Risman, 2017). Engagement goes beyond enrolling or graduating from a degree program. Additionally, of the literature reviewed, most work on gender differences in student engagement tend to generalize across content areas that is not math focused. This study is unique in its focus on student engagement (as defined by Fredricks et al., 2004) and gender identity(as a social structure) in the mathematics classroom at the college level.

Theory

Flow theory offers an effective lens for interpreting student engagement in that both flow and engagement describe states of total involvement in a task and involve internal motivation (Steel & Fullagar, 2009). From flow theory, student engagement is made up of interest, enjoyment (emotional & behavioral), and concentration (cognitive) (Shernoff, et al., 2003). Thus, the extent of students' engagement is based on these factors.

To understand gender identity as a social structure (Risman, 2017), this study adopts the stance that gender identity is a person's own sense of self by virtue of being part of a society. Gender is socially constructed in that societies have a set of gender categories that usually serve as a basis for the formation of gender identity. This perspective emanates from Tajfel and Turner's social identity theory (1986), which assumes individuals define their own identities as a result of societal norms. Gender as a social structure has psychological and social characteristics.

Methods

I addressed the research questions: (a) What characterizes student's engagement in an undergraduate introduction to proof course? (b) What are the different ways in which gender identity shape students' engagement experiences in this setting? To address these questions, I observed an introduction to proof class for five consecutive weeks. The participants in this study are students enrolled in the course. Fifteen undergraduate students: three of those who identify as women and twelve of those who identify as men volunteered for the study. The sessions I observed focused on the teaching of combinatorics. All students are assigned pseudonyms.

Sessions were video recorded. A demographic survey was administered to students, which asked questions such as "Please describe your gender identity?", "Do you believe that how you identify (gender) affects your experiences during classroom interactions? If so, please explain?". In-the-moment student engagement was captured through the experience sampling method (ESM) (e.g., Shernoff, et al., 2003). ESM data took student-reports of interest, enjoyment, concentrate (i.e. engagement), perceived skill and challenge.

Lastly, ESM responses and video data were used to develop protocols for stimulated recall interviews. Using thematic coding, data from this study were analyzed to understand the nature of student engagement and how gender identity shapes student engagement experiences.

Results and Discussions

I present themes associated with students' engagement that emerged from the stimulated recall interviews as students described both their personal high/low engagement moments. Finally, I focus on how gender identity influences these students' engagement.

Student Engagement

A theme associated with students' engagement from the stimulated recall interviews was the social norms of the classroom. The instructor presented the content with a very nontraditional approach. For instance, almost every single day, students presented their mathematical ideas on the board. Furthermore, students' ideas were valued in the classroom and were encouraged to initiate mathematical discussions. Students asserted that the nature of the classroom influenced their engagement. For instance, Bridget, who identifies as a woman, asserted that "seeing how they (students) thought to solve mathematical problems in different ways influenced my concentration positively." Being able to see the multiple ways to solve mathematical problems influenced her engagement. Nat, who identifies as a man, explained that "the fact that I was given an opportunity to describe the mathematics to my peers increased my interest." On the other hand, Ian, who identifies as a man, explained that since the nature of the class was not lecture based, his engagement was on the low side. As Bridget, explains seeing multiple ways of doing math influenced her concentration, her cognitive engagement is increasing. Nat's interest about verbalizing his mathematical reasoning positively influenced his emotional engagement. However, Ian's emotional engagement is on the low side as he does not enjoy the mathematical content being covered. Although Ian's emotional engagement was low, two of his peers explained how this student-centered environment fosters their emotional and cognitive engagement.

Group work was instrumental, in that all participants who volunteered for the stimulated recall interviews asserted that working with peers influenced their engagement positively. Bridget asserted that "I don't know, just working with my peers increased my interest and concentration." Nat explained that working with his classmates served as a competition for him to be cognitively engaged. During the stimulated recall interview, he explained,

> with the partners I have its kind of a motivation thing kind of use this as a competition type deal like to try match my partners' skills. if I was working individually, I don't think my concentration would be high for this mathematics class.

Ian explained that being a "privileged" male, he speaks less in group work when working with those women to enable them to express their ideas. He explained that working with his classmates positively influenced his cognitive and emotional engagement. He also explains that, "when working with classmates even though no one is an expert, they can still point that why did he (emphasis added) did that in this proof, so I think working with my classmates helps to concentrate."

Thus, from this study, there is a positive association between group work and their engagement. It is interesting to note how Ian did not wish to be the only speaker during a mixed gender group interaction to allow those he perceives as women to talk. Furthermore, it is more fascinating how he used a "male pronoun" for a hypothetical student, when describing his experiences working with classmates. This points out some connections between one's gender identity and engagement experiences.

The Role of Gender Identity in Student engagement

ESM data show that overall those identifying as women reported higher levels of engagement than those identifying as men. Aaron, who identifies as a man, reported being less engaged than his peers; whereas, Janet, who identifies as a cis woman, reported being most engaged. Aaron did not volunteer for the stimulated recall interview hence much cannot be inferred on the low engagement reported by him. Observations suggest his behavioral engagement aligned with his self-reported levels of engagement from the ESM survey. During the analysis of the initial demographic survey, Janet explained,

> I have a distinct memory of trying to take Calc 1 in summer and being the only woman in the room, and it being my first class not taught by a woman and feeling anxious about it. I don't think it was a very fair feeling, but I did feel it.

She did not volunteer for the stimulated recall interview, which might be explained by prior experience in her calculus 1 summer class and that possibly because I identify as a black man.

Five participants asserted that their gender identity influenced engagement. Ian, who identifies as a man, explains,

> If I am in a group interaction made up of different genders, the same actions done by a man and a woman will be definitely be viewed differently so I definitely think gender has an impact on how students engage with mathematics.

Edwin, who identifies as a man, explained how he thinks gender influences engagement in the mathematics classroom by indicating that being male has not "negatively affected" him as a learner of mathematics. Some students identifying as men said they did not think one's gender identity influenced experiences in mathematics courses; however, they would also say something to suggest otherwise. Although Nat explicitly acknowledged that gender did not influence one's engagement, his explanation indicated that gender actually influenced how he approaches group work, "when working with women I allowed them to express their mathematical ideas to get them involved in the discussion." Thus, we do see that those perceived to be women are on the disadvantage. For instance, Mike thinks men contribute more in the mathematics classroom. On the other hand, Bridget asserted one's gender did not influence their engagement while studying mathematics. In conclusion, some students indicated that their gender did influence how they engaged in the mathematics classroom.

Conclusion

This study investigated the nature of students' engagement in mathematics and how their gender identity shapes engagement experiences in the mathematics classroom at the college level. Some students indicated that their gender identity did influence how they engaged in the mathematics classroom, with those who identified as women more likely to be negatively affected. However, this perception was not borne out by the observation and ESM data. In the observation and ESM data, those who identified as women reported higher levels of engagement than those who identified as men. There is room for further exploration and research on gender as a social structure rather than biological, and much remains to be learned about the different ways in which gender identity shapes student engagement in mathematics.

End Notes

Students' gender identities were reported exactly as identified in the demographic survey.

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RECONCILING TENSIONS IN EQUITY DISCOURSE THROUGH AN ANTI-HIERARCHICAL (ANARCHIST) THEORY OF ACTION

CONCILIANDO LAS TENSIONES DEL DISCURSO DE EQUIDAD A TRAVÉS DE TEORÍA DE ACCIÓN ANTI-JERÁRQUICA (ANARQUISTA)

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In this text, we invite others into a different paradigm of thought and action than what has been historically represented at PME-NA. We surface anti-hierarchical theory of action as a macro-theory implicitly present across substantial discourse in equity in mathematics education, then explore some of the affordances and challenges to be considered in adopting anarchism as a theory of action for research and practice. Findings relate to (1) the (de)colonizing tension between focus/specificity and the reification of compartmentalization, (2) the concomitant tension of constructed (in)visibility, and (3) the pursuit of solidarity in the disruption of a violent status-quo. Implications for ongoing research and pedagogic praxis are shared.

Keywords: Equity and Justice, Systemic Change

The 42nd annual conference of the North American Chapter of Psychology in Mathematics Education invites us to "promote the exchange and enrichment of mathematics education research" by looking to its manifestation across "different cultures, places, and contexts" (PME-NA, 2020). The goal of this article is to outline and trace the value an anti-hierarchical (anarchist) theory might hold as a macro-theory undergirding and providing definition (and direction) to equity-oriented research. We aim, in short, to invite others into a different paradigm of thought and action than what has been historically represented at PME-NA. Our goal for each section of this text is not to be exhaustive, but to be illustrative of a different way of being and living. (Restivo, 1998)

What is Anarchism?

For the purposes of this paper, we place emphasis on anarchism as an articulation of ethics, as a mode of human organization with social self-determination, rooted in the experiencing of daily life (Suissa, 2010). Anarchism aims to maximize both individual autonomy and collectivist freedom, leading to the reduction of fixed hierarchies that systematically privilege some people over others. Individual autonomy is, perhaps, self-explanatory as a principle; it is a self-determination, the freedom of choice in one's own acts, with no form of external compulsion. Collectivist freedom is one in which the individual finds their freedom through voluntary association with other members of society, not isolation from them. Bakunin (1971) argued that man is only truly free when among equally free men; "the freedom of each is therefore realizable in the equality of all" (p. 76). Concisely, to embrace any form of anarchism is to express profound skepticism toward skewed, coercive, and exploitative power relations, and to reject all forms of oppression, including those of class, race, gender identity, religion, etc. (Lawler, 2019).

In short, anarchism is the absence of hierarchy. It imagines a relationship among people that minimizes if not eliminates coercive structures and interactions, taking seriously the possibility of an equal and free society, organized on core values of *cooperation*, *mutual aid*, and *freedom from hierarchy*.

How is Anarchism Currently Alive and Well in Mathematics Education and How Might Anarchism Resolve or Inform Tensions in Mathematics Education and Equity?

The proliferation and diversity of ways anarchism manifests in equity discourse may, at first, seem surprising, given the small number of publications that explicitly draw connections between anarchism and mathematics education (e.g. Lawler, 2019; Restivo, 1998, 2011; Wolfmeyer, 2012, 2017a, 2017b). However, the undergirding tenets of anarchism (e.g. cooperation, mutual aid, and freedom from hierarchy) exist as a common thread within mathematics education equity discourse, conceptually stitching a wide array of professional and political efforts. Here we foreground this thread, offering a silhouette or sketch of anarchism as a potentially valuable macro-theory in mathematics education equity research and practice.

Relations between people – absence of hierarchy. Education as a social activity is a widely held stance; many scholars argue that the field now understands the perspective that mathematical activity and learning is fundamentally social (Lerman, 2000). But the social is necessarily political, and thus the social contexts of mathematical knowledge construction, identity development, and of our work as researchers is best understood through examination of power differentials or hierarchies of this sociopolitical context (Gutiérrez, 2010/2013). Freire (1970/2000) provided useful models to reimagine the hierarchical relationship between the teacher and student so as to disrupt the perpetuation of oppression. These models defined a heterarchical relationship between teachers and students.

Through dialogue, the teacher-of-the-students and the students-of-the-teacher cease to exist and a new term emerges: teacher-student with students-teachers. The teacher is no longer merely the one-who-teaches, but one who is himself taught in dialogue with the students, who in turn while being taught also teach. They become jointly responsible for a process in which all grow. In this process, arguments based on "authority" are no longer valid (Freire, 1970/2000, p. 80)

Steffe and Thompson (2000) identified the problematic nature of the researcher-student relationship in any effort to make a claim about what the other knows. Steffe's Radical Constructivism took seriously the notion that knowledge is constructed, and thus ways of knowing are unique to each knower, viable in their experiential reality. It follows that the researcher must acknowledge their role as observer and work to create second-order models of the child's knowing—fully realizing this second-order model is merely one's own knowing, not a claim of truth about the child. Gutiérrez (2017) similarly disrupts hierarchies of knowing by demonstrating how indigenous ontologies see mathematics as a quality of all living beings. We have methodologies in mathematics education that have potential to disrupt the violence of uninterrogated hierarchies of knowing in our relationships with others (Lawler, 2012).

Disruption of (M)athematics. In the course of engaging in research or praxis in the area of mathematics education, one inevitably encounters a certain narrative of worship around the discipline, a mythology Lakoff and Núñez (2000) refer to as the *Romance of Mathematics*. In broad terms, the *Romance of Mathematics* is a mythology that perceives mathematics as objective, acultural, and beyond the vagaries of the human. The *Romance of Mathematics* constructs and propagates "...the mystique of the Mathematician with a capital M as someone who is more than mere mortal—more intelligent, more rational, more probing, deeper, visionary" (p. 340).

This mythology presents an immediate barrier to equity work. "If mathematics is objective, it makes no sense to be concerned with learners' cultures and lived experiences. If mathematical achievement can be accurately and fairly measured with standardized tests of routinized items, it makes no sense to develop more "subjective" assessments of mathematical understanding. And if mathematics is inherently too difficult for many to master, it makes no sense to try to teach all students rigorous aspects of the discipline" (Ellis & Berry III, 2005, p. 13). In response to this barrier, substantial research has sought, either directly or indirectly, to reveal the illusory nature of the myth (e.g.

Bowers, 2018, pp. 290–291). Some (e.g. Burton, 1999; Sinclair, 2009; Wells, 1990) have observed the powerful and subjective role aesthetic plays throughout the discipline, some (e.g. D'Ambrosio, 1985; Lipka et al., 2012; Meaney et al., 2013; Thomas, 1996) have observed the ways mathematical meaning-making have varied from culture to culture, and some (Burton, 1999; Joseph, 1987; Lakatos, 1976) have observed the ways (culturally localized) human interdiscursivity acts as the force by which mathematics is constructed, to name but a few directions researchers have taken in this vein.

Thus, we see again in this context that equity work is anarchic at its foundation—it opposes the socially constructed hierarchy of discipline. In foregrounding this anarchic thread, we see a means of tying these superficially disparate efforts together. Several of the above cited works (e.g. Lakatos, 1976; Wells, 1990) would not even typically be identified as works of equity, but in their valuing of the human and cooperative aspects of mathematics they nonetheless find a place in the greater social project of social justice. Through the paradigm of anarchism, we highlight a broad array of work that stands in opposition to the hierarchy of discipline.

Disruption of white supremacy. Public education within and beyond North America, in both theory and practice, is grounded in whiteness, (re)producing hegemonic social norms such as cultural deficit perspectives, colorblind racism, race neutrality, and meritocracy (Nicholson, 1998; Ladson-Billings, 2000; Sleeter, 2001). The problems here are not so superficial as simply having teacher education programs or rosters of practicing teachers that are predominantly white (though in some places this is a keenly felt tension); rather, the crux of the problem lies in the ways whiteness is normalized across these spaces. Subscribing to whiteness inherently supports institutionalized white supremacy because it tacitly (or overtly) reifies a system that has historically disadvantaged minoritized groups (Matias et al., 2016).

Equity work in mathematics education has tackled this ever-looming problem from a number of directions. Some work has focused on how positioning and identity manifest in the context of small-group interactions (Bishop, 2012; Langer-Osuna, 2011; Wager, 2014), while other work has focused on broader macro-narratives and bias (LópezLeiva & Khisty, 2014; Martin, 2011; Shah, 2017; Tiedemann, 2002). Some work has tackled the topic of power and domination (Battey & Leyva, 2016; Martin, 2009; Setati, 2008), while other work has interrogated cultural practices and pedagogy in mathematics education (Dominguez, 2011; Gutstein, 2012; Leonard, 2008; Lipka et al., 2005). Some work has considered the insular nature of mathematical language (Halliday, 1978; Herbel-Eisenmann, 2002; Pimm, 1987; Schleppegrell, 2007), while other work has broken tackled white supremacy from an intersectional lens (Bullock, 2017; Civil, 2002; Esmonde et al., 2009; Gholson & Martin, 2014). All this is to say that disrupting white supremacy has been one of the core social projects of equity in mathematics education, and though it has been tackled from many perspectives, they share at their foundation an anarchic valuing of the human's right to self determination and opposition to hierarchy.

The above equity projects share, at their core, opposition to racial hierarchy. In foregrounding the anarchic thread connecting these and other seemingly disparate spaces, we see opportunity. For, though we speak here of race in particular, it is difficult nigh impossible to extricate racial power structures from those power structures that are expressed in terms of gender, sexuality, ability, or socioeconomic status. As Roibeard (2015) said of feminism, "Feminism and anarchism are kissing cousins; feminism aims to abolish patriarchy, yet patriarchy does not stand alone and its abolition is intertwined with the abolition of all oppression. We cannot pick and choose which power structures we like and which ones we don't like; they are all connected, for patriarchy to truly be dismantled they all must be." In foregrounding the anarchic thread undergirding equity work, we see the potential to push back against compartmentalization and the interrelated hierarchies imposed thereon.

In foregrounding the anarchic thread, we see the opportunity to make visible the multiply marginalized (e.g. Crenshaw, 1991; Gholson, 2016), and to proceed in solidarity.

What Dangers and Limitations Must We Consider?

It is important at this juncture that we acknowledge and draw attention to potential dangers and limitations in operationalizing anarchism as a macro-theory. The vision of anarchism expressed throughout this text is a broad one, and there are many interpretations of anarchism which entail different assumptions and values, each having very real consequences in their adoption and use. An exhaustive overview of all of the potential risks of particular instantiations of anarchism would lead us far astray from our central goal of introducing and inviting others into a new paradigm of thought that has not seen explicit representation at PME-NA, so here we must satisfy ourselves with something smaller but more intentional—a scalpel rather than a sledgehammer. We draw particular attention to two substantial and complex areas of risk: Recolonization (e.g. Patel, 2016) and constructed (in)visibility (e.g. Crenshaw, 1991; Gholson, 2016). We do not select these risks for further conversation because they are more important than other risks, but simply because their risk is substantial and because they serve the purpose of suggesting the ways these dangers are deeply and complexly interwoven with the very affordances we spoke about above. Indeed, these two risks are complexly interwoven with each other and will consequently be treated in concert below. After all, one of the tools of colonization is compartmentalization (Patel, 2016), and it is precisely compartmentalization that gives rise to constructed invisibility, that trick of (social) cognition wherein the multiply minoritized are only "seen" in terms of one minoritized identity at a time.

We begin with the perennial tug-of-war between the inertias of *colonization and white supremacy* and conscious efforts of *decolonization and antiracism*. The ways humans continuously come into being, the ways we carry and perform ourselves, are living and breathing mixtures of affordances shaped and informed by the echoes of long-standing yet not sealed histories (Baldwin, 1963; Gordon, 1997). Phrased differently, subjectivities are inextricably entangled with material conditions, and social becomings enact and exact material consequences (Barad, 2007). Given these deep trajectories and the role of educational research in perpetuating settler-slave-Indigenous relationships, educational researchers are answerable to working to dismantle those structures (Patel, 2016).

Like all things, the common image of anarchism summoned by the public imagination is a Eurocentric and white image, commonly conjuring white (cis-male, heterosexual, …) European visages such as Max Stirner, Peter Kropotkin, or Henry David Thoreau. Restricting oneself to these visions of anarchism recolonizes, and (re)constructs many voices and bodies as invisible, hypocritically reifying hierarchy while nominally standing in opposition to it. A decolonizing and antiracist vision of anarchism must think outside these frameworks that mask white Western parochialism as universal and eternal verity (Grovogui, 2006). A decolonizing and antiracist vision of anarchism must expand beyond perspectives that recognize and respond to only one hierarchy—the hierarchy of class and capital. Thus, whatsoever vision of anarchism we might choose in any given moment or context should be grounded in visions constructed and expressed by diverse groups of thinkers and activists (Alston, 2003; Grovogui, 2006; Roibeaird, 2015).

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BEARING WITNESS TO MATHEMATICAL GHOSTS: THE ETHICS OF TEACHERS SEEKING JUSTICE

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Common mathematics education practices such as valuing speed and accuracy over creativity and reflection often cause mathematical trauma. Once established, this trauma haunts students, their teachers, and even a researcher in future mathematical experiences, calling for recognition and for a justice that can only be found through transformative engagement. This study integrates sociological theory, poststructural methods, and empirical data from an ethnographic study of veteran mathematics teachers to explore how teachers enact ethical relations by bearing witness to students' mathematical ghosts and, in doing so, make possible a more just future.

Keywords: social justice; marginalized communities; affect, emotion, beliefs, and attitudes

Common mathematics education practices such as tracking, valuing speed over reflection, and assigning repetitive procedural homework constitute a "slow violence" (Gutiérrez, 2018, p. 3) that (re)marginalizes students who have historically been marginalized in mathematics education due to racism and other forms of oppression, in part by denying students' agency (Lange & Meaney, 2011). Students who have been repeatedly subject to such mathematical violence carry trauma resulting from their prior experiences and also from the "ongoing, accruing impact and consequences of social malignancies such as racism" (Dutro & Bien, 2014, p. 23). In mathematics classrooms, then, "the past haunts the present" (Gordon, 2008, p. viii). If this is the case, how can mathematics teachers engage in ethical response? In other words, how ought they attend to the lasting impact of traumas perpetuated on students by hegemonic forms of mathematics teaching and learning within an already oppressive society?

Conceptual Framework: Haunting

In this paper, I view haunting as an apparition of students' mathematical trauma. Many scholars have probed the presence of ghosts in public schooling (e.g., Ewing, 2018; Lawrence-Lightfoot, 2003); here, I follow Gordon's (2008) framing of ghosts as "one way in which abusive systems of power make themselves known and their impacts felt in daily life, especially when they are supposedly over and done with" (p. xvi). In Gordon's conceptualization, ghosts call out for justice: for the damage that has been done by abusive systems of power to be addressed, not because it can be undone, but in order to "en[d] this history and se[t] in place a different future" (p. 66). In mathematics, then, ghosts might call out for transformative ways of teaching and learning that are no longer traumatizing, violent, or marginalizing.

Methods: Greeting the Ghost

The data for this study were collected ethnographically in a high school serving almost exclusively Latinx students eligible for free-and-reduced-price lunch. I followed two veteran mathematics teachers, Franck and Clark (all names are pseudonyms), in their Algebra 1 classes for one week each month across a full school year, taking extensive descriptive fieldnotes and writing in-process memos (Emerson et al., 2011). Since subjectivity is inevitable in qualitative research, I positioned myself as a partner for brainstorming and reflecting rather than minimizing my role, and was welcomed "as an outsider and as an insider" (email from Clark, 2/2019).

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As I observed, I was struck by the frequency and force with which Franck and Clark named a teacher who had taught many of their students the previous year: Mr. Montoya. Mr. Montoya's ghost first appeared to me in October, when, in an interview about building relationships with students, Franck repeatedly cursed Mr. Montoya for "the damage that he did to these kids." According to students, Mr. Montoya's pedagogy included requiring them to memorize conventions of mathematical notation regardless of whether they made sense, randomly calling on students, and shaming them for not having immediate correct answers; in an interview, a student explained to me that "he kept on trying to make us better each time but in a way that would make us feel like we weren't good." Mr. Montoya may have intended tough love but nevertheless created a mathematical culture of exclusion (Louie, 2017) through hegemonic practices that are especially routine in classrooms with marginalized youth (Gutiérrez, 2018).

Mr. Montoya's ghost provoked strong affects throughout the school year, illustrating the "living effects, seething and lingering, of what seems over and done with" (Gordon, 2008, p. 195). Despite no longer being in his class, students brought him up unsolicited when I asked them about their *current* mathematics classes, saying he was "rude," "really bad," and "he would pick on me." He haunted teachers' interviews and collaborative meetings: Franck called his methods "public abuse," Clark mentioned "kids that hate him," and another teacher, Abigail, said that hearing them talk about him made her both "want to cry" and "fight." As a researcher, I felt constantly alert to his name or the mere possibility that someone might be referring to him. As an instantiation of both the individualized and structural trauma carried in mathematics classrooms, Mr. Montoya's ghost offers an analytic opportunity to examine 1) how students, teachers, and researchers are haunted by histories of oppressive mathematics education, and 2) how we can reckon with ghosts as "pregnant with unfulfilled possibility" for change (Gordon, 2008, p. 183).

To greet the ghost of Mr. Montoya, I engaged in Jackson and Mazzei's (2012) "thinking with theory," reading theoretical perspectives and empirical data through analytic questions derived from and tied to both. The entangled nature of this method presses against forms of research that seek to classify and determine truths after data collection is "complete," instead honoring the ambiguous and emergent nature of any possible "truth." To do so, I attended to the "*flow* and *arrest* of thoughts" (Gordon, 2008, p. 65, italics original) in students' and teachers' talk to identify Mr. Montoya's presence, looking for "how a person translates his or her experience of historical trauma across time and space" (Zembylas, 2006, p. 315). I sought moments of wonder and surprise (MacLure, 2013) experienced by those haunted—students, teachers, and myself as researcher—rather than moments of clarity. I used these moments to examine what is known and what counts as reality, thus allowing ghosts to speak.

Preliminary Findings: Bearing Witness to Mathematical Ghosts

How does one listen when ghosts speak? Gordon (2008) suggests that by haunting, ghosts are "leading us somewhere… [calling for] something to be done" (p. 205), and that exorcising ghosts requires attending to their insistence on a future that is more just than the past (Yoon, 2019). Those who see hegemonic practices of teaching and learning mathematics as violent, then, are called by mathematical ghosts to *do* something. Zembylas' (2006) draws on Kelly Oliver and others to articulate "witnessing as an affective practice [and] an ethical and political project" (p. 316); I use this theorization to describe the somethings that Franck and Clark *do*. First, I present how Franck and Clark bear witness to mathematical ghosts by "see[ing] Others with loving eyes that invite loving response" (Oliver, 2001, p. 19). Then, I illustrate how they "wor[k] through rather than merely repeating the blind spots of domination" (Oliver, 2001, p. 218), being vigilant against the injustices that cause mathematical ghosts to appear (Zembylas, 2006).

Seeing Others with Loving Eyes. In the "state-sanctioned violence" (Yoon, 2019, p. 421) of public schooling, students who do not meet participation expectations are typically viewed as off task, disengaged, or noncompliant; as a researcher who has observed in hundreds of mathematics classrooms, I have seen many teachers react to students this way, and many students accustomed to this treatment. Both I and students, haunted by these assumptions, were consequently *arrested* when these assumptions were subverted. For the sake of space, I offer two brief examples from my data. First, when a student was unprepared to answer a question that Franck had given students ample time to answer independently and also discuss in their groups, Franck responded: "Oh you didn't have enough time. My bad." Similarly, when Clark noticed that several students had left blank his request for comments on a homework assignment, he said to the class, "I don't think you were being lazy. Maybe you just forgot and that's fine… it would be great if you left me two comments today but you don't have to." Mathematics classrooms are haunted by the assumption that students who do not participate how teachers expect them to are deficient. Franck and Clark challenged this so-called reality by offering generous interpretations that saw students with loving eyes instead of recycling damaging assumptions about them. In doing so, they opened up opportunities for future connection and reciprocity (Dutro & Bien, 2014).

Working Through Rather Than Repeating Domination. Franck and Clark, respectively, wanted students to have "an enjoyable five hours [each week]" and to "be at ease in my class." As a result, they grappled with whether particular pedagogical practices would contribute to or undermine this desire, even and especially those taken for granted as commonsense (Kumashiro, 2014) in other mathematics classrooms. Although they discussed many practices, here I share how they considered their participation expectations for students in light of mathematical ghosts summoned both in the guise of Mr. Montoya and by contemporary equity discourse.

Calling randomly on students to answer questions has long been commonsense practice in mathematics classrooms. Like many of their predecessors across decades and classrooms, several of the students I interviewed were haunted by memories of Mr. Montoya requiring that they stand at the whiteboard, in front of waiting classmates, until they could produce a correct answer. Clark bore witness to these mathematical ghosts, accounting for students' experiences and affects—what he called their "anxiety"—by offering ways for students to demonstrate their interest and ideas without repeating this exercise of teacher power:

> If you don't want to share [with the whole class], you can raise your hand and share with me as I'm walking around… you don't have to share in front of the class to get points… I don't want you to share for points; I want you to share for love. (classroom observation, 8/2018)

Ethics, however, demands engagement with the uncertainties and complexities in teaching. Upon hearing that one of their students associated being "randomly pick[ed] on" with "the way [my teacher] trusts me" in an interview that I conducted, Franck and Clark negotiated their surprise around how students interpret their actions (collaborative meeting, 2/2019):

Clark: This is interesting: 'because he really trusts you.' That's interesting. I would hesitate to pick students because I think they would feel—that's an interesting correlation, right?

[*Clark asks if I've seen comments like this before; I say I found it interesting too.*]

Franck: When you are doing the task, you are not particularly looking to solicit an answer, you're soliciting their thinking. 'Whatever you say is correct, even if it's wrong. It's okay, just tell me what you're thinking. You can't think wrong, even if the answer is wrong.' If we didn't do those things and if the problem has one answer and the kid says the wrong answer then they'll feel bad for being wrong…

Clark: I don't call on them randomly.

Franck: But you don't particularly call them for the ANSwer

Clark: No, I avoid that.

Franck: ‘Cause there's no point.
Clark: I would rather not you contribute to the result. Maybe, the process, or something, contribute to the process. Is just as important as contributing the result. So you trust them to contribute is maybe the way it could be seen. You're basically saying, like, the kids feel like you trust them to contribute something to the class. That feels like agency to me.

Asking students to publicly share answers is a common ritual in mathematics classes (Mehan, 1979) and a traumatizing one in classrooms like Mr. Montoya’s, but structured groupwork can also provide equitable access to mathematical learning opportunities (Cohen & Lotan, 1997). Even so, Franck wondered if insisting that students perform in specific ways repeats domination:

> I’m not sure if I’m sold on the fact that we need to make all the students in our class talk to each other… I’ll listen to the kids but I know like [student name] or certain kids in my class they don’t want to talk out loud and I don’t want to force them to. The philosophical questions I’m trying to reconcile are: what people are doing in their classrooms that’s deemed as equity and access [e.g. making students talk], to me it just seems annoying and you’re forcing it upon the kids… I’m still reconciling a lot of things. (interview, 2/2019)

Mathematical ghosts summoned by pedagogical practices taken for granted in both hegemonic and equity-oriented mathematics classrooms make students feel (uncomfortable, trusted, coerced) when teachers require them to speak in front of their classmates. Franck and Clark heed the affective hauntings of these ghosts and the power dynamics that they signal by contending with what forms of student participation they value and why, thereby starting to build towards a less traumatic, more transformative version of mathematics education.

Discussion: Stretching Toward the Horizon

Mr. Montoya is a named ghost who haunted Franck’s and Clark’s Algebra 1 classrooms during the school year I observed. He is also a stand-in for unnamed mathematical ghosts across contexts whereby students and teachers are haunted by histories of racialized oppression in public schooling, histories of hegemonic mathematics education, and their own personal histories of traumatizing mathematics experiences. I have begun to illustrate how Franck and Clark bear witness to mathematical ghosts by seeing their students with loving eyes and by working through rather than repeating domination. What happens when teachers bear witness to mathematical ghosts, rather than perpetuating the “abusive systems of power” that give rise to ghosts in the first place? I suggest that Franck and Clark are adopting an ethical stance in seeking to remedy past injustices by negotiating an alternative vision of mathematics education. The presence of mathematical ghosts attunes them to the violence done to students by hegemonic forms of mathematics education, and by transgressing—by attending to ghosts that might otherwise be ignored—they make possible a future that is different from the past.

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RURAL PARENTS' VIEWS ON THEIR INVOLVEMENT IN THEIR CHILDREN'S MATHEMATICS EDUCATION

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As rural parents can face barriers that are different from those encountered by suburban and urban parents, this study sought to assess the level of rural parents' satisfaction with their level of involvement in their children's mathematics education in a rural location in the Midwestern United States. Semi-structured interviews were conducted with seven parents from one rural school district. Parents' satisfaction with their level of involvement in their children's mathematics education correlated highly with their children's success in school mathematics. Another result found that often at least one of the rural parents in each household commuted a great distance each day from where their children attended school, but that the commuting parent was the parent most likely to help children with mathematics homework. Implications of this finding are discussed.

Keywords: Rural Education, Marginalized Communities, Communication

Purpose

Parental involvement has been shown to positively influence children's education (Sheldon & Epstein, 2005; Liu, Wu, & Zumbo, 2006). Rural parents can face barriers to involvement in their children's mathematics education that other parents do not. Therefore, the purpose of this study was to research the involvement of rural parents in their children's mathematics education. In particular, this study sought answers to the following research questions:

1. How satisfied are rural parents with their involvement in their children's mathematics education?
2. What level of involvement do they desire to have in it?
3. If they want more involvement, what hinders their involvement? On the other hand, if they want less involvement, in what ways would they like to be less involved?

Perspectives

This study was grounded in the previous work of Lareau, Civil, and Remillard and Jackson. First, Lareau (2000) discussed the idea of social capital. Essentially, some parents are able to interact more fluidly with mathematics teachers and the entire school setting than other parents are based on their previous experiences, perceptions of school mathematics and its purpose, and other related cultural values and norms. Indeed, some parents place themselves in perceived positions of authority over math teachers while others would never think of questioning a math teacher.

Next, Civil worked to help parents view themselves as funds of mathematical knowledge (Civil, Bratton, & Quintos, 2005; Civil & Bernier, 2006). Many parents have the knowledge to assist their children with school mathematics but simply are not confident in doing so. Regardless of their current mathematical knowledge, Civil et al. helped their participants see that they could both learn mathematics and facilitate mathematical learning at the same time.

Finally, Remillard and Jackson considered parents' involvement in their children's learning and schooling, both in terms of what the school could and could not see (2006). They considered a parent-centric view of parental involvement that consisted of three things: involvement in the

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children's learning, involvement in the children's schooling, and involvement in the children's school. While all three are important, only the last of the three is directly observable by the school.

Methods

Context

Rural places and people are not homogeneous. There is variety among rural people both in different localities and within the same locality. This study focused on rural people who live within a 45-minute drive of an urban center in the Midwestern United States. Originally, families were sought for this study that both a) lived outside the city limits of the town or towns in which the children attended school and b) lived at least five miles from all schools attended by the children at the time of the study. Selecting parents to interview in this manner would have eliminated both those who live in towns and near schools. It was hoped that what remained would constitute a population from which a characteristically rural sample may be have been drawn.

Participants

Parents were recruited with the assistance of principals and mathematics supervisors. The principals and mathematics supervisors were sent a letter detailing the project and asking for help to identify potential participants.

Two principals recommended thirteen people who agreed to give their contact information to me. After being contacted, seven parents eventually participated in the interview process. After the interviews began, it became clear that the principals recommended some people who did not satisfy the original participant criteria of living outside of town and at least five miles away from the schools. Upon analysis, the comments of the parents who lived closer to the schools than originally desired were quite similar to those of the parents who did satisfy the original criteria. Therefore, the inclusion criteria were relaxed.

One *severe* limitation of this study, though, is that many of the people recommended by the principals were employed somehow either with the school district in which their children lived or another school district in the area (see the table below). When studying parental involvement in children's mathematics education, this clearly produces an unwanted bias in the participant sample; however, on one hand, because of this it could be argued that we should expect to see the best parental involvement results possible with this sample. In a more unbiased sample, we would expect to see less involvement and more frustration among parents than in this sample of school-affiliated parents. Therefore, the results below may well indicate a best-case scenario for rural parental involvement, while the reality is likely worse. On the other hand, it could be argued that a more representative sample of rural parents may well be nothing like this sample of mostly school-affiliated parents. Problems like this are typical of parental involvement studies. For example, Civil and Bernier's (2006) study of 15 mothers contained several who were teacher's aids, some who were members of the local parent teacher organization, and some who were already regularly volunteering in the schools. Similarly, nearly half of Jackson and Remillard's (2005) sample of mothers were highly involved in their children's schools before the study. While this study's sample is somewhat flawed and non-representative, the results below could still inform our ideas of rural parental involvement in mathematics education. Below is a table profiling the participants.

Table 1: Participants

Name	Children's Age, Grade, Gender	Education	Occupation	Spouse Education	Spouse Occupation
Amanda	23 21, F 15, 9th, F 12, 7th, M	Bachelor's degree	Nurse		

Brad	15, 10th, F 12, 7th, M 9, 4th, M 8, 2nd, M 3, F	Bachelor's degree	Youth minister in town of children's school	Bachelor's degree	Stay at home mother
Chad	15, 10th, F 12, 7th, F 8, 3rd, M	Master's degree	Middle school teacher 31 miles away	Associates degree	Childcare professional at children's school
Denise	12, 7th, F	One year of college	Lunch lady at school in child's district	High school diploma	Retail manager 25 miles away
Elizabeth	19, M 17, 11th, M 14, 9th, M	Master's degree	Elementary resource room teacher in children's district	Bachelor's degree	Administrator 17 miles away
Fallyn	17, 11th, F 13, 8th, M	High school graduate	Secretary in children's school	Two years of college	Construction foreman 24 miles away
Gabby	19, F 17, 11th, F 14, 8th, F	Master's degree	Kindergarten teacher 8 miles away	Bachelor's degree	Computer specialist 58 miles away

Data Collection and Analysis

Semi-structured interviews were conducted with the seven parents. With permission, the interviews were audio recorded and transcribed. Direct and indirect questions were asked to assess the participants' thoughts with respect to the research questions. Responses were probed to make sure respondents were being understood as fully and correctly as possible. The researcher also attempted to ask follow up questions that enhanced the narratives being given by the interviewees. As the interviews were semi-structured, a few standard questions were asked, but the flow of the interview and the information gathered were largely left to the interviewees.

In conducting these interviews, I was thinking of parental involvement in the terms of Lareau, Civil, and Remillard and Jackson. From Lareau (2000), I considered whether parents were as involved as they would like to be and maintaining communication about mathematics education with the schools in a way with which they felt comfortable. From Civil, I wanted to assess whether parents viewed themselves as funds of knowledge (Civil & Bernier, 2006). Furthermore, from an ethnomathematics perspective, I wanted to know how well they felt their knowledge and use of mathematics aligned with their experiences of helping their children with schoolwork. Finally, from Remillard and Jackson (2006), I wanted to ask about the parents' involvement in their children's learning and schooling, both in terms of what the school could and could not see.

After the interviews were collected and transcribed, an open coding process began. The interviews were listened to in their entirety as notes were affixed to the transcripts summarizing what points were being made. Then, the transcripts and notes were inspected for common themes. When a theme was found, all instances of that theme were highlighted in all transcripts.

Results

The first two research questions were: How satisfied are rural parents with their involvement in their children's mathematics education, and what level of involvement do they desire to have in it? During data analysis, it became apparent that these two research questions could not be answered separately. When asked the first question, five of the seven parents interviewed said that they were satisfied with their level of involvement. For many of them, though, it appeared that they were satisfied with their level of involvement because between them and their spouses they were doing what needed to be done for their children to succeed. So, the apparent answer to the second research question seemed to be that they desired whatever level of involvement was necessary for the success of the children at school mathematics.

One parent who stated at the beginning of the interview that she was "absolutely not" as involved as she would like to be was Gabby. She said she wished she could be more directly involved, but was happy that her husband and a hired tutor were helping her daughters get what they needed:

> Gabby: My husband does a little more, but if we really get in trouble, we hire a tutor.
> Interviewer: How has the experience been so far with hiring the tutor? Was that pretty easy to do, to find somebody?
> Gabby: It was wonderful. Because I'm a teacher, it was pretty easy. I think it's probably more difficult for other people. I think it was wonderful if you can afford to do it. Especially being a teacher, I'm kind of done with that when I get home, and I don't really want to deal with it any more. So, it's really nice. I happily would pay somebody to do that for me.

The other parent who wanted more direct involvement was Elizabeth, a kindergarten teacher in the district where her children attend school. She said all of her children have struggled with math and that she probably should have been more involved. She cited concerns of both her perceived lack of mathematical ability beyond the elementary level and the time commitment it would take to be on top of communicating with mathematics teachers in a meaningful way as keeping her from being more involved.

Even among the five parents who said they were satisfied with their level of involvement in their children's mathematics education when directly asked, sometimes the interviews indicated that they would be happy to be less involved if they thought their children could succeed in class with less involvement on their part. Denise was one such parent:

> Interviewer: When you say you're really involved in your daughter's math education and working with her on the advanced math stuff, what does that mean? What does that take the form of?
> Denise: It means I have an A in math right now. <*laughs*> Just kidding. It means that, basically, she'll come home with her math book and a sheet of questions, and she'll say, "I don't know how to do any of these. <*laughs*> And I'll have to read over the book and the first few pages of the chapter, and do the examples, and then teach her the math…I feel like I'm in class every time she has homework, and I have to relearn seventh grade algebra piece by piece, and then teach it to her, and then do the problems together.

So, in this case, Denise expressed before the above excerpt that she was as involved as she wanted to be, but it might be more accurate to say that she was as involved as she thought she needed to be for her daughter to succeed. At several points in her interview, she expressed frustration that her daughter seemingly always came home not knowing how to do the problems on the homework, which often made her question what went on during her daughter's math class. Denise's willingness to be involved in the process of doing the homework at this high of a level could also have contributed negatively to her daughter's education by allowing her daughter to pay less attention in class. Still, she was doing what she thought she needed to do in order for her daughter to be successful.

This brings us to the third research question regarding what causes unwanted parental involvement and what hinders desired parental involvement from happening. While Denise's comments best illustrate the only type of desire for less involvement found during the interviews, several common themes emerged from the majority of the interviewees' comments about what hindered their involvement. First, most of the parents that were interviewed said either they or their spouse had already reached a point where they did not feel confident helping with or doing the mathematics that their children were doing in school, or that they anticipated reaching that point in the future.

Whether the parents had already felt like they could no longer help their children with mathematics homework depended chiefly on the age of their oldest child. Those parents with at least one child in or beyond high school spoke of it as something that had happened already, as Amanda, Fallyn, and Gabby did. Those parents whose oldest child was only in middle school anticipated it, though. Denise, whose only daughter is in seventh grade, said:

There seemed to be two potential causes for this split. First, the parent most likely to be interacting with the school either worked in the school or at least in the town where the school was located. The other parent generally did not work in the same town where the school was located, and often worked in another town some considerable distance away. This sort of long distance commuting is common in rural areas; however, given these commuting distances, it makes sense why the one parent would be communicating with the school more regularly than the other because of proximity. Elizabeth, who works in the local school but whose husband works nearly twenty miles away, was one example of this phenomenon:

Second, however, where this parent split existed, it fell directly on gender lines. The mothers were more likely to be working at the school or in the town of the school, and the fathers tended to work in another town. Also, among the parents interviewed, the two men were helping their children with mathematics homework. One of them noted that his wife wasn't comfortable doing so, or would often ask him questions afterward on the occasion that she did assist her children with math. The majority of the five women interviewed said that their husband was the primary mathematics homework helper in the home, particularly once the children were beyond elementary school. Chad, who is a certified mathematics teacher in another district over thirty miles away, but whose wife works in the local schools, was in this situation:

Conclusions

This study also attempted to determine what hinders and what facilitates rural parental involvement in mathematics education. One main finding was that while the geographically closer mothers were often the parents engaging in the most communication with the children's teachers, the fathers who were more confident in their mathematics ability but worked farther away were most often the parents assisting the children with their mathematics homework. Clearly this disconnect is not ideal. Having teacher communication with the parent most likely to help the children with the mathematics would be more helpful.

It would appear that there are two potential ways to close this disconnect. First, as Civil et al. (2002) and Jackson and Remillard (2005) have done, the mothers can be engaged in a way that results in them feeling more confident with their mathematics ability. In turn, they might be more likely to help their children with their mathematics homework. Recall Gabby, who helped her children with homework in other areas, but said she "must" stay away from helping her children in mathematics because of her lack of confidence at doing mathematics.

Second, perhaps more could be done to facilitate more communication between the fathers and the schools. Granted, the fathers tended to work farther away. This largely caused the lower level of communication with the schools and lower attendance at parent conferences. One tool that could potentially help this situation is the Internet. While work locations may well interfere with a rural father's ability to attend parent-teacher conferences held during the school day or shortly thereafter, the Internet is available at any time. Perhaps mathematics curricula authors and mathematics teachers could work together with rural fathers to find ways to engage with each other meaningfully using the Internet. This could be as simple as sending out a daily email as some of the interviewed parents said some teachers did. It could also be more broad-based in scope, with curricula authors developing modules for teachers to display on their websites for parents not familiar with the type of mathematics being taught or the way it is being taught. Also, while it is closing, the digital divide is still a matter to contend with for many rural families. While this is still the case, perhaps more low bandwidth Internet solutions should be studied.

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AN ETHNOGRAPHY OF RE/HUMANIZING (MATH) PEDAGOGIES AT A PREDOMINANTLY LATINX CALIFORNIA HIGH SCHOOL

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This ethnographic study of one predominantly Latinx-serving high school in California theorizes around the functioning of re/humanizing pedagogies layered through the school and the mathematics department as they challenged flattened social, academic, and mathematical identities. Findings revealed that math department practiced built from school-wide commitments, offering students de-flattened identities. However, challenges remained specifically related to the availability of advanced mathematics courses.

Keywords: culturally relevant pedagogy, equity and diversity, high school education, marginalized communities

Colonial projects flatten the histories and lived experiences of minoritized communities, presenting homogenized narratives that label, compare, and sort people into groups for the sake of resource extraction and economic gain (Mignolo, 2003; Said, 2012). Schools are critical sites for the reproduction of these flattened social categories, (Eckert, 1989; Willis, 1977). Mathematics education participates in this project by providing fodder for categories of intelligence through the political legitimacy of mathematics (Apple, 1993) and the proliferation of tracking available in K-12 mathematics education that consistently relegates minoritized students to lower tracks where they are denied quality learning opportunities (Oakes, 1990).

The persistent reproduction of inequitable mathematical learning experiences and outcomes for minoritized students is well documented (Martin 2013; Tate, 1994). Yet, calls for change most often center mathematics pedagogy and programming as sites for intervention without attention to school-wide culture and organization beyond mathematics itself (NCTM, 2018).

This paper presents findings from a school-based ethnography that examined the production and negotiation of identities of mathematical competence across one predominantly Latinx-serving high school in California. The study uses re/humanizing pedagogies as a lens for asking how identities of mathematical competence were produced and negotiated. The dual focus on both whole school and math department commitments and practices revealed both resonance and tensions. Findings demonstrate that school and math department commitments challenged the flattening of social, academic, and mathematical identities to varying degrees of success.

Humanizing Pedagogies and Rehumanizing Mathematics

I use humanizing pedagogies (Bartolome, 1994) as an umbrella term inclusive of those pedagogies that challenge the colonial flattening of minoritized students. Under this umbrella, and specific to this study I focus on pedagogies that center care and relationship (Noddings, 1988; Valenzuela, 2005), and those that embrace responsibility for providing opportunities for academic excellence and supports, especially attending to the most vulnerable or traditionally excluded (Ladson-Billings, 1995; Duncan-Andrade, 2005). Care and responsibility were central commitments of Sierra High School (SHS), the site of the study. Relations of care and deep responsibility for students were understood by SHS staff as challenges to the exclusion and neglect common to experiences of schooling for minoritized students.

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Specific to mathematics education, Gutiérrez (2018) poses eight dimensions of rehumanizing practice. SHS math department commitments closely resonated with two: participation/positioning and the broadening of mathematics. Rehumanizing mathematics practices position students as valued participants in mathematics itself, not simply as consumers. A broadening of mathematics includes understanding mathematics as a living practice that entails communication, reflection, visual approaches, and collaboration.

The dual frameworks of re/humanizing pedagogies and practices provided a lens through which to conceptualize the relationship between the school and math department in disrupting flattened social, academic, and mathematical identities to varied degrees of success.

Methods

Ethnography is an anthropological method for tracing cultural production and involving participant observation in the community of study over an extended period of time (Geertz, 1973). Ethnography was chosen for this study to enable mapping production and negotiation of intersecting social and mathematical identities at multiple layers.

Sierra High School, the site of the study was a predominantly Latinx-serving public high school in a small urban district in California. The school faced a history of racialized stigma as the only predominantly low-income and predominantly Latinx school in a highly segregated district. In the years immediately preceding and including the study, the school community was conscientiously working to provide a counter-narrative to this entrenched negative reputation.

Over two school years, the author acted as a participant observer across multiple school contexts including academic and non-academic spaces, with a focus on mathematics teaching and learning. All math teachers were observed and interviewed. Formal and informal interviews were conducted with students and either audio-recorded or captured in fieldnotes. Detailed fieldnotes were produced daily. Artifacts were collected including lesson plans, student work, school brochures, and math course enrollment data by student.

Analysis entailed bi-weekly review of field notes and artifacts and the production of an analytic memo (Emerson, Fretz & Shah, 2011). Analytic memos captured new and recurring themes and critical incidents that illuminated or contradicted details of a developing theme (Creswell & Poth, 2016). The themes of care, responsibility, and de-flattening were identified and elaborated through this process.

Findings

I use two metaphors from SHS – the Energy Bus and the sheep – to illustrate the SHS staff's commitments to care and responsibility. These school-level commitments permeated the mathematics department as well. A commitment to responsibility was expressed through the providing students an abundance of opportunities for both academic excellence and academic support. These opportunities also provided opportunities for relationships of care between teachers and students. Together, forms of responsibility and care functioned to challenge the flattening of social and academic identity categories, including but not limited to those of mathematical competence. However, challenges remained. Specifically, the availability of categories related to advanced math courses reproduced flattened math identities and racialized the distribution of these positions.

Sierra High School Commitments: The Energy Bus and the Sheep

When I initially approached the SHS math department about research I was enthusiastically welcomed to the "Energy Bus." The Energy Bus was a metaphor used by the staff to describe the dramatic commitment of energy to students that SHS staff undertook together. This metaphor permeated staff communication. The Google Drive that hosted teacher resources was named "The

Energy Bus." At the opening staff meeting of school year 2018-2019, new staff were introduced and enthusiastically commanded to "Hop on the Bus!" (Fieldnote 8.10.18).

At the same opening staff meeting the Principal shared a video from her summer trip to Ireland, where she visited a sheep farm. Staff knew that she was raised on a sheep farm, and the use of sheep as a metaphor was familiar. The video showed a tractor and a sheepdog working together to maneuver a herd of sheep across a road into pasture. The sheepdog was seen gently pursuing one sheep who had strayed from the group. After sharing the clip, Principal James asked, "Did anyone think of any likenesses?" Staff called out, "kindness," "patience," "encouragement" comparing the role of the sheepdog's to theirs as teachers. One staff member commented, "You gotta get them all, we're not going to leave any one on the side of the road." The principal concluded, "Love always goes way farther with our kids. They need clear boundaries and some sternness, but there has to be love in there." (Fieldnote, 8.10.18).

The Energy Bus and the sheep metaphors were lived in tandem at SHS, through the provision of a multitude of opportunities for academic and extra-curricular excellence as well as academic and social supports. Centrally organizing the joint commitment to academic rigor and academic support was the combination of two programs: Advancement Via Individual Determination (AVID) and the International Baccalaureate (IB) program. These two programs traditionally have distinct target audiences. An AVID teacher described AVID as supporting "students who could go to college but just didn't have the things they needed to get there" (Interview, 1.11.19). In contrast, IB coursework and the IB Diploma program provide an elite international certificate of advanced standing.

Students, while enthusiastic about the plethora of high quality academic and extra-curricular opportunities offered, consistently cited their teachers as the best thing about SHS. Students described teachers using words such as "supportive," "friendly," "nice," and "helpful." One student explained, "All the teachers I've had - they were always there when I needed help, so I think teachers are pretty amazing here" (Interview, 10.24.18, 11th grader). Another student said the best thing about SHS was "the connections I feel with my teachers. The more you build connections with them the more confident you feel to ask a question or share during class" (Fieldnote, 9.11.18, 12th grader).

The SHS Math Department: We Keep it Rollin'

SHS math teachers in particular were recognized by students and staff as providing multiple layers of support and care, making themselves available during lunch and after school, and providing opportunities for re-takes of assessments and submission of late assignments. As one guidance counselor said, "Our teachers are here, especially in the math department, all the time. They're here all the time working with kids that come to see them" (Interview, 10.12.18). Classroom observations captured interactions across all ten math teachers that reflected rapport with students – examples of evidence included the use of humor and making connections to students' lives within and beyond instructional contexts.

In accordance with NCTM's articulation of the tenets of high quality mathematics instruction (NCTM, 2014), the SHS math department was committed to student-centered and discourse-oriented approaches to teaching and learning mathematics. These approaches centered student participation and positioned students as authors of mathematical ideas, with students relying on each other for evaluation and revision of those ideas through peer conversation. These practices were in line with Gutiérrez's (2018) "participation/positioning" dimension of rehumanizing mathematics. Observations of all teachers evidenced that students worked in collaborative groups and engaged in rich mathematical tasks. The department used a set of curricular materials reflecting discourse and sense-making approaches to mathematics. The department collaborated to revise and extend the materials to support and challenge their own students where they saw the need, providing expanded notions of the discipline of mathematics, Gutiérrez's sixth principle.

De-Flattening: Successes and Challenges

The wide range opportunities available and the commitments of adults to building connections with students provided students multiple access points to into the school community and relatedly multiple opportunities for complex identities. For example, in a space with a majority Latinx student-body this meant that understandings of what it meant to be Latinx were taken for granted as multiple and complex - drum major, IB student, lead actor in the school play, football player, valedictorian, or community service club president.

The particular combination of opportunities and supports combined with the caring teacher relationships also served to broaden notions of academic excellence. For example, students at times were jointly enrolled in IB courses and in AVID. AVID, a symbol of extra support for reaching college, was widely respected at the SHS suggesting a de-flattening of notions of excellence in which making use of a wide range of academic supports was not equated with being a low-achiever or being unsuccessful.

In the context of mathematics, students also articulated complex ideas about mathematical competence where students consistently referenced participation, perseverance, and being able to explain to others as indicators of mathematical competence. While many students also shared notions of mathematical competence that included accuracy and speed, in line with the dominant narratives of success in math (Franks, 1990), the de-flattening of mathematical competence itself provided expanded opportunities for students to identify as people with mathematical competence.

One significant challenge to the de-flattening of identities through re/humanizing pedagogies at SHS was the fact that the multitude of programs made available a multitude of categories related to these courses and programs. While the aspirational vision of the school was one where every student benefited from both IB-level rigor and AVID-informed supports, the particular prestige of IB courses was recognized by students as a marker of comparative intelligence and status. In the math department specifically, Latinx students were underrepresented in IB courses and in their advanced prerequisites.

Discussion and Implications

School-wide commitments to care and an outpouring of energy through academic and extra-curricular programming was reflected in the mathematics department through student-teacher relationships and was extended through pedagogical commitments to student-centered, sense-making pedagogies that offered expanded views of mathematics and therefore more expansive and inclusive opportunities for students to identify people with mathematical competence.

Careful attention to the school context revealed the ways in which school level commitments shaped the aspects of rehumanizing mathematics practices taken up in the math department. Findings suggest that the rehumanizing math practices that math teachers develop will draw from whole-school commitments. At the same time, there may be tensions between the school-level conceptualization of humanizing pedagogies and enactment in the mathematics context. For example, mathematics education is one of the disciplines most susceptible to notions of comparative categories, including both advancement and remediation, such that the multitude of high quality opportunities at SHS, while largely celebrated by students and families, played into the reproduction of widely available of narrow, comparative, and racialized categories of math student.

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SUPPORTING THE WHOLE STUDENT: BLENDING THE MATHEMATICAL AND THE SOCIAL EMOTIONAL

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This case-study sought to describe instruction that supported students' mathematical learning (ML) and social-emotional learning (SEL). Transcripts from audio-recorded lesson observations and teacher interviews, field notes, and written teacher reflections were collected to answer two research questions: (1) In what ways does a high school mathematics teacher support ML and SEL during instruction? (2) How does the teacher characterize her attempts to provide social-emotional and mathematical supports during instruction? Preliminary findings suggest that strategies for revising ideas and handling errors support students ML and SEL.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Classroom Discourse; Equity and Diversity; High School Education

Traditionally, research in mathematics education has focused much of its attention on achievement. However, a growing body of research suggests that maintaining a focus on achievement ignores other key factors in understanding students' learning of mathematics, especially in discussion- and collaboration-oriented classrooms of the reform era (Bargagliotti, Gottfried, & Guarino, 2017; Battey & Levya, 2016; Horn, 2008). More specifically, researchers have highlighted the importance of understanding the social and relational aspects of particular learning contexts (Battey, 2013; Battey & Levya, 2013; 2016; Hackenberg, 2005; 2010; Moschkovich, 2002).

Across a variety of contexts, researchers have noted the importance of teacher-student relationships (Allexsaht-Snider & Hart, 2001; Averill, Anderson, Easton, Smith, & Hynds, 2009; Delpit, 2012; Hackenberg, 2005). These relationships are especially important – and complex – when students and teachers from different cultural backgrounds come together to pursue the learning of mathematics (Battey, 2013; Delpit, 2012; Hackenberg, 2010). Therefore, the primary goal of this study is to describe mathematics instruction that fosters students' development of positive relationships with individuals and mathematics within the classroom.

Theoretical Framework

Seeking to understand the social and relational elements of instruction is a complex undertaking (Battey, 2013; Hackenberg, 2010; Horn, 2008). A previous study (Gartland, 2019) identified instructional moves made by a third-grade mathematics teacher that supported students' mathematical learning (ML) *and* their social-emotional learning (SEL). Additionally, Bargagliotti, et.al. (2017) linked instructional choices made in kindergarten mathematics to students' learning *and* social-emotional development. However, I have yet to locate literature on supports for ML and SEL at the high school level. This study addresses that gap.

Supporting Mathematical Learning

Positive social interactions and relationships within the classroom are a necessary element of instruction. Thus, Battey's (2013) framework for *relational interactions* to help define what I consider to be instruction that supports ML. A relational interaction is "a communicative action or episode of moment-to-moment interaction between teachers and students, occurring through verbal and nonverbal behavior that conveys meaning and can mediate student learning" (Battey & Levya,

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2013, p. 981). A relational interaction can be categorized as: addressing behavior, framing mathematics ability, acknowledging student contributions, attending to culture and language, or setting the emotional tone (Battey, 2013). These categories take into consideration a teacher's Mathematical Knowledge for Teaching (Loewenberg Ball, Thames, & Phelps, 2008) and other instructional practices for focusing students on mathematical ideas (Battey, 2013). Thus, I consider instruction that supports ML to be the observable relational interactions and instructional decisions associated with meeting a particular mathematics learning goal.

Supporting Social-Emotional Learning

Instruction that supports students' SEL has been most systematically researched in the K-6 context (Weissberg & O'Brien, 2004). However, with increasing numbers of districts adopting SEL initiatives, the Collaborative for Academic, Social, and Emotional Learning (CASEL) has taken a leading role in working with schools to research and promote SEL best practices (Blad, 2015) at all levels of schooling. Across SEL curricula and policy documents, several key features emerged. Instruction that supports SEL in students can be defined as what a teacher does to promote self-awareness and self-management of feelings, positive identity development, and decision-making skills among students (CASEL, 2019; Elias & Moceri, 2012).

Blending Mathematical and Social-Emotional Supports

More research shows the benefits of SEL-focused interventions than on intentional blending of academic *and* social-emotional supports (CASEL, 2019; Weissberg & O'Brien, 2004). Even fewer studies directly link ML and SEL. In one such study, Bargagliotti, et.al. (2017) highlighted relationships between kindergarten mathematics instruction and academic and social-emotional outcomes. Although they noted several positive associations between mathematics instruction and *both* academic and social-emotional outcomes, they warned "readers against generalizing these findings to grade levels beyond kindergarten" (p. 27). In response to their concern, this study aims to explore similar phenomena at other grade levels.

Since little research exists that explicitly describes instruction that supports both ML and SEL, other literature informs potential areas of overlap. Real-time reports of academic-SEL integration in school districts (CASEL, 2018) as well as research on learning from errors in mathematics (Steuer, Rosentritt-Brunn, & Dresel, 2013; Zander, Kruetzman, & Wolter, 2014) point to instruction on the handling of errors as a means for improving discussion and collaboration. Futhermore, instructional strategies such as *rough-draft talk* (Jansen, Cooper, Vascarello, & Wandless, 2017) can provide students with opportunities to discuss not only errors but also incomplete mathematical thoughts in ways that enhance the learning experience. These findings highlight a space for potentially observing instruction that supports both ML and SEL.

Guided by these ideas, this study explored two research questions: (1) In what ways does a high-school mathematics teacher support ML and SEL during instruction? (2) How does the teacher characterize her attempts to provide social-emotional and mathematical supports during instruction?

Methods

Participant

This paper presents the preliminary findings from a case-study (Hatch, 2002) of one high-school mathematics teacher, Ms. Yang. She has 5 years of experience teaching at a large suburban high school, and she has taught versions of Geometry, Algebra, Pre-Calculus, and Calculus. Ms. Yang actively participates in professional development and continuing education focused on improving students' discussion of mathematics.

Data Collection

A variety of data was collected over the course of a high-school semester. Four audio-recorded observations of an 80-minute, semester-long, college-preparatory, integrated mathematics course were conducted. In addition to the audio-recordings, each observation also generated field notes, a transcript, and a written teacher reflection. Finally, following the final observation, Ms. Yang participated in a half-hour semi-structured interview (Strauss & Corbin, 1994).

Data Analysis

Data analysis is currently ongoing. Thus, data has been, and will continue to be, coded using multiple approaches. First, a priori codes informed by the literature were applied to the data. More specifically, data was coded for observed relational interactions (Battey, 2013), which constitute ML supports, and observed SEL supports (CASEL, 2019). Next, an open coding process was used to reveal any unanticipated themes or patterns (Hatch, 2002). Codes have been compared and grouped using a constant comparison approach (Strauss & Corbin, 1994). Observation data has been more thoroughly analyzed than the interview data at this point.

Preliminary Findings

The following sections present two emerging themes. First, rough-draft talk frequently stood out as an instructional practice that supported ML and SEL in Ms. Yang's class. Second, while reflecting on and discussing her own teaching, Ms. Yang consistently linked student feelings of comfort and confidence with high quality mathematics.

"I'm Not Looking for a Final Answer."

To the left of Ms. Yang's Smart Board was a section of decorated space dedicated to "Rough Draft Talk," in which paper cut-outs of hands giving a "rock-on" gesture contain examples of what constitutes rough-draft talk. On another wall, the "Rights of Learners" (Kalinec-Craig, 2017) were listed: students in this classroom always have the right to be confused, to make a mistake, to say what makes sense to them, to share unfinished thinking and not be judged, and to revise their thinking. These displays and the consistency with which she referred to aspects of rough-draft talk as conceptualized by Jansen, et. al. (2017) and the rights of the learner showed that they were an integral part of her teaching practice. Most importantly, instructional practices involving rough-draft talk emerged as the ones most frequently supporting both ML and SEL.

While coding the data for relational interactions and SEL supports, references to and the use of rough-draft talk appeared often. To investigate relationships between the three constructs further, excerpts that illustrated the principles for supporting rough-draft talk (Jansen, et.al., 2017) were selected. Then, any codes for ML supports (i.e. relational interactions) *and* SEL supports associated with the excerpts were noted. This process allowed connections to be drawn between key features of each construct. Table 1 summarizes the relationships between the principles for supporting rough-draft talk and observed instruction that was coded as a relational interaction *and* a support for SEL. Also included in the table is an excerpt illustrating each principle.

Table 1: Rough-Draft Talk as a Support for Mathematical and Social-Emotional Learning

Principles for Supporting Rough-Draft Talk	Corresponding ML Categories	Corresponding SEL Supports
1) Foster a culture supportive of intellectual risk taking	Framing mathematics ability; setting the emotional tone	Promoting decision-making skills

Example: "Can I see your graph? Would you be willing to share it on any of these boards? No pressure, no pressure…Oh! Snaps! Snaps! Hey, and that's your right, right? Share unfinished thinking. No one's judging you. Thank you for even starting this off."		
2) Promote the belief that learning mathematics involves revising understanding over time	Acknowledging student contributions; addressing behavior	Promoting self-awareness and self-management of feelings; promoting positive identity development
Example: "Do you know all the words for the different transformations? Can you rattle them all off? …How about this, if you don't know them just say left/right, up/down, big/small. Okay? Mirror. Whatever it is. Then we'll refine those words together."		
3) Raise students' statuses by expanding on what counts as a valuable contribution	Acknowledging student contributions; framing mathematics ability	Promoting positive identity development
Example: "Thank you! For being honest about that! How many of us felt that way? Like, 'the three minutes lapsed and I don't have any idea what is happening.' Hey! Look around the room. Everybody's in the same boat as you. You're good. Alright, so help each other out."		

"Be Brave. Be Kind"

A second finding is that when discussing her own instruction Ms. Yang tends to focus on decisions related to student discussions of mathematics. In describing those decisions, she almost invariably links the decisions to perceived student comfort and confidence. For example, Ms. Yang explained that "the students shared how it's not so much what their teachers say that makes them feel devalued, but it's what [their teachers] don't say when they present an answer that makes them feel like there was something inherently wrong with their response." In working to avoid those types of interactions, Ms. Yang expects herself and her students to live by the class motto: "Be brave. Be kind." She believes that deeper and higher quality mathematics discussion, and therefore learning, will take place only after students feel comfortable and confident in the classroom. This finding will likely be expanded as the analysis of the interview data is completed.

Discussion and Conclusion

Individually, these findings identify a particular instructional strategy that supports both ML and SEL and highlight the ways in which a high-school teacher characterizes her own attempts to provide such supports. Taken together, these findings suggest that teachers who attend to student comfort and confidence within the classroom are likely to be implementing instructional strategies that support both ML and SEL. This is significant because such limited research exists beyond the K-6 grade bands related to the integration of ML and SEL. More specifically, it provides a viable starting point for future research.

Limitations associated with the single-participant case-study design prevent generalization and establishment of any links between the support of ML and SEL and student learning outcomes. Additionally, this study excludes the student perspective. Thus, immediate future research will focus on collecting and analyzing student data to continue gaining a deeper understanding of instruction of this type.

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NON-BINARY PEOPLE'S VIEWS OF GENDER AND MATHEMATICS

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A considerable corpus of research exists about people's views of gender and mathematics. As this research is nearly always reported by binary participant groups (e.g., women/men), there is a gap in the research about the views of people with non-binary genders. We conducted a study in Canada and Australia about the general public's views of gender and mathematics. Here, we report on the findings specific to the non-binary participants in the study (n = 7). Participants generally were quite gender-egalitarian in their responses, demonstrated sound understanding of gender as a social construct, and avoided the use of "sex" language and binary language. We conclude by discussing considerations for conducting research with non-binary participants.

Keywords: LGBTQ; Gender and Sexuality; Affect, Emotion, Beliefs, and Attitudes

A great deal of research has been conducted about people's views of gender and mathematics, the vast majority of which has been undertaken with students, teachers, and parents (e.g., Denner, Laursen, Dickson, & Hartl, 2018; Moè, 2018; Nürnberger, Nerb, Schmitz, Keller, & Sütterlin, 2016). Although parents and teachers certainly play a substantial role in students' developing conceptions of gender and mathematics, it is also important to understand the broader context in which these conceptions form. Therefore, investigating the general public's views of mathematics provides crucial information about other views to which students are exposed.

There is a paucity of research about the general public's views of mathematics, and even less regarding the general public's views of gender and mathematics. One notable study about the latter topic was conducted by Forgasz and Leder in Australia, working with international collaborators in Canada, South Korea, Spain, and the United Kingdom (e.g., Hall, 2018; Forgasz, Leder, & Gómez-Chacón, 2012; Forgasz, Leder, & Tan, 2014). Participants were asked about their views of gender and mathematics, as well as related topics (e.g., science). All of the gender-related questions used in this study were worded in a binary manner (e.g., "Who are better at mathematics, girls or boys?"), and participants' genders were assumed, based on appearance, by interviewers.

Although the findings of this study are informative, we were concerned about the binary wording of the questions, as well as the gender attributions (Ryle, 2019) made by the interviewers. Therefore, we adapted Forgasz and Leder's instrument so that the questions were written in a non-binary manner, and participants were explicitly asked to identify their genders. We trialled this instrument in Canada and Australia with approximately 400 members of the general public.

Here, we report on findings specific to a participant group that is vastly under-represented in research: non-binary people. In studies regarding people's views of gender and mathematics of which we are aware, findings are presented by binary participant groups (e.g., girls/boys). Such groupings are indicative of binary conceptions of gender and therefore marginalize an entire gendered participant group and overlook their views and experiences.

Theoretical Perspectives

Working from a feminist and social constructivist stance, we view gender as a performative social construct that occurs on a spectrum, rather than in a binary (Butler, 1999; Ho & Mussap, 2019). Specifically, we conceive of gender as the "behavioral, social, and psychological characteristics"

(Pryzgoda & Chrisler, 2000, p. 554) of women, men, and non-binary individuals. The broader category of non-binary genders comprises several variants, such as pangender and genderqueer. Here, for simplicity and to reflect our participants' terminology, we use the term "non-binary" to refer to participants with genders outside the woman/man binary.

Since gender is a social construction, what is considered appropriate for each gender is subject to the specificities of time, place, and culture. Mathematics is a field that was historically and continues to be conceptualized as masculine in Western culture (Ernest, 1998; Leyva, 2017). Hence, exploring views of gender and mathematics remains a worthy research goal.

Research Design

The study was conducted in two large, comparable cities: one in Canada and one in Australia. People in four 'matched' public places (e.g., shopping mall in each city) were approached and asked to orally complete a brief questionnaire about their views of gender and mathematics. In the following sections, we describe the data collection instrument, participants, and analysis methods.

Data Collection Instrument

Our data collection involved the replication of some questions from the questionnaire used in the aforementioned study led by Forgasz and Leder, but we altered the gender-related questions to make them non-binary. For instance, instead of asking "Is it more important for girls or boys to study mathematics?", we asked "For which gender is it most important to study mathematics?" The purpose of changing the wording was twofold: 1) We did not want to provide binary gender options and 2) We wanted to make the wording sufficiently open-ended so that participants would use their own wording in their responses. As such, we were able to analyze the linguistic choices that the participants made in their responses, as we will later discuss.

The questionnaire had three sections. In the first section, participants provided demographic information (e.g., gender, age). Participants' genders were not assumed based on appearance; rather, participants were asked, "What is your gender?" In the second section, participants were asked five questions about their views of gender and mathematics (e.g., ability, importance) and prompted to explain their responses. In the third section, participants were asked three questions about their views of gender and related constructs (e.g., sex) and again prompted to explain their responses. Finally, participants were given the chance to provide any additional comments about gender and mathematics. Here, we report on findings from the second section of the questionnaire.

Participants

In total, 405 adult participants took part in the study: 195 from Australia and 210 from Canada. Due to the participants' inconsistent use of gender terminology (e.g., woman, man, genderqueer) and sex terminology (e.g., female, male), responses to the gender demographic question were combined into the following categories: women/females/etc., men/males/etc., and non-binary. Examples of 'etc.' responses were "girl" for the first category and "bloke" for the second category. Information about the participants is shown in Table 1, with percentages applying to columns.

Table 1: Australian and Canadian Participants, by Gender Group

Gender Group	Australian Participants	Canadian Participants
Women/Females/etc.	88 (45.1%)	109 (51.9%)
Men/Males/etc.	105 (53.8%)	96 (45.7%)
Non-Binary	2 (1.0%)	5 (2.4%)

There was a higher proportion of non-binary participants in Canada than in Australia. In total, seven (1.7%) of the participants across the sample were non-binary. This percentage is slightly higher than estimates (less than 1%) from larger questionnaires conducted with the general public in Australia and Canada (Australian Bureau of Statistics, 2018; Waite & Denier, 2019). Additional details about the non-binary participants are shown in Table 2.

Table 2: Demographic Information about Non-Binary Participants

Participant	Gender Response	Age Category	Language(s) Spoken at Home	Highest Level of Education Completed
AusNB1	"genderqueer… non-binary"	18–39	English	High school
AusNB2	"non-binary"	18–39	English	College*
CanNB1	"non-binary"	40–59	French	College
CanNB2	"other or third"	18–39	English	High school
CanNB3	"non-binary"	18–39	English	High school
CanNB4	"non-binary"	18–39	English	Undergraduate
CanNB5	"non-binary"	18–39	English/French	Undergraduate

**Note: In Australia and Canada, college is a post-secondary institution that typically offers career-focused programs.*

Data Analysis

The participants' responses to the open-ended questions were analyzed using emergent coding (Creswell, 2014). That is, all responses to a question (from the entire dataset) were read multiple times to get a sense of the data. Then, codes were created and applied to the responses. Participants' responses were also allocated a "sex/gender language" code (SGL code; e.g., sex, gender, mixed) and a "binary/non-binary language" code (BNBL code; e.g., binary, non-binary, no indication). For instance, if a participant responded, "It's equally important for boys and girls to study math," the response would be coded with a "gender" SGL code and a "binary" BNBL code. In contrast, a response of "Males, females, and people of mixed genders can do math equally well" would be given a "sex" SGL code and a "non-binary" BNBL code. Due to the small number of non-binary participants, only descriptive statistics (e.g., counts) could be calculated.

Findings

In the following sections, we provide details about the non-binary participants' views on gender and mathematics, based on their responses to five open-ended questions on these topics. Unless otherwise mentioned, no SGL or BNBL was used in the vast majority of responses.

Relationship Between Mathematics Ability and Gender

Participants were asked whether they believed that mathematics ability was related to gender, and, encouragingly, most (n = 5) did not. Participants cited that individual variability precluded this kind of relationship, as characterized by a comment that any observed ability difference "is due to socialization and it's not actually due to their innate abilities" (CanNB5). Of the participants with other viewpoints, one argued that there was "a very strong emphasis in males, especially to perform in mathematics" (CanNB2) and the other explained that "girls are better at maths because they…are raised to have more patience, are not expected to just be good at things automatically" (AusNB1). Interestingly, these participants also justified their positions using arguments about social practices, rather than inherent differences.

Change Over Time in this Relationship

Next, participants were asked if they believed that there had been a change over time in the relationship between mathematics ability and gender. Most (n = 5) expressed a belief that there had been a change over time, but that this change related to outcomes rather than actual ability. Participants noted that there has been increased opportunity in recent times for people with marginalized genders to access mathematics. AusNB1 explained that "education and employment in maths and sciences hasn't been accessible to women and people of other genders historically until very recently." The remaining participants indicated that they did not think there had been a change over time but did not elaborate on their reasoning. In terms of the use of SGL, the bimodal responses ($n = 3$ for each category) were gender language or no indication of SGL.

Perceptions of Parents' Views of this Relationship

Participants were asked if they believed that parents thought that mathematics ability was related to gender. The responses to this question were bimodal ($n = 3$ for each category): that it depended on the parents or that parents favoured boys. For example, CanNB1 suggested that parents' views depended on their cultural backgrounds, whereas AusNB1 argued that "obviously parents have gendered expectations of the vocations that their children will choose and I think they probably are more likely to expect boys to become engineers." While responding, some participants referenced their own parents' views, while other participants answered generally.

Perceptions of Teachers' Views of this Relationship

Participants were also asked if they believed that teachers thought that mathematics ability was related to gender. There was little consistency in the responses: three participants said that such views depended on the teacher, one thought that teachers favoured boys, one thought that teachers viewed all children equally, and two provided unclear responses. Some participants responded generally, while others extrapolated from their own experiences. For example, AusNB2 reported, "I just feel like they would always put the people that were males ahead of the class or think that they would do better" and shared a story of a teacher thinking that they cheated on a test because they earned 100%, while no boys did (AusNB2 identified as a girl at this point).

Gender and the Importance of Studying Mathematics

In the final question, participants were asked, "For which gender is it most important to study mathematics?" The modal response ($n = 5$) was that it was important for people of all genders to study mathematics. For instance, CanNB2 stated, "I believe all of them are equally important. I don't think that professions should be limited by gender." The other two participants 1) provided an unclear response and 2) stated that it was more important for women to study mathematics. While responding, four participants used non-binary language while one used binary language.

Concluding Remarks

In this report, we described the findings pertaining to the non-binary participants in our study of the general public's views of gender and mathematics. Generally, these participants held gender-egalitarian views and mixed perceptions of others' views (i.e., teachers, parents). With regards to language use, use of any SGL or BNBL by the participants was limited. However, when used, sex language was rare, and only one instance of binary language occurred. Although we cannot know for certain, it is reasonable to assume that as a result of their personal experiences exploring gender, these participants are more knowledgeable and understanding that gender is social construct. Indeed, non-binary people tend to use gender-related language that is more sophisticated than that used in general society, and they are more likely to use gender language and non-binary language than are people with binary genders (Hall & Jao, 2018a, 2018b; Matsuno & Budge, 2017).

Pervasive binary perspectives and structures of gender in society continue to marginalize non-binary people. We, in the mathematics education community, are not immune to such marginalizing practices. In the vast majority of mathematics education studies that include gender (either as a focus or simply as one of many demographic "variables"), researchers strictly involve binary gender groups. We hope that our study may serve as an example of a way to frame research, and collect and analyze data in a more inclusive way. In so doing, we hope to encourage other researchers to reflect on their own practices. It is only with our ongoing collective efforts that all members of our society will be included and represented in research.

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AN HISTORICAL EXPLORATION OF ACHIEVEMENT GAP RHETORIC: A CRITICAL DISCOURSE ANALYSIS OF FEDERAL EDUCATION LEGISLATION

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This article presents initial findings of how the ideas of achievement and the achievement gap are presenting throughout U.S. federal education legislation. Through the lens of Critical Race Theory and governmentality, I highlight the ways in which achievement is used in legislation as well as how that connects to discussions of race and equity in mathematics education. The discussion links to research on how current trends in language use perpetuate policy as performance as well as anti-Black sentiments within U.S. mathematics education. In conclusion, I join the calls for altering conceptions of what achievement means beyond the performance on assessments.

Keywords: equity & diversity; marginalized communities; policy matters; social justice

Mathematics education holds a unique position within the curriculum of K-12 education in the United States (U.S.) given that the National Assessment of Educational Progress (NAEP) has legislated periodic testing since 1969. This assessment requirement, together with legislated concern about the existence of achievement gaps in education, focuses attention on the disaggregated data of student test scores broken up by race. More specifically, the continued existence of a test score gap between Black and white students raises questions about how racism plays a role in legislation and what the focus of that legislation should be if not the gap between test scores.

The continued inability for Black students to obtain a proficient status on mathematics assessments (U.S. Department of Education, 2019) is manifested in mathematics classrooms through the master-narrative that racialized students, and Black students in particular, are unable to achieve in mathematics (Martin, 2009; Nasir, Atukpawu, O'Conner, Davis, Wischnia, & Tsang; 2009). This narrative exists within ideologies such as the myth that "mathematics is a white male subject" (Gutiérrez, 2008; Stinson, 2013) that lead to students having racialized experiences, where "the socially constructed meaning for race comes to be a deciding factor in who gets to do mathematics and who does not" (Martin, 2006, p. 223).

As a way to illuminate the entrenchment of racism within mathematics education teaching and practice, I focus on how the messages of the master-narrative and racialized experiences exist within legislation in the U.S. To that end, this paper aims to explore the beginnings of the achievement gap conversation through an historical exploration of U.S. federal education legislation in an attempt to question if a focus on the achievement gap actually maintains ideas of racial neutrality within policy, when really there should be a more pointed focus on race as it impacts education (Bonilla-Silva, 2014; Martin, 2003). In the end, this research shows that the continued removal of references to race, racism, and racialization as they relate to achievement is a continuation of an unequal and highly stratified education system based on race.

Theoretical Framework

There are two theories that I rely on to provide grounding for my research; they are Critical Race Theory (CRT) and governmentality. The importance of CRT stems from the ultimate goal to eliminate racial oppression while simultaneously working to rid society of all forms of subordination (Gutiérrez, 2013; Ladson-Billings & Tate, 1995; Solórzano, 1997, 1998; Tate, 1997; Yosso, Parker, Solórzano, & Lynn, 2004). CRT research in mathematics education uses the five elements of CRT to

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acknowledge how practices such as tracking and intelligence testing actively work against students of color (Berry, 2008). The second part of my theoretical framework is governmentality, which works with CRT to engage with policy documents and to uncover how the discourses of race have, in Foucault's (1991) terminology, disciplined our way of thinking about particular topics. In essence CRT and governmentality together seek to find, acknowledge, and name the ways in which power functions within the actions of mathematics classes in relation to education legislation.

Methodology

The methodology that I rely on to guide my analysis is historical ontology which allows for both a historical and philosophical analysis simultaneously. Essentially, historical ontology uses history, temporally, in an effort to understand how particular vocabulary can be used to limit how an idea is understood in the present (Hacking, 2002). Thus, looking at how a specific word is used in a particular time and place, and following its trajectory through time, it is possible to see how present ideas around that same word are constrained by the ways in which the word was used in the past. In this way, historical ontology works together with both CRT and governmentality to address issues of power through the use of vocabulary within legislation.

Results

In order to historically analyze vocabulary around achievement and the achievement gap, I used the historical record of U.S. federal education legislation starting with the Elementary and Secondary Education Act of 1965 (ESEA) including all of the subsequent reauthorizations of that Act. This includes the well-known reauthorizations such as No Child Left Behind (NCLB), as well as the Reagan era reauthorization which occurred within the Omnibus Budget Reconciliation Act of 1981. The method used to conduct this research is Critical Discourse Analysis (CDA) which provides a way to both search for and analyze underlying ideologies present within educational discourse (Fairclough, 2010). CDA allows for policy analysis to look beyond explicit rhetoric that exists within the policy documents to determine if present legislation is maintaining previous trends (Atkins & Wallace, 2012).

Achievement in the Legislation

As a way of exploring how achievement appears within the legislation, I searched through all of the reauthorizations for the words achievement and achievement gap. Table 3 presents the breakdown of those searches, together with a representation of how many sections address both the achievement gap and contain racial terminology to see if and where these ideas appear together.

Table 1: Individual Uses and Section References to Achievement and the Achievement Gap

	1965	1966	1968	1970	1972	1974		1976	1978	1981	1983	1988	1994	2002	2015	Total
Achievement (all uses)	3	1	1	5	9	10		1	33	1	0	65	110	535	215	989
Achievement gap, phrase	0	0	0	0	0	0		0	0	0	0	0	4	6	2	12
Achievement gap, sections incl. racial terminology	0	0	0	0	0	0		0	0	0	0	0	2+1	4+1	0	6+2

Note. Full sections that include racial terminology relating to one theme are counted as "1" and sections that include racial terminology relating to more than one theme are counted as "+1" to indicate any partial sections referring to a particular theme.

The word achievement on its own becomes more common in recent legislation, although it is present throughout most of the reauthorizations. A notable shift seems to have occurred with the 1978 reauthorization, likely due to the introduction of the NAEP examination to the legislation. Clearly with language referencing student test scores and discussing the level of achievement of students, the introduction of a new federal testing regime would account for the increased use of the word achievement in the 1978 reauthorization. The other interesting shift to note is the almost quintupling of the use of the word achievement from 1994 to 2002. Arguably, this increase could be largely due to the stated purpose of NCLB as desiring to pay closer attention to the achievement gap. However, this does not explain why there is such a drastic decrease immediately following NCLB where the legislation maintains a similar stated purpose.

Despite a proliferation of research after NCLB extremely concerned about testing requirements and achievement gaps, the phrase achievement gap originates legislatively in 1994.

Thus, according to historical ontology, the idea of the achievement gap presented in IASA feeds into the understanding of the achievement gap presented in NCLB. In addition, it is interesting to note how the use of achievement gap follows a pattern similar to the term achievement, in that the peak usage is in the 2002 reauthorization, followed by a drastic decrease in 2015.

The third and final line of Table 3 represents a thematic analysis of sections within the legislation that contain racial terminology pulled from my dissertation (Hawks, 2019). This data was created by first searching for racial terminology, then thematically analyzing each section. For this data the theme of achievement gap was used for all sections of the legislation that indicated that a funded program was meant to focus on either eliminating the achievement gap or increasing minority student achievement. One of my assumptions when I created this category was that testing requirements and achievement gap sections would have a significant correlation with racial terminology given concerns about racial testing disparities in research (Meier & Wood; 2004; Rothestein, 2004). However, as Table 3 shows, there are very few connections between the use of racial terminology and references to achievement within the legislation. In fact, comparing the 6+2 sections that use both racial and achievement gap terminology with the instances of the phrase achievement gap shows a lot of overlap between the two measures. This trend is especially true for NCLB where the only mentions of the achievement gap that are not also linked with racial terminology are two sections which set aside funding to present recognition and awards to schools that have made substantial gains in closing the achievement gap between student test scores. The overlap lends credence to Hilliard's (2003) conclusion that references to the achievement gap are implicitly referencing the *racial* achievement gap.

Discussion

One of the most intriguing elements of the achievement gap rhetoric is the simultaneous focus, and yet complete ignorance, of how race plays a factor in gauging achievement. For example, in a section of NCLB a definition of the achievement gap is proffered which identifies that one of the gaps of interest is the one between racial and nonracial students (P.L. 107-110, sec. 1503(d)(3)). This specific use of the term nonracial in relation to a definition of the achievement gap actually seems to suggest a self-correction within the legislation, simultaneously acknowledging how race plays a factor in the creation of the achievement gap while also indicating that there are those who do not fit within a "race" per se. The self-corrective nature of this turn of phrase and use of the term nonracial could be a further indicator of the anti-Black nature of U.S. education as theorized by Danny Martin (2019).

The biggest problem with race falling to the background of the overall legislation is that when we begin to talk about the achievement gap and how that impacts racialized students, we are unable to engage with how racism, and racialization play roles in how those scores have come into being. Essentially, highlighting the racial achievement gap puts a spot light on disaggregated student test scores which then reifies the master-narrative around which students are able, or capable, of achieving well in mathematics. Thus as mathematics educators responsible for student performance in the subject most often tested and highlighted in achievement gap rhetoric, by avoiding race in our research and teaching we inadvertently, or intentionally, are complicit in maintaining the structures that uphold racist ideologies. In a similar vein, Schick (2011) argues that moving from specific language around ensuring that racialized students do well to language around all students doing well, shifts the focus of policy away from the importance of considering how racialization impacts racialized students.

In the end, mathematics education and the achievement gap are inextricably linked through both rhetoric and practice and what this analysis shows is that attempts to understand discrepancies in educational attainment and achievement is completely without a race analysis of any kind. This is partially because the ways in which race and achievement are used together, or linked, in the legislation are so limited as to be almost meaningless. For example, the main stream mathematics education literature that strives to discuss student test scores, merely uses race as a category to assist in the disaggregation of data as a comparative measure (Harwell et al., 2007; Post et al., 2008; Price, 2010; Stiefle, Schwartz, & Chellman, 2007; Wei, 2012). This practice is also used in the presentation of scores for *The Nation's Report Card*, which is also the basis for claims of the existence of an achievement gap between Black and White student test scores in mathematics. At a very basic level, these practices ignore, or attempt to simplify, the extremely complex nature of the idea of "race" to a categorical comparison between groups of students. What occurs because of this ignorance or inability to engage with the realities of racism and racialization in K-12 schooling, is that the master-narrative is reified into existence and pseudo-scientific claims of hierarchies of intelligence are able to flourish unacknowledged in the background. Therefore, the danger of mathematics education policies and federal education legislation systematically removing references to race as they relate to achievement is the continuation of an unequal K-12 education system that is highly stratified based on race.

Conclusion

This conclusion suggests that if future legislation maintains the goal of eliminating the achievement gap(s) then it must be reframed to not only focus on the children of low-income families, but also the children of racialized families. Thus, if the goal of federal education legislation is actually to reduce or eliminate the achievement gap(s) there needs to be a stronger and more purposeful focus on issues that are impacting racialized students. Including the ways in which the legislation and policies have a tendency to refer to achievement as an individual characteristic rather than acknowledging the system of policies and assessments that define how achievement is to be understood. This includes, but is not limited to, acknowledging the historical ways in which racialized people have been systematically devalued, how that process continues in K-12 schooling today through tracking and testing requirements, and the importance of noting how the process of racialization treats students differently on both an individual and systemic level. Without these, and other measures to actively engage with how race impacts schooling, the systems that maintain the existence of the achievement gap(s) will continue unfettered. In conclusion, as Spencer (2009) suggests, a way forward is for legislation to redefine what it means to achieve, by renegotiating the focus from achievement to success, which includes the role of resistance as a response to the limitations of schooling.

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COMMUNITY MATHEMATICS PROJECT: TUTORING LOW-INCOME PARENTS TO MAKE SENSE OF MATHEMATICS

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Abstract: This research report explores the ways in which the Community Mathematics Project (CMP) supports underserved populations to learn mathematics in a large urban area. This project seeks for parents to have tools to teach their children mathematics at home. As part of a longitudinal study, in this report we explore the experiences of a student-parent when learning and re-learning mathematics with the supports of a researcher and a teacher-parent. Findings suggest that sense-making strategies supported the student-parent to make connections with prior knowledge to figure out new mathematical concepts. Further, the student-parent leveraged from this mathematical knowledge to use it in her daily life and to support her son learn mathematics at home.

Keywords: equity and diversity, social justice, culturally relevant pedagogy.

Objectives of the Study

Family and parent engagement in mathematics is a crucial practice that facilitates and enhances deep mathematical learning (Turner et al., 2012), yet many parents feel disconnected from schools and from their child's education in mathematics (Civil & Berneir, 2006; Mistretta, 2013). Thus, it is essential that teachers learn to leverage children's funds of knowledge (Moll et al., 1992) in mathematics instruction (Turner et al., 2016) to engage families and communities, but it is also imperative that parents feel connected to their child's learning. Indeed, parent participation in the early years of math education of children can increase academic development and math achievement (Cho, 2017).

When parents and families become a part of the goal-setting and deep learning, children benefit. This research report will describe preliminary findings from the *Community Mathematics Project* (CMP)[1], a collaborative endeavor that aims to address mathematics opportunity and achievement gaps that exist, especially as they relate to Hispanic and low-income students in the urban center of San Antonio, Texas. The project supports prospective elementary school teachers to attend to both in school (pedagogy that is centered on the use of student funds of knowledge) and out of school (to connect parents and communities to school/mathematics) opportunities through partnerships between a community college, a four-year institution of higher education, and community centers based in low-income neighborhoods, with a goal of sustainability through the identification of parent experts. This study focuses on a one-on-one tutoring program between a parent educator and a parent tutee.

The CMP is multi-faceted and includes various mechanisms for facilitating partnership between these entities. The community college and four-year institution work to align curricula to improve prospective teacher pedagogical content knowledge and theoretical foundations. Moreover, prospective teachers are provided with bridging opportunities as they matriculate to the four-year institution's teacher preparation program. In this program, students who have been identified work with faculty in a community center to provide mathematics tutoring to parents in low-income communities. This allows the prospective teacher to gain knowledge about community members and parents while engaging them in culturally relevant mathematics activities. Further, parents are

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empowered to connect everyday practices to the school mathematics that they will be learning, and can engage more readily with their student who is learning similar concepts. After one semester of this program in a community center, we envision identifying a "parent scholar" who can assist in sustaining the program once our prospective teachers move on. In subsequent semesters, then, we will work with additional community centers. This one-on-one tutoring aspect of the project will be the focus of this presentation, which will present data related to the following guiding question: What are the experiences of a parent tutee who participated in the CMP tutoring center?

Theoretical Framework

To answer the aforementioned research questions of CMP, there are two main theoretical frameworks that undergird our work. The first framework draws from the immense research that foregrounds the mathematical knowledge and expertise of parents and families. The research in the literature review is situated in frameworks that resist the deficit notion that parents, families, and young children do not engage in practices that connect to mathematical ideas and skills. Rooted within seminal work of Moll et al. (1992) and the research of Funds of Knowledge perspective, when mathematics teachers value the experiences and practices of parents and families, they can make more connections to authentic ways that children use mathematics at home and in their communities.

The second framework continues to dispel the myth that young children only learn mathematics in traditional classroom or pre-school settings—parents and families can play a role in how and in what contexts young children develop their mathematical knowledge and experiences (Berkowitz et al., 2015; Cho, 2017). That is, through collaborations with researchers and mathematics teachers, parents and families can have more opportunities to learn mathematics, which may be shared with their children at home. The work of Marta Civil (Civil, 2007; Civil, Bratton, & Quintos, 2005) and colleagues (Rodriguez, 2013; Téllez, Moschkovich, & Civil, 2011; Willey, 2008) offer examples of how parents and families can engage in learning new mathematics based on their existing experiences. When parents and families have more opportunities to strengthen their own mathematical knowledge by making more connections to new ideas and skills, they can also have more opportunities to engage in similar discussions with their young children; thereby further pushing back on the notion of learning only occurring in the classroom. As a result, a broader outcome of this project is to support more families, parents, and young children to build upon their existing knowledge while seeking new connections to more knowledge. The following sections will briefly discuss the existing literature and seminal scholars who have explored the notion of families of young children in the field of mathematics education. Given these frameworks, it is important to continue to study the integral role that parents and families play on how young children learn mathematics (Berkowitz et al., 2015; Civil, Díez-Palomar, Menéndez, & Acosta-Iriqui, 2008; Jackson & Remillard, 2005; Sheldon & Epstein, 2008) and their background knowledge that can be used as foundation for learning more mathematics.

Methods

Context. In an eight-week course, the researchers and a parent-tutor used aligned curricula to provide mathematics tutoring to parents and care givers seeking for them to have the tools to teach mathematics to their children at home. On this phase of this longitudinal study, the researchers worked with a Latino community in which parents spoke Spanish as first language (L1), they had limited knowledge of English, and in most cases parents held a high school diploma or less. The researchers provided supports in the form of co-teaching and content knowledge to a mother-tutor—Isabel [all names are pseudonyms]—who taught mathematics to parents in her community. Isabel's role was to promote sustainability of the program. From the twenty parents who started the program, we chose one of them—Ofelia—as a case study. We sought to understand teaching moves that could support Ofelia make sense of mathematics, and understand additional supports that Ofelia can benefit from to learn and to teach mathematics to her child.

Participants. Ofelia participated in an eight-week course in which the parent worked in a triad with the researcher and the parent expert Isabel. One of the researchers observed Ofelia at the community center as she received tutoring classes from Isabel. We sought to observe a parent whose educational background and characteristics represents the average parent of the Latino community.

At the time of the study, Ofelia was 44 years-old. She arrived to the USA 12 years ago by crossing the border. Ofelia has been married for 20 years and has 2 children—the oldest was 19 (attended a local community college), and the youngest was 10 (5^{th} grade). Ofelia went to school in Mexico and studied until 9^{th} grade. Ofelia's main job was cleaning house, however, at the time of the study, she was unemployed. Ofelia shared that she enrolled in the "*Latino Math*" program to support her youngest child to learn mathematics at home. Ofelia expressed that her youngest struggled with mathematics, and prior participating in the program, Ofelia did not have the mathematical knowledge to support him increase his mathematical achievement (Cho, 2015). Ofelia expressed that the differences between the way she learned mathematics in Mexico, and the way mathematics is taught in the USA presented challenges to support her child at home.

Data collection. Data collection took place throughout eight weeks, which was the duration of the course. For one hour each week, one of the researchers went to the community center and supported Ofelia in developing understanding of mathematical concepts along with the parent-instructor—Isabel. In class, the researcher used sense-making moves to support Ofelia relearn mathematical concepts, and also understand and figure out new ones.

Data sources include: 420 minutes of transcripts from classroom observations, 99 minutes of transcript from two interviews to Ofelia and Isabel, detailed field notes from 7 classroom observations, and artifacts such as pictures from Ofelia's work in her mathematics book.

Analysis. Data collection and analysis were iterative processes (Yin, 2014). Coding took place in four cycles. Before coding we adopted an interpretivist approach (Miles & Huberman, 1994) to capture the essence of the participant's sense-making process. Specifically, we coded the different ways in which the participant made sense of mathematics. We also coded on how the participant figured out content by building on prior knowledge. Because we acknowledged the participant's educational background, we coded for her studying strategies to support her in developing new ones.

Initial coding started by re-reading the data, but this time we looked for provisional themes and highlighted relevant quotes. Next, we used a descriptive coding process (Saldaña, 2013) which enabled us to analyze participant's sense-making and figuring out over time. A provisional list of codes emerged and we organized these codes into a "meta-matrix" (Miles & Huberman, 1994, p. 178). In other words, we assembled the descriptive data on a spreadsheet in chronological order. We used short phrases to find basic topics across the different data sources, and labeled each data source with codes. For instance, we focused on how different sense-making moves systematically supported Ofelia to learn the mathematics content and to figure out content without receiving explanations but from building on prior knowledge. Examples of these codes are: patterns, association, logical reasoning, look backwards, negotiate, think out loud, talk at every stage of problem solving, think of alternative ways to perform a task, and so on. To organize and refine the codes and themes we used "code mapping" (Saldaña, 2013, p. 194). In other words, we went through three iterations of analysis to reorganize the full set of codes into a list of 17 categories, to later condense those into 3 central themes: sense-making, figuring out, and studying strategies. These themes are unpacked in the following section.

Results

Working with Ofelia one-on-one for eight weeks enabled us to explore in-depth the supports that she needed to make sense of mathematics. Analysis of the data suggests that sense-making moves supported Ofelia in developing understanding of mathematical concepts to systematically figure out new ones. At the same time, learning about mathematics provided Ofelia with tools to support her 10-year-old son learn math at home, and to be confident to use math in her daily life.

Sense-making and figuring out. One of the challenges to support the development of *sense-making* was to bridge the disconnect between the way Ofelia was taught in Mexico and the way that mathematics is taught in the USA— "apprenticeship of observation" (Lortie, 1975, p. 61). In multiple opportunities Ofelia expressed the need to memorize mathematical procedures and time tables, and to be given answers away as opposed to making sense and finding answers for the mathematical problems. During our first sessions, Ofelia explicitly asked to be explained procedures and to be a passive listener. Ofelia would often give up when working on tasks and asked to be given answers. To move away from top-down learning, the researchers relied on sense-making moves such as building on prior knowledge and using logical reasoning. For example, Ofelia was working on factorization and the number she had was 75. Ofelia's first attempt was to factorize by 2, "¿Se le hace? 75 entre… Y es 5, entre 2 (giggle), 35. [Is this? 75 divided by… And it is 5, divided by, 35.]" In the previous lesson, I explained that numbers that end in 5 or 0 are multiple of 5. The researcher encouraged Ofelia to make connections with the prior lesson and to reflect on how to solve the problem. Ofelia revisited the previous lesson and reflected, "Ay, a ver (erase). ¿Entonces voy a dividir entre qué? Entre 5, a ver (thinking out loud and working). 5x5=25. ¿Entonces divide entre 5? Y luego me da 15. Y 3x5, pienso. [Oh, let's see. Then, I divide it between what? Between 5, let's see. 5x5=25. Then, divide it by 5? And I get 15. And 3x5, I think.]" As part of teaching Ofelia studying strategies, the researcher encouraged her to check her answers. She shared, "A ver, entonces pongo 5x3=15. A ver. Es 75. Let's see, then I write 5x3=15. Let's see. It is 75.]" In this example, we share one way in which we supported sense making by building on prior knowledge and reasoning, and avoided giving answers away and memorization.

Another sense-making move used was thinking out-loud. Thinking out-loud supported Ofelia to self-correct, negotiate, reflect, build from prior knowledge, and to think of multiple ways to perform a task. For instance, Ofelia worked on another factorization problem,

> Ofelia: 7x5=35 estoy repitiendo las tablas (giggle). 7x5=35; 7x6=42. Ah entonces es 42. Entonces me dijo según yo, aquí pongo el 42, lo resto. Serían 2 y aquí sería cero. Entonces bajo el 7, verdad? O ya voy mal? Ahora, 7x3=21; 7x4=28; sería 3. 7x3=21. Hago una restita, y luego son 6 y aquí son cero. Y el último lo bajo. ¡Yo ya no sabía que hacer con este híjole! (giggle). Okay, serían hay ya yai. 7x9=63. 9x7=63. ¿Será? Ya ya recordé (giggle).
>
> [*7x5=35 I am repeating timetables. 7x5=35; 7x6=42. Then, it is 42. Then according to me, here I write the 42, I subtract it. It would be 2 and here it would be zero. Then, I bring down the 7, right? Or am I wrong? Now, 7x3=21; 7x4=28; it is 3. 7x3=21. I subtract, and then here is 6 and here 0. And the last one I bring down. I didn't know what to do! Okay, it will be… hay ya yai.7x9=63. 9x7=63. Would that be? I I remembered*.]

In this example, Ofelia said timetables out loud as she looked to the answers. Then, she walked us through the procedure and because she was talking, she self-corrected, and she revisited previous exercises to solve this problem.

Changes in terms of sense-making were more evident by the fourth learning session. By then, Ofelia asked *not* to be told the answers to the problems and she started looking for alternative ways to solve them—*figure out*.

Mathematics in Ofelia's daily life. The learning sessions supported Ofelia not only to make sense of mathematics and to figure out. The learning sessions also supported Ofelia in developing confidence about using math in her daily life, and bonding with her son more as they studied mathematics together. By the end of the program, Ofelia shared how she started using mathematics outside the learning sessions,

Another important aspect of learning mathematics was that Ofelia could develop a better relationship with her son. Ofelia expressed, that learning mathematics allowed her to teach her son content related to his grade level and spend more time together,

Discussion and Conclusions

Secondly because Ofelia comes with an immense amount of mathematical experiences that she can draw from (as is what we posit is true for all parents, families, and young children who do mathematics) we bore witness to Ofelia actively leveraging her existing knowledge with her new knowledge and sense-making practices to advocate for herself during a purchase at a garage sale. Ofelia shows how when parents and families engage in mathematical sense-making while accessing connections to new mathematical knowledge, they also identify moments in which they can access and use this knowledge and practices in their daily lives. Moreover, Ofelia inspires us to help more parents and families to seek a wide number of opportunities for them to use their mathematical knowledge and sense-making practices to enact a sense of agency for fairness for themselves, their families, and communities (McGee & Spencer, 2015). The practice of engaging in sense-making while advocating for fairness and social justice is a line of research that can help the field to know more about the role of parents and families in the broader field of mathematics education (e.g., Mistretta, 2013; Rodriguez, 2013).

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GENDER DIFFERENCES IN NUMBER STRATEGY USE FOR STUDENTS SOLVING FRACTION STORY PROBLEMS

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Gender differences in fourth and fifth grade students' strategy use for a fraction story problem were investigated using multinomial logistic regression on a sample of 193 written student strategies. Gender was not a significant predictor of type of strategy used, in contrast to earlier studies finding that boys tended to use more abstract strategies whereas girls tended to use more concrete strategies or the standard algorithm.

Keywords: Gender and Sexuality; Elementary School Education; Number, Concepts, and Operations

Gender differences in mathematics have long been a topic of study in mathematics education (Fennema, 1974; Leyva, 2017). One particular focus of interest has been gender differences in strategy use, inspired in part by research in which Fennema, Carpenter, Jacobs, Franke, and Levi (Fennema, et al., 1998a, 1998b) found gender differences in students' strategies for story problems. In a longitudinal study with 38 girls and 44 boys in grades 1-3, they found no gender difference in the ability to solve addition and subtraction story problems and multidigit computations, yet significant difference in type of strategy used to solve these problems. Girls tended to use concrete solution strategies. Boys tended to use abstract solution strategies that "reflected conceptual understanding" (p. 11). The researchers argued these results indicated differences in the degree to which girls and boys had developed mathematics understanding.

Fennema et al. (1998a) invited interpretations of the results from four scholars: mathematics educator Judith Sowder; social psychologists Janet Hyde and Sara Jaffee; and feminist philosopher Nel Noddings. Sowder (1998) suggested these gender differences could reflect differences in preferences for explaining one's strategy (e.g., girls prefer to give explanations that are clear for others) and worried that students who use more abstract strategies are more likely to make sense in mathematics and have a better chance at succeeding mathematically. Hyde and Jaffee (1998) cautioned against interpreting female deficits based on findings. They suggested teachers could hold gender stereotypes (e.g., girls are compliant, boys are independent) and that those stereotypes were activated in teachers' interactions with students. Noddings (1998) suggested girls could be less interested in mathematics and noted that society does not show the same concern when boys demonstrate less interest than girls in other activities (such as early childhood education or nursing), which led to a critique of the social structure: "Do we approve of a social structure that values competence in mathematics over competence in child care?" (p. 18).

Other education researchers have also studied gender differences in strategy use (Carr & Davis, 2001; Carr, Jessup, & Fuller, 1997; Carr, Steiner, Kyser, & Biddlecomb, 2008) and framed their findings in a variety of ways. For example, Carr and Davis (2001) examined 84 students' use of strategy under both free and constrained choice. Under free choice, girls chose manipulative strategies while boys chose retrieval strategies, consistent with findings by Fennema and colleagues. Under constrained choice, they found that boys were able to use manipulative strategies, but girls were "not as capable" as boys in using retrieval. In their study, manipulative strategies were considered more concrete and retrieval strategies more abstract. In a cross national study, Shen, Vasilyeva, and Laski (2016) found gender differences in strategy use that mediated accuracy for

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students in the United States and Russia but not Taiwan, suggesting that differences could be attributed to instructional contexts, rather than inherent to girls and boys.

If gender differences in strategy use exist, they might reflect important differences in students' conceptual understanding (Fennema et al., 1996; Sowder, 1998), the enactment of gender stereotypes by teachers (Hyde & Jafeee, 1998), or simply differences in students' interests (Noddings, 1998). Whatever the source, attending to differences in strategy use is important, as they could reflect differences in opportunities to develop conceptual understanding. Further, because potential disparities in strategy use are not visible in standardized tests that do not differentiate between types of strategies used, investigating strategy use on a large scale requires analyses that account for the types of strategies girls and boys use to solve problems.

Building on the early study on gender differences in mathematics (Fennema et al., 1998a), this study investigates gender and strategy use in fraction story problems. Our study is different in important ways. (1) Our analysis is over 20 years after the publication of the study by Fennema and colleagues, and thus provides a glimpse of current gender dynamics in mathematics teaching and learning. (2) Our study is comprised of fourth and fifth grade students solving a fraction story problem, compared to the first, second, and third grade students and a focus on whole-number addition and subtraction in the original study. Like Fennema and colleagues (1998a) and the invited interpretations (Hyde & Jafeee, 1998; Noddings, 1998; Sowder, 1998), we are careful to consider the importance of strategy use and avoid framing these differences as reflective of inherent differences in ability. We investigated these differences for 193 fourth and fifth grade students by asking the following: Are there significant gender differences in strategy use for fourth and fifth grade students solving fraction story problems?

Methods

Sampling and Participants

Data for this analysis came from a larger professional development design study, focused on documenting and supporting the development of teachers' responsiveness to students' fraction thinking during instruction (Jacobs, et al., 2019). As part of this larger study, students from 50 different classrooms were administered a paper and pencil assessment at the beginning and end of the school year, to measure fraction problem solving and conceptual understanding. Items were open response. A rubric for scoring and coding student responses was developed, and for each item, all responses were triple coded until 85% (or higher) agreement was reached among coders, at which point, responses were single coded.

Data Sources and Analysis

For the current study, we focus on one item on the assessment administered at the end of the school year to 562 students, in grades 4 and 5. Of these, 244 student responses were coded as having a valid strategy, which means the student started with the given quantities and operated on those quantities in some justifiable way to reach an answer, and they could include small mistakes. The item consisted of the following story problem: *Allie has 1 6/8 sticks of butter. She needs a total of 5 1/8 sticks of butter to make cookies. How much more butter does Allie need so she can make cookies?*

Each of the responses was coded individually for type of strategy used. For this analysis, we focused on valid strategies (n = 193), including concrete strategies (n = 19), invented algorithms (n = 90), and the standard algorithm (n = 84). Strategies labeled as "other" (n = 24) or "none" (n = 27) were not included because they were not interpretable with respect to the research question. The final sample included 101 girls and 92 boys. These strategy codes and their frequencies in the sample are described and illustrated in Figure 1. Type of strategy served as our dependent variable.

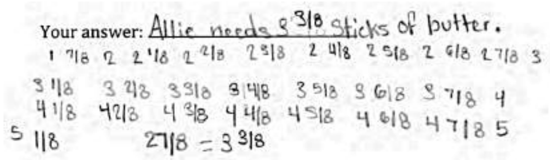
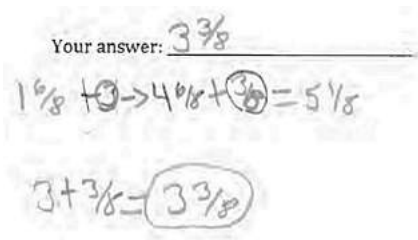
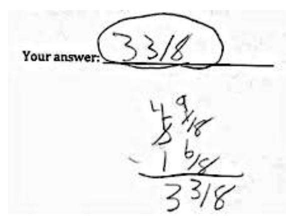
Type of Strategy		Example
Concrete Strategies Girls (n = 10) Boys (n = 9)	*Direct modeling:* strategies that represented all sticks and fractional sticks of butter individually. Usually these were notated with drawings. *Counting up/down by unit fraction*: Strategies that represented each individual group of 1/8 in the count in some way.	 Counting strategy
Invented Algorithms Girls (n = 45) Boys (n = 45)	Computation strategies that decompose the mixed numbers and/or fractions in some way and/or increment or decrement in "hops" (larger than a unit fraction) in some way.	
Standard Algorithm Girls (n = 46) Boys (n = 38)	Standard algorithms for subtraction in which a child uses knowledge of the standard algorithm procedure to determine the missing addend.	

Figure 1: Types of Valid Strategies and Examples Used in the Analysis

Because the dependent variable is categorical, we used multinomial logistic analysis. Concrete strategies and the standard algorithm were separately predicted against the reference category of invented algorithms. We chose invented algorithms as the reference category because if gender differences reflecting conceptual understanding were significant, we would expect to see an over representation of girls in either concrete strategies or the standard algorithm. Using invented algorithms as the reference category allowed comparison of both concrete strategies and the standard algorithm to invented algorithms.

We used three models to analyze students' strategy choice and gender. The first model was used to detect if gender significantly predicted strategy use across grade levels. The second model was used to detect if gender predicted strategy use in fourth or fifth grade. In the third, we added the interaction of grade and gender to the second model.

Results

Results from the statistical models are listed in Table 1. Odds ratios (and standard errors) of the three models show that gender was not a significant predictor of concrete strategy use or standard algorithm use. In the first model, there was some significance in the intercept, meaning students were less likely to use concrete strategies than invented algorithms ($p < 0.001$), but these differences were not based on gender. In the second model, the significance in the intercept remained, and there was some significance in grade level ($p < .05$), meaning fifth grade students were less likely to use concrete strategies compared to invented algorithms. Again, these differences were not based on gender. We included the interaction of gender and grade in the third model and did not detect significance in the interaction.

Table 1: Results of Statistical Models

	Odds ratio (and standard errors)		
Variable	1	2	3
Concrete strategy compared to invented algorithm			
Intercept	0.20 (0.37)***	0.33 (0.40)**	0.36 (0.41)*
Female (compared to M)	1.11 (0.51)	0.99 (0.51)	0.81 (0.58)
Fifth (compared to fourth)		0.22 (0.67)*	0.12 (1.10)
Gender X Grade			3.14 (1.39)
Standard algorithm compared to invented algorithm			
Intercept	0.84 (0.22)	0.64 (0.28)	0.64 (0.34)
Female (compared to M)	1.21 (0.30)	1.27 (0.31)	1.28 (0.45)
Fifth (compared to fourth)		1.63 (0.31)	1.64 (0.45)
Gender X Grade			1.00 (0.62)
	R^2= .00 χ^2 = 0.40, p > .05	R^2= .03 χ^2 = 12.43, p < .05	R^2= .04 χ^2 = 13.18, p < .05

N = 709; *p < 0.05; **p < 0.01; ***p < 0.001

Discussion and Conclusion

The implications of these findings are in the statistical insignificance of the dependent variable. We tested three models related to the research question: Are there significant gender differences in number strategy use for fourth and fifth grade students solving a fraction story problem? Because gender was not significant in any of the models tested, gender-based differences in strategy use were not indicated, a finding that is in opposition to previous findings. However, we focused only on a single item and two grade levels, which limits the scope of our findings. Further, our study was not longitudinal and cannot speculate on trends in development of strategy use and conceptual understanding. Research across multiple assessment items and grades is needed for a more complete examination of students' gender and strategy use in the domain of fractions. Finally, we noted that only 193 of the 562 students used a valid strategy, roughly a third of all students, which suggests that this was a difficult problem for the sample. Research on an item in which a greater proportion of students used a valid strategy is necessary to examine if and for what items the finding holds.

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SOCIAL NORMS CONDUCIVE TO WOMEN'S LEARNING IN INQUIRY-ORIENTED ABSTRACT ALGEBRA

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Gender-based inequities can arise within inquiry-oriented (IO) classes, affecting women's participation and achievement. Given that social norms can enable/constrain students' participation, it is pertinent for researchers to study social norms that are conducive to women's learning and how they can be fostered. In this paper, we explore social norms present in two IO abstract algebra classes with positive learning outcomes for women students. We found social norms related to working on tasks, giving contributions, and responding to others' contributions. We provide examples of these normative behaviors, discuss the instructors' roles in fostering these norms, and examine how these norms could promote gender-based equity in IO instruction.

Keywords: Gender, Inclusive Education, Classroom Discourse, Post-Secondary Education

Prior research has shown that active learning approaches to instruction lead to improved student achievement (e.g., Freeman et al., 2014). However, Eddy and Hogan (2014) argued that any instructional intervention will affect different student groups in varying ways. Johnson et al. (in press) found men and women had differential learning outcomes in an inquiry-oriented (IO) setting; men experienced greater achievement gains than women in IO abstract algebra classes. This gap was not present in the non-IO classes that served as the comparison group for Johnson et al.'s study. In this study, we selected two divergent cases from the Johnson et al. data base. In these two classes, women students had positive learning outcomes. We analyzed the social norms in each class, identified the instructors' roles in fostering these norms, and hypothesized ways in which these social norms may be conducive to women's learning.

Background

Given the interactive nature of Inquiry-Oriented Instruction (IOI), it is pertinent to consider how IOI settings align with aspects of equity. Gutiérrez (2002) outlined four dimensions of equity as access, achievement, identity, and power. These dimensions align with Cook et al.'s (2016) characteristics of inquiry-based classes, for access is given to students through inquiry pedagogies that encourage participation and peer involvement (Tang et al., 2017). Participation gives the opportunity to build knowledge, which can foster achievement and confidence in mathematics. Peer involvement may lead to a shift in a student's mathematical identity (Boaler & Greeno, 2000; Hassi & Laursen, 2015). There is also a shift of power from teachers to students in inquiry pedagogies. However, inequities may emerge with peer involvement. Group work may support privileged groups over marginalized groups by placing value on input given by students who are more likely to participate (Esmonde et al., 2009; Esmonde & Langer-Osuna, 2013). Smith et al. (2019) found evidence of instructors' gender bias, with men disproportionately contributing to class discussions and women disproportionately being asked to contribute in less sophisticated ways. Since women have been

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marginalized in mathematics classes, we investigate the social norms present in mathematics classes that may be conducive to women's learning.

Social norms describe the general rules and expectations for the teacher and students' roles within any classroom. They "characterize regularities in communal or collective classroom activity and are considered to be jointly established by the teacher and student as members of the classroom community" (Cobb & Yackel, 1996, p. 178). Norms are a person's "beliefs about [his/her] own role, others' roles, and the general nature" of classroom activity (Cobb & Yackel, 1996, p. 177). Social norms within a classroom can support or constrain students' participation, which can potentially impact equity in students' participation and achievement. We posit that certain social norms may provide students with access and opportunities for mathematical identity development. We believe this warrants research on the social norms present in IOI classes and the role of instructors in fostering these norms. We address the following questions: What social norms were present in two IO abstract algebra classes that had positive achievement outcomes for women students? What is the instructor's role in fostering these social norms?

Methods

We analyzed social norms present in two IO abstract algebra classes, taught by Dr. Carter and Dr. Ryan. Dr. Carter is a White man who taught at a large doctoral-granting institution in the Midwest US. Dr. Ryan is a White man who taught at a midsize masters-granting institution in the Northern US. Both instructors participated in a semester-long professional development focused on implementing IOI. Their women students had positive learning outcomes; their average scores on the Group Theory Content Assessment (Melhuish, 2015) were higher than the comparison sample of women. The same instructional unit involving the reinvention of definitions of isomorphism from the Inquiry Oriented Abstract Algebra curriculum (Larsen et al., 2016) was recorded for both instructors during weeks 4–6 in a 15-week semester.

We analyzed two 50-minute subsequent class periods for each instructor. We identified episodes of behavior that conveyed students' roles during class. We focused on how participants reacted to those behaviors. We wrote descriptions of these episodes, as well as memos (Maxwell, 2013) reflecting on how the behaviors in each episode seemed normative. We inferred a certain behavior was normative if it was common, classroom participants did not challenge one enacting that behavior, and/or class participants challenged a class participant when they did not comply with that behavior. These criteria were based on Clark et al.'s (2008) conditions for documenting the development of norms. We open coded these episodes (Miles et al., 2013), naming the norms present in these episodes. We also analyzed how the instructors fostered these class social norms.

Results

Norms for Students Working on Tasks

Working on new tasks individually. Dr. Carter explicitly stated his expectation for students to work on tasks individually, saying phrases such as, "I want you to do this on your own" and "Paper. Pencil. Human. Solo." Students behaved this way for the rest of the class. Each time they started a new task later in the class period, they worked on it individually before they talked to other members of their groups, without being prompted to do so. The commonality of this behavior was evidence of this being a social norm in Dr. Carter's class. He fostered this social norm by setting expectations for students' behavior.

Discussing tasks with group members. After students in both classes worked on tasks individually for a few minutes, the students commonly started discussing the tasks with other group members, often without prompting. After some students began discussing the task, Dr. Ryan reinforced this behavior by telling the class to "convene in our groups and discuss what you've done so far and any progress you've made." The entire class began to discuss their ideas within their groups. This behavior seemed normative, for once students had ideas from their work, they began talking. Dr. Ryan fostered this social norm by reinforcing his expectations for students' behavior, particularly when students did not yet comply with the expected behavior.

Norms for Students Providing Contributions

Sharing contributions. Dr. Carter and Dr. Ryan fostered this norm of students sharing contributions by explicitly leveraging students' contributions to inform the lesson. For instance, Dr. Carter's student, Jessica, shared a conjecture regarding necessary conditions for the correspondence between the elements in D_6 and elements in the group represented by the mystery Cayley table. Dr. Carter assigned a follow-up task to Jessica's conjecture, asking why a specific correspondence, which met the conditions Jessica specified, would not work for showing the mystery group was D_6. Dr. Carter said, "So Jessica, I am challenging your conjecture. I'm gonna put it up here [on the board] as well. It has merits! It has good merit." Dr. Carter reinforced his expectation that students should share their conjectures by giving the opportunity to share contributions, writing Jessica's conjecture on the board, assigning a follow-up task that leveraged her conjecture, and saying that her conjecture had "good merits."

Explaining their reasoning. Another social norm present in these classes involved students explaining their reasoning. Dr. Ryan demonstrated his expectation for this behavior by calling on a student, Matt, and telling him to "explain your reasoning, tell us about the mappings you found, how you found them." The instructor reinforced the social norm of explaining reasoning by stating his expectations for the student's response. Matt then complied with this social norm and was not challenged because he did not violate the norm. If students violated this social norm by giving contributions without explaining their reasoning, the instructors challenged them for doing so by asking questions. For instance, as Dr. Carter's class discussed a student's definition of isomorphic groups, which said "$a, b \in G.\ c, d \in H.\ a \cdot b = c * d$," Becky claimed, "I'm a smidge bothered by the equal sign...so I think that we need to define some kind of function that maps one to the other." Dr. Carter responded, "Wait, hold on, why are you a smidge bothered? I don't understand. What's wrong with the equal sign?" Dr. Carter fostered this norm of explaining reasoning by challenging students when they did not comply with it.

Explaining difficulties they experienced. Students were expected to share the difficulties they experienced with mathematical tasks. For example, a student, Mallory, volunteered to share what Dr. Carter called a "productive failure." Mallory shared her initial failed attempts for a homework problem. She explained how she got stuck on the problem, took a break from it, and later tried a new strategy. Dr. Carter asked, "your productivity in the failure is?" Mallory replied, "Well I learned about that strategy…I feel a little bit more resilient now 'cause I just learned to like try stuff… not be afraid to try new different things." Mallory's experience in sharing her productive failure could have been an instance in which she developed her identity as a mathematician. This showed her confidence in her problem-solving ability, despite her previous failed attempts. By having Mallory present her productive failure, Dr. Carter reinforced the idea that it is okay to fail because something productive might come from it, which fostered the norm.

Norms for Students Responding to Others' Contributions

We identified social norms of responding to other students' contributions, giving productive feedback, and being non-judgmental. Dr. Carter explicitly stated his expectation for this behavior at the beginning of class, saying "remember that we are in a non-judgmental phase in our lives right now, so keep your comments very productive." To demonstrate these social norms, consider the following episode of Dr. Carter's class where each small group presented their definition of isomorphic groups on whiteboards, displayed at the front of the class.

Dr. Carter: "$a, b \in G, \phi(a \cdot b) = \phi(a * b)$." What is going on here?
Madison: We would have to split up the phi. (This means $\phi(a \cdot b) = \phi(a) * \phi(b)$)
Dr. Carter: This is another operation, right? [pointing to $*$]….so G is defined as G and dot, right?….But I think there is something going on here as well. Right? There's this phi. Not sure what it is yet, but there's this kind of correspondence as well.
Dr. Carter then directed the students' attention to another definition on the whiteboard.
Dr. Carter: What's going on over here? "G is isomorphic to H if and only if there exists a homomorphism."
John: I think that it is relevant to mention the homomorphisms. I think to add to that, you have to say that there exists a homomorphism G to H and there also exists a homomorphism H to G.

Responding to other's contributions. In this episode, both students, Madison and John, responded to other students' definitions of isomorphic groups. This behavior was common throughout the class discussions. Dr. Carter gave students an opportunity to respond to a group's contributed definition by asking "what's going on over here?" This reinforced the social norm.

Providing productive feedback to contributions. Both students then responded by giving productive feedback to make the contributed definitions more precise. Dr. Carter fostered this norm by providing productive feedback, thereby modeling the behavior in his own response.

Being non-judgmental of contributions. These students responded in a non-judgmental way, by not commenting on the imprecision or the incompleteness of the contributed definitions; rather, they validated and elaborated on the other students' contributions. Dr. Carter fostered this norm of being non-judgmental of contributions by modeling that behavior in his responses.

Discussion

We explored the social norms present in two IOI classes that had positive learning outcomes for women. We hypothesize these norms may be conducive to women's learning and promote the equity dimensions of access and identity (Gutiérrez, 2002). Working on a task individually gives access to the opportunity of engaging in meaningful mathematics tasks. Discussing tasks with group members, sharing contributions, explaining reasoning, and responding to others' contributions gives women opportunities to develop their mathematical identity as they present and evaluate each other's ideas (Hassi & Laursen, 2015). Having students explain the difficulties they experienced and normalizing productive failure can promote women's confidence. Norms of being non-judgmental and giving productive feedback to peers' contributions can promote a positive learning environment where students can contribute without fear of judgment. Although researchers have documented deficiencies in women's mathematical confidence (e.g., Lubienski & Ganley, 2017), these social norms can foster women's confidence and give opportunities for access and identity development, which can lead to achievement and power (Gutiérrez, 2002).

We aimed to explicate the instructor's role in fostering positive social norms. Instructors in our study fostered social norms by establishing expectations for students' behavior, modeling the expected behavior, challenging students when they do not comply with those expectations, and reinforcing those behaviors by showing they are valued. Our study contributes to research on social norms by further explicating the instructor's role in fostering norms. Instructors can use these strategies to foster positive social norms in their own classes.

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DIFFERENCES IN MATHEMATICAL ABILITY BELIEFS BETWEEN TEACHERS AND MATHEMATICIANS IN HIGHER EDUCATION

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Existing stereotypical beliefs regarding mathematical ability as being innate and being associated with men more have severe consequences for female students' perceptions of their mathematical ability, their course-taking decisions, and eventually, their decision to enter and stay in STEM fields. Yet how such beliefs compare among educators at different educational stages needs more attention. In this study, we analyzed the beliefs held by K-8 teachers and mathematicians who had or were pursuing a doctoral degree in mathematics regarding whether mathematical ability is innate. We found significant differences between mathematics teachers and mathematicians in their beliefs about mathematical ability and in the underlying structure of their responses.

Keywords: teacher beliefs, mathematics ability beliefs, gender-specific ability beliefs

Gender disparities persist in the representation of women in mathematically intense STEM fields (National Center for Education Statistics, 2013; National Science Foundation [NSF], 2015). Some research has explored the extent to which these gender differences can be explained by widely held stereotypical beliefs and biases that are communicated to girls at an early age in social environments, harming their self-perceptions and academic performance (e.g., Rosenthal & Jacobson, 1968; Steele & Aronson, 1995; also see Ceci, Williams, & Barnett, 2009 and Wang & Degol, 2017, for reviews). Exposure to gender-specific beliefs and implicit biases is hypothesized to reinforce stereotypes that affect women's feelings of competency (or self-concept) in a specific domain (Correll, 2001; Greenwald et al., 2002), potentially dissuading them from pursuing careers in that domain. Thus, it is important to explore potential implicit and explicit messages students receive throughout their academic lives, especially from their teachers during elementary and secondary education as well as their instructors in postsecondary education, given that teachers' and instructors' opinions can have a substantial impact on their self-concept. As such, the objective of this study was to measure and compare teachers' and mathematicians' beliefs about mathematical ability.

To date, little research has compared stereotypical beliefs held by instructors at different stages of education. Prior studies have found that K-12 teachers' conceptions play a role in shaping their actions (for foundational studies, see Cooney, 1985; Ernest 1989; Thompson, 1984; 1992), that elementary and middle-school teachers sometimes believe that mathematical ability is fixed and innate (Copur-Gencturk, Thacker, & Quinn, in press; Chrysostomou & Philippou, 2010), they associate innate mathematical talent with boys more often than girls (Authors, 2019; Fennema, Peterson, Carpenter, & Lubinski, 1990; Tiedemann, 2000, 2002), and they stereotype mathematics as a male domain (see Li, 1999, for a review)—stereotypes that are also associated with those held by their students (Keller, 2001). As students transition from secondary to postsecondary education, young women aspiring to pursue STEM careers continue to be exposed to messages conveying that mathematical ability is innate (e.g., Leslie, Cimpian, Meyer, & Freeland, 2015; Meyer, Cimpian, & Leslie, 2015). Women also receive overtly gender-biased messages from their professors about their mathematical ability (Robnet, 2016) that may lead to gender differences in STEM self-concept (Boysen, 2009; Sax, 1994), suggesting that stereotypical messages might be passed on from professors and internalized by students through personal interaction. However, based on the existing evidence, it is difficult to assess how professors' beliefs compare with those of elementary and

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middle-school teachers given that few studies have directly measured professors' beliefs about mathematics, and those that do use scales that differ from the ones used at the elementary and middle-school levels.

Current Study

In the present study, we used the same set of questions with two different populations—K-8 mathematics teachers and mathematicians at universities—to investigate what beliefs these two groups held about mathematical ability and how their beliefs compared with one another. To our knowledge, no studies have compared whether the beliefs held by teachers at these different grade levels are different. We aimed to answer the following two research questions:

1. What are mathematics teachers' and mathematicians' beliefs regarding the role of raw ability, hard work, and gender in students' mathematical success?
2. How similar are the constructs underlying the responses of K-8 teachers and mathematicians to these questions about mathematical ability?

We leveraged the existing data gathered by Leslie and colleagues (2015) and then adapted the items used in their study to capture K-8 teachers' beliefs on the same issues. We argue that knowing the kinds of messages students receive across their academic lives has important implications for recognizing female students' perceptions of their ability and their available career trajectories.

Methods

We used existing data from the study by Leslie and colleagues (2015) along with a new data set we created from the survey responses of K-8 teachers. Leslie and colleagues distributed an online survey to experts across 30 disciplines from nine universities in the USA. Of this wider sample, 1,427 mathematicians were contacted, and 133 of them provided usable data (9.3%). Mathematicians were graduate students (45%), postdoctoral researchers (12%), and faculty members (43%) who were mostly female (83%. With regards to K-8 teachers, we collaborated with the Consortium for Policy Research in Education (CPRE) to send out our survey items to elementary and middle school teachers in a large school district in the USA. We restricted our analytical sample to those teachers who reported teaching mathematics and who answered all the survey items, which resulted in 412 teachers. Teachers were mostly female (89%), and taught grades K-2 (45%), grades 3-5 (38%) and grades 6-8 (17%).To make the comparison meaningful between the two groups, we revised the wording of the items used by Leslie and colleagues (2015) to make them relevant to elementary and middle school contexts (see Table 1 for original and updated items).

Analytical Approach

To answer the first research question, we examined descriptive statistics for each group separately and then ran independent *t*-tests for each item to investigate whether the differences in mean scores for these two groups were statistically significant. To answer our second research question, we explored and tested several factor model structures separately for each group to identify which structure fit the data better.

Results

We began by summarizing teachers' and mathematicians' responses to the survey items. As shown in Table 1, the survey responses indicated significant differences between mathematics teachers and mathematicians in their agreement with the statements that (a) being a top student in mathematics requires innate ability that cannot be taught, (b) innate ability is needed to be successful in mathematics, (c) that anyone can become a top student or scholar in mathematics with the right amount of effort and dedication, and (d) that males are more better at/more suited for mathematics than girls. Significance was Bonferroni corrected ($\alpha = .05/5$).

Table 1: Descriptive Statistics for the Item Responses of Mathematics Teachers and Mathematicians

	Mathematicians			Mathematics teachers				
Individual items	Mean (SD)	Min	Max	Mean (SD)	Min	Max	*t*	*p*
V1. Being a top student (scholar) of mathematics requires a special aptitude that just can't be taught	4.84 (1.78)	1	7	2.41 (1.40)	1	7	16.3	**<.0001**
V2. If you want to succeed in mathematics, hard work alone just won't cut it; you need to have an innate gift or talent	4.52 (1.74)	1	7	2.14 (1.18)	1	7	17.9	**<.0001**
V3. With the right amount of effort and dedication, anyone can become a top student (scholar) in mathematics	3.15 (1.88)	1	7	5.56 (1.50)	1	7	15.1	**<.0001**
V4. When it comes to mathematics, the most important factors for success are motivation and sustained effort; raw ability is secondary	4.44 (1.82)	1	7	4.81 (1.61)	1	7	2.22	.033
V5. Even though it's not politically correct to say it, boys are often better at mathematics than girls (men are often more suited than women to do high-level work in mathematics).	2.23 (1.69)	1	7	1.84 (1.22)	1	7	2.92	**.004**

Note. N = 133 for mathematicians, and N = 413 for mathematics teachers. To account for multiple tests, significance was Bonferroni corrected at α = .05/5. Item text that appears in parentheses indicates the version given to mathematicians.

To answer our second research question regarding the factors underlying these two groups' responses, we explored the same two-factor model for these five items in both mathematician group and mathematics teacher group, given that beliefs about innate mathematical ability and gender ability seemed to be two theoretically different constructs. Thus, we expected that in the two-factor model, the first four items (V1–V4) would load onto the first factor because they were designed to capture mathematics as a discipline that requires raw aptitude, and we expected the fifth item (V5) to load onto the second factor because it was designed to measure beliefs about gender-specific mathematical ability. An exploratory factor analysis (EFA) for the mathematicians' data supported the two-factor model structure. Conducting the EFA with two factors and a promax rotation (i.e., the factors were allowed to correlate), the factor loadings of the five items (V1–V5) on the first factor were 0.726, 0.765, 0.709, 0.746, and 0.014, and the factor loadings of the five items on the second factor were 0.032, 0.081, −0.031, −0.084, and 0.994. On the basis of the results of the EFA, we performed a confirmatory factor analysis (CFA) with the data from mathematicians, in which V1–V4 loaded onto the first factor and V5 loaded onto the second factor. The two-factor model fit was good for the mathematicians' data (CFI = .996; RMSEA = .033; SRMR = 0.032). The factor loadings on V2–V4 were 1.050, 1.000, and 0.987 (Bentler, 1990; Hu & Bentler, 1999).

We attempted to fit the same two-factor model to the mathematics teachers' data, but the model fit was poor (CFI = .692; RMSEA = .213; SRMR = 0.109). Because the same structure was not valid in both groups, the configural invariance test failed. This result implies that the mathematics teachers' data had a different structure. We then conducted an EFA of the mathematics teachers' data to explore the structure of the data. Only the first two eigenvalues were larger than 1 (i.e., 1.99 and 1.28), and a relatively large drop occurred after the first two factors. Therefore, with an EFA of two factors and a promax rotation, the factor loadings of the five items (V1–V5) on the first factor were 0.743, 0.652, 0.008, 0.043, and 0.493, and the factor loadings of the five items on the second factor were 0.021, 0.006, 0.995, 0.468, and −0.071. According to the results of the EFA, V1, V2, and V5 should load onto the first factor, and V3 and V4 should load onto the second factor. To confirm this structure, a CFA was conducted, and the model fit was good (CFI = .994; RMSEA = .034; SRMR = 0.021). The factor loadings of V2 and V5 onto the first factor were 0.710 and 0.525, and the factor loading of V4 onto the second factor was 0.650 (for model identification, the factor loadings of V1 and V3 were constrained to 1). Thus, the two items emphasizing the raw talent needed for success in mathematics and the item associating boys with higher mathematical ability formed one construct, whereas the two items emphasizing the role of hard work and dedication in mathematical success formed another scale for mathematics teachers.

Discussion and Conclusions

These results show that mathematics teachers and mathematicians seemed to hold different sets of beliefs regarding mathematics requiring innate ability, the role of hard work in success in mathematics, and female students' mathematical ability. Furthermore, the underlying structure for these two groups was not identical. The mathematicians seemed to think that mathematics required ability and hard work and that dedication would not lead to success; however, they also did not consider this ability as belonging only to men. In contrast, teachers seemed to differentiate effort and dedication as constructs separate from innate ability. Unlike mathematicians, K-8 teachers did not agree that mathematics was a subject requiring innate ability. Rather, they seem to think that hard work and dedication could lead to success in mathematics.

As mentioned, students' academic self-concept is shaped by the messages they receive from their social environment. Thus, our study suggests that students may be receiving mixed messages from their environments, which could contribute to changes in their self-concept at different stages of their education (e.g., Robnett, 2016; Sax, 2008; Wigfield et al., 1997). The elementary and middle school teachers seemed more likely to agree that mathematical ability is a malleable construct and that effort and hard work could lead to success in mathematics, whereas the mathematicians seemed to believe ability played a key role in success in mathematics. This finding, showing that elementary teachers' and mathematicians' beliefs were different, might explain why gender differences in self-concept shift and expand after elementary school and into postsecondary education, although causal evidence of this link is still needed. Additionally, such potentially drastically different messages between these two groups might severely affect students' self-concept in college, which could explain their shifting majors (e.g., Seymour & Hewitt, 1997). However, more research is needed on the impact of these different and contrasting messages on students' self-concept at different stages of their education. Our study suggests that close attention needs to be paid to the messages teachers and college instructors send so that female students avoid entering or have difficulty staying in STEM-related fields because of stereotypical beliefs their educators may have held.

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PRODUCTS OF WHITE INSTITUTIONAL SPACE: AN ANALYSIS OF WHITENESS IN ONLINE MATHEMATICS TASKS

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Research has shown that mathematics education is a white institutional space. Utilizing two frameworks, I interrogate an online mathematics curriculum for tenets of neutrality, impartiality, and color blindness. While several themes emerged from this analysis, for the sake of this paper, I highlight three themes: mathematics is portrayed as neutral, real-world scenarios are readily manipulated with data, and lack of cultural sensitivity. There is a value in dismantling mathematics education as a white institutional space because it broadens the opportunities for students to engage with mathematics in more authentic ways. I encourage other scholars to join in interrogating whiteness in mathematics education.

Keywords: Curriculum Analysis, Equity and Diversity, Teaching Tools and Resources

Purpose of the Study

In recent years, scholars have started to problematize how we research, teach, and learn mathematics to dismantle systems, pedagogies, and practices that privilege whiteness while oppressing minoritized students. Using Martin's (2013) framework of mathematics as a white institutional space, I analyze six lessons from the *Mathalicious*[1] curriculum to examine the extent to which the curriculum perpetuates white institutional spaces. Martin (2013) argues that mathematics education and its products (e.g., curricular materials) operate as a white institutional space that produce racial/ethnic educational inequities. White institutional spaces are characterized by

> (1) The numerical domination by whites and the exclusion of people of color from positions of power in institutional contexts, (2) the development of a White frame that organizes the logic of the institution or discipline (3) the historical construction of curricular models based upon the thinking of white elites, and (4) the assertion of knowledge production as neutral and impartial, unconnected to power relations," (Martin, 2013, p. 323).

As such, mathematical curricula are not neutral and often have connections to politics and inequities (Lesser and Blake, 2006).

While mathematics is often conceptualized as universal, culture-free, and based on a system of meritocracy, anthropologists have shown that mathematics is a product of the social, cultural, and economic needs, values, and norms of societies and cultures (Powell and Frankenstein, 1997). Simply put, mathematics is not neutral or objective (Battey, 2013; Gutierrez, 2013; Bishop, 1994). The research questions elucidated in this paper are the following:1) Does *Mathalicious*, treat mathematical knowledge as neutral and impartial, if so, how do we identify neutrality and impartiality? And 2) do mathematics tasks or lessons, such as *Mathalicious*, discuss societal phenomena unconnected from power relations, if so, how do we identify this discontinuity?

Theoretical Frameworks

For this study, I integrate two complementary theoretical frameworks: Martin's (2013) work on mathematics education as a white institutional space and Battey and Leyva's (2016), framework for understanding whiteness in mathematics education. Divided into three dimensions, Battey and Leyva's (2016) framework serves as a lens that functions to, "(a) systematically document how

[1] www.Mathalicious.com

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whiteness subjugates historically marginalized students of color and their agency in resisting this oppression, and (b) make visible the ways in which whiteness impacts White students to reproduce racial privilege," (Battey & Levya, 2016, p. 49). The dimensions of whiteness (institutional, labor, and identity) are divided into corresponding elements. The elements encompassed in institutional whiteness are ideological discourses, history, organizational logic, and physical space. The labor dimension includes cognition, emotion, and behavior. The final dimension, identity, encompasses academic (de)legitimization, co-construction of meaning, and agency and resistance.

Methodology

I choose to investigate *Mathalicious* because the mathematics lessons integrate Common Core State Standards (CCSS) and Standards for Mathematical Practices (SMP) as its foundation and use real-world contexts to help students build conceptual understanding in mathematics. Several prominent researchers frame mathematics education as an example of white institutional spaces due to the historical construction of curricular models based upon the thinking of white elites. CCSS serves as a racial project in that it "presents itself as a colorblind and universal effort with equity and social justice ends" (Martin, 2013, p. 326). Moreover, CCSS serves as a product of white institutional spaces, because the curriculum encourages assimilation, and is organized by a white frame of logic that is also inherent in the discipline (Martin, 2013).

Statistical literacy is an essential skill. Individuals need to be able to interpret, produce, and be critical consumers of data-based arguments. Thus, as statistical literacy, "is rooted in practices for participating in, critiquing, and (re)shaping structures and discourses in society that are crucial for critical citizenship in society," (Weiland, 2016, p. 988), educational stakeholders must critically investigate how we teach statistics. For the purposes of this paper, I examined six lessons in *Mathalicious* that dealt with statistics. Three of the lessons were from the One Variable Statistics unit, and the remaining three lessons were from the Bivariate Statistics unit. The One Variable Statistics lessons were "Good Cop, Bad Cop," "Police Academy," and "Distributive Properties." The Bivariate Statistics lessons were "Joy to The World," "Pic Me," and "Win At Any Cost."

As part of my analysis, I mapped the lessons from *Mathalicious* onto Battey and Leyva's (2016) framework. There are three dimensions in the conceptual framework: institutional, labor, and identity.

Results

While several themes emerged from this analysis, for the sake of this paper, I am highlighting those that specifically addressed the research questions presented above: mathematics is portrayed as neutral, real-world scenarios are readily manipulated with data, and lack of cultural sensitivity.

Results from the analysis indicate that, in some cases, mathematics was conceptualized and depicted as unconnected to power relations. For example, in the "Distributive Property" lesson, students examine income inequities in the United States. Educators utilizing the lessons are advised to stick to math and not be swayed by the nuances or real-world consequences connected to the data: "They are evaluating which subgroup made the most improvement in their income distribution, rather than trying to determine if the distributions are more equitable. As long as students flesh out their argument and support it with evidence from the data, there is no right or wrong answer" (p. 6). While students can address the problem in several ways, there is no discussion of how the income disparities have real consequences for communities. If students approach the problem in one direction, they will learn that white women had the most significant change in income distribution. If students examine percentage change, then they will conclude that Hispanic women are making the most progress. Students leave with the message that progress can be quantified with numbers. A second instance can be found r in "Good Cop, Bad Cop," where teachers are cautioned against allowing the conversation to deviate from the math (analyzing and interpreting graphical displays). The lesson notes state "While the discussion provides important context for the problem, it could also make it difficult to redirect the conversation back to the math," (p. 5). The message is clear: the data can be separated from the contexts in which they are collected.

In conjunction with the theme that mathematics is neutral, the *Mathalicious* lessons under analysis present conflicting messages about the ease with which real world data can be manipulated and modeled. In each of the lessons, the real-world data presented by *Mathalicious* is devoid of the mathematical complexities that exist in the real world. In the lesson, "Joy to the World," students learn that the data are not as perfect as the linear regressions suggest. Students are cautioned against making definitive conclusions about what increases happiness. Moreover, students learn that while mathematical models can be applied to data, the resulting models do not always tell the complete picture. However, in "Good Cop, Bad Cop" and "Police Academy," students are encouraged to use data to draw conclusions about policies. The data is presented as an easy model and complete enough to make, "decisions about the appropriateness and effectiveness of a policy meant to change that data," (Good Cop, Bad Cop Exemplar Response, p. 1). Thus, while it is important for students to understand that data and subsequent analyses may have limitations, there is a difference in messaging between these three lessons.

These lessons in *Mathalicious* demonstrate that real-world relevance does not equate to cultural relevance or sensitivity. Some fail to consider issues of race, gender, or how communities are impacted by structural or institutional racism. In "Good Cop, Bad Cop," the curriculum guide suggests that some instances of excessive force by police are understandable given that officers, "deal with people in public as well as prisoners in jail, and some officers do this often." Implicit in this message is that some people and communities need excessive policing. The message normalizes or justifies some instances of excessive force. Additionally, there is a disproportionate number of People of Color in jails. By justifying the use of excessive force in prisons, the curriculum also perpetuates that violence against Black and Brown bodies is sometimes necessary. In "Distributive Property," students make comparisons between different subgroups of the population. The comparisons are between white men and women and their "non-white" counterparts. Hispanic and Black men and women are collapsed into two categories (men and women) that imply their lack of whiteness.

Discussion

Of the six lessons examined for this project, five had indicators of or were related to whiteness. The lesson, "Win at Any Cost," did not have any direct connections to whiteness, although it could be argued that the lesson topic is tangential to whiteness. "Win at Any Cost," requires students to use data to make judgments about whether professional sports organizations (Major League Baseball, National Basketball Association, National Football League, and National Hockey League) are spending their money well when they secure talent. For the lessons that had connections to whiteness, the indicator that was observed the most amongst the lessons was, "Mathematics as neutral." This is directly related to the fourth tenet of Martin's framework, "the assertion that knowledge production as neutral and impartial, unconnected to power relations" (p. 323). The second most frequently occurring indicator was "Distribution of classroom and mathematical authority." I contend that this indicator is related to Martin's (2013) second and third tenet: The development of a white frame that organizes the logic of the institution or discipline and the historical construction of curricular models based upon the thinking of white elites. Mathematics curriculums position students differently in relation to mathematics, their peers, their teachers, other people, and their own experiences (Herbel-Eisenmann & Wagner, 2007). The *Mathalicious* curriculum is positioned as the mathematical authority. Additionally, the curricular model encompassed in *Mathalicious* is like other contemporary mathematics lessons and tasks.

It is troubling that a majority of the lessons had instances of whiteness for many reasons. This analysis reveals the pervasive nature of whiteness in contemporary mathematics curriculum materials and the potential for its use in the continued marginalization and exclusion of the experiences of children of Color. Second, there were many messages of mathematics being neutral. Mathematics as neutral and relationship with deficit discourses students in terms of their racial, gender, and cultural identities are two indicators of whiteness in mathematics education (Battey & Leyva, 2016). While *Mathalicious* does not broadcast the number of teachers or districts that utilize their curriculums, it has been featured in The Washington Post, Education Weekly, and at several mathematics education conferences, including National Council of Teachers of Mathematics and National Council of Supervisors of Mathematics. With such a broad publicity base and open access to their lesson, it is likely that *Mathalicious* has a broad reach.

To begin to dismantle systems of oppression in mathematics education, we must publicly interrogate the policies and practices that perpetuate inequities. This project is essential because, "An ideology of whiteness would then serve to position white people, white ideas, and white behaviors as more valued institutionally and in classrooms, which may not always be visible in terms of curriculum designers and policy developers," (Battey and Leyva, 2016, p. 55). This project provided an example of whiteness in mathematics lessons. More research is needed to examine how mathematics curriculum perpetuates and reinforces whiteness. To facilitate more research on mathematics lessons or tasks as a product of or connected to whiteness, analytic frameworks are needed.

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TENSIONS OF RESPONSIBILITY IN CRITICAL MATHEMATICS: DEFINING TEACHER RESPONSIBILITY AND STUDENT AGENCY

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Often even teachers who are committed to and plan to teach critical mathematics struggle to do so. To understand this struggle I studied discussions between seven preservice secondary mathematics teachers and myself. From these discussions I identified a key theme of responsibility for teaching CM. I argue that our understanding of responsibility was shaped by dominant discourses in ways that were incompatible with the goals of CM.

Keywords: Teacher Education-Preservice, Social Justice, Equity and Diversity

Since its introduction, critical mathematics[1] (Frankenstein, 1990; Gutstein, 2006) has gained interest. However, teachers often struggle to effectively teach CM (Bartell, 2013; Brantlinger, 2004; Rubel, 2017). Preparing teachers for social justice is a challenge of teacher education (Galman, Pica-Smith, & Rosenberger, 2010; Sleeter, 2001), especially mathematics education (de Freitas, 2008; Gutiérrez, 2009; Gutstein, 2006). This may stem from the difficulty of developing a critical understanding of “teacher” and “student.” Dominant discourses cast teachers as authoritative and students as passive.

Resistance to social justice in education, especially from White teacher candidates, has been linked to Whiteness (Aveling, 2002; Sleeter, 2001). This resistance is manifest even as teacher candidates disrupt Whiteness (Applebaum, 2010; Aveling, 2006; Hytten & Warren, 2003; McIntyre, 1997). Here I am concerned with how teachers who support CM struggle to understand a teacher role that is socially just. To this end, I present selections from discussions with a class of pre-service mathematics teachers. While dominant views bounded our discussions, there were moments when we began working towards discourses that reflected responsibility to students.

Theoretical Framework

The theorizing of CM has drawn on critical pedagogy (Frankenstein, 1990; Gustein, 2006). This perspective emphasizes the use of mathematics for social critique. Expanding this view, Gutiérrez (2012) describes four dimensions of equity in mathematics education: access, achievement, identity, and power. *Access* views the resources that students have available, including technology and quality instruction. *Achievement* is measured with grades and test scores. Both access and achievement generally leave the content unchanged. *Identity* means providing opportunities for students to draw on their linguistic and cultural resources, meeting their own standards, and understanding themselves and their world mathematically. *Power* addresses voice in the classroom (authority), mathematics for social critique, nature of mathematics, mathematical ways of knowing, and humanizing mathematics.

Discourse, Whiteness, & Responsibility

In focusing on discourse I draw from Gee (2005) and Fairclough (2001). Discourse includes how we speak and the things that accompany speech. While discourses do not create the physical being of a mathematics teacher, they define the categories of mathematics teacher, mathematics student, and mathematics. Discourses enable and constrain, particularly dominant discourses (Fairclough, 2001).

[1] I use CM broadly to include much of what is referred to as social justice and/or equity in mathematics.

The pressure felt to conform to discourses maintains power structures. In the United States dominant discourses are discourses of Whiteness. Whiteness Theory uncovers how discourses maintain and promote White power structures. Whiteness Theory operates on the assumption that the lives of all people in the U.S., in particular, are racially structured (Frankenberg, 1993; Frye, 1992). Dominant discourses of responsibility (and Whiteness) center around *individual* responsibility. For students, these discourses are used to justify the inequities in schools by suggesting that students or their families are responsible for their own failures (Gutiérrez, 2015). A key aspect of the dominant discourses of mathematics and Whiteness is the need to portray school mathematics as neutral. Thus mathematical achievement is constructed as individual skill and effort. Individual responsibility helps support the view of mathematics as neutral by justifying blaming students.

Methods

In this study I focus on the discourses seven pre-service secondary mathematics teachers and I used during their final course. The participants included four White males (Pseudonyms: Jeff, Karl, Gavin, and myself), two White females (Stella and Lisa), one Latina (Esperanza), and one Japanese-American female (Jane). All of our classes were recorded and transcribed. I then selected those transcripts when we discussed mathematics and social justice. In my analysis I used Critical Discourse Analysis (CDA) as defined by Fairclough (2001). I combine this approach to CDA with Walshaw's (2013) recognition that teachers as well as students are caught in dominant discourses, and Thompson's (2003) explanation of how White anti-racist educators can reinscribe Whiteness. As I analyzed our discussions on the roles of teachers, students, and their relationships I found us repeatedly circling around ideas of responsibility and blame.

Results and Discussion

Taking Responsibility

For progressive educators one logical response to dominant discourses would be to accept responsibility. However, accepting responsibility may become a means of taking power in the classroom and reinscribing Whiteness. These problems with taking responsibility were illustrated in Lisa's narrative. She carefully links her struggles to her choices, rather than to her students.

> Lisa: ok so . . . I had no idea what my students were learning at all ((Lisa explains some of what led up to this project)) and then I started having problems so I realized I didn't know what they knew and then I started to plan lessons anyway and this like divide grew in my class and you should see the scores for their final it's like half As half failing they're like there's like nobody in the middle it's like crazy so I think like people were getting it and I didn't really pay attention to them because I knew that people were struggling and we kept like repeating stuff and people got bored and then I realized that that was happening so I switched my focus and started paying attention to them but by that point people had given up on trigonometry in general and the kids that were sort of lost like still were and it just became like this huge thing.

Notice throughout how frequently "I" is used as the subject of the sentence. This positions Lisa as agent and powerful, even though most of these statements refer to her mistakes. The power of her decisions can create major problems. When students are brought up they are passive ("divide grew in my class", "people were getting it", or "people got bored"). There is little student agency shown, which positions them as powerless and blameless. Students are left out of their education and teachers are problematically positioned as solely responsible the class. Lisa has taken all authority and left none for students. While Lisa does not appear to do so, this could be consistent with a "lone-hero" (White) teacher who tries to "save" her students through social justice (Thompson, 2008). Not blaming students and taking responsibility feel like necessary steps towards social justice. It could meet Gutiérrez's (2009) call to mathematics teachers to "be in charge" in their classroom. However, Gutiérrez links this teacher-in-charge with the tension to "not be in charge" to balance classroom power. Lisa does not avoid responsibility, instead she takes all of the responsibility onto herself. This is one of the few instances where the discourse of responsibility was used to place responsibility on the teacher. However, this combination does not leave room for active student roles. The problem with this combination of discourses is that recognition of structural and historical factors is disallowed while students are denied participation.

Hinting at an Alternative: Responsibility to Students

Margonis (2015) suggests that teachers can opt out of dominant discourses of responsibility through responsiveness. Through responsiveness, instead of blaming students (as dominant discourses would have us do), we are open to the messages students send, and we have an ethical responsibility to be responsive to those messages. Responsiveness requires that we view our students as worth hearing *and meaningfully responding to*. There were times when, in our discussions, we proposed a different kind of responsibility.

Considering negative effects of CM. Towards the end of the semester we again had a class devoted to what it means to teach CM. This is one of the first times in this discussion that a student perspective is taken up and students are positioned as intelligent and capable. This change in perspective shifts how we position students and teachers.

> Lisa: I feel like we need to talk about some potentially negative consequences that could happen.
> Teacher: Ok. Good.
> Lisa: Because all I want to say when you ask that question is well they'll probably think that we understand them better and that we're on their side and all these positive things but there has to be a negative side
> Teacher: Is there risk in doing that
> Lisa: Yeah and maybe they're if they don't trust they're right to so definitely negative feelings to be had ((we discuss laws regarding undocumented students and in-state tuition))
> Jane: . . . so it was kind of like depressing I feel like when they said like well hey you have to pay way more to go to school in this state because you're undocumented but then I feel like it's kind of like all those all the other students who are like legal or whatever their tax dollars are going towards the state and that's the reason they get in-state tuition or whatever I don't know I feel like that that thought would be absolutely depressing . . .
> Lisa: Oh yeah
> Jane: like $60,000 more when you already don't have very much money and like with that one especially I couldn't see any solution or any benefit of telling it . . .

Lisa flips the perspective to say "if they [students] don't trust they're right". This positions students as intelligent and selective in whom they choose to trust. This also suggests that teachers may not always be worthy of their trust. These are both important in how we position teachers and students and momentarily disrupt the dominant discourse of teacher authority. Within the context of the discussion these are students of color and White teachers. Positioning students as intelligent and careful navigators of a racist school system is necessary to work *with* students. In response to Lisa's comments Jane brings up an example from our readings. Jane characterizes this lesson as "depressing". She implies that teachers may be unaware of what students think and feel. As a daughter of immigrants Jane may feel a particular connection to this example. She does not specify a subject position for undocumented students. However, she references the argument about "legal" students whose "tax dollars" *earn* them "in-state tuition". She implies that teachers need to judge what information to share with students. This manifestation of authority is an authority to judge what is needed by others, a key aspect of Whiteness (Frye, 1992), based on the implied lack of (racial) bias that Whites possess. Despite Jane's return to dominant discourses other students address the cost of college for undocumented students.

Karl: Yeah or yeah it was really discouraging um I guess in a way it could be in a way motivating to think about or at least at least they realize how much it would cost and it would be tough but that there are probably things they could do I guess.
Jane: Mhmmm
Esperanza: A way to follow up would be like as a teacher investigate scholarships . . .
Karl: Right
Jane: But I feel like that's another problem is that they don't usually offer that many scholarships to undocumented students.

Karl suggests that knowing could be "motivating". This suggests that students have resilience. He then adds that "they realize", positioning students as capable. Esperanza then points out that teachers could "follow up". Following up assumes that the teacher has taught the lesson. For her teaching the lesson is an assumed part of being a teacher. This framing of responsibility comes as we position students as capable. This suggests a balanced responsibility where teachers share and support and where students are capable of understanding and utilizing information. Esperanza suggests that "a teacher investigate scholarships" and so positions teachers as responsible *to* their students to do this work; work that may typically be framed as outside the role of a teacher. This is also part of teacher authority, however, in this case instead of judging, the teacher uses authority to provide information and resources. Positioning teachers as responsible-to is facilitated by positioning students as capable. Jane then points out the relative lack of scholarships for undocumented students. Her she characterizes lack of scholarships as a "problem", critiquing the system and repositioning undocumented students as innocent.

Conclusion

Dominant discourses are used to portray teachers as solely responsible for what happens in their classroom. As a result in our discussions of CM we often used these discourses to position individual teachers as *not* responsible to teach for social justice. However, taking responsibility for CM is also problematic. Instead I propose responsibility *to* students. This kind of responsibility requires that students be positioned as intelligent and capable and an understanding of shared authority. Positioning students and teachers in these ways can disrupt dominant discourses.

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EXPLORING CULTURE IN MATHEMATICS EDUCATION FROM THE PERSPECTIVES OF PRESERVICE TEACHERS OF COLOR

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Studies on culture, equity, and social justice issues in mathematics teacher preparation have called for preparing teachers to build on students' mathematical and cultural backgrounds. Yet few studies have examined the preparation of preservice teachers of color (PSTCs), especially those attending Historically Black Colleges and Universities (HBCUs) or Hispanic Serving Institutions (HSIs). We present preliminary findings from a cross-site research project documenting PSTCs' perspectives on culture in mathematics education. We analyzed PSTCs' engagement in a culture unit during their mathematics methods course and their expressed views on cultures other than their own. Our findings reveal that these PSTCs often defined culture based on nationality and would repeat dominant deficit discourses about minoritized students. We provide insights for the importance of culture discussions in mathematics.

Keywords: Equity and Culture, Teacher Education - Preservice, Elementary

Introduction

Mathematics education research has acknowledged the role of culture in teaching and learning, and specifically how mathematics classrooms create a context that legitimizes or invalidates various forms of knowledge (Nasir et al., 2008). Mathematics teacher educators (MTEs) have the opportunity to expose preservice teachers (PSTs) to the different ways students reason about and learn mathematics and teach them how to build on students' mathematical and cultural backgrounds. MTEs have incorporated into their courses culture activities such as classroom observations and lesson planning (Koestler, 2012), field placements and student interviews in diverse settings (Fernandes, 2012), storytelling to challenge PSTs'color blindness (Ullucci & Battey, 2011), and family and community exploration projects (Bartell et al., 2019; Zavala & Stoehr, 2019). These activities are designed to help PSTs become "culturally conscious" (Gay, 2010) by recognizing their own cultural socialization and understanding how this affects their attitudes and behaviors toward other ethnic group cultures in mathematics education.

Although previous literature on MTEs has explored opportunities in which PSTs engage in deeper discussions and reflections on the role of culture in their teaching and learning, most studies have reported on the experiences of White PSTs only. Few studies have examined the ways in which PSTs from culturally diverse backgrounds engage in and respond to similar activities. As Montecinos (2004) cautions, assumptions based on the experiences of White monolingual teachers do not translate to the experiences of all PSTs. There remains a need to understand how PSTs of Color (PSTCs) navigate their teacher education programs. MTEs would also benefit from learning about PSTCs' views on teaching mathematics and culture, enabling them to prepare all future teachers to teach equitably. Our work sought to answer the following research question: *What are PSTCs' perceptions of other cultures in the teaching and learning of mathematics?*

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Conceptual Framework

A discussion of the role of racism in mathematics education and its implications for teacher education programs is beyond the scope of this paper (Martin, 2009, 2019; Rousseau Anderson, 2019). However, highlighting the "structural phenomenon" of racism in education (Rousseau Anderson, 2019) affords us the opportunity to examine how messages about certain groups of students and their mathematical identities continue to permeate mathematics classrooms. Recent research on the mathematics learning experiences of minoritized students has found that students have had racialized experiences in which "socially and personally constructed meanings of race emerge as salient in interactional experiences related to mathematics" (Martin, 2019, p. 461). Thus, PSTCs are likely to have experienced mathematics classrooms as White institutional spaces in which non-white cultural knowledge may be positioned as inferior or lacking. This recognition highlights the need for all PSTs to learn to question and understand issues of power, race, culture, and identity in mathematics classrooms (Gutiérrez, 2015; Rubel, 2017).

In discussing how to prepare PSTs to enter culturally diverse communities, Bartell and Aguirre (2019) warn MTEs about the deficit perspectives some PSTs hold about children and their communities and their potential resistance to engaging with families and communities that are different from their own. Thus, preparing PSTCs must include recognizing and creatively responding to discourse that positions minoritized students as incapable and developing the PSTC's role as an advocate (Gutiérrez, 2015). This is especially important because when PSTCs become teachers, they may "carry problematic beliefs into the classroom and replicate the cultural alienation students of color experience in schools" (Kohli, 2014, p. 371). Unless PSTCs have been required to reflect on their own racialized schooling experiences and engage in conversations and activities focused on creating rehumanizing mathematics classrooms that counter deficit views of minoritized students, they may unintentionally perpetuate structural inequities in their mathematics classrooms (Chao et al., 2019; Zavala, 2017).

Methods and Data Sources

CAM Up! (Cultural Awareness in Mathematics Unit Project) is a cross-site research study that seeks to illuminate PSTs' interests, perspectives, and dispositions toward teaching mathematics to culturally diverse student populations. Three institutions serve as research sites: one Historically Black College and University (HBCU), one Hispanic-serving institution (HSI), and one Predominantly White institution (PWI). We purposefully selected these sites to include PSTs who are culturally diverse across racial, ethnic, language, socioeconomic, and geographical backgrounds.

We used a modified version of a cultural awareness unit (White et al., 2016) to explore the PSTCs' perspectives. The unit included three components: (1) an article critique paper; (2) audiotaped class discussions in which PSTCs share their article critiques, describe their own culture, examine stereotypes in mathematics education, and discuss culturally relevant math teaching strategies; and (3) a post-discussion reflection paper. Project data include recorded class observations, unit artifacts, and researchers' field notes.

In this paper, we present preliminary findings for 10 female PSTCs (HBCU=3, HSI=4, PWI=3) to convey key themes on culture that emerged across the three sites. These PSTCs were randomly selected from a larger group of 52 PSTCs. Four of the PSTCs self-identified as African American/Black, one as Asian, and five as Hispanic/Latina.

The findings presented here focus solely on our analysis of the article critique assignment. This assignment required PSTs to find, read, and write a critique of an article that focused on teaching mathematics to students from a cultural group other than their own. We intentionally did not define culture prior to the assignment so as not to influence their perspectives on culture.

For the first cycle of coding/data analysis, all authors read the PSTCs' article critiques, met, and created a list of holistic codes, keeping the research question in mind (Saldaña, 2016). These holistic codes captured the overall ways PSTCs positioned themselves in relation to the culture discussed in their selected article. For the second cycle, each author re-coded holistic categories individually using values coding to arrive at more precise categories that captured the values, attitudes, and beliefs represented in the data. This cycle uncovered how PSTCs perceived the culture of others with regard to teaching and learning math. Analyzing the critiques and unit artifacts allowed us to triangulate the data, yielding two emergent themes that identified the common mathematics discourses we discuss in our findings.

Findings

A preliminary analysis of the article critiques revealed that PSTCs have complex views of culture, as evidenced by which cultures they selected and why they made that choice. Several insights emerged into how PSTCs view the intersection of culture and mathematics teaching and learning. In this section, we describe the PSTCs' views on culture and their acceptance or rejection of dominant cultural views about minoritized students in mathematics education.

Views of Culture

Understanding and unpacking the participants' descriptions of culture shed light on how they view their own culture in the context of teaching and learning mathematics. The PSTCs described culture in three ways: *culture as nationality, culture as language acquisition*, and *culture as socioeconomic status (SES)*. Six PSTCs ascribed to *culture as nationality*, comparing the way mathematics was taught in their "American culture" to the way it was taught in the cultures of other countries. In their article critiques, the PSTCs identified cultural nuances that emerged in the descriptions of the mathematical teaching strategies. For example, one PSTC noted, "One way the Chinese culture is different from my American culture is based on differences in language structure."

Three PSTCs noted that language plays an important role in learning mathematics. PSTCs who ascribed to *culture as language acquisition* contrasted their use of "standard academic English" with the language of non-native English speakers. Moreover, some PSTCs discussed the role that language played in their own identity. For example, one PSTC stated, "The primary culture described in the article is that of English language learners (ELL) . . . growing up, I could not define myself as an ELL."

Several PSTCs also viewed *culture as SES*, suggesting in their critiques that students' socioeconomic status represented an aspect of culture that influenced their mathematics learning. One PSTC wrote, "I grew up in a middle- to upper-middle-class area, therefore I don't share the low-income aspect of the students' culture either." None of the10 PSTCs saw culture as the shared norms among a group of people.

Views of Mathematics and Culture

The PSTCs expressed interest in learning about various cultures because they recognized both the changing demographics in schools and the achievement gaps that persist. Although they wanted to learn about other cultures, however, the PSTCs were often unaware of asset-based teaching strategies. Many PSTCs expressed agreement with or repeated dominant deficit mathematics discourses about minoritized students in the article critiques. These discourses included: *language is a barrier, caring parents are involved in their child's education*, and *Asian teaching methods are better in math*.

Some PSTCs viewed speaking a primary language other than standard English as a barrier to learning mathematics in the U.S. One PSTC wrote, "Although I am Asian and grew up in a bilingual household, I was never classified as an ELL nor did I need any type of additional academic instruction due to language barriers." Another PSTC wrote, "English Language Learners (ELL) and African American English (AAE) speakers are groups who suffer greatly from culturally skewed word problems . . . [M]y family always spoke Standard English. Therefore, I never faced the barrier of making sense of the Standard Academic English that is used in schools." These PSTCs viewed language as a barrier to rather than a resource for mathematics teaching and learning. In describing strategies for teaching mathematics to English language learners, the PSTCs recommended removing the complex language from word problems instead of incorporating words from the students' cultures.

Some PSTCs accepted the deficit discourse that certain cultures value parental involvement and that without it, learning mathematics is more difficult. Moreover, these PSTCs suggested that parents who care will be actively engaged in their child's education. One PSTC identified parent support as a family value, noting, "Asians are more likely to listen to their parents in regard to educational and vocational decisions, due to an innate need to not disappoint or bring shame to the family. Family is at the center of their culture, something that was once essential to the African-American community and their survival but is sadly no longer valuable."

While the critiques of eight PSTCs reflected dominant narratives, two resisted stereotypes such as, "All Asians are good at math." These participants perceived culture as learned and adaptable. One PSTC expressed a desire to examine the Asian model minority stereotype more closely, stating,

> This theory jeopardizes the students who actually need help in academics . . . I particularly like that the author says students can respond using their 'own mathematical power.' I truly believe that everyone can achieve mathematics, but we just have our own 'super power' that we cannot project.

Discussion

This study's preliminary findings uncovered three views of culture expressed by PSTCs—culture as *nationality*, *language acquisition*, and *socioeconomic status*—that reflect the dominant narratives of culture in the contemporary U.S. We also uncovered two themes related to how PSTCs perceived math and culture: agreement with deficit narratives/stereotypes about how mathematics is taught and learned, and rejection of these narratives/stereotypes.

These findings suggest several implications for methods courses and the faculty who teach them. As MTEs, we are aware of the racialized mathematical experiences of PSTCs (Martin et al., 2019), and we believe it is incumbent upon us to guide PSTCs to reflect on the cultural footprint they bring to the mathematics classroom. Moreover, we must push back against narratives that assume PSTCs will automatically make cultural connections with students of color in their classrooms. We concur with Gist (2017) that teacher educators need to "value teacher candidates' cultural and linguistic diversity and understand how to draw on and develop their multicultural capital" (p. 930). Only then will we develop PSTCs' cultural consciousness, enabling them to dismantle deficit narratives in mathematics teaching by honoring and incorporating their students' assets and strengths.

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TRANSLANGUAGING TO ENSURE LATINX MATHEMATICS LEARNERS THRIVE

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This paper reports on a study designed to showcase Latinx bilingual children's linguistic and cultural resources for learning mathematics in an after-school mathematics club. Specifically, we examine the design of the activity system (Engström, 1999) and social interactions therein through a translanguaging perspective in which students leverage their language and culture to engage in mathematical learning. Our primary objective is to highlight some to the triumphs and struggles of bilingual children as they expand communicative practices and mathematical resources via interactions with bilingual facilitators and electronic communication with a math wizard, El Maga.

Keywords: Communication, Instructional activities and practices, Equity and diversity, Informal education

This paper presents a translanguaging perspective in which students use their agency to develop and affirm their Latinx bilingual mathematical identities in the context of an afterschool math club. Although research has disproven deficit-oriented beliefs that using native language during mathematics learning is unnecessary or harmful (e.g., Khisty & Chval, 2002; Moschkovich, 2007), there is still much to understand about the intricate processes by which Latinx bilingual students use their language and culture to engage in mathematical learning (Razfar, 2013). Therefore, our aim is to highlight how Latinx bilinguals leveraged their communicative practices and mathematical resources while communicating electronically with a math wizard. Specifically, we highlight critical features of a mathematics teaching and learning environment that supports Latinx bilinguals' translanguaging and what translanguaging affords learners. We use the term "bilingual" or "bilinguals" to center the reality that these students are multilingual, even if only emerging multilinguals. Additionally, we use the term "leverage" to reflect students' agency to capitalize on their communicative and linguistic repertoires (see Martinez, Morales, & Aldana, 2017, for a review of how this term is used by scholars).

Latinx Students and Mathematics

Latinx students are one of the fastest growing school age populations in the U.S. (NCES, 2016). Yet, Moll (2001) posits that classroom practices have continued to create a distance between Latinx students' language, cultural knowledge, and what they know academically. These systems persist in marginalizing, and thus not privileging linguistic, social, and cultural capital to help create dehumanizing school practices (Langer-Osuna, Moschkovich, Norén, Powell & Vazquez, 2016). Moreover, guiding texts like Principles to Actions: Ensuring Mathematical Success for All (NCTM, 2014) asserts that all students must have access and opportunity to study mathematics. As such, math classrooms have been encouraged to move from isolated seatwork to more social and verbal activities that require students and teachers to engage in more substantive mathematical discussions and collective practice (Bass & Ball, 2015). However, there is a concern that mathematics reforms may be in danger of ignoring the needs of Latinx students unless their needs are re-examined in light of the new demands of the mathematics classroom with its increased emphasis on communication and collaboration (Moschkovich, 2000). While this is a generalized perspective of classrooms with Latinx students, it nevertheless raises questions about marginalization and undervaluing Latinx students' learning resources in mathematics.

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Garcia, Ibarra Johnson, and Seltzer (2017) reminds us that all good education must begin with recognizing students' strengths that come from their own community's linguistic and cultural repertoire. Much of the research with Latinx students has focused on bilingual language learners. The research is often framed from a deficit perspective focusing on the relationship between students' proficiency in their first language and learning mathematics (e.g., Mestre & Gerace, 1986) or the obstacles faced by Latinx bilinguals learning mathematics across languages (English and Spanish) (e.g., Khisty, 1995). Deficit perspectives emerge when bilinguals' linguistic resources are ignored or forbidden in the classroom while only privileging the dominant school language (Langer-Osuna et al. 2016). These are de-humanizing practices with the sole purpose of controlling and dominating students' cultural identity and excluding it from the classroom and school (Gutierrez, 2017). Garcia (2017) argues that this view of language and academic discourse in schools acts as a barrier to knowledge (in our case, mathematical knowledge), only privileging those students whose linguistic repertoire mirrors the dominant school language.

Other studies have also shown that Latinx students use a wide variety of cultural resources to construct, negotiate, and communicate (spoken or written) about mathematics (Chval & Khisty, 2009; Varley Gutiérrez, Willey & Khisty, 2011). These resources include cultural knowledge (Gutiérrez, 2002), linguistic resources (e.g. mathematics register, mathematical discourse) (Celedon-Pattichis, 2003; Moschkovich, 2000), everyday experiences, life histories, and community funds of knowledge (Moll, 2001). Moreover, Razfar, Khisty and Chval (2011) advocates for a social-cultural model for language development in mathematics classrooms. They juxtapose social-cultural theory (SCT) against second language acquisition (SLA) models. In the SLA model, learners are perceived as passive recipients of mathematical knowledge proceeding in a linear developmental path and language is seen as an external tool. In contrast, SCT positions bilingual students as active agents in their language use, capable of working collaboratively, interacting, and communicating while grappling with challenging mathematical tasks. Given this disparity, a counter-narrative to common deficit perspectives, will allow us to illuminate the ways in which Latinx bilinguals encouragingly use their linguistic and other funds of knowledge while engaging their individual and collective agency to assert their identities as mathematics learners. Gutiérrez (2017) discusses such a rehumanizing perspective as one that positions the student as central to the meaning-making process while engaging in the practice of doing mathematics. This perspective refutes the imposing of standardized or normalized practices onto students, such as the routine expectation of students reproducing the teachers' idea of productive mathematical activity. Instead, a rehumanizing perspective fosters respect and dignity through privileging the viewpoint and experiences of the student, and the ways in which they develop personal understandings through their own disciplinary perspective on mathematics.

Translanguaging

We draw on translanguaging to reconceptualize bilingualism as a liberating and empowering communicative practice and resource capable of transforming learning the goes beyond students' transition to the dominate school language (MacSwan, 2017). Translanguaging is more than just a simple shift between two languages (i.e., English and Spanish). It is a complex and interrelated communicative practice that make up bilinguals' linguistic repertoire (Cenoz, 2017). Garcia (2017) posits, "…speakers use their languaging, bodies, multimodal resources, tools and artifacts in dynamically entangled, interconnected and coordinated ways to make meaning" (p. 258). Translanguaging classrooms are powerful spaces that take full advantage of and leverage students' linguistic repertoire to engage with complex content and texts, strengthen students' linguistic repertoire in academic contexts, draw on students' bilingualism for the purpose of expanding their ways of knowing, and support students' bilingual identities that counter English only ideologies (Garcia, Johnson, & Seltzer, 2017).

Given the paucity of studies that use a translanguaging framework to understand language practices of multilingual persons in mathematics classrooms, we argue that translanguaging re-conceptualizes the mathematics learning and language practices of Latinx bilinguals. Mazzanti and Allexsaht-Snider (2018) report on a study in a kindergarten classroom with a large Latinx population. They find that their use of English and Spanish occurred simultaneously as a way to make mathematical meanings. These meanings were not limited to just linguistic accomplishments, but also used symbolic representation of numbers and visual models of the number problems to leverage these students' communicative resources. Other studies have focused on spontaneous translanguaging of 12th grade Latinx bilinguals in an upper level math classroom. Morales (2004) used a translanguaging and perseverance framework to compare the collaborative efforts of two groups of students solving a challenging mathematical task. Results show that when students are given the freedom to explore mathematics via their dynamic bilingualism they are able to spontaneously and dialogically leverage communicative resources to help them persevere and overcome in-the-moment obstacles. Morales & DiNapoli (2019) further ague that these students' translanguaging practices reposition students as competent problem solvers and agents of their own learning while leveraging bilingual identities as learners of mathematics reflecting a rehumanizing perspective for all students in the group.

Maldonado, Krause, and Adams (2018) also believe that emergent bilinguals could participate in mathematics teaching and learning in ways that are rehumanizing to all participants in the classroom. They explore ways in which a 2nd grade dual language mathematics classroom built a translanguaging stance where teachers made choices to build on children's thinking while engaged in mathematics instruction that develops knowledge, dispositions, and builds on students' cultural, linguistic, and community funds of knowledge. They argue that mathematics teachers must nurture a translanguaging stance by (1) Respecting others ideas. (2) Committing to caring for others and bringing the community together. (3) Collectively working together for the good of the individual and group. And (4) engaging in a mathematical practice together.

Research Design, Context, and Methods

Our research was conducted in a large, urban school district in a very large Midwestern city in the United States where 85.6% of the students are classified as coming from low-income families, and 13.7% are categorized as "English learners". Our research site was James Dual Language School. James was made up of approximately 425 students from Pre-Kindergarten through the 6th grade. The school population consisted demographically of 99.4% Latino/a. Additionally, 98.3% of the students were eligible for the government's free or reduced lunch program, and 68% of the students were categorized as English language learners (ELLs) (School District Data, 2007).

The research reported here is based on data gathered in an after-school project called "Los Rayos de CEMELA" adapted after the work of The Fifth Dimension (Cole, 2006) and La Clase Mágica (Vásquez, 2003). The after-school project was designed to give Latinos/as experiences doing non-remedial mathematical activities including problem solving and playing mathematics games that were intended to enhance students' knowledge of probability and algebraic concepts. Students were encouraged to be self-directed, to work collaboratively, to verbalize their thinking, and to ask questions. One of the goals of the after-school project was to promote mathematical bi-literacy. All participants were encouraged to speak Spanish. Playfulness between the adults and children was a critical part of the interactions in the after-school project. In addition, students communicated electronically with a mathematics wizard, El Maga, who engaged students in bilingual conversations about their mathematical experiences.

Findings

Despite our original efforts to privilege Spanish in the after-school, social forces prevented participants from fully realizing the potential and capital of Spanish. We highlight these challenges and discuss adjustments made to the after-school club to more authentically draw on children's, and families', funds of knowledge, including their linguistic resources.

The children's gravitation towards English – but also El Maga's persistence using Spanish and commitment to supporting language growth – was also documented in children's correspondence with El Maga (an undergraduate student facilitator).

As we mentioned earlier, we were particularly interested in creating a space that privileged communication in Spanish, so we purposely wanted to include the parents during the second session. We considered that including the parents would naturally encourage the students to speak in Spanish. We asked the students to explain the rules of the game and to demonstrate how to play the game to the parents. When Rodrigo played the Counters Game during the second session, he worked with a group that was more Spanish dominant. He played with two other students, Margarita and Rafael, his mother Olga, two more mothers, and an undergraduate facilitator, Carlo. All three mothers were Spanish language dominant, and their command of English was limited. When Rodrigo played the Counters Game again, he was more confident and even took on the role as sort of a referee making sure everyone including the parents played by the rules.

The after-school club got a critical boost from the parent-participants. Parents served multiple roles, one of those being a bridge where Spanish moved from its familiar place in family and community spaces into the academic space where children frequently preferred to speak English. Parents, through their presence and eagerness to participate alongside children, established new norms where mathematical conversations were in Spanish. Furthermore, parents embraced their role as collaborators in the design of the community mathematics projects. Their insights contributed to mathematical activity where problem solving was done within and through rich community and cultural contexts, rather than trying to bring cultural contexts into pre-fabricated mathematical activities. The evidence suggests that this collaboration allowed children and parents to draw on meaningful – although subjugated and grossly under-acknowledged and under-utilized – forms of funds of knowledge.

Conclusions and Implications

It turned out that parents, because of their language backgrounds, are natural examples of translanguaging. Mathematics learning environments are well-served by the involvement of multilingual parents and community members because they re-shape the linguistic lanscape in a way that elevates language use and development, and likely meta-linguistic awareness, too. Palmer et al. (2014) argues further that much of the work involving bilingual students is framed around an ideology and policies of language separation that encourage teachers to create separate instructional spaces (physical and other) for bilingual students to communicate in either the dominant or minority language. These ideologies and school policies also contribute to separating parents and other community members from the teaching and learning context and, moreover, do not take into consideration the power of communicating across multiple languages to adequately support the learning of mathematics among bilingual Latinxs. We are only in the beginning phases of understanding the intersection of language ideologies, mathematics learning environments and language practices. This data implies that further research should examine the role of various multilingual actors and the context of the mathematical activity in order to determine the myriad ways they support Latinx learners to thrive mathematically where traditional classroom practices and schooling structures have largely failed.

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EQUITY AND JUSTICE:

POSTER PRESENTATIONS

MIM: MATHEMATICS EDUCATION RESPONSIVE TO DIVERSITY: A NORWEGIAN, CANADIAN AND AMERICAN RESEARCH COLLABORATION

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Keywords: Classroom Discourse, Cross-cultural Studies, Culturally Relevant Pedagogy, Equity and Diversity

MIM aims to promote education responsive to diversity through participatory research by developing and evaluating strengthening pedagogies. These are research based pedagogies building upon individuals' strengths and assets identified by examining past positive experiences; encouragement of hope and optimism and development of emotional satisfaction with the present (Seligman, 2002) hence moves away from cultural-deficit orientations and instead promotes achievement for all students.

Linguistic and cultural challenges are not new. Indigenous communities have experienced them for decades as a result of colonisation, as have children from non-dominant communities in other contexts. Tensions in education are intensified by language and cultural differences in times of large migration (Cenoz & Gorter, 2010). Classrooms with a high number of students from different migration waves, historically homogeneous communities with newcomers in their schools for the first time, and Indigenous schools with endangered languages are contexts that have been described in research, society and media as problematic due to race, gender, culture and religion, hence impacting *all* students.

The main objective is *to develop new scientific knowledge about how mathematics education may contribute to equity and social justice - and vice versa. At the heart of the research are students' and teachers' storylines.* Through juxtaposing Indigenous and migration contexts, we will further understand students' experiences and hence pedagogical possibilities, within Norway, Canada and the USA. We apply positioning theory (e.g., Harré & van Langenhove, 1999) to understand students' and teachers' experiences as it provides the required tools to understand *how* the people in an interaction may have different understandings both of the interaction and of the opportunities available to them within it. The storylines used by migrated and Indigenous students to interpret their mathematics classroom interactions and the role of mathematics in their life trajectories will be juxtaposed with the storylines used by the others in their classrooms and community. We have recently begun extending the field's understanding of the availability of storylines and identities in mathematics classrooms (Andersson & Wagner, 2019; Wagner, Andersson & Herbel-Eisenmann, 2019).

MIM is to be situated historically and culturally draw on and further examine research-based work that has shown to have a positive impact on groups who have been marginalized by policies and practices in educational contexts; and will be done, reflexively, in partnership with the peoples and communities who the work is supposed to benefit. MIM has all of these characteristics and will contribute to mathematics education work empirically, theoretically, methodologically, and practically.

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PROSPECTIVE MATHEMATICS TEACHERS' CONCEPTUALIZATIONS OF EQUITY

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Keywords: Equity and Diversity, Teacher Education - Preservice

Prospective mathematics teachers (PMTs) may have early and frequent opportunities to observe and teach in mathematics classrooms. What is often lacking, however, are their opportunities to examine and discuss inequities that exist in the classroom. This paper explores 30 PMTs' (19 from a university (U) and 11 from a community college (CC)) conceptions of equity, utilizing two of Gutiérrez's (2012) four dimensions of equity, namely access and power. Specifically, we investigated and compared their proposed responses to two hypothetical vignettes (Max, 2017) from mathematics department conversations regarding calculator usage and mathematical discourse.

Vignette 1-Ms. Lopez: I encourage some, but not all, of my students to use calculators in class. If I don't let these students use calculators, they can't contribute to the problem solving we're doing. Mr. Parker: I too have students who benefit from the use of calculators, but I think fairness is really important. Depending on the lesson, I either let all of my students use calculators of none of my students. This way no one ever feels cheated.

Vignette 2 -Ms. Booth: Because I know more mathematics than my students and they look to me as the expert, I do most of the talking. It's important for students to hear the correct uses of mathematical language so I model that as much as possible. Ms. Sutherby: Students will learn mathematics by using the language themselves, even if imperfectly, so I let them talk as much as possible.

Preliminary analysis revealed that the majority of the PMTs explicitly agreed with one teacher's approach (U = 84% and CC = 64% for access and U = 63% and CC = 45% for power). PMTs considered equality, creating more interactive learning environments, and classroom resources with respect to calculator usage. Most of the PMTs viewed fairness as equality and expressed a desire to create an interactive learning environment (U = 79% and CC = 55%) while less than 10% of the PMTs from each group favored only encouraging some students to use calculators. More PMTs from the community college (36%) preferred a balance of both approaches than those from the university (11%). The equality responses included: "I agree with Mr. Parker and think if I do allow students to use their calculators it should be everyone, so no one feels left out or as Mr. Parker said cheated out of the test" (CC). Modeling correct language and allowing the students to talk during class were both important factors in PMTs' views on mathematical discourse. In the discourse vignette, 63% and 45%, respectively of the PMTs supported having students do most of the talking, none of the PMTs encouraged the teacher-majority approach, and 37% and 55% of the PMTs preferred a balance of the two approaches. The student majority approach responses included: "Classroom conversation should be about creating new knowledge. … I want my students to feel comfortable about sharing their thinking, therefore, I want them to do most of the talking…" (U). Additionally, the elementary PMTs from CC were less likely to relinquish power regarding whose voice dominates in the mathematics classroom, however, all PMTs were aware of and thinking about some equitable issues.

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Access and power as seen through vignette responses. *School Science and Mathematics 117*(7-8), 286-294.

BLACKNESS AND WHITENESS IN APPALACHIAN MATHEMATICS CLASSROOMS

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Keywords: Equity and Diversity, Marginalized communities, Rural education, Social Justice

The inequitable experiences and outcomes for Black learners of mathematics is well documented in urban areas (Martin, 2012). There is less work on the construction of race in education in rural areas, specifically Appalachia. Because race intersects with multiple economic and social structures, it is necessary to attend to context when studying race in mathematics education (Ladson-Billings, 2005; Patel, 2016). This poster makes the case for a study of learning mathematics while Black in Appalachia and provides initial findings of research on Black students and White teachers of mathematics in West Virginia.

Critical race theory in education is based in the idea that racism is endemic to American society and its educational system. Also, American society was formed in and still continues to function based on property rights (Ladson-Billings & Tate, 1995). In Appalachia, property and economics is the focus of discourse in the region. Because this work in economics is centered on White rurality, racial issues are made invisible (Anglin, 2002). Race is also ignored in rural education as schools are sites of normative White cultures that lead to structural racism (Groenke & Nespor, 2010).

The theory also operates under the assumption that Whiteness is a form of property. However, this functions differently in Appalachia as White people in the region have been essentialized as "white trash" (Smith, 2004). This leads to a presumed "White innocence" in Appalachians and substitutes class issues for race issues making them invisible (Scott, 2009). As the vast majority of teachers in West Virginia are White, this has ramifications for the education of Black students in Appalachia. Particularly with labor strife rampant in Appalachian education, Black students are a "neglected minority within a neglected minority" (Cabell, 1985, p. 3). Critical whiteness studies are centered in the hyper-segregation in schools and explore the race consciousness of White teachers (Jupp, Berry, & Lensmire, 2016) Using critical Whiteness studies can provide a nuanced look at White teachers of Black students.

Critical race theory and Critical Whiteness studies provide insight into the way mathematics education functions in racialized ways. Mathematics is often viewed as a neutral, universal field free from politics (Gutierrez, 2013) and can serve as a "gatekeeper" to upper levels of mathematics and higher education (Moses & Cobb, 2002). STEM education is also a form of property and capital in American society (Bullock, 2017). Mathematics education is a White institutional space (Battey & Leyva, 2016), so it is essential to use these frameworks to study students and teachers in Appalachian mathematics classrooms.

This poster provides a framework for the study of Black learners and White teachers of mathematics in Appalachia. It also provides preliminary findings in a study of Black students' experiences learning mathematics and White teachers' perspectives on teaching students of color in the region. There is ample documentation of how race functions in urban mathematics classrooms. This work is designed to illuminate racist interactions and structures that oppress Black people in mathematics education in Appalachia which can lead to the disruption and dismantling of racism in rural education.

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MATHEMATICS PROBLEMS AND REAL WORLD CONNECTIONS: HOW POLITICAL IS TOO POLITICAL?

PROBLEMAS MATEMÁTICOS Y CONEXIONES CON EL MUNDO: ¿QUÉ TAN POLÍTICO ES DEMASIADO POLÍTICO?

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In this poster presentation the authors share the results of surveying preservice teachers (PST's) in The United States (US) and Uruguay, with different problem types following Simic-Muller et al. (2015) framework and work. US PST's consistently showed an inclination to family background or community practice problems, and a rejection of issues of injustice problems. Uruguay PST's data is not as clear as US data, but there are some points of contact with the US results.

Keywords: Problem Solving; Social Justice; Teacher Education – Preservice.

Mathematics has long been seen as a separate tool from everyday lives of children, disconnected from social issues. Teachers need to re-discover that connection and exploit it towards the academic and non-academic success of children in and out of school. Most important, teachers need to realize and accept that as Mathematics is a "weapon in the struggle" (Gutstein, 2012), teachers themselves are political actors (Gutierrez, 2013). One effective way to help children see the connection, and help teachers explore this "new" identity, is to teach Mathematics through problem solving. Given that Mathematics it is not a "culture free" content that can be taught in a space "politics free", then teachers must acknowledge the political power of Mathematics and their own political power, specially as they teach minoritized students. The authors investigate how pre-service teachers respond to different problem types, that even though are all "real life situations", are more or less committed to show Mathematics as a political tool to understand and change the world.

The questions the authors aim to answer are:

Do elementary pre-service teachers consider using problems for which the context requires an analysis of issues of injustice that may be difficult, but that are part of children's lives?

What are the explanations pre-service teachers provide to ground their decisions in regard to what contexts are or are not acceptable for teaching mathematics in elementary school?

The authors surveyed 21 US, and 33 Uruguay elementary preservice teachers (PST) taking their first mathematics methods course. The US PST's belong to the same section on a teacher preparation program located in the South of the United States, and the Uruguayan PST's belong to the same preparation program in a urban area.

Preservice teachers have strong opinions of what is and is not appropriate for elementary school children when learning mathematics. This is usually based on their understanding of the students. However, with background so different from their students and families, it is not clear how this is helping. They assumed that children would be scared if they "introduce" mathematical contexts like family separation at the border (in the US) or feminicide (in Uruguay). They also assume apple picking and soccer would be familiar and liked by the students. Yet we know children are already scared about those difficult topics. It is imperative that the PST's understand the issues that relate and are important to their students so they can draw on those to teach mathematics. The authors consider this a pilot study, and would like to repeat this experiment with a larger audience to be able to perform statistical analysis of the data. The results also suggests that they should conduct interviews with some of the participating PST's about their choices and explanations. In addition to that, the authors also would like to consistently use cognitively guided instruction (CGI) as a way to control mathematical complexity of problems.

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CONCEPTUALIZING ACCESS KNOWLEDGE FOR TEACHING MATHEMATICS

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Keywords: Equity and Diversity; Teacher Beliefs; Teacher Knowledge; Students with Special Needs

Drawing from critical approaches to disability such as neurodiversity and Disability Justice this paper focuses on developing understanding of teacher knowledge necessary for successful engagement of students with disabilities in meaningful mathematics. Students with disabilities have long been conceptualized through deficit frames within both special and mathematics education and denied opportunities to engage in sense-making (Lambert 2018). Utilizing data from a previous study (Lambert, Sugita, Yeh, Hunt, & Brophy, 2020), this paper is a call to reframe teacher knowledge in mathematics to understand the importance of *access knowledge for teaching mathematics*, knowledge about how math class and math learning feels to students with different *bodyminds* (Price 2011).

Disability Justice (Berne 2017) is an emerging political movement in the disability community that includes attention to intersectionality and embodiment. Access is understood not as not only being able to enter a space, but relational engagement within that space (Mingus 2017). The concept of access-knowledge comes from the work of Aimi Hamraie (2018) who analyzed how disabled maker culture and the founders of Universal Design redefined access-knowledge towards creative problem solving based on close understanding of user experience, as well as understanding the complexity and diversity within disability.

I define Access-Knowledge for Teaching Mathematics (AKTM) as knowledge about how math class and math learning feels to students with different bodyminds, including an approach to solving access issues for students that locates difficulties in classroom spaces, practices and school systems rather than within individual students. Solving access issues necessarily involves collaboration between the student and the teacher. Access is not just being able to enter a classroom, or be given alternative to particular forms of presentation, but also access to relationships, interaction and a feeling of safely that are at the core of student's experiences in school. This paper expands current scholarship in mathematics education teacher education by theorizing new forms of teacher knowledge necessary to include students with disabilities in meaningful mathematics.

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FACILITATING MATHEMATICS TEACHER EDUCATORS' CONVERSATIONS ON INEQUITIES IN MATHEMATICS CLASSROOMS

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Keywords: Equity, Teacher Education - Preservice, Inservice/Professional Development

Mathematics teacher educators (MTEs) have a responsibility to prepare and support mathematics teachers to build safe cultural spaces for students to learn mathematics. This requires making teachers aware of "national, state, district, and school contexts for educating students and be[ing] ready to engage in conversations to address inequitable learning experiences" (Association of Mathematics Teacher Educators, 2017, p. 23). One method of providing this preparation is to engage teachers in conversations and experiences that enable them to recognize and address inequitable learning experiences.

We used the casebook *Cases for Mathematics Teacher Educators: Facilitating Conversations about Inequities in Mathematics Classrooms* (White et al., 2016) in our three respective contexts to foster math educators' ability to identify and address inequities and to develop MTEs' ability to engage in conversations as inequities arise or are recognized. Guided by a situative framework (Putnam & Borko, 2000) and an inquiry stance (Cochran-Smith, 2003), we sought to answer the question: *How do mathematics teachers across the teacher development continuum respond to equity-related cases and engage in conversations about inequities in mathematics education?*

This cross-site research study was situated within teacher education and professional development settings. Participants included PreK-12 mathematics teacher leaders; graduate students preparing to become university-based MTEs; and preservice mathematics teachers. After the participants read the cases, we used common prompts to explore how they grappled with, embraced, and/or resisted equity-based dilemmas. The participants' written responses, recordings of the discussions, and facilitator notes were collected and analyzed (Saldaña, 2016). Preliminary data analysis revealed that all groups were eager to engage in conversations around the cases, which enhanced their ability to notice and analyze inequitable situations.

A case titled "Who Counts as a Mathematician?" led to rich conversations across all three groups. Several participants challenged traditional assumptions regarding who can be considered doers of mathematics, allowing us to address inequitable learning contexts that may result from stereotyping and tracking. The teacher leaders often had experiences similar to those of the case authors and were ready to disrupt inequities. However, some graduate students and preservice teachers found it difficult to unpack the complexities and nuances of the dilemmas. This finding suggests the developmental nature of educators' ability to see themselves in the cases, feel empowered to disrupt inequities, and be ready to facilitate conversations with other educators.

As MTEs, we must continue to grow and develop our ability to facilitate conversations about inequities in mathematics classrooms. This study allowed us to learn collaboratively across settings and adjust our practice. Our findings highlight the need to explore how educators develop an understanding of various scenarios in order to create equitable mathematics classrooms.

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MATHEMATICS IDENTITY AND SENSE OF BELONGING OF DEVELOPMENTAL MATHEMATICS STUDENTS

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Keywords: Developmental Math, Mathematics Identity, Equity

One of the significant challenges facing higher education is narrowing the educational attainment gap between students who are academically prepared and those who are not. Although the intention of developmental education is to help support underprepared students in achieving academic success, there have been disagreements among researchers on the effectiveness of achieving this goal (Goudas & Boylan, 2012). On one hand, developmental mathematics has the capability of providing the impetus that can propel students to their overall academic success. On the other hand, the long road the students have to go through in completing mathematics requirements causes many to give up before they can finish the sequence of courses (Rosin, 2012).

This study examines how developmental students' general and mathematical experiences help to shape mathematical identities they develop and how these identities in turn hinder or enhance their successful participation in mathematics. Also examined are the factors that influence students' mathematics identities after taking a developmental mathematics course. To this end, the following research questions guided this study of first year students taking a developmental mathematics course at a mid-sized, urban public university:

1. How do developmental mathematics students describe their mathematics identities?
2. What factors coalesce to influence students' mathematical identities after taking a developmental mathematics course?

Data was collected using pre-post surveys and semi-structured interviews. The analysis reported here is based on the data from the survey instrument. The statements in the survey were grouped into five aspects of mathematics identity; self-concept, self-efficacy, motivation, and anxiety, and value of mathematics. Qualitative data from the open-ended items of the instrument was systematically analyzed using grounded theory to uncover patterns and trends in participants' responses while descriptive statistics was calculated for the quantifiable portions of the surveys. Items were compared both within surveys and across surveys to identify correlations and trends, as well as to support qualitative themes.

Analysis revealed that differences of the overall mean scores of all five aspects of mathematics identity between females and males were not statistically significant. Further, students scored the lowest on self-concept while the highest score was on their perception of the value or importance of math in their lives. As Klinger (2004) pointed out there are many students who do not particularly enjoy mathematics and report a disliking for the subject (negative affect), even though they still respect the utility and importance of math in their future lives and careers. Also, the study revealed that students' self-efficacy and self-concept increased significantly over time.

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CRITICAL MATHEMATICS EDUCATION FOR SUB-SAHARAN AFRICAN YOUTH: TOWARDS EPISTEMIC FREEDOM

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Keywords: Ethnomathematics, equity and diversity, first nations and indigenous cultures, informal education

I draw on the findings from one study to show how Njo (pseudonym), a Gambian youth in the study shifted towards *Epistemic Freedom* during the course of our semester long co-exploration. I argue that when multiple knowledges are valued in mathematics spaces, students are able to see themselves as capable mathematics learners. Ndlovu-Gatsheni (2018) speaks about epistemicide as the killing of indigenous peoples knowledges, that occurred during colonialism and is still in place now. Thus, he introduced Epistemic Freedom as "democratising 'knowledge' from its current rendition in the singular into its plural known as 'knowledges'." In mathematics education, epistemic freedom pushes us to make room for multiple knowledges as this shifts towards what Ndlovu-Gatsheni termed "cognitive justice."

Using African epistemologies – Sankofa, Ubuntu, and the Fela Anikulapo-Kuti Music (FAM) – I sought to co-explore with five Sub-Saharan African youth, if and how they use mathematics in understanding, challenging, and disrupting social issues related to the African context. Ubuntu (Tutu, 1999) is a Southern African philosophy emphasizing that I am because we are. Sankofa (Dei, 2012) is from the Twi people in Ghana that asserts that we must look into our past before reflecting on the future. Lastly, I coined the FAM methodology from Fela's song *Teacher Don't Teach Me Nonsense* (Anikulapo-Kuti, 1986) which has three main facets including co-learning, disruption, and joy.

The research question guiding this inquiry is "what knowledges do African youth draw upon in their investigation of social issues and how might these knowledges advance our understanding of critical mathematics education?" In the spring of 2019, I partnered with five sub-Saharan African youth in an out-of-school context. Together, we met for approximately two hours once a week using a story-telling approach. I collected audio and video data from all sessions along with WhatsApp group messages. I ensured that we collectively designed the space with the overarching goal of making sense of social issues on the African continent. I was intentional in not centering mathematics but instead was interested in seeing if mathematical ideas emerged in thinking through social issues. I analyzed the data thematically while ensuring that I was guided by the stories told (Wilson, 2008).

In the poster, I will use seven narratives that reveal Njo's journey towards epistemic freedom as she began to claim authority in her mathematics education. These narratives reveal her identity formation, disruption of colonial discourses, awareness and valuing of multiple ways of knowing, and lastly, multiple ways of knowing in mathematics. Njo's journey showed how through looking at cultural artifacts and reflecting on her own experiences, she was able to value indigenous [mathematics] knowledge. Moreover, she asserted that if young children are given the opportunity to see the multiplicity of knowledges within their communities, perhaps they will enter formal mathematics spaces with more confidence and belief in their abilities.

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"YA ME CONFORME": RESISTING DOMINANT NARRATIVES IN MATHEMATICS CLASSROOMS

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Keywords: Equity, Resistance, Social Justice.

Mathematics is often portrayed as an apolitical, culture- and color-blind universal language (Gutiérrez, 2013; Marin, 2009; Shah, 2017). However, Recent work in mathematics education has called for a need in exploring the intersectionality between mathematics, learning, and student identities (Gutiérrez, 2002, 2013; Shah, 2017; Zavala, 2014). This is of particular interest for multilingual students since they are a growing and diverse population (de Araujo et al., 2018) at the intersection of race, language, culture, and immigration.

In this poster, we draw from Critical Race Theory and Latinx Critical Theory (Delgado Bernal, 2002; Solórzano & Delgado Bernal, 2001) as the basis for examining patterns of resistance by Latinx multilingual students in linguistically diverse high school mathematics classrooms. Resistance is a form of agency that can be used to explore how multilingual students negotiate and struggle with structures, and use those interactions to create meanings (Solórzano & Delgado Bernal, 2001). We were particularly guided by the following research question: How do students engage in resistance that are motivated by a critique of oppression and an interest in social justice?

Similar studies have analyzed Latinx experiences in mathematics high school classrooms in more diverse settings (e.g., Zavala, 2014). However, this study includes four ninth graders from one classroom at a more racially segregated setting, City High. City High is located on the US-Mexico borderlands and has a relatively large population of recent immigrant and transnational students. The four students included in this study represent a variety of Latinx experiences, from students who recently arrived to City High from Mexico to students who have lived around City High their entire life.

Counterstories (Solórzano & Yosso, 2002) of semi-structured interviews about mathematics and identity and classroom observations are shown to illustrate how students engage in resistance. Particularly, we found that students acted in both individual and collective forms of resistance that are at the intersection of Solórzano and Delgado Bernal's (2001) transformational and conformist resistance. *Transformational resistance* includes students who are motivated by social justice and explicitly critique oppression. *Conformist resistance* includes students who are motivated by a need for social justice but do not hold an explicit critique of oppression. *Resilient resistance* lays at the intersection of transformational and conformist resistance and includes students whose actions challenge oppression, but they may not explicitly challenge the nature of the oppressive structures (Yosso, 2000). For example, students may say things like "ya me conforme" and note that race and language do not have a direct impact on their educational experiences. Yet, they may also collectively organize (e.g., assigned group roles to each other, embrace translanguaging, and engaged in collaborative group work) as a way to negotiate, struggle, and make meaning within school structures.

More importantly, the four students in this study did not have direct instruction about critiquing oppression or fostering a motivation social justice, as is with critical pedagogy (e.g., teaching math for social justice). Yet, students showed steps toward a path of transformational resistance, providing evidence for critical consciousness that students bring into the classroom.

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CAPTURING THE HIGH GROUND IN LEARNING DISABLED MATHEMATICS EDUCATION RESEARCH

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Keywords: Equity, diversity, learning disabled, learners with difference, special education

Despite the mathematics-for-all mantra, few mathematics education (math ed) scholars have studied learning disabled (LD) students' mathematical learning (Xin et al., 2015). This extremely low number of math ed studies specifically on LD students is puzzling (considering that LD students comprise at least 5% of student populations) and unfortunate (because special education scholars—steeped in behaviorist/medical-deficiency paradigms—have dominated the research landscape and largely promulgated a dehumanizing, explicit-only instructional approach for LD students' mathematics learning that directly contradicts current reform-oriented approaches embraced by the math ed community). This special education explicit-only message is so pervasive that even mainstream reform-oriented math ed publications—like the National Council of Teachers of Mathematics' position statement on intervention (NCTM, 2011)—default to this explicit-instruction-is-best-for-LD-students belief. It is time for the field of math ed to "capture the high ground" by exerting more influence on the research narrative about what instructional methods are appropriate for LD students.

It is in this context that this mixed-methods research study investigates with fine-grained analysis the embarrassingly immature condition of math ed LD research. I examine the various tiers of math ed research publications (as defined by various authors, e.g. Toerner & Arzearello, 2012; Williams & Leatham, 2017) to describe the quantity and quality of LD math ed research by math ed scholars. For example, over the last 20 years, the two top-tier math ed journals, *Educational Studies in Mathematics* and the *Journal for Research in Mathematics Education* have only published six and five studies—respectively—on LD students' mathematics learning. Even the proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) demonstrates a paltry 1.7% ratio of LD to all studies, far below the 5% minimum threshold expected based on the number of LD students in mathematics classes. I also apply Glaser's (1965) constant comparative method to develop a theoretical matrix cataloging the various types of math ed publications that include LD issues (e.g., ones that include special education statistics without separating out LD students from those with physical handicaps). I conclude with concrete recommendations for mathematics educators to capturing the high ground of LD math ed research.

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A NARRATIVE INQUIRY OF *GEMS* WOMEN'S EXPERIENCE WITH STEM

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Historically, women are underrepresented in science, technology, engineering, and mathematics (STEM) fields (Hill, Corbett, & Rose, 2010). To inspire girls to participate in STEM disciplines, many researchers have found that a variety of interventions, such as female role models, hands-on activities, and single-sex learning environments, can increase girls' interest and shape their identities in STEM areas (e.g., Chen et al., 2011; Holmes et al., 2012; Tyler-Wood et al., 2012). National efforts to engage girls in STEM disciplines have had mixed reviews. Between 1993 and 2015, the number of women in computer and mathematical sciences occupations increased by 173%. However, these fields have attracted relatively more men, whose participation increased by 239%. Thus, the overall proportion of women has declined from 31% to 26% (National Science Board [NSB], 2018). The NSB (2018) also reported that five years after receiving their highest degrees across all science and engineering areas, only 18% of females remained in these fields compared with 33% of males.

Corbett and Hill (2015) pointed out that although many studies have concentrated on factors contributing to the entry of women into STEM fields, far fewer have examined the question of why women leave these fields, often after years of preparation, and what factors could support them to remain in the fields. Through exploring the lived experiences of the women who participated in the first *Girls Excelling Math and Science* (*GEMS*) Club–an ongoing afterschool STEM program begun 26 years ago–this study will investigate the group of women's experiences with STEM. The study is guided by the research question: How have the original *GEMS* Club members' experiences in *GEMS* influenced their education, career selection, and lives, both personally and professionally? Specifically, a) How have their experiences in *GEMS* impacted their identity, including mathematics, STEM, and gender identities? And b) How have their experiences in *GEMS* influenced their sustained interest, engagement, and participation with STEM?

Given the purposes of this study, narrative inquiry is the methodology and form of analysis in the study (Clandinin & Connelly, 2000). Data is collected through a questionnaire and interviews. The questionnaire collects demographic information and experiences in the *GEMS* club from all women who have participated in the first *GEMS* Club. Respondents who have detailed memories of their *GEMS* experiences are interviewed in a sequence of three semi-structured interviews with each participant, as suggested by Seidman (2013).

Data analysis consists of both analysis of the narrative and narrative analysis (Polkinghorne, 1995) identify common themes that emerge across stories from different participants. The narrative analysis focuses on the social environment that shaped the stories, particularly from a feminist standpoint (Brooks, 2007; Harding, 1991). Using thick, rich descriptions within the three-dimensional inquiry space (Clandinin & Connelly, 2000), the study represents each participant's life story by laying out the events, configuring them into episodes, and constructing contextual explanations by drawing from common themes across each participant's story.

The findings are used to create more effective informal STEM learning environments for girls, thereby empowering women in STEM.

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GEOMETRY AND MEASUREMENT

RESEARCH REPORTS

UNDERSTANDING INSTRUCTIONAL CAPACITY FOR HIGH SCHOOL GEOMETRY AS A SYSTEMIC PROBLEM THROUGH STAKEHOLDER INTERVIEWS

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This paper reports an ongoing effort to address the problem of instructional capacity for high school geometry from a systems improvement perspective. In an effort to understand the system that contains the high school geometry instructional capacity problem, we identified key stakeholders and conducted preliminary interviews to learn about the problem from their perspective. We use these interview data to describe the system in more detail and to identify six major factors contributing to the high school geometry instructional capacity problem.

Keywords: Geometry and Geometrical and Spatial Thinking, University Mathematics, Teacher Education - Preservice, Systemic Change

This paper describes emerging efforts to develop a networked improvement community to address the problem of instructional capacity for high school mathematics. The capacity problem as we see it is the following: While scholars' understanding of the knowledge teachers need for teaching has progressed to a point that this knowledge base could be used to inform teacher development efforts (Ball, Lubienski, & Mewborn, 2001), the volume of aspirants to secondary mathematics teacher education has been decreasing to a point that sustaining and improving university programs for initial mathematics teacher preparation can be challenging (Sutcher, Darling-Hammond, & Carver-Thomas, 2016). Traditional efforts at instructional improvement have started from the design of policies or practices (including curriculum implementation) that are believed to have the potential to solve problems, followed by evaluation efforts that seek to achieve main effects. Following such an approach might require action at the college level, seeking both for investments in the recruitment of teachers and implementation of curricular approaches for teacher preparation focused on the knowledge teachers need. Such efforts have been underway (e.g., U-Teach, https://uteach.utexas.edu/uteach-institute-seed-grant; see also Newton et al., 2010) and may help produce some improvement. At the same time, Bryk et al. (2015) have described such approaches as problematic on account of their top-down logic of improvement, relying on developers' own understanding of the problem and conception of the solution, which usually put a premium on implementation fidelity.

Bryk et al. (2015) propose a different approach to improvement that seeks to involve all actors within the system being improved in the process of problem formulation, system mapping, and improvement design aimed at reducing variability in outcomes. This seems particularly useful in a context where improvement design and implementation is likely to cross boundaries across systems (e.g., K-12, higher education) with different existing practices and norms, where the logic that might support improvement in one system (e.g., curriculum implementation at the K-12 level) may or may not be useful in the other (viz., higher education).

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Our paper documents an initial effort to undertake systemic improvement in secondary mathematics teaching knowledge by developing an understanding of the problem of instructional capacity for secondary geometry. While the former seems like a huge problem in general, addressing it systemically seems crucial. By this we mean understanding connections across the systems that participate in the problem, specifically university teacher preparation, K-12 teaching, and the policy level (see Figure 1). Thus, our first approximation looks at a smaller version of the problem of instructional capacity while still addressing it systemically. The choice of geometry is strategic as a way to simplify the more general problem of instructional capacity at the secondary level because geometry is largely contained in a single secondary course and is taken by almost all high school students. Likewise, most teacher preparation programs require their candidates to take a Geometry for Teachers (GeT) course. Furthermore, some progress has been made measuring mathematical knowledge for teaching geometry (Herbst & Kosko, 2014; Ko, 2019). Hence, reducing the investigation of the problem of developing instructional capacity in secondary mathematics to secondary geometry may allow us to maintain attention to systemic issues while not being overwhelmed by the sheer scale of the systems being investigated. In this paper, we outline the development of an understanding of the system that produces the current high school geometry (HSG) instructional capacity problem with the goal of further developing an inter-institutional community of university instructors of Geometry for Teachers courses.

Theoretical Framework: Systems Improvement Approach

Bryk et al. (2015) developed the concept of *networked improvement communities* by adapting an improvement science framework to the context of education research. In the early stages of the formation of a networked improvement community, Bryk et al. (2015) recommend developing a complete understanding of the "the system that produces the current outcomes" (p. 57), followed by systematic experiments in which members of the community implement small changes that have the potential to produce measurable differences in the target outcome. This requires that stakeholders identify a specific problem to be addressed and seek understanding about the larger system by asking "why" questions. Bryk et al. (2015) suggest using some diagrammatic tools to visualize who the stakeholders are and elicit their knowledge about the system to develop an accurate picture. Both a systems diagram (Figure 1), and what Bryk et al. (2015) call a fishbone diagram (Figure 2) for problem formulation are used.

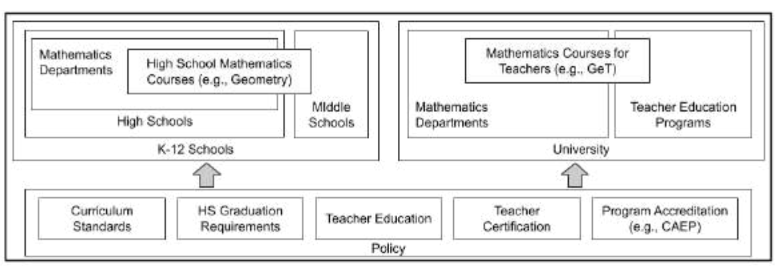

Figure 1. Schematic of the system that contains the HSG instructional capacity problem.

We developed the schematic in Figure 1 to determine who the potential stakeholders are in the problem of HSG instructional capacity. Material connections between stakeholders are indicated by overlapping rectangles. For example, the geometry courses for teachers overlap mathematics departments and teacher education programs because, while those courses are usually offered by mathematics departments, their existence relies on being required by teacher preparation programs. Students and teachers flow through the complex system: For example, high school students take geometry courses in high school; their mathematics experiences in school, for better or worse, prepare them for studying mathematics in college; those who decide to become mathematics teachers take college courses like Geometry for Teachers; if and when they get certified to teach, they are hired in K-12 districts where they might be teaching high school geometry. Likewise, some university students eventually become mathematicians who might be employed in mathematics departments and assigned to teach GeT courses. Mathematics teacher educators might also be observed in the system, inasmuch as they include former teachers and ordinarily participate in policy work such as standards development.

The *systems diagram* (Figure 1) is therefore useful as a way of identifying potential stakeholders in the problem of HSG instructional capacity. The diagram helps identify the institutional roles of people whose perspectives inform the problem. These include HSG teachers, secondary school or district leaders, HSG students, university Geometry for Teachers instructors, university mathematics department chairs, GeT students, faculty and administrators in secondary teacher education programs, recent university graduates, state or national policymakers, teacher assessment developers, and anyone making decisions about teaching certification and evaluation requirements at the state level. We surmise that improving the problem of instructional capacity for teaching high school geometry may require gathering representatives of all these institutions.

A *fishbone diagram* (Figure 2) provides a visual representation of the problem of focus for the networked improvement community (the head of the fish). Typically, these diagrams have five to six major bones, each representing a key factor that contributes to the problem, with more contributors (sub-bones) to each of those factors underneath. As our view of the HSG instructional capacity problem becomes more complex, the fishbone diagram serves as a tool to help us look for connections across stakeholders and determine potential levers for improvement.

One crucial component of a networked improvement community is a “hub” that serves to organize the information and activities of the network, including defining the system and eliciting the formulation of the problem. Our research team fills this role by gathering information from stakeholders in the system, collecting data, and disseminating the results back into the network. Members of our team have experience conducting research and teaching in secondary and higher education settings which supports our efforts to fulfill the role of network hub as well as collect and analyze the data reported in this paper.

Methods

We conducted semi-structured interviews with four sets of stakeholders from across the system in study; GeT instructors ($n = 42$), secondary school leaders ($n = 7$; 2 HS principals, 2 HS mathematics department chairs, 3 state/district instructional leaders), university mathematics department chairs ($n = 3$), and early-career mathematics teachers ($n = 3$). Recruitment and selection of participants varied by stakeholder group. The GeT instructors were already part of our project as members of an inter-institutional network made up of instructors of GeT courses located in mathematics departments at universities with teacher education programs. The secondary school leaders and early career teachers were recruited through a mass email to school leaders in a midwestern state that included an eligibility form to facilitate the screening process. We selected participants to ensure variability in administrative role, school and district size, and years of experience. The university mathematics department chairs were recruited through individual email requests from universities in a midwestern state that have a GeT course.

The interviews were conducted over video conference and recordings were transcribed for analysis. The interview protocol differed by stakeholder, but analysis of all interviews focused on connections between the data and our systems and fishbone diagrams. After the interviews had been conducted and analyzed, we reported our findings back to the network of GeT instructors through interactive online seminars.

Results

Each stakeholder group contributed to our understanding of the "hidden complexities" (Bryk et al., 2015, p. 14) within the complex system that contains the HSG instructional capacity problem. Within each stakeholder group, individual participants were able to offer insight into the particular ways in which those complexities manifested within their context. Participants vary in the number of years they have held a particular position within the system, and many of them also drew on experiences they had moving through the system (e.g., secondary school leaders that are former mathematics teachers who took a GeT course and were once students in HSG). In this section, we identify 6 main factors that contribute to the problem of HSG instructional capacity as indicated from interviews across stakeholder groups.

De-emphasizing the HSG Course

A contributing factor to the HSG instructional capacity problem is that the course has been de-emphasized by several stakeholder groups. Participants mentioned three underlying causes; (1) much of the HSG content is not needed to succeed in AP Calculus and AP Statistics, (2) HSG content is not rigorously assessed on the SAT, and (3) there is a lack of clarity on where and how HSG content should appear within the high school mathematics sequence (e.g., integrated into first- and second-year courses, as a fourth-year standalone course, etc.).

One district/state instructional leader shared that "the districts in [my] county right now are struggling with offering geometry, which is unfortunate, but they don't see an emphasis on the SAT" (CE). In addition, she identified an "importance of all students having success in Algebra 1 before they get to higher-level courses," which affords the Algebra 1 course more importance (and therefore more resources) within the high school mathematics sequence. This provides some evidence that the choice to de-emphasize HSG comes from pressure (or lack of pressure) at the school or district level. Policies at the state level may also be contributing to the deemphasizing of HSG within the system. "Messaging at the [state department of education] is that we don't have a course called geometry. Students have to demonstrate proficiency in the K-12 standards," some of which address geometry content (IF; district/state instructional leader).

There are a number of potential impacts on the HSG instructional capacity problem when schools, districts, and state education departments don't emphasize the HSG course. For example, while professional development specific to Algebra standards is frequently offered to inservice teachers, it is much less common to find Geometry-specific professional development.

HSG is not a Desirable Course to Teach

Teacher preparation programs set out to provide candidates with the knowledge, skills, and beliefs necessary to be successful secondary mathematics teachers, but multiple stakeholders identified disparities between the preparation candidates receive and the preparation they ultimately need. By far, the most common shortcoming in the preparation of HSG teachers is the overwhelming number who arrive on the job market with a lack of desire to teach geometry. Secondary school leaders at each level of administration told us, in one way or another, that "teachers don't like teaching [geometry]" (CK, Principal). One participant went so far as to say that teachers "were fearful of it" (IF, district/state instructional leader). Although there was one district that has had a steady stream of secondary mathematics teachers that view the course favorably, that was an exception to the trend.

While some stakeholders described the general phenomenon of teachers' lack of desire, some went further to identify underlying reasons for the undesirable nature of the HSG course; (1) compared to upper-level courses, HSG is not an ideal teaching assignment, (2) teachers are not comfortable with the HSG content, and (3) teachers hold an opinion that it is more difficult to plan for HSG than other mathematics courses. A district/state instructional leader shared that upper-level courses are more desirable for a number of reasons:

> There seem to be lots of people to volunteer at the upper-level courses to teach and not as many that want to teach Algebra and Geometry...I've been told those kids are easier to teach [in the upper-level courses], they come better prepared...The upper-level courses tend to have more stable classes, less kids that transfer in and transfer out...There seems to be a status thing, too, with teaching the upper-level classes where it's not seen as prestigious to teach Algebra or Geometry or an intervention class as it is to teach the upper-level classes. (ZM)

While some teachers express a desire to teach upper-level courses rather than HSG, others communicate discomfort with the content covered in HSG. A high school mathematics department chair described geometric proof as the most prominent stumbling block for teachers; "[A]s much as we can see what the process should be..., it's more difficult to teach that when it's pretty far from the other standards of math that are 'here's the process. That's it'" (RU). When one principal summarized experiences telling new hires that they would be assigned the HSG course, she said "[T]hey just cringe. They constantly tell me 'this is not my strong point.' And they try to blame it on the kids and say, 'the kids hate it'" (CI, Principal). One early career mathematics teacher even identified it as "kind of a beast to plan for—a lot of diagrams and whatnot and different notations—as far as making materials, it kind of seems overwhelming" (JL). Although none of these explanations for the lack of desire to teach HSG connect directly to university teacher preparation programs, we suggest that the structure of the system connecting universities to high schools, through the preparation and hiring of secondary mathematics teachers, may present levers for improvement.

Structure of University Teacher Preparation Coursework and Clinical Experiences

There are a number of factors that contribute to the HSG instructional capacity problem that are related to the structure of coursework and clinical experiences within university teacher preparation programs. Namely, there is wide variation in student teaching experiences, comparatively fewer opportunities to engage with geometry content, and a mismatch between the knowledge teachers gain and the knowledge they need on the job. Due to the wide variety of school contexts, teaching assignments, curriculum materials, and instructional styles, the student teaching experience often looks "wildly different, building to building, classroom to classroom, district to district" (ZM, state/district instructional leader), even for prospective teachers that come from the same program. On top of that variation in experiences, one principal reflected that "most student teachers do an Algebra [course]," and that if their mentor teacher has some sections of Geometry, "they don't really have them [take over] the geometry course until the end" (CI). If prospective teachers don't gain experience teaching Geometry in their clinical placements, they would need to rely on their university coursework for opportunities to think deeply about geometry content. If the GeT course is not required for certification, they may only have their own HSG experience to rely on. One HS mathematics department chair shared that "the most common response [to being assigned the HSG course] is 'I haven't done geometry since high school. I'm going to need to refresh on this'" (RI).

Even when prospective teachers do take a GeT course during their university coursework, the content is not always aligned to what prospective teachers feel they need when they begin teaching. Thinking back to his own university coursework, one principal shared that "there wasn't a whole lot of focus [in the GeT course I took] on what students would actually be learning in high school. It was more of exploring the higher levels of math. So I felt like there was a gap overall in my undergraduate experience of learning about subject matters that I would be teaching" (CK). One university mathematics department chair voiced support for that version of the GeT course, saying that prospective teachers "should know the subject matter a bit deeper than what their students do or what the textbooks cover" (RA). To him, that means that while HSG geometric proofs tend to be structured in two columns and contain a limited number of statements, "in a college course, they need to go one level deeper" (RA).

In addition to a focus on content that is beyond the high school curriculum, some GeT courses are taught in a style that runs counter to the instructional practices secondary mathematics teachers may be expected to enact when they are hired. An early-career mathematics teacher described the instructional style of his GeT course that was focused on spherical and hyperbolic geometry: “In the class itself, we never worked together. It was very lecture based. You sat there for an hour fifteen, got lectured to and left” (JL). This suggests that GeT courses at some universities are designed to support students’ content knowledge but do not attend to their pedagogical content knowledge. However, the secondary school leaders described the ideal qualifications for a HSG teacher in terms that blended content and pedagogical knowledge, indicating that they were not so cleanly distinct in practice. For example, a district/state instructional leader said that “we need someone who kind of has a vision of the content and what they feel is the best way to teach it” (ZM). Another described ideal applicants who are “comfortable enough with their own knowledge to be able to listen to or hear their students when their students are proposing or conjecturing and that they're flexible enough not to shut them off right away and say, no, that'll never work” (CE). The possibility that the GeT course does not prepare students to become the teachers that these leaders identify poses an issue that GeT instructors may have the power to influence. However, GeT instructors have a variety of backgrounds and visions for what the course should be.

Variability in Geometry Preparation Within Teacher Education Programs

Despite some common experiences across participants, there tends to be significant variability among GeT courses, across instructors and institutions. We have heard from many GeT instructors that the course is unique within the mathematics department offerings because of the diverse student population that enrolls. Many courses report a mix of prospective secondary mathematics teachers, mathematics majors, and students from other departments that are taking the course as an elective. The mix of students creates a tension for instructors who are not sure how to balance the rigor required for a mathematics degree with the practical needs of prospective teachers (see Milewski et al., 2019). One GeT instructor went so far as to say that prospective teachers “take the same hard math classes all the other math majors take… [but] they typically don’t like those courses. And some of them don’t see the value of it for being a teacher” (RU). Depending on the particular context of each institution, the instructor assigned to the GeT course may have a doctorate in mathematics or in mathematics education. One GeT instructor expressed that the course is “mathematically sophisticated enough that mathematicians should be teaching it” (MV), while at some institutions the mathematics department chair defers to the recommendation of a mathematics educator to determine who teaches the course. Moreover, there is not a set criteria for determining the particular instructor. According to the university mathematics department chairs, GeT instructors have been chosen depending on instructor preference (AR) or who taught it most recently (MA). Depending on which instructor is chosen, the course content can vary drastically in terms of the extent to which HSG topics are covered and pedagogical content knowledge gained. Lastly, the GeT course is not highly valued at some institutions “because this course is really not a crucial part of the mission of the math department” (DZ, GeT Instructor). As a result, fewer resources are devoted to the creation of a course syllabus and materials, and stewardship of the course is left to individual instructors. All of these factors contribute to the varied geometry preparation of prospective teachers which, in turn, adds to the HSG instructional capacity problem.

Lack of Communication Across Institutional Stakeholders

Each set of stakeholders holds responsibility for some aspects of the preparation and support of HSG teachers, but there is no single stakeholder that controls the entire system. In addition, the system does not contain structures to support communication and collaboration across groups of stakeholders. A university mathematics department chair reported that there was not direct communication between the mathematics and education departments. At his institution, an informal communication system emerged, but it was not systematic.

> Well, indirectly we get input because [an instructor] also teaches a course in the education department. And so she sort of brings us sometimes, 'no, okay—they are doing this, we might need to modify this course or something. But I don't actively ask them to. (MA)

At a different institution, a GeT instructor mentioned that "the education department and the math department don't collaborate together, as they're in separate departments—we send our students [to the school of education] to do a credential, but we are separate departments in that sense" (KC). Although that instructor seems to trust that their education department counterparts are providing the appropriate instruction in service of their shared goals, a high school principal shared that she was "not very confident with the universities" (CI) in terms of how they supported prospective teachers in gaining content knowledge.

If stakeholders did not have opinions about what could or should be occurring within other parts of the system, it would be reasonable to expect minimal communication across stakeholders. However, our interviews with secondary school leaders indicate that they have clear ideas about how their potential hires should be prepared. Looking across the levels of school or district leadership, we heard the desire for prospective teachers to have "some kind of experience with computer [based] instructional materials" (ZM; e.g., GeoGebra), to have experiences struggling with mathematics content (ZM), to take a curriculum course "where math teachers truly know the [state standards] or the common core [standards]" (CI) and how the standards build across grade levels (CK, IF), to learn strategies for teaching problem solving skills (RU), and to have support determining how geometry is used outside the classroom (CY). Without structures in place to incentivize communication across stakeholders, it is likely that efforts to better prepare HSG teachers will fall short of serving the local community's needs.

HSG Staffing Decisions Made for Subject-Generic Reasons

When hearing from the secondary school leaders, it became clear that these leaders often make staffing decisions based on generic factors rather than subject-specific considerations. For example, one district/state instructional leader shared that the choice of who teaches Geometry "doesn't seem to be about the mathematics at all. It seems to be more about the behavior management of those classes and who's got the right attitude, the growth mindset" (ZM). In addition, staffing decisions at secondary schools tend to be made by taking into consideration the preferences and requests of current mathematics teachers so that when a position opens up "there will be kind of a reshuffling in terms of who is teaching what" (CK, HS principal). Furthermore, several secondary school leaders explained that they fill open positions by posting secondary mathematics positions rather than HSG positions. This may be a result of the current teacher credentialing system: "I think as long as they have secondary mathematics credentials, then they're supposedly able to teach Geometry. So it would just be expected" (CE, district/state instructional leader). These hiring practices may be contributing to the HSG instructional capacity problem as declining instructional quality impacts high school students who then travel through the system before returning to secondary schools as mathematics teachers.

Discussion

By considering the interview data presented here in the context of the systems diagram (Figure 1), we develop a more nuanced view of the system that contains the HSG instructional capacity problem. Schools and districts post openings for secondary mathematics teaching teachers, but tend to place the new hires in Algebra 1 and Geometry courses because the current teachers within the department request to teach other courses. This means that Algebra 1 and Geometry courses tend to be taught by new hires, many of whom are just out of teacher preparation programs. Upon being hired, those new teachers will receive comparatively less geometry-specific professional development and support as a result of school, district, and state educators de-emphasizing the HSG course. In their university teacher preparation programs, these teachers were not guaranteed to have had opportunities to develop geometric pedagogical content knowledge through their coursework or clinical experiences. These connections across major bones in Figure 2 indicate that there is a need for a network that spans stakeholder groups to inform improvement efforts.

As we move forward with this work, we plan to use what we have learned from these interviews to create surveys that can be administered to a national sample within each stakeholder group. A larger sample is a critical way to address limitations to analysis that arise from making claims based on idiosyncratic experiences. We will consider how the preliminary interview data from one stakeholder group can inform efforts to craft survey questions for the other stakeholder groups. For example, secondary school leaders indicated that they sometimes look to see if applicants earned a passing grade in their GeT course, so we might like to ask how often GeT Instructors assign a non-passing grade in those courses. We are also interested in learning more about how preservice teachers' experiences in the GeT course might relate to their desire to teach HSG when they apply for jobs. In our role as the hub of this networked improvement community, we have a unique opportunity to disseminate the knowledge and perspectives of parts of the community across stakeholder groups in an effort to improve the instructional capacity of the high school geometry course.

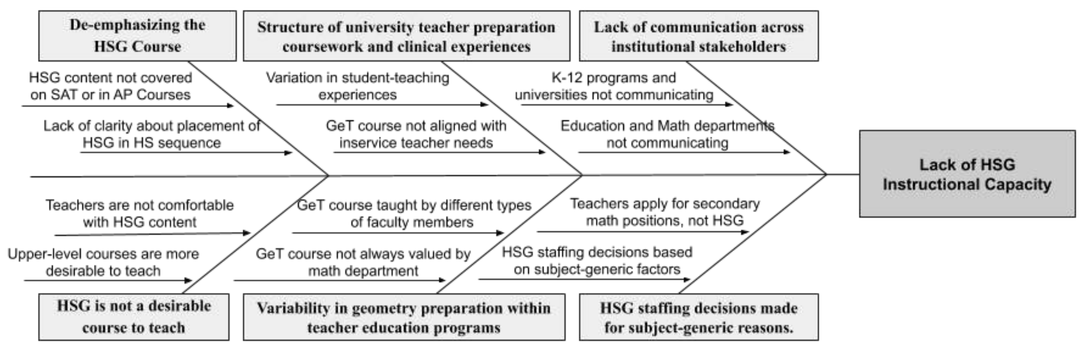

Figure 2. Fishbone diagram representing lack of HSG instructional capacity.

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DEVELOPMENT OF SPACE REASONING IN EARLY AGES THROUGH VARIATION ACTIVITIES

DISEÑO DE ACTIVIDADES PARA EL DESARROLLO DE RAZONAMIENTO ESPACIAL EN EDADES TEMPRANAS A TRAVÉS DE MANIPULATIVOS

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Spatial reasoning skills are necessary to perform activities at school, at work and in everyday life, in general. Different studies indicate the importance of its development at an early age, since it allows the reading of a three-dimensional world and its interpretation in two-dimensional representations. Our research focuses on the design of activities, using the Theory of Variation, to enhance spatial reasoning skills in seven to eight year-old students. In this document, we present characteristics of the design of an activity based on the use of pentominoes (two-dimensional puzzles). The results show that spatial reasoning skills develop when the following actions are favored: comparing, overlapping, rotating, moving, visualizing, and imagining movements, positions and locations of the pentomino pieces.

Keywords: Elementary School Education, Spatial Thinking, Representations and Visualization

Introduction and background

Spatial reasoning skills are necessary in human thinking and acting. The importance of developing these skills is reflected both in everyday life and in the school environment. In everyday life, authors such as Gonzato, Fernández & Díaz Godino (2011) recognize, for example, that the development of spatial reasoning is necessary to locate, move and read maps where bi-and three-dimensional objects may be present. Research also indicates that these skills are necessary for learning advanced mathematics (Mamolo, Ruttenberg-Rozen & Whiteley, 2015; Hallowell et al., 2015) and for the development in other areas such as technology, engineering, architecture (Arıcı & Aslan-Tutak, 2015; Van den Heuvel-Panhuizen, Iliade & Robitzsch, 2015), geography, computer graphics and visual arts (Clements & Sarama, 2011; Vázquez & Noriega Biggio, 2010). In terms of the relationship between spatial thinking and STEM (Science, Technology, Engineering, and Mathematics) performance, Uttal and Cohen (2012) suggested that spatial skills strongly predict student selection for studying the STEM subjects.

Although the importance of developing spatial reasoning skills in school is recognized, a review of the literature in mathematics education (Ortiz, 2018; Ortiz, Sacristán & Sandoval, 2019), shows a lack of studies for enhancing students' competences, abilities, thinking, or spatial reasoning. Ortiz (2018) analyzed articles in 12 mathematics education journals in English and Spanish, published between 2010 and 2016. She observed that only 4.7% of the articles focus on aspects of learning and teaching geometry (in contrast to those focused on arithmetic, algebra and calculus). Of this percentage, only 13% address aspects of spatial reasoning: some describe the importance of its development, others focus on identifying difficulties and some more present activity proposals.

Research focused on analyzing the consequences of a poor development of spatial reasoning, have identified difficulties in the visualization of 2D and 3D representations (Arıcı & Aslan-Tutak, 2015); in the turning movement of 2D and 3D objects (Pittalis & Christou, 2010); in understanding the meaning of area and volume formulas (Mamolo, Ruttenberg-Rozen & Whiteley, 2015); in the construction of 2D and 3D figures (Pittalis & Christou, 2010); in relating 2D and 3D representations

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(Dindyal, 2015); and in reading maps (Gonzato, Fernández & Díaz, 2011). These difficulties have been found to affect other disciplines and work areas that involve interpreting representations to solve tasks (Francis & Whiteley, 2015).

In relation to the design of activity proposals, the research recognizes objectives categorized by educational level: in preschool and elementary school, students are expected to recognize two-dimensional representations (drawings) of real objects (Hallowell, Okamoto, Romo & La Joy, 2015; Van den Heuvel-Panhuizen, Iliade & Robitzsch, 2015); in secondary school, students are expected to manipulate 3D objects, either with dynamic geometry or tangible materials (Arıcı & Aslan-Tutak, 2015; Gómez, Albaladejo & López, 2016), and be able to draw conjectures of events related to a specific context.

There are also few longitudinal studies focused on describing what type of materials, activities, or teaching/learning proposals promote the development of spatial reasoning skills. As Davis, Okamoto and Whiteley (2015) point out, further research is required in this regard.

In this sense, the study on which this document is based, investigated how to provide, through the design of activities involving 2D and 3D representations, learning opportunities for 7 to 8 year-old students to develop their spatial reasoning skills. To answer this, we designed six activities based on the Theory of Variation (Marton & Pang, 2006), which were implemented in a public school located in a marginalized area of Mexico City.

This document presents the design of the first activity that involves isometric movements in the plane, using puzzles with pentomino pieces. Next, we describe some elements of the Theory of Variation on which we based the activity design; we then provide a brief analysis of how this design can promote the development of students' spatial reasoning.

Theoretical perspective: Variation as a tool for the design of the activities

In the Theory of Variation (Marton & Pang, 2006; Ling-Lo, 2012) learning happens when a difference is experienced between two things or between two parts of the same thing (Marton & Pang, 2006), that is, when the learner manages to *discern characteristics* and *critical aspects* of some *learning object* (Orgill, 2012; Runesson, 2005). An object of learning is "a specific insight, skill, or capability that the students are expected to develop" (Marton & Pang, 2006, p.194).

Ling-Lo (2012) distinguishes the critical aspects of a learning object from its critical characteristics: "a critical aspect refers to a dimension of variation, whereas critical feature is a value of that dimension of variation" (p. 65). This can be better understood through an example: if the learning object is the cube, some critical aspects to discern can be the dimensions of the *shape* or *number of its faces*; and it has, as critical characteristics (the values of the dimensions), the fact that *each face is a square (congruent between themselves)* and the fact that *it has exactly six faces.*

Methodology

In our study, we carried out a teaching experiment framed in the research design paradigm (Cobb & Gravemeijer, 2008). A teaching experiment involves a cyclic process of design, implementation, and analysis of a sequence of activities (Steffe & Thompson, 2000) to improve and refine it. Derived from the literature review and the Theory of Variation, the following aspects were considered for the design of the sequence: the use of different manipulatives (pentominoes, blocks, SOMA Cube, Lego pieces) and digital technologies (Lego Designer); construction activities with dimension variation (starting with 2D, continuing with 2D-3D dimension changes); and variation in the characteristics of the representations in the printed materials (number of divisions in the different pentominoes, and colors or grayscale for the representations of 3D objects). The first activity is described in detail in the following section.

Our study had two cycles: in the first one, we designed a sequence of six activities to be carried out in 13 sessions; we analyzed the results of its first implementation for improving the design in the second cycle, in which we had the same number of activities but implemented in 15 sessions.

In both implementations, 7-8-year-old students participated. In the first cycle, we had eight participating students and in the second, 26 (a complete group of third grade). For the data collection, we used two video cameras to record what happened during each class, complemented by field notes. These records were used to plan subsequent sessions, as well as to perform a retrospective analysis of the experiment. In the second implementation, the first author of this paper acted as teacher-researcher, and together with the students, they agreed on the organization of the activities: student autonomy, materials to use (manipulatives and worksheets –with brief and clear guidelines on the activity) and ways of working (individual, in pairs, teams or as a whole group). When managing her class, the teacher-researcher carried out different actions: she did a recap of the aim of the previous session, mentioned the objective of the current session, asked questions and clarified doubts to the teams, specified the mathematical vocabulary used to describe spatial actions, and coordinated the closing plenary discussion with the entire group.

In order to analyze the development of spatial reasoning as a result of the implementation of the sequence of activities, we used a diagnostic pre and post test, as well as observations of the cognitive and movement actions of the students during the activities.

Description of some of the activities

In our design, and according to the Theory of Variation, each activity has a specific *learning object* and is made up of two or three tasks. An assignment can be carried out in one or two class sessions. *Variants* and *invariants* of this learning object are required for each task. *Critical aspects* and *characteristics* describe those elements of spatial reasoning that students are expected to develop. Next, we present the design of the first activity –which focuses on recognizing, visualizing and performing isometric movements in the plane using two-dimensional representations (puzzles)–; we describe the elements of the Theory of Variation involved in it and then analyze some results of the activity implementation.

An activity focused on isometric movements of two-dimensional representations

The *learning object* of this activity is rotation and translation (isometric movements) in the plane and in space, when using pentominoes (12 pieces; see Figure 1.a).

a. b.

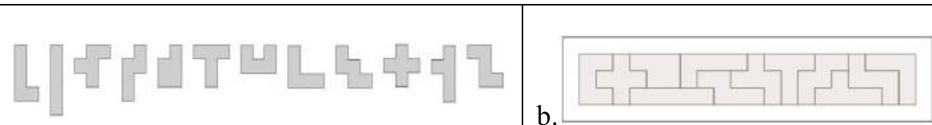


Figure 1: a. The pentomino pieces. b. Rectangle made up of pentomino pieces.

Students have to solve *different* puzzles (the *variants*) formed with the *same* pentomino pieces (the *invariants*); this requires promoting the ability to discern that, even though the shape and perimeter change, the area is conserved. In this activity there are four *critical aspects*: comparison and visualization, localization, turns, and organization of the pieces. The *critical characteristics* related to each critical aspect are presented in Table 1.

Table 1: Description of the critical characteristics

Critical aspects	Critical characteristics
Comparison and visualization	• Recognizing the shape of each piece. • Comparing the shapes of the pieces in order to assemble them. • Recognizing, in the 2D representation, the pieces that make up each puzzle (whether they are explicit or implicit). (Ideas of congruence through immediate perception and superposition). • Imagining what the union of the pieces represents in the given configuration. (The whole).
Localization	• Locating, in the 2D representation, the location of each piece. • Describing the relative position of each piece using terms of proximity (near, far) and direction (up, down, right, left).
Turns	• Rotating and flipping pieces to complete a given setup.
Organization of the pieces	• Organizing the pieces to achieve the requested configuration. • Dividing the puzzle into sections. • Assemble the pieces that make up a section.

This activity has two tasks (T1 and T2). In T1, each student has 12 pieces of pentomino to assemble them on a rectangular frame of 3 × 20 units, in such a way that it covers it completely, without overlapping pieces (see Figure 1.a). In this task, students need to recognize the shape of each piece of the pentomino and compare them.

In order to promote the use of language, in the assembly of the puzzles, students are encouraged to interact and support each other with ideas or suggestions, with the restriction of only giving verbal indications or gestures. And at the end of the T1 task, a group discussion is carried out, with the following guiding questions: *¿Do all the puzzles have the same number of pieces? ¿Are the pieces themselves, different or the same? ¿Why are they different or the same? ¿How could we differentiate them?*

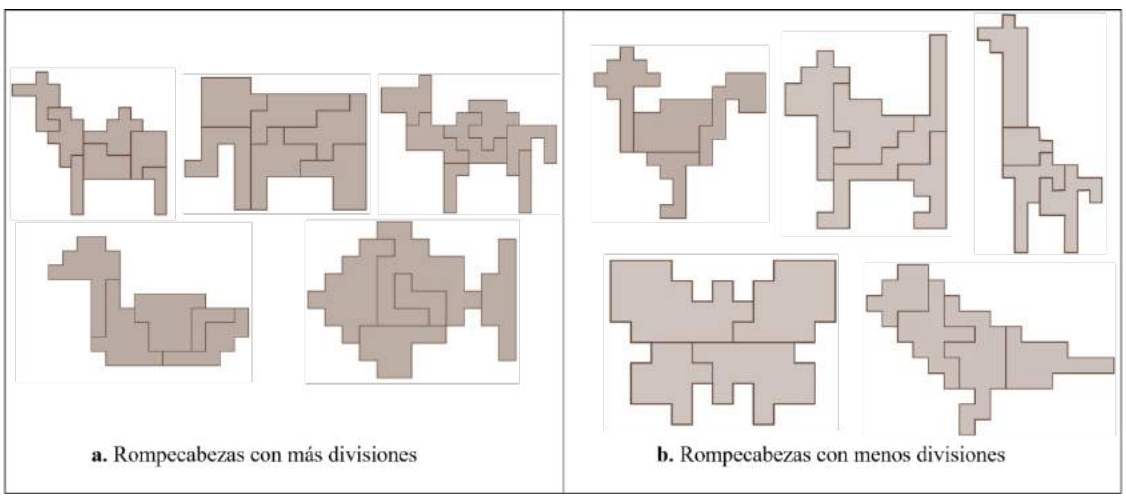


Figure 2: Representations of the puzzles to assemble, as given to the students in T2.

In task T2, each student must assemble an assigned puzzle in the shape of an animal (see Figure 2), for which clues are given regarding the shape of the pieces that compose it: some puzzles (Figure 2.a) show some subdivisions congruent to the pentomino pieces; while in others, only four divisions are shown, none of which is congruent to any part (Figure 2.b). Students put together the puzzles with the most divisions and then the puzzles with the least divisions. In this sense, different levels of cognitive difficulty are considered (from more guided to less guided).

Results of Activity 1

In the implementation of T1, the students worked for two hours and all managed to *recognize the shape of the pentominoes* and identified how they fit into the rectangular frame, without any leftover or missing space.

For T2, two sessions were required, each lasting two hours. In the first of these sessions, students put together the puzzles with more divisions; students who had difficulties, were given a sheet with a printed replica of the assembled puzzle, showing the pieces in real size (in a 1:1 scale) (see Figure 3.a). The intention was for all students to recognize the shape of each pentomino piece, its location and position in the puzzle. In the second session, students put together puzzles with fewer divisions. Those who finished first helped their teammate (see Figure 3.b); to do this, they used the assembled puzzle (scale 1:1), and gave instructions to their partner regarding the orientation and position of each piece.

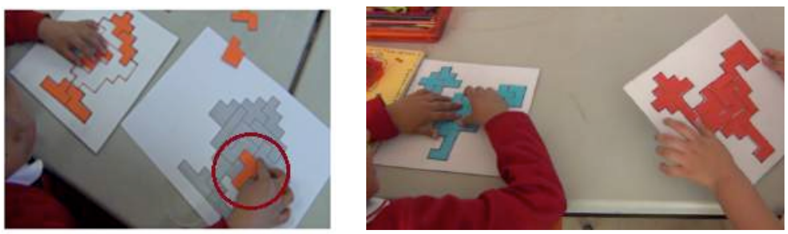


Figure 3: Examples of student work during task T2.

From overlapping pieces to visualizing spaces

During T1 and T2 we identified three strategies for assembling the puzzles: overlapping, trial-and-error, visualization-and-imagination. When they used the replica to assemble the puzzle, the most common way was to overlap the pentomino pieces unto the congruent spaces on the replica (see the part indicated in Figure 3.a) to later transfer (carry out a translation) of the pieces unto the puzzle.

When assembling the puzzles with more divisions, initially students used *trial-and-error* to place the pieces (see Figure 4.a); later, they *visualized* if any of the pieces had the shape of a section in a space of the puzzle that still needed to be completed (see Figures 4.b and 4.c).

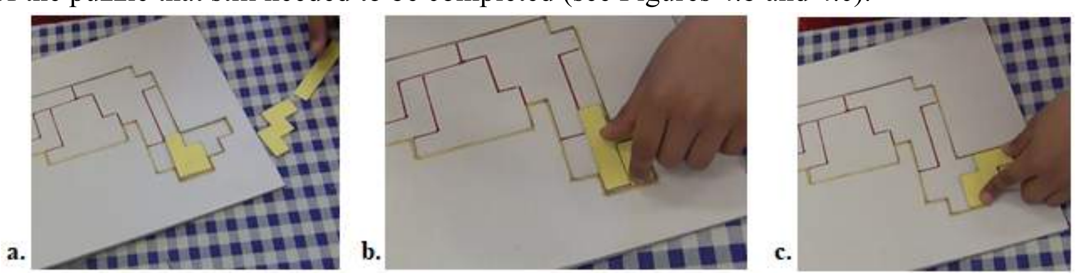
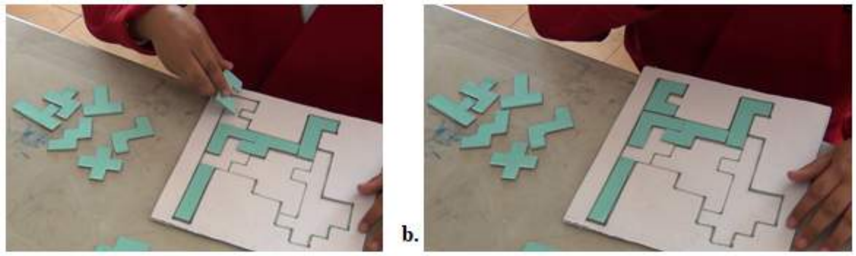


Figure 4: Location of the pentominoes by trial and error.

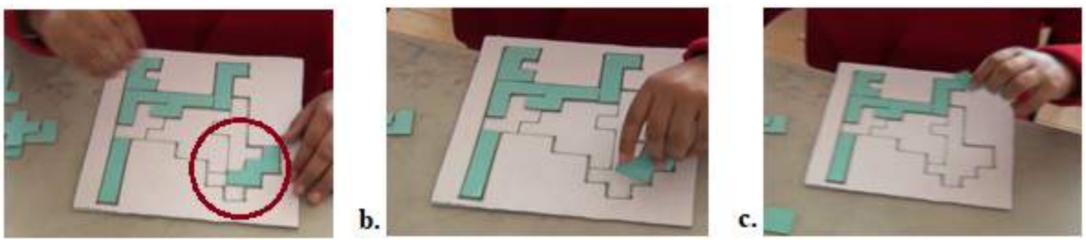
For putting together the puzzles with fewer divisions, the students generated a strategy: to *first locate the pentominoes at the edges of the puzzle*, since it was easier to recognize the congruence of the shape of the edges with the shape of some of the pentomino pieces (see Figure 5); thus, they postponed placing the pieces of the center of the puzzle.



a. b.

Figure 5: Assembling puzzles with fewer divisions in T2.

In putting together these puzzles, not only was it necessary to match the shape of the pentomino pieces with those of the puzzle, but also to compare the missing shapes of the puzzle with the remaining pentomino pieces. For example, Figure 6 shows that the student placed a pentomino in a space (marked section in Figure 6.a,), because its shape partially coincides with that of that space in the puzzle; however, since he did not find another piece that would complete the space, as can be seen in Figures 6.b and 6.c, the student discarded the piece that he had initially placed.



a. b. c.

Figure 6: Matching puzzle pieces in T2.

Identifying the locations of the pentomino pieces and the different ways in which they fit into the puzzles, allowed a refinement of strategies. At first it was a trial-and-error activity; and it was through the comparison between the remaining spaces of the puzzle and the shape of the pieces. that students could *imagine-and-visualize* the congruence between the divisions of the puzzle and the assembly of pentomino pieces.

Critical aspects of isometric movements: from translation and rotation, to turn

In T1 and T2, students were able to locate, immediately or not, a pentomino piece in the puzzle depending on whether the orientation of the piece and the congruent space were the same or different. When it was the same, the students moved (*translated)* the piece without difficulty (see Figure 7).

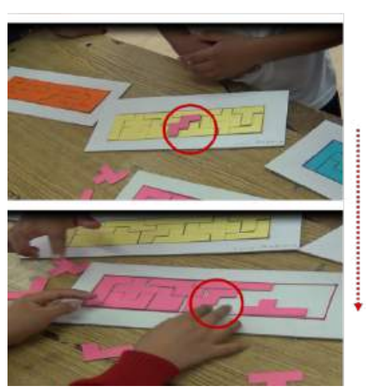


Figure 7: Translation movement.

In the case where the orientation of the piece and the missing space were different, students *rotated* the pieces, with no apparent difficulty, through a movement carried out on the plane (see Figure 8).

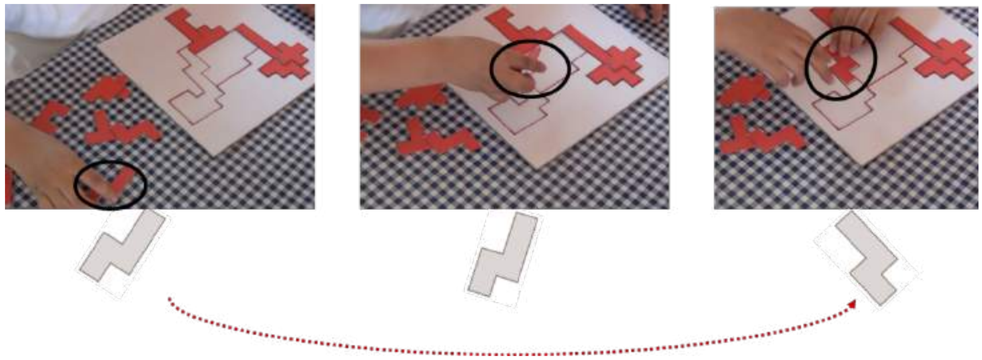


Figure 8: Rotation movement in the plane.

A challenge for the children was the action of turning. If the location of a piece in the puzzle required a *rotation* movement in space, and if a student did not immediately recognize how to locate it, he would first rotate it in the plane and then *visualize* whether, with a turn of the piece in the 3D space, it would fit into the puzzle (see Figure 9).

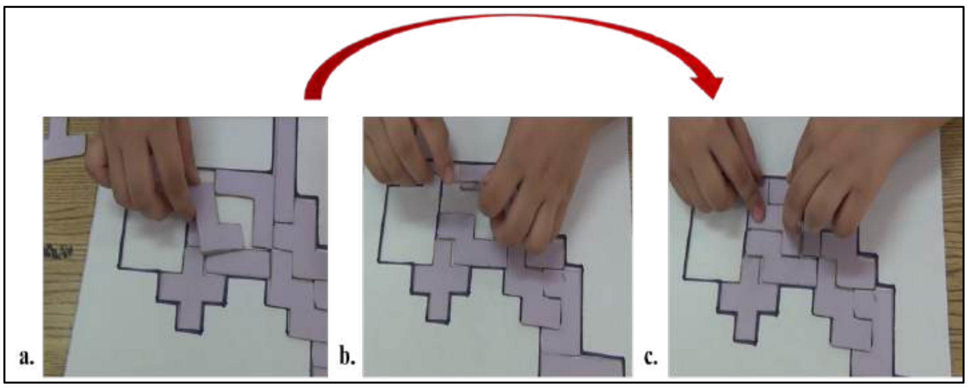


Figure 9. Rotation movement in space.

All of the students, through rotation and translation movements, achieved the required assembled puzzles. Experimenting in putting together these puzzles allowed them to *identify subconfigurations* (one figure contained in another) and *visualize* the congruence of a composite shape (a union) formed by joining several pentomino pieces (the parts). It should be noted that the movement actions, when locating a piece, are not linear. In the examples presented in this section, we illustrate how students, in order to place a piece in the puzzle, perform a combination of the following actions: they manipulate the piece to *translate* it, *visualize* its *orientation*, *rotate* and/or *turn* it, in order to finally *locate* it in the empty space in the puzzle.

The implementation of this activity reflects a development of the spatial reasoning of the students, since, thanks to an activity design that included a *variation of the puzzles, a variation of the positions and orientations of the pentomino pieces,* and an *invariance in the number and shape of the pieces*, they went from *locating the pieces by trial and error*, to *visualizing* the location of the pieces in the puzzles. The two tasks led the students to *rotate* and *translate* the pieces of the pentomino, find a *correspondence* of each of the pieces according to the shape that they identified in the drawing, *compare* the remaining spaces with the unused pieces, and *visualize* and imagine the composition. Furthermore, they recognized that there are several puzzle shapes that can be put together using the same number of pentomino pieces.

Discussion and final remarks

In the design of the activity reported here, three elements were considered in order to develop spatial reasoning in the students: i) The use of the same pentomino pieces, in different puzzles; ii) a gradual change in the level of difficulty of the puzzles (from more to fewer divisions); and iii) the difficulties, reported in the literature, when putting together puzzles (e.g., turning in the 3D space).

Through the first element, students were given the opportunity to recognize that, even when the puzzles were different, they could always be assembled using the same number of pieces; they noted also that the location, orientation and direction of the pentomino pieces varied. The variation of puzzles and invariance of the pentomino pieces could assist these students in creating meanings for the concept of area, which according to Mamolo, Ruttenberg-Rozen and Whiteley (2015), is a common difficulty for students.

The gradual increased complexity of the activities promoted the development of visualization processes through the composition of pieces. When students assembled puzzles with more divisions, they could see the *congruence* of remaining puzzle spaces with the pentomino pieces. But when they had to put together puzzles with fewer divisions, they had to *discern* which of the pieces, when put together, would complete a section of the puzzle divisions. In this process, we found that children *developed construction strategies,* such as: starting the assembly by identifying and locating the pieces at the edges that is, going from the outside in); and assembling the puzzle in sections according to the divisions.

Both in the design and in the development of the activities, possible difficulties were anticipated that a 7-8-year-old student could face, as reported in the literature (Arıcı & Aslan-Tutak, 2015; Gonzato, Fernández & Díaz, 2011). If students did not identify the pentomino pieces that would go in a certain section, a replica was provided to help them in solving the puzzle. This helped the children to *establish congruences* between pentomino pieces and *perform isometric transformations such as rotation and translation* with the pieces.

The previous results show the potential of activities using 2D representations to support the development of spatial reasoning skills in 7-8-year-old children. The use of tangible material, in accordance with what is mentioned by Gutiérrez (1991), allowed students to experiment and recognize the movement and shapes of the pentomino pieces and two-dimensional puzzles. In addition, identifying the position and orientation of the pentominoes in the puzzles required students to *compare, overlap, rotate, translate, visualize,* and *imagine* the shapes of the pieces in the various puzzles.

Gonzato, Díaz-Godino and Neto (2011) have found that, if these experiences are provided in the first years of schooling, children acquire greater abilities to build constructions and mentally manipulate figures in the plane and in the 3D space. Therefore, we consider that these types of experiences that promote cognitive actions (visualizing, imagining) and movement (comparing, overlapping, rotating, translating), allow students to make sense and understand a world presented through two-dimensional representations in different areas of knowledge and in daily life.

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DISEÑO DE ACTIVIDADES PARA EL DESARROLLO DE RAZONAMIENTO ESPACIAL EN EDADES TEMPRANAS A TRAVÉS DE MANIPULATIVOS

DEVELOPMENT OF SPACE REASONING IN EARLY AGES THROUGH VARIATION ACTIVITIES

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Las habilidades de razonamiento espacial son necesarias para desempeñar actividades en la escuela, en el trabajo y en la vida cotidiana en general. Diferentes estudios señalan la importancia de su desarrollo en edades tempranas, pues permite la lectura de un mundo tridimensional y su interpretación en representaciones bidimensionales. Realizamos un estudio enfocado al diseño de actividades, usando la Teoría de la Variación, para potenciar habilidades de razonamiento espacial en estudiantes entre siete y ocho años. En este documento presentamos características del diseño de una actividad basada en el uso de pentominós (rompecabezas bidimensionales). Los resultados muestran desarrollo en habilidades de razonamiento espacial cuando se favorecen acciones como comparar, superponer, rotar, trasladar, visualizar e imaginar movimientos, posiciones y ubicaciones de las piezas del rompecabezas.

Palabras clave: Educación primaria, pensamiento espacial, representaciones y visualización.

Introducción y antecedentes

Las habilidades de razonamiento espacial son necesarias en el actuar y el pensar del ser humano. La importancia del desarrollo de dichas habilidades se refleja tanto en lo cotidiano como en el ámbito escolar. En la vida cotidiana, autores como Gonzato, Fernández y Díaz Godino (2011) reconocen, por ejemplo, que el desarrollo de razonamiento espacial es necesario para ubicarse, desplazarse y leer mapas donde objetos bi- y tridimensionales pueden estar presentes. En relación con lo escolar, investigaciones señalan que estas habilidades son necesarias para el aprendizaje de matemáticas avanzadas (Mamolo, Ruttenberg-Rozen y Whiteley, 2015; Hallowell et al., 2015) y el desenvolvimiento en otras áreas como tecnología, ingeniería, arquitectura (Arıcı, y Aslan-Tutak, 2015; Van den Heuvel-Panhuizen, Iliade y Robitzsch, 2015), geografía, computación gráfica y artes visuales (Clements y Sarama, 2011; Vázquez y Noriega Biggio, 2010). En términos de la relación entre el pensamiento espacial y el rendimiento en STEM (por sus siglas en inglés, correspondientes a Ciencia, Tecnología, Ingeniería y Matemáticas), Uttal y Cohen (2012) exploraron y sugirieron que las habilidades espaciales predicen fuertemente la selección de estudiantes para estudiar las áreas STEM.

Si bien se reconoce la importancia de desarrollar habilidades de razonamiento espacial en la escuela, una revisión de la literatura en educación matemática (Ortiz, 2018; Ortiz, Sacristán, Sandoval, 2019) refleja una falta de estudios para potenciar competencias, habilidades, pensamiento o razonamiento espacial. Ortiz (2018) analizó artículos en 12 revistas en inglés y español de educación matemática, publicados entre 2010 y 2016. Observó que solo el 4.7% de los artículos se enfoca en aspectos del aprendizaje y enseñanza de la geometría (en contraste con los enfocados a aritmética, álgebra y cálculo). De este porcentaje, solo el 13% aborda aspectos de razonamiento espacial: unos describen la importancia de su desarrollo, otros se centran en identificar dificultades y algunos más presentan propuestas de actividades.

Las investigaciones centradas en analizar las consecuencias del poco desarrollo del razonamiento espacial han identificado dificultades en la visualización de representaciones 2D y 3D (Arıcı y Aslan-

Tutak, 2015); en el movimiento (giros) de objetos 2D y 3D (Pittalis y Christou, 2010); en la comprensión del significado de fórmulas de área y volumen (Mamolo, Ruttenberg-Rozen y Whiteley, 2015); en la construcción de figuras en 2D y 3D (Pittalis y Christou, 2010); en la conexión entre representaciones 2D y 3D (Dindyal, 2015); y en lectura de mapas (Gonzato, Fernández y Díaz, 2011). Se ha encontrado que estas dificultades inciden en otras disciplinas y áreas de trabajo que implican interpretar representaciones para resolver tareas (Francis y Whiteley, 2015).

En relación con el diseño de propuestas de actividades, en la investigación se reconocen objetivos por nivel educativo: en preescolar y primaria se pretende que los estudiantes reconozcan representaciones bidimensionales (dibujos) de objetos reales (Hallowell, Okamoto, Romo y La Joy, 2015; Van den Heuvel-Panhuizen, Iliade y Robitzsch, 2015); en secundaria se pretende que manipulen objetos tridimensionales, ya sea con geometría dinámica o material tangible (Arıcı y Aslan-Tutak, 2015; Gómez, Albaladejo y López, 2016), y, de esa manera, logren conjeturar sucesos relacionados con algún contexto determinado.

También son pocos los estudios longitudinales enfocados en describir qué tipo de materiales actividades o propuestas de enseñanza/aprendizaje propician el desarrollo de habilidades de razonamiento espacial. Como lo precisan Davis, Okamoto y Whiteley (2015) se requiere mayor investigación al respecto. En este sentido, el estudio en el que se basa este documento investigó ¿cómo proporcionar, a través del diseño de actividades que involucran representaciones bi- y tridimensionales, oportunidades de aprendizaje a estudiantes de siete a ocho años de edad para el desarrollo de habilidades de razonamiento espacial? Para responder a este cuestionamiento, se diseñaron seis actividades con base en la Teoría de la Variación (Marton y Pang, 2006), que se implementaron en una escuela pública ubicada en una zona marginada de la Ciudad de México.

En este documento se presenta el diseño de la primera actividad que involucra movimientos isométricos en el plano, mediante el uso de rompecabezas con piezas pentominó. A continuación se describen elementos de la Teoría de la Variación en los que se basa el diseño de la actividad; posteriormente, se realiza un breve análisis de cómo este diseño promueve el desarrollo del razonamiento espacial de los estudiantes.

Perspectiva teórica: La variación como herramienta para el diseño de actividades

En la Teoría de la Variación (Marton y Pang, 2006; Ling-Lo, 2012) el aprendizaje sucede cuando se experimenta una diferencia entre dos cosas o entre varias partes de una misma cosa (Marton y Pang, 2006), es decir, cuando el aprendiz logra *discernir características* y *aspectos críticos* de algún *objeto de aprendizaje* (Orgill, 2012; Runesson, 2005). Un objeto de aprendizaje, en esta teoría, es "una idea introspectiva (insight), una habilidad o una capacidad específicas que se espera que los estudiantes desarrollen" (Marton y Pang, 2006, p.194).

Ling-Lo (2012) distingue los aspectos críticos de un objeto de aprendizaje, de sus características críticas: "un aspecto crítico se refiere a una dimensión de la variación, mientras que una característica crítica es un valor de esa dimensión" (p. 65). Esto se puede entender mejor mediante un ejemplo: si el objeto de aprendizaje es el cubo, algunos aspectos críticos a discernir pueden ser las dimensiones de *forma* o *número de sus caras*; y tiene como características críticas (los valores de las dimensiones) que *cada cara es un cuadrado (congruentes entre sí)* y que *tiene exactamente seis caras*.

Metodología

En nuestro estudio se llevó a cabo un experimento de enseñanza enmarcado en el paradigma de investigación de diseño (Cobb y Gravemeijer, 2008). Un experimento de enseñanza implica un proceso cíclico de diseño, implementación y análisis de una secuencia de actividades (Steffe y Thompson, 2000) para mejorarla y refinarla. Derivado de la revisión de la literatura y de la Teoría de la Variación, para el diseño de la secuencia se consideraron los siguientes aspectos: uso de diferentes manipulables (pentominós, bloques, Cubo SOMA, piezas de Lego) y tecnologías digitales (Lego Designer); actividades de construcción con variación en la dimensión (comenzando con 2D, continuando con cambios de dimensión 2D-3D); y variación en las características de las representaciones en los impresos (cantidad de divisiones en los diferentes pentominós, y colores o escala de grises para las representaciones de objetos 3D). En el siguiente apartado se describe con detalle la primer actividad.

Nuestro estudio tuvo dos ciclos: en el primero se diseñó una secuencia de seis actividades llevadas a cabo en 13 sesiones; se analizaron los resultados de su primera implementación y los resultados sirvieron para mejorar el diseño en el segundo ciclo, terminando con el mismo número de actividades pero implementadas en 15 sesiones. En ambas implementaciones participaron estudiantes entre los siete y ocho años. En el primer ciclo participaron 8 estudiantes y en el segundo, 26 (un grupo completo de tercer grado). Para la toma de datos se utilizaron dos cámaras de video y se registraron en notas de campo lo sucedido durante cada clase. Estos registros se usaron para planear las sesiones posteriores, así como para realizar un análisis retrospectivo del experimento. En la segunda implementación, la primera autora de este escrito fungió como profesora-investigadora y junto con los estudiantes acordaron la organización de las actividades: autonomía de los alumnos, material a utilizar (manipulables y hojas de trabajo –con indicaciones breves y claras sobre la actividad) y formas de trabajo (individual, en parejas, equipos o grupo completo). En la gestión de la clase, la profesora-investigadora realizó diferentes acciones: recapituló el objetivo de la sesión anterior, mencionó el objetivo de la sesión, hizo preguntas y aclaró dudas en los equipos, precisó el uso de vocabulario matemático para describir acciones espaciales y coordinó la discusión plenaria de cierre con todo el grupo.

Para analizar el desarrollo del razonamiento espacial a raíz de implementar la secuencia de actividades, se usó una prueba diagnóstica aplicada como pre y post test, así como observaciones de las acciones cognitivas y de movimiento de los estudiantes durante las actividades.

Descripción de algunas actividades

En nuestro diseño, cada actividad tiene un *objeto de aprendizaje* determinado y está conformada por dos o tres tareas. Una tarea puede realizarse en una o dos sesiones de clase. Para cada tarea se precisan *variantes* e *invariantes* de dicho objeto de aprendizaje. Los *aspectos* y *características críticas* describen aquellos elementos del razonamiento espacial que se espera desarrollen los estudiantes. A continuación se presenta el diseño de la primera actividad –la cual se centra en reconocer, visualizar y realizar movimientos isométricos en el plano usando representaciones (rompecabezas) bidimensionales—, se describen los elementos de la Teoría de la Variación involucrados en ella y se analizan algunos resultados de su implementación.

Una actividad centrada en movimientos isométricos de representaciones bidimensionales

El *objeto de aprendizaje* de esta actividad está enfocado en movimientos isométricos en el plano (rotación y traslación) y en el espacio, al usar pentominós (12 piezas; ver Figura 1a).

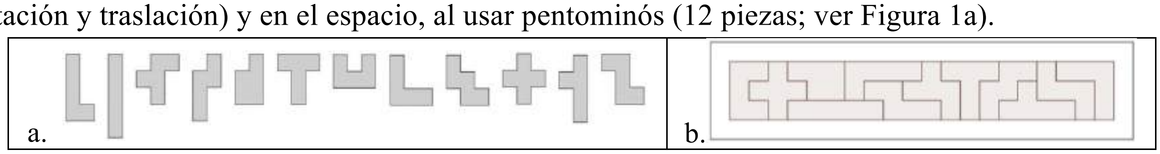

Figura 1: a. Las piezas pentominó. b. Rectángulo formado por piezas pentominó.

Los alumnos resuelven *diferentes* rompecabezas (las *variantes*) armados con las *mismas* piezas de pentominó (las *invariantes*); esto supone promover el discernimiento de que, aunque cambia la forma y el perímetro, hay conservación del área. En esta actividad son cuatro los *aspectos críticos:*

comparación y visualización, localización, giros y organización de las piezas. Las *características críticas* relacionadas a cada aspecto crítico se presentan en la Tabla 1.

Tabla 1. Descripción de las características críticas

Aspectos críticos	Características críticas
Comparación y visualización	Reconocer la forma de cada pieza. Comparar las formas de las piezas para ensamblarlas. Reconocer, en la representación 2D, las piezas que componen cada rompecabezas (estén explícitas o implícitas). (*Ideas de congruencia por percepción inmediata y superposición*). Imaginar lo que representa la unión de las piezas en la configuración dada. (*El todo*).
Localización	Ubicar, en la representación 2D, el lugar de cada pieza. Describir la posición relativa de cada pieza usando términos de proximidad (cerca, lejos) y de dirección (arriba, abajo, derecha, izquierda).
Giros	Rotar y voltear piezas para completar una configuración determinada.
Organización de piezas	Organizar las piezas para lograr el ensamble solicitado. Dividir el rompecabezas en secciones. Ensamblar las piezas que conforman una sección.

Esta actividad tiene dos tareas (T1 y T2). En T1 cada estudiante tiene 12 piezas de pentominó para ensamblarlas sobre un marco rectangular de 3 unidades × 20 unidades de tal manera que lo cubra completamente, sin sobreponer piezas (ver Figura 1a). En esta tarea los estudiantes necesitan *reconocer* la forma de cada pieza del pentominó y *compararlas*.

Para favorecer el uso del lenguaje, se promueve la interacción entre los estudiantes para apoyarse en el armado de los rompecabezas con ideas o sugerencias, con la restricción de sólo dar de indicaciones verbales o gestos. Y en el cierre de la tarea T1 se hace una socialización para responder las siguientes preguntas guía: *¿Todos los rompecabezas tienen el mismo número de piezas? ¿Las piezas entre sí, son diferentes o iguales? ¿Por qué son diferentes o iguales? ¿Cómo las podríamos diferenciar?*

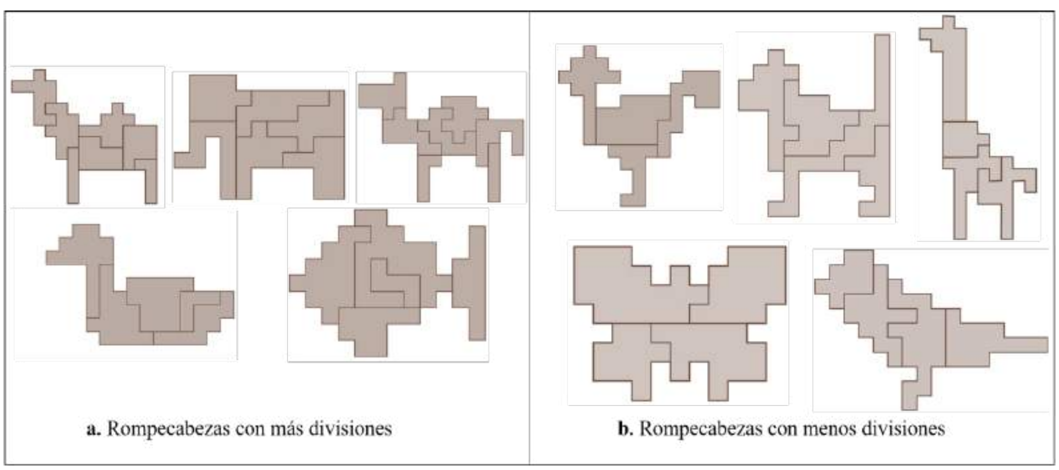

Figura 2: Representaciones, dadas a los alumnos en T2, de los rompecabezas a construir.

En la tarea T2, cada estudiante debe armar un rompecabezas asignado con la forma de un animal (ver Figura 2), para el cual se dan pistas respecto a la forma de las piezas que lo componen: unos rompecabezas (Figura 2.a) muestran algunas subdivisiones congruentes a las piezas pentominós; mientras que en otros, solo se muestran cuatro divisiones, ninguna de las cuales es congruente a alguna pieza (Figura 2.b). Los estudiantes arman primero los rompecabezas con más divisiones y después aquellos con menos divisiones. En este sentido se consideran diferentes niveles de dificultad cognitiva (de más guiado a menos guiado).

Resultados de la Actividad 1

En la realización de T1 los estudiantes trabajaron durante una sesión de dos horas y todos lograron *reconocer la forma de los pentominós* e identificaron cómo se encajaban en el marco rectangular, sin que sobrara o faltara algún espacio.

Para T2 se requirieron dos sesiones, cada una de dos horas. En la primera de estas sesiones, los alumnos armaron los rompecabezas con más divisiones; a quienes se les dificultaba, se les proporcionaba una hoja con una réplica del rompecabezas ya armado, mostrando las piezas en tamaño real (escala 1:1) (ver Figura 3.a). La intención era lograr que todos reconocieran la forma de cada pieza pentominó, su ubicación y posición en el rompecabezas. En la segunda sesión, los estudiantes armaron rompecabezas con menos divisiones. Quienes terminaron primero ayudaron a su compañero (ver Figura 3.b); para ello, usaron el rompecabezas armado (escala 1:1), y dieron indicaciones a su compañero respecto a la orientación y posición de cada pieza.

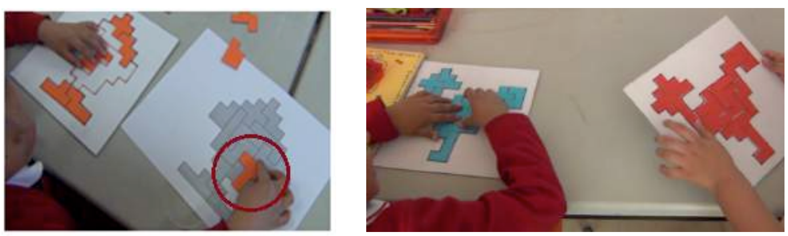

Figura 3: Ejemplos del trabajo de los estudiantes durante la tarea T2.

De la superposición de piezas a la visualización de espacios

Durante T1 y T2 identificamos tres estrategias para el armado de los rompecabezas, superposición, ensayo-y-error, visualización-e-imaginación. Cuando usaron la réplica para el armado del rompecabezas, lo más usual fue hacer uso de la superposición de las piezas de pentominó en los espacios congruentes sobre la réplica (ver parte señalada en la Figura 3.a) para, después, trasladar dichas piezas al rompecabezas.

Al armar los rompecabezas con más divisiones, colocaron las piezas de pentominó, por *ensayo-y-error* inicialmente (ver Figura 4.a); y luego, fueron *visualizando* si alguna de sus piezas tenía la forma de alguna sección en un espacio del rompecabezas que faltaba por armar (ver Figuras 4.b y 4.c).

a. b. c.

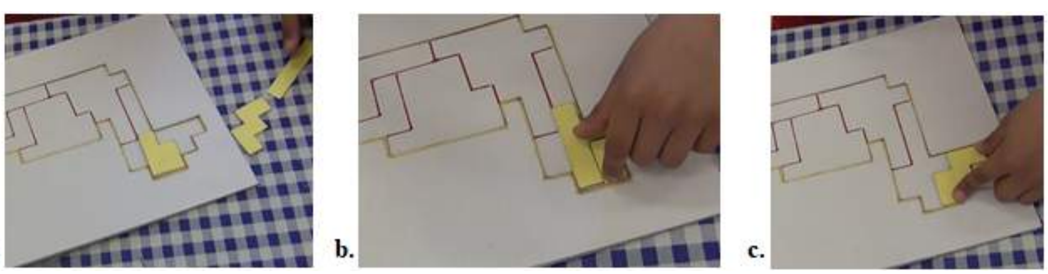

Figura 4: Ubicación de los pentominós por ensayo y error.

Para el armado de rompecabezas con menos divisiones los estudiantes generaron una estrategia: *ubicar primero los pentominós de los bordes del rompecabezas*, pues era más fácil reconocer la congruencia de la forma del borde con algunas formas de las piezas pentominó (ver Figura 5); así posponían la colocación de las piezas del centro del rompecabezas.

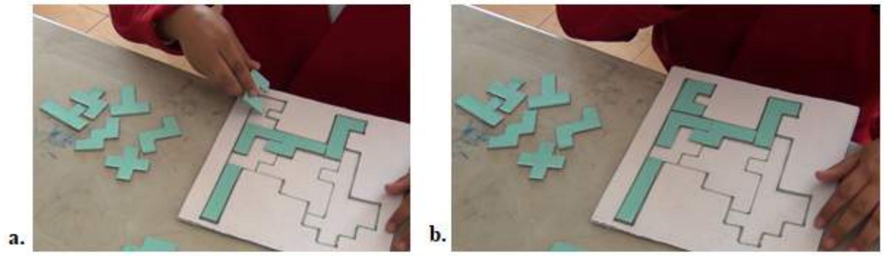

Figura 5: Armado de rompecabezas con menos divisiones en T2.

En el armado de estos rompecabezas no solo era necesario hacer corresponder la forma de las piezas pentominó con las del rompecabezas, sino comparar las formas faltantes del rompecabezas, con las piezas restantes de pentominó. Por ejemplo, en la Figura 6, se puede observar que el estudiante ubicó en un espacio (sección señalada en la Figura 6.a,) una pieza pentominó, porque su forma coincide parcialmente con la de ese espacio del rompecabezas; sin embargo, como no encontró otra pieza que completara el espacio, en las Figuras 6.b y 6.c se aprecia que el estudiante descarta la pieza colocada inicialmente.

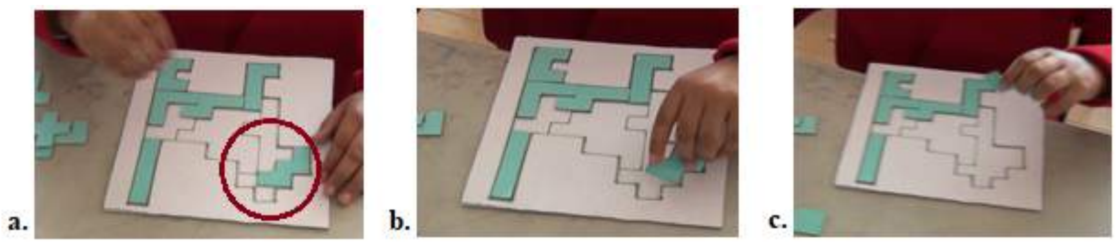

Figura 6: Haciendo corresponder las piezas en el rompecabezas en T2.

La identificación de la ubicación de las piezas pentominó y las diferentes maneras como éstas encajaban en los rompecabezas permitió el refinamiento de estrategias. Al principio fue una actividad de ensayo-y-error; y fue a través de la comparación entre los espacios sin armar del rompecabezas y la forma de las piezas, que los estudiantes fueron *imaginando-y-visualizando* la congruencia entre las divisiones del rompecabezas y la unión de piezas pentominó.

Aspectos críticos de movimientos isométricos: de la traslación y la rotación, al giro

En T1 y T2 los estudiantes ubicaban en el rompecabezas una pieza pentominó inmediatamente o no dependiendo de si la orientación de la pieza y del espacio congruente era igual o diferente. Cuando era igual, los estudiantes *trasladaban* la pieza sin dificultad (Figura 7).

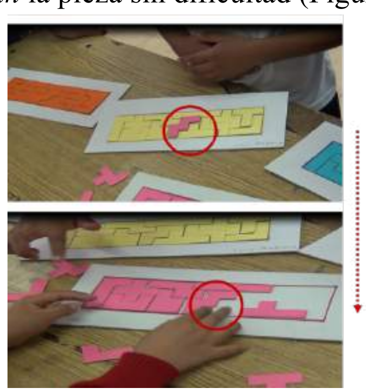

Figura 7: Movimiento de traslación.

En el caso de que la orientación de la pieza y el espacio faltante fueran diferentes, *rotaban,* sin dificultad aparente, las piezas con un movimiento realizado sobre el plano (ver Figura 8).

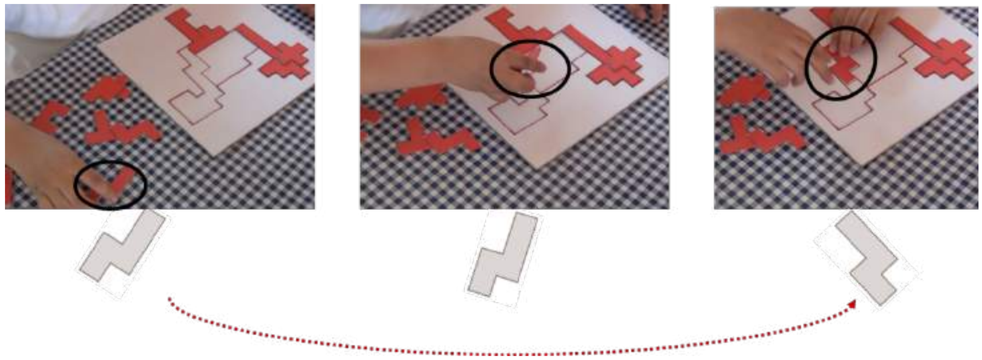

Figura 8: Movimiento de rotación en el plano.

Un reto para los niños fue la acción de girar. Si la ubicación de la pieza en el rompecabezas requería realizar un movimiento de *rotación* en el espacio, y si algún estudiante no reconocía de inmediato cómo ubicarla, primero la rotaba en el plano y luego *visualizaba* si con un giro de la pieza en el espacio le permitía encajarla en rompecabezas (ver Figura 9).

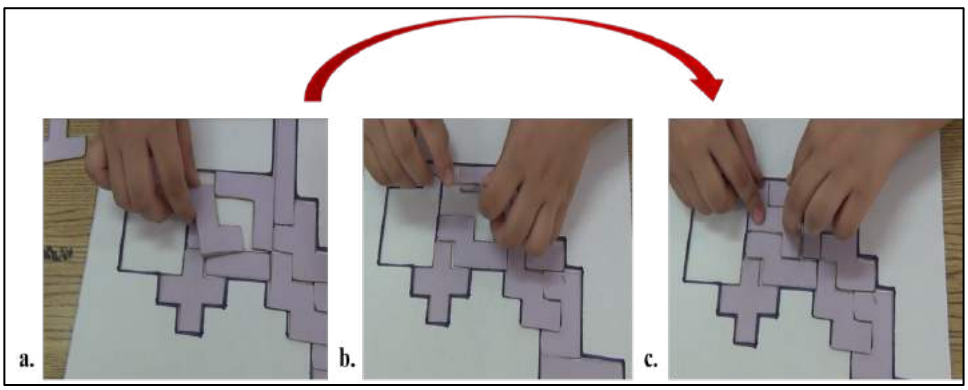

Figura 9. Movimiento de rotación en el espacio.

Todos los estudiantes, a partir de movimientos de rotación y traslación, lograron los ensambles solicitados. Experimentar con el armado de estos rompecabezas, les permitió *identificar subconfiguraciones* (una figura contenida en otra) y *visualizar* la congruencia de una forma compuesta (un ensamble) obtenida al unir piezas de pentominó (sus partes). Cabe señalar que las acciones de movimiento, al ubicar alguna pieza, no son lineales. En los ejemplos presentados en este apartado ilustramos cómo los estudiantes, para colocar en el rompecabezas alguna pieza, realizan una combinación de las siguientes acciones: la manipulan para *trasladarla*, *visualizan la orientación* de la pieza, la *rotan* y/o *giran* para finalmente *ubicarla* en el espacio vacío del rompecabezas.

La implementación de esta actividad refleja un desarrollo del razonamiento espacial de los estudiantes, pues, al incluir en el diseño de las actividades: la *variación de rompecabezas, variación de posiciones y orientaciones de las piezas* pentominó, e *invariancia en el número y forma de las piezas*, pasaron de *ubicar las piezas por ensayo y error,* a *visualizar* la ubicación de las piezas en los rompecabezas. Las dos tareas llevaron a los estudiantes a que *rotaran* y *trasladaran* las piezas del pentominó, *correspondieran* una a una las piezas dadas con la forma que identificaban en el dibujo *compararan* los espacios faltantes con las piezas sin usar y *visualizaran* e imaginaran su composición. Además, reconocieron que hay varias formas de rompecabezas que se pueden armar con la misma cantidad de piezas pentominó.

Discusión y reflexiones finales

En el diseño de la actividad reportada aquí, se consideraron tres elementos a fin de desarrollar razonamiento espacial en los estudiantes: i) El uso de las mismas piezas de pentominó en diferentes rompecabezas; ii) un cambio progresivo en el nivel de dificultad para armar el rompecabezas (de más a menos divisiones); y iii) las dificultades reportadas en la literatura para armar los rompecabezas (e.g., girar en el espacio 3D).

Con el primer elemento se proporcionó a los estudiantes la oportunidad de reconocer que, aun cuando los rompecabezas fueran distintos, siempre podían ser armados con el mismo número de piezas; además, notaron que la ubicación, orientación y sentido de las piezas pentominó también variaban. La variación de rompecabezas y la invarianza de las piezas pentominó podrían ayudarle a estos alumnos en la compresión del significado de área, que según Mamolo, Ruttenberg-Rozen y Whiteley (2015), es una dificultad común en los estudiantes.

El cambio gradual en la complejidad de las actividades promovió el desarrollo de procesos de visualización a través de la composición de piezas. Al armar rompecabezas con más divisiones, los estudiantes podían *ver la congruencia* de espacios faltantes del rompecabezas con las piezas de pentominó. Pero cuando armaban rompecabezas con menos divisiones, ellos debían *discernir* cuáles de las piezas, al unirlas, completaban una sección de las divisiones del rompecabezas. En este proceso, encontramos que los niños *generaron estrategias de construcción* tales como: iniciar el armado identificando y ubicando las piezas de los bordes (es decir, yendo de afuera hacia adentro) y armar el rompecabezas por secciones según las divisiones.

Tanto en el diseño como en el desarrollo de las actividades se previeron posibles dificultades a las que un estudiante de 7-8 años podía enfrentarse, según lo reportado en la literatura (Arıcı y Aslan-Tutak, 2015; Gonzato, Fernández y Díaz, 2011). Si el estudiante no identificaba las piezas de pentominó que debían ir en determinada sección, se le proporcionaba una réplica que le ayudara en la resolución del rompecabezas. Esto ayudó a los niños en el *establecimiento de congruencias* entre piezas pentominós y la *realización de transformaciones isométricas como rotación y traslación* de las piezas.

Los resultados anteriores muestran el potencial de actividades con representaciones bidimensionales para apoyar el desarrollo de habilidades de razonamiento espacial en niños con edades entre 7-8 años. El uso de material tangible, en concordancia con lo mencionado por Gutiérrez (1991), permitió a los estudiantes experimentar y reconocer el movimiento y las formas de las piezas de pentominó y los rompecabezas bidimensionales. Además, la identificación de la posición y orientación de los pentominós en los rompecabezas requirió que los estudiantes *compararan, superpusieran, rotaran, trasladaran, visualizaran* e *imaginaran* las formas de las piezas con los distintos rompecabezas.

Gonzato, Díaz-Godino y Neto (2011) han encontrado que, de darse estas experiencias desde los primeros años de escolaridad, los niños adquieren mayores capacidades para hacer construcciones y manipular mentalmente figuras en el plano y en el espacio. Por lo que, consideramos que este tipo de experiencias que promueven acciones cognitivas (visualizar, imaginar) y de movimiento (comparar, superponer, rotar, trasladar), permiten a los estudiantes la lectura y comprensión de un mundo presentado a través de representaciones bidimensionales en distintas áreas de conocimiento y en el cotidiano.

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INTERPRETIVE MODEL OF THE CONCEPTUALIZATION OF THE CONGRUENCE OF POLYGONS (MICP)

MODELO INTERPRETATIVO DE LA CONCEPTUALIZACIÓN DE LA CONGRUENCIA DE POLÍGONOS (MICP)

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The document presents a set of categories for the analysis of the conceptualization of the congruence of polygons - a central theme in school mathematics - and details the application of the analytical tools used, derived from Grounded Theory, in this construction. This set of categories is called 'Interpretive Model of the Conceptualization of Polygon Congruence' (MICP). This model emerged from the interpretive analysis of empirical data recollected during the investigation. The MICP categories can be used by teachers or researchers to cover different didactic objectives (e.g., interpret the resolution of tasks with congruence content; prepare student profiles or identify their difficulties. See Peña, 2019) and it is relevant because it does not seem to exist in the literature a similar model that covers the previously stated objectives.

Key words: Geometry, research methodologies.

Problem formulation and research question

The criteria of congruence and the notion of congruence are important and fundamental subjects for geometry, both for the geometry taught from the basic levels of education and for geometry as a branch of mathematics. In different versions of Euclidean geometry - proposed by Euclid, Legendre or Hilbert and which present a deductive organization with different degrees of formalization - the notion of congruence and the criteria of congruence emerge from the beginning of axiomatization. In the case of Euclid's *Elements,* according to the Heath (1908) version, the notion of congruence is introduced in Common Notion 4 (Heath, 1908) and the 'criteria of congruence of triangles' are found in propositions I.4; I.8 and I.26 (Heath, 1908). In the case of Legendre geometry (Legendre, 1984), congruence is discussed at the beginning of axiomatization and the congruence criteria for triangles are shown in propositions VI, VII, and XI. Something similar occurs in the formalization of the geometry proposed by Hilbert (1996) where the congruence criteria are also included at the beginning of the work. This allows us to suppose that in disciplinary geometry, particularly in the different versions of euclidean geometry, it is required from the beginning of axiomatization to make use of the notion of congruence and the criteria of congruence of triangles.

Just as the notion of congruence and the criteria of congruence are necessary for disciplinary geometry, they are also essential in school mathematics education. To verify this statement, curriculum for Mexico and Colombia were reviewed. In Mexico, the Secretary of Public Education (SEP, 2017) introduces in its study plans the notion of congruence and the criteria of congruence of triangles from the third year of primary - where it is requested to compare geometric figures and establish uniqueness (p.314) -, up to the third year of secondary -level at which the triangle congruence criteria are expected to be determined and used (p.315). In Colombia, the Ministry of National Education (MEN, 2006) proposes "to recognize the congruence of figures" (p.80) in the 1st and 3rd grade; "Identify and justify the congruence between figures" (p.82) between 4 ° and 5 °; make use of figure congruence to solve problems for 6th and 7th (p.84) and study the properties of congruence for 8th and 9th (p.86). All of this basically refers to the congruence of triangles.

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Given the weight and scope that congruence has not only in disciplinary geometry but also in school geometry, tools are required to cover a series of didactic objectives related to this concept, among others: interpret the task resolutions, with congruence contents, produced by the students; elaborate student profiles and identify some of their conceptual difficulties regarding this notion; analyze the statements of the tasks and develop didactic sequences on the subject. However, in the mathematics education literature, no studies were found that provide these analysis tools for the case of triangles, and even less so for the congruence of polygons, which is the central theme of the research presented here.

When the authors of this writing did documentary research on mathematical education studies focused on polygon congruence issues, didactic proposals were found for teaching the topic (e.g. Carbó and Mántica, 2010; Piatek-Jimenez, 2008; Zakiz & Leron, 1991 to name a few). However, these works lack, for example, a systematic analysis of the possible difficulties in learning and teaching aspects related to the congruence of polygons; they also lack theoretical support that could justify the order of exposition of topics in a didactic sequence on congruence and that allow outlining possible profiles that account for the level of understanding that students have of this concept.

This document sets out a set of categories - which has been called the 'Interpretive Model of the Conceptualization of the Congruence of Polygons' (MICP) (Peña, 2019) - which is intended to help cover (albeit preliminary) the shortcomings mentioned above.

Methodology and application of the analytical tools used

For the construction of the MICP, some of the principles that rule the Grounded Theory (GT) were followed in the version by Corbin and Strauss (2015), although the study was not intended to achieve the ultimate objectives of the GT (i.e., build theory). What follows is an outline of the category construction process and some central ideas of that construction process.

In GT interpretive categories are based on empirical data collected during research and do not emerge from a theoretical framework given in advance (Corbin & Strauss, 2015). Following this general principle of the GT, the construction of the MICP was carried out, always taking care that the categories were oriented by empirical data.

Initially, the empirical data was fractured, from which patterns were constructed based on which conceptual labels were generated. In a return to the empirical domain, the authors verified that these conceptual labels adequately represented the data. Then a comparative analysis was made between the conceptual labels. From a synthesis process carried out in the conceptual domain, categories were generated. Constant comparisons, asking questions and preparing memos and diagrams, analytical tools that are part of the TF methods, were involved in all these processes (Corbin & Strauss, 2015). At a later time - which Birks and Mills (2011) call 'intermediate coding' - each of the categories was deepened and subcategories were defined and processes of logical ordering were carried out synchronously between them, making use of the idea or reification proposed by Sfard and Linchevski (1994) (paragraphs below the authors explain this point).

Subsequently, a theoretical sampling was carried out (Corbin & Strauss, 2015) that allowed to provide new properties and dimensions to the constructed categories. At a subsequent stage, these categories were confronted with the levels proposed by the researchers Van Hiele (trad. in 1984). Finally, and with the support of some of the reification ideas proposed by Sfard and Linchevski (1994) and by Wenger (2001), modifications were made at the conceptual level. This gave rise to the MICP that is exposed in this document, which has properties of a conceptual order according to the definition given by Corbin and Strauss (2015).

Literary references

As it was told, for the construction of the MICP some of the ideas proposed by Wenger (2001) and by Sfard and Linchevski (1994) were used. Wenger (2001) expresses that, to signify their daily actions and practices and their experience in the world, the members of the communities carry out reification processes. The idea of reification is generally used by Wenger to refer to the process of shaping our experience by producing objects that translate this experience into one thing (p.84). For example, writing a law, creating a recipe for a cake, or proving a theorem are reification processes in which a certain social, gastronomic, or mathematical experience is shaped or "materialized." Thus, in these processes of objectification, some aspects or characteristics of a practice become objects - the law, the cake or the theorem - objects that can be treated as if they were material and concrete elements, even when they are not. Once these objects are constituted, we perceive them as if they existed in the world, as if they had a reality of their own. This is very clear with mathematical concepts and scientific structures. They are usually seen as if they had (and as if they had always had) an independent existence.

Reification, Wenger (2001) argues, can refer to both a process, or a practice, and its product, that is, the object that results from and reflects that practice. In fact, these objects are the basis for new processes, which will give way to new objects. This consideration, applied to the field of epistemology -consideration according to which in the construction of individual and historical knowledge there is an iteration of processes that give rise to objects, which are part of new processes- underlies the organization of the categories that exposed in this document.

Sfard and Linchevski (1994) retake the ideas of reification theory for the analysis of the construction of algebraic knowledge, both historically and at the student level. In particular, they propose that, starting from a set of processes A, a first object A is generated, from which another set of processes B is carried out, in order to build an object B, which serves to carry out a set of processes C that give rise to an object C. This development allows to build increasingly abstract objects.

Wenger's ideas of reification and the interpretation made by the aforementioned authors of these ideas, based on constructivist positions, are part of the philosophical framework of the authors of this writing. This philosophical framework guided the interpretive work and the methodology from which the empirical data recovery methods were derived. However, in this work, these literary references were included in advanced stages of the analysis, when there was already a set of categories that described the data. With these literary references we worked only at the conceptual level: the categories were reorganized and renamed, assigning them much more eloquent and appropriate names, thus gaining generality, systematization, and abstraction. Peña (2019) describes the use of literary references in the construction of the MICP and details of its construction.

Methods of collecting empirical data

Eleven third-year high school students (14 to 15 years old) from a public school in Mexico City participated in the research. These students had already studied the topic of triangle congruence. Four questionnaires were applied. For questionnaire 1 and 2, the first two work sessions were arranged. The objective was to provide students with tools to work on the concept of geometry congruence (congruence as superposition, geometric constructions, basic notions of geometry). For the solution of questionnaire 3 and 4, 4 class sessions were arranged; the objective in these questionnaires was to collect information on the way in which the students understood the congruence criteria for triangles and polygons. His empirical ideas on congruence were used as empirical data for the construction of the MICP. In Peña (2019) a detailed description of the battery of questionnaires is presented.

Results

In the following, the MICP is presented and each of its phases is exemplified.

Table 1. Interpretive Model for the Analysis of Polygon Congruence

Process	Object
Phase 1. Empirical intrafigural idea of congruence	
Process	Object
	In this phase, triangles, some of the components of the triangle (sides, angles, base, area, perimeter) and an idea of congruence seen as a property of the components of a triangle have been conceptualized; the conceptual object in this case is the intrafigural notion of congruence. In this phase it is usual to see representations of equilateral or isosceles triangles since in this type of triangles there are sides and angles congruent to each other.
Phase 2. Empirical interfigural idea of congruence. Congruence as a property of triangles	
Process	Object
In this phase pairs of triangles are compared to determine their possible congruence. To do this, the congruence between the components of the triangles is used (a process supported in the previous phase). To determine the interfigural congruence, empirical processes are performed such as the superposition of the triangles (where the decomposition of the figure into its parts is not necessary), or the comparison of measurements of pairs of sides or corresponding angles in the triangles.	In this phase the congruence relations occur between two specific triangles. The conceptual object in this Phase is an interfigural notion; In this phase, the students do not yet take it as an object of reflection. Although there is an idea of congruence, it only makes sense when associated with specific triangles. In this phase the congruence has a purely applicative character and is not an object of reflection as such, i.e., the student's reflection does not appear to be deliberately directed toward congruence as such. The notion of congruence grammatically functions as a predicate.
Phase 3. Initiation of congruence as an object of reflection	
Process	Object

In this phase, the processes and actions carried out respond to a need to consider triangles in a general way and not only as concrete cases. In this context, students carry out cognitive processes in which they reflect directly on the congruence related to these generic objects. In this phase, they change the focus of attention: from the concrete triangles they direct their interest towards congruence as an object of reflection; in these processes they seek to characterize the congruence of triangles by appealing to their components (sides, angles, base, area ...). Although the interest or need of the students in this phase is to get rid of the concrete cases, the lack of conceptual tools leads them to return to the use of empirical methods of representation and verification of congruence; however, unlike the previous phase, the specific triangles they use are considered by the student as general representations of the triangles.	In this phase the student begins to conceive of congruence as an object on which the student reflects and with which he associates properties. In this Phase, for example, there are those answers where the student makes explicit reference to some characterization of the congruence. In this phase two sub-phases are distinguished: If the properties with which students seek to characterize the congruence are not mathematically relevant to define it, the answer is located in Phase 3.1. In this case, the characterizations that the students make of the congruence of triangles are unsystematic, inconsistent, and usually made with atypical components (base, height, perimeter, area). If the characterizations about congruence are relevant, the answer would be in phase 3.2. In this case, these characterizations are usually coherent and systematic, and usually typical components are used (sides and angles). Grammatically, in this phase, congruence already plays the role of noun.
Phase 4. Reflection on properties of congruence as an object of reflection	
Process	Object
In this phase, reflection processes occur that no longer focus only on congruence but on the properties associated with congruence (which emerged in the previous phase). It is usual to find processes where sufficient conditions (or minimum conditions) are discriminated (either correctly or incorrectly, from the mathematical point of view) to guarantee congruence. This is done by the student with the understanding that it is possible to limit, depending on certain conditions or reasons, the number of components to establish the congruence of two triangles. In the case in which sufficient conditions that are correct are chosen, they coincide with what in school mathematics is known as the congruence criteria (SSS, SAS and ASA).	In this phase, sufficient conditions are associated with congruence, which are based on a definition of congruence that occurred in the previous phase. These are new properties related to this conceptual object, which make it much more general and more solid, although in this Phase it is still tied to empirical considerations. Two sub-phases are also identified in this phase: If the characterizations that the students carry out are erratic or inconsistent (e.g. when there are cases of consistency criteria such as AA, SSSA, SSSAA) it is associated with Sub phase 4.1. On the other hand, if the minimum conditions proposed by the students to guarantee consistency are mathematically correct (e.g. the consistency criteria), it is in Sub phase 2
Phase 5. Processes of deductive support.	
Process	Object
Students use deductive processes, based on definitions and general properties, to answer the why of certain properties of congruence (e.g., sufficient conditions for congruence of polygons) and other properties of congruence that they can conjecture or anticipate.	The mathematical concept of congruence is consolidated, as a general and abstract concept, by supporting some of its properties in deductive arguments.

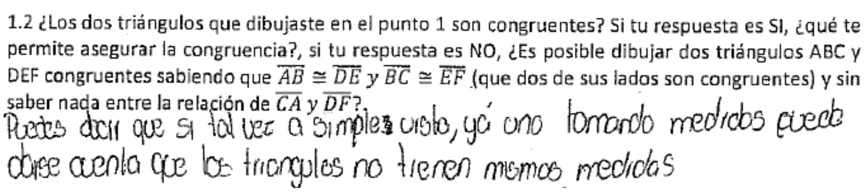
To exemplify the phases of the MICP, in the following some productions of the students are presented. Although examples of each phase are offered, the categorization of the students' responses was not carried out considering these responses in isolation. In the interpretation of each answer, the response patterns identified in the course of each student's production were taken into account as an analysis criterion.

Phase 1. Empirical intrafigural idea of congruence

Figure 1. Student response 2 to question 1 of questionnaire 3

The answer of figure 1 is located in this phase since the student shows to be aware that the triangle can be separated into components (sides); proof of this is that he names the sides and gives them a measurement. In this case, the congruence relations that are observed are only intra-figural, which can be verified in the fact that he only directs his gaze towards the congruence of sides that are part of the first triangle (side A and side C).

Phase 2. Empirical inter-figural idea of congruence. Congruence as a property of triangles.


1.2 ¿Los dos triángulos que dibujaste en el punto 1 son congruentes? Si tu respuesta es SI, ¿qué te permite asegurar la congruencia?, si tu respuesta es NO, ¿Es posible dibujar dos triángulos ABC y DEF congruentes sabiendo que $\overline{AB} \cong \overline{DE}$ y $\overline{BC} \cong \overline{EF}$ (que dos de sus lados son congruentes) y sin saber nada entre la relación de $\overline{CA}$ y $\overline{DF}$?

Puedes decir que si tal vez a simple vista, yá uno tomando medidas puede darse cuenta que los triangulos no tienen mismas medidas

Figure 2. Student response 1 to question 1.2 of questionnaire 3

The answer in figure 2 is found in phase 2 since the student explicitly shows that he is thinking of two particular triangles; this is seen when the student says "by taking measurements you can see that the triangles have the same measurements". Furthermore, he evaluates congruence through an empirical comparison method, which he calls “at a glance”.

Phase 3. Reflection on consistency.

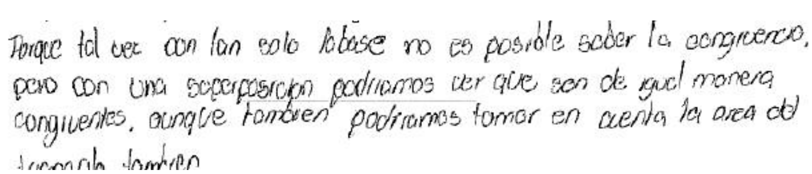
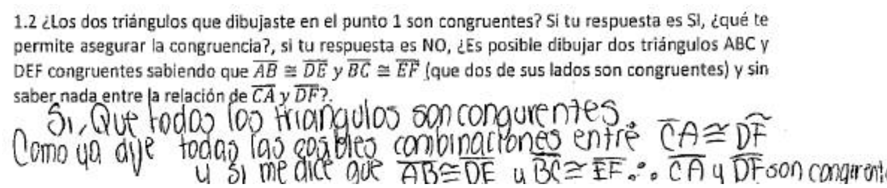

Porque tal vez con tan solo la base no es posible saber la congruencia, pero con una superposición podriamos ver que son de igual manera congruentes, aunque tambien podriamos tomar en cuenta la area del triangulo tambien

Figure 3. Student 1 response to question 1.4b of questionnaire 3

Figure 3 presents a student's answer to the following question: what other minimum data should we have to ensure the congruence of two triangles if we already have a pair of corresponding congruent sides? As a result, the student does not seem to propose sufficiency criteria (as expected); on the contrary, he proposes a way to characterize congruence in terms of necessity, the latter being a constant in many of his other responses. On the other hand, it seems that the student lacks conceptual tools to support his hypothesis, so he must resort to a known field, the superposition, even when he tries to characterize the congruence far from the empirical verification methods. In this case, the student was in subphase 3.1, since his characterizations turn out to be atypical (since he makes use of the base of the triangles).

Phase 4. Reflection on properties of congruence as an object of reflection

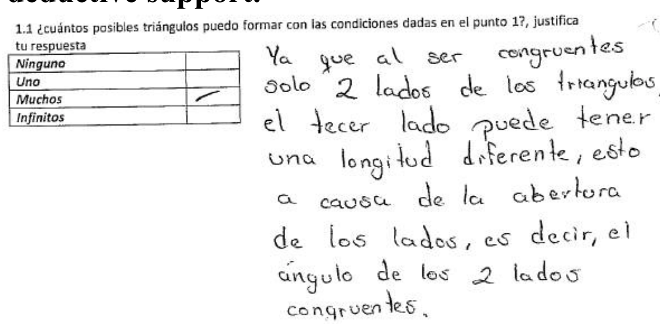

1.2 ¿Los dos triángulos que dibujaste en el punto 1 son congruentes? Si tu respuesta es SI, ¿qué te permite asegurar la congruencia?, si tu respuesta es NO, ¿Es posible dibujar dos triángulos ABC y DEF congruentes sabiendo que $\overline{AB} \cong \overline{DE}$ y $\overline{BC} \cong \overline{EF}$ (que dos de sus lados son congruentes) y sin saber nada entre la relación de $\overline{CA}$ y $\overline{DF}$?

Si, Que todos los triangulos son congruentes.
Como ya dije todas las posibles combinaciones entre $\overline{CA} \cong \overline{DF}$
y si me dice que $\overline{AB} \cong \overline{DE}$ y $\overline{BC} \cong \overline{EF}$ ∴ $\overline{CA}$ y $\overline{DF}$ son congruent

Figure 4. Student response 6 to question 1.2 of questionnaire 3

The answer of figure 4 corresponds to this level since the student shows to have characterized certain sufficient criteria for polygon congruence, although in this particular case those criteria are erroneous. The student shows to believe that it is possible, having $\overline{(AB)} \cong \overline{(DF)}$ and $\overline{(BC)} \cong \overline{(EF)}$ ensure that $\overline{(CA)} \cong \overline{(DF)}$, this shows that he understands, at least in one level basic, the sufficient conditions, i.e., is aware that it is possible to exclude components in the congruence of triangles and still continue to ensure congruence.

Phase 5. Processes of deductive support.


1.1 ¿cuántos posibles triángulos puedo formar con las condiciones dadas en el punto 1?, justifica tu respuesta

Ninguno	
Uno	
Muchos	✓
Infinitos	

Ya que al ser congruentes solo 2 lados de los triangulos, el tecer lado puede tener una longitud diferente, esto a causa de la abertura de los lados, es decir, el ángulo de los 2 lados congruentes.

Figure 5. Student response 7 to question 1.1 of questionnaire 3

The answer in figure 5 is in phase 5 since the student presents a deductive argument that supports the congruence criterion for SAS triangles.

Comments and conclusions

The MICP is a preliminary model. It is hoped, in future works, to be able to further refine the model by adding properties to the phases. In future works, the authors of this document will seek to identify the types of arguments presented in each phase. The MICP is considered to be a model that can be useful for teachers, researchers and people involved with education in mathematics, since it allows analyzing the productions of students, their possible profiles in relation to performance in congruence tasks (Peña, 2019), in addition to suggesting hypotheses about possible trajectories of the construction of the concept of "congruence" in the classroom.

The authors of the document have considered important to detail the process of construction of the MICP categories, because that constructive process allows readers to get a closer look at the construction of conceptual categories, whose details are often omitted in the research literature in mathematics education. The authors of the writing are aware that the work involved in the construction of the categories is sometimes enormous, that the data is often overwhelming and that there will be many moments when the researcher (in training and already trained) must reformulate and discard ideas that had already developed. However, even with all the problems involved in constructing theoretical models or conceptual categories, the process that this entails leads, from our point of view, to basically understanding what it involves doing a certain type of research.

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INTERPRETIVE MODEL OF THE CONCEPTUALIZATION OF THE CONGRUENCE OF POLYGONS (MICP)

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En el documento se presenta un conjunto de categorías para el análisis de la conceptualización de la congruencia de polígonos -tema central en la matemática escolar- y se detalla la aplicación de las herramientas analíticas empleadas, provenientes de la Teoría Fundamentada, en esa construcción. A ese conjunto de categorías se le llama 'Modelo interpretativo de la conceptualización de la congruencia de polígonos' (MICP). Este modelo surgió como resultado de la interpretación de datos empíricos recuperados durante la investigación. Las categorías del MICP pueden ser empleadas por profesores o investigadores para cubrir distintos objetivos didácticos (e.g., interpretar la resolución de tareas de contenidos de congruencia; elaborar perfiles de estudiantes o identificar sus *dificultades. Ver Peña, 2019) y resulta relevante porque no parece existir en la literatura un modelo semejante que cubra los objetivos antes planteados.*

Palabras clave: Geometría, metodologías de la investigación.

Planteamiento del problema y pregunta de investigación

Los criterios de congruencia y la noción de congruencia son temas importantes y fundamentales para la geometría, tanto para la que se enseña desde los niveles básicos de educación como para la disciplinar. En distintas versiones de la geometría euclidiana -propuestas por Euclides, Legendre o Hilbert y que presentan una organización deductiva con distintos grados de formalización- la noción de congruencia y los criterios de congruencia surgen desde los inicios de la axiomatización. En el caso de *Los Elementos* de Euclides, conforme a la versión de Heath (1956), la noción de congruencia se introduce en la Noción Común 4 y los 'criterios de congruencia de triángulos' se encuentran en las proposiciones I.4; I.8 y I.26. En el caso de la geometría de Legendre (Legendre, 1984), se habla de la congruencia en los inicios de la axiomatización y se muestran los criterios de congruencia para los triángulos en las proposiciones VI, VII y XI. Algo similar ocurre en la formalización de la geometría propuesta por Hilbert (1996) en donde también se incluyen los criterios de congruencia al inicio de la obra. Lo anterior permite suponer que en la geometría disciplinar, particularmente en las diferentes versiones de la geometría Euclidiana, se requiere desde los inicios de la axiomatización hacer uso de la noción de congruencia y de los criterios de congruencia de triángulos.

Así como la noción de congruencia y los criterios de congruencia resultan necesarios para la geometría disciplinar, de igual forma resultan imprescindibles en la educación matemática escolar. Para verificar esta afirmación se revisaron planes de estudio de México y de Colombia. En México, la Secretaria de Educación pública (SEP, 2017) introduce en sus planes de estudio a la noción de congruencia y los criterios de congruencia de triángulos desde tercero de primaria -donde se solicita comparar figuras geométricas y establecer unicidad (p.314)-, hasta tercero de secundaria -nivel en el que se espera que se determinen y usen los criterios de congruencia de triángulos (p.315). En Colombia, el Ministerio Nacional de Educación (MEN, 2006) propone "reconocer a la congruencia de figuras" (p.80) en el 1° y el 3° grado; "identificar y justificar la congruencia entre figuras" (p.82) entre 4° y 5°; hacer uso de la congruencia de figuras para resolver problemas para 6° y 7° (p.84) y el estudio de las propiedades de la congruencia para 8° y 9° (p.86). Todo esto hace referencia básicamente a la congruencia de triángulos.

Dado el peso y el alcance que la congruencia tiene no solo en la geometría disciplinar sino en la geometría escolar, se requieren herramientas que permitan cubrir una serie de objetivos didácticos relacionados con ese concepto, entre otros: interpretar las resoluciones de tareas de contenidos de congruencia elaboradas por los alumnos, elaborar perfiles de estudiantes e identificar algunas de sus dificultades conceptuales sobre esa noción; analizar los enunciados de las tareas mismas y elaborar secuencias didácticas sobre el tema. No obstante, no parecen haber trabajos en investigación en educación matemática que brinden estas herramientas de análisis para el caso de los triángulos, y menos aún, para la congruencia de polígonos, que es el tema central de la investigación que aquí se expone.

Al hacer investigación documental sobre reportes de estudio de educación matemática centrados en temas de congruencia de polígonos, solo se encontraron propuestas didácticas para la enseñanza del tema (Carbó y Mántica, 2010; Piatek-Jimenez, 2008; Zakiz y Leron, 1991 por citar algunos). Estos trabajos carecen, por ejemplo, de un análisis sistemático de las posibles dificultades en el aprendizaje y la enseñanza de temas afines a la congruencia de polígonos; carecen también de sustentos teóricos que pudieran justificar el orden de exposición de tareas en una secuencia didáctica sobre la congruencia y que permitan delinear posibles perfiles que den cuenta del nivel de comprensión que los alumnos poseen de este concepto.

En este documento se expone un conjunto de categorías -al que se le ha llamado 'Modelo interpretativo de la conceptualización de la congruencia de polígonos' (MICP) (Peña, 2019)- que tiene la intención de ayudar a cubrir (aunque sea de manera preliminar) las carencias antes planteadas.

Metodología y aplicación de las herramientas analíticas empleadas

Para la construcción del MICP se siguieron algunos de los principios que rigen la Teoría Fundamentada (TF) en la versión de Corbin y Strauss (2015), si bien el estudio no tenía como propósito alcanzar los objetivos últimos de la TF (i.e., hacer teoría). En lo que sigue se esboza el proceso de construcción de las categorías y algunas ideas centrales.

En la TF las categorías interpretativas se basan en los datos empíricos que se recolectan durante la investigación y no emergen de un marco teórico dado de antemano (Corbin & Strauss, 2015). Siguiendo este principio general de la TF, se realizó la construcción del MICP cuidando siempre que las categorías estuvieran orientadas por los datos empíricos.

En un primer momento se fracturaron los datos empíricos, a partir de lo cual se construyeron patrones con base en los cuales se generaron etiquetas conceptuales. En un regreso al dominio empírico, se comprobó que dichas etiquetas representaran a los datos. Enseguida se hizo un análisis comparativo entre las etiquetas conceptuales. A partir de un proceso de síntesis llevado a cabo en el dominio conceptual, se generaron categorías. En todos estos procesos estuvieron involucradas las comparaciones constantes, el planteamiento de preguntas y la elaboración de memos y diagramas, herramientas analíticas que forman parte de los métodos de la TF (Corbin & Strauss, 2015). En un momento posterior -que Birks y Mills (2011) denominan de 'codificación intermedia'- se profundizó en cada una de las categorías y se definieron subcategorías y sincrónicamente se realizaron procesos de ordenamiento lógico entre ellas, haciendo uso de las ideas propuestas por Sfard y Linchevski (1994).

Posteriormente, se realizó un muestreo teórico (Corbin & Strauss, 2015) que permitió dotar de nuevas propiedades y dimensiones a las categorías construidas. En una etapa subsecuente, estas categorías se confrontaron con los niveles propuestos por los investigadores Van Hiele (trad. en 1984). Finalmente, y con apoyo de algunas de las ideas de cosificación propuestas por Sfard y Linchevski (1994) y por Wenger (2001), se realizaron modificaciones a nivel conceptual. Esto dio lugar al MICP que se expone en este documento, el cual posee propiedades de un ordenamiento conceptual de acuerdo con la definición que dan Corbin y Strauss (2015).

Referentes Literarios

Para la construcción del MICP se usaron algunas de las ideas propuestas por Wenger (2001) y por Sfard y Linchevski (1994). Wenger (2001) expresa que, con el fin de significar sus acciones y prácticas cotidianas y su experiencia en el mundo, los miembros de las comunidades llevan a cabo procesos de cosificación. La idea de cosificación la emplea Wenger de manera general para referirse al proceso de dar forma a nuestra experiencia produciendo objetos que plasman esta experiencia en una cosa (p. 84). Por ejemplo, redactar una ley, crear una receta para un pastel o demostrar un teorema son procesos de cosificación en los que se da forma o se 'materializa' una cierta experiencia social, gastronómica o matemática. Así, en estos procesos de cosificación se convierten en objetos -la ley, el pastel o el teorema- algunos aspectos o características de una práctica, objetos que pueden ser tratados como si fueran elementos materiales y concretos, aún y cuando no lo sean. Una vez constituidos esos objetos, los percibimos como si existieran en el mundo, como si tuvieran una realidad propia. Esto es muy claro con los conceptos matemáticos y las estructuras científicas. Se suelen ver como si tuvieran (y como si siempre hubieran tenido) una existencia independiente.

La cosificación, sostiene Wenger (2001), puede hacer referencia tanto a un proceso, o una práctica, como a su producto, es decir, al objeto que resulta y es reflejo de esa práctica. De hecho, esos objetos son la base para nuevos procesos, mismos que darán paso a nuevos objetos. Esta consideración, aplicada al terreno de la epistemología -consideración conforme a la cual en la construcción del conocimiento individual e histórico se da una iteración de procesos que dan lugar a objetos, que forman parte de nuevos procesos- subyace a la organización de las categorías que se exponen en este documento.

Sfard y Linchevski (1994) retoman las ideas de la teoría de la reificación para el análisis de la construcción de conocimientos algebraicos, tanto a nivel histórico como a nivel del estudiante. En particular, ellas proponen que, partiendo de un conjunto de procesos A, se genera un primer objeto A, a partir del cual se realiza otro conjunto de procesos B, para así construir un objeto B, mismo que sirve para realizar un conjunto de procesos C que dan lugar a un objeto C. Este desarrollo permite construir objetos cada vez más abstractos.

Las ideas de cosificación de Wenger y la interpretación que hacen las autoras antes citadas de esas ideas, basadas en posturas constructivistas, forman parte del marco filosófico de los autores de este escrito. Ese marco filosófico orientó el trabajo interpretativo y la metodología de la cual se desprendieron los métodos de recuperación de datos empíricos. Sin embargo, en este trabajo esos referentes literarios se incluyeron en fases avanzadas del análisis, cuando ya se contaba con un conjunto de categorías que describían los datos. Con esos referentes literarios se trabajó sólo a nivel conceptual: se reorganizaron las categorías y se renombraron, asignándoles nombres mucho más elocuentes y adecuados, con lo que se ganó generalidad, sistematización y abstracción. En Peña (2019) se describe el empleo de los referentes literarios en la construcción del MICP y detalles de su construcción.

Métodos de recolección de los datos empíricos

En la investigación participaron 11 estudiantes de tercero de secundaria (14 a 15 años) de una escuela pública de la ciudad de México. Estos alumnos ya habían estudiado el tema de congruencia de triángulos. Se aplicaron cuatro cuestionarios. Para el cuestionario 1 y 2 se dispusieron las dos primeras sesiones de trabajo. El objetivo fue brindar a los estudiantes herramientas para trabajar sobre el concepto de congruencia de geometría (congruencia como superposición, construcciones geométricas, nociones básicas de la geometría). Para la solución del cuestionario 3 y 4 se dispusieron 4 sesiones de clase, el objetivo en estos cuestionarios era el recolectar información sobre la manera en la que los estudiantes entendían los criterios de congruencia para triángulos y polígonos. Como datos empíricos para la construcción del MICP se utilizaron sus ideas sobre la congruencia. En Peña (2019) se presenta una descripción detallada de la batería de cuestionarios.

Resultados

En lo que sigue se presenta el MICP y se ejemplifican cada una de sus fases.

Tabla 1. Modelo Interpretativo para el análisis de la Congruencia de Polígonos

Fase 1. Idea empírica intrafigural de congruencia	
Proceso	Objeto
	En esta fase se han conceptualizado los triángulos, algunos de los componentes del triángulo (lados, ángulos, base, área, perímetro) y una idea de la congruencia vista como una propiedad de los

	componentes de un triángulo; el objeto conceptual en este caso es la noción intrafigural de la congruencia. En esta fase es usual ver representaciones de triángulos equiláteros o isósceles ya que en este tipo de triángulos hay lados y ángulos congruentes entre sí.
Fase 2. Idea empírica interfigural de congruencia. la congruencia como una propiedad de triángulos	
Proceso	Objeto
En esta fase se comparan parejas de triángulos para determinar su posible congruencia. Para ello se recurre a la determinación de la congruencia entre los componentes de los triángulos (proceso que se respalda en la fase anterior). Para la determinación de la congruencia interfigural, se realizan procesos empíricos como la superposición de los triángulos (en donde no es necesaria la descomposición de la figura en sus partes) o la comparación de medidas de pares de lados o ángulos correspondientes en los triángulos.	En esta fase las relaciones de congruencia se dan entre dos triángulos concretos. El objeto conceptual en esta Fase es una noción interfigural; en esta Fase, todavía no se reflexiona sobre ese objeto. Aunque hay una idea de congruencia, solo cobra sentido cuando se le asocia a triángulos concretos. En esta fase la congruencia tiene un carácter puramente aplicativo y no es objeto de reflexión como tal; i.e, no parece que la reflexión del estudiante se dirija deliberadamente hacia la congruencia como tal. La noción de congruencia gramaticalmente funge como predicado.
Fase 3. Inicio de la congruencia como objeto de reflexión	
Proceso	Objeto
En esta fase los procesos y las acciones que se llevan a cabo responden a una necesidad de considerar a los triángulos de manera general y no solo como casos concretos. En este contexto, los alumnos realizan procesos cognitivos en los que reflexionan directamente sobre la congruencia relacionada con esos objetos genéricos. En esta fase, ellos cambian el foco de atención: de los triángulos concretos dirigen su interés hacia la congruencia como un objeto de reflexión; en estos procesos buscan caracterizar a la congruencia de triángulos apelando a sus componentes (lados, ángulos, base, área…). Aunque el interés o necesidad de los alumnos en esta fase es desprenderse de los casos concretos, la falta de herramientas conceptuales los lleva a regresar al uso de métodos empíricos de representación y verificación de la congruencia; sin embargo, a diferencia de la fase previa, los triángulos concretos que utilizan son considerados por el alumno como representaciones generales de los triángulos.	En esta fase el alumno empieza a concebir a la congruencia como un objeto sobre el cual el alumno reflexiona y al cual le asocia propiedades. En esta Fase, por ejemplo, se encuentran aquellas respuestas en donde el alumno hace referencia explícita a alguna caracterización de la congruencia. En esta fase se distinguen dos sub fases: Si las propiedades con las que busca caracterizar a la congruencia no son matemáticamente relevantes para definirla, su respuesta se ubica en la Fase 3.1. En este caso, las caracterizaciones que realizan los alumnos de la congruencia de triángulos son asistemáticas, incoherentes y usualmente realizadas con componentes atípicos (Base, altura, perímetro, área). Si las caracterizaciones sobre la congruencia son relevantes, su respuesta se ubicaría en la fase 3.2. En este caso, esas caracterizaciones suelen ser coherentes y sistemáticas, y usualmente se acude a componentes típicos (Lados y ángulos). Gramaticalmente, en esta Fase la congruencia juega ya el papel de sustantivo.
Fase 4. Reflexión en torno a propiedades de la congruencia como objeto de reflexión	
Proceso	Objeto
En esta fase se dan procesos de reflexión que ya no solo se centran en la congruencia sino en las propiedades que se asocian a la congruencia (que	En esta fase se asocian a la congruencia condiciones suficientes, las que se basan en una definición de la congruencia que se dio en la fase previa. Se trata de

surgieron en la fase anterior). Es usual encontrar procesos en donde se discriminan (ya sea correcta o incorrectamente, desde el punto de vista matemático) condiciones suficientes (o condiciones mínimas) para garantizar la congruencia. Esto lo hace el alumno con la comprensión de que es posible acotar, atendiendo a ciertas condiciones o razones, la cantidad de componentes a considerar para establecer la congruencia de dos triángulos. En el caso en el que se eligen condiciones suficientes correctas, coinciden con lo que en la matemática escolar se conoce como los criterios de congruencia (LLL, LAL y ALA).	nuevas propiedades relacionadas con este objeto conceptual, que lo hacen mucho más general y más sólido, aunque en esta Fase todavía esté atado a consideraciones empíricas. En esta Fase se identifican también dos subfases: Si las caracterizaciones que realizan los alumnos son erráticas o incoherentes (e.g. cuando se dan casos de criterios de congruencia como AA, LLLA, LLLAA) se asocia a la Sub fase 4.1. Por otro lado, si las condiciones mínimas que proponen los alumnos para garantizar la congruencia son matemáticamente correctas (e.g. los criterios de congruencia) se ubica en la Sub fase 2
Fase 5. Procesos de sustentación deductiva.	
Proceso	Objeto
Los alumnos recurren a procesos deductivos, basados en definiciones y propiedades generales, para dar respuesta al por qué de ciertas propiedades de la congruencia (e.g., condiciones suficientes para la congruencia de polígonos) y a otras propiedades de la congruencia que ellos pueden conjeturar o anticipar.	Se consolida el concepto matemático de congruencia, como un concepto general y abstracto, al sustentar algunas de sus propiedades en argumentos de tipo deductivo.

Para ejemplificar las fases del MICP, en lo que sigue se presentan algunas producciones de los estudiantes. Aunque se ofrecen ejemplos particulares de cada fase, la categorización de las respuestas de los alumnos no se realizó considerando esas respuestas de manera aislada. En la interpretación de cada respuesta se tomó en cuenta, como criterio de análisis, los patrones de respuesta identificados en el transcurso de la producción de cada alumno.

Fase 1. Idea empírica intrafigural de congruencia:

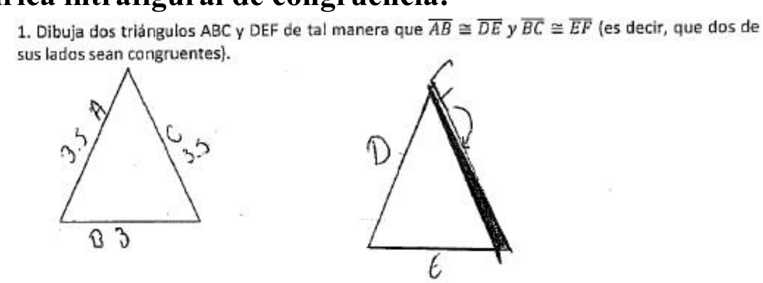


Figura 1. Respuesta del alumno 2 a la pregunta 1 del cuestionario 3

La respuesta de la figura 1 se ubica en esta fase ya que el alumno muestra ser consciente de que el triángulo se puede separar en componentes (lados); y es que él nombra los lados y les da una medida. En este caso, las relaciones de congruencia que se observan solo son intrafigurales, lo cual se puede constatar en el hecho de que él solo dirige su mirada hacia la congruencia de lados que forman parte del primer triángulo (el lado A y el lado C).

Fase 2. Idea empírica interfigural de congruencia.

1.2 ¿Los dos triángulos que dibujaste en el punto 1 son congruentes? Si tu respuesta es SI, ¿qué te permite asegurar la congruencia?, si tu respuesta es NO, ¿Es posible dibujar dos triángulos ABC y DEF congruentes sabiendo que $\overline{AB} \cong \overline{DE}$ y $\overline{BC} \cong \overline{EF}$ (que dos de sus lados son congruentes) y sin saber nada entre la relación de $\overline{CA}$ y $\overline{DF}$?

Puedes dcir que si tal vez a simple vista, ya uno tomando medidas puede darse cuenta que los triangulos no tienen mismas medidas

Figura 2. Respuesta del alumno 1 a la pregunta 1.2 del cuestionario 3

La respuesta de la figura 2 se encuentra en la fase 2 ya que el alumno muestra explícitamente estar pensando en dos triángulos particulares; esto se ve cuando el alumno dice "tomando medidas puede darse cuenta que los triángulos tienen mismas medidas". Además, evalúa la congruencia a través de un método de comparación empírico, que él llama "a simple vista".

Fase 3. Reflexión en torno a la congruencia.

Porque tal vez con tan solo la base no es posible saber la congruencia, pero con una superposicion podriamos ver que son de igual manera congruentes, aunque tambien podriamos tomar en cuenta la area del triangulo tambien

Figura 3. Respuesta del alumno 1 a la pregunta 1.4b del cuestionario 3

En la figura 3 se presenta la respuesta de un alumno a la siguiente pregunta ¿qué otros datos mínimos debemos tener para asegurar la congruencia de dos triángulos si ya se tiene un par de lados correspondientes congruentes? Como resultado el estudiante no parece proponer criterios de suficiencia (como se esperaba); por el contrario, él propone una manera de caracterizar a la congruencia en términos de necesidad, esto último es una constante en muchas de sus otras respuestas. Por otro lado, parece que el alumno carece de herramientas conceptuales para soportar su hipótesis, por lo que debe recurrir a un campo conocido, la superposición, aun cuando intenta caracterizar a la congruencia lejos de los métodos empíricos de verificación. En este caso particular el alumno se ubicó en la subfase 3.1, pues sus caracterizaciones resultan ser atípicas (ya que hace uso de la base de los triángulos).

Fase 4. Reflexión en torno a propiedades de la congruencia.

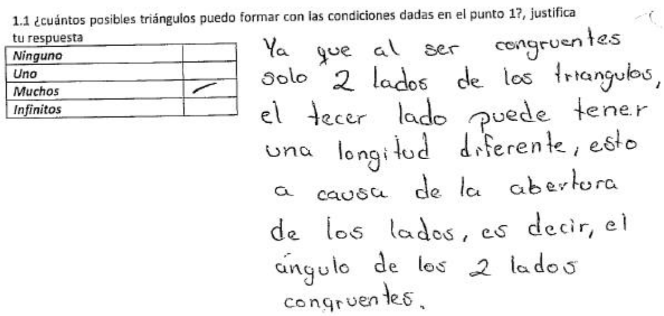
1.2 ¿Los dos triángulos que dibujaste en el punto 1 son congruentes? Si tu respuesta es SI, ¿qué te permite asegurar la congruencia?, si tu respuesta es NO, ¿Es posible dibujar dos triángulos ABC y DEF congruentes sabiendo que $\overline{AB} \cong \overline{DE}$ y $\overline{BC} \cong \overline{EF}$ (que dos de sus lados son congruentes) y sin saber nada entre la relación de $\overline{CA}$ y $\overline{DF}$?

Si, Que todos los triangulos son conguruentes. Como ya dije todas las posibles combinaciones entre $\overline{CA} \cong \overline{DF}$ y si me dice que $\overline{AB} \cong \overline{DE}$ y $\overline{BC} \cong \overline{EF}$ $\therefore$ $\overline{CA}$ y $\overline{DF}$ son congruentes

Figura 4. Respuesta del alumno 6 a la pregunta 1.2 del cuestionario 3

La respuesta de la figura 4 corresponde a este nivel ya que el alumno muestra haber caracterizado ciertos criterios de suficiencia de congruencia de polígonos, aunque para este caso particular dichos criterios sean erróneos. El alumno muestra creer que es posible, teniendo $\overline{AB} \cong \overline{DF}$ y $\overline{BC} \cong \overline{EF}$ asegurar que $\overline{CA} \cong \overline{DF}$, esto muestra que comprende, al menos en un nivel básico, las condiciones de suficiencia, i.e es consciente de que es posible prescindir de componentes en la congruencia de triángulos y aun así seguir asegurando la congruencia.

Fase 5. Procesos de sustentación deductiva.


1.1 ¿cuántos posibles triángulos puedo formar con las condiciones dadas en el punto 1?, justifica tu respuesta

Ninguno	
Uno	
Muchos	✓
Infinitos	

Ya que al ser congruentes solo 2 lados de los triangulos, el tecer lado puede tener una longitud diferente, esto a causa de la abertura de los lados, es decir, el ángulo de los 2 lados congruentes.

Figura 5. Respuesta del alumno 7 a la pregunta 1.1 del cuestionario 3

Se ubica la respuesta en la figura 5 en la fase 5 ya que el alumno presenta un argumento deductivo que sustenta al criterio de congruencia para triángulos LAL.

Comentarios y conclusiones

El MICP es un modelo preliminar. Se espera, en trabajos futuros, poder seguir afinando el modelo agregando propiedades a las fases. Particularmente, en trabajos próximos, los autores de este documento buscarán identificar los tipos de argumentos que se presentan en cada fase. Se considera que el MICP es un modelo que puede ser útil para profesores, investigadores y personas involucradas con la educación en matemática, pues permite analizar las producciones de los estudiantes, sus posibles perfiles con relación al desempeño en tareas de congruencia (Peña, 2019), además de que sugiere hipótesis sobre trayectorias posibles de la construcción del concepto de "congruencia" en el aula.

Los autores del documento han considerado importante detallar en este escrito el proceso de construcción de las categorías del MICP, porque ese proceso constructivo -cuyos detalles se suelen omitir en la literatura en investigación en educación matemática- le permite a los investigadores mirar de manera crítica y a profundidad en los temas de su estudio y les permite generalmente ir más allá de lo que está escrito. Los autores del escrito están conscientes que el trabajo que supone la construcción de las categorías es a veces enorme, que en muchas ocasiones los datos abruman y que habrá muchos momentos en donde el investigador (en formación y ya formado) debe reformular y descartar ideas que ya había desarrollado. Sin embargo, aun con todos los problemas que conlleva la construcción de modelos teóricos o categorías conceptuales, el proceso que esto supone lleva, desde nuestro punto de vista, a comprender básicamente lo que implica hacer un cierto tipo de investigación.

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CHILDREN'S DURATIONAL ORGANIZATION OF EVEREYDAY EXPERIENCES: A MATHEMATICAL PERSPECTIVE OF A LINGUISTIC STUDY

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How do children reason about the durations of daily experiences? Following Tillman and Barner's (2015) linguistic study, three children (age five, six, and seven) were asked to organize four everyday activities from the shortest duration to longest duration: watching a movie, brushing their teeth, sleeping at night, and eating lunch. After creating their "timeline", each child was asked why they ordered the events as they did. By allowing the children the opportunity to reflect on common experiences and explain how they reasoned about durations, we can begin to recognize how children understand time as a quantity. Their responses showed that reflections on lived durational experiences were heavily influenced by physical acts, such as speed of actions or movement of the sun. These findings were consistent with past research on children's conception of physical time (Long & Kamii, 2001; Piaget, 1969).

Keywords: Measurement, Elementary School Education, Cognition

Tillman and Barner (2015) presented a series of linguistic experiments to explore young children's development of durational language. Through these experiments, Tillman and Barner theorized that children hear and use durational words, such as minute and second, but are unable to define these words with precise meanings, such as a minute being made of 60 seconds. The authors' asserted, "a lag [exists] between production and comprehension of duration words" (Tillman & Barner, 2015, p. 58), a point that would be supported by mathematics researchers and educators (Earnest, 2015; Harris, 2008; Kamii & Russell, 2012; Piaget, 1969). Thus, Tillman and Barner's research does not seem to lie solely within the field of linguistics, but also mathematics education research.

Durational words are commonly used in everyday English, for example, a parent telling their child "just a second" or "hang on a minute". During casual conversation these durations are not used as quantified elapsed time intervals, but rather are used as informal estimates of general wait times (Tillman & Barner, 2015). How such informal intervals are understood and used by children is an underrepresented area in mathematics education research. The value of such research would be to establish what temporal conceptions elementary students might bring with them to the mathematics classroom prior to formal time instruction.

Theoretical Framework

Currently, the Common Core State Standards Initiative (CCSSI) places time instruction within the Measurement and Data strand beginning in first grade. According to Common Core, first graders are taught to, "Tell and write time in hours and half-hours using analog and digital clocks" (CCSSI, 2020). There are no precursory standards that establish children's understanding of what hours and minutes are, so it could be inferred that the creators of Common Core believe these durational conceptions are either unimportant or unnecessary for clock reading and time telling, or already known prior to first grade (age six).

This lack of prerequisites seems counter to how Common Core has structured other forms of measurement. According to Common Core, kindergarteners should be able to, "Describe measurable attributes of objects, such as length or weight" (CCSSI, 2020) and, "Directly compare two objects with a measurable attribute in common" (CCSSI, 2020). These standards are addressed prior to learning the tools for such measurement. In other words, kindergarteners are explicitly taught about

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length-based attributes before being taught how to use the tools to measure them. For example, a kindergartener is taught to compare two lengths of string in order to tell which is longer. This comparison is intended to later bring about strategies for measuring, such as aligning starting points or considering left-overs—both of which are important to consider when measuring time (D. Earnest, personal conversation, May 27, 2019).

Unlike measuring lengths, where quantities can be physically compared against one another as described by the Common Core standards, measuring duration requires the creation of a hypothetical, iterable unit (Piaget, 1969)."The only method of [creating a "mobilized" durational unit] is to reproduce the physical phenomenon whose course (motion) was [the] duration" (Piaget, 1969, p. 67). So, when making judgements about the duration of common experiences, an individual must mentally reconstruct their experience, then dissociate time from perceptive influences, such as effort, emotion, or velocity.

The mental process of distinguishing time from spatial influences is referred to as the operationalization of time (Piaget, 1969). To construct *operational time*, one needs to coordinate succession (the consecutive sequence of events) and duration (the intervals of and between events). This, unlike *intuitive time*—which is based on spatial perceptions—means the individual understands time is continuous, homogeneous for all individuals, and uniform in its measurement (Russell, 2008). For example, a child who conceived of time intuitively would believe that as they walked faster, time moved faster. When reasoning operationally about time, this child would know that their actions have no impact on the duration of their experience.

Piaget (1969) contended that children were able to reason operationally about time by the age of nine (around fourth grade). Long and Kamii (2001), however, found it was not until sixth grade (around the age of 11) and Russell (2008) argued that it was not until eighth grade (around the age of 13). Irrespective of which of these studies might demonstrate an accurate age for the operationalization of time, none place the necessary reasoning for time measurement at first grade (age six), as described by Common Core (CCSSI, 2020) and Tillman and Barner's (2015) study. I hope this study may begin illustrating how children in the early elementary years think about duration based on their lived experiences.

Methodology

This multiple case study (Yin, 2003) followed a modified investigation from Tillman and Barner's (2015) *Experiment 3*, which explored how children placed familiar events on a figurative "timeline". During *Experiment 3*, Tillman and Barner asked children, age five to seven, and adults, to place whole numbers, everyday experiences (e.g., watching a movie, washing hands), units of time (e.g., seconds, hours), and timed durations (e.g., four minutes, two hours) on an open number line. Participants were not asked about their reasoning for their placements, instead a quantitative analysis was completed on each age group.

To understand how children might reason about different durations from reflections on their daily experiences, I asked three children (age five, six, and seven) to organize four activities from shortest to longest amount of time, following the last study of *Experiment 3*. Each child was asked to reflect on why they arranged the events as they did, so that I could complete a qualitative analysis.

Participants and Procedure

To represent the same youth population used by Tillman and Barner (2015), three children: Kris (age five), Sam (age six), and Casey (age seven) were interviewed. All three participants were from the same elementary school in a large suburban city in the western United States. Each child, and their parents, consented to participate in this interview, as part of a larger study on how children reason about durational experiences.

Kris, Sam, and Casey were video, and audio recorded during one-on-one interviews. The interview protocol was modified from Tillman and Barner's (2015) *Experiment 3*. Each child was given four cards with pictures and words of common activities: watching a movie, brushing their teeth, sleeping at night, and eating lunch. These activities slightly differed from those used in *Experiment* 3 in order to better relate to the sample population. Each child was asked to arrange the cards along a continuum from what took the shortest amount of time to the longest amount of time. After completing their "timeline", each child was asked to explain why they organized the cards as they did.

Following each interview, field notes were taken, and each interview was transcribed. Transcripts captured words, hesitations, and actions of each child. All transcripts were member checked by the child and their parents prior to analysis.

Data Analysis

Given the exploratory nature of this research, and the open-endedness of the responses, I used a constant comparison analysis (Glaser & Strauss, 1967). Reasonings were coded inductively, then codes were compared to note any similarities, differences, or apparent progression between and across participants.

The focus of this analysis was not on the accuracy of the child's organization, but rather the durational reasoning presented during their explanation. For example, Casey, age seven, ordered the events: brushing teeth as the shortest duration, then sleeping at night, then eating lunch, then finally watching a movie as the longest duration. This order was not the same as either of the other two participants, nor was it accurate by the actual average length of each activity. However, Casey's explanation displayed a different interpretation of what sleeping at night meant, which explained how her durational order made sense.

After initial coding, themes were created, hypothesizing possible attributes being attended to during the three children's durational reasoning, such as effort exerted, standard units of time, and physical indicators of passing time. These themes were compared against previous studies of children's conceptions of time.

Findings

Tillman and Barner's (2015) quantitative results showed that children in all three age groups (five, six, and seven years old) performed fairly poorly when organizing the durations of common events. The authors' asserted that given these results, "it seems highly unlikely that children's learning of the rank ordering of duration is mediated by knowledge of the approximate durations of events (e.g., that children learn 'an hour' by mapping it to events described as 'an hour', and noting the duration of those events)" (Tillman & Barner, 2015, p. 68). In other words, when children organize familiar experiences, it does not seem that they are using an understanding of the actual duration of each activity.

Through the interviews, I found that at some point, each of the three children explained their sequence of events using standard durational units (e.g., minutes, hours). However, when asked how they knew these units, none of them were able to justify the actual durations. Rather, other explanations for the durations were given, such as how enjoyable the experience was or what their effort to complete the activity.

Kris: Age Five

Kris began by sorting the activities into two columns, what he called the "short amount of time side" (brushing teeth and watching movie) and the "long amount of time side" (sleeping at night and eating lunch). When asked if he felt that it took the same amount of time to watch a movie as brush your teeth, he said that one was shorter, and pointed to the brushing teeth card. Similarly, when questioned about sleeping at night and eating lunch, he said sleeping at night was longer. This progression can be seen in Table 1.

Table 1. Kris’s Organization Process and Reasoning

Initial Order	“Short amount of time” Brushing teeth Watching a movie		“Long amount of time” Sleeping at night Eating lunch	
Final Order	Brushing teeth	Watching a movie	Eating lunch	Sleeping at night
Reasoning	“Like five minutes.”	“You can watch much.” “Like 15 minutes.”	“About 10 minutes.”	“About 30 minutes.”

Kris’s initial ordering of the events was more of a classification than a seriation, where he “put everything that [was] alike together” (Piaget, 1985, p. 100) under the umbrellas, “short” or “long” amounts of time. One question that I overlooked during the interview was why he chose to group these events in such a way. Instead, I focused on his sequencing of events from the original *Experiment 3* (Tillman & Barner, 2015). Though I do feel this was a missed opportunity to better understand his overall durational reasoning.

After creating his final “timeline”, Kris began by explaining his reasoning from the middle of the events, with watching a movie. He stated that it was placed in the middle because “you can watch much”. Here, Kris seemed to be equating the amount of activity (“much”) with duration, a common conception in early time reasoning (Piaget, 1969).

As Kris continued his explanation, he switched his reasoning from activity of an event, to standard units of time—explaining that brushing teeth was five minutes, eating lunch 10 minutes, and sleeping at night 30 minutes. I asked Kris about his use of these specific durational words, which prompted him to add that watching a movie took “like 15 minutes”. From this, it seems that Kris knows that the word “minutes” can be used to explain lengths of time, an understanding highlighted by Tillman and Barner (2015). However, looking at how he now ordered the minutes of each event (5, 15, 10, 30), the value does not align with the chronological duration.

I asked Kris specifically about the chronology of the durations, comparing the order of the numbers given versus the order the events were placed. This comparison seemed to confuse Kris, as highlighted in the following excerpt.

A (Author): You said this was five minutes [taps brushing teeth card], 15 [watching movie card], 10 [eating lunch card], 30 [sleeping at night car], is that the right order?
K (Kris): I don’t know.
A: Do numbers go 5, 15, 10?
K: No/
A: Do they go [flipped watching movie and eating lunch cards] 5, 10, 15?
K: No. [furrows his brow and shakes his head]
A: [Puts cards back] Okay, so you think it takes you longer to eat than watch a movie?
K: Yup!

Kris seemed confused by the rearranging of the cards, and focused more on his perceived duration of each event rather than the numbers he had assigned to the minutes of each event. This aligns with Tillman and Barner’s (2015) findings that overall, the five-year-old participants performed better on organizing the duration of familiar events than on organizing timed durations (such as nine seconds or two minutes). There may also be a connection to Kris's concept of number, although more study would be needed to make such a claim.

Sam: Age Six

Sam quickly organized the four events: first, brushing teeth on the far left (shortest duration), then eating lunch, watching a movie, and finally sleeping at night on the far right (longest duration), as shown in Table 2. She was very deliberate about pulling the cards in increasing durational order, moving from shortest to longest activity, and did not verbalize her reasoning during this process, as the other two participants did.

Table 2. Sam's Organization Process and Reasoning

Final Order	Brushing teeth	Eating lunch	Watching a movie	Sleeping at night
Reasoning	"That's two minutes." "I sometimes count it of a second and then made it up to two minutes." "30 seconds in a minute, 60 seconds in two minutes."	"Pretty easy and you can sometimes shove it in your mouth."	"It's not longer than, if you would start like in the middle of the night it wouldn't even be in the middle of the night, it would be, like, before the middle of the night."	"Would be like all the night, it'd be like all the day."

When asked to explain why she chose to order the events as she did, Sam started her explanation with watching a movie, similar to Kris. However, where Kris explained the duration of a movie through action, as "watching much", Sam explained the duration compared to the time of day one might watch a movie (night) and the progression of time chunked as "the night". Sam went on to compare watching a movie being the middle of the night to sleeping at night being all night and all day. This reasoning seemed to align with the use of start and end points in measurement reasoning (D. Earnest, personal conversation, May 27, 2019; Kamii & Russell, 2012; Piaget, 1969), where darkness and light serve as figural endpoints. Because in Sam's reasoning she could watch a movie that both starts and ends in the dark, but sleeping at night starts in the dark and ends in the day (light), the duration of sleep is longer than watching a movie.

From using physical indicators of time (sun-up versus sun-down), Sam changed her durational measure to explain eating lunch through effort and activity. This use of action to explain time is a common characteristic of reasoning intuitively about time (Piaget, 1969). When asked how "shoving food in her mouth" changes the amount of time the activity takes, Sam responded by saying:

S (Sam): You would just get a handful and shove it in your mouth. [pretends to quickly shove handfuls of food in her mouth]
A (Author): How would that change how long it took you to eat your lunch?
S: Um, uh [looks up, hesitates for six seconds] I don't know.

Sam seemed to correlate her actions to the amount of time that something can take to complete, but she could not reason how or why. Piaget (1969) explained that young children are unable to conserve velocity, believing that quicker actions equate to more time. Sam seemed to have moved past this reasoning. When she modeled eating quickly to explain how eating your lunch is a shorter activity than watching a movie, she seemed to demonstrate the inverse relationship between action and duration (i.e., moving faster results in a shorter duration).

Finally, when asked why she put brushing teeth at the far-left side of the timeline, Sam quickly responded, "cause that's two minutes". This was the first time Sam used standard units. This was also the first time during Sam's explanation that she did not hesitate in her response. I asked how she knew it took two minutes to brush her teeth, which cause her to pause repeatedly and stumble through an explanation of counting two minutes, which consisted of counting 30 seconds (which she explained was a minute) twice.

From other interviews I have conducted with children of Sam's age, "two minutes" has been repeatedly given as the duration of brushing teeth. Through many of these other interviews, children have explained that it is two minutes because: "That's what my mom told me", "That's what the dentist said to do", and "That's how long my toothbrush counts to". I cannot say if any of these accounts explain Sam's reasoning, however, with her inaccurate calculation of 30 seconds to a minute and the immediacy of her initial response, I would conjecture that she has been told by an outside source that brushing teeth takes two minutes.

Casey: Age Seven

Casey's process of organizing the events was much slower than the other two participants. She began by moving watching a movie, eating lunch, and sleeping at night to the right side (longer durations) and brushing teeth to the far left (shorter duration). She then arranged the three "longer duration" activities along the "timeline", as seen in Table 3.

Table 3. Casey's Organization Process and Reasoning

Initial Order	Brushing teeth		Watching a movie Eating lunch Sleeping at night	
Final Order	Brushing teeth	Sleeping at night	Eating lunch	Watching a movie
Reasoning	"It's just easy so you can go really quick."	"Takes a long time because I'm not that tired."	"I'm a really slow eater."	"Like an hour, that's why it's really long."
		"Not as long much time as eating."		

From the beginning, Casey seemed to focus on the effort she took to complete each activity as an explanation for the amount of time it took to complete. Casey began by stating that brushing her teeth was "easy, so you can go really quick". This reasoning seemed different than Sam's description of eating being "easy and you can sometimes shove it in your mouth", in that Casey explained that *because* it is easy you can go fast. This causal relationship seems to demonstrate a more advanced conception about the inverse relationship between velocity and duration, and was reaffirmed by Casey later in the interview, as shown below.

A (Author): If you brush your teeth fast, what does that do to the amount of time it takes you to brush your teeth?
C (Casey): Less, fast, it takes less amount of time because you go fast.
A: So faster makes less amount of time?
C: Yes.
A: If you were eating your lunch and you wanted it to take longer, what could you do?
C: Like eat really slow.

Prior to this exchange, Casey had explained that she was a slow eater when she compared sleeping at night as taking a long time but not as much time as eating. This comparison may seem illogical, since for most people sleeping at night takes longer than eating, however, I believe that Casey interpreted sleeping at night as the act of falling asleep not as the entire duration of being asleep. Thus, Casey described falling asleep as taking a "long time" because she's "not that tired".

When Casey then compared the duration of falling asleep against eating lunch, Casey seemed to, "Directly compare two objects with a measurable attribute in common" (CCSSI, 2020). In other words, Casey took the attribute time and compared the two events. And while I would argue that to Casey, time is the effort she exerts to complete an activity—not necessarily the operationalized time Piaget (1969) discussed—she still seemed to be correlating two events with this common attribute to order their lengths.

Similar to Sam, when I asked Casey about the duration of the final event, watching a movie, she flipped her reasoning from effort (easy, really slow) to standard units (an hour). Once again, Casey compared the events by stating, "[watching a movie is] like an hour and none of these stuff (points to other three cards) takes like an hour, so that's why it's really long". This is interesting, though, because unlike her previous comparison of two efforts, this is comparing effort against a standard duration. Unfortunately, I did not ask how she knew that a movie took an hour, however, this mixing of temporal conceptions may indicate some transition in reasoning, from action being a proxy for duration to time being a standardized quantity.

Discussion

From a linguistic framework, Tillman and Barner (2015) concluded that "the lexical category that children form for duration words is not a simple grouping of these words, but rather a structured, ordered scale that reflects some knowledge of the relative temporal magnitudes of the words" (p. 73). This scale, Piaget (1969) might have argued, results from the operationalization of time—from intuitive perceptions of time to the coordination of temporal and spatial relations.

Across the three interviews, Kris, Sam, and Casey demonstrated varied conceptions about time measurement as they organized the four events (watching a movie, brushing their teeth, sleeping at night, and eating lunch) along their figural timelines. But, by providing the opportunity for these children to share their reasoning, several common themes arose that echoed past mathematical research, most notably the use of action as a proxy for duration.

Piaget (1969) noted "that to primitive intuition, time is simply the 'prolongation of activity'" (p. 60) and was representative of pre-operational thinking. Kris, Sam, and Casey all explained time through activity, where "watching much" or "eating slowing" justified the placement of their duration. This attention to activity may indicate an intuitive perception of time, which, for this age range, would align with past research (Long & Kamii, 2001; Piaget, 1969; Russell, 2008).

Additionally, both Sam and Casey went further to explain how ease of activity created shorter durations (i.e., it's easy so it doesn't take much time). Perceptively, though, I believe that Sam and Casey were using the term "easy" as a substitute for "routine". Some activities such as waiting for paint to dry or being a passenger on a long road trip, might be considered "easy" as they involve little effort, however, the actual duration of these activities could be quite long. For Sam and Casey, the act of eating lunch or brushing teeth, may be so routine that it has created an instance of "temporal compression" (Flaherty, as cited by Evans, 2004, p. 736), where low levels of stimuli cause low levels of information processing, resulting it time feeling like it passes quickly. Conversely, when Casey described falling asleep at night as "[taking] a long time because I'm not that tired", it seems she was describing a "protracted duration" (Flaherty, as cited by Evans, 2004, p. 737), where the event felt long, despite the fact that she situated it on the shorter duration side of her timeline. Both temporal compression and protracted durations are based on perception, thus, indicative of intuitive reasoning of time.

Beyond the focus on activity, all three children used standard units of time to explain their sequence of events. This was not unexpected for two reasons. First, Common Core places time instruction in first grade (CCSSI, 2020), which both Sam and Casey had experienced (as a first and second grader, respectively). And second, much of what young children understand about time is the result of what their parents and other adults have told them (Earnest, 2018; Lareau, 2011; Piaget, 1969). This was most evident by Sam's description of brushing her teeth taking two minutes with no real explanation of why. However, for all three children, the actual durations they understand for these standard units cannot be fully analyzed given the current data. Kris seemed to understand that minute was a word to describe time, but he sorted the durations as 5, 15, 10, 30 minutes and was not able to reason about the ordering of the numbers. Casey conceptualized an hour as being a long amount of time, but never explained what an hour meant beyond this. There is clearly more to learn about how children reason about these durational units as measurements of time and their everyday activities.

Tillman and Barner (2015) presented a broad quantitative analysis of how children order durational words and experiences. Many of their findings aligned with previous mathematical research on children's conceptions of time (Earnest, 2015, 2018; Harris, 2008; Kamii & Russell, 2012; Piaget, 1969). It is encouraging to see common findings across fields of research, linguistics and mathematics.

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GEOMETRY AND MEASUREMENT:

BRIEF RESEARCH REPORTS

ACKNOWLEDGING NON-CIRCULAR QUANTIFICATIONS OF ANGULARITY

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In recent years, researchers have advocated for measuring angles by measuring circular arcs (i.e., circular quantifications of angularity). Leveraging results from a teaching experiment with ninth-grade students, I demonstrate the existence of non-circular quantifications of angularity, which have not previously been acknowledged in existing empirical research or standards.

Keywords: Cognition, Geometry and Geometrical and Spatial Thinking, Measurement

Angularity is an important geometric attribute throughout K–12+ curricula. It arises in many contexts including classifying shapes, congruence, similarity, transformations, construction, proof, coordinate systems, and trigonometry. Despite this prevalence, few studies have investigated how students reason about angularity (Smith & Barrett, 2017). At the undergraduate level, researchers have argued that robust quantifications of angularity are critical for trigonometry (Akkoc, 2008; Moore, 2013). However, research on angularity with high school students is especially scarce. This presents a problem. In fact, Moore (2013) noted, "future studies that investigate secondary students' quantification of angle measure are needed…" (p. 243). To this end, I conducted a teaching experiment with ninth-grade students to understand how they quantified angularity. In this report, I elaborate the quantifications of angularity indicated by two students and consider implications of these results.

Theoretical Components and a Hypothesis

This study was informed by principles of quantitative reasoning (Thompson, 1994; 2011). A quantity is an individual's conception of a measurable attribute of an object or situation; quantities are mental constructions consisting of three interrelated components: (a) an object, (b) an attribute, and (c) a quantification. A quantification involves a collection of mental operations that an individual could carry out to measure an attribute or interpret a measurement value in a given context. For example, upon assimilating an angle model an individual might establish a goal of determining how open the angle model is in degrees; alternatively, an individual might be asked to consider how to make a one-degree angle. In these instances, the collection of mental operations activated would be components of the individual's quantification of angularity.

Following Thompson's (2008) first-order conceptual analysis, Moore (2013) elaborated that quantifying angularity involves (a) considering a circle centered at an angle's vertex, (b) making a multiplicative comparison of two lengths (e.g., arc length and circumference), and (c) recognizing this ratio is invariant across all possible circles centered at the angle's vertex; for example, a one-degree angle "subtends 1/360 of the circumference of any circle centered at the vertex of the angle" (p. 227). This approach is compatible with the CCSSM standards, where angle measure is explicitly introduced in Grade 4. I refer to these quantifications of angularity as *circular quantifications of angularity* because they leverage multiplicative comparisons of arcs and other circular lengths (e.g., circumference, radius, etc.). Circular quantifications of angularity yield coherent interpretations for angle measure across standard units of angular measure; thus, such quantifications of angularity are productive, particularly for the study of precalculus mathematics and beyond. However, circular quantifications of angularity are sophisticated, and angle measure is introduced relatively early in curricula. Therefore, it is reasonable to question: Is it possible for students to quantify angularity in

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other ways? Might these other quantifications support students in later constructing circular quantifications of angularity?

When one discusses the measure of an angle, one is describing the size of the interior of the angle (Hardison, 2019). The major hypothesis investigated in the present study was that students might establish productive non-circular quantifications of angularity by enacting extensive quantitative operations on angular interiors. Extensive quantitative operations are operations that introduce units (Steffe, 1991). In length and area contexts, Steffe & Olive (2010) provide numerous examples of such operations including iteration (imagining making and uniting copies of an established unit to produce a composite whole) and partitioning (imagining the simultaneous production of equal-sized parts within an established whole).

Methods

The data and analyses presented in subsequent sections are drawn from a teaching experiment (Steffe & Thompson, 2000; Steffe & Ulrich, 2013) conducted over an academic year in the southeastern U.S. with four ninth-grade students. At the time of the study, all students were enrolled in a first-year algebra course. The overarching goal of the teaching experiment was to investigate how the students quantified angularity and how these quantifications changed throughout the study (see Hardison, 2018); the author served as teacher-researcher for all teaching sessions. Throughout the study, students engaged in mathematical tasks involving rotational angle models (e.g., rotating laser) and non-rotational angle models (e.g., hinged wooden chopsticks). Each student participated in 13–15 video-recorded sessions, which were conducted individually or in pairs approximately once per week outside of their regular classroom instruction; each session was approximately 30 minutes in length. The records of students' observable behaviors (e.g., talk, gestures, written responses, etc.) were analyzed in detail via conceptual analysis (Thompson, 2008; von Glasersfeld, 1995). In this report, the activities of two students, Bertin and Kacie, are foregrounded to illustrate the existence of non-circular quantifications of angularity and to evidence that the construction of circular quantifications of angularity can be supported by non-circular quantifications.

Data and Findings

The results in the following sections are structured around the analysis of four purposefully selected examples of mathematical interactions with Bertin or Kacie.

Angular Repetition and Iteration

To establish models for students' ways of reasoning at the onset of the teaching experiment, students were asked to solve a variety of tasks. One such task involved two pairs of hinged wooden chopsticks: one short pair that could be freely adjusted and one long pair which was fixed. Each student was asked to set the short pair of chopsticks to be four times as open as the long pair of chopsticks. When presented with this task, Bertin proceeded by immediately tracing four adjacent copies of the long chopsticks on a piece of paper and setting the short pair of chopsticks to contain these four adjacent copies.

I refer to Bertin's physical actions as angular repetition. Through angular repetition, Bertin produced an angle model four times as open as the given angle model. Because Bertin engaged in angular repetition without hesitation, I infer he imagined uniting adjacent copies in visualized imagination prior to his physical actions. In other words, the immediacy of Bertin's activities suggests an anticipation indicative of the mental operation of angular iteration. Nothing in Bertin's observable activities indicated that Bertin was considering circular arcs as he solved this task; instead, the figurative material subjected to angular iteration was the interior of the given angle model. Because this task was from his initial interview, Bertin's way of reasoning was previously established and not engendered by the teacher-researcher. Thus, Bertin's activities indicated he may have constructed a non-circular quantification of angularity prior to the study.

One-Degree Angles

Three months later, Bertin was asked how to make an angle with a measure of one degree. Bertin replied, "If you get a ninety-degree angle [*gestures a right angle*], you can divide that into nine so it would be like ten degrees each, and then you can divide each one of those into ten, but it would need something like really really small to write with." The gesture Bertin enacted indicated he first brought forth a familiar angular template in visualized imagination, specifically a right angle. His response also indicated he had assigned this right-angle template a measure of 90°, thereby positing it as a composite unit. Bertin then imagined partitioning the right angle into nine 10° parts, each of which he subsequently partitioned into ten 1° subparts. Thus, Bertin indicated producing 90 one-degree angles within a right angle in visualized imagination. As in the previous example, Bertin's way of reasoning did not leverage circles or arcs; instead, Bertin demonstrated he had established a normative conception of a one-degree angle via extensive quantitative operations enacted on the interior of a familiar angular template.

Contraindication of a Circular Quantification of Angularity

Five months into the study, Bertin was presented with tasks involving central angles to determine whether he had constructed a circular quantification of angularity. In one task, he was asked to determine the measure of a central angle in degrees, given that the length of the green subtended arc was one inch and the green circle's circumference was six inches (Figure 1 left).

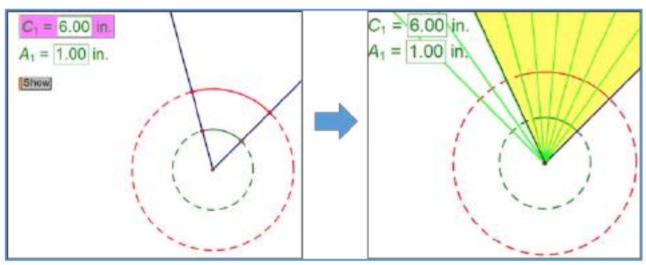

Figure 1: A Central Angle Task (left) and a Model of Bertin's Solution (right)

After an approximately 10-second pause, Bertin tentatively responded, "like seventy," and explained that he "kind of based off of ninety degrees" as he dragged the cursor to form a right angle containing the given central angle. When pressed for how he might precisely determine the measure, Bertin indicated with the cursor that he imagined partitioning the right angle into ten-degree parts; he then counted how many of these parts were contained in the central angle's interior. The green lines in Figure 1 (right) approximate how Bertin dragged the cursor to indicate ten-degree parts. Afterwards, Bertin reiterated, "it's like around seventy somewhere."

Bertin's activities indicated his reliance upon a non-circular quantification of angularity and were remarkably similar to his production of one-degree angles: he started with a familiar template (a right angle); posited this right-angle template as a 90° composite unit; and he partitioned it into nine 10° parts, which he leveraged to solve the task at hand. Bertin's solution is commendable; however, notably absent are any reference to the given measures for the arc or circumference. Thus, Bertin's activities contraindicate a circular quantification of angularity.

Evidence That Non-Circular Quantifications Can Support the Circular Counterpart

To illustrate that non-circular quantifications of angularity might support the construction of the circular counterpart, I present and analyze Kacie's activities on a final interview task involving a central angle. At this point, Kacie had developed a non-circular quantification of angularity similar to

Bertin's. In particular, Kacie had established the following way of reasoning: if *n* adjacent copies of an angular section exhaust a full angle, then the angle has a measure of $360°\div n$. The central angle task from Kacie's final interview involved determining the measure of a blue central angle subtending a green arc 3.47 cm long in a circle of circumference 22.83 cm. Kacie's reasoning is described in the transcript, which has been edited for brevity.

T: How would you determine the measure of the blue angle?
K: Um, [*11s pause*]. You could subtract, um, the three point four seven and the twenty two point eighty three. And that might give you your measurement. Because that's what that angle is like that – well, no. Just kidding. [*16s pause*]. Yeah. I guess you could subtract.
T: And what would that subtraction tell you?
K: Um [*4s pause*]. No! Wait. You could do twenty two point eight three divided by – wait, no. Yeah. Divided by three point four seven and that would give you the number of times the angle would go around the circle. And then you could do … three hundred sixty divided by that number and then that would give you the measurement of the angle.
T: Can explain why that works?
K: … well twenty two point eighty three divided by three point forty seven … would give you a number of how many times the green arc could go around the circle. And then that would give you how many times the blue angle would need to go to the circle to reach back to its starting point. And then if you did three hundred and sixty divided by the number of times the blue angle needed to go around it would give you the measurement

Kacie used the known arc length as a unit for measuring the known circumference. She considered the quotient of these lengths (i.e., $22.83\div3.47$) without enacting the numerical division and interpreted this quotient as how many times the green arc "could go around the circle." Kacie also interpreted this quotient in terms of the central angle, which indicated she mentally united the arc and the central angle and was subjecting these united objects to the same mental operations. Having established the number of adjacent copies of the central angle needed to exhaust a full angle, Kacie relied on a previously established way of reasoning to solve the task. In short, Kacie was able to solve this task involving arc length by leveraging the non-circular quantification of angularity she had previously established.

Discussion, Conclusions, and Implications

Bertin and Kacie developed powerful non-circular quantifications of angularity reliant upon (a) establishing mental templates for familiar angles, (b) positing these familiar templates as composite angular units, and (c) making and measuring other angles via the application of extensive quantitative operations to angular interiors. These non-circular quantifications of angularity have not previously been identified and celebrated in empirical literature. Such quantifications are productive and should be recognized in classrooms and curricular standards along with circular quantifications. I hypothesize non-circular quantifications of angularity naturally precede, and are necessary for constructing, circular quantifications of angularity. Future studies are needed to investigate this hypothesis; however, Bertin's spontaneous angular repetition during the initial interview and his description of one-degree angles evidence that non-circular quantifications can precede circular quantifications, and Kacie's activities evidence that non-circular quantifications of angularity can support the circular counterpart. Additional research is needed to determine the prevalence of circular and non-circular quantifications of angularity at various grade levels.

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PROSPECTIVE HIGH SCHOOL MATHEMATICS TEACHERS' USES OF DIAGRAMS AND GEOMETRIC TRANSFORMATIONS WHILE REASONING ABOUT GEOMETRIC PROOF TASKS

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The purpose of the study was to examine prospective teachers' uses of diagrams and approaches to congruence while solving proof tasks. Eight prospective high school mathematics teachers were given two proof tasks to solve at the beginning and end of a mathematics education course. Analysis revealed that at the beginning of the course preservice teachers' approached congruence proofs using a perceptual or correspondence approach and interacted and used a descriptive mode of interaction with diagrams. At the end, their approaches to congruence included more instances of transformations and measures and their interactions with diagrams included fewer uses of the descriptive mode and more instances of representational and functional modes.

Keywords: Geometry and Geometrical and Spatial Thinking; Reasoning and Proof; Representations and Visualization

Introduction and Related Literature

The study of geometry in high school is often students' first experiences with conjecturing, justification, and formal proof. Most mathematics standards recommend that students be familiar with different approaches to proof that include synthetic, analytic, and transformational methods (Coxford, 1991; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Yet many teachers have not had experiences using geometric transformations to write formal proofs. There is also research that suggests students' and teachers' interactions with diagrams can support their conjecturing and proving activities (Herbst, 2004; Gonzalez & Herbst, 2009; Chen & Herbst, 2013).

There is significant research related to students' and teachers' abilities to write formal proofs. In a recent research synthesis, Stylianides, Stylianides, and Weber (2017) identified three perspectives on proof abstracted from the literature: "proving as problem solving, proving as convincing, and proving as a socially embedded activity" (p. 239). We adapted a proving as problem solving perspective in which participants were presented four proof tasks to solve in a task-based interview setting. Within this perspective, Selden and Selden (2013) make distinctions between the formal-rhetorical part of a proof and the problem-centered part of the proof. The former focuses on the logical sequencing of steps when writing a formal proof while the latter refers to the creative problem solving that is involved in considering how one might go about proving a conjecture. The problem-centered part of proving is similar to the identification of a proof plan (Melis & Leron, 1999) or proof idea (Reiss, Heinze, Renkl, Grob, 2008) that occurs prior to the writing of a formal proof. It is within this area that we focus our analysis on describing how preservice teachers interact with diagrams while constructing proofs involving congruence.

Gonzalez and Herbst (2009) investigated high school students' conceptions of congruence and then identified perceptual (PERC), correspondence (CORR), transformational (TRANS), and measure preserving (MeaP) conceptions of congruence. The perceptual conception is one that relies on visual information provided in a diagram to determine if two objects appear congruent. The correspondence conception is one in which two objects are congruent if corresponding sides and angles are congruent. The transformational conception uses properties of geometric transformations to map one geometric object to another. The measuring conception relies on measures of objects to determine if

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

they are congruent. The tasks selected for the current study could be approached by students holding any of these four different conceptions of congruence.

Conceptual Framework

Building on the work of Duval (1995), Herbst (2004) proposed four modes of students' interaction with diagrams as empirical (EMP), representational (REPR), descriptive (DESC), and generative. With empirical interactions, the actor has proximal, physical experiences with diagrams. The actor's operations on diagrams (measuring, looking, drawing) is limited to actual properties of the physical drawing. This identifies the diagram as an object; that is, a diagram is taken as a figure without semiotic mediation (Chen & Herbst, 2013). In the representational mode, the actor uses distal physical experiences to make depictions about the diagram and the diagram is seen as a sign of the object. Herbst (2004) also suggests two other modes of interactions, descriptive and generative, to characterize the role of diagrams in the process of proving. In the descriptive mode the actor sets up a distal relationship with a diagram while making statements that could be read off the diagram. Also students use visual perception when they are doing proofs and verify this perception by additional symbols like hash marks or arcs. This mode is a hybrid mode that students use both visual perception to make conjectures like the empirical mode and also see diagrams as symbols to justifying their statements like the representational mode when proving (Chen & Herbst, 2013). Conversely, within the generative mode, students make sensible changes that are not originally given and make "reasoned conjectures" in predicting and making hypotheses about the figure. Students interact in proximal relationship and work generatively with diagrams by using definitions and properties of the geometric objects as well as making changes. Gonzalez and Herbst (2009) proposed the functional mode (FUNC) of interaction to define students' interactions with dynamic geometry diagrams. They describe how students relate outputs and inputs when they use the dragging feature of the dynamic geometry software. Within this mode the combination of dragging and measuring provides students opportunities to explore relationships. Students may also check invariants when making changes to the diagram by dragging and set up the same relation between several diagrams. The purpose of this study was to examine preservice teachers' (PT) interactions with diagrams as they solved proof tasks that were amenable to synthetic or transformational approaches.

Context and Methods

The current study took place at a large public university. Eight preservice (PT) high school mathematics teachers (four males and four females, identified as S1-S8) enrolled in a senior level mathematics education course agreed to participate. Approximately three weeks of the course were devoted to the study of transformations, congruence, and similarity. An emphasis on proof and justification was included throughout the course which addressed number (real, complex), rates of change, functions (linear, exponential, logarithmic), and statistics. The participants were required to solve three tasks at the beginning and four tasks (three were the same) at the end of the semester. At the beginning, PTs were provided iPads with the ShowMe app (interactive whiteboard app) and asked to record themselves solving the tasks. At the end, PTs were invited to participate in task-based interviews; they were provided with the same materials and technology they had used in class. For this paper, analysis of the first two tasks is provided. These tasks were selected and adapted from high school mathematics curriculum and prior research that emphasized transformational and synthetic approaches to proof. Task 1 was adapted from the Mathematics Vision Project Secondary II Curriculum (Module 5, page 16, https://www.mathematicsvisionproject.org/secondary-mathematics-ii.html). Task 2 was modified from professional development materials created by Jim King that were used to prepare teachers to teach congruence using a transformation approach (Figure 1).

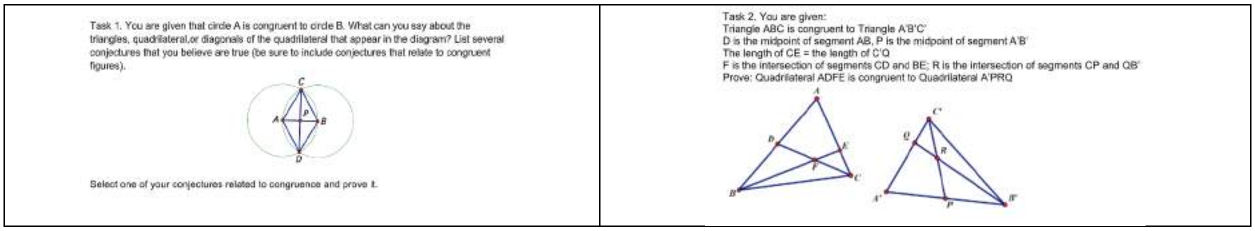

Figure 1. The first two geometry tasks used in the study

For each of the eight PTs, video recordings of their work on the first two tasks from the beginning and end were reviewed and coded to characterize their approach to proving congruence and coded to identify how they interacted with diagrams. The data were analysed by the researchers independently in line with the theoretical framework. Afterwards researchers discussed and agreed on the codes for trustworthiness and consistency.

Results

The most common conception of congruence involved a combination of the CORR with PERC based reasoning. At the beginning, only one PT (S4) used a TRANS approach on the first task. This PT and one other (S6), changed the placement of the second triangle in the second task for ease of determining which sides and angles corresponded to one another. Although this repositioning involved a rotation, this strategy was not used to justify why the quadrilaterals were congruent and thus not coded as a TRANS approach. At the end three PTs (S3, S4, S8) used a TRANS approach. Also only two PTs chose to use dynamic geometry in solving the first task (S1, S8). When examining PTs' interactions with diagrams, we note that there were 11 instances of the DESC, two instances of the REPR, and three instances of the GENE at the beginning. At the end, there was a greater variety in the types of interactions with diagrams. There were seven instances of the DESC, three instances of the REPR, three instances of GENE, and two instances of FUNC mode.

On the first task, all teachers at the beginning made conjectures about the two equilateral triangles and the four congruent right triangles created by the circles. Three of the teachers (S1, S4, S8) made conjectures about the quadrilateral and among these only S4 used reflections in his proof. Most of the PTs noticed that the sides of the quadrilateral are radii of the two circles and used that information to prove triangles congruent. The PTs who made conjectures about the quadrilateral proved it was a parallelogram (S1) or a rhombus (S4, S8). Almost all PTs (except S4) built their conjectures based on the PERC. Even if teachers approached the first task by PERC, they also used a CORR (coded as PERC-CORR). Only S4 utilized the TRANS during the reasoning process at the beginning. None of the teachers used dynamic geometry at the beginning. At the beginning teachers generally interacted with figures DESC, but there are GENE (S2, S4) and a REPR instances (S8). At the end S1 and S8 tried to prove their conjectures for Task 1 using dynamic geometry. Especially, S1 measured all the line segments and used the drag test to justify his/her conjectures (congruent triangles) and S8 used reflections as well as dragging. Also there was one MeaP, three instances of PERC, two instances of CORR, and two instances of TRANS conceptions. From the point of interaction with diagrams, PTs' interactions have varied at the end as three instances involve a DESC, two instances of REPR, one instance of GENE, and two instances of FUNC mode of interaction with diagrams.

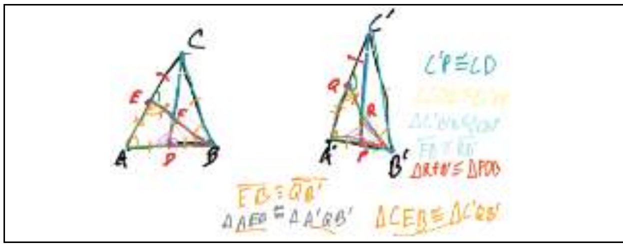

Figure 2: Example of correspondence approach

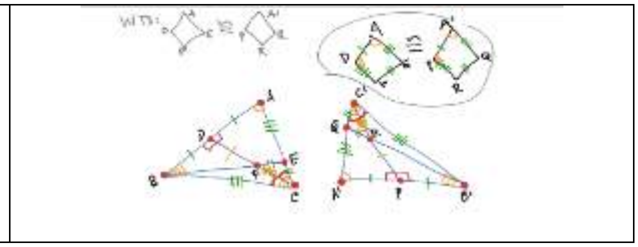

Figure 3: Example of generative mode

On the second task, at the beginning most teachers (S1, S2, S5, S6, S8) used a CORR approach combined with the PERC. They often started the task by marking the given information and using the fact that the two triangles are congruent to identify corresponding parts. In the process of constructing their proof they often made inferences about congruent objects from the diagram using visual perception. At the end, there were no instances of a completely PERC and three instances of the use of TRANS (S3, S4, S8), one of which was combined with a CORR. PTs' descriptions of each of the TRANS described the transformation (e.g., rotation, translation) and stated where points would be mapped, but did not specify a center and angle of rotation or a translation vector (Figure 4). Four PTs used a combination of perceptual and correspondence approaches and one PT decided to skip this question.

Discussion

When looking across the eight participants and two tasks implemented before and after the course we observed that there were no instances of EMP interactions. This is not surprising since most observations of empirical interactions with diagrams occur before high school (Herbst, 2004). Although there was no change in the number of GENE, only one participant was the same and two new participants used this mode. The number of REPR modes increased by one and the number of DESC decreased from 11 to 7. The appearance of the FUNC mode was identified in participants who used dynamic geometry. Analysis of the conceptions of congruence across participants shows more variation in the ways PTs reason. While at the beginning participants wrote proofs that primarily used PERC and CORR approaches, at the end participants used MeaP when using dynamic geometry and used transformations more often.

The identification of the interaction between PERC and CORR approaches to congruence was useful to the researchers in describing how PTs engaged in proof problems using an approach with which they were familiar (correspondence), but when unsure about how to continue made inferences from the diagram based on visual information to proceed with the proof. Their proof idea, proving two figures congruent using a CORR approach, was correct, but it was in the details of formalizing that idea that they encountered challenges. While Selden and Selden (2013) make distinctions between the formal-rhetorical part of a proof and the problem-centered part of the proof the challenge experienced by many of our participants seemed to lie somewhere between these two activities. While in many cases they understood how to go about solving the proof problem, it was in the details of logically moving from one step to the next where they encountered challenges. This aspect of proof writing might be worth examining in future research.

Acknowledgments

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INTERACTIONS WITH GEOMETRIC FIGURES: A CASE WITH GRADUATE STUDENTS

INTERACCIONES CON FIGURAS GEOMÉTRICAS: UN CASO CON ESTUDIANTES DE LICENCIATURA

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Geometric figures can play a mediating role in students' reasoning, solving a problem, or justifying the truth of a proposition. These objects have been studied in various investigations, where the interactions that students have with figures that are constructed by the teacher or presented in a textbook have been analyzed. In this report, we describe an example of the interaction that undergraduate students had with figures constructed by us, prior to solving a task; and constructed by them during said resolution. The results show a tendency towards the empirical operations of the students on the figures with which they interacted; and the need to incorporate auxiliary elements in the figures, such as labels or segments, in order to solve the proposed tasks.

Keywords: Geometry and Geometrical and Spatial Thinking; Representations and Visualization; Reasoning and Proof; Student-figure interaction; Empirical operations

In this work we show part of an investigation with undergraduate students, in which the role of a geometric figure during the resolution of different tasks was characterized. Taking into account the work of Herbst and Arbor (2004), we describe the modes of interaction that the students had with the figures, according to the way in which they were presented: accompanied by a proposition before carrying out a task; or constructed by the student during the resolution of the task.

Literature review

Various studies have analyzed the characteristics and functions of geometric figures during the process of solving a task. Below we show a description of some investigations that have been relevant in the study of the geometric figure, and the main ideas that have emerged around this topic.

Spatial and conceptual properties of a geometric figure

A geometric figure is considered by Fischbein (1993) as a mental representation where spatial and conceptual properties of a geometric object interact. Spatial properties are those that have to do with the shape, size and position of the figure; while the conceptual properties are related to the abstract idea of a geometric object, with characteristics such as ideality and perfection.

The heuristic role of the geometric figure

Duval (1995) suggests that the geometric figure can play a heuristic role in the reasoning of an individual to solving a task. From the point of view of the author, the representation of a geometric figure can help find the main idea of a proof or support the solution of a problem.

To describe the heuristic role of the geometric figure, Duval (1995) defines four forms of apprehension: perceptual, sequential, descriptive, and operational. Particularly, the heuristic role of a figure can be presented with the operative apprehension, since this refers to the modification of a figure in such a way that ideas for the solution of a task are produced.

Difference between drawing and geometric figure

Laborde (1996) points out that a drawing is a material identity on a support (paper, for example), which acts as a signifier of a theoretical referent. In this sense, the author defines the geometric figure as a set of pairs, whose components are a geometric reference and one of the possible drawings that represent it. Relations between the drawing and its referent are constructed and interpreted by those who produce or reads the drawing, so the meaning of a geometric figure is determined by who interacts with it.

In this investigation we use the term geometric figure when we refer to the graphic representations of objects or geometric situations, which are described in a proposition.

Research Problem

Many of the studies regarding the geometric figure analyze the interactions that an individual has with figures that are constructed and presented by a teacher or in a textbook, leaving aside the interactions with figures that are constructed by students during the resolution of a task. The purpose of this research is to investigate and describe the different interactions that undergraduate students had with geometric figures presented in two different ways: presented by us and constructed by them.

Theoretical framework

The analysis of the obtained results was made considering what was established by Herbst and Arbor (2004) regarding the modes of interaction between a subject, a diagram and a geometric object. The authors use the term diagram equivalently to what we refer to as a geometric figure; while by geometric object they refer to the referent represented in the diagram. The modes of interaction that had an impact on the results obtained are described below:

Empirical: is that interaction where a subject performs physical operations on a diagram (for example, measuring, observing or incorporating new elements). The arguments presented about these actions are restricted by the characteristics of the diagram and the properties of the instruments used (ruler, compass, etc.). In this sense, the subject considers the diagram as an equivalent of the geometric object it represents, allowing it to communicate its results based on the operations it performs.

Representational: in this mode of interaction, the subject interprets the geometric object through the graphic representation of the diagram, limiting himself to considering only the characteristics of said object established in a proposition. With this interaction the diagram does not provide additional knowledge about the geometric object, but only acts as a sign of it. Unlike the empirical interaction, the characteristics of the diagram are not restricted by physical operations, but by the geometric knowledge of the subject to interpret and represent what is established in a proposition.

The remaining modes of interaction are descriptive and generative. The first has to do with the simultaneous interpretation of the signs shown in a diagram with the properties established in a proposition, while the second suggests the joint construction of a diagram and a deductive proof. However, these modes of interaction are not reflected in the results presented here.

Method

This section describes the characteristics of the research participants, as well as the way the tasks implemented in the research were presented.

Participants

The research participants were students between 20 and 22 years old, who were chosen considering the experience suggested by their respective careers (BUAP, 2011), particularly in geometry courses.

Implemented Tasks

The tasks were applied in two different ways: a group was presented with tasks whose content was a proposition and a figure that represented it; while another group was presented with the same tasks, but showing only the proposition and suggesting that the respective figures were constructed. In each task, it was requested to justify a property of the figure based on the characteristics described in the proposition.

Results

The following results reflect significant aspects of students behavior, when interacting with pre-constructed figures and with figures constructed by them.

Results about the task with figure

The task that was implemented was the following: in the figure (Illustration 1) the parallelogram ABCD is presented, from which it is known that the points M and N are midpoints of the sides DC and AB respectively. How do you justify that the AM and CN segments are parallel?

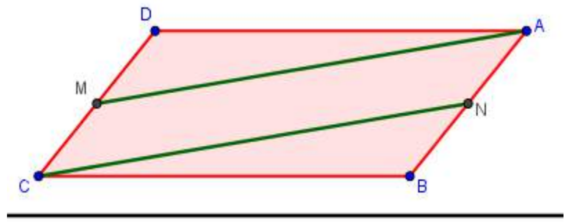

Illustration 1. Figure that accompanied the proposal of the task.

A student traced over the figure the segment MN and considered that it was the perpendicular bisector of the segments DC and AB (Illustration 2). This action was the starting point for the justification of the student's response.

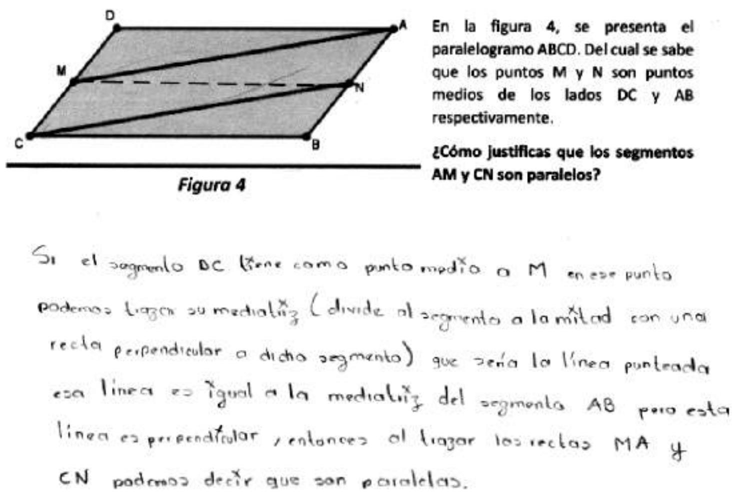

Illustration 2. Trace made on the figure presented.

The student explicitly manifests knowledge about the properties of a perpendicular bisector, in particular, the perpendicularity of this line with the segment it divides. However, the student assumes this property without any justification and without relating it to the properties of the figure shown. The perception of the figure leads the student to affirm the parallelism of the AM and CN segments, without this affirmation having a logical support.

The form of interaction between the student and the figure is totally empirical, since the arguments are restricted by the line that he drew in the figure and his observation to argue the justification.

Results about the task without figure

The wording of the task that was only composed of one proposition was as follows: A parallelogram of vertices A, B, C and D, has as midpoints of two opposite sides the points M and N respectively (M is the midpoint of CD and N is the midpoint of AB). How do you justify that the MA and NC segments are parallel?

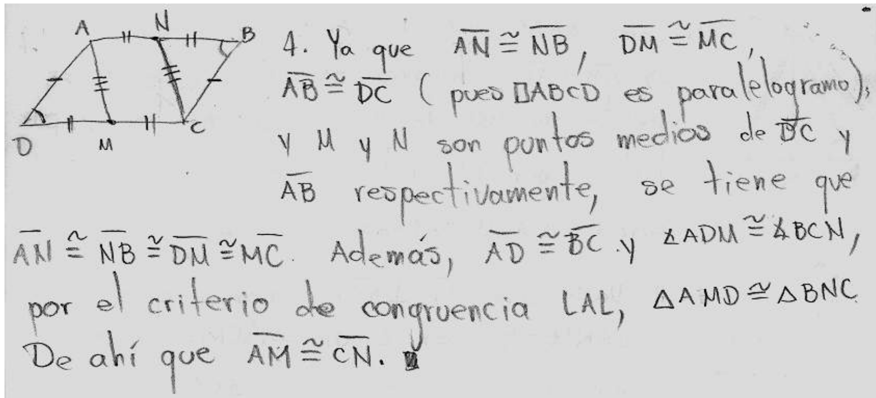

Illustration 3. Figure and student response.

Illustration 3 shows the response of a student who identifies a pair of triangles within the parallelogram that he constructed. The student argues the congruence of these triangles to justify his answer. However, the answer was inconclusive since the student did not mention that the congruence of the triangles results in the parallelism of the AM and CN segments.

The interaction of the student with the figure is representational type, since only considered the properties of the geometric object described in the proposal, to construct his figure. In addition, the characteristics of the figure constructed are restricted by the geometric knowledge of the student and not by empirical actions.

Conclusions

The figure as an object for support in solving geometric tasks involves various aspects that must be considered, one of them is the way a student interacts with a figure during the resolution of a task.

The results that we present indicate a tendency towards the empirical interaction of the students with the figures that accompany a proposition, mainly, it reflects a need to incorporate new elements on the figures. In the tasks that contained only one proposition, a need to externalize the mental image of the geometric object that was described was reflected. The representation of this object can help to keep a check on the arguments that are established to justify a property.

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INTERACCIONES CON FIGURAS GEOMÉTRICAS: UN CASO CON ESTUDIANTES DE LICENCIATURA

INTERACTIONS WITH GEOMETRIC FIGURES: A CASE WITH GRADUATE STUDENTS

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Las figuras geométricas pueden jugar un papel mediador en el razonamiento de los estudiantes, al resolver un problema o justificar la veracidad de una proposición. Estos objetos han sido estudiados en diversas investigaciones, donde se han analizado las interacciones que los estudiantes tienen con figuras que son construidas por el profesor o presentadas en un libro de texto. En este reporte, describimos un ejemplo de la interacción que tuvieron estudiantes de licenciatura con figuras construidas por nosotros, previo a la resolución de una tarea; y construidas por ellos durante dicha resolución. Los resultados muestran una tendencia hacia las operaciones empíricas de los estudiantes sobre las figuras con las que interactuaron; y la necesidad de incorporar elementos auxiliares en las figuras, como etiquetas o segmentos, para poder resolver las tareas propuestas.

Palabras clave: figura geométrica, interacción estudiante-figura, operaciones empíricas

En este trabajo mostramos parte de una investigación con estudiantes de licenciatura, en la que se caracterizó el papel que juega una figura geométrica durante la resolución de diferentes tareas. Tomando en cuenta el trabajo de Herbst y Arbor (2004), describimos los modos de interacción que tuvieron los estudiantes con las figuras, según la forma en que estas fueron presentadas: acompañada de una proposición antes de realizar una tarea; o construida por el estudiante durante la resolución de la tarea.

Revisión de Literatura

En diversos estudios se han analizado las características y las funciones de las figuras geométricas durante el proceso de resolución de una tarea. Enseguida mostramos una descripción de algunas investigaciones que han sido relevantes en el estudio de la figura geométrica, y las principales ideas que han surgido en torno a este tema.

Propiedades espaciales y conceptuales de una figura geométrica

Fischbein (1993) considera que una figura geométrica es una representación mental donde interactúan propiedades espaciales y conceptuales de un objeto geométrico. Las propiedades espaciales son aquellas que tienen que ver con la forma, tamaño y posición de la figura; mientras que las propiedades conceptuales están relacionadas con la idea abstracta de un objeto geométrico, con características como idealidad y perfección.

El papel heurístico de la figura geométrica

Duval (1995) sugiere que la figura geométrica puede jugar un papel heurístico durante el razonamiento de un individuo al resolver una tarea. Desde el punto de vista del autor, la representación de una figura geométrica puede ayudar a encontrar la idea principal de una prueba o apoyar la solución de un problema.

Para describir el papel heurístico de la figura geométrica, Duval (1995) define cuatro formas de aprehensión: perceptiva, secuencial, descriptiva y operativa. Particularmente, el papel heurístico de una figura se puede presentar con la aprehensión operativa, ya que esta se refiere a la modificación de una figura de tal forma que se produzcan ideas para la solución de una tarea.

Diferencia entre dibujo y figura geométrica

Laborde (1996) señala que un dibujo es una identidad material sobre un soporte (papel, por ejemplo), la cual actúa como un significante de un referente teórico. En este sentido, la autora define a la figura geométrica como un conjunto de pares, cuyos componentes son un referente geométrico y uno de los posibles dibujos que lo representan. Las relaciones entre el dibujo y su referente son construidas e interpretadas por quien produce o lee el dibujo, así, el significado de una figura geométrica es determinado por quien interactúa con ella.

En esta investigación utilizamos el término figura geométrica cuando nos referimos a las representaciones graficas de objetos o situaciones geométricas, las cuales se describen en una proposición.

Problema de investigación

Muchos de los estudios referentes a la figura geométrica analizan las interacciones que un individuo tiene con figuras que son construidas y presentadas por un profesor o en un libro de texto, dejando de lado las interacciones con figuras que son construidas por los estudiantes durante la resolución de una tarea. El propósito de esta investigación es indagar y describir las diferentes interacciones que estudiantes de licenciatura tuvieron con figuras geométricas presentadas de dos diferentes formas: presentadas por nosotros y construidas por ellos

Marco teórico

El análisis de los resultados obtenidos se hizo considerando lo establecido por Herbst y Arbor (2004) respecto a los modos de interacción entre un sujeto, un diagrama y un objeto geométrico. Los autores utilizan el término diagrama de forma equivalente a lo que nosotros nos referimos como figura geométrica; mientras que por objeto geométrico se refieren al referente representado en el diagrama. Los modos de interacción que tuvieron incidencia en los resultados obtenidos se describen a continuación:

Empírico: es aquella interacción donde un sujeto realiza operaciones físicas sobre un diagrama (por ejemplo, medir, observar o incorporar nuevos elementos). Los argumentos que se presentan sobre estas acciones son restringidos por las características del diagrama y las propiedades de los instrumentos que se utilizan (regla, compás, etcétera). En este sentido, el sujeto considera al diagrama como un equivalente del objeto geométrico que representa, lo que le permite comunicar sus resultados con base en las operaciones que realiza.

Representacional: en este modo de interacción el sujeto interpreta al objeto geométrico a través de la representación gráfica del diagrama, limitándose a considerar únicamente las características de dicho objeto establecidas en una proposición. Con esta interacción el diagrama no aporta un conocimiento adicional sobre el objeto geométrico, sino que actúa únicamente como un signo de este. A diferencia de la interacción empírica, las características del diagrama no están restringidas por operaciones físicas, sino por el conocimiento geométrico del sujeto para interpretar y representar lo establecido en una proposición.

Los modos de interacción restantes son el descriptivo y generativo. El primero tiene que ver con la interpretación simultanea de los signos mostrados en un diagrama con las propiedades establecidas en una proposición, mientras que el segundo sugiere la construcción conjunta de un diagrama y una prueba deductiva. Sin embargo, estos modos de interacción no están reflejados en los resultados que aquí presentamos.

Método

En esta sección se describen las características de los participantes de la investigación, así como la forma en que fueron presentadas las tareas implementadas en la investigación.

Participantes

Los participantes de la investigación fueron estudiantes de entre 20 y 22 años, quienes fueron elegidos considerando la experiencia que sugerían sus respectivas carreras (BUAP, 2011), particularmente, en los cursos de geometría.

Tareas Implementadas

Las tareas se aplicaron de dos formas distintas: a un grupo se le presentaron tareas cuyo contenido era una proposición y una figura que la representaba; mientras que a otro grupo se le presentaron las mismas tareas, pero mostrando únicamente la proposición y sugiriendo que se construyeran las respectivas figuras. En cada tarea se solicitó justificar una propiedad de la figura con base en las características descritas en la proposición.

Resultados

Los siguientes resultados reflejan aspectos significativos del comportamiento de los estudiantes, al interactuar con figuras preconstruidas y con figuras construidas por ellos.

Resultados sobre la tarea con figura

La tarea que se implementó fue la siguiente: en la figura (Ilustración 1) se presenta el paralelogramo ABCD, del cual se sabe que los puntos M y N son puntos medios de los lados DC y AB respectivamente. ¿Cómo justificas que los segmentos AM y CN son paralelos?

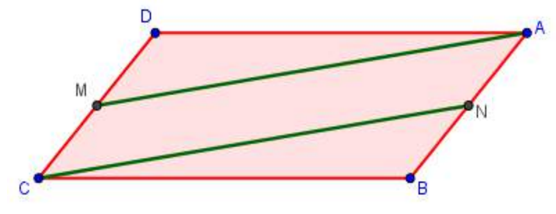

Ilustración 1. Figura que acompañó a la proposición de la tarea.

Un estudiante trazó sobre la figura el segmento MN y consideró que era la mediatriz de los segmentos DC y AB (Ilustración 2). Esta acción fue el punto de partida para la justificación de la respuesta del estudiante.

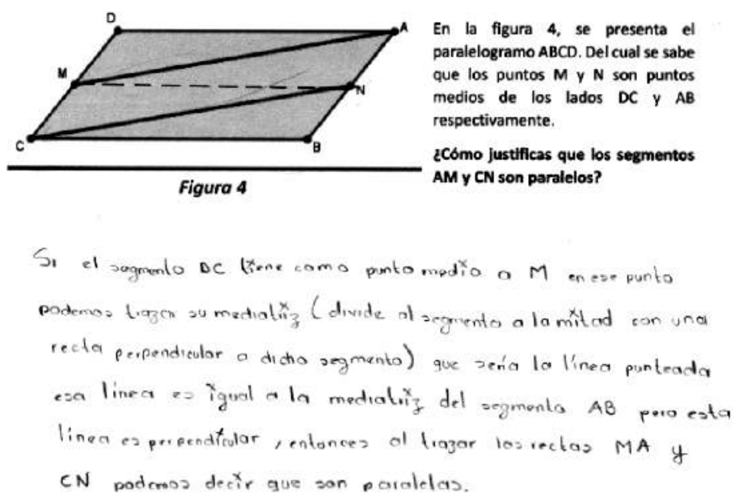

Ilustración 2. Trazo realizado sobre la figura presentada.

El estudiante manifiesta explícitamente el conocimiento sobre las propiedades de una mediatriz, en particular, la perpendicularidad de esta recta con el segmento que divide. Sin embargo, asume esta propiedad sin ninguna justificación y sin relacionarla con las propiedades de la figura mostrada. La percepción de la figura conduce al estudiante a afirmar el paralelismo de los segmentos AM y CN, sin que esta afirmación tenga un sustento lógico.

El modo de interacción entre el alumno y la figura es totalmente empírico, ya que los argumentos están restringidos por la línea que trazó en la figura y su observación para argumentar la justificación.

Resultados sobre la tarea sin figura

La redacción de la tarea que solo estaba compuesta por una proposición fue la siguiente: Un paralelogramo de vértices A, B, C y D, tiene como puntos medios de dos de sus lados opuestos a M y N respectivamente (M es punto medio de CD y N es punto medio de AB). ¿Cómo justificas que los segmentos MA y NC son paralelos?

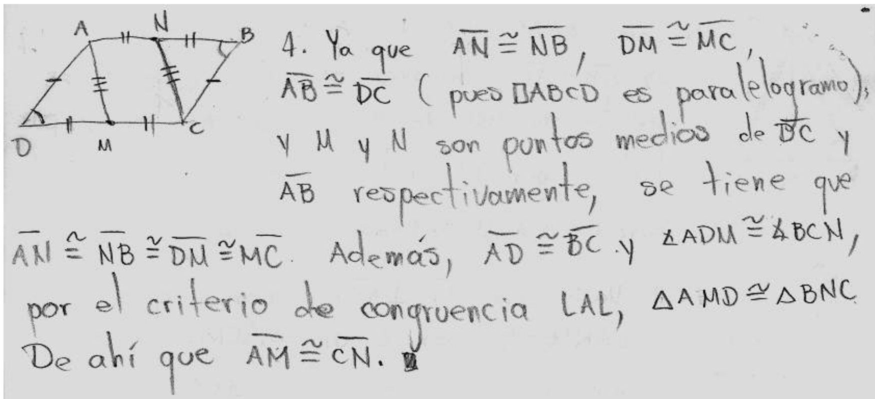

Ilustración 3. Figura y respuesta del estudiante.

En la Ilustración 3 se observa la respuesta de un estudiante quien identifica un par de triángulos dentro del paralelogramo que construyó. El estudiante argumenta la congruencia de estos triángulos para justificar su respuesta. Sin embargo, la respuesta quedó inconclusa ya que no mencionó que la congruencia de los triángulos trae como consecuencia el paralelismo de los segmentos AM y CN.

La interacción del estudiante con la figura es de tipo representacional, ya que solo consideró las propiedades del objeto geométrico, descritas en la proposición, para construir su figura. Además, las características de la figura construida están restringidas por el conocimiento geométrico del estudiante y no por acciones empíricas.

Conclusiones

La figura como objeto para el apoyo en la resolución de tareas geométricas involucra diversos aspectos que deben ser considerados, uno de ellos, es la forma en que interactúa un estudiante con una figura durante la resolución de una tarea.

Los resultados que presentamos señalan una tendencia hacia la interacción empírica de los estudiantes con las figuras que acompañan a una proposición, principalmente, se refleja una necesidad de incorporar nuevos elementos sobre las figuras. En las tareas que contenían solo una proposición se reflejó una necesidad de exteriorizar la imagen mental del objeto geométrico que se describía. La representación de dicho objeto puede ayudar a llevar un control sobre los argumentos que se establezcan para justificar una propiedad.

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USING DESIGN-BASED TASKS TO TEACH AREA MEASUREMENT

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Geometric measurement is a critical domain that is difficult for many students. The focus of this study was to determine if the incorporation of design processes into instructional activities for area measurement may enhance engagement and learning of students from low-resource, historically marginalized communities. We adapted activities from a learning trajectory for area measurement, prompting Grade 3 students to integrate knowledge of arrays, multiplication, and area measurement. Results suggest the design focus prompted students' integration of knowledge of space and number by engaging in novel representations of designed objects and by prompting multiplicative thinking.

Keywords: Design Experiment, Elementary School Education, Geometry and Geometrical and Spatial Thinking, Learning Trajectories

We employed design and measurement tasks to teach mathematics across complex cultural contexts. Bishop (1988) considered mathematics a poly-cultural activity. He said people in many cultures engage in six fundamental mathematical activities to develop mathematical knowledge: counting, locating, measuring, designing, playing and explaining. Similarly, "…science learning can be understood as a cultural accomplishment" (National Research Council [NRC], 2012, p. 283). The Common Core Standards for Mathematics recommend students use geometry to solve design problems (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). The Next Generation of Science Standards (NGSS Lead States, 2013) suggest constructing explanations and designing solutions to prepare students for STEM fields. At Grades 3 – 5, students should, "Generate and compare multiple possible solutions to a problem based on how well each is likely to meet the criteria and constraints of the problem" (NGSS Lead States, 2013, p.46). We explored design work as a means to engage mathematical learning across cultural groups among students.

We selected area measurement and design problems as tools to support mathematics instruction across cultural contexts. We used a learning trajectory (LT) for area measurement (Barrett, Clements, & Sarama, 2017) to develop plausible assessment and instructional tasks (Barrett, Cullen, Behnke & Klanderman, 2017; Barrett & Battista, 2014; Battista, 2006, 2012; Sarama & Clements, 2009). The Cognitively-Guided Instruction group (Carpenter & Fennema, 1992; Fennema, Carpenter, & Franke, 1997) benefited from productive adaptations of tasks (Brown & Campione, 1996). Likewise, we drew on students' community funds of knowledge to adapt our tasks (Celedòn-Pattichis et al., 2018; Wager & Carpenter, 2012).

We had two goals: (a) determine if using design work to adapt instructional activities from an LT for area measurement would enhance learning and engagement, and (b) find whether design processes support mathematical learning. We expected the tasks to help students establish area units as cognitive tools for measuring space, through multiplication or addition. By anticipating spatial collections of units, students might extend skip counting and transition toward multiplicative reasoning in an array structure. We sought to promote students' use of arrays as models to measure area. We expect to suggest a model for improving the development of asset-based LTs that bridge cultural, community-based practices among elementary students. This was our rationale for adapting existing LT instructional tasks to (1) feature design processes, and, (2) integrate multiplication operations, arrays and area measurement.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Method

Participants were a convenience sample of twenty-two Grade 3 students in an urban Midwest classroom and their teacher. Their school district consists of approximately 13,000 students (20.1% White, 57.7% Black, 11.3% Hispanic). Approximately 68% of the students at the school receive free or reduced lunch and 8% are English learners. We used a written assessment adapted from the LT (Barrett, Clements, & Sarama, 2017, pp. 105-115), with classroom observation to identify four levels of thinking among 22 students: Physical Coverer and Counter (3 students), Complete Coverer and Counter (5), Area Unit Relater and Repeater (9) and Initial Composite Structurer (3). We targeted these levels of thinking in design work within area measurement tasks.

Instruction Design Cycle

What we report here is the feasibility study phase of a design experiment (Middleton, Gorard, Taylor & Bannan-Ritland, 2008). This phase is meant to evaluate an intervention through qualitative methods such as observations, interviews, and case studies to determine what aspects of the intervention work and those that need improvement.

Prior to instruction, the researchers observed and helped students in the classroom to build familiarity and rapport. Later, we interviewed students in focus groups. We asked them how they may already use mathematics outside of the classroom to count, locate, measure, design, play or explain (Bishop, 1988). We conducted three lessons during one week of school in the Fall of 2019. Each lesson was led by one of the authors, with assistance from the classroom teacher. Each lesson began with whole-class discussion of a complex measurement question on area. The first lesson was an adaptation of patio tasks targeting LT levels often found among Grade 3 students (Barrett et al., 2017, p. 133-137). We set a designing task using a novel problem, to find the number of buses that could fit in a parking lot. Day 2, we asked them to design and draw a parking lot to fit a given number of cars or buses. Finally, on Day 3, we asked students to design and draw a park for pets, to provide room for a given number of dogs to move around freely for exercise. The teacher and researchers surveyed students' progress by assisting students who asked questions and posing questions to students while they worked. Students worked independently at first, and later in teams of two or four. The researchers kept field notes. Student work was collected for analysis. At the end of each lesson, the researchers reflected on what occurred in the class to develop the goal and a focal task for the subsequent lesson.

We analyzed the students' work in two ways. First, we examined all three tasks to determine strategies used to solve each. We asked ourselves how students made use of arrays or units in developing solutions. For the second and third tasks, we examined whether the students met the constraints of the task in the process of designing a solution.

Results

For the sake of the paper, we only discuss the results of the third day of instruction. We presented students with an image of a dog kennel. In the image it showed that a 6 x 6 foot square was an adequate area for a dog to run around in. We mapped out the square area on the floor of the classroom so students could see the space, walking around inside the mapped-out region to show the space needed per dog. We gave students a 1.5 x1.5 inch square cutout piece of cardboard. We told the students it represented the space that one dog needed to move around. The task we posed for them was to design a rectangular dog park (Constraint 1) that had enough room for 24 dogs (Constraint 2). At the end of the lesson, we had students present their designs. Table 1 shows the strategies students used in creating their dog parks.

Table 1: Strategies Used in Designing Dog Park

Strategies	Number of Students
Consistent Units with Some Use of Arrays	3
Consistent Units with Grouping	7
Consistent Units with no Grouping	7
Inconsistent Units No Clear Spatial Arrangement	1
No Use of Units Shown	4

Ten students made use of grouping and arrays to create their dog parks. Most students (n = 17) made use of consistent units in their designs. In dealing with the constraints students sometimes met both constraints (see Table 2), but still they were not successful in the total design project. For example, some students created a rectangular area that had room for more than 24 dogs. Other students accounted for the space for 24 dogs but appointed another rectangle to be the actual dog park.

Table 2: Met Dog Park Constraints

Constraints	Number of Students
Area of Park Design for 24 Dogs	6
Dog Park Rectangle	4
Both Constraints Met	6
Neither Constraint Met	6

Conclusion

Given the brief span of the intervention we conducted, we were not expecting students to move on to a new level of the learning trajectory (LT) for area measurement strategies. Rather, we used the LT levels as a rubric to find a suitable instructional level given students' exhibited knowledge of area measurement. Our findings with these design-focused tasks suggest students were creating designs and engaging with area measurement tasks that involved multiplication schemes in productive ways which is in keeping with AURR levels. This finding suggests design-centered tasks of this type offer ways of supporting student thinking and of observing their reasoning at these particular levels of a LT for area. This may provide a way of improving the instructional task descriptions as the LT is modified to broaden its impact on a wider range of students in various contexts and communities.

Furthermore, the design process of instruction appears helpful in focusing students' attention on the meaningful association among arrays, multiplication operations with number, and the measure of rectangular shapes. By engaging contexts that fit with our observations about the students' own experiences, we apparently gained access to familiar stories from their daily routines and community-based language for spatial quantity. The teacher was pleased to note that several students who typically did not engage in mathematics stayed engaged with the tasks for as long as they did. More work is needed to find what motivated this level of investment in the tasks.

Dealing with design constraints had mixed results from our vantage point; some students did not address any of the constraints, although other students successfully addressed one or more constraints. Nevertheless, students in Grade 3 demonstrated the capability to address design constraints related to measurement and space. The design emphasis, with the integration of measurement, multiplication schemes, and arrays as tools appears to be a viable way to adapt learning trajectory-based activities for area measurement. The lesson outcomes indicate promise that students in Grade 3 can engage in design activities with constraints related to multiplicative reasoning using skip-counting and grouping schemes. The interaction among these schemes may have prompted students to engage in the quantitative reasoning by access to their knowledge of such contexts as scanning to find whether a parking lot is empty, partly filled, or full. We believe the prominence and meaningfulness of the context provided a way for the mathematics of area measurement to be addressed as an integrated part of instruction on multiplication and arrays. This is consistent with work in statistics education showing the importance of linking context knowledge to the statistical schemes for organizing and reporting on data in such a context (Langrall, Nisbet, & Mooney, 2006).

We believe further design cycles may need to draw out a more comprehensive analysis of the multiplication processes and the arrays as tools for measuring the capacity of a parking lot to hold cars. We plan further work with the same students to have them redesign a dog park to meet constraints related to the area measurement and to the shape of the region (by requiring a rectangle). We also expect to include further ways to prompt students to check their own design by using a grouping scheme for collections of units. This could focus them more on iterating squares to fill space, and link to arrays and measuring area. The process of testing, designing, retesting and redesigning are vital STEM skills for students to develop (https://stem.getintoenergy.com/stem-skills-list/). The redesign process is important as we learn to extend the instructional tasks found in learning trajectories (e.g., the area LT) to different communities.

Ideally, teachers will use similar design-based tasks to adapt and work with their students in different community contexts. The principles of designing, measuring and describing, taken from analyses across a wide range of culture and communities by Bishop (1988) may productively inform both teachers and researchers who want to adapt learning trajectories for other content areas. Our findings suggest that designing, describing and measuring may be productive ways of engaging students as young as grade 3 in substantive mathematical projects. This may support them as they learn the structural advantages of noticing or setting up arrays to support area measurements and multiplication operations.

Acknowledgments

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SELF-INSTRUCTIONS FOR APPLYING WRITING IN GEOMETRY PROBLEM RESOLUTION

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This work reports a teaching experiment, which explores the use of writing as a metacognitive tool in high school Geometry problem solving. We develop a qualitative research study, to explore how explicit writing directives can help students to understand, organize and monitor the steps involved in the different phases of a cycle of activities for Geometry problem solving in ninth grade.

Keywords: Problem Solving; Metacognition; High School Education; Geometry and Geometrical and Spatial Thinking

Introduction

In my professional experience as a teacher I have observed lack of comprehension in Mathematics concepts and procedures in students as well as disorganized strategies when solving problems. Such disorganization is observed when students start solving the problem without clearly identifying the data, nor the procedure and the developed rationale to get an answer. Also, the notes they make in the process are disorganized and lack a systematic feature.

From 2006 to 2013, the Mexican Ministry of Education set a nationwide multiple choice evaluation focused to language, mathematics and other subjects from the national syllabus for each school cycle in basic education (ENLACE). From 2015 up to now, such evaluation changed only to language and mathematics (PLANEA). Such evaluations, known by their acronyms in Spanish, serve as a diagnostic for basic education in Mexico, however, a noticeable side effect was that many teachers and students, looking for the improvement in their school rank, developed strategies for success in multiple choice answers, demeaning their skills to face open ended questions in mathematics.

Theoretical framework

This experiment is focused in Veenman (2012) identified learning process, as areas where metacognitive skills are mainly developed: reading, problem solving, learning by discovery and writing, also, we considered Hyde's (2002) research with primary school students where reading and math problems solving are mixed.

Veenman (2012) describes metacognitive skills as the regulation of cognitive process, this means, the acquired capacity for supervision, orientation, direction and control of one`s learning behavior and problem solving. Actually, metacognitive skills are the learning activities and the main determiners to learning results.

Hyde (2006) follows the guidelines of cognitive psychology and uses the braiding term to state that language, thinking and mathematics can be braided into one entity, achieving though connections among those three important processes, a stronger, more lasting and powerful result than if worked individually. With the braiding term it is suggested that the three components are inseparable, with mutual and necessary support. He assures that the stronger the connections among the related ideas are, the deeper and richer the understanding of the concept will be.

Hyde (2006), states that the braiding context benefits kids to imagine, visualize and connect mathematics into context. He assures that this model has been used efficiently in the instruction of small classes involving the teacher's support. The questions are useful to discuss the problem orally, so students interiorize such questions to use them by themselves for later tasks.

After reviewing Veenman's (2012) work, which distinguishes between the metacognitive knowledge and the metacognitive skills to focus their development in science teaching, as well as Hyde's (2006) research, who applies the Braiding Model in primary education to solve problems in mathematics, we consider some useful elements in their work to design our teaching experiment that uses writing from self-instructions as a metacognitive tool when solving geometry problems.

Methodology

In our teaching experiment, students were given self-instructions shown as simple questions, to help them develop the activities at the starting, during and after solving the problem (Table 1). This was intended to gradually introduce the students so, through writing, they started analyzing the given information in the problem, continued with the necessary rationale and representations to interpret the problem situation, and finally they followed the adequate procedures, verifying through justification that they had the right answer realizing how they go to it.

In the teaching experiment, 12 problems selected by the Ministry of Education in the State of Jalisco were selected to practice for the State's Olympics in Mathematics in Primary and Secondary schools (OEMEPS), since these problems require reasoning and creativity in students, they must work with descriptions, explanations and justifications when solving them, and different paths could be followed for the answers.

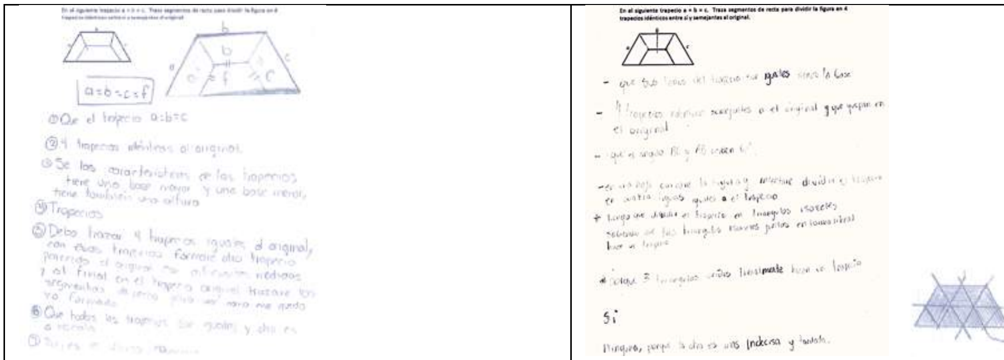
These problems were solved in 20 sessions, 45 minutes each, by 10 students from 9th grade in Mexico City, and they were highly motivated working for their admission exams for High School. The first four sessions, self-instructions were explained to them from examples in collective problem solving. In the rest of the sessions, students answered individual worksheets for 12 problems with self-instructions written on the blackboard as a reminder. The last four sessions, different solving processes for each problem were exteriorized and each student defended his/her answers. The written productions for data analysis are the sources as well as the notes from the teacher-researcher.

Table 1: Veenman's connections between self-instructions and metacognitive skills.

Self-instructions designed to use writing when solving problems.		Veenman's Learning Activities representing metacognitive skills.
Start	What information am I given in the problem?	Reading.
	What do I need to find?	Task analysis,
	What knowledge do I have about the topic?	Activation of previous knowledge.
During	How am I going to do it?	Planning.
	Which steps should I follow?	Follow or change the plan.
	Which drawings can help me get to the solution?	Note taking.
After	How do I justify my answer?	Performance evaluation.
	Is this the only way to get to the answer?	Summing up.
	Which other ways could be used?	Process Reflection.

Results

Results are presented according to the self-instructions order for the teaching experiment. Such self-instructions guided the research and oriented students to work in the activities at the staring of the problem, then the activities during its solving and finally, activities after the problem solving. Students followed self-instructions with their own presentation style, for instance, one student organized through a subsequent number, another uses a hyphen to each (Fig 1b).



a b

Figure 1. Complete answers by two students.

Activation of previous knowledge was given through students answers to the first three questions at the starting of the problem solution, which is the process of remembering the information kept in their minds, relate it to the identified information and deciding which process they will use to obtain the answer, as we see during the problem solving.

"that trapezoid a = b = c" "4 trapezoids identical to the original" *"I know the trapezoids features, they have a major base and a minor base, and they also have height".*	*"The trapezoid sides are the same, except the base"* *"4 identical trapezoids similar to the original and that fit in the original" That angles BC and AB are 60º".*
a	b

Figure 2. Two students' answers at the starting of the task.

During the problem solving students come up with the same answer for the questions "How am I going to do it?" and "Which steps should I follow?" When describing their procedures organization and clearly relate the ideas for the process (Figure 3a and 3b).They also rely on their notes and the trapezoids they draw out form the step description (note taking) or in the marks in the given drawings in the task along with the drawings they sketch as well as written signals (a, b, c, f).Together, descriptions and drawings guided them to follow the process step by step and get the right answer. (Fig. 1a, 1b).

Once the problem solution is obtained, the last questions lead them to think about the process they had applied when describing their justification as well as to look for other ways to solve the problem, however, in this problem none of them identified another procedure to obtain the answer.

"I must draw 4 trapezoids equal to the original; with them I will form another trapezoid similar to the original with different measures and at the end of the original trapezoid I will draw the line segment to see how it looks once I finish". "All trapezoids are the same and the other is escalated". a	*"On a piece of paper I will trace the figure and I will try to divide the trapezoid into four equals to the trapezoid". "I must divide the trapezoid into isosceles triangles knowing that the isosceles triangles together aligned make a trapezoid"* b

Figure 3. Two students' answers when developing the task.

Conclusions

With the use of this learning experiment we noticed, first of all, self-instructions are indeed an action plan, they guide students step by step during all the problem solving process, and secondly, when students wrote down the justification to their answers they carefully reviewed the steps they followed. This means, not only did they analyze if they got the correct answer but also recognized the successful steps to solve the problem, this means that writing helped them understand the solving process, guided by the self-instructions given at the beginning of the experiment.

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AUTOINSTRUCCIONES PARA APLICAR LA ESCRITURA EN LA RESOLUCIÓN DE PROBLEMAS DE GEOMETRÍA

SELF-INSTRUCTIONS FOR APPLYING WRITING IN GEOMETRY PROBLEM RESOLUTION

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Este trabajo reporta un experimento de enseñanza, que explora si el escribir de forma sistematizada puede ser una herramienta que les facilite la resolución de problemas a los estudiantes de educación básica. Desarrollamos un estudio de investigación cualitativa, para indagar como las directivas de escritura de manera explícita pueden ayudar a los estudiantes a entender, organizar y controlar los pasos implicados en las distintas fases de un ciclo para la resolución de problemas de geometría en el noveno grado de educación.

Palabras clave: Autoinstrucciones, Escritura. Resolución de Problemas.

Introducción

A lo largo de mi experiencia profesional como docente he observado una deficiente comprensión de conceptos y procedimientos matemáticos por parte de los estudiantes, así como estrategias desorganizadas en la resolución de problemas. Esta desorganización se observa cuando ellos comienzan a resolver el problema pero no distinguen claramente los datos, ni los procedimientos y el razonamiento que desarrollan para llegar a la respuesta. De la misma manera las anotaciones que realizan en el proceso son desordenadas y carecen de un carácter sistemático.

De 2006 a 2013, el Ministerio de Educación Pública aplicó en todo el país una evaluación de opción múltiple guiada en lenguaje, matemáticas y otras asignaturas del plan de estudios nacional en cada ciclo escolar de la educación básica (ENLACE). A partir del 2015 a la fecha cambio la evaluación enfocada unicamente a lenguaje y matemáticas (PLANEA). Estas pruebas conocidas por sus siglas en español, ayudan a diagnosticar la educación básica en México, pero un efecto secundario no deseado de este tipo de valoración, fue que muchos profesores y estudiantes, en sus esfuerzos por mejorar las marcas de sus escuelas, desarrollan estrategias para lograr el éxito en pruebas de opción múltiple, en detrimento de las habilidades para hacer frente a preguntas abiertas en matemáticas.

Marco teórico

Este experimento está centrado en los procesos de aprendizaje que han sido identificados por Veenman (2012) como las áreas donde mayormente se desarrollan las habilidades metacognitivas: lectura, resolución de problemas, aprendizaje por descubrimiento y escritura, además consideramos las indagaciones con estudiantes de primaria efectuadas por Hyde (2006), donde se combina la lectura y la resolución de problemas de matemáticas.

Las habilidades metacognitivas las describe Veenman (2012) como la regulación de los procesos cognitivos, es decir la capacidad adquirida de la supervisión, orientación, dirección y control de la propia conducta en el aprendizaje y la resolución de problemas. En sí las habilidades metacognitivas son las propias actividades de aprendizaje y son el principal determinante de los resultados en el mismo.

Hyde (2006) se guía por los principios de la psicología cognitiva y utiliza el término de trenzado para indicar que el lenguaje, el pensamiento y las matemáticas pueden ser entrelazados en una sola entidad, logrando que al hacer conexiones entre estos tres procesos importantes, el resultado sea más fuerte, durable y poderoso que si se trabajará cada uno de forma individual. Con el término trenzado sugiere que las tres componentes son inseparables de apoyo mutuo y necesario. Afirma que cuanto más fuerte son las conexiones entre las ideas relacionadas más profunda y más rica es la comprensión del concepto.

Hyde (2006) hace hincapié que el contexto del trenzado beneficia a los niños para imaginar, visualizar y conectar las matemáticas con el contexto. Afirma que este Modelo se ha utilizado con eficacia en la instrucción de una clase con grupos pequeños y con el apoyo del maestro. Las preguntas son eficaces para discutir el problema en grupos pequeños así como las estrategias de representación en el lenguaje oral, de esta manera los estudiantes comienzan a internalizar estas preguntas para utilizarlas por si mismos durante las tareas posteriores.

Una vez revisado el trabajo de Veenman (2012), que hace una distinción entre el conocimiento metacognitivo y las habilidades metacognitivas para orientar el desarrollo de éstas en la enseñanza de las ciencias; así como las investigaciones de Hyde (2006) quien aplica el Modelo del Trenzado en educación primaria, para la resolución de problemas en matemáticas; consideramos algunos elementos útiles de dichos trabajos para hacer el diseño de nuestro experimento de enseñanza que consiste en utilizar la escritura a partir de autoinstrucciones como herramienta metacognitiva en la resolución de problemas de geometría.

Metodología

En nuestro experimento de enseñanza se dieron autoinstrucciones a los estudiantes, presentadas como preguntas sencillas, para auxiliarlos en el desarrollo de las actividades del inicio, durante y después de la resolución del problema (Tabla 1). Lo anterior con la intención de inducirlos gradualmente para que a través de la escritura comenzaran un análisis de la información proporcionada en el problema, continuarán con la exploración y representaciones necesarias para interpretar la situación del problema y finalmente siguieran los procedimientos adecuados, verificando por medio de la justificación que llegaron a la respuesta correcta y se dieran cuenta cómo lo hicieron.

En el experimento de enseñanza, se seleccionaron 12 problemas propuestos por la Secretaría de Educación de Jalisco para el entrenamiento de las Olimpiadas Estatales de Matemáticas en Educación Primaria y Secundaria (OEMEPS) debido a que estos problemas requieren razonamiento y creatividad del estudiante, en los cuales se deben trabajar descripciones, explicaciones y justificaciones en su resolución y se pueden seguir diferentes caminos para obtenerla.

Estos problemas fueron resueltos en 20 sesiones de 45 minutos cada uno, con un grupo de 10 estudiantes de noveno grado de una escuela en la Ciudad de México, quienes estaban altamente motivados trabajando para preparar sus exámenes de admisión para bachillerato. En las primeras cuatro sesiones se explicaron las autoinstrucciones a los estudiantes a partir de ejemplos en la resolución de problemas trabajando colectivamente su aplicación. El resto de las sesiones los estudiantes contestaron las hojas de trabajo individual para los 12 problemas, con las autoinstrucciones expuestas en el pizarrón como recordatorio. Las últimas 4 sesiones se exteriorizaron los diferentes procesos de solución en cada problema y cada estudiante defendío sus respuestas. Las fuentes de datos para el análisis son las producciones escritas de los estudiantes, y las notas de campo del profesor-investigador.

Tabla 1: Conexión entre las autoinstrucciones y las habilidades metacognitivas de Veenman

Autoinstrucciones diseñadas para utilizar la escritura en la resolución de problemas		Actividades de Aprendizaje representativas de las Habilidades Metacognitivas (Veenman)
Inicio	¿Cuáles son los datos que me da el problema?	Lectura
	¿Qué necesito encontrar ?	Análisis de la tarea
	¿Qué conocimientos tengo acerca del tema?	Activación de los conocimientos previos
Durante	¿Cómo le voy hacer?	Planificación
	¿Qué pasos voy a seguir?	Seguir el plan o cambiar el plan
	¿Qué dibujos me pueden ayudar para llegar a la solución?	Toma de notas
Después	¿Cómo justifico la respuesta que encontré?	Evaluación del desempeño
	¿Es el único camino que se puede seguir para llegar a la respuesta?	Recapitular
	¿Qué otras formas puedes aplicar?	Reflexión sobre el proceso

Resultados

La presentación de los resultados va de acuerdo con el orden de las autoinstrucciones diseñadas para el experimento de enseñanza, las cuales fueron la guía de la investigación y orientaban a los estudiantes a trabajar las actividades al inicio de la resolución del problema, enseguida las actividades durante la resolución y por último las actividades después de la resolución del problema. Los estudiantes siguieron las autoinstrucciones dando cada uno su estilo de presentación, por ejemplo mientras que un estudiante les da un orden por medio de un número subsecuente (Figura 1a), otro simplemente utiliza un guión para cada una (Figura 1b).

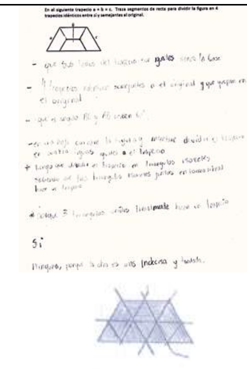
 a	 b

Figura 1. Respuestas completas de dos estudiantes

A través de las respuestas de los estudiantes a las tres primeras preguntas del inicio de la resolución del problema (Figura 2a y 2b) se dio la activación de los conocimientos previos, que es en sí el proceso de recordar la información registrada en su memoria, relacionarla con la información que identificaron y tomar la decisión de que proceso aplicarán para obtener la respuesta como lo vemos durante la resolución del problema.

"que el trapecio a = b = c" "4 trapecios idénticos al original" "Se las características de los trapecios, tienen una base mayor y una base menor, tienen también una altura". a	"Que sus lados del trapecio son iguales, menos la base" " 4 trapecios idénticos semejantes a el original y que quepan en el original " "Que el ángulo BC Y AB miden 60º". b

Figura 2. Respuestas del inicio de la resolución de dos estudiantes

Durante el proceso de solución del problema los estudiantes brindan una misma respuesta, para las preguntas ¿cómo le voy hacer? y ¿qué pasos voy a seguir?, al describir la organización de sus procedimientos y relatar con claridad las ideas para el proceso (Figura 3a y 3b). Se apoyan también en las anotaciones y representaciones de los trapecios plasmadas por ellos fuera de la narración de los pasos (la toma de notas) o las marcas en los dibujos dados en el problema en conjunto con los dibujos que ellos elaboraron así como las señales escritas (a, b, c, f). En conjunto las descripciones y los dibujos les orientaron para seguir sus procedimientos paso a paso y tener la solución de manera correcta (Figura 1a y 1b).

"Debo trazar 4 trapecios iguales al original, con esos trapecios formare otro trapecio parecido al original con diferentes mediadas y al final en el trapecio original trazaré los segmentos de recta para ver como me quedo ya formado". *"Que todos los trapecios son iguales y otro es a escala".* a	*"En una hoja calcaré la figura y intentare dividir el trapecio en cuatro iguales iguales a el trapecio". "Tengo que dividir el trapecio en triángulos isosceles sabiendo que los triángulos isósceles juntos en forma lineal hacen un trapecio" "Porque 3 triángulos juntos linealmente hacen un trapecio".* b

Figura 3. Respuestas del desarrollo de la resolución de dos estudiantes

Después de obtener la solución del problema, las últimas preguntas los guiaron a reflexionar sobre el proceso que habían aplicado anteriormente al narrar su justificación y a buscar otros caminos que los llevaran a la solución del problema, aunque en este problema ninguno de ellos identificó otro procedimiento para llegar a la respuesta.

Conclusiones

Con la aplicación del experimento de enseñanza nos percatamos, en primer lugar, que las autoinstrucciones son en sí un plan de acción, las cuales guían al estudiante paso a paso durante todo el proceso de la resolución del problema y en segundo lugar, que cuando ellos escribieron la justificación de sus respuestas revisaron detenidamente los pasos que siguieron. Esto es, no solo analizaron si llegaron a la respuesta correcta, sino que reconocieron qué pasos les resultaron exitosos en la resolución, es decir, la escritura les ayudó a entender el proceso de solución, orientados por las autoinstrucciones dadas al inicio del experimento.

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DIFFICULTIES TO JUSTIFY GEOMETRIC PROPOSITIONS WHEN SOLVING LOCI PROBLEMS WITH GEOGEBRA

DIFICULTADES PARA JUSTIFICAR PROPOSICIONES GEOMÉTRICAS AL RESOLVER PROBLEMAS DE LUGARES GEOMÉTRICOS CON GEOGEBRA

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A study developed with engineering students from the University of Sonora, Mexico, about their difficulties to justify geometric propositions is presented here. The work is framed within an Analytical Geometry course designed with GeoGebra as support, and refers the topic of loci. The theory of geometric paradigms is used to explain these difficulties and the study reveals that the geometry education of students entering University is mainly limited to natural geometry, showing a precarious familiarity with the methods of natural axiomatic geometry.

Keywords: Geometry and Geometrical and Spatial Thinking, Instructional activities and practices, Technology

Introduction

To explore the extent to which engineering students can justify geometric propositions, five teaching sequences on the subject of geometric loci have been designed. The sequences have been designed to be implemented with GeoGebra, taking as reference the methods used by Descartes (1954, pp. 50-55) in the presentation of his hyperbolograph, to identify the geometric relationships of the construction and find the equation of the hyperbola. The activities were proposed in an Analytical Geometry course taught to 35 Civil Engineering students during the first semester of 2018.

Theoretical elements

Students' responses were analyzed within the framework of the geometric paradigm theory proposed by Houdement and Kuzniak (2003); According to these authors, under the term elemental geometry, three distinct types of paradigms coexist that lead to the distinction of three types of geometry: *natural geometry*, *natural axiomatic geometry* and *formalist axiomatic geometry*. The first two geometries are described below, as they are those used in the present work, as written by Kuzniak (2006):

Natural geometry can be seen as an empirical science in which objects are closely linked to reality, which can be measured and physically compared. The propositions in this geometry are validated experimentally as they depend on the sensory perception of them.

In *natural axiomatic geometry,* geometric objects are only approximations to reality and are defined by the axioms of classical Euclidean Geometry. The propositions here are validated by demonstrations constructed from euclidian axioms and previously proven geometric results.

In the design of the teaching sequences, we attempt to recover the ideas developed by Descartes on geometric loci, particularly those in the construction of the hyperbolograph (Descartes, 1954, pp. 50-55). From Descartes's method we retake two main principles:

1. Identification of the quantities that change and those that remain constant during the plotting of the curve and the relationships between them, as shown in Arcavi (2000).
2. Deduction requires the use of the methods of natural axiomatic geometry, since it is based on the properties of similar triangles, established as theorems in this geometry.

Both characteristics seem desirable to us in the mathematical formation of an engineer, the first because of the importance it has in algebraic modeling of geometric situations and the second because it facilitates the explanation and argumentation of geometric results.

Design and implementation of didactic sequences

Based on the ideas of Descartes, Arcavi and Kuzniak, and the use of GeoGebra, we have developed a series of didactic sequences on the notion of loci. The main goal is for engineering students to justify some elementary geometric results with arguments typical of natural axiomatic geometry.

There is a total of five sequences on which the students worked over a two-week period, where each student had a computer with GeoGebra installed and worked independently. In all the sequences, the students are first offered a construction in GeoGebra, in which the Cartesian axes are omitted so that students can concentrate on the elements of the construction and the relationships that manifest by varying the elements of the figures. As an example, the following table summarizes the objectives and characteristics of sequence 2.

Table 1 Characteristics of Sequence 2

Objectives: a) Identify the relationships between quantities, which are preserved by varying k and presenting geometric arguments about their veracity. b) Verify that the curve plotted by P is a circumference and geometrically justify this result. c) Algebraically represent the relationships identified in (a). d) Find the equation in terms of s and t of the circumference of Figure 1b, and in terms of x and y in Figure 1c.		
Characteristics of the constructions		
AB is a fixed line, *CBD* is an angle of value *k*, *BE* is the bisector of *CBD*, *AP* is parallel to the bisector traced from *A*, and *P* is the intersection between *AP* and *BP*.	The construction is the same as in Figure 1a, but it adds a perpendicular to *AB* from *P* that intersects *AB* in *Q*, also labels segment *BQ* as *s* and *QP* as *t*.	The construction is the same as shown in Figure 1b but has been translated and rotated so that the AB line matches the *X-axis* and to and *B* match the origin.
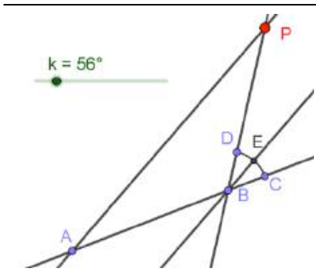	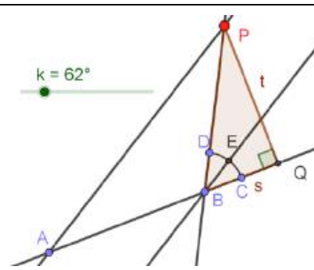	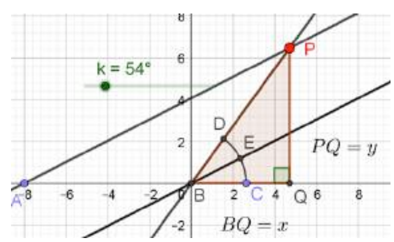
Figure 1a	Figure 1b	Figure 1c

This sequence begins with the GeoGebra construction from Figure 1a, in which the student is asked to explore the construction by varying the slider *k*, which controls the CBD angle, and to note that the point P describes a circumference when moving the construction. The tasks here are intended to systematize exploration: that those quantities (measures of angles and segments) that change are first distinguished from those that remain constant, and then that the students establish the possible relationships between these quantities, mainly equality relationships. The work with this construction concludes by requesting the students to geometrically justify the detected relationships. The sequence concludes with the introduction of Cartesian coordinates to be used as a reference (Figure 1c) and algebraically express the curve drawn by point *P* when the parameter *k* is changed.

Analysis

Throughout the development of these activities, we have focused our attention on the questions in which students are asked to justify some geometric proposition, since our main interest is to observe the nature of the arguments they built. We will focus our analysis on the answers to these questions in sequence 2.

In this activity, having explored the construction shown in Figure 1a, and once they have established which angles remain the same when dragging *k*, students seamlessly conclude that the triangle *ABP* is isosceles. As can be seen in the following responses, their difficulties begin when they try to justify some of the results obtained. Item (d) of the worksheet asks them to justify why does the triangle remains isosceles and item (e) asks them to justify why does the segment *BP* works as the radius of the circle, i.e. why does its magnitude not change. The construction provides data on the angles and on the segment *AB*, it was expected that they would conclude that the triangle *ABP* is isosceles on the basis that two of their angles always remain the same and that they could argue that *as a consequence* their opposite sides AB and BP should always be the same.

Four types of responses were detected, illustrated below:

In the first type we place the students who have offered answers whose forms of argumentation are inconsistent, in which arguments that appear to be located in Paradigm 2 are mixed with others based on facts observed with little or no relation to what is asked to justify. In this case we locate the following answers:

Table 2 Example of type 1 answers

d)	It is isosceles since two angles remain the same, since they are co-corresponding angles with others and thus we can determine if they are equal or not.
e)	Because it acts as the radius of a circle that which always has the same magnitude and is linked to a center as in the current case.

In a second case, another student uses the isosceles triangle definition to justify that the triangle is isosceles. And when he tries to justify why the segment *BP* maintains its constant magnitude, he uses its visual perception as a source of argumentation.

Table 3 Example of type 2 answers

d)	Because it has two equal angles and two equal sides, which are characteristics that made it an isosceles triangle
e)	Because the construction makes the segment AP to always be of fixed length.

It is clear that the students located in this case do not recognize the hypotheses of the problem and confuse the causes with the effects on the construction, indicating that they do not base their response on a deductive reasoning and are therefore located in Paradigm 1.

In the third type of responses the students mix Paradigm 1 ways of arguing with arguments specific to Paradigm 2, this is the case of the following response, which provides an answer to item (d) that was independent from the observation, but does not make it clear that equality of the angles implies equality of the sides, instead the student responds to item (e) very clearly that in the case of an isosceles triangle with a fixed side, the other side must have a fixed length. We consider that the answer to the first item does not correspond to Paradigm 2, but its response to the second item fits in the ways of arguing specific to this paradigm.

Table 4 Example of type 3 answers

d)	Despite of its movement there is still two equal sides and two equal angles.
e)	Segment *AB*'s length is 8 and as it is an isosceles triangle it's the same as *BP*, then, despite the movement, they will measure the same and stay constant.

Finally, the fourth type of response includes those that can be located in Paradigm 2, where it is observed that the student recognizes the data provided as hypotheses from which he can build his arguments. The following two responses illustrate this case.

Table 5 Example of type 4 answers

d)	Because it has two equal internal angles and therefore the opposite sides to the angles are equal.
e)	Since the triangle is isosceles, it forces the segment *BP* to remain the same length as *AB*, and as *AB* never changes, neither does *BP*.

Conclusions

Of the 35 students who were part of the group, 8 of them were able to present arguments to justify propositions, which can be located in Paradigm 2, those represented in the Type 4 responses. The rest, to some extent, showed arguments of Paradigm 1. This proportion illustrates a serious problem, showing that most students are unable to use deductive arguments to justify geometric propositions. We emphasize that their responses indicate that they can identify geometric facts, but they cannot present deductive arguments to explain them.

These results motivate us to, in the future, study the characteristics of the teaching of Geometry at the pre-university level, where the source of some of the difficulties observed could be found. Finally, we want to add that the use of GeoGebra has been attractive to students, but this has not been reflected in improving the nature of the arguments put forward by students.

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DIFICULTADES PARA JUSTIFICAR PROPOSICIONES GEOMÉTRICAS AL RESOLVER PROBLEMAS DE LUGARES GEOMÉTRICOS CON GEOGEBRA

DIFFICULTIES TO JUSTIFY GEOMETRIC PROPOSITIONS WHEN SOLVING LOCI PROBLEMS WITH GEOGEBRA

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Se presenta aquí un estudio desarrollado con estudiantes de Ingeniería de la Universidad de Sonora, México, sobre sus dificultades para justificar proposiciones geométricas. El trabajo está enmarcado en un curso de Geometría Analítica diseñado con apoyo de GeoGebra y se refiere al tema de lugares geométricos. Se utiliza la teoría de los paradigmas geométricos para explicar estas dificultades y el estudio revela que la formación en geometría de los estudiantes que ingresan a la Universidad, está limitada principalmente a la geometría natural, mostrando una precaria familiaridad con los métodos de la geometría axiomática natural.

Palabras clave: Geometría y Pensamiento Geométrico y Espacial, Actividades y Prácticas de Enseñanza, Tecnología

Introducción

Con el propósito de explorar hasta qué punto los estudiantes de ingeniería pueden justificar proposiciones geométricas, se han diseñado cinco secuencias didácticas sobre el tema de lugares geométricos. Las secuencias han sido diseñadas para desarrollarse con GeoGebra y tomando como referencia los métodos usados por Descartes (1954, pp. 50-55) en la presentación de su hiperbológrafo, para identificar las relaciones geométricas de la construcción y encontrar la ecuación de la hipérbola. Las actividades fueron propuestas en un curso de Geometría Analítica impartido a 35 estudiantes de Ingeniería Civil durante el primer semestre de 2018.

Referencias teóricas

Las respuestas de los estudiantes se analizaron en el marco de la teoría de paradigmas geométricos propuesta por Hedemount y Kuzniak (2003); según estos autores bajo el término geometría elemental, coexisten tres tipos distintos de paradigmas que conducen a distinguir tres tipos de geometría: la geometría natural, la geometría axiomática natural y la geometría axiomática formalista, se describen a continuación las dos primeras geometrías, que son las que se utilizan en el presente trabajo, según la versión de Kuzniak (2006):

La *geometría natural* puede verse como una ciencia empírica en la que se trabaja con objetos muy ligados a la realidad, que pueden ser medidos y comparados físicamente. Las proposiciones aquí pueden validarse experimentalmente porque dependen de la percepción sensorial que se tiene de ellas.

En la *geometría axiomática natural* los objetos geométricos son solamente aproximaciones a la realidad y son definidos por los axiomas de la Geometría Euclidiana clásica. Las proposiciones aquí son validadas mediante demostraciones construidas a partir de los axiomas euclidianos y de resultados geométricos previamente demostrados.

En el diseño de las secuencias didácticas, se han intentado recuperar las ideas desarrolladas por Descartes sobre lugares geométricos, en particular aquellas que están presentes en la construcción del hiperbológrafo (Descartes, 1954, pp. 50-55). Del método utilizado por Descartes, rescatamos dos características que serán tomadas en cuenta para el diseño de las secuencias:

1. La identificación de las cantidades que cambian y las que permanecen fijas durante el trazado de la curva y de las relaciones entre ellas, tal como se muestra en Arcavi (2000).
2. La deducción exige el uso de los métodos propios de la geometría axiomática natural, puesto que está basada en las propiedades de los triángulos semejantes, establecidas como teoremas en esta geometría.

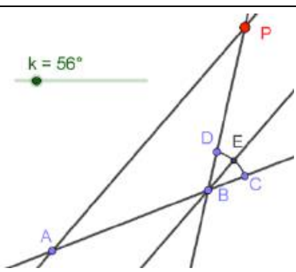
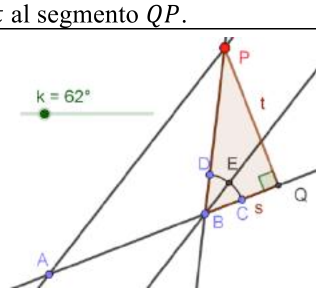
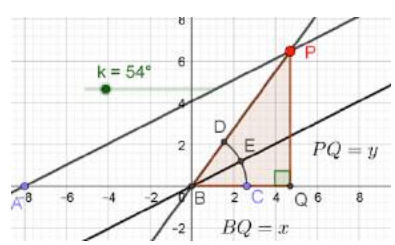
Ambas características nos parecen deseables en la formación matemática de un ingeniero, la primera por la importancia que tiene en la modelación algebraica de situaciones geométricas y la segunda porque facilita la explicación y la argumentación de resultados geométricos.

Secuencias y aplicación

Retomando las ideas de Descartes, Arcavi y Kuzniak y apoyándonos en el software GeoGebra, hemos elaborado una serie de secuencias didácticas sobre la noción de lugar geométrico. La idea es que los estudiantes de ingeniería justifiquen algunos resultados geométricos elementales con argumentos propios de la *geometría axiomática natural*.

Las secuencias fueron cinco en total y los estudiantes las desarrollaron a lo largo de dos semanas, en un centro de cómputo en el que cada quien contó con una computadora con el software GeoGebra instalado. En todas las secuencias se ofrece primeramente al estudiante una construcción en GeoGebra, en la que se omiten los ejes cartesianos para que los estudiantes se puedan concentrar en los elementos de la construcción y las relaciones que se manifiestan al variar elementos de las figuras. Como ejemplo, la siguiente tabla resume los propósitos y características de la secuencia 2.

Tabla 1 Características de la secuencia 2

<table>
<tr><td colspan="3">Propósitos: a) Identificar las relaciones entre las cantidades, que se conservan al variar k y presentar argumentos geométricos sobre su veracidad. b) Verificar que la curva trazada por P es una circunferencia y justificar geométricamente este resultado. c) Expresar algebraicamente las relaciones identificadas en a). d) Encontrar la ecuación en s y t de la circunferencia de la Figura 1b) y en términos de x y y en la Figura 1c).</td></tr>
<tr><td colspan="3">Características de las construcciones</td></tr>
<tr><td>AB es una recta fija, CBD es un ángulo de medida k, BE es la bisectriz del ángulo CBD, AP es la paralela a la bisectriz trazada por A y P es el punto de intersección de AP y BP.</td><td>La construcción es la misma que en la Figura 1a), pero se ha trazado desde P , una perpendicular a AB que interseca a AB en Q y se ha etiquetado como s al segmento BQ y como t al segmento QP.</td><td>La construcción es mostrada en la Figura 1b), pero se ha trasladado y rotado para que la recta AB coincida con el eje X y B coincida con el origen.</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>Figura 1a</td><td>Figura 1b</td><td>Figura 1c</td></tr>
</table>

En esta secuencia, se inicia con la construcción de la Figura 1a, en la que se han trazado las rectas AB y BD (de tal forma que BD forme un ángulo de k grados con la recta fija AB), la bisectriz del ángulo CBD y una recta paralela a la bisectriz EB, que pasa por A. Luego se pide al estudiante que explore la construcción haciendo variar el deslizador k, el cual controla el ángulo CBD, y que al mover la construcción observe que el punto P describe una circunferencia. Las tareas aquí pretenden sistematizar la exploración: que se distingan primero aquellas cantidades (medidas de ángulos y segmentos) que cambian de las que permanecen constantes, durante el movimiento y luego que establezca las relaciones posibles entre estas cantidades, principalmente las relaciones de igualdad. El trabajo con esta construcción concluye con la solicitud al estudiante de que justifique geométricamente las relaciones detectadas. La secuencia concluye con la introducción de coordenadas cartesianas para tomarlas como referencia (Figura 1c) y expresar algebraicamente la curva trazada por el punto P, cuando se hace variar el parámetro k.

Análisis de resultados

A lo largo del desarrollo de estas actividades, hemos concentrado la atención en aquellas preguntas en las que se solicita a los estudiantes justificar alguna proposición geométrica, porque nuestro interés principal es el de observar la naturaleza de las argumentaciones construidas por los estudiantes. Concentraremos nuestro análisis en las respuestas a estas preguntas en la secuencia 2.

En esta actividad, después de haber explorado la construcción mostrada en la Figura 1a y una vez que han establecido cuáles son los ángulos que permanecen iguales al arrastrar k, los estudiantes concluyen sin dificultades que el triángulo ABP es isósceles. Como podrá verse en las respuestas siguientes, sus dificultades empiezan cuando intentan justificar algunos de los resultados obtenidos. En el inciso d) se les pide que justifiquen por qué el triángulo se mantiene isósceles y en el inciso e) se solicita que justifiquen por qué el segmento BP funciona como radio del círculo, es decir por qué su magnitud no cambia. La construcción ofrece datos sobre los ángulos y sobre el segmento AB, se esperaba entonces que llegaran a la conclusión de que el triángulo ABP es isósceles partiendo de que dos de sus ángulos permanecen siempre iguales y pudieran argumentar que por lo tanto sus lados opuestos AB y BP debieran ser siempre iguales.

Se detectaron cuatro tipos de respuestas, que ilustraremos a continuación:

En un primer caso ubicamos a los estudiantes que han ofrecido respuestas cuyas formas de argumentación han resultado poco coherentes, en los cuales se mezclan argumentos que parecieran ubicarse en el Paradigma 2, con otros basados en hechos observados con poca o nula relación con lo que se pide justificar. Ubicamos en este caso las respuestas siguientes:

Tabla 2 Un ejemplo de respuestas de tipo 1

d)	Es isósceles ya que 2 ángulos permanecen iguales ya que son ángulos correspondientes con otros y así podemos determinar si son iguales o no.
e)	Porque actúa como el radio de un circulo el cual siempre tiene la misma magnitud y está anclado a un centro, así como ocurre en el presente círculo.

En un segundo caso, el estudiante recurre a la definición de triángulo isósceles para justificar que el triángulo es isósceles. Y cuando intenta justificar por qué el segmento BP mantiene su magnitud constante, recurre a su percepción visual como fuente de argumentación.

Tabla 3 Un ejemplo de respuestas de tipo 2

d)	Porque tienen dos ángulos iguales y dos lados iguales por lo tanto son características por la cual se conforma un triángulo isósceles.
e)	Porque la construcción obliga a que el segmento AP siempre se mantenga fijo.

Es claro que los estudiantes ubicados en este caso no reconocen las hipótesis del problema y confunden las causas con los efectos en la construcción, lo cual indica que no basan su respuesta en un razonamiento deductivo y por lo tanto se ubican en el Paradigma 1.

En un tercer tipo de respuestas se observa que el estudiante mezcla formas de argumentar del Paradigma 1 con argumentos propios del Paradigma 2, éste es el caso de la respuesta siguiente, en la se ofrece una respuesta al inciso d) que se ha desprendido de la observación, pero no deja en claro que la igualdad de ángulos implica la igualdad de lados, en cambio responde al inciso e) con mucha claridad al respecto de que tratándose de un triángulo isósceles con un lado fijo, el otro lado deberá tener una magnitud fija. Consideramos que la respuesta al primer inciso no corresponde al Paradigma 2, pero su respuesta al segundo inciso cae en las formas de argumentar propias del este paradigma.

Tabla 4 Un ejemplo de respuestas de tipo 3

d)	A pesar de su movimiento siguen siendo con dos lados iguales y tiene ángulos iguales.
e)	El segmento AB mide 8 y al ser triángulo isósceles mide igual que BP, entonces a pesar del movimiento medirán lo mismo y se mantiene constante.

Y el cuarto tipo de respuesta son las que propiamente pueden ubicarse en el Paradigma 2, en donde se observa que el estudiante reconoce los datos proporcionados, como hipótesis a partir de las cuales puede construir sus argumentaciones, las siguientes dos respuestas ilustran este caso.

Tabla 5 Un ejemplo de respuestas de tipo 4

d)	Porque tiene dos ángulos iguales y por lo tanto dos lados iguales que son los opuestos a los ángulos iguales.
e)	Porque al ser triángulo isósceles obliga a que el segmento BP permanezca de la misma magnitud del AB, y como AB nunca cambia, este tampoco.

Conclusiones

De los 35 estudiantes que integraban el grupo, 8 de ellos pudieron presentar argumentos para justificar proposiciones, que pueden ubicarse en el Paradigma 2 y que aquí hemos catalogado como respuestas de Tipo 4. El resto, en mayor o menor medida, mostraron utilizar las formas de argumentación propias del Paradigma 1. Esta proporción ilustra un problema serio, al mostrar que la mayor parte de los estudiantes se encuentran imposibilitados para usar argumentos deductivos al justificar proposiciones geométricas; destacamos que sus respuestas indican que pueden identificar hechos geométricos, pero no pueden presentar argumentos deductivos para explicarlos.

Los resultados nos motivan a estudiar en el futuro las características de la enseñanza de la Geometría en el nivel pre-universitario, en donde pudiera encontrarse la fuente de algunas de las dificultades observadas. Queremos agregar por último que el uso de GeoGebra ha resultado atractivo para los estudiantes, pero esto no se ha reflejado en mejorar la naturaleza de las argumentaciones presentadas por los estudiantes.

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SPATIAL IMAGINATION TO WORK ON 3D SPACE USING A FIGURAL DEVICE

IMAGINACIÓN ESPACIAL PARA TRABAJAR OBJETOS EN EL ESPACIO 3D USANDO UN DISPOSITIVO FIGURAL

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Currently, the spatial imagination is not explicitly developed in schools, it will be necessary to interpret 3D objects that are represented in 2D. To develop the ability of 3D spatial imagination, it is necessary to consider the spatial location and relative position between objects. Relating sets of points with appropriate variations of triads of type (x, y, z) promotes this ability. In this ongoing research, the numerical variation of the entries associated with the graphic objects provides indicators of location and relative position of points, lines or planes to high school students. We find selective numerical variation, supports the development of spatial imagination, whose repeated use will train a habit in orientation, which must be worked to become transparent.

Keywords: Geometry and Geometrical and Spatial Thinking

According to research related to the analysis of syllabus and textbooks of middle and higher education, for the study of topics such as vector spaces, an explicit introduction to the development of spatial imagination is not considered necessary, which then is a disadvantage in graphic interpretations. However, in these syllabus there is a graphic approach that tries to be solved through the graphic point-ordered pair relationship, which does not seem to be sufficient for the type of tasks proposed. At the same time, 3D representations are scarce and the problems associated with interpretation are not addressed because they are used as an illustration and not as a mathematical object; due to that situation, students do not have the opportunity to go beyond a formal education that emphasizes algebraic treatment, because they do not have graphical representations (Van Dormolen, 1986).

On the other hand, students show a natural propensity to identify more easily some directions over others in tasks of graphic interpretation in the case of graphs in representations of 3D spaces, which contributes to disorienting the student in tasks of spatial imagination (Cohen , 2001). Other sources of conflict arise in the interpretation of perspective, due, among other things, to the fact that we must reorganize the information to understand that this interpretation is present and that the graphic representation is subordinate to it, that is, "seeing and knowing" are two different things, according to Parzysz (1988). Preference to identify lines or sets of lines in a general way can also be an obstacle if there is not a direct instruction to determine their orientation (Bakó, 2003).

Because the interpretation of the perspective or 3D depth in a plane is a work in which the visual information must be decoded, from our point of view it is necessary to have indicators that guide the spatial imagination to carry out an adequate treatment of 3D graphic objects.

Some learning proposals to work with 3D space, link the figural properties of drawing with its mathematical characteristics through the use of orthogonal projections, which have a constructive and prescriptive foundation, that is organized through parallelism and perpendicularity (Parzysz , 1988), and goes in the direction of establishing spatial indicators for students to improve their spatial imagination. Therefore, we consider it of great importance to provide such indicators that guide students for the interpretation of 3D space in terms of locating the position of points as well as sets of points in their relative positions.

Theoretical framework

In learning mathematics, the use of signs as semiotic instruments (Vigotsky, 1981a) that allow knowledge to be mediated, provides adequate conditions for students to develop reflective practice (Radford, 2006, 2012); therefore, it is important to associate mathematical ideas such as graphics, in terms of a practice that uses semiotic instruments, particularly when we establish relationships between algebraic objects and graphics, to integrate the interpretation of 3D graphics into a single body of knowledge, as in this case.

On the other hand, mathematical objects such as graphs, have a visual and abstract character, so it can be considered as a figural concept (Fischbein, 1993), mathematical objects that can be thought of in a double status, as objects and as concepts. In particular, geometric objects are figural concepts, since they reflect spatial properties (shape, position, magnitude) and at the same time, they possess conceptual qualities such as ideality, abstraction, generality, perfection, among others (Fischbein, 1993), and apparently this is also the case for graphs.

Because the figural properties are ostensive, that is "they are in front of the eyes", they allow us to differentiate their location and their relative position, in addition to the fact that their position is the only property of graphic objects, because they are represented in homogeneous spaces of representation where the only quality is their position (Nemirovsky, 2001). We are able to lean on these properties to develop spatial imagination, which in our case includes both, the ability to locate points, as well as the ability to establish the relative position between graphic objects that include points, lines and planes.

In order to establish spatial indicators for the spatial imagination, we will be using a treatment on the position of points of the type (x, y, z) which by means of the numerical variation of their coordinates, describes points and sets of associated points as points, lines and planes, which depending on the situation take on the role of spatial indicators. Used in this way, the triads become signs that associate properties of algebraic description and spatial position, which makes them what we have named, a figural device, when used for this dual purpose. Besides that, due to its perceptual nature, its repeated use would allow students to develop habits of thought (Cuoco, Goldenberg and Mark, 1996) until they acquire a form of transparency (Roth, 2003) in their location and relative position.

Next, we propose what are our research hypotheses:

1. The spatial imagination can be developed by positional approximations by means of the numerical variation of the triad (x, y, z) associated with the points that can describe lines and planes, as well as the hability to establish their relative position.
2. The acquisition of habits of thought plays a central role in the acquisition of the ability of spatial imagination, if the variation of the triads is associated with the position of the objects described.

Methodology

In particular, in the case of visual interpretation of three-dimensional space, it is possible to make use of certain types of figural devices when we use the triad (x, y, z) that has its origin in the representation of points in three-dimensional spaces. Based on their position properties, we explore the relationship between each coordinate of the triad (x, y, z) and those of the points in space.

The activity carried out establishes that each of the coordinates refers to a specific orientation in space, which makes it possible to make constant some of the values of the coordinates to vary one or two of them or to vary all three. This allows the coordinate to be associated with the relative position of the points and point sets, using the axes as an orientation reference.

We proceeded with two different environments for locating points and sets of points. On the one hand, in a physical environment we made use of a manipulable (Godino, 1998) for the realistic representation of the 3D space, where we used a model consisting of three slightly transparent plastic sheets, of different colors, assembled in such a way that they represented the eight octants, and the joints of each sheet formed the axes of space. For the second environment, graphical representations were used in GeoGebra.

The developed procedure was based on establishing a location relationship supported by the numerical variation in the triad coordinates (x, y, z), asking for the point or set of points that were described from the variations made and the coordinates with constant values in the two environments.

Implementation

The implementation of this research was carried out in two groups of the sixth semester of high school: group A, consisted of 20 students and group B, consisted of 36, all of them 17 and 18 years old and none of them with a notion of 3D space. They were given a class where they were introduced to three-dimensional space, explaining concepts and placing triads in different octants. After this, we did a case study with the student Elías from group B, and we did a follow-up interview that we describe below.

First phase: physical model and location of areas associated with the signs. It was a reminder of the positional meaning that the triad can acquire, as in cases where the points have the following signs: a) (±, -, ±) b) (+, +, +) c) (+, - , ±) d) (-, +, ±) e) (+, ±, ±) f) (-, -, -). The work was accompanied by the comparison of the location of the triad that Elías made first in his worksheets, varying the possibilities of orientation depending on the signs. He was also asked for suggestions on points with specific properties.

Second approach: Variations. The researchers proceeded to make variations, first, we set a coordinate allowing the variation in the two remaining positions, so that the sets of points described formed planes to reflect on the properties of the lines formed by the intersections of parallel and perpendicular planes at this stage, particularly with the canonical planes. Then we set a pair of coordinates allowing the variation of the missing position, so that the sets of points described now formed lines, parallel to one of the axes, as in the cases of (-1, 2, z), (1, y, 3) or (x, 2, -1), among others; the sets were located in the space provided by GeoGebra.

We present an extract from the interview with Elías, that allowed the researchers to see the use of the triad as a figural concept in the case of the variation of two coordinates with the other one fixed; we asked him to mention 5 triads in which the first coordinate had the same value and the second and third varied, those points were graphed in the software (Figure 1) and he was asked:

INV: What do you observe about these points? (Figure 1)
Elías: Now they are all facing each other.
INV: Would they be on some plane? Would all the points be inside this? (indicates the region of the points)
Elías: From the plane? Yes.
INV: What plane would it be?
Elías: It would be (5, y, z)
After Elías said (5, y, z), the plane was graphed as in Figure 2 to verify:

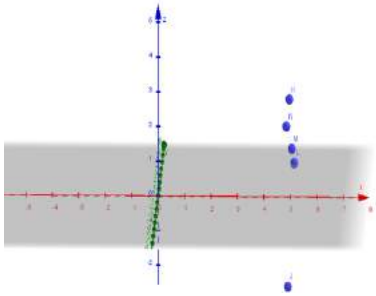

Figure 1: Graphed points

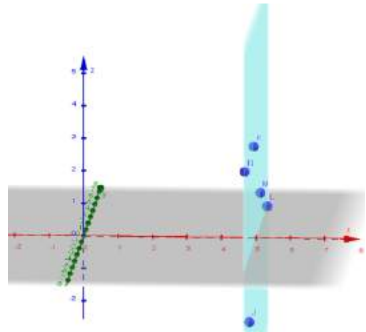

Figure 2: Plane (5, y, z)

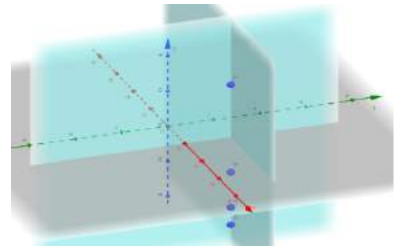

Figure 3: Planes (2, y, z) y (x, 2, z)

Subsequently, the student was asked to mention four triads in which the first and second coordinates were fixed and the third varied, those points were graphed in GeoGebra and Elías was able to visualize that they were "on top of each other", that is "there is only one line left" and there could be two planes - (2, y, z) and (x, 2, z) - that could go through that line (Figure 3).

Results.

In the group stage of the experiment, students were able to have a first adequate approach to three-dimensional space, this was observed qualitatively from the variational management they gave to triads with different signs, placing them correctly in one of the eight octants of space, and by making sense of the "new" axis, the Z axis. In addition, although the manipulative was only used at the beginning to determine the orientation and the octants, in the interview the student Elías used it and expressed that this model made it easier for him to locate the triads, that is, it allowed him to better organize his thoughts. We saw this reflected at different times: when he identified each graphed plane and related them to their corresponding triads; when he was asked to mention five triads where only the first one was fixed and the other two varied, managing to visualize the plane in which they were located; and finally, when he chose different triads, with the first two coordinates fixed and varying the third, to visualize the intersection of two planes that form in them, and these, in their role as a figural device led him to also identify the line generated with those triads. Achieving an approximation to the transparency of the particularly chosen triads.

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IMAGINACIÓN ESPACIAL PARA TRABAJAR OBJETOS EN EL ESPACIO 3D USANDO UN DISPOSITIVO FIGURAL

SPATIAL IMAGINATION TO WORK ON 3D SPACE USING A FIGURAL DEVICE

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Actualmente en las escuelas no se desarrolla explícitamente la imaginación espacial, esta será necesaria para interpretar objetos 3D que se representan en 2D. Para desarrollar la habilidad de la imaginación espacial en 3D es necesario considerar la ubicación espacial y la posición relativa entre objetos. Relacionar conjuntos de puntos con variaciones adecuadas de triadas del tipo (x, y, z) promueve esta habilidad. En esta investigación en curso, la variación numérica de las entradas asociadas a los objetos gráficos proporciona indicadores de ubicación y posición relativa de puntos, rectas o planos a los estudiantes de bachillerato. Encontramos que la variación numérica selectiva apoya el desarrollo de la imaginación espacial, cuyo uso reiterado tiende a formar un hábito en la orientación, el que debe ser trabajado para convertirse en transparente.

Palabras clave: Geometría y Pensamiento Geométrico y Espacial

De acuerdo con investigaciones relacionadas con el análisis de programas de estudio y libros de texto de nivel medio superior y superior, para el estudio de temas como el relativo a los espacios vectoriales no se considera necesaria una introducción explícita al desarrollo de la imaginación espacial, lo que luego es una desventaja en la interpretación gráfica. Sin embargo, en dichos programas hay un enfoque gráfico que pretende ser resuelto a través de la relación punto gráfico-pareja ordenada, que no parece ser suficiente para el tipo de tareas que se proponen. Al mismo tiempo las representaciones 3D son escasas y no se atienden las problemáticas asociadas a la interpretación porque son usadas como una ilustración y no como un objeto matemático, por lo que los estudiantes no tienen oportunidad de ir más allá de una educación formalista que enfatiza el tratamiento algebraico, debido a que no cuentan con representaciones gráficas (Van Dormolen, 1986).

Por otro lado, los estudiantes manifiestan una propensión natural a identificar con más facilidad unas direcciones sobre otras en tareas de la interpretación gráfica en el caso de gráficas en representaciones de espacios 3D, lo que contribuye a desorientar al estudiante en tareas de imaginación espacial (Cohen, 2001). Otras fuentes de conflicto se presentan en la interpretación de la perspectiva debido, entre otras cosas, a que debemos reorganizar la información para entender que esta interpretación está presente y que la representación gráfica se supedita a ella, es decir "ver y saber" son dos cosas distintas al decir de Parzysz (1988). También puede ser un obstáculo la preferencia para identificar unas rectas o conjuntos de ellas de manera general si no hay una instrucción ex profeso para determinar su orientación (Bakó, 2003).

De manera que la interpretación de la perspectiva o profundidad 3D en un plano es un trabajo en el que se debe decodificar la información visual, desde nuestro punto de vista es necesario contar con indicadores que orienten la imaginación espacial para llevar a cabo un tratamiento adecuado de los objetos gráficos en 3D.

Algunas propuestas de aprendizaje para trabajar con el espacio 3D ligan las propiedades figurales del dibujo con sus características matemáticas a través del uso de proyecciones ortogonales, las cuales tienen un fundamento constructivo y prescriptivo, el que es organizado a través del paralelismo y la perpendicularidad (Parzysz, 1988), lo que avanza en la dirección de establecer indicadores espaciales a los estudiantes para mejorar su imaginación espacial. Por ello, consideramos de gran importancia proporcionar dichos indicadores que orienten a los estudiantes para la interpretación del espacio 3D en términos de ubicar la posición de puntos así como de conjuntos de ellos en sus posiciones relativas.

Marco Teórico

En el aprendizaje de la matemática, el uso de los signos como instrumentos semióticos (Vigotsky, 1981a) que permitan mediar el conocimiento proporciona condiciones adecuadas para que los estudiantes desarrollen una práctica reflexiva (Radford, 2006, 2012); por ello, es importante asociar las ideas matemáticas como la gráfica en términos de una práctica que hace uso de instrumentos semióticos, en particular cuando establecemos relaciones entre los objetos algebraicos y los gráficos para integrar en un solo cuerpo de conocimiento la interpretación de la gráfica 3D, como en el caso que nos ocupa.

Por otro lado, los objetos matemáticos como la gráfica tienen un carácter visual y abstracto, por lo que puede ser considerada como un concepto figural (Fischbein, 1993), objetos matemáticos que pueden ser pensados en un doble estatus, como objetos y como conceptos. En particular, los objetos geométricos son conceptos figurales, ya que reflejan propiedades espaciales (forma, posición, magnitud) y al mismo tiempo, poseen cualidades conceptuales como idealidad, abstracción, generalidad, perfección, entre otras (Fischbein, 1993), y al parecer este también es el caso de las gráficas.

En tanto que las propiedades figurales son ostensivas, esto es "están frente a los ojos", nos permiten diferenciar su ubicación y su posición relativa, además de que su posición es la única propiedad de los objetos gráficos, debido a que son representados en espacios de representación homogéneos donde la única cualidad es su posición (Nemirovsky, 2001). Estamos en condiciones de apoyarnos en estas propiedades para desarrollar la imaginación espacial, que en nuestro caso incluye tanto la habilidad para localizar los puntos, así como la de establecer la posición relativa entre objetos gráficos que incluyen puntos, rectas y planos.

Con el objeto de establecer indicadores espaciales para la imaginación espacial, estaremos haciendo uso de un tratamiento sobre la posición de puntos del tipo (x, y, z) que mediante la variación numérica de sus coordenadas nos describen puntos y conjuntos de puntos asociados como puntos, rectas y planos, los cuales según sea la situación toman dicho papel de indicadores espaciales. Usadas así las triadas se transforman en signos que asocian propiedades de descripción algebraica y de posición espacial, lo que las convierte en lo que hemos dado en llamar un dispositivo figural cuando es usado con este doble propósito. Además de que, por su naturaleza perceptual, su uso reiterado permitiría a los estudiantes desarrollar hábitos de pensamiento (Cuoco, Goldenberg y Mark, 1996) hasta adquirir una forma de transparencia (Roth, 2003) en la localización y la posición relativa de ellos.

A continuación, proponemos las que son nuestras hipótesis de investigación:

1. La imaginación espacial puede ser desarrollada por aproximaciones posicionales mediante la variación numérica de la triada (x, y, z) asociada a los puntos que pueden describir rectas y planos, así como pueden establecer su posición relativa.

2. La adquisición de hábitos de pensamiento juega un papel central en la adquisición de la habilidad de imaginación espacial si se asocia la variación de las triadas con la posición de los objetos descritos.

Metodología

En particular, en el caso de la interpretación visual del espacio tridimensional, es posible hacer uso de cierto tipo de dispositivos figurales cuando usamos la triada (x, y, z) que tiene su origen en la representación de los puntos en el espacio tridimensional. Con base en sus propiedades de posición exploramos la relación entre cada coordenada de la triada (x, y, z) y las de los puntos sobre el espacio.

La actividad desarrollada establece que cada una de las coordenadas se refiere a una orientación específica en el espacio, lo que permite hacer constantes algunos de los valores de las coordenadas para variar uno o dos de ellos o variar los tres. Esto permite asociar la coordenada con la posición relativa de los puntos y conjuntos de puntos, usando a los ejes como referencia de orientación.

Procedimos con dos entornos distintos para la localización de puntos y conjuntos de puntos. Por un lado, en un entorno físico hicimos uso de un manipulable (Godino, 1998) para la representación realista del espacio 3D, donde usamos un modelo formado por tres láminas de plástico ligeramente transparente, de diferentes colores, ensambladas de tal forma que representaban los ocho octantes, las uniones de cada lámina formaban los ejes del espacio. Para el segundo entorno se usaron representaciones gráficas en GeoGebra.

El procedimiento desarrollado se basó en establecer una relación de localización apoyada en la variación numérica en las coordenadas de la triada (x, y, z), preguntando por el punto o conjunto de puntos que eran descritos a partir de las variaciones hechas y de las coordenadas con valores constantes en los dos entornos.

Puesta En Marcha.

La puesta en marcha de esta investigación se llevó a cabo en dos grupos de sexto semestre de bachillerato: el grupo A, conformado por 20 estudiantes y el grupo B, conformado por 36, todos ellos de 17 y 18 años y ninguno con noción del espacio 3D. Se les dio una clase donde se les introdujo al espacio tridimensional, explicando conceptos y ubicando triadas en diferentes octantes. Después de esto, hicimos un estudio de caso con el estudiante Elías del grupo B, le hicimos una entrevista de seguimiento que describimos a continuación.

Primera fase: modelo físico y localización de zonas asociadas a los signos. Se trató de un recordatorio del significado posicional que puede adquirir la triada como en los casos en que los puntos tienen los siguientes signos: a) (±, -, ±) b) (+, +, +) c) (+, -, ±) d) (-, +, ±) e) (+, ±, ±) f) (-, -, -). El trabajo se acompañó de la comparación de la ubicación de la triada que Elías hizo primero en sus hojas de trabajo variando las posibilidades de orientación dependiendo de los signos. También se le pidieron sugerencias sobre puntos con propiedades específicas.

Segunda aproximación: Variaciones. Las investigadoras procedimos a hacer variaciones, en primer lugar, fijamos una coordenada permitiendo la variación en las dos posiciones restantes, de manera que los conjuntos de puntos descritos formaron planos para reflexionar en las propiedades de las rectas formadas por los cruces de planos paralelos y perpendiculares en esta etapa, en particular con los planos canónicos. Luego fijamos un par de coordenadas permitiendo la variación de la posición faltante, de manera que los conjuntos de puntos descritos ahora formaron rectas paralelas a alguno de los ejes como en los casos de (-1, 2, z), (1, y, 3) o (x, 2, -1), entre otros; los conjuntos fueron ubicados en el espacio proporcionado por GeoGebra.

Presentamos un extracto de la entrevista con Elías que a las investigadoras nos permitió ver el uso de la triada como concepto figural en el caso de la variación de dos coordenadas con la otra fija; le pedimos que mencionara 5 triadas en las que la primera coordenada tuvieran el mismo valor y la segunda y la tercera variaran, se graficaron esos puntos en el software (Figura 1) y se le preguntó:

INV: ¿Qué observas de estos puntos? (Figura 1)
Elías: Ahora todos están uno frente a otro.
INV: ¿Estarían sobre algún plano? ¿Todos los puntos estarían adentro de esto? (señala la región de los puntos)
Elías: ¿Del plano? Sí.
INV: ¿Qué plano sería?
Elías: Sería (5, y, z)

Luego de que Elías dijo (5, y, z), se graficó el plano como en la Figura 2 para comprobar:

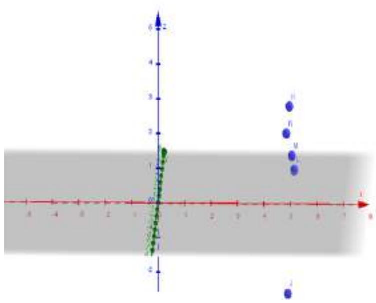

Figura 1: Puntos graficados

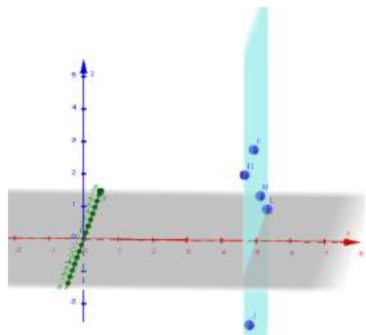

Figura 2: Plano (5, y ,z)

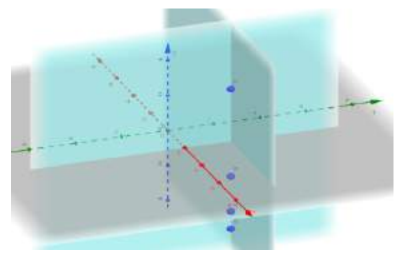

Figura 3: Planos (2, y, z) y (x, 2, z)

Posteriormente, se le pidió al estudiante que mencionara cuatro triadas en las que la primera y la segunda coordenadas fueran fijas y la tercera variara, se graficaron esos puntos en GeoGebra y Elías pudo visualizar que estaban "uno encima del otro", es decir que "queda una línea nada más" y que podrían ser dos planos - (2, y, z) y (x, 2, z) - los que podían pasar por esa línea (Figura 3).

Resultados.

En la etapa grupal del experimento se logró que los estudiantes tuvieran un primer acercamiento adecuado al espacio tridimensional, esto se observó de forma cualitativa a partir del manejo variacional que le daban a las triadas con diferentes signos ubicándolas correctamente en alguno de los ocho octantes del espacio y al darle sentido al "nuevo" eje, el eje Z. Además, aunque el manipulable sólo se usó al inicio para determinar la orientación y los octantes, en la entrevista el estudiante Elías hizo uso de él y expresó que este modelo le facilitaba ubicar las triadas, es decir, le permitía una mejor organización de sus pensamientos. Lo cual vimos reflejado en diferentes momentos: cuando identificó cada plano graficado y los relacionó con sus triadas correspondientes; cuando se le pidió que mencionara cinco triadas donde solo la primera quedara fija y variaran las otras dos, logrando visualizar el plano en el que se encontraban; y finalmente, cuando escogió diferentes triadas con las dos primeras coordenadas fijas y variando la tercera, para visualizar la intersección de dos planos que se forman en ellas, y estos en su papel de dispositivo figural propiciaron que también identificara la recta generada con esas triadas. Logrando una aproximación a la transparencia de las triadas elegidas particularmente.

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ENHANCING SPATIAL ABILITIES THROUGH EXPOSER TO COMPUTER-AID DESIGN PROGRAMS

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Spatial ability is important in the learning and understanding of mathematics and has been recognized as an indicator of mathematics achievement and STEM success. The present study aims to investigate how experiences with computer-aid design programs can enhance student's spatial ability measured by mental rotation skills. Quantitative data were collected before and after intervention using the redrawn Vandenburg and Kuse Mental Rotation Test by Peters et al. (1995). A paired sample t-test and 95% confidence intervals indicated a statistically significant difference between observed pre and post test scores. The calculated Cohen's d effect size of 0.63 indicated the CAD intervention had a positive impact on students' mental rotation skills. It can be concluded that utilizing these technologies can aid in developing and improving spatial abilities which can lead to improved mathematics achievement and STEM success.

Keywords: Spatial Thinking, STEM, Teaching Tools, Technology

Introduction and Literature Review

Improving mathematics science, technology, engineering and mathematics (STEM) success is a very relevant issues in research and education. Spatial ability has been recognized as a predictor of adult success in STEM areas across several longitudinal studies that followed a large population of both normative and intellectually gifted individuals from adolescence to adulthood (Shea, Lubinski, & Benbow, 2001; Wai, Lubinski, & Benbow, 2009). In addition, researchers have found that spatial ability provides validity to mathematical and verbal reasoning abilities yet in education there is little emphasis on the development of spatial abilities (Basham & Kotrlik, 2008). Psychologists and education researchers have been interested in the connection between spatial ability and mathematics achievement since the mid-1900's (Bishop, 1980) and prior research has supported this link (e.g. Carr et al., 2019; Casey, Nuttal, & Perzris, 1997; Ganley & Vasilyeva, 2011; Gilligan, Flouri, & Farran, 2017; Hawes, Moss, Caswell, Seo, & Ansari, 2019; Rabab'h & Veloo, 2015; Rutherford, Karamarkovick, & Lee, 2018; Verdine et al., 2014). Despite the acknowledgement and confirmation of the connections between spatial ability and STEM success, specifically mathematics achievement, there is little research on the development of spatial abilities. In the current study, researchers investigated how the implementation of computer-aided design software and 3D printing class with adolescents is associated to growth in spatial ability measured via mental rotation skills.

Mental Rotation. Spatial ability is often organized into three categories: (1) spatial perception, (2) spatial visualization, and (3) mental rotation (Linn & Peterson, 2004). All three categories are important for learning and understanding of mathematics and can improve students' problem solving and reasoning skills. In the current study mental rotation is measured as an indicator of spatial ability.

Mental rotation of objects is a fundamental spatial ability that affects several aspects of life. Mental rotation is primarily associated with the skill to mentally rotate images or objects into particular orientations. The ability to mentally rotate objects includes the visual inspection and mental simulation of the object's rotation in space (Hegarty & Waller, 2005). Mental rotation ability is important for success in several academic and career fields, especially the science, technology, engineering, and mathematics (STEM) domains (Károlyi, 2013). Mental rotation skills are utilized in performing many everyday tasks such as rearranging furniture, packing the car trunk, navigating a map, and parking a car, among many other examples academic and job tasks in non-STEM careers.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Mental rotation is especially useful in several career areas (e.g. architecture, industrial design, engineering, sculpting, surgery, kinesiology, dentistry, and aviation). Interventions and activities that develop and improve mental rotation skills could lead to increased success in STEM and related domains. Computer-aided design (CAD) programs and 3D printing activities is an excellent way to utilize advanced technologies to develop and improve students' spatial abilities.

CAD Programs. Computer-aided design programs are utilized to create detailed three-dimensional models and two-dimensional drawings. Computer-aided design programs are most commonly used by engineers for drafting, designing, and developing diverse and complex machinery components (Sharma & Dumpala, 2015). They are also widely used by designers due to its ability to offer the creation of intricate designs (Martin & Velay, 2012) in a way that is more accessible to others. Computer-aided design programs and manufacturing tools, such as 3D printers, are ever-present in current product commercialization environment and students entering this environment need to be practiced in using such tools (Johnson & So, 2015). The utilization of CAD programs in 3D printing and design classes showed positive influences on student's mathematics skills, real-life skills, interests and motivation (Kwon, 2017). There are several types of CAD programs on the market such as *TinkerCAD*, *SketchUp*, *SolidWorks*, and *AutoCAD*, all of which are used create 2-dementional renderings of 3-dementional designs.

CAD and Mental Rotation. Spatial ability and mental rotation in children are often neglected in early education but can be promoted through experiences with three-dimensional modeling programs (Matthews & Geist, 2002). Students' mental rotation has been identified as a predictor of success in STEM domains and computer-aided design (CAD) programs have overcome barriers to spatial expression (Chang, 2014) that has been an essential tool in engineering education (Chester, 2007). Mental rotation skills could be developed and improved through the use of certain technologies (e.g. CAD software and 3D printing) which can lead to improved mathematics achievement and STEM success.

Methodology

In the present study a quasi-experimental study was conducted to explore the relationship between the utilization of CAD programs and students' mental rotation skills. To determine how the implication and use of CAD programs in a classroom influenced student's mental rotation skills, data were collected during a week-long CAD intervention. The current study was guided by the following research question: How will student's mental rotation skills be influenced after experiences and utilizing computer-aid design programs in a 3D printing class?

Participants. The sample was comprised of 1 middle school student and 24 high school students who attended a one-week STEM summer camp at a research-intensive university. The ethnic backgrounds of the sample included 15 Caucasians (60%), 7 Hispanics (28%), 1 African Americans (4%), and 2 whom did not disclose ethnicity (8%). The sample is comparable to the United States population with a noted difference that African Americans were slightly underrepresented in the sample. The sample included 5 females (20%) and 20 males (80%), females were underrepresented in the sample regardless of the level of comparison.

Instrument. The *Mental Rotations Test* was used to assess participants mental rotation skills (Peters et al., 1995) and adaptation of the original paper and pencil *Mental Rotations Test* by Vandenberg and Kuse (1978). The test contained 24 items with five 3D drawings of cubical figures per item. Each item contained one target figure on the left and four answer choices on the right. The participants were to identify which two of the answer choices were identical to the target but rotated along the y axis. The two other answers were mirror-images of the target and thus could not become identical to the target by rotation. The test was given in two parts, the participants were given three minutes to complete the first 12 questions, a two-minute break, and three minutes to complete the remaining 12 questions. Two undergraduate students recorded participants answers into an excel spreadsheet which was then checked by a senior doctoral student. Recorded answers were then scored by the instructor. One point was given for each correct answer, and one point is subtracted for each incorrect answer yielding a maximum of 48 points. The internal consistency for this sample was measured using Cronbach's (1951) alpha coefficient; score reliability was high with a 0.84 for the pre-test and 0.89 for post-test. All participants were tested both before and after the intervention at the same time by the same test administrator and instructions.

CAD Programs Used for Instruction. In the present study two different 3D CAD programs (TinkerCAD and SketchUp) were introduced and utilized. These two CAD programs were chosen because they both are available free TinkerCAD, owned by Autodesk, is a free, online 3D modeling program that runs in a web browser and is known for its simple interface and ease of use. TinkerCAD allows users to start their designs with 3D geometric primitives (Avila & Bailey, 2016) that can be combined and manipulated. Geometric primitives are basic geometric shapes (e.g. sphere, cube, cylinder, pyramid) that can be assembled with others to construct more complex shapes (Boubekeur, Kaiser, & Ybanez Zepeda, 2019). Starting with these basic 3D shapes provides a much simpler mode to create complex shapes and objects. TinkerCAD provides an easy, early training ground to introduce solid modeling and 3D printing to a younger or less experienced students.

SketchUp, owned by Trimble Inc., is a 3D modeling computer program used for a variety of applications including but not limited to architectural, interior design, mechanical engineering, and video game design. SketchUp starts with 2D geometric primitives (e.g. point, line, plane, circle) and allows users to push or pull them into 3D objects (Avila & Bailey, 2016). SketchUp provides a platform for users to sketch and create 3D designs with much more creativity, precision, detail, and complexity than TinkerCAD. SketchUp could be considered an intermediate to advanced CAD program but is still user friendly and accessible. Several SketchUp packages are offered at different price points but a free web-based version is now available. The free version was used in the present study.

Intervention

During a one-week STEM summer camp participants were randomly assigned and placed in a 3D printing class. The class met for one and a half hours each day for one-week: Monday through Friday. The class was instructed and facilitated by a third year female Mathematics Education PhD student who had previous training in 3D printing and Sketch-Up. She also had two years of experience (two projects per school year) utilizing a 3D printer, TinkerCAD, and Sketch-Up in high school Geometry classes (on level and pre-AP). On the first day participants took the pre-test and listened to an introduction to 3D printing, the engineer design process, and were given guidelines for the first project *Make your own Trophy* (see Table 10. They created TinkerCAD accounts, explored the software, and created their trophy. Day two the instructor introduced SketchUp and guided participants through how to use the available tools. They then received three open-ended final projects to choose from.

Table 1: Project descriptions

Project Title	Description
Make your own trophy	Create an award for yourself to receive at the end of the camp that is less than 5"x5"x5", has your name, and consist of at least 10 shapes.
Fusion of Art and Function	Design and create something that is both fun/interesting to look at and is functional.

Project Title	Description
Historical Structure	Choose and replicate any historical structure of your choice.
Product Prototype	Invent the next big thing; design and create a prototype of a brand-new product.

For the final project participants were given the option to work in TinkerCAD or SketchUp; which required participants to self-assess their skill level and choose a platform for which they were comfortable using. It was unknown to the instructor what prior knowledge, skills, and/or experiences participants would have entering the class, therefore a beginner program and an intermediate/advanced program were offered so participants of various skill levels would remain actively engaged by attempting to avoid disengagement caused by frustration, boredom of a steep learning curve of the CAD program. Only two participants chose to work in SketchUp. Day three and four were project workdays where the participants worked independently on their final projects. During this independent work time their trophies were being printed and participants were taken in small groups to the 3D printer room to see them in action. The instructor provided them with additional information about the printer and printing procedures then answered any questions they had. On the fifth day participants finished their final project and shared their design.

Data Analyses

Quantitative data were collected pre and post intervention to assess participants mental rotation skills using the Mental Rotations Test - Form A (Peters et al., 1995). First, data were analyzed using a paired sample *t*-test to investigate the statistical differences between mental rotation skills before and after the intervention. Prior to conducting the paired sample *t*-test, Q-Q plots and box plots of the pre-test and post-test scores were analyzed to assess score distribution and check for outliers. Second, to provide a visual representation of the results, 95% confidence intervals (CIs) for the pre and post test scores were examined. The 95% CI indicates that if the study was conducted an infinite number of times, the calculated point estimate would be captured 95% of the time. Using 95% CIs provides a visual depiction of the preciseness of the estimate and a direct comparison model for other similar studies (Thompson, 2002). Finally, Cohen's *d* effect size estimates were computed to quantify the magnitude of the difference between pre and post test scores. Due to the small sample size of the present study *t*-test results may be skewed therefore effect size is more suitable for the given data. Effect size indices are valuable in quantifying the effectiveness of an intervention because they are unitless, making them comparable across studies, and do not depend on sample size (Sullivan & Feinn, 2012). Reporting effect size is an important practice in order to report and interpret results in a trustworthy way that is usable for scholars and practitioners (Thompson, 1999a). In addition, reporting effect sizes, even for results that are not statistically significant, aids in compelling researcher to think meta-analytically and provides grist to possible future meta-analyses (Cumming & Finch, 2001; Thompson, 1999b; Thompson, 2001). Statistical package SPSS 25 was used in the aforementioned data analyses.

Overall, the results indicated that mental rotation skills, as measured by the instrument, were improved by the CAD intervention. No class time was spent on reviewing mental rotations or practicing with similar diagrams. Therefore, direct instruction in the concept is not a potential threat to validity.

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GEOMETRY AND MEASUREMENT:

POSTER PRESENTATIONS

"A SQUARE IS NOT LONG ENOUGH TO BE A RECTANGLE": EXPLORING PROSPECTIVE ELEMENTARY TEACHERS' CONCEPTIONS OF QUADRILATERALS

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Developing knowledge of the characteristics and properties of two- and three- dimensional geometric shapes are considered key content for elementary-aged students to learn. Quadrilaterals, one of the conceptually complex topics in geometry, have a range of attributes such as side length, angle measure, relationships between opposite/adjacent angles/sides, length and bisection of diagonals, etc. Previous research corroborated that transitioning from the introductory information about attributes of these shapes to mentally manipulating and proving the properties is challenging. Many teachers across elementary through senior grades, struggle with geometry content, and consequently, so do their students (Bhagat & Chang, 2015). Research reports that prospective teachers have difficulty in identifying specific shape types based on properties, rather than visual recognition (Burger & Shaughnessy, 1986) and struggle to determine other properties or define the relationships between various quadrilaterals. Drawing on Tall and Vinner's (1981) description of concept-image and concept definition, this study aims to add to the existing literature by answering the research question: *What conceptions do prospective elementary mathematics teachers have of the attributes of the family of quadrilaterals?*

The participants in this study were 19 prospective elementary teachers (2 male and 17 female) between 17-24 years of age enrolled in a content course designed to support knowledge of geometry and measurement in the elementary curriculum using a problem-solving approach. Data sources include the PSTs responses on a 26-item (15 closed and 11 open-ended) geometry tests. Findings showed that PSTs reasonings seemed to be fixed towards certain images rather than mental manipulation of the attributes and concept definitions (Vinner, 1991). For instance, PSTs considered rectangle as a figure with two long parallel sides and two shorter parallel sides and four 90° angles, while defined parallelogram as a slanted shape. PSTs tended to over-generalize the attributes of certain common quadrilaterals showing a continuous exposure to a geometrical shape in specific orientation (e.g., square having a horizontal base). While responding to the question, 'do the diagonals of kite bisect each other?', most of the PSTs responded incorrectly having generalized the properties of the diagonals of a square, rectangle, and rhombus.

The results of this study highlight the importance of emphasizing the two-way interaction between concept image and concept definition by explicitly showing interactions between theoretical and practical application of the learned ideas. Creating the definitions of quadrilaterals based on understanding of attributes (Fujita & Jones, 2007; Usiskin & Griffin, 2008), and exploring the inclusive relationship between various quadrilaterals (Fujita, 2012) may support PSTs in developing conceptions using non-prototypical examples. In this regard, time, high-quality tasks, and appropriate scaffolding are essential to strengthening understanding of quadrilaterals specifically, and geometric concepts more broadly.

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GROWTH IN MATHEMATICAL UNDERSTANDING AND SPATIAL REASONING WITH PROGRAMMING ROBOTS

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This poster describes how programming robots might support both the development of spatial reasoning and growth in mathematical understanding using interpretive video analysis of two Grade 4 students' attempts to program their robot to follow a pentagon.

Theoretical Perspective. We argue that programming robots to move could lead to growth in mathematical understanding and contribute to developing spatial reasoning. We draw on Pirie and Kieren's (PK) (1994) model of growth in mathematical understanding which describes modes of engagement with mathematical concepts as seven distinct levels with increasing abstraction. We suspect that spatial reasoning is essential to all modes, but that it is especially relevant to the first three elements (*primitive knowing*, *image making*, and *image having*).

"[S]patial reasoning … refers to the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects" (Bruce et al., 2017, p. 147). Davis et al. (2015, p. 141) attempted to collect the many competencies and habits associated with spatial reasoning into a model that represents the emergent complexity of spatial reasoning skills as co-evolved and complementary nature of the mental and physical actions.

Research Question. We questioned how programming robots might provide children with opportunities to gain mathematical understanding and develop spatial reasoning.

Data Collection Techniques and Analyses. Consistent with Knoblauch et al.'s (2013) notions of interpretive video analysis, we reviewed and selected one video based on instances of observable spatial engagement from 9 months of weekly videos collected of 32 Grade 4 students in 2 classrooms. In this video, a pair of students is attempting to program an EV3 LEGO Mindstorm robot to trace the third vertex of a pentagon having previous success following the first two straight-turn segments. We identified spatial elements in the two students' interactions according to Davis et al.'s (2015) framework while they engaged in determining how to steer their robot to travel around the 108^{o} vertex. We then analysed levels of mathematical understanding according to the PK model.

Summary of Findings. In this video one can observe the children working with many aspects of spatial reasoning and mathematical understanding. Drawing upon Davis et al.'s (2015) elements of spatial reasoning, the students were simultaneously INTERPRETING, [DE]CONSTRUCTING, MOVING, SITUATING, ALTERING and SENSATING. In the video, we observed the pair engage in how the distance the robot turns relates to the number of wheel rotations. The mathematical concepts included additive thinking, angles, properties of shape, measurement (distances, robot turns), multiplicative thinking (number of wheel rotations), pattern recognition, and direct proportion. Students' growth in understanding dynamically progressed between *primitive knowing*, *image making*, and *image having*. Our findings highlight how programming robots could support both the inner modes of PK's growth in mathematical understanding and contribute to developing spatial ability.

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FUNKY PROTRACTORS CREATED BY PROSPECTIVE TEACHERS

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Angles and angle measure are important and frequently leveraged concepts throughout school mathematics curricula. Yet, relative to other quantities like length, area, and volume, very little scholarly literature addresses how students and teachers understand angle measure (Smith & Barrett, 2017). From the scant extant literature, it is clear that developing productive conceptions of angle measure is non-trivial for students and teachers alike (Akkoc, 2008; Lehrer, Jenkins, & Osana, 1998; Smith & Barrett, 2017). In the U.S., individuals' challenges in quantifying angularity may be partially attributed to instructional approaches that (a) overemphasize the use of conventional protractors to measure angles and (b) fail to address how the design of these conventional tools renders them appropriate for measuring angles (Moore, 2012). This is especially problematic given that well-prepared beginning teachers must be skilled in explaining how to select appropriate tools for particular mathematical goals (Association of Mathematics Teacher Educators, 2017).

To occasion conversation and reflection about angular measurement and protractors in our geometry courses for prospective teachers, we designed tasks involving a collection of non-standard tools that might be used to measure angles. We refer to these tools as *funky protractors* (Hardison & Lee, 2020a). For each funky protractor we designed, we altered one or more features to differentiate it from a conventional protractor (e.g., uncommon shape, equally spaced linear or angular intervals, non-standard angular unit of measure, etc.). We intentionally designed some funky protractors to be valid tools for measuring angles and others to be invalid; in previous implementations, we have asked prospective teachers to determine which funky protractors are valid tools for measuring angles and to justify their decisions. Thus, funky protractor tasks are the angular analogue of the "strange ruler" tasks others have used to promote critical thinking about linear measure (Dietiker, Gonulates, & Smith, 2011). Elsewhere, we have discussed prospective teachers' decisions regarding the validity of funky protractors, as well as the strategies they leveraged to support their decisions (Hardison & Lee, 2020b, this volume).

In this poster presentation, we report on an extension of the funky protractor tasks, which we implemented with prospective middle and secondary teachers enrolled in one section of a geometry content course at a large public university. After evaluating the validity of four funky protractors and engaging in a whole-class discussion, prospective teachers were asked to create two of their own funky protractors: one that would be a valid tool for measuring angles and one that would not be a valid tool for measuring angles. We present examples of the funky protractors that prospective teachers created and analyses of these items. In particular, we (a) summarize how successful prospective teachers were in creating valid and invalid tools for angular measurement, (b) describe the features prospective teachers manipulated when designing their own funky protractors, and (c) discuss prospective teachers' perspectives on the pedagogical utility of funky protractors.

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GESTURES IN GEOMETRY: HOW DO GESTURES CONTRIBUTE TO ENGAGEMENT AND VOCABULARY ACQUISITION THROUGH GAME PLAY?

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Keywords: Embodiment and Gesture; Geometry and Geometrical and Spatial Thinking; Instructional activities and practices

Gestures have been shown to improve students' abilities to process new mathematics concepts (Goldin-Meadow et al., 2009) and reduce cognitive load (Ping & Goldin-Meadow, 2010). Producing gestures related to mathematical concepts have been shown to support reasoning as learners ground their understanding of math concepts in body-based movement (Walkington et al., 2014). Such findings show the affordances of gesturing to create meaningful representations in mathematics. However, less is known about how giving students the option to gesture in activities impacts engagement and vocabulary acquisition for geometry concepts.

We adapted a modified version of the game Taboo, designed for high-school geometry students (Carter, 2015), where players take turns being the hint-giver with 60 seconds to verbally describe geometry terms on cards (e.g., *intercept*, *parallelogram, function*). The hint-giver creatively describes each term, avoiding the taboo words (e.g., intercept: *cross, axis, graph*).

A pilot study was conducted to examine whether the option to gesture while playing Geometry Taboo may contribute to engagement, performance, and vocabulary acquisition for high school geometry students. The study took place in two tenth grade geometry classes. The first day, 25 participants completed a paper-and-pencil timed pretest that required matching the 25 terms from the Geometry Taboo cards with pictures. The second day, students played the game in small groups for two rounds consisting of each player in the group having one turn to describe as many geometry terms as possible to their group. One group was able to use *speech and gestures* to describe terms on the cards; the other group was restricted to *speech only* hints. The next day, students completed a mirroring posttest and online survey about their experience.

An ANCOVA, controlling for pretest performance revealed no significant differences in posttest scores by condition but overall students improved slightly from pretest (M = .34, SD = .19) to posttest (M = .48, SD =.23). Next, an ANCOVA predicting total points earned by each student during the game, controlling for pretest found no significant differences but there was a trend that students in the *speech-and-gesture* condition (M = 8.2 points, SD = 4.2) scored more points than students in the *speech-only* condition (M = 6.0 points, SD = 2.9), p = .08. This suggests that the option to gesture may make describing terms easier rather than relying on speech alone. Surveys suggested largely positive perceptions of the game; 21 students responded that they would like to play again in class.

We draw limited conclusions from this pilot study. The exposure to the intervention may have been insufficient; more time playing could have led to increased learning. However, differences in game performance and student input suggest that the option to gesture makes the game easier and accessible for students, which in turn, could impact engagement and learning.

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NAVIGATING COMPLEXITIES IN DEFINITIONS OF LENGTH AND AREA

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Deficiencies in elementary students' conceptual understanding of spatial measurement have persisted, emerging through educational research (e.g., Kamii & Kysh, 2006) and national assessments (e.g., National Assessment of Educational Progress [NAEP]). Investigating several decades of results from the NAEP, Kloosterman, Rutledge, and Kenney (2009) described persistent measurement deficiencies. Research suggests that elementary students struggle with conceptual understanding of spatial measurement (i.e., length, area, volume) and graduating preservice teachers (PSTs) often share their struggles. For example, elementary students struggle in understanding distinctions between area and perimeter and relationships between their measures (e.g., Bamberger & Oberdorf, 2010; Barrett & Clements, 2003; Woodward & Byrd, 1983). The intuitive expectation that measures of perimeter and area always increase or decrease together is an enduring, commonly held misconception (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998; Tan Sisman & Aksu, 2016). PSTs, soon to be teaching such concepts, have shown similar misconceptions (e.g., Ma, 1999; Livy, Muir, & Maher, 2012; Wanner, 2019).

We examined definitions related to length and area measurement in 11 textbooks specifically developed for use with preservice elementary teachers in mathematics content courses. Our selection of the textbooks was guided by Raven (2006) and represents a wide range of textbooks that vary in organization, coverage of topics, and attention to pedagogy. The books are written by mathematicians, mathematics educators, or both.

Two researchers adapted and clarified an existing framework to code definitions of spatial measurement in elementary curricula with respect to selected aspects (Gilbertson, He, Satyam, Smith, & Stehr, 2016). We identify the coding unit, a definition, as a focused description of meaning, set apart from other text. We captured definitions of length and area using the textbook index and scanning relevant sections. Two researchers independently coded each definition and met to compare coding and resolve discrepancies.

Based on Stehr and He (2019), we used a four-step measurement process: (1) select an object and measurable attribute, (2) select a unit of measure, (3) compare the attribute of the object with the unit, and (4) express the measure. We provide our analytical frameworks and findings in the poster. In the first step of the measurement process, select an object and an attribute of that object to be measured. A measurable spatial attribute is a characteristic of an object that can be quantified, has dimensionality, takes up space, and often has clear boundaries. To select a unit of measure in the second step, note that the unit could be standard or nonstandard, a reproducible unit that tessellates space, using parts of a unit as needed, and may be be continuous or discrete. In the third and fourth steps, the measure of an attribute is expressed by comparing the attribute to the unit to determine the number of units and parts of units that cover or fill the space without leaving gaps or overlaps. The comparison may include procedural tool use. The final measure of an attribute is expressed as a multiple of the standard or nonstandard unit.

The goal in analyzing textbook definitions and finding variation is not necessarily to point out gaps or failings, because textbooks may add to definitions through tasks or other text. We focus attention on the ways definitions could be written at multiple levels of sophistication and with careful choice of aspects, hoping to open a larger discussion.

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A RESEARCH APPROACH ON THE ROLE OF SPACE IN THE CONSTRUCTION OF CONIC SECTIONS

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The results of a literature review of an ongoing research about the construction of the solid conic section and its transition to plane conic section are presented. The review was done in Mathematics Education and History of Geometry Teaching with a main emphasis on the construction of the solid conic as a cutting of the cone, these treatments of conics absent in the curriculum can give meaning to the current treatments of school mathematics. The review concludes with five essential elements that will define the type of study, the study object, and the position of our research in the field.

Keywords: Geometry and Geometrical and Spatial Thinking, High School Education.

The initial research object for the development of the literature review was *construction of the conic as a cone's cutting and its transition to the plane*. As a first point, one of the problems reported by Mathematics Education research is the absence of conic section's geometric treatments in High School Education, because the conic section's algebraic treatments is in Analytical Geometry class (Pérez-Moguel, 2018; Salinas & Pulido, 2017; Contreras, Contreras & García, 2003); nevertheless, the meanings of these notions are linked to the Plane and Space Geometry.

In this way, History of Geometry Teaching recognizes: school mathematics are based on the mathematics of the 17th century (Barbin, 2008; 2012; Dennis, 2009); in particular, the solid conic section's construction isn't since 1905 in the curriculum (Barbin, 2012; 2008), and it's replaced by a narrative about the cuttings of a cone made by Apollonius of Perga (Fried, 2007; 2001), and then to define them on the plane from the foci and directrix (as the case may be), without any link to the Apollonius' cuttings (Salinas & Pulido, 2017). Indeed a second point, we identify conics as "a perennial notion with many properties, many theories and contexts, geometric and algebraic approaches, relations between plane geometry and space" (Barbin, 2008, p. 157); therefore, we synthesize the chronology of the study and development of this notion: *Solid Conic Section; Plane Conic Section; Analytical Conic Section* (Coolidge, 1968; Bartolini Bussi, 2005; Bongiovanni, 2007).

As a third and more important point, the research of Pérez-Moguel (2018) will be a fundamental antecedent because her historical-epistemological study of parabola's geometrical construction as a cone's section, she identifies a series of *actions* and *activities* that encourage us to ask ourselves about the practices associated to construction of the solid conic section and its passage to plane conic section. Among these actions, Pérez-Moguel (2018) highlights the transition between 3D and 2D dimensions in solid parabola's construction, coinciding with Salinas and Pulido (2017) who consider that spatial ability is fundamental in the construction of solid conic sections. Therefore the new object of investigation, modified after the review will be: *spatial processes, and practices associated to solid conic section's geometrical construction relative to the cone's cutting, and its transition to the plane conic section,* in the original text *Apollonius of Perga: Conics*.

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INSTRUCTIONAL LEADERSHIP, POLICY, AND INSTITUTIONS/SYSTEMS

RESEARCH REPORTS

ELEMENTARY MATHEMATICS TEACHER AGENCY: EXAMINING TEACHER AND ECOLOGICAL CAPACITY

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We investigated how 53 elementary teachers interpreted the impact of the contexts in which they work on their mathematics instruction, and what those interpretations reveal about the agency individual teachers were able to achieve. Latent class analysis revealed two distinct classes, with teachers in one class perceiving that their contexts had a greater and more supportive impact than teachers in the other class. Interviews of four elementary mathematics specialists then revealed that the extent to which agency was achieved depended on not only their evaluations of the constraints and affordances of their contexts, but also their past experiences and future goals.

Keywords: teacher agency; policy; elementary mathematics specialists; latent class analysis

Although decades of policy have sought to limit teacher agency through, for example, highly prescriptive curriculum and accountability regimes (e.g., Biesta, 2010), discourses in mathematics education have emphasized how teachers exert agency in their specific enactments of broader policy (National Research Council, 1997; OECD, 2005). Theories of agency suggest that mathematics teachers have always interpreted and responded to policies, even those designed to limit agency, based on their experiences and frames of references (e.g., O'Day, 2002; Osborne et al., 1997; Zancanella, 1992). Discourses focused on agency, however, raise questions about what it might mean for teachers to be agents and the extent to which teachers can achieve agency. In this paper, we share a mixed methods study that investigates how elementary mathematics teachers achieve agency in their unique contexts. In particular, we focus on how elementary teachers interpret the impact of the contexts in which they work on their mathematics instruction and what those interpretations reveal about the agency they are able to achieve.

Theoretical Framings & Related Literature

We view agency as a temporal process informed by the past (iterational dimension), oriented towards the future (projective dimension), and achieved in the present (practical-evaluative dimension) (Emirbayer & Mische, 1998). In other words, teachers build upon past experiences and understandings to refashion and appropriate patterns of behaviors. Motivated to create a future that is different from the past and present, teachers generate possible trajectories of action. Although agency is tied to the past and future, it can only be achieved in the present as teachers make judgements based on evaluations of the constraints and affordances of their contexts. An implication is that, in response to present problems, teachers who are able to draw upon a greater repertoire of past experiences or form a wider range of alternative futures might achieve greater levels of agency (Priestley et al., 2015).

However, agency is not simply a quality of teachers; it is a dynamic interplay between both individual efforts and ecological conditions (Biesta & Tedder, 2007). Carried out in concrete situations, agency is achieved as teachers engage with their ecological contexts. Teachers may

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achieve agency in one situation but not another, and that may depend on the availability of social, cultural and economic resources.

Prior studies on teacher agency have highlighted the importance of both teacher capacity and ecological capacity. Regarding the former, research suggests that teachers' experiences and beliefs play an important role in the achievement of agency (Sloan, 2006; Vähäsantanen, 2015). A wide range of past experiences may enhance agency by allowing teachers to see alternatives to the present, while strong beliefs about student learning enable teachers to develop a broader set of aspirations (Priestley, 2011; Priestley et al., 2012). In contrast, when teachers' discourses and goals are framed in terms of policy (e.g., meeting accountability expectations), projective elements of agency are reduced because teachers' potential to envision alternative futures is narrowly defined by the constraints of policy (Biesta et al., 2015).

Regarding the latter, research suggests that the ecological contexts in which teachers work influence the extent to which teachers are able to achieve agency. Teachers' evaluation of the professional obligations of their contexts may limit the actions they take towards projected goals (Priestley et al., 2012). For example, teachers working in contexts where standardized outcomes are highly valued may feel pressured to forgo ambitious instructional practices for those that are better suited for meeting accountability expectations (e.g., those focused on developing procedural fluency). However, access to ecological resources, such as professional relationships with administration and other teachers, may foster agency by supporting teachers to develop their practice, take risks, and see alternative futures (Coburn & Russell, 2008; Priestley et al., 2013).

Our study builds upon existing research by examining how elementary mathematics teachers - a group not yet investigated in the research on teacher agency - are able to achieve agency in their unique ecological contexts. We expand beyond the individual case study methodology commonly used in studies on teacher agency to also include quantitative analyses of surveys reporting the extent to which teachers evaluated their contexts as impacting their mathematics instruction. Specifically, we investigated the following questions: 1) how do elementary mathematics teachers interpret the impact of their ecological contexts on their mathematics instruction? and 2) what do different interpretations reveal about the agency individual teachers are able to achieve? Though the practical-evaluative (present) dimension is foregrounded, the judgments mathematics teachers make about the affordances and constraints of their contexts are influenced by the projective (the instructional goals teachers have for the future) and iterational dimensions (the past experiences they draw upon to achieve those goals).

Methods

Study Context

This study originates from a larger multi-year project focused on the beliefs, knowledge, practices, and student achievement for certified elementary mathematics specialists (EMSs) (McGatha et al., 2017). Among 55 participating teachers, there were 24 EMS and 31 comparison teachers that were recruited from the same schools (or districts) and same grade levels as the EMS teachers. A variety of data were collected for the larger project, including teacher surveys, measures of teacher knowledge, and observations of teachers' instructional practices. In addition, eight EMS teachers were selected as case study participants and each participated in five semi-structured interviews that were audio-recorded and transcribed.

Data & Participants

For the present study, we focused on a set of items from the teacher survey that asked participants about the impact of 14 items on their mathematics instruction. The 14 items were: 1) current state standards; 2) district curriculum frameworks; 3) district and/or school pacing guides; 4) state testing/accountability policies; 5) district testing/accountability policies; 6) textbook/program selection policies; 7) teacher evaluation policies; 8) students' motivation, interests, and effort in mathematics; 9) students' reading abilities; 10) community views on mathematics instruction; 11) parent expectations and involvement; 12) principal support; 13) time for you to plan; 14) time available for your professional development (see Figure 1 for survey directions). These items were completed by 53 teachers (23 EMS and 30 non-EMS). In addition, we analyzed the interviews of four case study EMS teachers: Amy, Denise, Emma, and Mary. Selection of the cases is further discussed in the Data Analysis section.

What is the extent of the impact of each of the following on your mathematics instruction?					Indicate the nature of the impact			
No impact (1)	Some impact (2)	Great impact (3)	Not applicable (4)	Don't know (5)	Mostly inhibits effective instruction (1)	Mostly supports effective instruction (2)	Both inhibits and supports effective instruction (3)	Not applicable (4)

Figure 1: Survey Directions

Amy and Denise taught at the same school in a district with five K-4 elementary schools. Part of the instructional day at this school included a math intervention time where students engaged in Rocket Math, a fluency program, with the stated goal of improving state standardized test scores. The district's curricular program was Math in Focus and teachers were provided a pacing guide that suggested how much time to spend on each topic.

Emma and Mary taught at different schools within a district that served 13 elementary schools. The district's curricular program was Go Math and, as in Amy and Denise's school, teachers were provided a suggested pacing guide. The district administration also encouraged teachers to engage students in weekly problem solving, though this was taken up by teachers in various ways, which will be further discussed in the cases of Emma and Mary.

Data Analysis

Using the 14 survey items described above, we employed latent class analysis (LCA) with the poLCA package in R (Linzer & Lewis, 2011) to identify groups of teachers who perceived different impacts of their ecological contexts (i.e., the 14 items) on their mathematics instruction. To create binary variables for LCA, we first created a holistic score for each item combining 'extent' and 'nature' of impact (e.g., great impact and mostly inhibits =1; great impact and mostly supports =5). Based on exploratory factor analysis, we consolidated the 14 items into 6 factors. These were named by their 'type': standards (items 1, 2), textbook/pacing guide (items 3, 6), accountability policies (items 4, 5, 7), students/community (items 8, 9, 10, 11), principal (item 12), and time (items 13, 14). To dichotomize each factor, we calculated the average score of the items and coded "supportive" (average > 3) as 2 and "inhibitive or mixed" as 1. Then we conducted the LCA analysis using the factor scores for each participant. After 100 iterations of 2-class and 3-class models, we selected the 2-class model because both the Bayesian information criterion (BIC) and the Akaike information criterion (AIC) were minimized (2-class: BIC = 422.575, AIC = 396.961; 3-class: BIC = 443.926, AIC = 404.520).

We selected two teachers from each latent class: Amy and Emma from the first class and Denise and Mary from the second class. As described earlier, Amy and Denise taught at the same school and Emma and Mary taught in the same district. These teachers were selected because they worked in similar ecological contexts and, comparing across the two classes, we were able to explore how they interpreted the impact of their contexts on their instruction differently and how those interpretations influenced their achievement of agency. These four teachers participated in five semi-structured interviews that elicited their vision and goals for teaching mathematics, supports available for and challenges anticipated in enacting that vision, the resources available for teaching (e.g, curriculum materials), the influence of state-mandated standardized assessments, and their understanding and implementation of mathematics teaching standards. Our analysis attended explicitly to how the iterational (e.g., prior experiences; understanding of math standards), projective (e.g., vision and goals for teaching) and practical-evaluative (e.g., resources, supports, and challenges for enacting vision; influence of standardized assessments) dimensions informed how each teacher was able to achieve agency.

Results

Classes of Impact on Mathematics Instruction

The LCA analysis revealed two distinct latent classes: SUPPORTED (Ecological Factors Supported Instruction) and MIXED (Ecological Factors Inhibited and Supported Instruction). Teachers in SUPPORTED (45% of teachers), on average, perceived that their ecological contexts had a greater and more supportive impact on their mathematics instruction than those in MIXED (55% of teachers). For example, teachers in MIXED had a 0% probability of reporting that accountability policies (state testing, district testing, and teacher evaluation) supported their mathematics instruction, compared to an 87% probability among teachers in SUPPORTED. The three factors that had the greatest difference in perceived support between the two classes are accountability policies, standards, and principal support. Figure 2 shows, for each class, the probability of a teacher reporting that a particular factor supported their mathematics instruction.

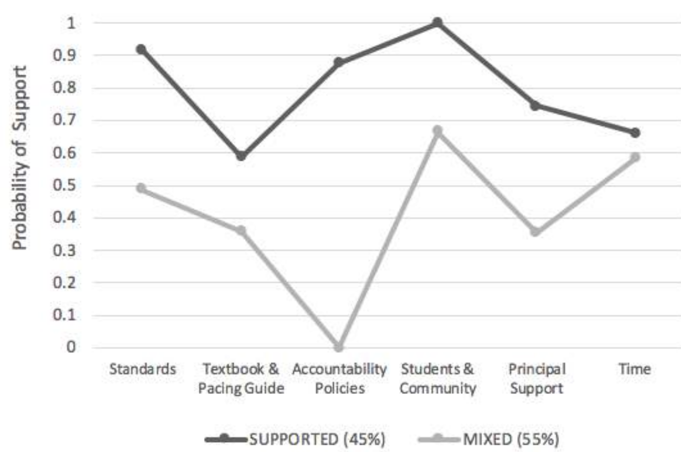

Figure 2: Conditional item probability plot for two classes

Cases of Amy and Denise (Same School)

Amy was an instructional coach for several years, and after completing an EMS program, returned to the classroom as a third-grade teacher. Illustrative of the SUPPORTED class, Amy conveyed little concern with regard to the district's accountability policies and pacing guide. For example, Amy perceived that the only stress regarding state testing was that "the computers didn't work the way they were supposed to." Amy's evaluation may explain why she did not use the test preparation materials like her partner teacher did:

> When my kids started doing that packet and I watched them, I'm like, "This is crap. I am not doing this. Stop…" The conversations were so much better than making them do 25 problems on their own. And my partner did it the traditional way…My principal is cool with

> it. He was like, "I don't want you to spend the whole month prep on it. They don't need it. They can think and that's the main thing."

This agentic activity - deciding not to engage students in test preparation materials - was supported by Amy's principal, who accepted her justification that her students were able to think and reason mathematically. Amy's decision to engage in mathematical conversations rather than practice problems also reflected her vision of mathematics instruction and her goals for student learning. Making specific references to the Standards for Mathematical Practice, Amy explained, "…it's so important for kids to be able to interchange numbers and to problem solve. And I think, honestly, the math practice standards are probably something that gets skipped over so much, and those are so important…Like taking a problem, making sense of it."

Drawing upon her goals and vision for mathematics learning, Amy was also able to achieve agency when she decided to devote extra time for Calendar Math in place of Rocket Math: "I kinda talked to my principal about it. I would like next year, instead of doing the Rocket Math, it's so very, it's a basic, it's a procedure, is what it is.…instead of doing Rocket Math with the group, I would like to teach Calendar [Math] to my second group. Because I add so much." Perceiving support from her principal, Amy saw Calendar Math as an alternative future to Rocket Math that better reflected her vision for instruction by developing a deeper understanding of mathematics concepts and providing students opportunities for reasoning and sense-making.

Amy's achievement of agency drew to a large degree upon her capacity to enact her goals for student learning and develop a wide repertoire for maneuvering her school context. In other words, she was able to draw upon her past experiences and imagine alternative futures to test preparation and Rocket Math. Supported by resources (e.g., the principal), and unconstrained by accountability policies, Amy was able to achieve a relatively great deal of agency.

At the same school as Amy, Denise taught fourth grade. The interviews took place during her fifth year of teaching. Illustrative of the MIXED class, Denise associated a great deal of stress and risk with the district pacing guide, which she described as constraining what and how she teaches: "I think it'd be more free on your pacing of how you teach and then you can do fun projects but it's like, 'oh, we can't do that cause it's gonna take a week to do that. And it's gonna put us behind,' and so to me I always feel like it goes back to that pacing guide." In this excerpt, Denise attended to how the pacing guide constrained her agency to engage her students in fun projects and hands-on activities. Such learning opportunities figured prominently in Denise's broader description of her vision for mathematics teaching and learning:

> I want them to have, to know the vocabulary, to be able to use it and just, and that just comes with understanding. I'd see like presenting the lesson but then we have our hands-on activity, like we're doing things together like as we're working though the lesson and understanding concepts, they're doing it with me or they have their boards and they're writing it out.

Unlike Amy, whose vision emphasized sense-making and engaging in mathematical practices, Denise viewed mathematics learning as participating in interactive activities to practice vocabulary and procedures after teacher demonstration.

Denise also felt pressured by state tests, stating "I think that's the pressure that's put on you to do well because that's what's reflected in the school on the state test." She saw students' standardized performance as reflecting on her performance as a teacher, which is a stark contrast to Amy who only described stress related to the computer testing system. Influenced by her evaluation of the pressures of her ecological context, Denise engaged in substantial test preparation, stating "I'll go back and like go lower so that way we can walk our way up but going back in that re-teach piece like a month - like with review, of course we review before the [state standardized] test but it's like being better about, okay, you did fractions last month, let's do like a bell ringer right now." In this, Denise describes that she prepares for the state test by reteaching previously taught content, sometimes starting at a "lower" level. She even wishes that she had more time throughout the year to spiral back on prior knowledge.

Denise acted similarly in response to students' lack of advancement in their Rocket Math fluency program. She said that:

> I pulled back from that [fluency] because it wasn't working. It's like for some of them - don't get me wrong. Like my lower kids though it wasn't working for them because it's like part of it they don't put in the work… it's like you get to the sixth time they're not passing it and then it's like okay, so let's go back re-teach.

Denise's solution to the perceived problem of low math fluency was to reteach and review. By blaming students for their lack of effort, she also engaged in deficit discourses about students ("my lower kids…don't put in the work") that relieved her of responsibility for student learning. These cases illustrate that her agency was constrained by a lack of past experiences and beliefs that would allow her to imagine alternatives to reteaching and reviewing.

Constrained by the pacing guide and accountability policies of her ecological context, Denise seemed to enact a self-limiting form of agency framed by short-term goals and discourses focused around achievement and fluency. And without a range of past experiences or ambitious instructional goals to draw upon, Denise's repertoire for maneuvering her context was limited.

Cases of Emma and Mary (Same District)

Emma and Mary both taught fourth grade, but at different elementary schools within their district. A representative of the SUPPORTED class, Emma perceived that her context's accountability policies positively impacted her mathematics instruction. Though she acknowledged stress and pressure associated with standardized testing, she perceived that her context supported her in meeting such accountability expectations. For example, she explained that the curriculum was aligned with state standards, that instructional coaches taught students test-taking strategies, and that the district's weekly problem solving provided test preparation throughout the school year. For Emma, such policies supported her instructional goals, which were focused on achievement and proficiency: "My goal, always talk to them about improvement. That no matter where you start--for example, I had a student last year that started at common assessment for the first quarter at 19%, and then she got to 45%."

Emma's beliefs and goals for student learning are reflected in her understanding of problem solving. She explained that problem solving included highlighting and underlining key words to figure out which operation to use. Emma described that in her class,

> We would give a problem at the beginning of the week, and then the same type of skill problem at the end of the week…During the problem-solving time we would meet with that group that was struggling. Then, the last day of the week we would do it again and see how they improved.

For Emma, "problem solving" did not primarily involve making sense of problems and reasoning about numbers and concepts, but rather practicing and acquiring answer-getting skills.

Though Emma perceived that her ecological context supported her mathematics instruction, she seemed to enact a form of agency limited by her goals and conceptions of proficiency and achievement. Specifically, Emma did not achieve agency in ways that afforded students opportunities to meaningfully engage in conceptual understanding of mathematics, as her repertoire of iterational beliefs and projective futures was constrained.

Unlike Emma, Mary perceived that her context negatively impacted her mathematics instruction. Representative of the MIXED class, Mary perceived that accountability policies and the pacing guide constrained her instruction, stating that

> Because you know you're behind in your timeline and you know that students are going to be assessed on all these skills and you worry...Do I really want them to be able to just know how to get the right answer from rounding, or do I want them to really understand the number sense behind it?

Mary's achievement of agency was constrained by the pressures of standardized testing, limiting her ability to maneuver between district policies and her own goals for student sense-making.

Though Mary's agency was constrained by her ecological context in some instances, she was able to achieve agency in others. For example, Mary was able to take up an informal leadership role in her building to present to peers about problem solving. In describing the goals of her presentation, she stated that:

> Just because you're getting the kids a word problem does not mean that they are participating in problem solving because I remember student teaching within the district six years ago and there was Word Problem Wednesday and the teacher did the word problem up on the board for the kids and then the kids did the word problem that was exactly the same but with different numbers.

In this excerpt, Mary achieves agency in challenging, in front of her peers, the view that word problems imply problem solving: in particular, if students are mimicking the teachers' solution, they are not truly engaged in problem solving. Ironically, Mary's counterexample of problem-solving describes Emma's approach. Drawing upon her experiences and goals for student learning, Mary was able to achieve agency in promoting a problem solving that emphasized student reasoning and sense-making.

Though Mary was able to build upon iterational and projective dimensions to achieve agency in some instances (e.g., promoting problem solving opportunities), her repertoire for maneuvering her ecological context was constrained by practical-evaluative dimensions in other instances (e.g., rushing to cover the assessed material). In other words, though she was able to imagine a sort of problem solving that aligned with her goals for student learning, her achievement of agency was constrained by accountability policies.

Discussion & Conclusion

Our study contributes to the research base on teacher agency by revealing how elementary mathematics teachers perceive and achieve agency differently, even though they may share some ecological conditions. Using LCA, we found two unique classes: teachers in SUPPORTED perceived that their contexts had a greater and more supportive impact on their mathematics instruction than those in MIXED. Interviews then allowed us to investigate cases where agency was achieved differently within and between these classes. Drawing upon the theoretical literature (Biesta & Tedder, 2007; Emirbayer & Mische, 1998), these cases revealed the temporal nature of agency; in particular, how teachers' evaluations of their ecological contexts' constraints and supports dynamically interacted with their iterational experiences and projected goals (see Table 1).

Table 1: Summary of temporal dimensions of four case study teachers

	SUPPORTED Class	MIXED Class
Same School	Amy (achieved great deal of agency) • Iterational: beliefs about student learning focused on sense-making and mathematical practices • Practical-evaluative: felt supported by principal & unconstrained by context • Projective: able to imagine alternatives to test preparation and Rocket Math	Denise (constrained and self-limiting form of agency) • Iterational: beliefs about student learning focused on fluency and achievement • Practical-evaluative: felt constrained by pacing guide and accountability • Projective: unable to imagine alternatives to reteaching for test preparation and fluency
Same District	Emma (self-limiting form of agency) • Iterational: beliefs about student learning focused on proficiency and achievement • Practical-evaluative: perceived context as supporting goals • Projective: unable to imagine problem solving that focuses on reasoning and sense-making	Mary (constrained agency) • Iterational: beliefs about student learning focused on sense-making and problem-solving • Practical-evaluative: felt constrained by pacing guide and accountability • Projective: able to imagine problem solving that aligns with goals and vision

Emma's case suggests that feeling supported by one's ecological context is not sufficient for achieving agency, especially when teachers – even certified EMSs - lack ambitious goals and visions for mathematics instruction. Foregrounding the iterational and projective dimensions of the cases of Emma and Denise raises an important implication for teacher education: the need for teachers to have strong professional discourses about mathematics teaching and learning beyond those framed by policy (Biesta et al., 2015). Attention to the practical-evaluative dimension reveals that Mary's agency was constrained by her context's accountability policies as they conflicted with her goals for student learning. Such factors were not as constraining for Amy as she had more personnel resources (e.g., principal) to draw upon. This suggests a second implication for policy: the need to build ecological capacity. Mary's case illustrates how a teacher - one with experiences and visions aligned with ambitious mathematics teaching - can achieve agency in some situations and not others, depending on the availability of social, cultural and economic resources (Priestley et al., 2013).

If policies are to promote teacher agency, our findings suggest a need for building both teacher capacity and ecological capacity. This includes not only attending to the repertoire of past experiences and future trajectories mathematics teachers are able to draw upon, but also the ways their ecological contexts constrain and support their mathematics instruction. And, in our view, mathematics educators are especially well positioned to advocate for such policies.

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INSTRUCTIONAL LEADERSHIP, POLICY, AND INSTITUTIONS/SYSTEMS:

BRIEF RESEARCH REPORTS

USING TRANSCRIPT ANALYSIS TO PREDICT STUDENTS' SELF-REPORTED HAPPINESS IN ELEMENTARY MATHEMATICS CLASSROOMS: METHODOLOGICAL CONSIDERATIONS

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In this study, we tested the extent to which researchers with classroom experience can predict students' happiness and engagement in elementary math classes. The research group analyzed lesson transcripts and hypothesized which classrooms students rated as high-engagement/high-happiness. We sorted the teachers into high or low groups with 50% accuracy. We suggest four possible explanations for the group's inability to accurately guess: (1) students and adults have different views of which classroom practices will generate student happiness, (2) failure to consult literature on the conceptualization of happiness in elementary aged children, (3) existence of a halo effect and (4) contextualized relationships in classroom environments matter. To conclude, we suggest methodological improvements to increase the probability of identifying high engagement practices in elementary math classrooms via transcript analysis.

Keywords: affect, emotion, beliefs, and attitudes; elementary school education; policy matters

Introduction

Students' experiences in the classroom matter. We see two components of the student experience as especially important: levels of academic achievement and happiness. Drawing on the conceptual framework of Talebzadeh & Samkan (2011), student happiness is associated with several factors related to school performance of both students and teachers. Psychologists note that student happiness affects the school environment and can increase students' performance on measures of academic achievement as well as socio-emotional growth; outcomes often touted as key goals of education (Suldo, 2016) Additionally, there is a relationship between student happiness, teacher happiness, and learning (Blazar & Kraft, 2017). These three factors interact in different ways depending on the student's level of happiness. In general, happy students perform better academically and socially (Parrish & Parrish, 2005; Quinn & Duckworth, 2007). Although Parrish and Parrish (2005) did not look at measures of student achievement specifically, they did find that as students' happiness increases, students also increase in collaborative learning, respect, and enjoyment of school. Quinn & Duckworth (2007) found that students ages 10-12 who had higher levels of subjective well-being "went on to earn significantly higher final grades after controlling for IQ" (p. 3). Additionally, Quinn & Duckworth (2007) noted that students with higher levels of subjective well-being were more successful than their peers at raising their level of academic achievement. More broadly, researchers of happiness note that happiness is often linked with personal well-being (e.g., Graham, Powell, Thomas & Anderson, 2017; Price, Allen, Ukoumunne, Hayes & Ford, 2017). Because elementary school students spend upwards of a fourth of their day at school, understanding how students conceptualize happiness in the classroom is important for their overall well-being (Graham, Powell, Thomas & Anderson, 2017). If researchers have the ability to identify high happiness classrooms using transcripts, there may be opportunities for school-level actors to provide specific pedagogical support to increase students' self-reported measures of happiness in their

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elementary math class. Furthermore, understanding which instructional components predict student happiness will support future efforts to validly measure and improve student happiness at the classroom level. Given the importance of student happiness in the classroom, this paper investigates the ability of a research team to identify elementary math classrooms where students rated their experience as high engagement/high happiness versus classrooms where students rated their experience as low engagement/low happiness. Using survey data at the student level, transcript review, and video observations, this is the first study of U.S. public schools that provides guidance regarding the methods researchers should use to gauge students' happiness and engagement in elementary math classrooms. In the subsequent sections, we explain our original investigation to predict students' level of engagement/happiness using three lesson transcripts, why our findings aligned with previous research regarding the use of transcripts to predict a student's class perception, and we provide suggestions to improve the use of transcripts to predict students' conceptualizations of happiness in the elementary math classroom.

Investigation & Methods

We began with a group of twelve elementary math teachers, all of whom participated in a larger study that randomly assigned fourth and fifth-grade teachers to student rosters in four East Coast school districts in the United States (Blazar, 2015). This subset of teachers was selected based on high value-added scores, which allowed the investigation to focus on students' engagement/happiness while holding increases in students' academic achievement levels on standardized tests constant. Before beginning the project, researchers learned that six of the teachers received high ratings for engagement/happiness and six teachers received low ratings for engagement/happiness. Engagement/happiness ratings came from a student survey.

For each teacher, the research group reviewed three lesson transcripts and three video recordings of the class over one academic year. Due to IRB restrictions, two senior research group members had access to video recordings and transcripts of each class. The other four members of the research group only had access to the transcripts of each class. Each week researchers were randomly assigned two of a particular teacher's three videos or transcripts to review. Researchers met in person each week to discuss features of lessons that would align with their perceptions of high engagement/happiness or low engagement happiness. Given the thin literature base regarding the identification of student engagement/happiness via transcript and video review, the group used an open coding system. Prior to the weekly meeting, researchers worked independently to identify the most salient features of a teacher's practice that would align with students' conceptualization of their elementary math classroom as either a high engagement/happiness space or a low engagement/happiness space. At the end of each meeting, one researcher wrote a memo regarding the prominent features of the teacher's practice that the researchers hypothesized would increase or decrease students' engagement/happiness in math class. The research team also offered a guess about the engagement/happiness level of students in the class. The research group discussed their guesses until arriving at a consensus. The guesses were recorded in a spreadsheet.

At the conclusion of the coding and guessing process for all twelve teachers, the faculty advisor shared with researchers the survey results from each teacher's students. The research group was not successful at guessing whether students would rate a particular teacher as either high engagement/high happiness or low engagement/low happiness. In the end, only 50% of the predictions aligned with student perceptions of engagement and happiness in the math classroom (Table 1).

Table 1: Research Team Guess Spreadsheet

Teacher Identification Number	Happiness/Engagement Score via Student Survey	Researcher Guesses
12002	High	Low
12006	Low	Low
12008	Low	Low
12020	Low	High
13102	High	Low
14019	Low	Low
14030	High	High
14040	High	Low
14060	Low	High
14117	High	High
14127	Low	High
11070	High	High

The transcripts and the video did not provide the research team with the level of context required to accurately assess the dynamics in the classroom that would align with a student's perception of happiness/engagement in their elementary math class (Miles & Huberman, 1994; McCormack, 2000). As predicted by other studies that relied on transcript review, with limited context the research team was unable to accurately guess students' level of engagement/happiness.

Identification of Methodological Issues

After reviewing the original investigation, we propose four methodological issues that decreased the probability that the research group could correctly identify whether students rated the classroom as high or low engagement/happiness: (1) students and adults have different views of which classroom practices will generate student happiness, (2) failure to consult literature on the conceptualization of happiness in elementary aged children, (3) existence of a halo effect and (4) contextualized relationships in classroom environments matter.

Differing Conceptualizations of Happiness Related Classroom Practice

The literature on student happiness indicates that students and teachers may approach features of a happiness and engagement from different perspectives, which may indicate why a group of researchers were unable to view the transcripts from the student perspective. In a mixed-methods study, Tenny (2011) found that while the themes that emerged in her literature review also emerged in her findings, additional themes emerged that were not present in the literature. Based on the review of the literature, Tenny (2011) expected the following to impact student happiness: appropriate level of challenge, level of academic support, engagement and enjoyment through hands on meaningful, and collaborative activities, and positive relationships. Students also stated that physical and mental breaks, frequency of testing and homework were also important factors in their happiness. While researchers are capturing salient features of what contributes to classroom happiness for students, their adult objectivity can serve as a hindrance to them being able to identify components of classroom instruction that are important to young children. Holder and Coleman (2008) state that happiness in children may be different than that of adults because children lack the cognitive maturity and life experiences that influence the happiness of adults. In their study of how well-being is conceptualized and practiced in schools, Graham et al. (2017) found that teachers and students differed in their responses. In addition, the student-reported teacher actions that contributed to well-being were divergent among elementary- and secondary-aged students (Graham et al., 2017).

Failure to Consult Literature on Elementary Student's Conceptualization of Happiness

The research team did not consult the literature on student engagement/happiness in elementary school classes. Instead, the team reviewed the literature for studies that specifically focused on elementary school math classrooms and student engagement and happiness; finding none, the team decided to conduct the qualitative analysis using an open coding system. In retrospect, the literature on the elementary student engagement/happiness broadly could have provided the initial investigation with a stronger approach. Without a framework, the group brought their personal experiences and conceptualizations of student happiness to the transcript review process. Examining the classroom memos, the researchers lacked a clear definition regarding what features of a classroom would be indicators of a student's perception of a specific classroom as high engagement/happiness versus low engagement/happiness (Merriam, 1998).

Possibility of Halo Effect

The halo effect could have significantly biased our results. Students who score higher on standardized tests might be more likely to rate their teachers and classrooms higher in engagement/happiness (Egalite & Kisida, 2018). While we knew all the teachers in the study had substantial improvements in how their students performed on standardized testing, we were not privy to students' academic success. Without knowing the students' baseline math scores, it was unclear if students made substantial gains in the year of the study or if students had experienced greater gains in the years previous to joining the classroom of study. As a result, it may be that teachers rated as high engagement were rated so because of students' experiences and growth in math classes the year before and had limited relationship with the current teacher's actual day-to-day practices.

Relationships Matter (students-to-student & teacher-to-student)

The literature on student happiness indicates that relationships in the school are important to the way students conceptualize happiness. During our initial investigation, the research group used data that did not capture the relationships in the classroom. As a result, the group could not determine the peer-to-peer effects in a specific classroom nor could the group accurately gauge the level of connectivity between teachers and students. Our use of transcripts and videos without interacting with teachers and students provided insufficient data regarding the level of relationships in the classroom; thus the relationship between student connectivity to their peers or teacher was missing (Miles & Huberman, 1994; McCormack, 2000). To be clear, we are not saying that teachers are unimportant; rather, the relationship among teachers and students underscores the academic and social processes in the classroom, and more emphasis should be placed on these relationships when understanding how teachers impact student outcomes--whether academic or those relating to student well-being (Blazar & Kraft, 2017).

Recommendations

Our findings indicate that using transcripts to capture a student's conceptualization of happiness/engagement is complicated. If researchers, administrators, and teachers are interested in measuring student's happiness and engagement in elementary classrooms they should consider adhering to the following strategies:

1. Stimulated recall interviews with students (Davis, 1989). This method allows children to impart insights on the thought process and behavior of other children and "provide us with a point of view to which we apparently have lost direct access." (Davis, 1989, p. 39).
2. Measure a student's level of happiness and engagement over more than one time period.

3. Researchers should attempt to collect contextual information such that researchers can utilize classroom, teacher, and student information to identify practices that support student engagement/happiness (McCormack, 2000).

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INSTRUCTIONAL LEADERSHIP, POLICY, AND INSTITUTIONS/SYSTEMS:

POSTER PRESENTATIONS

A COLLABORATIVE SELF-STUDY TO FOREFRONT ISSUES OF IDENTITY AND EQUITY IN MATHEMATICS METHODS COURSES

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Keywords: Teacher Educators, Teacher Education-preservice, Equity and Diversity

This self-study is a collaboration of Mathematics Teacher Educators (MTEs) with the goal of raising issues of identity and equity within their elementary mathematics methods courses. A common problem of practice surfaced among the MTEs of how to support prospective teachers (PTs) in their development of seeing students' mathematical strengths. The research question was: How can MTEs collaboratively work towards addressing issues of identity and equity across varied institutional contexts?

Self-study involves the systematic studying of the self as a teacher within a context, aimed at improvement (LaBoskey, 2007) and is still emerging as a means for MTEs to study their own practice (Suazo-Flores et al., 2018). We sought to explore pedagogical practices in our methods courses to support PTs in seeing the mathematical strengths of PK-6 students. We selected an article by Skinner, Louie and Baldinger (2019) as a common course reading and developed a protocol that included pre/post PT reflective prompts around the article's strategies for seeing students' mathematical strengths. In order to examine our pedagogical practices we gathered and analyzed the following: (a) positionality statements, (b) lesson plans, including the selection of discussion facilitation questions, (c) post-implementation reflections from the MTEs and PTs, and (d) recordings and notes from our monthly meetings. Reflecting on these data offered insight into pedagogical changes for future course iterations.

Engaging in iterative cycles of practice, reflection, and change allowed us to continually learn from each other and modify our instruction. Based on insights from our colleagues and our own self-reflections, issues of practice to be taken up in future iterations were identified. These included the need to (a) directly address power and privilege with our PTs, (b) model and discuss trusting elementary students with challenging mathematics tasks, and (c) interrogate systemic issues in mathematics teacher preparation, such as purposeful field placements, and PT and elementary student assessment tools. Holistically, the self-study process helped us to develop collective terminology and refine our understanding and use of equity-based practices encompassing various mathematics education organizations' definitions and position statements. Through self-study we supportively and collaboratively pushed each other to reflect on our own teaching with and through equity-based pedagogies, recognizing that our enactment is vital for PTs who teach mathematics for equity and access (Chao et al., 2014).

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A MODEL FOR MATHEMATICS INSTRUCTIONAL IMPROVEMENT AT SCALE

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In this poster, we propose a model for school math instructional improvement that is adaptable to local settings and the organizations and practitioners in them. Different school districts have different problems of practice, and thus adaptive integration of interventions is important as they go to scale—as Penuel et al. (2011) find, successful "scaling up" depends on local actors who make continual, coherent adjustments to interventions as they make their way through various levels of an organization. Indeed, school- and district-level infrastructures that are not optimally designed to support instructional improvement can constrain professional development (PD) efforts to improve the effectiveness of the existing teaching force (Spillane & Hopkins, 2013). Similarly, school districts have been shown to influence the ways in which schools and school leaders implement a wide range of improvement efforts at the school level, thus helping or hindering such implementation (Honig & Rainey, 2014).

The model we propose is particularly designed to improve teachers', teacher leaders', and administrators' understanding of effective math teaching and learning, and to enhance the organizational capacities of schools and districts to support such improvements in math. The model is grounded in a Design-Based Implementation Research process involving collaboration between researchers, and district and school personnel to co-develop math PD from district through teacher levels. The components are: (1) gathering information about problems of practice collaboratively identified by districts, schools, and the research team, and developing related goals; (2) designing and implementing coherent PD that is aligned with identified problems of practice; and (3) engaging in iterative cycles of development, implementation, and revision to productively adapt the model to changing conditions. The iterative redesign process enhances the productive adaptation of the model, allowing it to be effective at scale.

In this poster, we will present our preliminary findings from the first cycle of iterative co-design of the model with stakeholders in four different school districts, including design considerations and challenges that emerged from the co-design process. In doing so, our aim is to make a significant contribution to the knowledge base regarding the process of organizational change in educational settings, effective teacher and administrator PD in math, and researcher-local stakeholder collaboration.

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EVOLUTION OF ELEMENTARY MATH LEADERS' COLLABORATIVE PLANS FOR SCHOOL-LEVEL CHANGE

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Keywords: Teacher Education – Inservice, Professional Development, Instructional Leadership, Elementary School Education

Research is needed to better understand how elementary teachers develop and make progress toward enacting and supporting new visions of mathematics instruction while transitioning to an informal leadership role (Chval et al., 2010). This study follows 13 elementary mathematics teachers within a Midwestern United States school district who are pursuing Elementary Mathematics Specialists (EMS) certification through fellowships provided via a project funded by the National Science Foundation. The fellows, working in six school-based teams, were asked as a part of their program to create and maintain action plans: revisable documents outlining their evolving visions and plans for improving mathematics teaching in their schools. In analyzing these plans, we investigated the question: What initiatives do the groups plan to implement in their buildings, and how do these plans evolve over the course of the school year?

Initial data included focus group interviews with each team and all four iterations of the action plans. Each school team submitted a revised version of their action plan monthly as a part of their EMS course. In order to name the ways in which each team evolved throughout the course of the project, we defined the following components of the fellows' plans: the scope of the initiative (within-own-classroom, grade-level teams, schoolwide, district-wide), focus (challenging perceptions of student competency, building a positive mathematical culture, supporting student identity formation through instructional practices), and medium (collaboration between mathematics leaders, grade-level collaboration, professional development, teacher observation, class restructuring).

A cross-case comparison revealed multiple trajectories for the groups of fellows. While five of six school-based teams named goals for their own classroom instruction in the first iteration of their plan, not one team applied a schoolwide lens. However, on the fourth iteration, all six teams were seen to employ a schoolwide lens. Interestingly, we found that only one school had maintained the same focus on "eliciting student thinking/providing student feedback" from the first to fourth version of their action plan, and that particular team additionally named that focus at a schoolwide level on the fourth iteration.

We also examined various factors of support (district, administrative, colleagues) and identity (leadership and confidence in mathematical content knowledge) as potential impacts on the trajectories followed by each school team. For example, one school team began to apply a schoolwide scope due to an empowering administrator who asked the fellows to develop professional development for staff, while another team adopted a schoolwide scope because the team perceived their peers' conceptions of student competency as deficit-based. Future analysis will continue to monitor the evolution of fellows' action plans for the duration of the larger project.

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CONTINUOUS IMPROVEMENT LESSON STUDY WITH MATHEMATICS TEACHER EDUCATORS

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Keywords: Teacher Educators, Curriculum Analysis, Teacher Knowledge

The purpose of this research study was to consider how a group of early career mathematics teacher educators (MTEs), from across the U.S., who participated in a continuous improvement lesson study (Berk & Hiebert, 2009), learned to better support their elementary preservice teachers' (PSTs) learning. In this poster, we share the continuous improvement lesson study process we used and discuss changes made to our researched lesson throughout the process. We also share the final version of the lesson and call on other MTEs to utilize continuous improvement lesson studies.

Methodology

This study took place across institutions and adopted the continuous improvement framework (a type of lesson study) which presents a model of curriculum development through studying one researched lesson with cycles of planning, enactment, analysis and revision (Berk & Hiebert, 2009). Four MTEs participated in this study over a period of five semesters in which either all or some of the MTEs taught the lesson. Once the lesson was finalized, we sought to answer the following research questions: What was the nature of the changes made to the researched lesson throughout the continuous improvement lesson study process? How did these refinements better support our PSTs? To answer these questions, all lesson changes were mapped for each iteration of the lesson and rationales for the changes were gathered from transcribed group meetings and individual MTE reflections. In addition, we gathered evidence of our PSTs' thinking, both written and oral, to document how PSTs' thinking changed and contributed to the changes made. Breaking the lesson down by components and lesson iteration, we were able to investigate each component's effectiveness. We used open coding and coded MTE written reflections and transcripts for places we discussed changes and the rationales for those changes.

Results and Implications

Eight structural changes (those changes made on how the lesson was structured) from the initial lesson to the final version were identified. We will share the changes ranging from which elements of a case-study to present to the PSTs at different points in the lesson to the types of questions we asked to better elicit PSTs' noticing of children's single digit multiplication thinking (Jacobs, Lamb & Philipp, 2010). As early career MTEs, we found that our commitment to developing a researched lesson following the continuous improvement framework deepened our sense of belonging to the MTE community, acted as an important means of professional development, and because of our collaborations, we were able to orchestrate better discussions. We were able to push our PSTs' learning to higher levels, more so than we could have done on our own.

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MATHEMATICAL KNOWLEDGE FOR TEACHING

RESEARCH REPORTS

SURVEY OF PRESERVICE TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE FOR STUDENTS' MULTIPLICATIVE REASONING

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Multiplication and division are vital topics in upper level elementary school. A teacher's pedagogical content knowledge (PCK) influences both instruction and students' learning. However, there is currently little research examining teachers' PCK within this domain, particularly regarding professional education of future teachers. To help address this need, the present paper presents an initial validity argument for a survey of preservice teacher's PCK for multiplication and division.

Keywords: Teacher Knowledge; Number Concepts and Operations

Overview & Purpose

Multiplicative reasoning is a critical concept in upper elementary school (grades 3-5) that facilitates student reasoning of later mathematics concepts (Hackenberg, 2010). Whole number multiplication and division is formally introduced in grades 3-5 (CCSSI, 2010), leading to their inclusion in initial licensure mathematics methods courses for early childhood, elementary, and middle grades preservice teachers (PSTs). However, there is limited research on PSTs' professional knowledge in this area (Thanheiser et al., 2014). Such literature tends to focus on PSTs' understanding of the content (Harkness & Thomas, 2008; Menon, 2003), and often conveys a large portion of novice teachers lack sufficient understanding of declarative knowledge related to multiplication and division. Yet, the professional knowledge needed to teach mathematics, or Mathematical Knowledge for Teaching (MKT), involves more than a deep understanding of the content (Hill et al., 2008b). Pedagogical Content Knowledge (PCK) "goes beyond knowledge of subject matter," in that it is a "particular form of content knowledge that embodies the aspects of content most germane to its teachability" (Shulman, 1986, p. 9). Indeed, there is evidence to suggest that PCK for mathematics is more sophisticated than content knowledge (Copur-Gencturk et al., 2019), but there is relatively little study of PSTs' PCK for multiplication and division of whole numbers (Thanheiser et al., 2014). One reason for this is the relative difficulty in defining and creating measures of PCK (Copur-Gencturk et al., 2019; Hill et al., 2008a). In our own work, we sought such a measure to gauge the effect of a teacher education initiative. The lack of a measure of PSTs' PCK for multiplication and division, therefore, fueled our need to create such a measure. Thus, the purpose of this study is to construct an initial validity argument for a survey of preservice teachers' pedagogical content knowledge for elementary children's multiplicative reasoning.

Background Literature & Theoretical Perspectives

Pedagogical Content Knowledge

This study reports on the design and initial validation of an MKT assessment of whole number multiplication and division. Current assessments of MKT have focused on either specific courses, such as Geometry or Algebra I (Herbst & Kosko, 2014; McCrory et al., 2012) or a wide range of content within a single mathematical domain, such as numbers and operations (Hill et al., 2008a). For example, McCroy et al. (2012) developed an instrument to test teachers' mathematics-teaching-knowledge of Algebra, constructing items specific to student reasoning of algebra problems. McCrory et al.'s (2012) definition of mathematics-teaching-knowledge is similar to PCK, as it includes knowing a student's mathematical reasoning and understanding possible misconceptions. Similar to McCrory et al. (2012), Herbst and Kosko (2014) developed items to investigate MKT in Geometry teachers by constructing items based on students reasoning and approach to geometry problems. Hill et al. (2008a) also created items focusing on PCK, but the majority of their assessment is focused on both common and specialized content knowledge for teaching. While the aforementioned efforts for designing PCK items have met some success, when scholars have designed MKT assessments for specific concepts, such as fractions, the focus tends to be on content knowledge, and not PCK (Izsák et al., 2019).

Although prior research provides useful contributions to the field, the lack of specified focus on aspects of PCK in measurement development has led to underspecification of the domain both within and beyond our focus on multiplication and division. Analyzing items from two different MKT measures, Copur-Cencturk et al. (2019) note that "what constitutes PCK and how PCK differs from [specialized content knowledge] SCK are not well articulated… We need a more in-depth understanding of teachers' instructional strategies that help their students learn and how teachers' knowledge of students' thinking is revealed in mathematics instruction and informs their teaching" (p. 494). Hill et al. (2008a) suggest the problem is two-fold in that there is a lack of research on teachers' PCK and that "the field has not developed, validated, and published measures to assess" (p. 373) such knowledge. Since Hill et al.'s (2008a) writing this statement, items assessing PCK have been successfully written and validated. However, these are typically couched in an overarching assessment of MKT (Depaepe et al., 2015; Herbst & Kosko, 2014). By contrast, this paper focuses explicitly on PCK. Shulman (1986) defines aspects of PCK as:

> An understanding of what makes the learning of specific topics easy or difficult: the conception and preconceptions that students of different ages and backgrounds bring with them to the learning of … frequently taught topics and lessons. If those preconceptions are misconceptions … teachers need knowledge of the strategies ... in recognizing the understanding of learners (p. 9).

There are two primary subdomains of PCK: Knowledge of content and students (KCS) and knowledge of content and teaching (KCT). KCS is defined by Ball et al. (2008) as the knowledge of knowing students as well as knowing the mathematical framework. Within this domain of PCK it is required that teachers know how a student is going to think through a problem and anticipate what problems students will find daunting and confusing (Ball et al., 2008). In contrast, Ball et al. (2008) defined KCT as having the knowledge of how to effectively teach combined with the knowledge of the mathematical subject matter. Teachers with a high level of KCT can use various models to illustrate a concept to students at varying stages of learning (Ball et al, 2008).

Both subdomains have been successfully assessed within the literature. Hill et al. (2008a) developed an assessment to identify KCS and was, to an extent, successful. The findings suggest that in order to answer an item pertaining to a common student error, student understanding, common student developmental sequences, and common student computations a teacher must possess content knowledge (CK) and KCS (Hill et al., 2008a). McCroy et al. (2012) suggested a framework to develop an assessment to measure KCT outside of Algebra by establishing the difference of math knowledge and teaching knowledge. In addition, Herbst and Kosko (2014) constructed an instrument to measure KCS and KCT as well as common content knowledge (CCK) and specialized content knowledge (SCK). The items developed to assess KCS in teachers "probe[d] for their knowledge of students' conceptions and errors in tasks" pertaining to geometry (Herbst & Kosko, 2014, p. 41). Their assessment was able to detect that experienced teachers were more successful at identifying student conceptions/misconceptions than less experienced teachers. The KCT items Herbst and Kosko (2014) constructed followed the same trend; experienced teachers were better able to determine appropriate tasks and examples to effectively illustrate a concept in comparison to less experienced teachers (Herbst & Kosko, 2014). These prior efforts at constructing PCK items, in the context of MKT as a larger construct, informed our own efforts at item design.

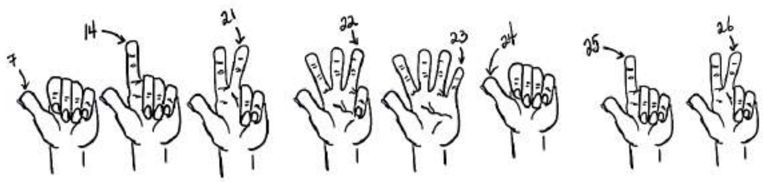
The present paper focuses on KCS of Multiplication and Division and follows item design recommendations of Ball et al. (2008) and Herbst and Kosko (2014). Specifically, items were written to assess teachers’ knowledge of students’ conceptions and errors in whole number multiplication and division for grades 3 to 5. In the next section, we describe this process in detail.

Development of the PCK-MaD Assessment

In this section, we describe the development of items for an assessment of Pedagogical Content Knowledge for Multiplication and Division (PCK-MaD). Items for the initial version of PCK-MaD were designed specifically to assess the KCS dimension of Ball et al.’s (2008) MKT framework. We anticipate including KCT items in a later version of the assessment but sought to focus on KCS as an initial step. Following recommendations from prior work in this area (Ball et al., 2008; Herbst & Kosko, 2014), we designed items focusing specifically on variations in upper elementary school children’s conceptions of multiplication and division. To do this, we focused on grades 3-5 Common Core Standards for Mathematics on multiplication and division standards (CCSSI, 2010) as a means of identifying key concepts to write items. Next, we conducted a literature review of mathematics education research on these and related concepts that described the nature of children’s reasoning. We paired this review of research with a review of practitioner resources (Battista, 2012; Van de Walle et al., 2019).

Figure 1 provides an example item to help illustrate this process of item design, writing, and revision. The item, designated M01, was designed to assess teachers’ knowledge of children’s developmental skip-counting, and aligns with CCSS standard 3.0A.A.1 specifying that children need to interpret products of whole numbers. Variations of skip-counting have been observed by researchers, including a phenomenon where students begin to miss certain skip-counts (Mulligan & Mitchelmore, 1997; Sherin & Fuson, 2005). Steffe (1994) describes this as a point where children are beginning to compose iterable units, counting with whole numbers other than 1, but that this action is still very dynamic for the child. Rather, the composite whole number has not been fully abstracted for the child, and as they attempt to skip-count, they may lose track between coordinating the unit to be skip-counted and coordinating the number of skip-counts. In Figure 1, we illustrate this form of reasoning with a context of multiplying 7 and 8, and an illustration of skip-counting with one’s fingers. Distractors were included to represent other points in learning progressions described for practitioners (Battista, 2012; Van de Walle et al., 2019). For example, Battista (2019) describes repeated addition as distinct from uncoordinated skip-counting. It is also a distinction that may be difficult for some PSTs to observe, making option #4 a useful distractor. Although Figure 1 provides a final version of item M01, multiple revisions occurred as language and figures were reviewed and critiqued by the project team.

Ashley was asked to multiply 7 and 8. She counted on her hands to solve the task (illustrated below):



Based on Ashley's actions, which statement is the best assessment of her reasoning?

①	Ashley skip counted by sevens to solve 7 times 8, but mixed up her skip-counts with the number of skip-counts she was using.
②	Ashley skip counted by sevens to solve 7 times 8, but began skip-counting with a different number part-way through.
③	Ashley, with the aid of her fingers, decomposed 7 times 8 into two smaller problems.
④	Ashley, with the aid of her fingers, used repeated addition to multiply 7 and 8.

Figure 1. Example PCK-MaD item assessing for understanding of children's developmental skip-counting.

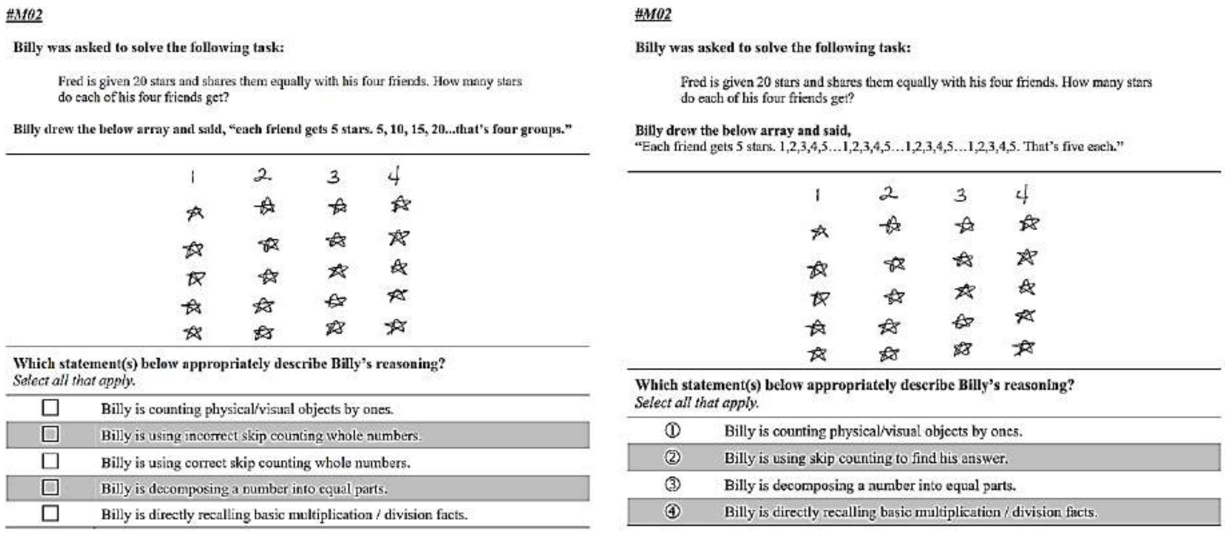
After all items had been successfully vetted by project team members, we conducted cognitive interviews with two elementary math coaches who were widely recognized by the field for their expertise. Cognitive interviewing is a process in which a participant engages in a one-on-one interview to complete an assessment. After completing each item, the participant is asked what they thought the item was asking them to do, what they answered, and why they answered the way they did (Karabenick et al., 2007). For the PCK-MaD, cognitive interview data was used to examine whether items were interpreted as intended, and whether rationales for responses focused on aspects of students' mathematical thinking (i.e., KCS). Each interview was roughly 2 hours in length, but within this time through feedback on items was given.

Item M15, Figure 3, illustrates an example of a question that was *not* altered based on the feedback from the two math coaches. By contrast, Figure 2 depicts an example that was drastically modified due to the constructive criticism. This item was originally designed to be multiple response but was modified to become a multiple-choice item. In addition, the language of the item pertaining to the sample students reasoning was revised to be clearer of the intended thought process due to the discrepancy of responses from the two expert teachers:

Expert Teacher #1: "Billy is decomposing items into equal parts."

Expert Teacher #2: "It looks like he is counting visual items by one"
In response to why she answered that way:
Expert Teacher #2: "Because of the fact that all twenty were represented by stars so it looks like he counted 1, 2, 3, 4, 5."

The different responses illustrated to us that the item stem was unclear, and some of the options may have been interpreted in ways we did not intend. The reasoning for "Billy" was modified to be clearer of his mathematical reasoning of counting the stars and changed to a multiple-choice item to not distract the users further. Unfortunately, the cognitive interviews also resulted in one item being removed completely from the assessment due to the overall confusion of the participants. Some items were revised with very minor adjustments (missing punctuation, a typo in an image, etc.). Following cognitive interviews, 9 items received some revision (minor to moderate), 4 items remained as-is, and 1 item was fully removed.



Original Question Revised Question

***Figure 2*: PCK-MaD questions M02 example of revision after cognitive interviews.**

Another outcome of our cognitive interviews was a realization of the cognitive demand of several items. Evidence from the literature suggests that KCS items may be more difficult than other MKT domains (Copur-Cencturk et al., 2019; Herbst & Kosko, 2014), and we found that many of our KCS items were indeed more difficult. Therefore, we created three additional items, following cognitive interviews, in an effort to have KCS items with an easier difficulty level.

Following the framework of Herbst and Kosko (2014) items were developed to measure KCS in pre-service teachers. Revision of the items based on the feedback from the expert teachers resulted in the pilot PCK-MaD assessment. The adjustments made to the items added to clarity and refined the level of difficulty of the language. However, to properly vet the items and gather further validity of the assessment, we collected pilot data from preservice teachers (PSTs) enrolled in a teacher education program. This process served to collect validity evidence for an initial validity argument for the PCK-MaD.

Method

Sample and Procedure

Participants included 58 PSTs, with 47 preparing to become elementary teachers (grades K-3 with an endorsement option for grades 4-5) and 11 preparing to become middle grades teachers (grades 4-9). Participants were in the latter half of their teacher education (31 juniors; 27 seniors). The majority of junior participants were elementary PSTs (n=27) preparing to take the first of two mathematics methods courses. Although these participants had some pedagogical coursework and field experience, they hadn't received formal education on PCK for multiplication/division. Four juniors were middle grades PSTs who had completed the first of two mathematics methods courses. Senior elementary PSTs (n=20) had completed two mathematics methods courses, with several field-based assignments relating to multiplicative reasoning across grades K-3. All participating elementary PSTs expressed their intent to complete an additional mathematics methods course focusing on grades 4-5, but none had completed this course at time of data collection. The majority of middle grades PSTs (7 of 11) were seniors and were enrolled in the second of two mathematics methods courses in their program.

Analysis and Results

Given the early stage of developing our PCK survey, the present paper examines validity evidence from test content and response processes. Validity evidence for *response processes* refers to "whether test takers are, in fact, reasoning about the material given instead of following a standard algorithm applicable only to the specific items on the test" (AERA et al., 2004, p. 15). Wolf and Smith (2007) suggest that psychometric measures can be used to assess the degree that the theoretical rationales for item content align with response processes. Therefore, to examine evidence for response processes in the present paper, we conducted a classical item analysis to examine the internal reliability of items and the resulting measure, and to examine the relative difficulty of those items in comparison with one another.

The PCK survey included 15 questions, with six questions conveyed in a multiple-response (i.e., select all that apply) format. For example, question M15 presents six different student algorithms and asks the survey respondent to select those that used all partial products (see Figure 3). This effectively conveys six different items for M15. Thus, for the 15 questions we examined, there were 41 items, due to the six multiple response questions. Our initial item analysis model, including all 41 items, resulted in a Cronbach's alpha coefficient of .47. For surveys and piloted assessments such as the one in this paper, the typically accepted threshold is at or near *.70* (Nunnally & Bernstein, 1992). Therefore, we examined the point-biserial correlations for each item to identify candidates for removal. Point-biserial coefficients below .30 are considered to not meaningfully contribute to the total score, possibly due to variance in response (Crocker & Algina, 2006). Rather than remove all such items, it is customary to remove one item at a time, so that the remaining items' point-biserial coefficients can be recalculated for a new model. In addition to identifying particularly low coefficients, items are examined in the context of their theoretical contributions to the model, as well as evidence from cognitive interviews and/or written work on the surveys. For example, the sixth item on question M15 had an initial point-biserial coefficient of .021 (see Figure 3). The low coefficient essentially flagged the item for review. We then considered evidence from our cognitive interviews in which unfamiliarity with the lattice method and how it functioned mathematically resulted in incorrect responses. Thus, this option for question M15 was removed. A similar process took place for all iterations of item analysis. Our final model included 21 items, from nine questions, with a Cronbach's alpha coefficient of .68. This suggests at least 68% of the variance in responses is due to the measured construct (PCK for multiplication and division). Point-biserial coefficients for most items were above or near the .30 threshold. Item difficulty for the remaining items ranged from .20 (20% of the sample answered correctly) to .90 (90% of the sample answered correctly), with a mean score of 14.88 (SD = 3.14, Range = 5 to 16).

Validity evidence for *test content* considers how well assessment content represents PCK for children's multiplicative reasoning, and how well this content aligns with interpreting PSTs' scores (AERA et al., 2004). To analyze this, we will examine the intended purpose of the assessment (i.e., to measure the effect of teacher education) using an independent samples t-test for PCK scores of juniors and seniors. Results were statistically significant (t = 2.686188, df = 56, p = .00933), indicated that senior PSTs had higher PCK scores (15.8674) than junior PSTs (13.6690). To ensure the comparison between junior and senior PSTs was not influenced by major, we examined the difference between elementary and middle grades PSTs' scores and found no statistically significant difference (t = 0.673, df = 56, p = .503). We also examined whether PSTs who had a field placement in grades 3-5 would have higher PCK scores and found no statistically significant difference between those with and without such field experience (t = 0.396, df = 56, p = .743). Considered collectively, these results suggest that, for participants in the current study, the PCK survey distinguishes between PSTs who are earlier or later in their teacher education program. Such a difference does not appear to be due to intended licensure (elementary or middle grades) or having grades 3-5 field experience.

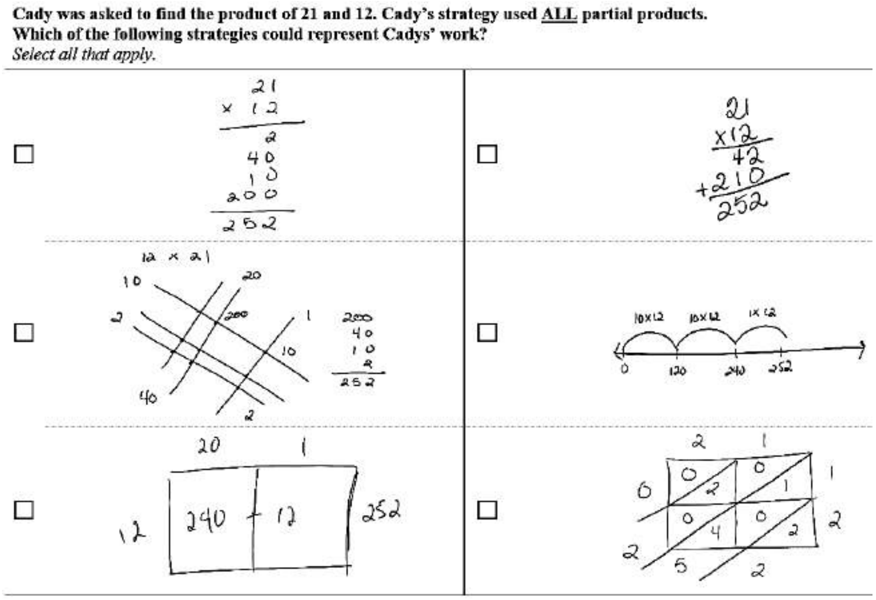


Figure 3: **PCK-MaD that assesses children's strategy-use with partial products.**

Discussion

To our knowledge, there is no prior assessment for PCK of multiplication and division for whole numbers. This study reported on the initial piloting of our assessment of PST's knowledge of content and students (KCS) with a focus on multiplicative reasoning. Given the need for additional research on PSTs' PCK for multiplication and division (Thanheiser et al., 2014), development of a measure for this domain has the potential for informing the field in this regard. The findings of this study suggest that our survey can distinguish between PCK scores of PSTs at different levels of teacher education (i.e., senior vs. junior). On one hand, this provides useful validity evidence for the PCK-MaD's ability to distinguish between PSTs at different points in their teacher education. However, this finding also lends support for the effectiveness of teacher education programs at developing PSTs' PCK. Both implications of this particular finding, while useful, should be interpreted with caution as the current study represents an initial pilot of an assessment and involves a sample from a particular teacher education program.

Psychometric data from the PCK-MaD item analysis and data from the two cognitive interviews suggest that the piloted assessment does measure the intended construct. However, future research is needed to improve the assessment. Results suggest initial support for an assessment of KCS, but additional items focusing on KCT should be developed. Further, results here focus predominately on responses from PSTs, suggesting a need to examine responses from inservice teachers to establish a better understanding of normative KCS in the field. Despite the early stage of this work, results suggest that the PCK-MaD may be used as-is for assessing the effect of teacher education initiatives.

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TEACHERS' ATTENTION TO AND FLEXIBILITY WITH REFERENT UNITS

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In this study, we explored teachers' attention to and flexibility with referent units as well as how teachers' understanding of referent units is related to their performance on other fraction concepts and their professional background. By using data collected from 246 U.S. mathematics teachers in Grades 3–7 where fractions are taught, we found that teachers' attention to and flexibility with referent units were moderately related. Whereas some teachers' professional background variables could explain their flexibility with referent units, none of the variables was linked to their attention to referent units. Furthermore, both teachers' attention to and flexibility with referent units seemed to be associated with their performance on other fraction concepts.

Keywords: Rational Numbers, Teacher Knowledge

Fractions are critical content in the upper elementary and middle grades curriculum (e.g., Common Core State Standard Initiatives [CCSSI], 2010). Despite teachers' computing well on fraction arithmetic, they usually struggle with understanding fractions conceptually (e.g., Izsák, 2008). For instance, teachers can confuse problem situations asking for division by a fraction with those asking for multiplication by a fraction (e.g., Ma, 1999) or overgeneralize rules for whole numbers to fractions such as division makes numbers smaller (Jansen & Hohensee, 2016).

Several scholars have argued that such difficulties with understanding fractions might be related to the whole number bias (e.g., Vamvakoussi, Christou, & Vosniadou, 2018), whereas others have argued that not understanding number magnitude may be the underlying reason (e.g., Siegler, 2016). Scholars in mathematics education have also brought up referent units (RU), which are critical, yet overlooked, for having a conceptual understanding of fractions (e.g., Izsák, Orrill, Cohen, & Brown, 2010). Empirical work has provided support for the importance of RU (e.g., Izsák, Jacobson, and Bradshaw, 2019). For example, Izsák et al. (2010) analyzed 201 U.S. middle grades teachers' responses to a set of items and found two classes that distinguish the teachers based on their understanding of RU. In a recent study that analyzed 990 U.S. middle grades teachers' responses to a multiple-choice assessment, Izsák et al. (2019) found that teachers who were proficient in RU tended to perform better on the remaining components of reasoning about fractions.

Although past research has provided insights into teachers' understanding of RU, it has focused heavily on such understanding in fraction multiplication and division situations, given that RU change during the process (e.g., Izsák et al., 2019). Thus, these studies capture teachers' *flexibility with* RU, which can be defined as "a teacher's ability to keep track of the unit to which a fraction refers . . . and to shift their relative understanding . . . as the referent unit changes" (Lee, Brown, & Orrill, 2011, p. 204). Although fraction multiplication and division situations provide an invaluable opportunity to examine whether teachers can identify referent units correctly and think accordingly as the referent unit changes, we argue that RU are important in any fraction concept. Our argument is grounded in the view that understanding RU also includes *attention to* RU, even in less explicit situations. To illustrate what we mean by *attention to* RU, when comparing fractions, creating equivalent fractions, and performing fraction operations such as fraction addition and subtraction, the same referent unit is used for the fractions involved. For instance, when two fractions are added, both fractions refer to the same whole. Thus, *attention to* RU could capture another characteristic of teachers' understanding of RU.

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In sum, although prior work has provided evidence for the importance of RU in understanding fractions, we still know little about the relationship between different characteristics of RU. In particular, we hypothesized that in addition to flexibility with RU, attention to RU is an important characteristic of teachers' understanding of RU and, in general, of their overall performance on fractions. To test our hypothesis, we created two constructed-response problems, one capturing teachers' *attention to* RU in a fraction comparison situation and the other capturing teachers' *flexibility with* RU in a fraction multiplication situation involving a visual representation. By using data collected from 246 U.S. in-service teachers who were teaching mathematics in Grades 3–7, we examined the relationship between teachers' performance on these two problems and the extent to which teachers' professional background was related to their responses to these two problems. Finally, we explored how teachers' responses to these two problems were related to their overall performance on a fractions measure. We aimed to answer the following research questions:

1. To what extent do teachers pay attention to RU?
2. To what extent do teachers demonstrate flexibility with RU?
3. What is the relationship between teachers' attention to and flexibility with RU?
4. What aspects of teachers' professional background are related to their attention to and flexibility with RU?
5. To what extent are teachers' attention to and flexibility with RU, along with their professional background, associated with their overall performance on fractions?

Our study contributes to the current literature in three significant ways. First, prior work has not focused on the relationship between teachers' understanding of different characteristics of RU. Thus, by examining the relationship between teachers' attention to and flexibility with RU, we aimed to contribute teachers' understanding of RU and fraction operations. Second, limited research (Izsák et al., 2019) has investigated the relationship between teachers' professional background and their understanding of RU. Thus, knowing the extent to which teachers' professional background is associated with their attention to and flexibility with RU will have implications for mathematics teacher education. Finally, by investigating the relationship between teachers' understanding of RU and their performance on a fractions measure, we aimed to provide further evidence for how teachers' understanding of RU might be linked to their overall performance on fractions.

Theoretical Framework

Referent units can be defined as units number refer to in mathematical situations. Although it is possible for teachers and students to perform algorithms correctly without relying on RU, a conceptual understanding of fractions requires one to explicitly attend to the units and to be aware of the units in these situations (Philipp & Hawthorne, 2015). Let us illustrate the RU in two different problem situations:

1. Which fraction is larger: 1/3 or 1/2?
2. One serving of yogurt is 1/3 of a cup. For one meal, Amanda ate 1/2 of a serving. How many cups of yogurt did Amanda eat?

In the first problem, the answer can be found by finding a common denominator for both fractions and noticing that 2/6 is smaller than 3/6. However, the comparison makes only sense if both fractions refer to the same unit. Thus, attention to RU is necessary to develop a conceptual understanding in situations where the referent unit stays the same. In this way, teachers can overcome several misconceptions such as the larger the denominator, the larger the fraction or adding across numerators and denominators (Newton, 2008). In the second problem, however, the numbers refer to different units. Whereas 1/3 and the product, 1/6, refer to 1 cup, 1/2 refers to one serving, which is 1/3 of a cup. When performing the standard algorithm, the answer, 1/6, can be found by multiplying across numerators and denominators. On the other hand, a conceptual understanding of fractions requires showing flexibility with RU by understanding that the RU for 1/2 and 1/3 are different and thinking accordingly as the referent unit changes. Therefore, partitioning the serving size into two parts and shading one part is needed to show 1/2 of 1/3 (Figure 1b). Because the problem asks for the number of cups, the referent unit of 1/6 then becomes 1 cup, the whole rectangle (Figure 1c).

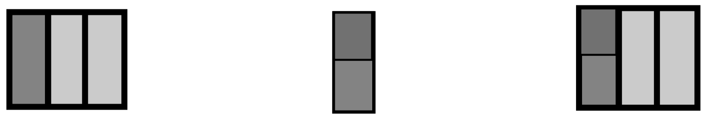


Figure 1: (a) 1/3 of the rectangle; (b) 1/2 of the 1/3; (c) 1/6 of the rectangle

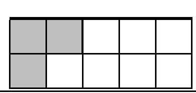
Most prior work on RU has focused on teachers' understanding of fraction multiplication and division, and reported both future and in-service teachers' struggle with RU (e.g., Baek et al., 2017; Izsák, 2008; Izsák et al., 2019; Lee, 2017; Webel et al., 2016). Much of this research used fraction multiplication and reported teachers' reliance on the overlapping method, which uses the same referent unit for the multiplier, multiplicand, and product. These studies have acknowledged that using the overlapping method either results in incorrect answers or causes mostly step-by-step algorithms instead of conceptual understanding about what it means to multiply two fractions.

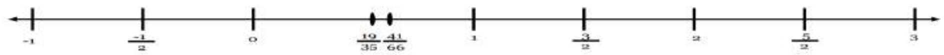
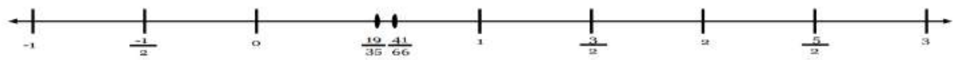
Methods

The data were collected from 246 in-service mathematics teachers in Grades 3–7 across 21 states in the United States. Teachers in our sample were mostly female (84%) and White (68.1%). In addition, 25.2% of the teachers had a master's degree, 77% of them were teaching mathematics in Grades 3–5, and 23% were teaching mathematics in Grades 6–7. While 70.3% had traditional certification, 19.3% had a credential in mathematics, and 52.5% were fully certified.

As seen in Table 1, the fractions measure used in this study consisted of a set of six items adapted from prior research (e.g., Siegler, 2015), the DTMR survey (Izsák et al., 2019) and the Teacher Education and Development Study in Mathematics (TEDS-M) survey (Tatto et al., 2012), and teacher education resources (Van de Walle, Karp, & Bay-Williams, 2019). We also administered the background survey (Izsák et al., 2019) and collected information regarding the professional background of our sample.

Table 1: Fractions measure items

Key concept	Item
Attention to RU	Is it possible for 1/3 to be greater than 1/2? Explain your thinking.
Equivalent fractions	In the figure, how many **MORE** small squares need to be shaded so that 4/5 of the total number of small squares are shaded? Explain your answer. 
Comparing fractions	For each set of fractions, put < , >, or = to make the statement true. $\frac{9}{21}\ \frac{15}{21}$ $\frac{31}{57}\ \frac{23}{57}$ $\frac{61}{44}\ \frac{33}{44}$ $\frac{20}{17}\ \frac{20}{33}$ $\frac{49}{48}\ \frac{49}{47}$ $\frac{71}{60}\ \frac{71}{52}$

Estimating the sum of fractions	The fractions $\frac{19}{35}$ and $\frac{41}{66}$ have been placed on a number line. **Without computing**, please estimate the sum of $\frac{19}{35}+\frac{41}{66}$ by placing a dot on the number line where you think the sum would be found. Explain your answer. 
Flexibility with RU	This item cannot be displayed because it is currently part of the DTMR survey (Izsák et al., 2019). We provided a drawn rectangle and asked teachers to model fraction multiplication and explain their answer.
Estimating the quotient of fractions	The fractions $\frac{19}{35}$ and $\frac{41}{66}$ have been placed on a number line. **Without computing**, please estimate the quotient of $\frac{41}{66}\div\frac{19}{35}$ by placing a dot on the number line where you think the quotient would be found. Explain your answer. 

We independently coded the items on attention to and flexibility with RU. The agreement was over 90% for each item. We classified teachers' responses to the item on attention to RU into three categories: no attention to RU, partial attention to RU, and full attention to RU. Specifically, teachers assigned to the first category did not refer to any RU implicitly or explicitly in their explanations. The second category included teachers who were using the same referent unit. The third category captured teachers who responded that the answer depended on the referent unit. We also classified teachers' responses to the item on flexibility with RU into three categories: no flexibility with RU, partial flexibility with RU, and flexibility with RU. The first category included teachers who did not demonstrate flexibility with RU at all in their responses such as "I am unsure how to model that the product of 1/3 × 1/4 is 1/12." The second category included teachers who used the overlapping method such as "She should draw two vertical lines to divide the rectangle into 3 equal-sized parts across, then shade in one of the vertical rectangles. The shaded piece that is overlapped demonstrates the 1/12." The third category included teachers who demonstrated flexibility with RU by keeping track of the units with explanations such as: "She should divide the picture into 3 equal-sized pieces vertically and show that 1/3 of the 1/4 is 1/12 of the whole." We also scored the remaining four fraction items and the agreement was greater than 90%.

To report teachers' attention to and flexibility with RU, we computed the percentages of responses in each category. To investigate the relationship between teachers' attention to and flexibility with RU, we used a Pearson chi-square test. We also computed the correlation between these categories by using gamma statistics, given that the categories for each problem were ordinal. To investigate the relationship between teachers' responses to the referent unit problems and their professional background variables, we ran a separate ordinal logistic regression for each problem. Finally, to examine the relationships among teachers' overall performance on other items of the fractions measure, their attention to and flexibility with RU, and the professional background variables, we ran a linear regression in which the total score was predicted by teachers' attention to and flexibility with RU and the aforementioned background variables.

Results

Teachers' Attention to Referent Units

As shown in Figure 2, 54.5% of the teachers demonstrated attention to RU by responding that 1/3 could be greater than 1/2, depending on the referent unit. For instance, one teacher explained that "If I am comparing two different-sized objects, then 1/3 may be greater than 1/2." On the other hand, 19.9% of the teachers demonstrated partial attention to RU by reporting that 1/3 could not be greater than 1/2 and by explicitly using the same referent unit to justify their responses. Furthermore, 25.6% of the teachers did not demonstrate attention to RU (Figure 2). Specifically, 57% of these teachers did not provide any explanation that showed why 1/3 could not be greater than 1/2, whereas 25.4% of the teachers constructed equivalent fractions in their explanations. For example, one teacher wrote "To easily compare these fractions, you can find common denominators, 2/6 and 3/6. The one half will always be greater than the one third." Lastly, 17.6% of the teachers either made factual statements in their explanations without mentioning any referent unit or they converted fractions into percentages by reporting that 1/3 and 1/2 means 33% and 50%, respectively.

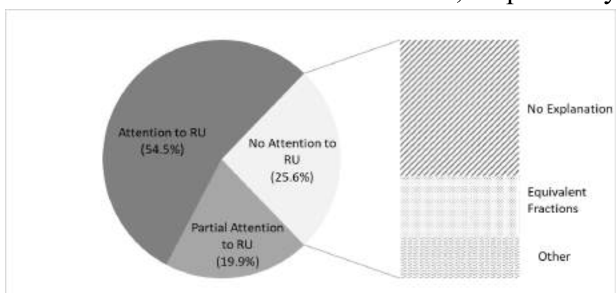


Figure 2: Teachers' performance on the item that measured their attention to RU

Teachers' Flexibility with Referent Units

Teachers' responses to the flexibility with RU item suggested that only 11.8% of the teachers demonstrated flexibility with RU (Figure 3). Those teachers reported that the referent unit for 1/4 was the entire rectangle and that the referent unit for 1/3 was 1/4 of the rectangle (i.e., the shaded part), not the entire rectangle. They also pointed out that 1/12 was 1/3 of the 1/4 rectangle. For example, one teacher explained "divide the picture [1/4 of the given rectangle] into 3 equal-sized pieces vertically and show that 1/3 of the 1/4 is 1/12 of the whole." On the other hand, the remaining teachers (88.2%) appeared to struggle demonstrating flexibility with RU. In particular, 44.3% of the teachers demonstrated partial flexibility with RU by relying on the overlapping method. They did not specify different RU for 1/3 and 1/4, and their explanations implied that for both 1/3 and 1/4, they considered the entire rectangle as their referent unit. For instance, one teacher explained that "Divide the rectangle vertically into 3 equal-sized parts and shade in one part. The overlapping part between the horizontally shaded part and vertically shaded part (one square) is 1/12." Unlike the aforementioned two categories, 43.9% of the teachers did not demonstrate any flexibility with RU. Those teachers did not appear to consider any referent unit, and they did not provide explanations for each fraction.

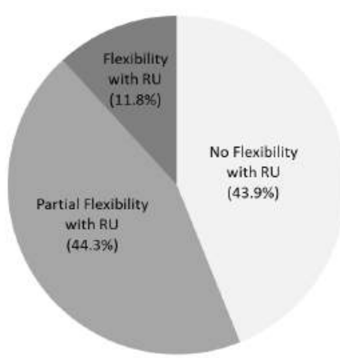


Figure 3: Teachers' performance on the item measuring their flexibility with RU

Relationship between Attention to and Flexibility with Referent Units

We found a significant, but moderate relationship between relationship between teachers' attention to and flexibility with RU ($\chi^2(4) = 13.3$, $p = .01$; $G = .35$). As shown in Figure 4, 60.3% of the teachers who did not pay attention to RU failed to demonstrate flexibility with RU, whereas 35.1% of the teachers who paid attention to RU failed to demonstrate flexibility with RU.

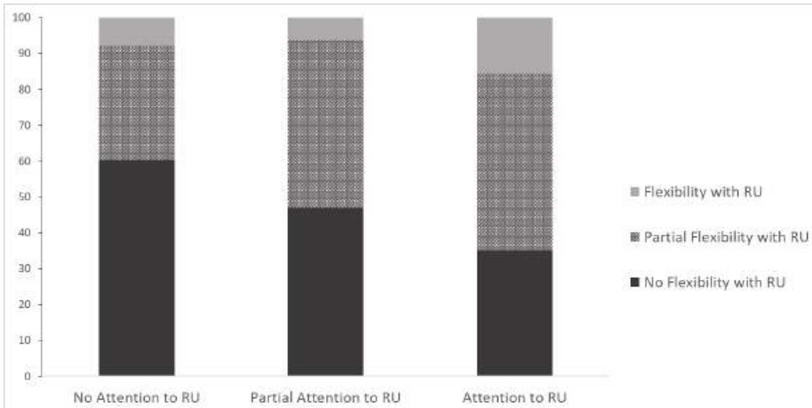


Figure 4: Teachers' performance on for different levels of attention to RU

Relationship Between Understanding of Referent Units and Professional Background

We also examined the relationships between teachers' understanding of RU and their various professional background variables. As shown in Table 2, none of the variables for teachers' background was associated with their attention to RU, whereas middle grades teachers and traditionally certified teachers showed more flexibility with RU compared with upper elementary and non-traditionally certified teachers. For example, the odds of middle grades teachers showing flexibility with RU was 2.67 times higher than that of elementary grades teachers ($p = .001$). This means that middle grades teachers were 2.67 times more likely to demonstrate flexibility with RU than elementary grades teachers.

Table 2: Logistic Regression of Probability of Attention to and Flexibility with RU

Teachers' professional background	Attention to RU	Flexibility with RU
Number of mathematics content courses (3 or more)	0.880 (.23)	0.667 (.18)
Number of mathematics methods courses (3 or more)	1.084 (0.32)	1.088 (0.32)
Fully certified teachers	0.875 (0.24)	0.923 (0.25)
Middle school mathematics teachers (Grades 6 & 7)	0.983 (.30)	2.666** (.81)
Traditionally certified teachers	1.265 (0.38)	2.098* (.64)

Note. Odds rations shown. Standard errors are in parentheses. *$p < 0.05$, **$p < 0.01$.

Relationship Between Knowledge of Referent Units and Fractions

As shown in Table 3, teachers' attention to and flexibility with RU significantly predicted their overall performance on the fractions measure. Specifically, when teachers' attention to and flexibility with RU were entered into the model separately, teachers who paid attention to RU significantly outperformed those who did not pay attention (effect sizes of .41 and .58 for the partial attention to and attention to referent unit categories, $p = .035$ and $p < .0001$). Similarly, those who demonstrated partial flexibility or flexibility with RU also performed significantly better on the fractions measure compared with those who did not show flexibility with RU (effect sizes of .52 and .60 for teachers who were in the groups showing partial flexibility and flexibility with RU, $p < .0001$ and $p = .007$).

Table 3: Teachers' performance on fractions predicted by their Attention to and Flexibility with Referent Units and Professional Background

	Attention to RU	Flexibility with RU	Attention to and flexibility with RU, and professional background
Attention to and flexibility with RU			
Partial attention to RU	0.140* (.067)		0.142* (.064)
Attention to RU	0.198*** (.053)		0.175*** (.052)
Partial flexibility with RU		0.171*** (.047)	0.105* (.048)
Flexibility with RU		0.198** (.073)	0.115 (.072)
Professional background			
Number of mathematics content courses			−0.042 (.046)
Number of mathematics methods courses			0.059 (.50)
Fully certified teachers			−0.024 (.046)
Middle school mathematics teachers			0.206*** (.053)
Traditionally certified teachers			0.039 (.052)

Note. $N = 238$ for all models. The numbers in parentheses are standards errors. $*p < 0.05$, $**p < 0.01$, $***p < 0.001$.

Finally, when teachers' attention to and flexibility with RU were included in the model along with their professional background variables, teachers who demonstrated partial attention to RU or those who demonstrated full attention to RU still performed better than those who did not demonstrate attention to RU (effect size of .43 and $p = .028$ for the partial attention to RU category; and effect size of .53 and $p = .001$ for the full attention to RU category). However, teachers' flexibility with RU did not seem to be significantly correlated with their overall performance on the fractions measure. This is possibly because of the correlation we reported earlier between teachers' professional background variables and their flexibility with RU. Of these variables, the only significant predictor of teachers' overall performance was being a middle grades teacher. Indeed, the difference between elementary and middle grades teachers' performance was an effect size of .63, $p < .001$. Other variables, such as the number of courses or being fully certified, did not link to their overall performance on fractions.

Discussion

In the present study, we examined U.S. in-service teachers' attention to and flexibility with RU and the relationship between these two characteristics, along with how teachers' understanding of RU was linked to their professional background and performance on the fraction items. We found that although about half of the teachers paid attention to RU, only 12% of the teachers showed flexibility with RU, which suggests that showing flexibility with RU is a more difficult concept to grasp. Our findings regarding teachers' flexibility with RU are similar to those from prior work (e.g., Lee et al., 2011; Webel et al., 2016). Furthermore, in alignment with past research (e.g., Izsák, 2008; Lee et al., 2011; Webel et al., 2016), teachers in our study commonly used the overlapping method to model fraction multiplication, indicating these teachers' difficulty with making sense of fraction multiplication.

Furthermore, our findings suggest a significant, but moderate relationship between teachers' attention to and flexibility with RU. These results may provide initial evidence that these items capture different characteristics of teachers' understanding of RU. It is interesting that teachers' performance on the item measuring flexibility with RU was associated with the teachers' preparation route, whereas the item measures attention to RU was not associated with any teacher background indicators. This may be because teacher education programs focus more on modeling fraction multiplication and division, given that many studies on future teachers have focused on fraction multiplication (e.g., Baek et al., 2017).

In a similar vein, it is important to point out that the number of mathematics content and methods courses was not associated with teachers' attention to and flexibility with RU. In an extensive review, Olanoff et al. (2014) reported an urgent need for research that finds ways to improve future teachers' understanding of fractions. The present study suggests that emphasizing attention to RU in teacher preparation programs, even when the referent unit stays the same, could help future teachers improve their understanding of fractions.

Our findings also underscore the importance of teachers' attention to and flexibility with RU in relation to their performance on other fraction concepts. In particular, teachers who paid attention to RU performed better than those who did not. Similarly, teachers who demonstrated flexibility with RU performed better on other fraction concepts than those who did not demonstrate such flexibility. Furthermore, when both attention to and flexibility with RU were included together, in addition to teachers' professional background variables, teachers who paid attention to RU or those who used the overlapping method for fraction multiplication performed better on the remaining items of the fractions measure than did those who did not pay attention to RU or those who showed no flexibility with RU. However, teachers who showed full flexibility with RU did not perform well compared with those who did not show any flexibility after adjusting for attention to RU. In sum, these findings also confirm the importance of teachers' understanding of RU in their mastery of other fraction concepts.

Acknowledgments

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TEACHERS' REVIEW OF TASKS AS A TOOL FOR EXAMINING SECONDARY TEACHERS' MATHEMATICAL KNOWLEDGE FOR TEACHING

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One important aspect of teaching is reviewing tasks in preparation for instruction. The goal of the multicase study of four secondary teachers was to examine the interplay between their mathematical knowledge for teaching [MKT] and what they attend to when reviewing a mathematics task. We engaged secondary mathematics teachers in a semi-structured, clinical interview focused on a non-routine mathematical task involving exponential growth. The results suggest experienced teachers may not explicitly attend to learning opportunities in their review of a task, and their own mathematical work contributes to their anticipation of student work and thinking. This work highlights how researchers focused on MKT can use clinical interviews as a tool for extracting and describing a teacher's MKT.

Keywords: Mathematical Knowledge for Teaching; Instructional Activities and Practices; Curriculum Enactment

Mathematical knowledge for teaching [MKT] is the "knowledge needed to carry out the work of teaching mathematics" (Ball, Thames, & Phelps, 2008, p. 395). Ball et al. (2008) clarify that *teaching* mathematics includes "everything that teachers must do to support the learning of their students" (p. 395). As such, this work includes planning, instruction, and assessment. Others in the literature suggest, in agreement, that pedagogy, the curriculum, and teachers' mathematical understanding are interconnected (e.g., Davis & Simmt, 2006; Sullivan, Knott, & Yang, 2015). For example, Sullivan et al. (2015) argue that "tasks do not exist separately from the pedagogies associated with their use nor are the pedagogies independent of the task" (p. 84).

One aspect of the mathematical work of teaching is reviewing mathematical tasks for potential use during instruction (Ball, 2017). From selection to implementation, teachers' use of mathematical tasks impacts the types of learning opportunities their students experience (e.g., Stein, Grover, & Henningsen, 1996). Sullivan et al. (2015) note that tasks allow students the opportunity to experience mathematical concepts and ideas. They claim that "the role of the teacher is to select, modify, design, redesign, sequence, implement, and evaluate tasks" (p. 83). Furthermore, "in planning and teaching, the role of the teacher is to identify potential and perceived blockages, prompts, supports, challenges, and pathways" (Sullivan et al., 2015, p. 86).

The purpose of this research study was to explore the aspects of a teacher's MKT elicited when reviewing a mathematics task. Specifically, we claim that aspects of a teacher's MKT including specialized content knowledge [SCK], knowledge of content and teaching [KCT], and knowledge of content and students [KCS] (Ball et al., 2008) become evident during task analysis. To investigate our claim, we pursued the following research question: What aspects of a secondary mathematics teachers' MKT can we describe from their review of a nonstandard exponential functions task? Our focus on exponential functions answers a call for increased research considering topics at the secondary level (Speer, King, and Howell, 2015).

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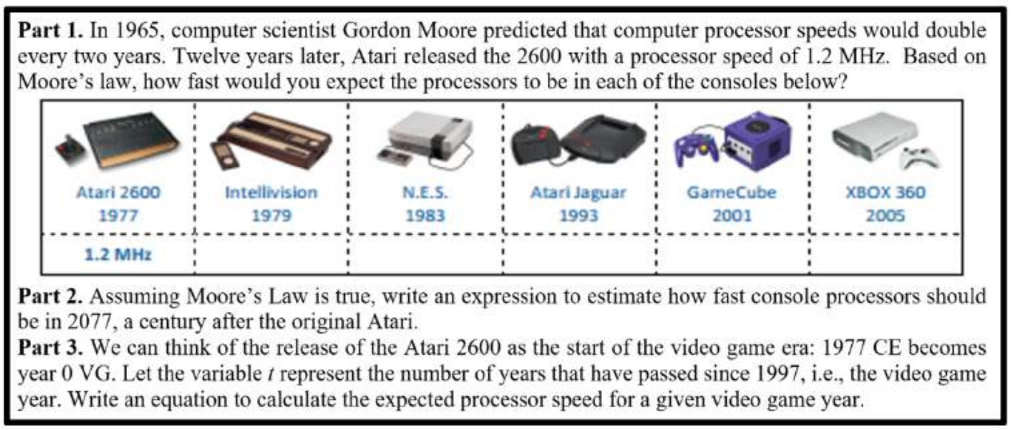
Methods

The study was part of a larger research study focused on the work of 16 high school teachers from the Great Plains region of the United States while they were teaching exponential functions in courses ranging from Algebra I to Precalculus. The research team identified the teachers as "highly effective" based on recommendations from the teachers' administrators or peers. The teachers had 5 to 25 years of teaching experience.

Data collection for the larger study consisted of multiple stages. First, a member of the research team conducted classroom observations of five lessons pertaining to exponential functions. In addition to observations, the teachers engaged in pre- and post-lesson interviews focused on the entire set of lessons as well as pre- and post-lesson interviews for each lesson. Second, a member of the research team administered a pre- and post-lesson assessment measuring the teachers' students' understanding on exponential functions topics. Finally, each teacher engaged in a semi-structured, clinical interview, called the *MKT Interview*, which focused on the teachers reviewing two non-standard exponential functions tasks. In this paper we focus on the first task of the MKT Interview, the *Xbox Xponential* task (see Figure 1).



Part 1. In 1965, computer scientist Gordon Moore predicted that computer processor speeds would double every two years. Twelve years later, Atari released the 2600 with a processor speed of 1.2 MHz. Based on Moore's law, how fast would you expect the processors to be in each of the consoles below?

Atari 2600 1977	Intellivision 1979	N.E.S. 1983	Atari Jaguar 1993	GameCube 2001	XBOX 360 2005
1.2 MHz					

Part 2. Assuming Moore's Law is true, write an expression to estimate how fast console processors should be in 2077, a century after the original Atari.

Part 3. We can think of the release of the Atari 2600 as the start of the video game era: 1977 CE becomes year 0 VG. Let the variable *t* represent the number of years that have passed since 1997, i.e., the video game year. Write an equation to calculate the expected processor speed for a given video game year.

Figure 1: The Xbox Xponential task (modified from Mathelicious, 2015).

During a semi-structured, clinical interview, a member of the research team asked the teacher to review the *Xbox Xponential* task. The teachers were prompted to articulate the mathematical opportunities to learn the task had the potential to support if used during classroom instruction. During the interview, the researchers collected video data of the teacher engaging with the task. Data analysis focused on how teachers approached the task mathematically and the descriptions they used to express the ways they thought students would interact with the task. We chose the Xbox task because it supported students to think about key mathematical ideas related to exponential functions and contained nonstandard elements. Specifically, the task provided opportunities for students to think about how a change in the independent variable other than one unit impacts the change in the dependent variable and how to capture that change in a table, an expression, and an equation. We anticipated that learners would work through calculating specific values using a multiplicative relationship between dependent and independent variables using recursive reasoning (part one), generalizing their work into an expression for a specific year (part two), and creating a general equation for the relationship (part three). This work provided the opportunity for students to learn about: the connections between multiplicative growth and exponential functions, the connections between repeated multiplication and exponents, and the importance of defining independent and dependent variables.

The Cases

Guided by an interpretive theoretical perspective (Creswell, 2013), we selected four teachers who engaged with the Xbox Xponential task during the clinical MKT interview in this multicase study (Stake, 2006). As part of our theoretical perspective, our participants developed "subjective meanings of their experiences," and our role as researchers was to interpret our participants' responses by relying "as much as possible on the participants' views" (Creswell, 2013, pp. 24-25). However, our analysis included interpretations and observations to investigate the teachers' MKT. The individual cases of Helen, Abby, Frankie, and Molly provided contrasting views of the Xbox Task and their cross-case analysis provided evidence of their collective MKT.

Helen

For part one of the task, Helen immediately identified the gaps in time and indicated them with arrows and increments from cell to cell in the table. For example, from 1983 to 1993, she drew an arrow and wrote "+10" to indicate the 10-year gap in time. She noted that all of the gaps in time are even which made "them all nice and easy" because "you [do] not have to deal with what happens if it is not an even number." After determining the gaps in time, Helen used a recursive strategy to find the values for the processing speeds by multiplying the previous value by a power of two based on the number of two-year intervals. For part two of the task, Helen initially wrote $y = 1.2 \cdot 2^{100}$. After moving on to part three of the task, she returned to part two to modify her answer to $f(100) = 1.2 \cdot 2^{100/2}$ which she simplified to $f(100) = 1.2 \cdot 2^{50}$. Finally, for part three of the task, she wrote her answer as $f(t) = 1.2 \cdot 2^{t/2}$. Helen completed all parts of the task as designed before addressing what learning opportunities were possible. With the exception of Helen's comment about the gaps in time in the table for part one being "nice and easy," Helen did not comment on her thinking while completing the task. It is interesting to note that Helen was able to attend to the two-year gaps and intervals in part one of the task, but she did not initially attend to them in part two of the task. It was only after moving on to part three that she returned to correct her answer.

With respect to the sequence of tasks, Helen claimed that "it's always easier for students to handle the numeric at first, especially if it's their first introduction to exponential functions." Starting with the table supports this view. She believed that her students would be able to "reason their way through" the table by attending to the two-year intervals and gaps in time. While some students might use a "brute force" strategy of repeatedly multiplying by two, Helen hoped that the table would motivate her students "to start thinking of another way." By doing that, she claimed that this "would allow them to make that jump into actually formalizing [the context] and writing it algebraically in general." Helen sees exponents as a tool students could use to complete the task. Helen noted that the sequence of tasks supports students at all levels to be able to complete the task since "not all students would be able to start [with part three]." The sequence gives all students "a path towards getting the ultimate goal of the [task] which would be coming up with [the function in part three]." Throughout these comments Helen is primarily focusing on students' completion of the task and the prior knowledge they need for successful completion of the task.

Helen highlighted that repeated multiplication is "a huge aspect of why we use exponential functions." She viewed the table as potentially supporting her in teaching students "about the behavior of exponential functions." Specifically, "the idea that you are doubling every certain number of years." In contrast to her previous comments, Helen, is now focused on student thinking about a key mathematical idea. Her comments stemmed from her view that the doubling is occurring every two years is "an interesting twist from the [tasks] that [her students] may be used to seeing." She hypothesized that her students would not recognize that they would need to divide the value in the exponent by two which is the same mistake she initially made when completing the task herself.

Abby

For part one of the task, Abby filled in the table by repeatedly multiplying the value in the previous table cell by two, ignoring the gaps in time. After completing the table in this way, she moved on to part two. She first wrote the equation, $f(x) = 1.2(2)^{x-1}$. Then, when she attempted to define her variable, x, she realized her error in part one. She then returned to part one of the task by identifying the size of the gaps in the years: +2, +4, +10, +8, and +4 respectively. Then, for each of the table cells, she identified the number of two-year intervals and multiplied the previous values by two raised to the exponent related to that number of intervals. For example, from 1983 to 1993 she multiplied 4.8 by 2^5 in order to calculate the value associated with 1993, or ten years after 1983. Later in the interview, Abby admitted that she did not attend to the years and assumed that they were "nice equal" intervals.

After completing the table for the second time, Abby returned to her earlier work where $f(x) = 1.2(2)^{x-1}$ was already written on the paper. Abby then defined x as "every two years." After asking herself, "does that work?" and checking values, Abby erased the original exponent $x - 1$ as well as her definition for the variable x. Looking back at the table, Abby changed the definition of the variable x to "# of yrs since 1997" and changed the exponent to $x/2$. Again, Abby checked her work by evaluating using her equation and comparing the values with the table in part one of the task. After reading the prompt for part three of the task, Abby noted that she "should read the question" for part two of the task. For her answer, Abby wrote and calculated $f(100) = (1.2)(2^{50}) = 1.251\times 10^{15}$ for part two of the task. She then moved on to part three of the task where she wrote $f(t) = 1.2(2)^{t/2}$ and defined the variable t as "# of yrs since 1997." Abby appeared to skim the task initially and completed what she assumed the parts of the task were asking. It was only after moving on to subsequent parts did Abby identify that she may have made an error. This suggests that Abby assumed that the task followed the format of (1) fill in the table for consecutive values, then (2) write an equation to model the context.

Abby noted that the overall sequence of tasks allowed students to access the mathematics at different levels. First, she liked "that we can model it numerically with the table." This is something that the students "could get just with a little calculator work and it is something that [the students] could actually conceptualize." Second, she noted that the task is an easy context to understand which would motivate students to complete it. Finally, the sequence of tasks "stair steps" students through the parts towards an answer and "gives them a method to check their work as they go." Specifically, the table allows students to see that they should divide by two in part two of the task as well as provides them with a way to check their function in part three of the task. Overall, Abby viewed the goal of the task as formulating a model which she claimed is done in part three of the task. She did not articulate any mathematical ideas that students would have the opportunity to think about through engagement with the task.

Abby noted that she could use the task to highlight the connections between the context and the actual real-world data. She noted that the task "would definitely teach [the students] to pay attention to their data." This mirrors the mistake Abby made while completing the task. Using this task, she might ask students to determine whether Moore's Law is true. Abby emphasized the use of graphing and using the table to check answers as strategies she would use to teach the task as well as a strategy for students to complete the task; again, highlighting Abby's focus on what students will do.

Frankie

Frankie only completed portions of the task for herself, and only after being prompted to do so by the interviewer. Frankie never filled in values for the table in part one of the task. However, she noted that, in order to complete the table, a rule like 1.2×2^x would be helpful and "is almost demanded" by the task design. In part two of the task, Frankie used her previously identified rule as a guide to find the answer $S_{2077} = 1.2 \times 2^{100/2}$. Frankie viewed part three of the task as a more formal version of the rule she constructed earlier. Specifically, the answer to part three of the task was $S(t) = 1.2 \times 2^{t}$ with the caveat that the *t* needed to be "adjusted." That is, "if *t* is the number of years, then you also have to have the idea that it's going to be divided by two." As a result, she rewrote her answer as $S(t) = 1.2 \times 2^{t/2}$.

Frankie highlighted the gaps in time for the various gaming systems. She noted that her students would likely need experience with problems that contain varying gaps in time to be successful in completing the task. In particular, she believed that her students' "tendency would be to just double and not pay attention to the years and how far apart they [are]." For those students who did attend to the gaps in years, Frankie expressed concern that they would get "hung up" on the 10-year jump. She believed that the table "is not necessarily going to get [the students] to an expression." To complete the table, the students might need "some intermediate values" in the table in addition to "really think[ing] about how the doubling is happening and how many doublings would take place." Frankie highlighted that students do not encounter tasks with interval gaps in other areas of the curriculum with the exception of linear functions. As a result, it would not be something that the students would immediately notice.

Frankie did not see a connection between part one of the task and part two of the task which might stem from the fact that she never completed the task as designed. As noted before, Frankie used a rule to support writing the answer to part two of the task. She claimed that her students would need to have "recogniz[ed] that doubling piece" in order to find the answer to part two of the task. While Frankie acknowledged that her students would need to have identified that the doubling was occurring over a two-year span, she did not connect part one of the task as supporting this realization in her students. If students successfully identified the doubling was occurring every two years, she believed some of her students would simplify their answers to part two of the task to "be two to the 50th power." Other students, she believed would simply raise to the 100th power without dividing by two. In her experience, her students, "when given something like 1.2×2^{x}, [the students] know to put a number in there and to get something." She noted that her students do not consider how "the power is changing." Throughout her comments Frankie is focused on the aspects of the task students are familiar or unfamiliar with and how they will respond.

Frankie viewed the purpose of the task as writing an example of a known formula type which matches the given data. She argued that "adjust[ing]" the power in part three of the task was the trickiest part of the overall task. She believed that her students would be able to work through parts one and two of the task "pretty well" and would not necessarily need to recognize "that any number of years is going to have to have [an] adjustment." While Frankie thought that students at "several levels" could be successful, she was concerned about students getting "caught up in the information" which would "keep them from moving forward."

Molly

Prior to starting the task Molly said that she had seen the task before when looking for material. When Molly began working on part one, she noted the varying gaps in years given in the table and said "this would give students an opportunity to start thinking about doubling periods." Molly suggested that she might have structured the table to start at year 0 and use the number of years since 1977 instead of using the actual years. As she continued to discuss student thinking around the table Molly created a table with "# of doubling periods" as the independent variable which she completed for three doubling periods. She transferred her work from the created table to the table provided in the task. Even though Molly said that she would present the information differently, she liked the way the task designers chose to present the information. Molly thought that the structure of the table would cause students to think about doubling periods as opposed to years. She said, "I think they're just thinking about doubling periods and they're working with exponents without realizing that's what they're working with." Molly did not complete the remainder of part one and it did not appear that Molly completed any of part the task for herself; rather, she the work she did was in service of illustrating the thinking she expected students to engage in.

When she began part two, Molly underlined "write an expression" and "a century." While discussing why the task designers included "a century," Molly initially seemed to misinterpret the task, she said "a century, ok and then they've given them another little hint here, so they're telling them that even though 2077 isn't, oh no, I'm sorry I was going back to 1965, so maybe that's what a student would do too." It seemed as if Molly had not thoroughly read the task and assumed that the initial processor speed was given for the year 1965. After clarifying the instructions Molly wrote the expressions $1.2(2)^{100/2}$ and $1.2(2)^{50}$. As with part one, Molly's work appeared to be in service of illustrating potential student thinking. She said the thinking in part two was similar to the thinking required for part one because students needed to focus on the number of doubling periods. She expected students to write $1.2(2)^{50}$ instead of $1.2(2)^{100/2}$ because "they're not going to want to see a fraction there so they would think about, it's doubled, I had to multiply it by two, fifty times." Molly said students might struggle with seeing how to come up with a pattern based on the starting point because they have to move from doing the problem recursively in part one to needing to base their expression off of a starting point in part two.

For part three, Molly underlined "write an equation for the expected processor speed for a given year" and then wrote $y = 1.2(2)^{(t-1977)/2}$ while explaining that part two can be generalized by starting with an original value and "multiplying by two a whole bunch of times." Molly explained the exponents as subtracting 1977 from the year you are looking for and then dividing by two. The equation mimics Molly's process in calculating the processor speeds. Only after the interviewer asked Molly a question which prompted her to reread the task did she write $y = 1.2(2)^{t/2}$ as her final answer for part three. Again, indicating that Molly did not thoroughly read the task but worked off of assumptions about the nature of the task. Molly summarized the overall structure of the task as, "from scaffolding from something that's very simple, just coming up with number answers … I need fifty years in the future, so I'm not going to be able to just keep extending my chart to get that … and so now I needed to take this and extend it to something where I can generalize." Molly said that this sort of task structure was common.

Cross Cases Analysis and Discussion

With this research study, we sought to describe aspects of a teacher's MKT based on what they attend to when reviewing a mathematics task for potential use during instruction. We claim that the previous case summaries provide rich opportunities to explore and describe aspects of the teachers' MKT including specialized content knowledge [SCK], knowledge of content and teaching [KCT], and knowledge of content and students [KCS] (Ball et al., 2008).

Ball et al. (2008) identified SCK as the "mathematical knowledge not typically needed for purposes other than teaching" (p. 400). This knowledge includes the ability to identify and interpret student mathematical work as well as the ability to make "features of particular content visible to and learnable by students" (Ball et al., 2008, p. 400). We chose this task partially because the relationship "doubling every two years" has the potential to make important mathematics visible to students. Specifically, doubling indicates that the relationship is exponential while the "every two-year period" adds a unique difficulty when compared to tasks the teachers would typically use. In the previous summaries, we saw that all of the teachers noted that "doubling every two years" was a significant part of the task. The ways in which the teachers spoke about the significance of "doubling every two years" revealed interesting aspects of their SCK

Helen saw the gaps in the years as something students needed to attend to in order to complete the table. She anticipated that students may not realize the number of years passed needed to be divided by two. She hoped the gaps would encourage students to use exponents as a push towards writing a function. Here she seemed to be implying something about student thinking but this was not explicit. When Helen talked about doubling, she explicitly discussed student thinking saying that students will understand something about the nature of exponential functions. Here we see that Helen's SCK allowed her to see the usefulness of the two-year gaps as supporting her students towards using exponents and writing an equation.

In contrast, Molly discussed the two-year gaps as doubling periods. Unlike the other teachers Molly did not anticipate the two-year gaps to be an issue for students. Rather she described students as finding the number of doubling periods for the two- and four-year gaps without necessarily realizing that they are dividing the number of years by two. Molly believed that students would realize that two years is one doubling period and that four years is two doubling periods. Molly thought that students might have more difficulty in thinking about the number of doubling periods for the ten-year gap. From Molly's interview, it is clear she saw the structure of the table as a key aspect of the task because it supports students in thinking about doubling periods. Molly's identification of aspects of the task which makes visible the key mathematical idea of doubling periods of exponential functions is an important component of SCK.

KCT "combines knowing about teaching and knowing about mathematics" (Ball et al., 2008, p. 401). For example, Ball et al. (2008) noted that teachers must "choose which examples to start with and which examples to use to take students deeper into the content" (Ball et al., 2008, p. 401). With respect to the task design, Abby, Molly, and Helen commented that they liked how the sequencing of the task provided scaffolding towards the equation and that starting with numerical calculations was easier for students. They all saw the goal of the task as doing something (writing a function) rather than thinking about some key mathematical ideas. This suggests that the three teachers KCT includes the idea that students can develop models of exponential growth from exploring values in a table, then calculating a larger value, and finally writing a formalized equation.

In contrast to Abby, Helen, and Molly, Frankie did not see the first part of the task as supporting students in completing the later parts of the task. For example, she identified the ten-year gap as especially significant as an obstacle that could prevent successful completion of the task. Frankie claimed that having a rule is almost required to complete the table and frequently discussed prior experience students would need to be successful in completing the task. Frankie's view of the purpose of the task was slightly different from Abby, Helen, and Molly. Where Abby, Helen and Molly saw the purpose as writing an equation based on the work done to create the table, Frankie saw the purpose as fitting a known equation type to the situation. This suggests that Frankie's KCT, as elicited by the task and the interview, does not include the same construct that the other teachers' have. Frankie's view of the purpose may be a consequence of her choice to write an equation to solve the first part of the task instead of solving the task as written. Her reliance on her equation to complete the table may have prevented her from seeing the ways that the table supported students to write an expression in part 2 and an equation in part 3.

Ball et al. (2008) defined KCS as the "knowledge that combines knowing about students and knowing about mathematics" (p. 401). This knowledge includes an understanding of the content so that the teacher can identify what has the potential to be confusing, challenging, easy, motivating, and interesting for students (Ball et al., 2018). In the preceding cases we saw that Helen, Abby, and Molly all wrote answers at some point during their work on the task that did not address the prompt as written, but rather what they assumed the prompt was. . For example, some errors stemmed from not realizing that the years in the table do not all have a gap of two or writing an expression with 100 as an exponent. All three teachers corrected their errors as they progressed through the task. Then, the teachers anticipated students encountering difficulty with the aspects of the tasks in which the teachers themselves made mistakes. This suggests that an aspect of the teachers' KCS is based on and includes knowledge of their own errors with respect to the task, indicating they possess the mathematical understanding to complete the task as written.

One of the most striking differences among how teachers reviewed the Xbox task surfaced when considering the coherence between the teachers' doing of the task and the ways students would think about and engage in the task. Molly focused on the way students would need to think about the mathematical concepts inherent in the task and only completed aspects of the tasks as a way to articulate student thinking. Unlike Molly, Abby and Helen completed the tasks for themselves before talking about students. When Abby and Helen discussed students, they tended to focus on describing what students may do and struggle with, but they did not articulate *why* or what thinking would create the struggle. Frankie, in contrast, did not complete the task as designed and discussed things not related to student mathematical thinking. Of the four teachers, Molly showed the greatest integration between the mathematics of the task and the students' thinking related to the task. That is, this task review assessment elicited evidence of Molly's SCK/KCS for exponential functions that we were not able to elicit from Helen or Abby.

Conclusion

The MKT interview was useful for gathering insight into teachers' MKT related to KCS, KCT, and KCC. The insight comes from doing an interview that allowed for teachers to articulate their own thinking in regards to a non-traditional task rather than selecting predetermined answers. We hypothesize that the interactions within an MKT interview provide a more genuine representation of MKT and call for further investigation.

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INDUCTIVE REASONING IN MATHEMATICS TEACHERS WHEN RESOLVING GENERALIZATION TASKS

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This study reports an analysis of inductive reasoning of Mexican middle school mathematics teachers, when solving tasks of generalization of a quadratic sequence in the context of figural patterns. Data was collected from individual interviews and written answers to generalization tasks. Based on Cañadas and Castro's inductive reasoning model, we found that most of the teachers followed four stages to obtain a general rule: observation of particular cases, search of patterns, conjecture formulation and generalization.

Keywords: Reasoning and Proof, Algebra and Algebraic Thinking, Inductive Reasoning, Figural Pattern, Generalization, Teacher.

Introduction

Inductive reasoning is a thought process that leads to the discovery of general rules by the observation and combination of specific instances (Polya, 1994). It is considered to be an important route to develop critical thinking and student´s ability to solve problem situations, to generalize different mathematical patterns and for mathematics learning (Sosa Moguel, Aparicio Landa, & Cabañas-Sánchez, 2019; Castro, Cañadas, & Molina, 2010; Papageorgiou, 2009; Haverty, Koedinger, Klahr, & Alibali, 2000). In addition, it also contributes on the route to make mathematical proof. However, students face difficulties with the formal validation processes. Some of these difficulties are linked to their reasoning skills and their capabilities to make and understand proofs immediately. For this, an adaptation process is required, as well as a logical progression in the development of their reasoning, from closer everyday reasoning to the concrete, to more abstract mathematical reasoning (Castro, Cañadas & Molina, 2010). Thus, it is recognized that induction fosters the development of these types of abilities, from the formulation of conjectures and their formal proof to guarantees of the veracity of the conjecture (Cañadas & Castro, 2007). In this regard, the National Council of Teachers of Mathematics [NCTM] (2000) stresses the need to develop middle school students' proficiency in using inductive (and deductive) reasoning to examine patterns and structures in order to identify regularities and make and evaluate conjectures about possible generalizations, in linear or quadratic patterns. Based on these demands on students, mathematics teachers are implicitly linked. To do so, they need the ability to help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning techniques to confirm or refute those conjectures (NCTM, 2000). According to Brodie (2010), "teachers can, through questions and prompts, try to provoke learners into thinking in particular ways and support them to compare, verify, explain, and justify their conjectures" (p. 45). In addition, Ball and Bass (2003) say that, teachers need abilities in providing resources to the students to allow them to develop these skills and to use environments that make this possible.

On the other hand, most of the inductive reasoning research that studies it as a thought process and as a generator of knowledge in generalization tasks context, mainly refer to linear relationships. Few have been focused on quadratic ones (Kirwan, 2017), both in training and in-service teachers. This article examines the inductive reasoning of Mexican middle school mathematics teachers, when solving tasks of generalization of a quadratic sequence.

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Theoretical Framework

Inductive reasoning is the human thought action that produces statements and reaches conclusions, starting with the observations of particular cases until arriving at a generality (Cañadas, 2002). It is a cognitive process that contributes to the advancement of knowledge, where more information is obtained than is provided by the initial data with which that process begins. Inductive reasoning in this study is analyzed from the inductive reasoning model proposed in Cañadas and Castro (2007). This model is based on Polya´s (1966) steps, Cañadas´s empirical work (2002) and Reid´s (2002) stages.

Inductive Reasoning Model

The inductive reasoning model is made up of seven stages. They are presented in an ideal order. They start with the observation of particular cases and end with the generalization. Not all these stages necessarily occur. In the following we describe these stages:

Observation of particular cases. The starting point of inductive reasoning is the experience with particular cases.

Organization of particular cases. The use of different strategies to systematize and facilitate work in particular cases.

Search of patterns. Some regularity or behavior is detected. Patterns are considered as something that is repeated regularly (Stacey, 1989), their recognition allows the development of the ability to generalize. There are different types of patterns: numerical, pictorial, figural, computational procedures or repetitive patterns (Amit & Neria, 2008).

Conjecture formulation. It is a statement about all possible cases, based in particular ones but with an element of doubt. This statement seems reasonable, but the validity needs to be validated. It has not been convincingly validated and it is not yet known that there are many examples that contradict it, nor is it known that it has any false consequence (Mason, Burton, & Stacey, 1988).

Conjecture validation. At this stage, there is an attempt to validate the conjecture for new specific cases, but not in general.

Generalization. Mathematics patterns are related to a general rule, not only to some cases. Based on a conjecture which is true for some particular cases, and having validated such conjecture for new cases (conjecture validation), students might hypothesize that the conjecture is true in general. Generalization is the deliberate extension of reasoning or communication beyond considered cases, recognizing and explaining their similarity (Kaput, 2008).

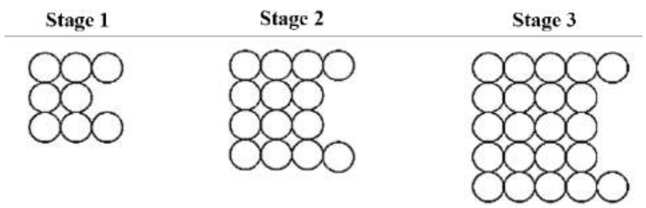
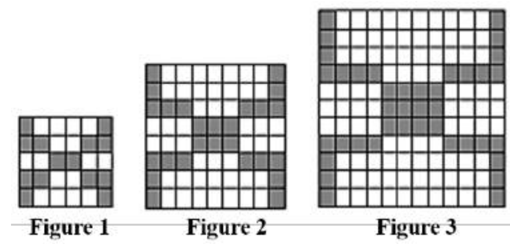
General conjectures justification. At this point, a formal proof can provide the final justification that guarantees the truth of the conjecture.

Method

This research is a qualitative and interpretative. It was carried out with sixteen middle school mathematics teachers (nine women and seven men) with between 5 and 14 years of teaching experience in public schools in Mexico. They were voluntarily involved in this study through inductive reasoning workshops in the context of mathematics education congresses. The participating teachers had professional qualifications as middle school teachers (Nine a mathematics bachelor's degree, two in Mathematics Education and five in Telesecundaria), so they all studied mathematical concepts such as sequences and linear an quadratic sequences. The selection criterion of the participants was to have experienced as a third grade teacher the teaching quadratic sequences using numerical and figural patterns. Taking into account the above, two task were designed on the generalization of quadratic sequences (see figure 1). The patio tile task adapted from Kirwan's study (2017) and the frog quadratic pattern task from Rivera's study (2013).

These tasks consist of increasing patterns with arithmetic progression of order 2 in the natural numbers. Teachers worked individually for 20 to 30 minutes on the tasks. After the analysis of their answers, six of them were interviewed, to understand in greater depth their inductive reasoning process and also because they had shown different ways of reasoning.

Table 1: Quadratic sequence tasks used in the research.

Task 1: The patio tile task. In the building of a patio, circular stones of equal size are placed. To observe the progress of the construction, you take a photo of the patio by stage.  a. How many circular stones have been placed for the sixth stage if the construction of the patio is carried out in the same way? Justify your answer. b. How many stones are there in stage 50? Justify your answer. c. How can you find the number of circular stones for any number of stages? Explain your answer	Task 2: The frog quadratic pattern task. Observe at the sequence of the following figures. Extensively justify the solution process in each of the questions.  a. How many gray squares are there in figure 5? b. How many gray squares are there in figure 7? c. How many gray squares are there in figure n?

Results

In this study the inductive reasoning stages most of the teachers followed were: Organization of particular cases, search of patterns, conjecture formulation and generalizing. Less frequently, the organization of particular cases and the conjecture validation. None of the teachers showed the involvement of the conjecture justification step.(see Table 2).

Table 2. Stages of inductive reasoning followed by mathematics teachers

Task	Stages						
	Observation of particular cases	Organization of particular cases	Search of patterns	Conjecture formulation	Conjecture validation	Generalization	General conjectures justification.
T1	11	5	12	7	2	11	0
T2	9	4	11	8	6	8	0

Observation and organization of particular cases

In task 1, eleven teachers observed particular cases and in task 2, nine. Most identified the number of objects at each stage of the sequence. In some cases, their work consisted of strategic counts based on the decomposition of the figures. The visualization of the figural pattern was fundamental to help teachers identify some configurations at the given stages. Other teachers relied on counting to establish the k-th terms and identify the type of sequence. Although it is recognized that teachers worked with particular cases, at least 3 of them faced difficulties in advancing to the following stages.

The teachers who used the organizing the particular cases stage (see Table 2) used double-entry tables, in which they established a correspondence between two variables, the number of objects in the figural pattern, with the number of the stage or figure, in the context of the demands of the tasks. The tables were represented in two ways, vertical and horizontal. This last form was used by the teachers who used the differences method in their inductive process to identify the type of sequences.

Search for Patterns

The teachers who identified the pattern resorted to two ways of proceeding (see Table 3):

A) Figure decomposition: The objects of the figural pattern were perceived as basic configurations: squares and / or rectangles. From this, a useful mathematical structure was associated to explain and justify the behavior of these objects.

B) Differences method: They identified the recurrence pattern between the k-th terms of the sequence associated with the figural pattern. Subsequently, they worked with the first difference and recognized that it is not constant, then the second differences were determined and observed that it is constant. Thus, they derived that the sequence is quadratic. Finally, they find the coefficients of the sequence of the form $ax^2 + bx + c$, to establish the general rule of quadratic sequence.

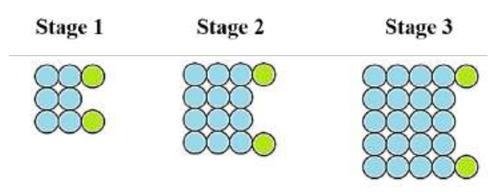
Formulation and justification of the conjecture

The teachers analyzed the behavior of the objects in the figures (figural pattern). At first, they made their conjectures through additive and multiplicative structures, then they transformed them into an algebraic structure, in terms of n (demand for the task) or with another variable. Few justified their conjectures. Of those who did, it was of the algebraic type and they validated it with particular cases.

Generalization

The results show that eleven of the sixteen teachers managed to construct the general rule that explains the behavior of the figural pattern in task 1 and eight teachers in task 2. Those who generalized without making conjectures used the difference method to recognize the type of sequence (quadratic) and based on that, they determined the general rule associated with the figural pattern. The generalization constructed by those who formulated a conjecture, verified its veracity. In some teachers, this process was carried out with the particular cases proposed and in others, through new cases, such as near and far terms. This process consisted of evaluating the conjecture, in correspondence between the quantity of the objects and the number of the stage or figure. Two ways of expressing the general rule were recognized, one algebraically and the other verbally.

Table 3: Ways of proceeding for the teacher when solving generalization tasks

Ways of proceeding	Task	Examples
Figure decomposition	T1	 Based on this way of perceiving the objects of the figural pattern in T1, they constructed a general rule associated with a multiplicative and additive structure: $(n+1)(n+2)+2$

<table>
<tr>
<td></td>
<td>T2</td>
<td>
Figure 1 Figure 2
Based on this way of perceiving the objects of the figural pattern in T1, they constructed a general rule associated with a multiplicative and additive structure: $4(1+2n)+n(n+1)$</td>
</tr>
<tr>
<td rowspan="2">Differences method</td>
<td>T1</td>
<td>k-th terms 8 14 22 32
First difference 6 8 10
Second difference 2 2
Based on this way of perceiving the objects of the figural pattern in T1, they constructed a general rule associated with an algebraic expression: x^2+3x+4
From the algorithm of:
$a \ +b+c=8;\ 3a+b=6;\ 2a=2,$ they found the coefficients of the sequence.</td>
</tr>
<tr>
<td>T2</td>
<td>k-th terms 14 26 56 74
First difference 12 14 16
Second difference 2 2
Based on this way of perceiving the objects of the figural pattern in T1, they constructed a general rule associated with an algebraic expression: x^2+9x+4
From the algorithm of:
$a \ +b+c=14;\ 3a+b=12;\ 2a=2,$ they found the coefficients of the sequence.</td>
</tr>
</table>

Discussion and conclusion

This article examined the inductive reasoning of Mexican middle school mathematics teachers, when solving quadratic sequences generalization tasks. From a methodological point of view the tasks encouraged teacher´s work in near and far stages in order to build and validate a conjecture which explains the behavior of the involved figural pattern and to establish it as the general rule. Although as was mentioned above, not all of them showed the validation of their conjectures. On the other hand, it was recognized that not all of these stages necessarily occurred in the inductive reasoning processes carried out by teachers.

With regard to solving tasks in figural patterns generalization, teachers showed the inductive process as a strategy. The study has reported that working with well-defined figurative patterns favors that they be interpreted as configurations in a certain way (Rivera, 2010). In this sense, the teacher's task in this study consisted of involving a significant generalization of patterns. This study also recognizes that the objects of some figural patterns are complex to interpret, even when their construction is well defined. As an example, task 2 where the figural pattern involved more than one object, gray and white squares, in relation to the figure number. The task required the teacher to represent algebraically the behavior of the variable, that is, the gray squares which make up a rectangular figure, also made up of white squares. The distribution of gray squares was not related to a specific geometric figure. These variables influenced teachers' difficulties to recognize the behavior of gray squares.

In line with Rivera (2010), in this study the context of generalization of figural patterns engaged teachers in the coordination of perceptual and symbolic inferential abilities, more specifically, the figural pattern that is high in Gestalt goodness, because it tends to have a well-defined structure that has easily discernible parts and a balanced, and harmonious form of the pattern, which allowed most of the teachers to specify an algebraically useful formula. Naturally, the teachers proceeded in a different way when working with the objects of the figural pattern of each task.

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RAZONAMIENTO INDUCTIVO EN MAESTROS DE MATEMÁTICAS AL RESOLVER TAREAS DE GENERALIZACIÓN

INDUCTIVE REASONING IN MATHEMATICS TEACHERS WHEN RESOLVING GENERALIZATIONS TASKS

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Este estudio reporta un análisis del razonamiento inductivo de maestros mexicanos de matemáticas de secundaria, al resolver tareas de generalización de una sucesión cuadrática en un contexto de patrones figurales. Los datos fueron recolectados de entrevistas individuales y respuestas escritas de tareas sobre generalización. Con base en el modelo de razonamiento inductivo de Cañadas y Castro, encontramos que la mayoría de los maestros siguieron cuatro etapas para obtener una regla general: observación de casos particulares, búsqueda de patrones, formulación de conjeturas y generalización.

Palabras clave: Razonamiento Inductivo, Patrón figural, Generalización, Maestro.

Introducción

El razonamiento inductivo es un proceso del pensamiento que conduce al descubrimiento de reglas generales mediante la observación y la combinación de instancias específicas (Polya, 1994). Se considera una ruta importante para desarrollar el pensamiento crítico y las habilidades de los estudiantes para resolver situaciones problemáticas, para generalizar diferentes patrones matemáticos y para el aprendizaje de las matemáticas (Sosa Moguel, Aparicio Landa y Cabañas-Sánchez, 2019; Castro, Cañadas, & Molina, 2010; Papageorgiou, 2009; Haverty, Koedinger, Klahr y Alibali, 2000). Además, también contribuye en la ruta para hacer pruebas matemáticas. Sin embargo, los estudiantes enfrentan dificultades con los procesos formales de validación. Algunas de estas dificultades están relacionadas con sus habilidades de razonamiento y sus capacidades para hacer y comprender pruebas de inmediato. Para ello, se requiere un proceso de adaptación y seguir una progresión lógica en el desarrollo de su razonamiento, desde un razonamiento cotidiano más cercano al concreto, hasta un razonamiento matemático más abstracto (Castro, Cañadas y Molina, 2010). Así, se reconoce que la inducción fomenta el desarrollo de este tipo de habilidades, a partir de la formulación de conjeturas y su prueba formal para garantizar la veracidad de la conjetura (Cañadas y Castro, 2007). Al respecto, el Consejo Nacional de Maestros de Matemáticas [NCTM] (2000) enfatiza la necesidad de desarrollar la competencia de los estudiantes de secundaria en el uso del razonamiento inductivo (y deductivo) para examinar patrones y estructuras para identificar regularidades; establecer y evaluar conjeturas sobre posibles generalizaciones, en patrones lineales o cuadráticos. Sobre la base de estas demandas en los estudiantes, los maestros de matemáticas están vinculados implícitamente. Para hacerlo, necesitan de la habilidad para ayudar a los estudiantes a hacer, refinar y explorar conjeturas sobre la base de evidencia y usar una variedad de técnicas de razonamiento para confirmar o refutar esas conjeturas (NCTM, 2000). Según Brodie (2010), "el maestro puede, a través de preguntas y sugerencias, tratar de provocar que los estudiantes piensen de manera particular y ayudarlos a comparar, verificar, explicar y justificar sus conjeturas" (p. 45). Además, Ball y Bass (2003) afirman que los maestros necesitan habilidades para proporcionar recursos a los estudiantes que les permitan desarrollar estas destrezas y utilizar entornos que lo hagan posible.

Por otro lado, la mayoría de las investigaciones de razonamiento inductivo, que lo estudian como un proceso de pensamiento y como un generador de conocimiento en el contexto de tareas de generalización, se refieren principalmente a relaciones lineales. Pocos se han centrado en los cuadráticos (Kirwan, 2017), tanto en la formación como en el servicio de los docentes. Este artículo examina el razonamiento inductivo de los maestros mexicanos de matemáticas de la escuela secundaria, al resolver tareas de generalización de una sucesión cuadrática.

Fundamentación Teórica

El razonamiento inductivo es la acción del pensamiento humano que produce afirmaciones y llega a conclusiones, comenzando con las observaciones de casos particulares hasta llegar a una generalidad (Cañadas, 2002). Es un proceso cognitivo que contribuye al avance del conocimiento, donde se obtiene más información de la que proporcionan los datos iniciales con los que comienza ese proceso. El razonamiento inductivo en esta investigación se analiza a partir del modelo de razonamiento inductivo propuesto en Cañadas y Castro (2007). Este modelo se basa en los pasos de Polya (1966), el trabajo empírico de Cañadas (2002) y las etapas de Reid (2002).

Modelo del razonamiento inductivo

El modelo de razonamiento inductivo consta de siete etapas. Se presentan en un orden ideal. Comienzan con la observación de casos particulares hasta la generalización. No todas estas etapas ocurren necesariamente. A continuación describimos estas etapas:

Observación de casos particulares. El punto de partida del razonamiento inductivo son las experiencias con casos particulares.

Organización de casos particulares. Uso de diferentes estrategias para sistematizar y facilitar el trabajo con casos particulares.

Identificación de patrones. Se reconoce alguna regularidad o comportamiento. Los patrones se consideran como algo que se repite con regularidad (Stacey, 1989), su reconocimiento permite el desarrollo de la habilidad para generalizar. Existen diferentes tipos de patrones: numéricos, pictóricos, figural, procedimientos computacionales o patrones repetitivos (Amit & Neria, 2008)

Formulación de conjeturas. Es una afirmación sobre todos los casos posibles, basados en los particulares, pero con un elemento de duda. La conjetura es una afirmación que parece razonable, pero cuya veracidad no ha sido validada. No se ha validado de manera convincente y aún no se sabe que haya ejemplos que lo contradicen, ni se sabe que tenga alguna consecuencia falsa (Mason, Burton , & Stacey, 1988).

Validación de las conjeturas. En esta etapa, se intenta validar las conjeturas para nuevos casos específicos, pero no en general.

Generalización. Los patrones matemáticos se relacionan con una regla general, no solo con algunos casos. Con base en una conjetura que es cierta para algunos casos particulares, y habiendo validado dicha conjetura para casos nuevos (validación de la conjetura), los estudiantes podrían hipotetizar que la conjetura es verdadera en general. La generalización es extender deliberadamente el razonamiento o comunicación más allá de los casos considerados, reconociendo y explicando su similitud (Kaput,2008).

Justificación de las conjeturas. En este punto, una prueba formal puede proporcionar la justificación final que garantiza la veracidad de la conjetura.

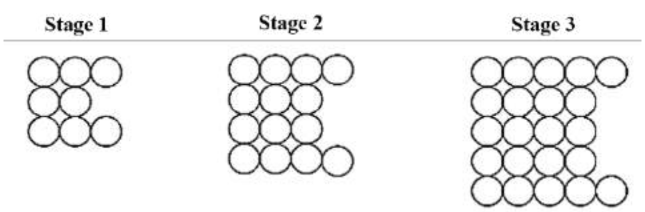
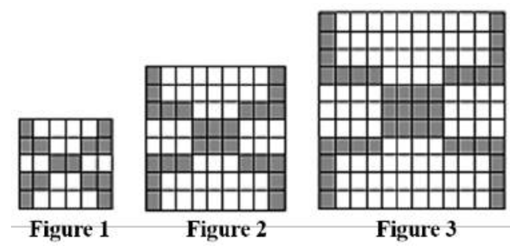
Método

Es una investigación cualitativa e interpretativa. Se llevó a cabo con dieciséis maestros de matemáticas de secundaria (nueve mujeres y siete hombres) con 5 y 14 años de experiencia docente en escuelas públicas de México. Participaron voluntariamente en esta investigación a través de talleres de razonamiento inductivo en el contexto de congresos de Educación Matemática.

Los maestros participantes tenían calificaciones profesionales como maestros de escuela intermedia (nueve una licenciatura en matemáticas, dos en educación matemática y cinco en telesecundaria), por lo que todos estudiaron conceptos matemáticos como sucesiones y sucesiones lineales y cuadráticas. El criterio de selección de los participantes era que hubiesen experimentado como maestro de tercer grado la enseñanza de sucesiones cuadráticas utilizando patrones numéricos y figurales. Teniendo en cuenta lo anterior, se diseñaron dos tareas sobre la generalización de sucesiones cuadráticas (ver figura 1). La tarea de mosaico de patio se adaptó del estudio de Kirwan (2017) y la tarea de patrón cuadrático de rana del estudio de Rivera (2013).

Estas tareas consisten en patrones crecientes con progresión aritmética de orden 2 en los números naturales. Los maestros trabajaron individualmente durante 20 a 30 minutos en las tareas. Después del análisis de sus respuestas, se entrevistó a seis de ellos para comprender en mayor profundidad su proceso de razonamiento inductivo y también porque habían mostrado diferentes formas de razonamiento.

Tabla 1: Tareas de sucesiones cuadráticas utilizadas en la investigación

Tarea 1*: Tarea de la piedra del patio*. En la construcción de un patio, se colocan piedras circulares de igual tamaño. Para observar el avance que sigue la construcción, se toma una foto al patio por etapa  a. ¿Cuántas piedras circulares se han colocado para la sexta etapa si en la construcción del patio se avanza de la misma manera? Justifica tu respuesta. b. ¿Cuántas para la etapa 50? Justifica tu respuesta. c. ¿Cómo se puede hallar la cantidad de piedras circulares para cualquier número de etapa? Describe ampliamente tu respuesta.	**Tarea 2: *Tarea del patrón cuadrático de la rana*.** Observa la secuencia de las siguientes figuras. Justifica ampliamente el proceso de solución en cada una de las preguntas.  a. ¿Cuántos cuadrados grises conforman la figura 5? b. ¿Cuántos cuadrados grises conforman la figura 7? c. ¿Cuántos cuadrados grises conforman la figura n?

Resultados

En este estudio, las etapas de razonamiento inductivo que siguió la mayoría de los maestros fueron: Organización de casos particulares, Búsqueda de patrones, formulación de conjeturas y generalización. Con menos frecuencia, la organización de casos particulares y la validación de conjeturas. Con respecto al paso de justificación de conjeturas, ninguno de los maestros mostró haber estado involucrado (ver Tabla 2).

Tabla 2. Etapas del razonamiento inductivo que siguen los maestros de matemáticas

Tarea	Etapas					
	Observación de casos particulares	Organización de casos particulares	Búsqueda de patrones	Formulación de la conjetura	Validación de la conjetura	Generalización	Justificación de la conjetura

T1	11	5	12	7	2	11	0
T2	9	4	11	8	6	8	0

Observación y Organización de Casos Particulares

En la tarea 1, once maestros observaron los casos particulares y en la tarea 2, nueve. La mayoría, identificó la cantidad de objetos en cada etapa de la sucesión. En algunos casos, este trabajo consistió de conteos estratégicos, basado en la descomposición de las figuras. La visualización del patrón figural fue fundamental, ya que contribuyó a que los maestros reconocieran determinadas configuraciones en las etapas dadas. Otros maestros, se apoyaron del conteo para establecer los términos k-ésimo e identificar el tipo de sucesión. Si bien se reconoce que los maestros trabajaron con casos particulares, al menos 3 evidenciaron dificultades para avanzar a las etapas siguientes.

Los maestros que recurrieron a la organizaron los casos particulares (ver tabla 1) se apoyaron de tablas de doble entrada, en las que establecieron una correspondencia entre dos variables, cantidad de objetos del patrón figural, con el número de la etapa o figura, en el marco de las demandas de las tareas. Las tablas fueron representadas de dos formas, vertical y horizontal. Esta última forma, fue utilizada por los maestros que se apoyaron del método de diferencias en su proceso inductivo, para identificar el tipo de sucesión.

Identificación del Patrón

Los maestros que identificaron el patrón, recurrieron a dos formas de proceder (ver tabla 2):

A. **Descomposición de la figura:** Percibieron los objetos del patrón figural como configuraciones básicas: cuadrados y/o rectángulos. A partir de ello, asociaron una estructura matemática útil para explicar y justificar el comportamiento de estos objetos.
B. **Método de diferencias:** Identificaron el patrón de recurrencia entre los términos k-ésimo de la sucesión asociada con el patrón figural. Posteriormente, trabajaron con la primera diferencia y reconocieron que no es constante, luego se determinaron las segundas diferencias y observaron que es constante. Por lo tanto, derivaron que la sucesión es cuadrática. Finalmente, encuentran los coeficientes de la sucesión de la forma $ax^2 + bx + c$, para establecer la regla general de la sucesión cuadrática.

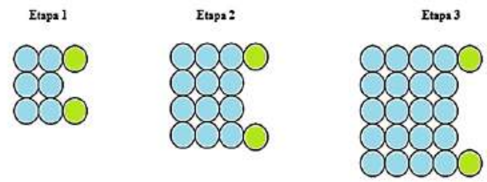
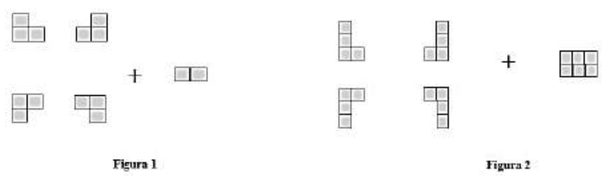
Formulación y Justificación de la Conjetura

Los maestros percibieron el comportamiento de los objetos en las figuras (patrón figural). Al principio, hicieron estas conjeturas a través de estructuras aditivas y multiplicativas, luego las transformaron en una estructura algebraica, en términos de n (demanda de la tarea) o con otra variable. Pocos justificaron sus conjeturas. De los que lo hicieron, fue el tipo algebraico y lo validaron con casos particulares.

Generalización

Los resultados evidencian, que once de los dieciséis maestros, lograron construir la regla general que explica el comportamiento del patrón figural en la tarea 1 y ocho, en la tarea 2, esto es, generalizaron. Los que generalizaron sin formular conjeturas, usaron el método de diferencias para reconocer el tipo de sucesión (cuadrática) y con base en ello, determinaron la regla general asociada al patrón figural. La generalización construida por quienes formularon una conjetura, la verificaron para comprobar su veracidad. En algunos, este proceso fue realizado con los casos particulares propuestos y en otros, mediante nuevos casos, como términos cercanos y lejanos. Este proceso consistió en evaluar la conjetura, en correspondencia entre la cantidad de los objetos con el número de la etapa o figura. Se reconocieron dos formas de expresar la regla general, una de forma algebraica y otra, verbal.

Tabla 3: Formas de proceder de los maestros al resolver tareas de generalización

Formas de proceder	Tarea	Ejemplos
Descomposición de la figura	T1	 Con base en esta forma de percibir los objetos del patrón figural en T1, construyeron una regla general asociada a una estructura multiplicativa y aditiva: $(n+1)(n+2)+2$
	T2	 Con base en esta forma de percibir los objetos del patrón figural en T2, construyeron una regla general asociada a una estructura multiplicativa y aditiva: $4(1+2n)+n(n+1)$
Método de diferencias	T1	Términos k-ésimo 8 14 22 32 Primera diferencia 6 8 10 Segunda diferencia 2 2 Con base en esta forma de percibir los objetos del patrón figural en T1, construyeron una regla general asociada a una expresión algebraica: x^2+3x+4 A partir del algoritmo de $a+b+c=8$; $3a+b=6$; $2a=2$, hallaron los coeficientes de la sucesión.
	T2	Términos k-ésimo 14 26 56 74 Primera diferencia 12 14 16 Segunda diferencia 2 2 Con base en esta forma de percibir los objetos del patrón figural en T2, construyeron una regla general asociada a una expresión algebraica: x^2+9x+4 A partir del algoritmo de $a+b+c=14$; $3a+b=12$; $2a=2$, hallaron los coeficientes de la sucesión.

Discusión y conclusiones

Este artículo examinó el razonamiento inductivo de los maestros mexicanos de matemáticas de secundaria, al resolver tareas de generalización de sucesiones cuadráticas. Desde un punto de vista metodológico, las tareas alentaron el trabajo del maestro en etapas cercanas y lejanas, para construir y validar una conjetura que explica el comportamiento del patrón figural involucrado y establecerlo como la regla general. Aunque como se mencionó anteriormente, no todos evidenciaron la validación de sus conjeturas. Por otro lado, se reconoció que no todas estas etapas ocurrieron necesariamente en los procesos de razonamiento inductivo llevados a cabo por los maestros.

Con respecto a la resolución de tareas en la generalización de patrones figurales, los maestros mostraron el proceso inductivo como una estrategia. La investigación ha documentado que trabajar con patrones figurales bien definidos favorece que sean interpretados como configuraciones de cierta manera (Rivera, 2010). En este sentido, la tarea del maestro en esta investigación consistió en involucrarse en una generalización significativa de los patrones. Este estudio también reconoce que los objetos de algunos patrones figurales son complejos de interpretar, incluso cuando su construcción está bien definida. Como ejemplo, la tarea 2. El patrón figural involucraba más de un objeto, cuadrados grises y blancos, en relación con el número de la figura. La tarea demandó representar algebraicamente el comportamiento de la variable, es decir, los cuadrados grises que conforman una figura rectangular, también compuesta de cuadrados blancos. La distribución de cuadrados grises no está relacionada con una figura geométrica específica. Estas variables influyeron en las dificultades de los maestros para reconocer el comportamiento de los cuadrados grises.

En línea con Rivera (2010), en esta investigación, el contexto de generalización de patrones figurales involucró a los maestros en la coordinación de habilidades inferenciales perceptivas y simbólicas, más específicamente el patrón figural que es alto en la bondad de Gestalt, porque tiende a tener una definición bien definida, estructura que tiene partes fácilmente discernibles, forma equilibrada y armoniosa del patrón, lo que permitió a la mayoría de los maestros especificar una fórmula algebraicamente útil. Naturalmente, los maestros procedieron de manera diferente al trabajar con los objetos del patrón figural de cada tarea.

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AN APPROACH TO DENSITY IN DECIMAL NUMBERS: A STUDY WITH PRE-SERVICE TEACHERS

UN ACERCAMIENTO A LA DENSIDAD EN LOS NÚMEROS DECIMALES: UN ESTUDIO CON PROFESORES EN FORMACIÓN

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Researchers, who have studied the understanding of the density property in the set of decimal numbers, have shown that the student uses the property of the discrete of natural numbers to solve tasks related to density. So, a restructuring of concepts is necessary, that is, a conceptual change from "the discrete" to "the dense". This report presents evidence from ten pre-service teachers from Mexico City that this change can initiate through the implementation of a didactic sequence. The pre-service teachers managed to visualize that a decimal number can be found in an interval. Consequently, they were able to conceive an infinity. However, several of them persisted with the idea of the existence of a successor in the set of decimal numbers.

Keywords: Teacher training, rational numbers, Concepts of numbers and operations, Mathematical knowledge for teaching

The challenges that students face when solving situations related to the density property of rational numbers have not been an easy process. When a student concludes elementary school, he believes that there is no other number between two decimal numbers. For example, in Argentina, fifth graders (10 years old) think that between 4.2 and 4.3 there is no other decimal (Broitman, Itzcovich & Quaranta, 2003). Ávila (2008) reports that only 10% of students in elementary school answer correctly the exercises of density property raised in the national evaluations in Mexico.

After elementary school, students still have the same thinking about the discrete property of natural numbers during middle school. This can be confirmed in the research carried out by Neuman (1998). He exposes that German seventh graders in elementary education (13 years old) associated common properties with natural numbers to solve questions related to the density of fractions. Hart, in 1981 (as cited in Widjaja, Stacey & Steinle, 2008), stated that between 22% and 39% of students from 12 to 15 years old thought that there were eight, nine or ten decimal numbers between 0.41 and 0.42 (for example, the nine decimal numbers in the order of thousandths: 0.411, 0.412, 0.413, 0.414, 0.415, 0.416, 0.417, 0.418, and 0.419).

The students in college and pre-service teachers are in the last stage of schools. In the United States, future teachers think there is a finite number of intermediate numbers in an interval and a rational number has a successor (Tirosh, Fischbein, Graeber & Wilson, 1999). In Indonesia, half of a population of pre-service teachers considered a finite amount of decimal numbers between two given numbers (Widjaja et al., 2008). Most of the 62 pre-service teachers and more than a tenth of 71 students specialized in mathematics at the university level in Finland used properties corresponding to the natural number system on tasks associated with the density property of fractions (Merenlouto & Lehtinen, 2004).

This problem seems to persist from elementary school therefore the attention was focused on pre-service secondary teachers; in this level the properties of decimal numbers including the density property are studied in more depth. It should mention that this document is an extension of a brief research report presented by Suárez-Rodríguez and Figueras in 2019.

Objectives

Considering the need to overcome the difficulty that a pre-service teacher has about the property of density in the decimal numbers' set, a didactic sequence was proposed (see Suárez-Rodríguez, 2017) to generate a *metaconceptual awareness*. In the process of conceptual change, the student begins to assume a metaconceptual awareness when he is aware that their assumptions and beliefs are hypothetical and limit how you interpret the information you learn (Vosniadou, 1994). Therefore for the present research, the following aims were established: 1. Identify and write explanatory framework — different ways of expressing an individual's interpretations (Vamvakoussi, Vosniadou & Van Dooren, 2013) — that pre-service teachers make about the density property of decimal numbers when they have a metaconceptual awareness, and 2. Analyze the actions of the participants in solving the activities if a conceptual change is promoted.

Theoretical Framework

Carey, in 1987, proposes the cognitive-developmental approach to Conceptual Change, in which she explains how childhood cognition works and develops alternatives to initiate a process of conceptual change. The author points out that the knowledge of an individual's concept, initially, is linked to a *naive theory* (it consists of explanations of innate concepts or concepts learned from everyday experience related to science). An example of a naive theory, or naive idea, in mathematics is one in which the student believes that there are only nine numbers between 1.2 and 1.3, namely: 1.21, 1.22, 1.23, 1.24, 1.25, 1.26, 1.27, 1.28, 1.29 (Suárez-Rodríguez, 2017). Once the knowledge is associated with naive theories, this will be linked with others forming a knowledge system in the student (Carey, 1987).

Vosniadou (1994) proposes that naive theories constitute the *mental models* of the human being and that these models are the first step to initiate a process of conceptual change. For Vosniadou, a mental model is a representation that the student forms about ideas that he or she acquires, either through experience or instruction, and that is accompanied by presuppositions. When a student assumes metaconceptual awareness these models change, the *synthetic models* are produced and represent the students' attempts to reconcile the culturally accepted scientific views with the presuppositions of their naive theories (Vosniadou, 1994). Finally, conceptual change requires changes in the presuppositions and beliefs that the student must make in his representations so that he can access the *scientific concepts*, and thus achieve an understanding of the concepts learned (Vosniadou , 1994; Vosniadou, Vamvakoussi, and Skopeliti, 2008).

In the educational field of mathematics, Stafylidou and Vosniadou (2004) and Vosniadou and Verschaffel (2004) often use explanatory frameworks instead of mental models or synthetic models. The first explanatory frameworks that the student carries out are the first reflections that form a coherent and solid structure about what he learns along with his presuppositions and beliefs (Vosniadou and Verschaffel, 2004). Continuing the idea to these researchers, we can say that the initial explanatory frameworks are the first thoughts a student makes about what he perceives, what he sees, what he plays and explains in his own way, in his words. In the context of a learning process likewise, the expression conceptual change is a process of restructuring concepts when the student is learning information that is not compatible with his knowledge built up to now so the student must make a resignification of concepts. This is how the understanding of the density property involves a process of conceptual change, gradually, a restructuring from "the discreet" to "the dense", as argued by Suárez-Rodríguez and Figueras (2019).

A conceptual change: the restructuring of the discrete to the dense

Ni and Zhou (2005) point out in their research three possible causes that make the student has difficulties to answer tasks associated with rational numbers when he uses knowledge about the natural number. A first cause is related to the innate, that is if the property of the discrete of the set of natural numbers is innate in nature. The second cause is related to the teaching of the properties of the set of natural numbers and the set of rational numbers. And finally, the relationship between student´s learning the concepts and learning the symbology of those concepts. Faced with this, Vamvakoussi and Vosniadou (2012) devised a strategy using the analogy between stretching an "elastic band" and the density property. The student, as he stretches the elastic band, observes that there is more space between "the two points drew"; therefore, there are more "imaginary points". Thus, the students (between 13 and 17 years old) achieved to imagine a similar relationship between the stretch of the garter and the property of density. In the research performed by Vamvakoussi and Vosniadou with 15-year-old students, in 2004, different explanatory frameworks were identified and described, related to a thought linked to finite or infinite quantities of rational numbers in an interval (see Table 1).

Table 1: Characterization of thinking about the quantity number of numbers in an interval

Naive thinking about the discrete	It is thought that there is no other number between two *consecutive false* rational numbers. Vamvakoussi and Vosniadou (2004) created this expression to refer that exists a successor of a rational number.
Advanced thinking about the discrete	It is thought there is a finite quantity of numbers between two consecutive false rational numbers.
Mixed thinking between discrete and dense	In some cases, it is thought that between two rational numbers there is an infinity of numbers; and in other cases, it is thought there is a finite number of intermediate numbers.
Naive thinking about the dense	It is understood that there is an infinity of numbers in an interval, but this situation is not justified by using the density property. The symbolic representation of the extremes of an interval influences the way of thinking; it is believed there can only be an infinite number of decimal numbers between decimals and an infinity of fractions between fractions, but not an infinity of fractions between decimals or otherwise.
Advanced thinking about the dense	There is a sophisticated understanding of the density property; that is, it is understood that there is an infinite number of numbers between two rational numbers, regardless of their symbolic representation and this is justified through the use of the density property.

Methodology

Participants

The population studied was 10 pre-service teachers in mathematics at the secondary basic education of an institution in Mexico City, in 2017. One of these participants was 36 years old and the others were between 18 and 23 years old.

Educational experimentation design

The implementation of the didactic sequence with the pre-service teachers focused on one purpose: to generate metaconceptual awareness. Vamvakoussi and colleagues (2013) indicate that conceptual change through instruction is a slow and gradual process because it not only involves the reorganization of conception but also of the entire knowledge system. However, the authors describe that a conceptual change can be achieved gradually with the following criteria: (a) an in-depth exploration of the concepts to learn, (b) to take into account the student's prior knowledge, (c) to facilitate a metaconceptual awareness, (d) to provide meaningful experiences, and (e) to encourage the use of various representations, whether graphic, written, or made with digital resources.

Considering the previous criteria and the backgrounds that shape the problem about around the understanding of the density property by the student, we designed and elaborated of the activities of the didactic sequence. The sequence was structured in two stages. The first stage is to recognize the participants' first explanatory frames through a paper and pencil questionnaire (diagnosis) and individual interviews. The second stage was elaborated with individual and group activities to identify and analyze their actions, in turn, their explanatory frameworks. This stage was carried out in four sessions about 1. The perception of the dense in concrete materials. 2. Addition and subtraction of decimals, 3. Localization of decimals in intervals, and 4. Comparing decimal numbers.

Results

Results of the questionnaire as diagnosis (first stage)

The pre-service teachers answered an 11-question questionnaire which showed the first explanatory frameworks related to the categories proposed by Vamvakoussi and Vosniadou (2004) about the property of the discrete and the property of density. Below are the responses of several participants to some questions, as well as some of their comments expressed in the individual interviews to find out the justification for their answers.

Naive thinking about the discreet. Some pre-service teachers considered that the extremes of an interval are consecutive and therefore they assured there cannot another number in this interval. Figure 1 shows the case of Amanda. She carried out the process of a fractional representation to a finite decimal writing, and she argued that both ends of the interval are consecutive.

5. Can you find decimal numbers and or fractions between 0.49 and 1/2? Write your answer.

No porque $\frac{1}{2}$ es 0.5 el siguente de 0.49

Amanda shows that 1/2 = 0.5 = 0.50 is the successor of 0.49. She mentions in the interview "0.49 and 1/2 are consecutive".

Figure 1: Naive thinking about discrete (example)

Advanced thinking about the discreet. The finite subdivision process in an interval was one of the initial explanatory frameworks the participants operated to confirm the existence of a finite of decimal numbers in a range. The notion that only numbers on the order of tenths — at most hundredths — are decimal numbers may be influenced by responses related to finite sets within an interval. For example, Fabiola evokes a finite process, considering a certain number of decimal places of a number to affirm that there are nine decimal places in the given interval; therefore, the idea of false consecutive numbers underlies (see Figure 2).

2. How many decimal numbers are there between 1.2 and 1.3?

Si nueve

Fabiola wrote that there were 9 numbers in the range. She said there were only 1.21, 1.22, 1.23, 1.24, 1.25, 1.26, 1.27, 1.28, and 1.29 (as she expressed in the interview).

Figure 2: Advanced thinking about discrete (example)

It is worth mentioning that some pre-service teachers have an initial explanatory framework that a successor is a number greater than itself. Perhaps, Isabella thinks the successor of any number is a larger number, as in her example: natural 7, 8 (See Figure 3).

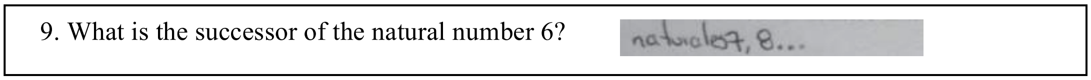

Figure 3: Belief of the successor's existence as a larger number (example)

To finish this first stage of the didactic sequence, it is concluded that half of the pre-service teachers tend to have combined thinking between the discrete and the dense, while three of them apparently have advanced thinking about the discrete and the rest tend to have naive thought related to the discreet.

Results of the activities of the didactic sequence (second stage)

The following paragraphs describe some actions and explanatory frameworks of the teachers in training who interacted with activities in the didactic sequence.

Performances related to the perception of the dense in concrete materials. The purpose of the first activities of the didactic sequence is that the future teacher conceives the notion of an infinity of numbers in an interval through an "infinity of points" in a geometric context. They reported that the more the elastic band was stretched (Vamvakoussi & Vosniadou, 2012), there could be more space; therefore, more imaginable points. The same occurs when blowing up a balloon. The participants pointed out that the more the balloon was blown up, the more space there would, be therefore, more imaginary points.

Performances identified with addition and subtraction with decimal numbers. For this second session, participants answered to two activities that emerged from the activities made by Broitman et al. (2003). The objective of the activities is the recognition of skills in writing different decimal numbers. In one of the activities, the teacher-researcher writes the number *1.5* on the board, then the pre-service teacher writes the greatest number of addends and that the total sum approaches or equals 10. Some participants revealed the use of numbers until two decimal places in this first experience, for example, Karen's explanatory framework. She used numbers with two decimal places, even considering the number 0 for the hundredths, as can be seen in Figure 4. Karen only writes down the numbers between 0 and 1. She first writes the sequence *0.5*, *0.4*, *0.3,* and then registers *0.25*, *0.75, 0.80,* and *0.15*; that is, she decomposes to hundredths. Then she writes *0.20*, *0.70*, *0.40,* and *0.30* and recognizes that zero can have the hundredths position. Perhaps she did not perceive the equivalence between 0.3 and 0.30 as well as 0.4 and 0.40 since the activity required different numbers.

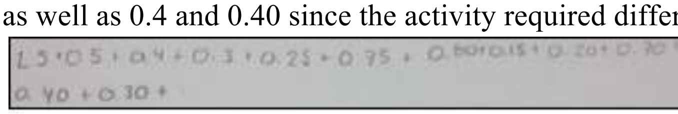

Figure 4: Karen's record on the addition and subtraction activity

Nicolás' explanatory framework was based on registering numbers up to six decimal places, but apparently, he had a strategy (see Figure 5). He wrote down the number *0.000001* and then *0.000009*, and in the following rows the position of the digit 9 changes. Nicolás wrote 9 in the position of the millionths, then in the position of the hundred-thousandths, then in the ten-thousandths, the thousandths, the hundredths, and finally, in the tenths. The process that Nicolás followed was to multiply 9 by $1/10^n$, where n is a natural number, starting with the millionth position (that is, $n = 6$). The sum of the numbers that Nicolás wrote is 1. This participant used numbers up to millionths and in the diagnostic questionnaire showed examples related to advanced thinking associated with the discrete. This fact highlights a process of a gradual conceptual change since he recorded numbers with up to two decimal places in some questionnaire responses.

Figure 5: Nicolás' record on the addition and subtraction activity

Performances linked to locating decimal numbers in intervals. The two activities in this session were developed from one made by Brousseau (1981) that seek to locate numbers in an interval, as well as find intervals for a given number. In one of the activities, future teachers should look for the interval in which the number "thought" by a colleague is found and pose questions to find that interval. The ends of this interval must be decimal numbers whose decimal places are consecutive. Figure 6 shows an example of a participant who wrote the intervals his colleagues mentioned and marked with an "X" those that did not correspond. Olga considered the numerical representation until ten-thousandths in the hidden number, 28.9306. She had shown evidence of naive thinking concerning discrete in the development of the diagnostic questionnaire. It seems, that Olga has been doing a restructuring of concepts because she believed there were only numbers up to two decimal places. Perhaps, she used the numerical representation of the order of the ten-thousandths as a consequence of the socialization of the previous activities (related to additions and subtractions).

Performances associated with comparing decimal numbers. The two comparison activities, in the last session of the didactic sequence, are part of a study carried out by Castillo in 2015, in Mexico. Understanding the density property of decimals from the decimal comparison property is the aim of the activities. In one of the activities, each future teacher must complete a decimal numbers' series. Figure 7 shows the work made by that one of the five participants carried out and completed the sequence correctly. His strategy consisted in detailing the last digits of the numbers that appear to "modify" them and, in other cases, to "add digits" without altering the ordering.

Some teachers in training did not perceive the equivalence between decimals, for instance, Olga, Isabella, Oscar, and Amanda (see Figure 8). Olga added a zero in the ten-thousandth position of the number 30,871. Isabella and Olga added two zeros in the hundred-thousandth and millionth positions of the number 30.8712. While Oscar added a zero in the position of one hundred-thousandths of the number 30.8721, and Amanda added a zero to the end of the expression 30.87125.

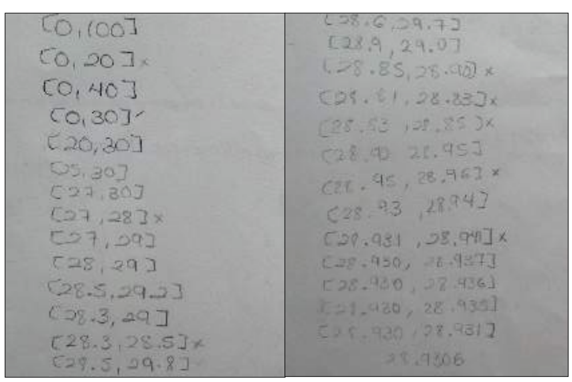

Figure 6: Future teacher's record on the location activity

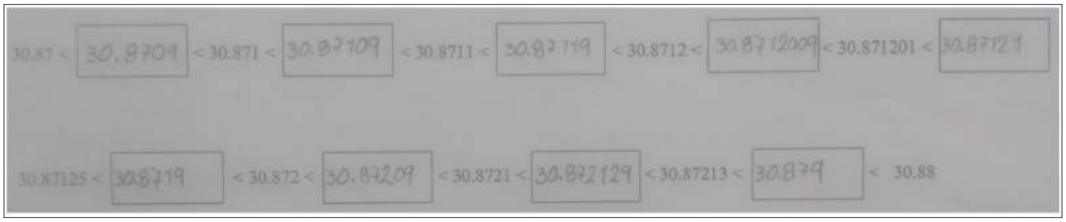

Figure 7: Future teacher's record on the comparison activity

Figure 8: Records of four participants in the comparison activity

After completing the individual test, the future teachers wrote down the numbers written on their worksheets on the board (see Figure 9). In order for the pre-service teacher to achieve a metaconceptual awareness that the property of density helps to visualize the fact that there is no successor in the set of decimal numbers, it was shown that between pairs of false consecutive ones it is found at least a decimal number, therefore an infinity. For example, in Figure 9 (see oval) that observed between the pair of false consecutive *30.8711* and *30.8712* are located the least seven decimal numbers, larger than the first and smaller than the second: *30.87112*, *30.871103*, *30.87119*, *30.871105*, *30.871102*, *30.87115* and *30.871106*

Finally, the teacher-researcher asked the participants if there were other strategies to find intermediate numbers in this activity. Nicolás mentioned "the arithmetic mean". The teacher-researcher accomplished a brief example with a pair of numbers from the activity to show the arithmetic mean helps to find intermediate numbers in an interval.

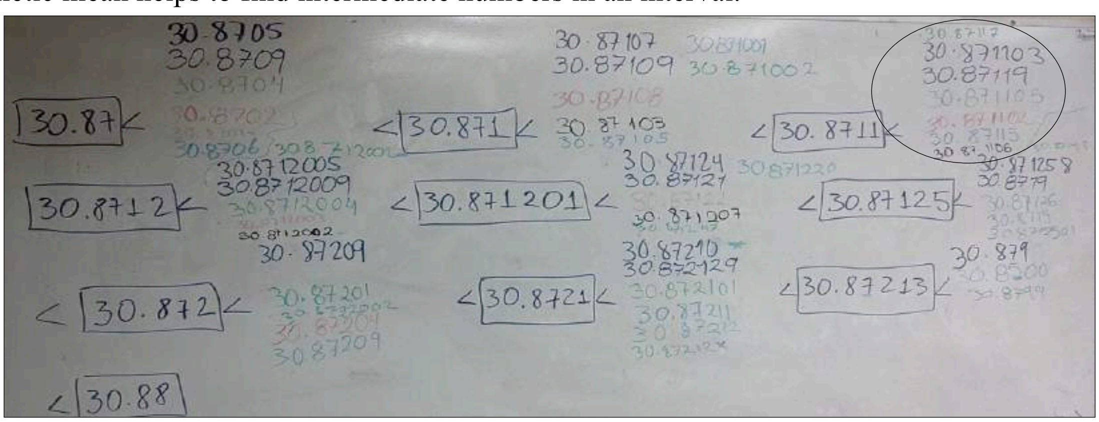

Figure 9: Participants' annotations in the comparison activity

Conclusions and implications

The socialization of the activities of the didactic sequence with the pre-service teachers promotes an approach to the understanding of the density property of the decimal numbers, and, in consequence, a process of conceptual change from the discrete to the dense. In the development of the diagnostic questionnaire, the ten future teachers had evidenced examples of thought associated with the property of the discrete of natural numbers. During the implementation of the activities of the didactic sequence, the ten participants achieved to extend the number of decimal places, a strategy that allowed them to locate numbers in an interval. However, three future teachers questioned the mediation about the existence of a decimal's successor; they even included in their explanatory framework the existence of a successor to a decimal number as a larger number.

Developing the writing of decimal expansion of a number is considered a task that may help the future teacher, or a student in general, in understanding the series' concept. Sums of arithmetic or geometric progressions with infinite terms are series' examples. A number with periodic decimal writing expresses an approximation of a rational number, which is the limit value of rational, for example, the limit value of the decimal expansion 0.0123123123... is 41/3330. Likewise, the development of writing infinite decimal expansions that cannot be expressed as a fraction could help the student to understand the irrational number' concept. Finally, as indicated by Suárez-Rodríguez and Figueras (2019), the didactic sequence is an example of a teaching model that may be of interest to in-service teachers who could start studying the density property in their classroom's lessons.

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UN ACERCAMIENTO A LA DENSIDAD EN LOS NÚMEROS DECIMALES: UN ESTUDIO CON PROFESORES EN FORMACIÓN

AN APPROACH TO DENSITY IN DECIMAL NUMBERS: A STUDY WITH PRE-SERVICE TEACHERS

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Investigadores que han estudiado la comprensión de la propiedad de densidad en el conjunto de los números decimales han mostrado que el estudiante usa la propiedad de lo discreto de los números naturales para resolver tareas relacionadas con la densidad. Por lo que es necesaria una reestructuración de conceptos, es decir, un cambio conceptual: de "lo discreto" a "lo denso". En este informe se presentan evidencias de diez profesores en formación, de la Ciudad de México, de que se puede iniciar ese cambio a través de la puesta en marcha de una secuencia didáctica. Los profesores en formación lograron visualizar que en un intervalo se puede encontrar un número decimal, en consecuencia, lograron concebir una infinidad. No obstante, varios de ellos persistieron con la idea de la existencia de un sucesor en el conjunto de los números decimales.

Palabras clave: preparación de maestros en formación, números racionales, conceptos de números y operaciones, conocimiento matemático para la enseñanza

Los desafíos que enfrentan los estudiantes para resolver situaciones relacionadas con la propiedad de densidad de los números racionales no han sido un proceso sencillo. Un estudiante que concluye la educación primaria cree que no hay otro número entre dos decimales. Por ejemplo, en Argentina, los estudiantes de 5to de primaria (10 años de edad) piensan que entre 4.2 y 4.3 no hay otro decimal (Broitman, Itzcovich, y Quaranta, 2003). Por otro lado, Ávila (2008) informa que solo el 10% de los estudiantes de la educación primaria responde correctamente ejercicios vinculados con esta propiedad en los exámenes nacionales en México.

Concluida la educación básica primaria los estudiantes siguen teniendo un pensamiento concerniente con la propiedad de lo discreto del conjunto de los números naturales durante la educación básica secundaria. Como prueba de ello se encuentra la investigación forjada por Neuman (1998) en la que él expone que estudiantes alemanes, de 7° de la educación elemental (13 años de edad), asociaron propiedades afines con los números naturales para resolver preguntas relacionadas con la densidad de fracciones. Hart, en el año 1981 (como se citó en Widjaja, Stacey, y Steinle, 2008), afirmó que entre el 22% y el 39% de los estudiantes de 12 a 15 años de edad pensaban que había ocho, nueve o diez números decimales entre 0,41 y 0,42 (por ejemplo, los nueve números decimales del orden de los milésimos: 0.411, 0.412, 0.413, 0.414, 0.415, 0.416, 0.417, 0.418 y 0.419).

En la última etapa escolar se encuentran los estudiantes universitarios, así como los que se preparan para ser profesores. En Estados Unidos, futuros profesores piensan que hay una cantidad finita de números intermedios en un intervalo y que un número racional tiene un sucesor (Tirosh, Fischbein, Graeber, y Wilson, 1999). En Indonesia, la mitad de una población de profesores en formación considera una cierta cantidad de números decimales entre dos dados (Widjaja et al., 2008). Una mayoría de 62 profesores en formación y más de una décima parte de 71 estudiantes que se especializan en matemáticas a nivel universitario, en Finlandia, utilizaron propiedades correspondientes con el sistema de los números naturales en tareas asociadas con la propiedad de densidad de las fracciones (Merenlouto y Lehtinen, 2004).

Esta problemática pareciera persistir desde la escuela primaria, por ello la atención se enfocó hacia los profesores en formación de secundaria, nivel en el cual se estudia con más profundidad las propiedades de los números decimales, entre ellas la propiedad de densidad. Cabe mencionar que el presente documento es una extensión de un breve informe de investigación presentado por Suárez-Rodríguez y Figueras en el año 2019.

Objetivos

Atendiendo a la necesidad de superar la dificultad que tiene un estudiante para profesor acerca de la propiedad de densidad en el conjunto de los números decimales, se pretendió poner en marcha una secuencia didáctica (ver Suárez-Rodríguez, 2017) con la finalidad de generar una *conciencia metaconceptual*. En el proceso de cambio conceptual, el estudiante empieza a asumir una conciencia metaconceptual cuando es consciente de que sus presuposiciones y creencias son hipotéticas y limitan la forma en que interpreta la información que va aprendiendo (Vosniadou, 1994). Por tanto, para la presente investigación se plantearon los siguientes objetivos: 1. Identificar y describir los marcos explicativos –distintas formas de expresar las interpretaciones de un individuo (Vamvakoussi, Vosniadou, y Van Dooren, 2013)– que hacen los futuros profesores sobre la propiedad de densidad de los números decimales cuando tienen una conciencia metaconceptual , y 2. Analizar las actuaciones de los participantes en la resolución de las actividades en caso de que se promueva un cambio conceptual.

Marco teórico

Carey, en el año 1987, propone el enfoque de desarrollo-cognitivo del Cambio Conceptual, en el que ella explica cómo actúa la cognición infantil y desarrolla alternativas para iniciar un proceso de cambio conceptual. La autora señala que el conocimiento de un concepto de un individuo, inicialmente, está ligado a una *teoría ingenua* (aquella que consiste en explicaciones de concepciones innatas o concepciones aprendidas de la experiencia cotidiana relacionadas con las ciencias). Un ejemplo de una teoría ingenua, o idea ingenua, en matemáticas, es aquella en la que el estudiante cree que solo hay 9 números entre 1.2 y 1.3, a saber: 1.21, 1.22, …, 1.29 (Suárez-Rodríguez, 2017). Luego de que el conocimiento esté asociado con teorías ingenuas, este se va relacionando con otros formando un sistema de conocimientos (Carey, 1987).

Vosniadou (1994) propone que las teorías ingenuas constituyen los *modelos mentales* del ser humano y que estos modelos son el primer paso para iniciar un proceso de cambio conceptual. Para Vosniadou, un modelo mental es una representación que forma el alumno sobre ideas que adquiere, bien sea por experiencia o por instrucción, y que va acompañada de presuposiciones. Cuando un estudiante asume una conciencia metaconceptual estos modelos van cambiando, se producen los *modelos sintéticos*, que representan los intentos que hace el estudiante para reconciliar las opiniones científicas culturalmente aceptadas con las presuposiciones de sus teorías ingenuas

(Vosniadou,1994). Finalmente, el cambio conceptual requiere de cambios en las presuposiciones y creencias que el aprendiz debe realizar en sus representaciones para que pueda acceder a los *conceptos científicos*, y así lograr una comprensión de los conceptos aprendidos (Vosniadou, 1994; Vosniadou, Vamvakoussi, y Skopeliti, 2008).

En el ámbito educativo de las matemáticas, Stafylidou y Vosniadou (2004) y Vosniadou y Verschaffel (2004) suelen usar marcos explicativos en lugar de modelos mentales o modelos sintéticos. Los primeros marcos explicativos que hace el alumno son las primeras reflexiones que forman una estructura coherente y sólida sobre lo que aprende junto con sus presuposiciones y creencias (Vosniadou y Verschaffel, 2004). Siguiendo la idea a estos investigadores, se puede decir que los marcos explicativos iniciales son los primeros pensamientos que hace un estudiante acerca de lo que percibe, de lo que ve, de lo que interpreta, y lo explica a su manera, con sus propias palabras. Así mismo, en el contexto de un proceso de aprendizaje, la expresión cambio conceptual es un proceso de reestructuración de conceptos cuando el estudiante está aprendiendo información que no es compatible con sus conocimientos construidos hasta el momento, por lo que el estudiante debe realizar una resignificación de conceptos. Es así como la comprensión de la propiedad de densidad conlleva un proceso de cambio conceptual, de manera paulatina, una reestructuración de "lo discreto" a "lo denso", como lo sostienen Suárez-Rodríguez y Figueras (2019).

Un cambio conceptual: la reestructuración de lo discreto a lo denso

Los autores Ni y Zhou (2005) en su investigación señalan tres posibles causas que hacen que el alumno tenga conflictos para responder tareas asociadas con los números racionales cuando usan conocimientos acerca del número natural. Una primera causa se relaciona con lo innato, es decir, si la propiedad de lo discreto del conjunto de los números naturales es de naturaleza innata. La segunda se relaciona con la enseñanza de las propiedades del conjunto de los números naturales y la del conjunto de los números racionales. Por último, la relación entre el aprendizaje de los conceptos que el niño adquiere y el aprendizaje de la simbología de dichos conceptos. Ante esto, Vamvakoussi y Vosniadou (2012) diseñaron una estrategia usando la analogía entre el estiramiento de una "liga elástica" y la propiedad de densidad. El alumno a medida que estira la liga observa que hay más espacio entre "dos puntos que están dibujados", por ende, hay más "puntos imaginarios". Efectivamente, los estudiantes (entre 13 y 17 años de edad) lograron concebir una relación de semejanza entre el estiramiento de la liga y la propiedad de densidad.

En la investigación forjada por Vamvakoussi y Vosniadou, en el año 2004, se identificaron y se describieron distintos marcos explicativos concernientes a un pensamiento vinculado con cantidades finitas –o infinitas– entre dos números racionales por estudiantes que tenían entre 15 y 17 años de edad (ver Tabla 1).

Tabla 1: Caracterización del pensamiento sobre la cantidad de números en un intervalo

Pensamiento ingenuo sobre lo discreto	Se piensa que no hay otro número entre dos números racionales *consecutivos falsos*. Esta expresión la acuñaron Vamvakoussi y Vosniadou (2004) para referir que existe un sucesor de un número racional.
Pensamiento avanzado sobre lo discreto	Se cree que hay un número finito de números entre dos números racionales consecutivos falsos.
Pensamiento compuesto entre lo discreto y lo denso	En algunos casos se piensa que entre dos números racionales hay una cantidad infinita de números, y en otros, que hay un número finito de números intermedios.
Pensamiento ingenuo sobre lo denso	Se comprende que hay una infinidad de números en un intervalo, pero no se justifica la situación usando la propiedad de densidad. La representación simbólica de los extremos de un intervalo influye en la forma de pensar; se cree que sólo puede haber una infinidad de números decimales entre decimales y una infinidad de fracciones entre fracciones, pero no una infinidad de fracciones entre decimales, o al contrario.

Pensamiento avanzado sobre lo denso	Hay una comprensión bastante sofisticada de la propiedad de densidad, es decir, se pone de manifiesto que se entiende que entre dos números racionales hay una infinidad de números independientemente de su representación simbólica, y se justifica con la propiedad de la densidad.

Metodología

Participantes

La población estudiada fue de 10 profesores en formación en matemáticas de la educación básica secundaria de una institución en la Ciudad de México, en el año 2017. Uno de estos participantes tenía 36 años de edad y los nueve restantes entre 18 y 23.

Diseño de la experimentación educativa

La puesta en marcha de la secuencia didáctica con los profesores en formación se centró en una finalidad: generar conciencia metaconceptual. Vamvakoussi y colegas (2013) indican que el cambio conceptual a través de una instrucción es un proceso lento y gradual, porque no solo involucra la reorganización de una concepción sino de todo un sistema de conocimientos. No obstante, los autores describen que se puede lograr un cambio conceptual paulatinamente con los siguientes criterios: (a) una exploración profunda de los conceptos a aprender, (b) tener en cuenta el conocimiento previo del estudiante, (c) facilitar una conciencia metaconceptual, (d) proporcionar experiencias significativas, y (e) fomentar el uso de diversas representaciones, ya sean gráficas, escritas, o hechas con recursos digitales.

Teniendo en cuenta los anteriores criterios y los antecedentes que reúnen la problemática en torno a la comprensión de la propiedad de densidad por parte del alumno, se procede al diseño y elaboración de las actividades de la secuencia didáctica. La secuencia se estructuró en dos etapas. La primera para reconocer los primeros marcos explicativos de los participantes a través de un cuestionario (diagnóstico) de papel y lápiz y entrevistas individuales. La segunda etapa se elaboró con actividades individuales y grupales para identificar y analizar sus actuaciones, a su vez, sus marcos explicativos. Esta etapa se llevó a cabo en cuatro sesiones sobre: 1. La percepción de lo denso en materiales concretos, 2. Adición y sustracción de decimales, 3. Localización de decimales en intervalos, y 4. Comparación de números decimales.

Resultados

Resultados del cuestionario como diagnóstico (primera etapa)

Los profesores en formación atendieron a un cuestionario de 11 preguntas el cual arrojó los primeros marcos explicativos relacionados con las categorías propuestas por Vamvakoussi y Vosniadou (2004) sobre la propiedad de lo discreto y la propiedad de densidad. A continuación se muestran las respuestas de varios participantes a algunas preguntas, así mismo se muestran algunas manifestaciones dadas por ellos en las entrevistas individuales con el ánimo de conocer la justificación de sus respuestas.

En Pensamiento ingenuo sobre lo discreto. Hubo estudiantes para profesor quienes consideraron que los extremos de un intervalo son consecutivos y por ello aseguraron que no puede haber otro número en dicho intervalo. En la Figura 1 se muestra el caso presentado por Amanda. Ella realizó el proceso de una representación fraccionaria a escritura decimal finita, y usó el argumento de que ambos extremos del intervalo son consecutivos.

5. ¿Puedes encontrar números decimales y/o fracciones entre 0.49 y 1/2? Justifica tu respuesta.

No porque $\frac{1}{2}$ es 0.5 el siguiente de 0.49

Amanda indica que 1/2 = 0.5 = 0.50 es el sucesor de 0.49. Ella menciona en la entrevista que “0.49 y 1/2 son consecutivos”.

Figura 1: Pensamiento ingenuo sobre lo discreto (ejemplo)

En Pensamiento avanzado sobre lo discreto. El proceso de subdivisiones finitas en un intervalo fue uno de los marcos explicativos iniciales que usaron los participantes para afirmar la existencia de una cantidad finita de números decimales en un intervalo. La concepción de que solamente los números del orden de los décimos –a lo mucho de los centésimos– son los números decimales, puede estar influida en las respuestas relacionadas con conjuntos finitos dentro de un intervalo. Por ejemplo, Fabiola evoca un proceso finito, toma en cuenta una determinada cantidad de cifras decimales de un número para afirmar que hay nueve números decimales en el intervalo dado, por ende, subyace la idea de los números consecutivos falsos (ver Figura 2).

2. ¿Cuántos números decimales hay entre 1.2 y 1.3? Si nueve

Fabiola registró que hay 9 números en el intervalo. Ella dice que solo se encuentran 1.21, 1.22, 1.23, 1.24, 1.25, 1.26, 1.27, 1.28 y 1.29 (manifestaciones en la entrevista).

Figura 2: Pensamiento avanzado sobre lo discreto (ejemplo)

Cabe mencionar que algunos profesores en formación tienen un marco explicativo inicial de que un sucesor es un número mayor que él. Posiblemente, Isabella piensa que el sucesor de cualquier número es un número mayor, como su ejemplo: naturales 7, 8 (Ver Figura 3).

9. ¿Cuál es el sucesor del número natural 6? naturales 7, 8...

Figura 3: Creencia de la existencia de un sucesor como número mayor (ejemplo)

Para finalizar esta primera etapa de la secuencia didáctica se concluye que la mitad de los estudiantes a profesor tienden a tener un pensamiento combinado entre lo discreto y lo denso, mientras que tres, al parecer, tienen un pensamiento avanzado sobre lo discreto y los dos restantes tienden a tener un pensamiento ingenuo afín con lo discreto.

Resultados de las actividades de la secuencia didáctica (segunda etapa)

En los siguientes párrafos se describen algunas actuaciones y marcos explicativos de los profesores en formación quienes interactuaron con actividades de la secuencia didáctica.

Actuaciones vinculadas con la percepción de lo denso en materiales concretos. La finalidad de las primeras actividades de la secuencia didáctica es que el futuro profesor conciba la noción de una infinidad de números en un intervalo a través de una “infinidad de puntos” en un contexto geométrico. Ellos refirieron que entre más se estiraba la liga elástica (propuesta de Vamvakoussi y Vosniadou, 2012) podía haber más espacio, por ende, más puntos imaginables. De la misma manera sucedió con el inflamiento de un globo. Los participantes señalaron que entre más se inflaba el globo habría más espacio, en consecuencia, más puntos imaginarios.

Actuaciones identificadas con la adición y la sustracción con números decimales. Para esta segunda sesión los participantes respondieron a dos actividades que surgieron de las actividades hechas por Broitman y colaboradores (2003). El objetivo de las actividades es el reconocimiento de destrezas en la escritura de números decimales distintos. En una de las actividades la docente-investigadora escribe el número *1.5* en el pizarrón, luego los estudiantes para profesor escriben la mayor cantidad de sumandos y que la suma total se aproxime o se iguale a 10. El uso de números hasta dos cifras decimales se puso de manifiesto en esta primera experiencia por algunos participantes, por ejemplo, el marco explicativo de Karen. Ella utilizó números con dos cifras decimales, incluso tuvo en cuenta el número 0 para las centésimas, como se puede apreciar en la Figura 4. Karen únicamente anota números comprendidos entre 0 y 1. Ella primero escribe la secuencia *0.5*, *0.4*, *0.3*, y luego registra *0.25*, *0.75, 0.80* y *0.15*, es decir, descompone hasta centésimas. Enseguida ella anota *0.20*, *0.70*, *0.40* y *0.30* y reconoce que el cero puede tener la posición de las centésimas. Al parecer, ella no percibió la equivalencia entre 0.3 y 0.30, así mismo, 0.4 y 0.40, puesto que la actividad requería números diferentes.

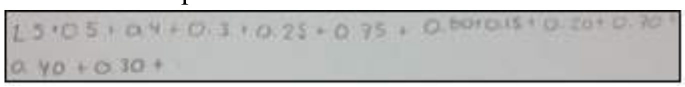

Figura 4: Registro de Karen en la actividad de adición y sustracción

El marco explicativo de Nicolás se basó en el registro de números hasta con seis cifras decimales, pero al parecer tenía una estrategia (ver Figura 5). Él anota el número *0.000001* y después *0.000009*, y en las siguientes filas va cambiando la posición del dígito 9. Nicolás ubica el 9 en la posición de las millonésimas, luego en la posición de las cienmilésimas, las diezmilésimas, las milésimas, las centésimas, y finalmente en las décimas. El proceso que realizó Nicolás fue multiplicar 9 por $1/10^n$, donde *n* es un número natural, comenzando por la posición de las millonésimas (es decir, *n=6*). Se observa que la suma de los números que Nicolás escribió es 1. Este participante, quien en el cuestionario-diagnóstico mostró ejemplos relacionados con el pensamiento avanzado asociado con lo discreto, utilizó números hasta millonésimos. Este hecho pone en evidencia un proceso de inicio de cambio conceptual de manera paulatina, puesto que él registró números hasta con dos cifras decimales en algunas respuestas del cuestionario.

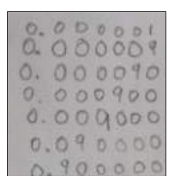

Figura 5: Registro de Nicolás en la actividad de adición y sustracción

Actuaciones identificadas con la localización de números decimales en intervalos. Las dos actividades de esta sesión se elaboraron a partir de una hecha por Brousseau (1981) que tienen la intención de localizar números en un intervalo, así como encontrar intervalos para un número dado. En una de las actividades los futuros profesores deben buscar el intervalo en el que se halla el número "pensado" por un compañero y van elaborando preguntas con el fin de encontrar dicho intervalo. Los extremos de este intervalo deben ser números cuyas cifras decimales sean consecutivas. En la Figura 6 se muestra un ejemplo de un participante quien escribió los intervalos mencionados por sus compañeros e iba marcando con una "equis" cuáles no correspondían. La representación numérica hasta los diezmilésimos en el número escondido, 28.9306, fue tomada en cuenta por Olga. Ella había mostrado evidencias de un pensamiento ingenuo concerniente con lo discreto en el desarrollo del cuestionario-diagnóstico. Olga, al parecer, ha estado haciendo una reestructuración de conceptos, pues ella creía que solo existían números hasta con dos cifras decimales. Es posible que ella haya usado la representación numérica del orden de los diezmilésimos como consecuencia de la socialización de las actividades anteriores (relacionadas con adiciones y sustracciones).

Actuaciones asociadas con la comparación de números decimales. Las dos actividades de comparación, de la última sesión de la secuencia didáctica, se derivaron de una tarea hecha por un estudiante para profesor de una investigación realizada por Castillo en el año 2015, en México. La comprensión de la propiedad de densidad de los decimales a partir de la propiedad de comparación de decimales constituye el propósito de las actividades. En una de ellas, cada profesor en formación debe completar una ordenación de números decimales. En la Figura 7 se evidencia la labor realizada por uno de los cinco participantes que completaron la secuencia de manera correcta. Su estrategia fue detallar los últimos dígitos de los números que aparecen allí para "modificarlos", y en otros casos, para "añadir dígitos o cifras" sin alterar la ordenación.

Algunos estudiantes para profesor no percibieron la equivalencia entre decimales, son los casos de Olga, Isabella, Oscar y Amanda (ver Figura 8). Olga agrega un cero en la posición de las diezmilésimas de la cantidad 30.871. Isabella y Olga agregaron dos ceros en las posiciones cienmilésimas y millonésimas del número 30.8712. Oscar agregó un cero en la posición de las cienmilésimas del número 30.8721. Y Amanda agregó un cero al final de la expresión 30.87125.

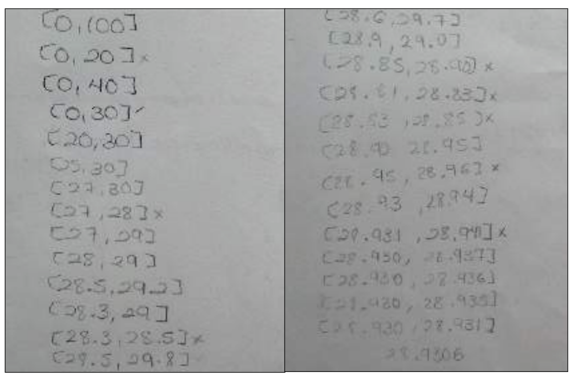

Figura 6: Registro de un profesor en formación en la actividad de localización

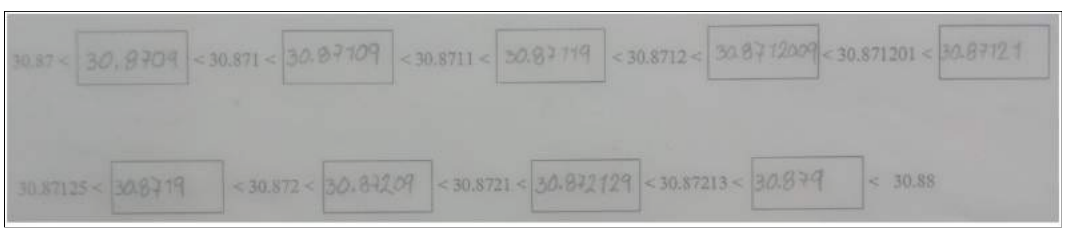

Figura 7: Registro de un futuro profesor en la actividad de comparación

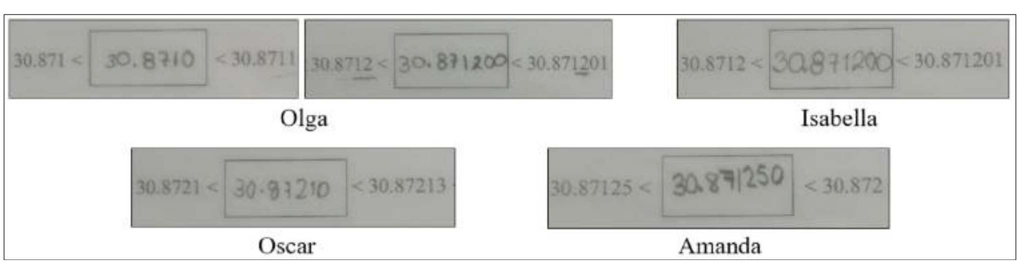

Figura 8: Registros de cuatro participantes en la actividad de comparación

Concluida la prueba individual, los futuros profesores anotaron en el pizarrón los números escritos en sus hojas de trabajo (ver Figura 9). Con el objetivo de que el estudiante para profesor lograra una conciencia metaconceptual de que la propiedad de densidad ayuda a visualizar el hecho de que no existe un sucesor en el conjunto de los números decimales, se mostró que entre pares de consecutivos falsos se halla al menos un número decimal, en consecuencia, una infinidad. Por ejemplo, en la Figura 9 (ver óvalo) se observa que entre el par de consecutivos falsos *30.8711* y *30.8712* se localizan al menos siete números decimales, mayores que el primero y menores que el segundo: *30.87112*, *30.871103*, *30.87119*, *30.871105*, *30.871102*, *30.87115* y *30.871106*. Finalmente, se cuestionó a los participantes si había otras estrategias para hallar números intermedios en esta actividad. Nicolás mencionó “la media aritmética”. Se realizó un breve ejemplo con un par de números de la actividad para mostrar que con ella se puede hallar números intermedios en un intervalo.

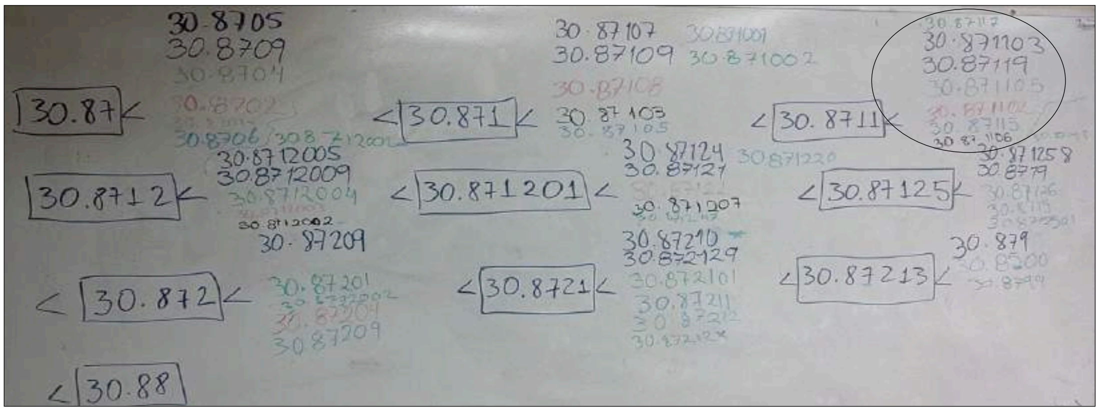

Figura 9: Anotaciones de los participantes en la actividad de comparación

Conclusiones e implicaciones

La socialización de las actividades de la secuencia didáctica con los profesores en formación promueve un acercamiento a la comprensión de la propiedad de densidad de los números decimales, en consecuencia, un proceso de cambio conceptual: de lo discreto a lo denso. En el desarrollo del cuestionario-diagnóstico, los diez profesores en formación habían evidenciado ejemplos de pensamiento asociado con la propiedad de lo discreto de los números naturales. Durante la puesta en marcha de las actividades de la secuencia didáctica, los diez participantes lograron extender la cantidad de cifras decimales, estrategia que les permitió ubicar números en un intervalo. Sin embargo, la mediación sobre la existencia de un sucesor de un decimal fue cuestionada por tres profesores en formación, ellos aun incluían en su marco explicativo la existencia de un sucesor de un número decimal como un número mayor.

Se considera que el desarrollo de la escritura de una expansión decimal de un número es una tarea que posiblemente puede ayudar al profesor en formación, o a un estudiante en general, en la comprensión del concepto de series. Las sumas de progresiones aritméticas o geométricas con infinitos términos son ejemplos de series. Un número con escritura decimal periódica expresa una aproximación de un número racional, que es el valor límite de dicho racional, por ejemplo, el valor límite de la expansión decimal 0.0123123123… es 41/3330. De igual manera, el desarrollo de la escritura de expansiones decimales infinitas que no se pueden expresar como fracción podría ayudar al estudiante en la comprensión del concepto de número irracional. Finalmente, como lo indican Suárez-Rodríguez y Figueras (2019), la secuencia didáctica es un ejemplo de un modelo de enseñanza que puede ser de interés para profesores en servicio quienes podrían iniciar el estudio de la propiedad de densidad en sus aulas de clases.

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EXPERIENCED SECONDARY TEACHERS' DECISIONS TO ATTEND TO THE INDEPENDENT VARIABLE IN EXPONENTIAL FUNCTIONS

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We report our findings and perspective to document the knowledge exhibited by three experienced high school teachers in their instructional decisions for lessons on the equation of an exponential function. We describe the nature of the mathematical ideas and connections teachers promoted in discourse and the decisions that supported the emergence and connections of the mathematics. Despite similarities in the structure of the mathematical activities, differences existed in the ideas that emerged in the three teachers' discussions regarding the relationship between the exponent value and the independent variable. We describe links between collections of teacher decisions to their influences on the mathematics discourse.

Keywords: Teacher Knowledge; Classroom Discourse

Introduction and Background Literature

This study aims to contribute to understanding the nature and quality of mathematics teachers' decisions as a means of describing teachers' knowledge for teaching mathematics in practice. The field widely accepts that teachers' knowledge strongly relates to their effectiveness (e.g., Charlambous & Hill, 2008). Expanding on the work to document and assess a cognitive perspective of teachers' mathematical knowledge for teaching, a call exists to integrate conceptualizations of teachers' knowing and their actions in the classroom (Depaepe et al., 2013). Reviewing literature, Stahnke et al. (2016) categorized studies of teacher knowing in action by the situation-specific processes investigated, namely perception, interpretation, and decision-making (Blömeke et al., 2015). Stahnke et al. concluded decision-making is the most challenging for pre-service teachers (PSTs). Meanwhile, decision-making of experienced teachers is tacit, effortless, and based on sophisticated networks of schema (e.g., Shavelson & Stern, 1981). To inform preparation of PSTs, we sought to learn from experienced secondary teachers by inquiring into their decisions in teaching exponential functions topics.

Despite observations of more powerful ways of understanding exponential growth (e.g., Confrey & Smith, 1994), high school curricula often introduce exponential functions through tasks that facilitate making a correspondence between a quantity growing by repeated multiplication and another related quantity (Davis, 2009). Defining exponential growth by repeated multiplication provides a potentially useful entry point (Weber, 2002); however, the metaphor is insufficient for explaining the meaning of expressions such as $2^{2/3}$ (Davis, 2009). In action, learners may reason about changes in the y-value without attending to the y-value's relationship to the x-value (Ellis et al., 2016) and therefore struggle to connect the repeated multiplication to the closed form of an equation (Davis, 2009). The closed form of an equation, when developed, can represent a correspondence view of the function. That is, one builds a rule to represent the relationship between an x-value and its associated y-value in the form of an algebraic equation $y = f(x)$. We sought to describe how teachers work within their available resources and constraints (Schoenfeld, 2011) to facilitate students meeting teachers' learning goals for understanding of equations of exponential functions.

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Theoretical Framing

We view knowledge or knowing from an enactivist epistemology (Maturana & Varela, 1992). Enactivism stems from evolutionary biology and conceptualizes an organism interacting and co-evolving with its environment. An organism "knows" within the environmental context if it acts in a way that is fitting and effective for the context (Maturana, 1988). Therefore, knowing or cognition is not a *thing* a person holds but "acting in a world that emerges in the doing itself" (Maheux & Proulx, 2015, p. 212). Knowledge is the body of effective behaviors and the underlying cognition that engender one to perceive the situation and categorize which behaviors are effective (Varela, Thompson, & Rosch, 1991). Learning is "a reciprocal activity — the teacher brings forth a world of significance with the learners" (Towers et al., 2013, p. 425).

In classroom mathematics discourse, there is both "*doing* something (some thing) recognizable as *mathematics*, but also producing *mathematics* as this thing that we are *doing* when we do what we do" (Maheux & Proulx, 2015, p. 215). The mathematics is the "world of significance" that the teacher brings forth with the learners by implementing a plethora of decisions both to set up the environment and to respond to (and with) the students. The mathematical ideas (i.e., concepts, patterns, principles, procedures, relationships) that emerge are not isolated entities. They are connected to and built up from other ideas with forms of coherence and structure fitting for the doers of the mathematics. As Towers et al. (2013) indicate, enactivism prompts observing "the relationship between things in a mathematical environment (ideas, fragments of dialogue, gestures, silences, diagrams, etc.), rather than to what each of those things might mean or represent in their own right and for the individual generating them" (p. 425). We conceptualize the mathematics as the emerging ideas in the discourse of the mathematical activity and the connections made to build up and connect the new ideas from and to other ideas. We define *knowing for teaching mathematics* as the teacher decisions to perturb the learning environment and to participate with students to influence the emergence of mathematics in ways they deem effective for student learning.

Interested in describing experienced teachers' knowing for teaching exponential functions enacted in whole class discourse (WCD), we sought to describe the nature of the mathematical world that emerged as well as the teacher activity that supported its emergence. We describe the nature of the mathematics in terms of the emergent ideas and the connections, consistency, and justifications offered in the discourse. Our research investigated: With respect to the *equation of an exponential function,* what is the nature of the mathematical ideas promoted in WCD and what instructional decisions supported the mathematics to emerge and connections to develop?

Methods

As part of a larger study, we collected data from 16 high school teachers engaged in the teaching of exponential function topics in the courses of Algebra I, Algebra II, College Algebra, and Pre-Calculus. All teachers were identified as highly effective and experienced by their administrators or peers and had obtained master's degrees. The corpus of data included classroom observations and interviews regarding teacher instruction. This study focused on the WCD of three teachers, Gabe, Evelyn, and Abby who had 28, 19, and 24 years of teaching experience respectively. Gabe and Evelyn taught College Algebra while Abby taught IB Math 3 (equivalent to Algebra II). We focused on these three teachers because we perceived surface-level similarities in their lesson structures for introducing the equation of an exponential function or geometric sequence.

We transcribed the classroom observations and partitioned each lesson into smaller segments of episodes and sequences (Wells, 1996) based on transitions of classroom tasks. Transcribed interviews included images of documents the teacher referenced during the interview when appropriate. Using the classroom videos and transcripts we developed concept maps representing the mathematics in WCD, noting connections made between the mathematical ideas (Leinhardt & Steele, 2005). Looking across the concept maps in three teachers' classrooms, we identified three common themes: the role of the independent variable, the relationship between the recursive multiplication and exponential form, and the definition of exponential. We created narratives describing the emerging mathematics and contributing teachers' decisions for each teacher and theme (i.e., nine narratives in total). We used comparative methods to identify contributing decisions and teacher actions in cross case analyses of the three teachers.

Findings

These findings focus on the first theme that emerged during the analysis of the classroom discourse surrounding writing the equation of an exponential function; specifically, the relationship between the value of the independent variable and the exponent of the algebraic expression.

Gabe Narrative

To write each of the three equations, students were told to complete a table of values given at least four consecutive entries and $\Delta x=1$ (see Figure 1). Students only needed to determine the *y*-intercept (i.e., the value when $x=0$) for y_5, the table included the *y*-intercepts for y_6 and y_7. Once students identified multiplying by two to move down the entries in the table (y_5), Gabe reviewed using exponents by leading students through going from 1 to 16 in the table via repeatedly multiplying by two. Gabe asked students how 1(2)(2)(2)(2) could be re-written, thus encouraging them to recall their work with exponents. After writing $1(2)^4$ Gabe asked students for the exponent for y_5. In using an example (y_5) students were told and then reminded (in y_6) that the exponent represents repeated multiplication, so x was the exponent in the general equation. The following excerpt from the WCD highlights Gabe's implicit connection between using exponents in the equation for y_5 due to repeated multiplication and the exponent being x in the equation.

T: So, this <the (2)(2)(2)(2) > would be two to what power? You said something power.
S: Three... fourth.
T: To the fourth power. <writes $1(2)^4$ under the expression 1(2)(2)(2)(2)>
T: So, what we're doing each time is we're multiplying by two, what's our exponent going to be?
S: x
T: Just x. <writes $y_5=1(2)^x$ > So, that's the equation for the first one. [E1:S4:L13-18]

In discussing writing the equation for y_6 Gabe stated, "[n]ow when we write it in this form, it's what we're multiplying by each time because that's what the exponent represents, a series of multiplications" [E1:S7:L3]. When writing the equation for y_7, the exponent was written but not mentioned. During notes, when introducing $y=ab^x$, Gabe defined the exponent saying, "[a]nd then our exponent's the number of times that we're going to be doing it" [E2:S1:L1].

Due to the structuring of the task (i.e., having students write equations from a table of values void of context), defining the independent variable was not needed. Rather, x was implicitly defined as being the exponent because the exponent represents repeated multiplication. Additionally, a need did not exist for making an explicit correspondence between defining the independent variable and stating that x was the exponent. In moving from $1(2)^4$ to asking students what the exponent would be for y_5, Gabe focused exclusively on the y_5 column and did not discuss that the four in the exponent connected to the row corresponding to $x=4$. In fact, the x column of the table was not included on the note sheet that students were given (see Figure 1).

y_5	y_6	y_7
	486	
	162	1000
2	54	1100
4	18	1210
8	6	1331
16		

Write equations for functions 5-7.

Exponential functions:

$y = ab^x$ or $y = ar^t$ or $y = a(1+r)^t$

Compound Interest Formula $A = P\left(1 + \frac{r}{n}\right)^t$

Figure 1: Re-creation of a portion of the note sheet Gabe created and gave to students

Evelyn Narrative

Evelyn introduced writing the equation of an exponential function in two lessons involving the discussion and then summary of three tasks (see Table 1). The prompts for the tasks provided the value of a quantity at a point in time and information that the quantity grew by a multiplicative factor over a set time period (i.e., each day or each year). In small groups, students determined the value of the quantity at other points of time, utilizing recursive multiplication or division. In WCDs, Evelyn oriented students to represent their computations as numerical expressions in a table and then generalize to an equation.

On the One Grain of Rice task, students worked to find how many grains of rice a girl would have on the thirtieth day if she started with one grain of rice and the number of grains doubled each day. As Evelyn predicted prior to the lesson, students reached different answers depending on whether they labeled the starting value Day 1 or Day 0. She considered having students compare the effects of labeling the staring value as Day 1 or Day 0, but in class she chose to have the class come to a consensus in choosing to state the girl received one grain of rice on Day 1, meaning the point (1,1) was in the data set. In WCD, Evelyn noted there could be another choice of creating that point as (0,1). The choice would affect the final answer but not their process. Rather than discussing the effect of the choice, the focus shifted to representing the situation and the students' computations in a table.

The tables created for the One Grain of Rice and Social Media WCDs captured the repeated multiplications used to calculate the values. Evelyn led students to rewrite the expressions as repeated multiplication and then exponential expressions. By recognizing a pattern down the right column of the table, the class generalized that to calculate the value for any point in time multiply the starting value by the growth factor some number of times.

To generalize beyond the table, Evelyn asked students to find expressions for larger values in the table (i.e., the number of users in year 2052). The class discerned a relationship between the exponents in the expressions for the dependent variable and the value of the time variable. For example, students noticed the exponent of the expression for a given year could be found by subtracting five from the years since 2000. The class looked across the two columns to generalize the relationship between the value of the independent variable and the computational exponential expression to find the number of users (or grains of rice) associated with the value of the time variable. Therefore, they developed equations to find the value of the dependent quantity in year x or day d.

Evelyn then presented the contexts for each of the three tasks and the equations they found for each situation reminding students how they defined the independent variable (Table 1) and then replaced each of these with "x." The class made observations that each equation involved a time period and that the exponent was some sort of time period. Evelyn then presented the general form, $y = ab^x$ and the class discussed the role of each parameter. The meaning of x was given as "some time period" and was not defined as the value of the independent variable.

Table 1. Three task summary provided by Evelyn

One Grain of Rice Started with one grain of rice and doubled each day.	Social Media Started with 3.2 million users and tripled each year after 2005.	Fruit Flies Started with 5 flies and they quadruped each day of vacation.
Grains of Rice = 2^{day-1}	#Social Media Users = $3.2(3)^{year-2005}$	#Fruit Flies = $5(4)^{\#days}$

Abby Narrative

Before this observation, students spent time solving and presenting their solutions to the domino skyscraper task (http://threeacts.mrmeyer.com/dominoskyscraper/) which posed the question, "If you wanted to topple over a domino the size of a skyscraper, how many dominoes would you need?" Students were told, "a smaller domino can topple a domino that is up to 1.5 times larger in every dimension" and that the first domino was 5 mm tall. In four small groups, students generated solutions for several skyscrapers by guessing and checking, creating a table, and using an exponential equation. Abby began this class by shifting the conversation from the solution to the task to the equations the students generated.

Abby asked Group 4 to present their equation $y=5(1.5)^x$ and table for the domino task, telling them to define their variables and connect their table to their equation. They explained why their equation was $y=5(1.5)^x$, where x represented the domino number and y represented the height of the domino, and connected it to their table. After the presentation, the class worked in small groups to "make a very clearly defined table. Identifying your variables, alright? And matching it up with your equation, alright? You want to make sure your equation matches up" [E1:S2:L1].

In a small group discussion, Abby asked the students why their equation $u_n=5\cdot1.5^{n-1}$ differed from the one presented. The students offered that the difference of "minus 1" in the exponent was due to their choice to label the initial domino of height 5 the first domino instead of the zeroth domino in their table. The teacher engaged in a similar discussion with another group which had the equation $= \frac{450,000}{1.5^x}$. Although there is not more WCD on this point, Abby engaged with two of the four groups focusing on why their equation was different from the one presented. In these group conversations with Abby, students explained how they constructed their table, how it was different than the table presented and how that impacted their equation, with particular attention to how the independent variable was defined.

In WCD, Abby returned to this theme when developing with students the equation for the general term of a geometric sequence. She began by highlighting the equation $y=5(1.5)^x$ and its associated table. She connected it with their work on geometric sequences by noting that the equation generated a geometric sequence (the y column in the table), that 5 was the initial value, and 1.5 was the common ratio. She pointed out that this table labeled the initial domino as the zeroth term, but that the convention for geometric sequences was to label the initial term (domino) as the first term.

Abby highlighted the work of the group who produced the equation $u_n=5\cdot1.5^{n-1}$, indicating that their first domino was 5 and that it aligned with the initial term of a geometric sequence being called the first term. The class established that the difference in the exponents between the two equations reflected the difference between starting with $x=0$ and $x=1$. Abby connected this to transformations explaining the difference as one equation being the other shifted to the right one. Through an interactive discussion, the students connected the equation $u_n=5\cdot1.5^{n-1}$ to the general term of a geometric sequence of the form $u_n=u_1\cdot r^{n-1}$, where n is the term number, r is the common ratio, and u_1 is the first term. Abby concluded this episode with the comment, "These are equivalent. <Pointing to $y=5(1.5)^x$ and $u_n=5\cdot1.5^{n-1}$.> It's just a matter of defining your variable. Where your starting point is. But they are really equivalent equations" [E4:S6:L10].

Cross-Case Analysis

Differences existed in the ideas that emerged in the three teachers' discussions regarding the relationship between the exponent value and the independent variable in the development of the exponential equations. For Gabe, the idea of the connection between the x-variable and exponent was minimal. In both discussion of a specific function (y_5) and the general forms, Gabe's class indicated the value of the exponent corresponded to the number of multiplications by the constant multiplier and the exponent was x. No connection was made between x as the number of multiplications and the x column of the table. The x column was only referred to when finding the y-intercept as the value of y that corresponded to an x-value of zero and consistently indicated the place where they start doing the repeated multiplications.

For Evelyn, consistency existed in providing intellectual need and opportunity to make a connection between the expression that represents the y-value and the corresponding value of the time variable in each of three contextualized tasks. The class developed tables by thinking about changes in the independent variable by one, implicitly attending to the x-value but focusing on the relationship between values in the y-column. Evelyn asked students to write an expression for a large x-value, skipping values in the table. Thus requiring students to look between columns of the table and generate the relationship between the x-value and the exponent value. They used this observed relationship to write the final equation for the functions in question. These conversations emerged in the particular discussions of generating the equations from tables but did not emerge in the end discussions regarding the general form. In fact, Evelyn noted in her interview that finding the value of the exponent would depend on the specific problem context. Specifically, she said, "[u]m, so to see that form of the starting value, the base and then that the exponent relies on whatever the context of the problem is" [Pre-Int Obs2 08:35]. We did not see a connection made between the exponent value as a transformation of the independent variable and the exponent as counting the number of multiplications.

The student groups in Abby's class created their own equations to model the situation of toppling dominos. Consistently, Abby directed students to check their equations with their table; thus the exponents of the student equations could be modified to account for the values of the corresponding domino number (independent variable). Ideas emerged from the particulars of individual groups making different choices in their work on the same task. The students' equations differed based on how they defined the starting value (i.e., Domino 0 vs. Domino 1). Abby built from the varied approaches of the students to motivate the generalizations providing standard language and definitions as needed. The result was a final general idea that the exponent of the general term of the sequence differed based on the labeling of the term number. Abby connected the idea to a horizontal transformation of functions.

The notion that different equations exist dependent upon defining the independent variable emerged in both Evelyn's and Abby's class due to the opportunity for students to create their own tables modeling the situation. The contexts of the domino and grain of rice tasks did not specify that the starting domino or day corresponded to a specific value of the independent variable; therefore, students made different choices. Abby allowed student groups to develop their own equations and made sure students matched all representations of the situation (sequence, table, context, equation, and later graph). She chose to have multiple groups present their solutions; therefore, the class saw three equations meant to capture the same relationship. The different equations presented a need for Abby to provide some closure to the idea. Students in Evelyn's class did not create equations on their own, but they made different choices in how they labeled the first value. Evelyn chose to lead the WCD of creating the equation and opted to label the independent variable as most of the students did. Not all students saw how the equation might look different based on defining of the independent variable.

All three classes developed and filled in tables, initially, through applying repeated multiplication. To write the equation, Evelyn and Abby's classes attended to the relationship between the x-value and the y-value or exponents of the expression of the corresponding y-value. Evelyn facilitated attention to this relationship by asking students to skip values in the table to find the y-value associated with a large x-value. Abby asked students to check to see if their equation was correct by paying attention to input-output correspondence in the tables.

In Gabe's class, it is unknown if and how students attended to the x-column of the table to determine the equation due to the nature of the tables presented. The values of x in the tables presented were equal to the number of multiplications. Referring to the x-column was not necessary since the rows of the table increased in Δx values of one and the tables provided a row corresponding to $x = 0$. When Gabe asked students what the exponent of the equation should be and a student said, x, it was unclear if x referred to the corresponding input for an output in a single row of a table, if x was a generalization for counting the number of multiplications, or if the exponent was x due to prior knowledge that exponential equations have an x in the exponent.

All three teachers included WCDs toward the end of the lessons providing general forms of the equations of exponential functions (or geometric sequences). Table 2 summarizes the language teachers used for these WCDs. Gabe provided his informal language tied to the process of finding the equations in the table. Evelyn asked students to compare and notice similarities of the structures of the three equations generated. Students described the parameters using their own language which tied to the class's previous mathematical activity. Using the explored domino sequence and equations as examples of the parameters to introduce vocabulary, Abby provided formal definitions of term, term number, and common ratio prior to introducing the general term. Abby's generalizing discussion was the only one which described the exponent of the equation as a transformation of the independent variable.

Table 2. Language and origin of language when defining a general exponential form.

Gabe	Evelyn	Abby
$y = ab^x$ *a:* "a-riginal", y-intercept, value at 0 b: What you're multiplying by *x:* How many times you multiply	$y = ab^x$ *a:* starting value *b:* what you're multiplying by each time period *x:* some sort of time period *y:* total amount of stuff	$U_n = U_1 r^{n-1}$ U_1: First term *r:* Common ratio n: Term number U_n: nth term
Teacher-provided language	Student provided language	Teacher-provided language

Discussion and Conclusions

The enactivist lens prompted us to not only notice single ideas but the relationships and connections among utterances in the discourse which emerged from the nature of the mathematical activity in the room: the activity of creating equations to describe the data of the tables. While the structures of these lessons are similar, attending to connections among ideas as facilitated in the mathematical activity allowed us to notice if and how the thread of the role of the independent variable was integral to the activity. Viewing from an enactivist lens, the teachers' knowing is seen from the mathematical worlds they facilitate to emerge in that context.

Gabe provided tasks incorporating functions and representations of those functions where implicit minimal attention to the x-column or relationship between the x and y values was sufficient. Labeling the exponent as x worked for every equation. He presented the general forms of the equation in a way that aligned with the taught process which included simple procedures and easy-to-remember language. The mathematics of Gabe's classroom was largely characterized by closed-form mathematics, one-word responses, and narrow examples facilitating single ideas to emerge sequentially. We posit Gabe's facilitated instruction prioritizing a mathematics that might be easy for students to replicate without error.

Evelyn asked questions encouraging students to notice and test patterns (e.g., patterns in the relationship between the values of the independent variable and the exponent in the expression for the corresponding y-value). Connections made between ideas existed within the tables which were artifacts of the activity of the mathematical discourse. The inductive reasoning repeated at a higher level when the class compared the structures of the three generated equations and $y=ab^x$. The generalizations about exponential functions were tied to the idea of multiplying a starting value repeatedly to find a total rather than formalizing a relationship between the independent and dependent variable. Evelyn accepted the language and definitions offered by the students rather than providing formal language. Her decision-making prioritized students making and testing generalizations based on the inductive reasoning *inspired by* the collective mathematical activity.

In Abby's class, the attention to the relationship between the independent variable and the equation was grounded in the class practice of reconciling the multiple representations of the growing quantity (context, table, multiple versions of equations, and graphs). Prior to generating the general form of a geometric sequence, Abby provided vocabulary for the relevant parameters of the domino task which corresponded to finding the value of term *n.* Her decisions positioned students as doers of mathematics, enabling students to identify the pertinent mathematical concepts. The decisions positioned her to provide a shared formal language connecting the class' mathematical activities of representing the domino task, making sense of others' representations, and making generalizations about properties of geometric sequences.

We viewed teacher knowing through the decisions teachers make as they engaged in activity (including mathematical activity) with their students to promote the emergence of a mathematical world. Taking this lens freed us from concerning ourselves with the individual actions of the teacher and each student to notice the nature of the activity (pedagogical and mathematical) which seemed to facilitate the emergence and connections among ideas. This work suggests developing teachers as decision makers by engaging PSTs in considering and a deliberate analysis (Brown & Coles, 2011) of the mathematical worlds afforded by collections of teacher moves.

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MATHEMATICAL KNOWLEDGE FOR TEACHING:

BRIEF RESEARCH REPORTS

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A CLASSROOM EXPERIENCE: VECTOR CONCEPT

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According to several studies, working with vectors is a problem for students from different educational levels. In this document we propose a Planeacion Didáctica Argumentada (PDA) to address the vector concept at lower secondary school level, emphasizing its features (magnitude, direction and sense) such that, it facilitates the formalization of the mentioned concept in a mathematical context. After implementing the PDA, we evidence that with the right instruction, students can develop skills to work with vectors. After analyzing the results, is possible to establish some aspects that must be considered while teaching vectors, for example: the aspects of the teacher, students, curriculums and the previous physics and math necessary contents.

Keywords: Teaching Activities and Practices, Interdisciplinary Studies.

Introduction

Vector is a mathematical object whose operation goes further than conventional numerical treatment (Zea, 2013); in Physics it can be characterized by displacement, speed, acceleration and force. Working with vectors in Physics at an early stage will allow students to develop skills and have a better understanding of the topic when they are working with a mathematical context later on. (Poynter &Tall, 2005). There are investigations that have focused on identifying difficulties when working with vectors (Knight, 1995; Nguyen & Meltzer, 2003; Flores et al., 2007; Flores et al., 2008; Mora, 2011; Barniol & Zavala, 2014 and Barrera et al., 2016); the common denominator of these investigations is that students have difficulties mainly with the concepts of scalar quantity and vector quantity, magnitude and direction of a vector, etc. Knight (1995) and Nguyen & Meltzer (2003) agree that students' misconceptions about vectors is the absence of a clear idea about vector. However, despite the fact that the most recurring difficulties of students are known and have been reported to occur at the basic level (Knight, 1995), didactic proposals do not specify how to work with the vector concept at the basic level and it is not known what could enable the formalization of this concept as a mathematical object.

Consequently, the objective of this document was to design and implement a didactic proposal to address the notion of vector at lower secondary school level, emphasizing its characteristics: magnitude, direction and sense. And due to the educational context, the Planeación Didáctica Argumentada (PDA) model was used as a methodology to achieve the objective.

Method

The actions that were carried out to during the research are described below.

The curriculum

The Natural Sciences curriculum of basic education in Mexico was consulted to know the lower secondary school topics related to vectors, as well as to find out if the curriculum suggests any specific treatment for this concept and with this, compile elements to include in the didactic proposal, such as: the curricular standards, the didactic approach, the competitions, etc.

The results of the review indicated that the curriculum does not consider a specific topic on the vector concept, therefore no expected learning was found for this concept; however, it is stated that: "the student must be able to represent forces with vectors and add them by the parallelogram and polygon methods."

The textbooks

The textbooks of the following authors were reviewed: Barragán (2013); Chamizo (2014); González, Lluis and Pita (2014) and Cuervo (2015), to see the characteristics of activities involving vectors. This information was taken as a starting point for the design of activities in the didactic proposal and, to take the aspects that the textbooks do not deepen and that are necessary to address vectors.

We note that the vector concept is implicit in the topic of "motion" to represent displacements. In addition, in the topic of force, it is necessary to use vectors for the graphical representation and the graphical addition of forces. The revised textbooks lack of graphic examples and activities that promote vector learning, which is natural since the curriculum does not mention a particular type of work with the vector concept. We also note that different definitions of vector and the concept of direction are provided (based on the straight line that the vector contains, relative to a positive angle, and based on a reference angle).

Conceptual aspects for research

Through the reviewed literature, the fundamental concepts for studying vectors were identified, such as: magnitude, scalar magnitude, vector magnitude, vector, features of a vector (magnitude, direction and sense). These concepts were addressed in the activities of the didactic proposal.

The definition of vector depends on the context in which is being worked, in this case it was used in the context of mechanics, since it is the one that fits the contents of the basic level textbooks reviewed. So: Vector is a quantity that has magnitude, direction and sense, according to Dávila and Pajón (2015). In addition, the arrow was used as a vector representation (Hibbeler, 2004).

Planeación Didáctica Argumentada (PDA)

In this research, the Planeación Didáctica Argumenta (PDA) was used as a methodology to design and implement the activities on the vector concept. The PDA enables students to learn content of their interest, and teachers allow them to review what has been done and what is achieved (Sánchez, 2016).

To design the PDA, the following elements were considered (Sánchez, 2016): internal and external context of the class, diagnosis of the class, preparation of the class plan (in three moments: beginning, development and closure), material resources, organization of the group, space, time, assessment and the argumentation of the planning (justification of the chosen teaching strategies).

The elements mentioned above were organized in a planning format, which can be designed at the teacher´s discretion, provided that the essential characteristics of the PDA are considered. The following describes how the PDA was implemented.

Description of the internal and external context. The PDA was implemented in a secondary school with an eighth grade group of 16 students (9 girls and 7 boys) between the ages of 13 and 14, during the 2016-2017 school period. The institution is located in Zumpango del Río, Guerrero, Mexico.

Group diagnosis. At the time of the implementation of the PDA, the group examined was in the fourth school term and had no Physics teacher. Through an interview, a substitute teacher stated the existence of difficulties in mathematics, a lack of participation in class by students, and the fact of there had not been any work with vectors.

Students´ previous knowledge. Evidence was found in class notes that students had worked with displacement, velocity, and acceleration, but there was no mention of the vector characteristics of these concepts.

Class plan. Six activities were designed to be carried out in three phases: initiation, development and closure. In activities 3, 4 and 5, we worked with the graphic representation of vector, and in activities 1, 2, 3 and 6 with the definition of vector. Prior to the implementation of the activities, we worked with angle measurement and the use of the protractor. The material resources, organization, space and time were proposed accordingly to the characteristics of each activity.

To assess learning, a checklist with insufficient, sufficient and outstanding scales was used to determine the performance of the students based on the objectives of each activity. As an example, Activity 3 ("Jimena goes home and to school") is presented below. A time of 90 minutes was proposed to solve this activity following 4 objectives:

1. Establishing the graphical representation of vector quantities using arrows. Use the vector notation $\vec{X}$.
2. Applying the characteristics of a vector: magnitude (the distances that Jimena walks), direction (horizontal, vertical, inclined) and sense (up-down, left-right), based on the type of route Jimena takes.
3. Defining the vector as a quantity that has magnitude, direction, and sense.
4. Recognizing and express events around us that can be represented with vectors.

Planning Argument. The activities involved contexts known by the students and in accordance with the curriculum guidelines; for their resolution, inexpensive and easily available materials were required. The scenarios were chosen according to the needs of each activity and in the spaces of the educational institution. Individual, team and group participation was encouraged in order to contribute ideas to formalize concepts. The checklist for the evaluation record allowed continuous observation of the progress of students in the development of tasks, actions, procedures, skills and attitudes.

Results

The analysis of results was carried out based on the suggested objectives for each activity and taking into account a scale to assess performance (insufficient, sufficient and outstanding). Below are some results of Activity 3.

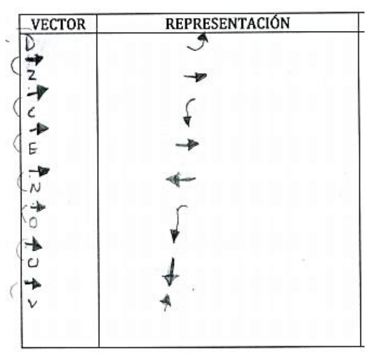

Figure 1: Student result who drew curves instead vectors.

The students managed to describe Jimena's route in words, considering the distance (magnitude), direction and sense; however, they did not discern that Jimena walked twice in the same direction but in a different route. Five students correctly used an arrow (vector) to represent the route, while two others drew "curves" instead of vectors (Figure 1).

To express the features of the distance traveled by Jimena, two students mistakenly determined the direction as "inclined" of some vectors and the sense as "lying down" instead of referring to "left or right". Three students correctly determined the magnitude, direction and sense of the vectors (see Figure 2).

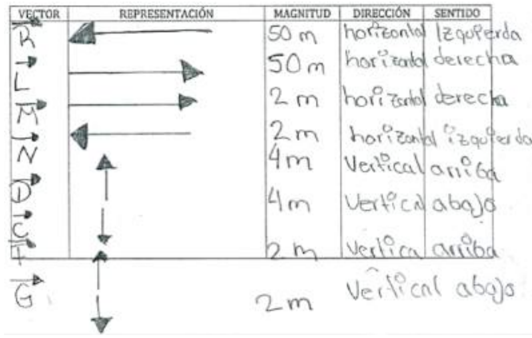

Figure 2: Student who obtained outstanding performance in determining the characteristics of a vector.

To achieve the formalization of the definition of vector, first, we mention that examples of everyday situations can be represented with vectors. Later, with the teacher's guidance, the vector concept was defined, and its features were also defined (Figure 3).

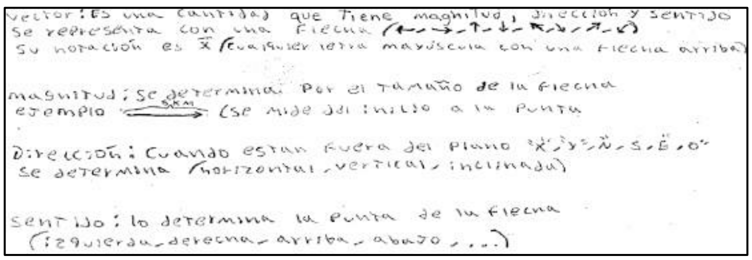

Figure 3: Student evidence in relation to the definition of vector and description of its features.

Conclusions

The reviewed literature made it possible to know aspects that students require to be able to work with vectors, know strategies that have been implemented and the possible difficulties that they may face.

The implementation of the PDA allowed us to reflect on the role of the teacher, since the fact of designing a class plan requires constant precision and adjustments in their practice, as well as identifying opportunities for improvement. In particular, this work made possible to make connections between Physics, Mathematics and everyday life.

This document suggests developing topics that are easier for students prior to working with vectors, involving different representations or contexts in the same topics, as we believe that those will allow us to know in more detail the learning styles of students. We propose, for the assessment, to implement the use of an evidence portfolio, in addition to the checklist, so that the student takes into account all the activities.

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UNA EXPERIENCIA DE CLASE: CONCEPTO VECTOR

De acuerdo con varios estudios, trabajar con vectores es un problema para estudiantes de diferentes niveles educativos. En este documento proponemos una Planeación Didáctica Argumentada (PDA) para abordar el concepto vector en el nivel secundario, enfatizando sus características (magnitud, dirección y sentido) de tal manera que, se facilite la formalización del mencionado concepto en un contexto matemático. Luego de implementar la PDA, evidenciamos que con la instrucción adecuada, los estudiantes pueden desarrollar habilidades para trabajar con vectores. Luego de analizar los resultados, es posible establecer algunos aspectos que deben ser considerados en la enseñanza de vectores, por ejemplo: los aspectos del profesor, estudiantes, plan de estudios y los conocimientos previos de física y matemática necesarios.

Palabras clave: Actividades y prácticas docentes, Estudios interdisciplinarios.

Introducción

El vector es un objeto matemático cuyo operatividad va más allá de todo tratamiento numérico convencional (Zea, 2013), en Física se puede caracterizar en desplazamiento, velocidad, aceleración y fuerza. Trabajar con vectores en Física en una etapa temprana permitirá a los estudiantes desarrollar habilidades y tener una mejor comprensión del tema cuando posteriormente trabajen en un contexto matemático. (Poynter yTall, 2005). Existen investigaciones que se han centrado en identificar las dificultades al trabajar con vectores (Knight, 1995; Nguyen y Meltzer, 2003; Flores et al., 2007; Flores et al., 2008; Mora, 2011; Barniol y Zavala, 2014 y Barrera et al., 2016), el común denominador de estas investigaciones es que los estudiantes tienen dificultades principalmente en los conceptos de cantidad escalar y cantidad vectorial, magnitud y dirección de un vector, etc. Knight (1995) y Nguyen y Meltzer (2003) coinciden en los estudiantes no tienen una idea clara sobre vector. Sin embargo, a pesar de que se conocen las dificultades más recurrentes de los estudiantes y que se ha informado que estas ocurren en el nivel básico (Knight, 1995), no se ha especificado en propuestas didácticas cómo trabajar con el concepto vector en el nivel básico, lo que podría posibilitar la formalización de este concepto como objeto matemático.

En consecuencia, el objetivo de este documento fue diseñar e implementar una propuesta didáctica para abordar la noción de vector en el nivel secundaria, destacando sus características: magnitud, dirección y sentido. Y debido al contexto educativo, se utilizó el modelo de Planeación Didáctica Argumentada (PDA) como metodología para lograr el objetivo.

Método

Las acciones que se llevaron a cabo durante la investigación se describen a continuación.

El Plan de Estudios

Se consultó el Plan de Estudios de Ciencias Naturales de educación básica en México para conocer los temas relacionados con vectores señalados en el nivel secundaria, así como conocer si el plan de estudios sugiere algún tratamiento específico para el concepto vector y con ello, recopilar elementos a incluir en la propuesta didáctica, tales como: los estándares curriculares, el enfoque didáctico, las competencias, etc.

Los resultados de la revisión indicaron que el plan de estudios no considera un tema específico sobre el concepto vector, por lo que no se encontraron aprendizajes esperados sobre este concepto, sin embargo, se afirma que: "el alumno debe ser capaz de representar fuerzas con vectores y sumarlos por los métodos del paralelogramo y del polígono ".

Los libros de texto

Se revisaron los libros de texto de los siguientes autores: Barragán (2013); Chamizo (2014); González, Lluis y Pita (2014) y Cuervo (2015), para conocer las características de las actividades que involucran vectores. Esta información se tomó como punto de partida para el diseño de actividades en la propuesta didáctica y, para tomar los aspectos que los libros de texto no profundizan y que son necesarios para abordar el concepto de vector.

Observamos que el concepto de vector está implícito en el tema del "movimiento" para representar los desplazamientos. Además, en el tema de la fuerza, es necesario utilizar vectores para la representación gráfica y la suma gráfica de fuerzas. Los libros de texto revisados carecen de ejemplos gráficos y actividades que promuevan el aprendizaje de vectores, lo cual es natural ya que el plan de estudios no menciona un tipo particular de trabajo con el concepto de vector. También observamos que se proporcionan diferentes definiciones de vector y el concepto de dirección (basadas en la línea recta que contiene el vector, en relación con un ángulo positivo y con base a un ángulo de referencia).

Aspectos conceptuales de la investigación

A través de la literatura revisada, se identificaron los conceptos fundamentales para el estudio de vectores, tales como: magnitud, magnitud escalar, magnitud vectorial, vector, características de un vector (magnitud, dirección y sentido). Estos conceptos fueron abordados en las actividades de la propuesta didáctica.

La definición de vector depende del contexto en el que se esté trabajando, en este caso se utilizó en el contexto de la mecánica, ya que es el que se ajusta a los contenidos de los libros de texto de nivel básico revisados. Entonces: Vector es una cantidad que tiene magnitud, dirección y sentido, según Dávila y Pajón (2015). Además, se usó la flecha como representación gráfica de vector (Hibbeler, 2004).

Planeación Didáctica Argumentada (PDA)

En esta investigación se utilizó la Planeación Didáctica Argumenta (PDA) como metodología para diseñar e implementar las actividades sobre el concepto vector. La PDA permite a los estudiantes aprender contenidos de su interés, y a los profesores les permiten revisar lo hecho y lo logrado (Sánchez, 2016).

Para diseñar la PDA se consideraron los siguientes elementos (Sánchez, 2016): contexto interno y externo de la clase, diagnóstico del grupo, elaboración del plan de clase (en tres momentos: inicio, desarrollo y cierre), recursos materiales, organización del grupo, espacio, tiempo, evaluación y la argumentación de la planeación (justificación de las estrategias docentes elegidas).

Los elementos mencionados anteriormente se organizaron en un formato de planeación, que puede diseñarse a criterio del profesor, siempre que se consideren las características esenciales de la PDA. A continuación se describe cómo se implementó la PDA.

Descripción del contexto interno y externo. La PDA se implementó en una escuela secundaria con un grupo de octavo grado de 16 alumnos (9 mujeres y 7 hombres) de entre 13 y 14 años, durante el período escolar 2016-2017. La institución está ubicada en Zumpango del Río, Guerrero, México.

Diagnóstico del grupo. En el momento de la aplicación de la PDA, el grupo examinado estaba en el cuarto bimestre del período escolar y no contaba con profesor de Física. A través de una entrevista, un profesor suplente manifestó la existencia de dificultades en matemáticas, la falta de participación de los estudiantes en clase y el hecho de no haber trabajado con vectores.

Conocimientos previos de los estudiantes. En las notas de clase se encontró evidencia de que los estudiantes habían trabajado con desplazamiento, velocidad y aceleración, pero no se mencionaron las características vectoriales de estos conceptos.

Plan de clase. Se diseñaron seis actividades para ser desarrolladas en tres fases: inicio, desarrollo y cierre. En las actividades 3, 4 y 5 se trabajó con la representación gráfica de vector, y en las actividades 1, 2, 3 y 6 con la definición de vector. Previo a la implementación de las actividades, trabajamos con la medición de ángulos y el uso del transportador. Los recursos materiales, la organización, el espacio y el tiempo se propusieron de acuerdo con las características de cada actividad.

Para evaluar los aprendizajes se utilizó una lista de cotejo con escalas insuficientes, suficientes y sobresalientes para determinar el desempeño de los estudiantes en función de los objetivos de cada actividad. A modo de ejemplo, a continuación se presenta la Actividad 3 ("Jimena se va a casa y a la escuela"). Se propuso un tiempo de 90 minutos para resolver esta actividad siguiendo 4 objetivos:

1. Establecer la representación gráfica de cantidades vectoriales mediante flechas. Usar la notación de vector $\vec{X}$.
2. Aplicar las características de un vector: magnitud (las distancias que recorre Jimena), dirección (horizontal, vertical, inclinada) y sentido (arriba-abajo, izquierda-derecha), según el tipo de ruta que tome Jimena.
3. Definir vector como una cantidad que tiene magnitud, dirección y sentido.
4. Reconocer y expresar eventos que nos rodean que puedan representarse con vectores.

Argumentación de la planeación. Las actividades involucraron contextos conocidos por los estudiantes y de acuerdo con lo establecido en el plan de estudios, para su solución se requirieron materiales económicos y de fácil acceso. Los escenarios fueron elegidos de acuerdo a las necesidades de cada actividad y en los espacios de la institución educativa. Se incentivó la participación individual, en equipo y grupal con el fin de aportar ideas para formalizar conceptos. La lista de cotejo para el registro de evaluación permitió la observación continua del progreso de los estudiantes en el desarrollo de tareas, acciones, procedimientos, habilidades y actitudes.

Resultados

El análisis de resultados se realizó con base a los objetivos propuestos para cada actividad y teniendo en cuenta una escala para evaluar el desempeño (insuficiente, suficiente y destacado). A continuación se muestran algunos resultados de la Actividad 3.

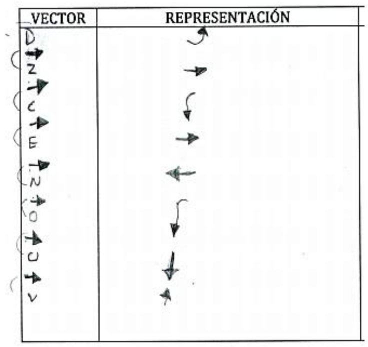


Figura 1: Los estudiantes dibujaron curvas en lugar de vectores.

Los estudiantes lograron describir en palabras la ruta de Jimena, considerando la distancia (magnitud), dirección y sentido, sin embargo, no distinguieron que Jimena caminó dos veces en la misma dirección pero en una ruta diferente. Cinco estudiantes utilizaron correctamente una flecha (vector) para representar la ruta, mientras que otros dos dibujaron "curvas" en lugar de vectores (Figura 1).

Al expresar las carcaterísticas de la distancia recorrida por Jimena, dos estudiantes determinaron erróneamente la dirección como "inclinada" de algunos vectores y el sentido como "acostado" en lugar de referirse a "izquierda o derecha". Tres estudiantes determinaron correctamente la magnitud, la dirección y el sentido de los vectores (ver Figura 2).

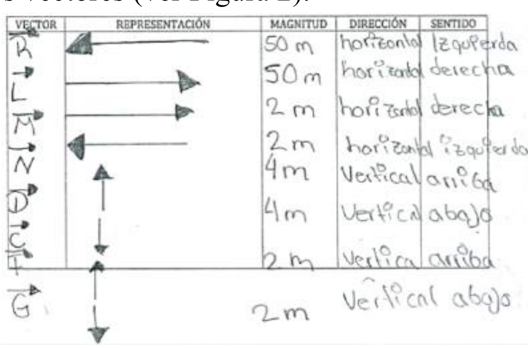
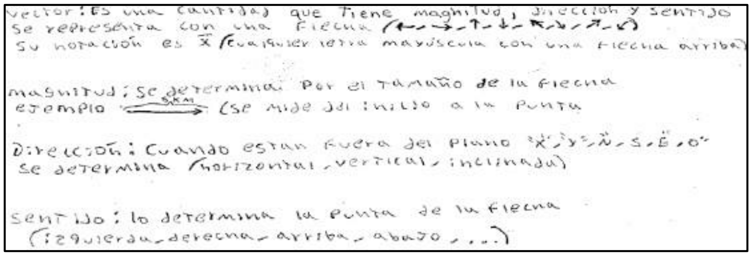


Figura 2: Estudiantes que obtuvieron desempeño destacado al determinar las características de un vector (actividad 3).

Para lograr la formalización de la definición de vector, primero se mencionaron ejemplos de situaciones cotidianas se pueden representar con vectores. Posteriormente, con la orientación del docente, se definió el concepto de vector y también se definieron sus características (Figura 3).



vector: Es una cantidad que tiene magnitud, dirección y sentido
se representa con una flecha (←, →, ↑, ↓, ↖, ↘, ↗, ↙)
su notación es $\vec{x}$ (cualquier letra mayúscula con una flecha arriba)

magnitud: se determina por el tamaño de la flecha
ejemplo $\xrightarrow{5KM}$ (se mide del inicio a la punta

Dirección: Cuando estan fuera del plano "x", "y", "N", S, E, O"
se determina (horizontal, vertical, inclinada)

sentido: lo determina la punta de la flecha
(izquierda, derecha, arriba, abajo, ...)

Figura 3: Producción de un estudiante en relación a la definición de vector y la descripción de sus características.

Conclusiones

La literatura revisada permitió conocer aspectos que requieren los estudiantes para poder trabajar con vectores, conocer las estrategias que se han implementado y las posibles dificultades que pueden enfrentar.

La implementación del PDA nos permitió reflexionar sobre el rol del docente, ya que el hecho de diseñar un plan de clase requiere precisión y ajustes constantes en su práctica, así como identificar oportunidades de mejora. En particular, este trabajo permitió establecer conexiones entre Física, Matemáticas y la vida cotidiana.

Este documento sugiere desarrollar temas que sean más fáciles para los estudiantes antes de trabajar con vectores, así como involucrar diferentes representaciones o contextos en los mismos temas, ya que creemos que lo anterior nos permitirá conocer con mayor detalle los estilos de aprendizaje de los estudiantes. Proponemos, para la evaluación, implementar el uso del portafolio de evidencias además de la lista de cotejo, con la finalidad que el alumno tenga en cuenta el total de actividades.

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EXPLORING THE INTERWOVEN DISCOURSES ASSOCIATED WITH LEARNING TO TEACH MATHEMATICS IN A MAKING CONTEXT

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In this paper, we aim to explore how prospective elementary mathematics teachers (PMTs) learn to teach mathematics through their engagement in a pedagogically informative Making experience. Grounded in a commognitive perspective, we define learning to teach mathematics as changes in any of these four discourse activities: mathematizing and identifying (Heyd-Metzuyanim & Sfard, 2012), in addition to pedagogy and designing. We present our analysis of one PMT's discourse activity, exposing Making as an effective venue for provoking all four discourses and revealing their intertwined nature, further illustrating that one's identity is as central to learning to teach mathematics as is their learning of mathematics, pedagogy, and design. We conclude with a discussion of the implications of these findings for the research and practice of teacher education.

Keywords: Teacher Knowledge; Teacher Education - Preservice; Affect, Emotion, Beliefs, and Attitudes; Technology

Teacher knowledge literature continues to evolve, with recent conceptualizations building on previous characterizations of distinctive knowledge domains in order to promote a wider focus on their integration (Scheiner, Montes, Godino, Carrillo, & Pino-Fan, 2019). In the current study, we adopt this perspective by viewing *teachers as learners* and foregrounding their identities (Sfard & Prusak, 2005) in order to recognize what affective, interpersonal, and social matters can bring to this conversation. That is, by honoring the interrelationship between the learning of mathematics and the learners themselves, we hope to move beyond the "static, explicit and objective" (Scheiner, et al. 2019, p. 161) outlooks on knowledge to recognize the blended nature of knowing (Scheiner, 2015). And because our teachers are designing manipulatives to share with children with the intention of promoting their mathematical learning, the promise of this approach is suggested by the proposition that teachers' "invention[s] of 'objects-to-think-with'... [offer] the possibility for personal identification" (Papert, 1980, p. 11).

Adopting a communicational perspective on learning (Sfard, 2008), our objective is to explore the premise that learning to teach mathematics can be seen as changes in discursive activities that include narratives about mathematics and identity. The following question guides this research: *As practicing and prospective elementary mathematics teachers Make new manipulatives and corresponding tasks to support the teaching and learning of mathematics, what might their discourses reveal about the epistemology of learning to teach mathematics?*

Theoretical Framework

Our theoretical framing is organized around the learning theories of *commognition* and *constructionism*. Commognition encompasses both interpersonal "communication" and individual "cognition" (Sfard, 2007, p. 570). Discourse, with its affective and social aspects, is central to commognition, and learning is seen through changes in discourse (Heyd-Metzuyanim & Sfard, 2012). The constructionist perspective adds a dimension of participation in a discourse community

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

with a view toward the learning that can happen during the process of making a shareable object (Harel & Papert, 1991).

Heyd-Metzuyanim & Sfard (2012) take a commognitive perspective to frame mathematics learning as the interplay between talking about mathematical objects (*mathematizing*) and talking about participants of the discourse (*identifying*). Sfard (2008) defines discourse as "a special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with (re-)actions" (p. 297). Discourse can include speech, gestures, and visual mediators (e.g., graphs, symbols, manipulatives) (Sfard, 2008). From there, identity is viewed as a collection of "narratives about individuals that are reifying, endorsable, and significant" (Sfard and Prusak, 2015, p. 16), and identity discourse is viewed as integral to the learning of mathematics [see also Graven & Heyd-Metzuyanim (2019)]. We supplement mathematizing and identifying with two additional forms of discourse that also may be relevant to the learning of mathematics: *pedagogy* (narratives about teaching and learning) and *designing* (narratives about design decisions). Thus, this framework provides us with a lens through which to study how the process of making a manipulative can provoke the four discourse activities of mathematics, pedagogy, design and identity, and help us to see the intertwined nature of a teacher's learning.

Methodology

This project is part of a larger study that aims to test and refine the hypothesis that a pedagogically genuine, open-ended, and iterative design experience centered on the Making and sharing of a physical manipulative for mathematics learning would be formative for the development of practicing and prospective elementary mathematics teachers' (PMTs') inquiry-oriented pedagogy. That study took place in the spring of 2019 in a graduate-level mathematics course for PMTs at a mid-sized university in the northeastern United States. Thirteen participating students comprised ten groups (yielding ten projects). The PMTs were tasked with designing and 3D printing a manipulative that would be shared with a child to support their meaningful learning of mathematics. Written assignments provided autobiographical information of the PMTs' experiences as mathematics students, as well as reflections on clinical interviews they conducted throughout the semester. Snapshots of the PMTs designs in progress are included in the data, as are the physical "printouts" of their manipulatives and video recordings of the course's design sessions.

For this project, we took an exploratory case study approach (Yin, 2009) that focuses on "Moira," a PMT whose initial design was a tool intended to simulate the "keep change flip" algorithm for fraction division. She thought this tool would make fraction division meaningful by providing a concrete representation in which a child could *physically* "keep" the dividend, "change" the division symbol, and "flip" the divisor. However, the course's teacher educator pushed back on Moira's idea by asking her, "When dividing fractions, *why* do you flip the second fraction and multiply?" In re-action to this prompt, Moira becomes intent on figuring out "why we flip the second," a move that signals a change in her mathematical discourse. In a subsequent session, we noticed that she deviates from this intention, opting instead for a new fraction tool design that could support meaningful comparisons of fractions with a broader age-range of students. Effectively, her new design takes familiar fraction strips and connects them end to end to turn them into eight partitioned rings that can be stacked vertically on a cylindrical pedestal.

We chose Moira as an exploratory case because the change in her mathematical discourse constitutes learning, but we also sought to understand this learning through the lenses of the other discourses. Accordingly, we invited her back after the course ended for a voluntary, follow-up, semi-structured and task-based interview (Ginsburg, 1997). In addition to helping us understand Moira's rationale for abandoning her earlier fraction division design, we viewed the manipulative she had made for fraction comparison as an instance of her design discourse and sought to use it to assess her understanding of fraction division. This interview was video-recorded and added to the corpus of Moira's data, along with written artifacts from the interview. That data was then analyzed through the conceptual lenses of the four discourse activities: identifying, mathematizing, pedagogy, and designing.

Results

In this section, we present two central results from the follow-up interview. The first result concerns Moira's decision to change her tool design, and our analysis of this choice through the discourses of mathematizing [M], Pedagogy [P], Designing [D], and Identifying [I]. Moira reflects, "I wanted to make something that could be interpreted in many different ways [*M/P/D*], that wasn't something that I was just forcing them to, like, all right, you have to use it this way. I wanted it to be able to be manipulated [*M/P/D/I*]." As she considers her initial "keep, change, flip" tool, she articulates, "You basically were just, like, flipping the fraction upside down in my initial tool and ... it was just not useful [*M/P/D*] ... So I decided to switch to comparing fractions and then I came up with this [fraction comparison tool] [*M/D/I*]."

These reflections reveal how Moira's initial decision to abandon her fraction division design is not just about mathematizing, but also about identifying: as a teacher, it is important to her that her students have the opportunity to develop their own ways of thinking about fractions with a tool that can be used in a variety of ways. Moira acknowledged that the pedagogy promoted by the instructor in the classroom was also part of her decision to change her design:

> Moira: Well, [the change of design] was because we were talking and you [the teacher educator] said, "you're just teaching them how to – you're just giving them a way to solve the problem." And I realized, you're right ... It wasn't helping them learn how to do a problem [M/P/D/I].

By switching to a design for comparing fractions, Moira can participate in the discourse endorsed in the teacher education classroom and honor the teacher she wants to be.

A second result related to Moira's learning emerges from the interviewers' awareness that her current tool *could* be used to make sense of fraction division and a question about whether Moira realizes this capability in her tool. The interviewers ask her about this possibility, prompting Moira's in-the-moment reflections: "½ divided by 2. ½, this divides it into two equal parts, and I know this equals fourths, so this is ¼" [*M/I*]. Then, in investigating 1 divided by 1/3, Moira takes the 1 and 1/3 ring, guesses the answer is 3, and says, "I know I can do it, and I'm seeing it, but I don't know how to describe it" [*M/I*]. Moira is using her tool to make sense of this problem when the interviewers prompt her to explain whole number division (e.g., 6 divided by 3). As Moira reasons through whole number examples [*M*], she exclaims, "Oh! So, so, if I am dividing 1 by ⅓, there are three thirds in 1, so it's 3! Yes! You can do division with these … Wow! Fractions make so much sense now" [*M/I*].

Although Moira's reflections on the ½ and (later) ⅓ examples seem the same, the shift from her use of a partitive conception of division to a measurement one gives her sought-after language to "describe" her tool's utility in her understanding of fraction division. Moira's mathematical discovery is intertwined with an expression that reveals how emotionally invested she is in this realization. The moment culminates in self-reflection: "Honestly, I'm so impressed with myself [*I*]. I did not think that it had this capability. I thought it was only for comparing fractions [*M/P/D/I*]. So we've learned something today, haven't we all…" [*I*].

As she uses her tool to think through fraction ideas [*M*], Moira comes to recognize its potential not only for her own learning, but also for teaching fraction division in a way that aligns with her identity as a teacher [*P/I*]. Moira's body language and energy substantiate her enthusiasm for this discovery. Finally, the whole experience leads her to identify herself as part of a community of learners who can struggle and reason as part of a sense-making process.

Conclusion and Implications

As in a woven tapestry, learning to teach mathematics weaves together four threads or discourses that are unique to a PMT's discursive experiences and particular to a learning community where inquiry pedagogy is promoted. In this sense, to characterize Moira's learning to teach mathematics as a complex structure of discursive activities interwoven in dialectical unity is to illuminate the brilliance of a tapestry threaded by what she wants to teach (mathematizing), how she wants to teach it (pedagogy), decisions about what resources to make available (designing), and the kind of teacher she wants to be (identifying). Zooming in on that tapestry might provide a view on a single thread of Moira's understanding of fraction division, but focusing on a single thread obscures the others with which it is interwoven. Collectively, these threads contribute to a more intellectually honest depiction of the "organic whole" (Scheiner, 2019, p. 165) that is learning to teach mathematics.

This project set out to explore the proposition that learning to teach mathematics can be credibly conceived as changes in mathematizing, identifying, pedagogy, and designing discourses. Our analysis of data related to Moira's experiences making a physical manipulative for sharing with a child reveals how her experiences provoked all four discursive activities, and revealed the intertwined nature of these discourses. This finding resonates with a view of mathematics teacher learning that emphasizes the blending and transformation of constituent knowledge domains into emergent knowing. It also resonates with an acknowledgment of the complex dynamics of mathematics teacher knowledge in action (Scheiner, 2019).

Our study also establishes that identity is as central to learning to teach mathematics as is the learning of mathematics, pedagogy, and design. The ensuing changes of discourse have revealed that although sometimes viewed as distinct, teacher learning domains are inherently connected to a PMT's identity. In light of research by Pratt and Noss (2010), its centrality can be understood in the context of a design project carried out in a Maker community where PMTs were engaged in creative activity, leveraging their personal experiences and invoking personal design decisions, reflections, and articulations. All in all, implications of this finding speak to the potential of interdisciplinary experiences like the design experience as venues for the meaningful learning of learning to teach mathematics within teacher preparation coursework.

Acknowledgments

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ELEMENTARY TEACHERS' DISCOURSE ABOUT MATHEMATICAL REASONING

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Even though mathematical reasoning [MR] is at the heart of several elementary curricula around the world, very little is known about the meanings given to MR by teachers. In this paper, adopting a commognitive perspective (Sfard, 2008), we aim at better understanding the different meanings teachers give to MR. To do so, we used the Mathematical Discourse for Teaching framework (Cooper and Karsenty, 2018) to analyze elementary teachers' discourse about MR. Through individual and collective interviews, we gathered the data. We coded the data by first highlighting the vocabulary used to give meaning to mathematical reasoning by the teachers and, secondly, by identifying the utterances linked to their Mathematical Discourse for Teaching. Analyses revealed that elementary school teachers' discourse about MR is coherent with the prescribed curriculum.

Keywords: Teacher knowledge, Elementary School Education, Reasoning and Proof, Mathematical Knowledge for teaching.

This paper presents an analysis of teachers' discourse about mathematical reasoning [MR]. The MR is at the heart of several curricula around the world. In Quebec, where this project takes place, it is one of the three competencies of the elementary and secondary school curriculum (MEQ, 2001)[1]. According to Loong, Vale, Bragg and Herbert (2013), primary school teachers feel confused or uncertain about the task of defining Mr. Likewise, the meanings given to MR could play an important role in how teachers approach it in class (Stylianides and Ball, 2008). Taking a commognitive perspective, we aim at describing the discourse about MR of elementary teachers. In doing so, we want to better understand how MR can be fostered in classrooms from the teacher perspective.

What do we know about Mathematical Reasoning discourse at elementary level?

Despite a growing interest in MR and teachers' practices, very little is known about the meanings given to MR by teachers as well as how they promote its development in the classroom. Clarke, Clarke and Sullivan (2012) asked 104 elementary school teachers which MR related terms, from a given list, they frequently used in math class. Only four terms—explaining, justifying, proving and reasoning—were chosen by more than 50% of the teachers. To evaluate a professional development [PD] that aims at fostering MR in elementary classrooms, Herbert, Vale, Bragg, Loong and Widjaja (2015) explore the different meanings given to MR by teachers from Australia and Canada (Vancouver) involved in the PD. Their analysis highlights seven meaning categories that elementary teachers may attribute to MR: 1) thinking; 2) communicating; 3) solving problems; 4) validating thinking; 5) forming conjecture, 6) using logical arguments for validating conjectures; and, 7) connecting different mathematical aspects. Those categories emerged from the discourse developed during the PD. However, what about terms used and meaning given by teachers who never participate in this kind of PD? This study aims at investigating this question.

[1] The three competencies are 1) to solve a situational problem related to mathematics, 2) to reason using mathematical concepts and processes and 3) to communicate by using mathematical language.

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Mathematical Discourse for Teaching

From a commognitive perspective, cognition and communication are two aspects of the same ontology, i.e. discourse. Discourses are constituted of keywords, visual mediators, rules of discourse, routines and generally endorsed utterances (Sfard, 2008). Knowledge and practices are two aspects of the discourse associated with the teaching of mathematics (Cooper and Karsenty, 2018). So, in the same way, the investigation of the meaning that mathematical reasoning can take in elementary school can be done by considering the teachers' discourse, rooted in practice.

Cooper (2014) reformulated the Mathematical Knowledge for Teaching framework of Ball et al. (2008) from a commognitive perspective: Mathematical Discourse for Teaching [MDT]. As the MKT framework, the MDT framework is divided into two types of discourse: Mathematical discourse and pedagogical discourse. Mathematical discourse [MD] consists of common content discourse (the mathematical discourse that is common to a large portion of educated society), specialized content discourse (mathematical discourse that is typical of teachers of mathematics) and discourse at the mathematical horizon (patterns of mathematical communication that are appropriate in higher grade levels). Pedagogical content discourse [PCD] consists of discourse about content and teaching, discourse about content and students and discourse about the curriculum and resources.

Adopting this framework, we can reformulate our aims as: What are the keywords, visual mediators, rules, routines and generally endorsed utterances that constitute MDT of elementary teachers in relation to MR?

Some methodological insights

The data used in this paper came from a larger project that aims to document how MR is defined and fostered by elementary and secondary teachers. Six elementary teachers with 2 to 16 years of experience participated in one 60 minutes individual interview (Pseudonymes: Martine, Gisèle, Aurélie, Jeanne, Alice, Agathe). Five of them participated in a 120 minutes collective interview. All interviews were video or audio recorded.

Three different moments constituted the individual interview. First, the interviewer asked the participant to recall a moment of her teaching or to present a task that she gives her students in which MR would be promoted. This allowed us to stay in an area known to the participant. Furthermore, it informed about the learning environment that teachers considered favourable to the development of MR. Then, two examples of tasks including one with a student's solution were presented to the participant. The participant was then invited to decide on the possibility for a student solving these tasks to develop MR or not and to justify their answer. If the answer was positive, she was asked to describe the possible reasoning processes in their own words. Finally, to close the meeting, the participant was invited to give in a few sentences her definition of MR.

The collective interview sought to encourage exchanges between practitioners around MR so as to bring out the discursive elements shared by them. The first part of the interview aimed at defining MR. The interview therefore began with the question that had ended the individual interview: "How do you define mathematical reasoning in a few words or sentences?" This was followed by an activity where participants constructed a conceptual map with vocabulary words widely used to define MR during individual interviews or in the literature. The second part of the interview was to see how the teachers could reinvest the conceptual map to comment on students' written work. Finally, the group interview ended once again by offering each participant the opportunity to add something related to their definition of MR.

In order to analyze the data, the videos and audiotapes were viewed/listened to repeatedly, and transcribed (Powell, Franscisco & Maher, 2003). Using Nvivo software, a first layer of coding made it possible to highlight the keywords used to give meaning to MR by the teachers. A second layer of coding made it possible to identify what seem as endorsed utterances linked to MDT for this group of teachers. By focusing mainly on keywords and endorsed utterances, discourse about the curriculum and resources [DCR] was particularly highlighted by the analysis.

Mathematical Discourse for Teaching and Mathematical Reasoning

When asked specifically to define MR, teachers use different keywords that seem to come from a common discourse about MR. For example, MR is linked to logic and argumentation, which are two words used to define reasoning in general dictionaries. But other keywords are more specific. For Martine, MR is to communicate and understand why. Similarly, Aurelie defines MR as explaining why. For Agathe it is also to communicate but to explain how. Gisele refers to MR as applying concepts and explaining what you have done. Alice used the metaphor of the toolbox. For her, MR is knowing when to use your toolbox, justifying and identifying and extrapolating patterns. Jeanne refers to organizing, thinking and making sense.

In relation to the PCD of the participants, generally endorsed utterances are usually embedded in DCR. Here are two illustrative examples.

Analyzing, making choices, applying, justifying and Mathematical Reasoning

In the teachers' discourse, analyzing, making choices, applying and justifying are important aspects of MR. Most teachers refer to those terms and they usually do so specifically by referring to the evaluation grid provided by school boards and based on MELS (2011) document.

> Aurelie: it's the evaluation criteria. 30 points for the analysis, 50 points for applying it … The last evaluation criterion is justifying, with 20 points.
>
> Gisele: Once I understand, I have analyzed the problem, then I have to make choices in what I know and what I think that will help me to reason with it.

This grid also *renders MR processes a linear structure* in the teachers' discourse as illustrated by Gisele's utterance above. Moreover, it is possible to draw a parallel between the grid and Pólya's problem-solving model: 1) understand the problem that is similar to analyzing; 2) develop a plan or make choices; 3) implement the plan or apply; and finally, 4) verify or justify. This is what Agathe feels in connection with MR:

> Agathe: Listen… I have the impression that reasoning with MR, well, this is the old one … this is the old problem solving from 15 or 20 years ago.

The criteria for assessing MR competency therefore play an important role in the discourse on MR. This role contributes to blurring the discourse on problem solving and MR.

Problem Solving and Mathematical Reasoning

In addition to being used as a quasi-synonym for MR, problem solving takes three other meanings for the teachers. First, it's a pedagogical method that can foster MR. Second, it's a competency evaluated with a particular type of task. Third, it's the type of task that evaluates problem solving. Those last two meanings are embedded as for both, problem solving is seen as more global and complex than MR.

> Alice: Well, that's why it's interesting to teach with problems too. So, not to make problems after the concepts, to bring the concepts with the problems.
>
> Martine: Well, the link I do between solving and reasoning is that… In fact, well, a situation to learn and evaluate, it should be complex. So, for sure, every child can have a different answer. Then, he [the child] uses the concepts uh that we include in reasoning [competency] because there is a need and then goes and solves it. So, I think reasoning is like a prior to solving [competency] because it is part of knowledge.

We can link those elements of discourse to the type of tasks used by teachers to evaluate each competency. Both types of task, namely situational problem and problem using application (MEQ, 2001), have different characteristics. The wording of the former is longer, includes a context, many steps to do, many concepts and processes to use and, as the students have to make choices in the data in order to solve the task, many solutions are possible (Lajoie & Bednarz, 2012). The wording of the latter is usually shorter, with one or two steps and the students have to choose the concepts and processes needed to solve it.

Discussion and Conclusion

Similarly to Herbert et al. (2015), the meanings given to MR by those 6 elementary teachers are broad and manifold. Likewise, as shown by the partial analysis presented, those meanings are tied to and somehow limited by teachers’ DCR. In fact, the discourse found in the Quebec curriculum (MEQ, 2001), just like the one in math education literature, is also quite blurry (author, year). As MR is a competency which must be assessed with a specific grid and so-called *problem using application* task, this grid greatly colours their whole discourse. However, the evaluation criteria seem to favour a linear vision of the MR activity. Although questioned by the teachers, this linear vision of MR conveyed by the grid could limit learning opportunities for students as it gives the impression that MR is a series of steps to follow. Moreover, the criteria make it difficult for teachers to differentiate problem solving competency from MR otherwise than by the type of task used.

Like the discourse in the mathematics education community, the discourse of teachers is formed from a set of discourses related to different fields: educational institutions, psychology, pedagogy, mathematics education. These discourses are sometimes incommensurable. Thus, as they point out, enriching and clarifying the vocabulary related to the MR, whether in the curriculum or in training could, among other things, open up new possibilities for developing it in the classroom:

> Agathe: But that proves that we have to have a common language… We teach mathematics differently depending on our understanding.

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LANGUAGE: A HIDDEN RESOURCE IN PREPARING BILINGUAL PRE-SERVICE TEACHERS

LENGUAJE: UN RECURSO OCULTO EN LA FORMACIÓN DE MAESTROS BILINGUES

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We worked to identify how the availability of two languages, Spanish and English, facilitates the teaching of mathematics when students' mathematical thinking forms the basis of instruction. To this end we studied seven teachers in training, one expert teacher and seven bilingual elementary students working on learning and teaching fractions. Using a theoretical framework focused on the use of one-on-one interviews, translanguaging and responsive teaching, we identified two aspects of language in the teaching of mathematics in the bilingual classroom: (1) Language is a hidden resource that bilingual teachers possess and (2) the maintenance and furthering of linguistic abilities should not be isolated from the teaching of mathematics. We argue that these two aspects are situated and integrated into the practice of teaching and in this way should be part of the preparation of bilingual teachers.

Key Words: Elementary School, Bilingual Teacher Preparation Programs, Teacher Educators

Our article presents a study we conducted based on our interest to understand more deeply how to support the preparation of bilingual teachers, particularly those bilingual in Spanish-English. The study was conducted in the context of an extracurricular program with bilingual 5th grade students (Spanish-English). This extracurricular program focused on teaching fractions. Our focus on fractions has two theoretical foundations: (1) children's mathematical thinking (Empson & Levi, 2011; Carpenter et al., 2014; Jacobs & Empson, 2016), and (2) how students' mathematical thinking supports teacher training (Hunt et al., 2019; Krause & Maldonado, 2019).

Our work focuses on understanding what factors influence the linguistic negotiation between teacher and student during the teaching of mathematics. Specifically, our research focuses on identifying these factors when teacher educators and elementary students interact during the teaching and learning of fractions. In this specific context, this research answers the following question: How does the availability of two languages, Spanish and English, facilitate the teaching of mathematics when students' mathematical thinking is used as the basis for fraction instruction?

Theoretical Background

Recent proposed reforms in the field of mathematics education promote environments that encourage student participation in mathematical reasoning, problem solving, and the use of common sense when learning mathematics (National Council of Teachers of Mathematics, 2014). All these reforms have a research foundation that comes from monolingual classrooms. To date, there is little evidence to suggest that reforms that work in the monolingual context could also be carried out in bilingual contexts in the same manner. For example, decades of research confirm that instruction based on students' mathematical thinking improves student learning (Fennema et al., 1996; Simon & Schifter, 1993; Saxe et al., 1999; Jacobs et al., 2007). However, these investigations do not mention, or consider, the importance and influence of language in the way students express their mathematical ideas. Furthermore, as bilingual teachers we often have to adapt the use of language. This particular practice adds a level of difficulty in teaching due to the idiosyncrasy of how different speakers learn languages in multicultural contexts (Urciuoli, 1985).

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Theoretical Framework

Next, we present the concepts that were considered as theoretical foundations for our study.

One-on-One Interviews as an Approximation of Practice

Grossman et al. (2009) presented a theoretical framework for teaching practice that includes three main components: approximation, representation and decomposition. Our work focuses on one of these central practices: approximation of practice. The teaching practice approach is defined by Grossman et al. (2009) as a way to provide opportunities for representation and experimentation through simulations of aspects of the teaching practice. Grossman et al. (2009) also make the case that engaging in approximation of practice allows the trainee to engage in repetition. Repetition allows pre-service teachers to gain fluency with common teaching moves, so that they can place greater attention on more nuances and individualized aspects of practice.

In our study we designed an approximation of practice by developing an after-school program where BPSTs worked on one-on-one interviews (Ginsburg, 1997) with a bilingual elementary student.

Translanguaging as a means of communication in the Bilingual Classroom

In addition to working in spaces where approaches to educational practice are available to BPSTs, we also focused our study around translanguaging. García et al. (2017) define a classroom where translanguaging can be observed as a "space built collaboratively" by teachers and students, where each has their own linguistic practices. In this space the main goal is to teach and learn in “deeply creative and critical ways” (p.2). Maldonado et al. (2018) have pointed out the lack of studies that focus on this practice when teaching or learning mathematics. They argue that it is necessary for mathematics educators to cultivate translanguaging and build classrooms in which teachers and students work and enrich the practice and culture of the language, involve families and communities as sources of knowledge, and create a democratic classroom that co-creates knowledge (Maldonado et al., 2018). We used this lens to understand the language practices of BPSTs and their students in order to understand the development of BPSTs instructional practices.

Responding in the Moment to the Mathematical Ideas of the Students

To illustrate how translanguaging is positioned during mathematics instruction and how BPSTs became involved in translanguaging while teaching mathematics, we used the theoretical framework of Jacobs and Empson (2016), responsive teaching. They conceptualize responsive teaching as a “type of teaching in which teachers’ instructional decisions about what to pursue and how to pursue it are continually adjusted during instruction in response to children’s content-specific thinking, instead of being determined in advance" (p.1). Their work establishes a framework of questions that identifies four categories of instruction: (a) ensure that the student understands the context of the problem, (b) explore details of the strategy the student uses to solve the problem, (c) encourage the student to consider other strategies, and (d) connect the student's mathematical thinking with symbolic notation.

These three principles provided the theoretical basis in the design of the extracurricular program in our study and in the data analysis.

Methods and Data Analysis

For the analysis we followed the parameters of an exploratory case study (Yin, 1984).

Data Collection

The data for this study comes from an extracurricular program for bilingual 5th grade students. BPSTs worked once a week for approximately 30 minutes at a time. We recorded a total of 20 sessions. We also held three interview sessions with each of the BPSTs. The first interview was conducted at the beginning of the extracurricular program, the second during the middle of the semester and the third at the end of the extracurricular program.

Participants

Seven BPSTs, one bilingual Latina teacher with more than 12 years of experience, and seven 5th grade students participated in this study. All teachers identified themselves as Latinx. All seven BPSTs had taken (the previous semester) the bilingual mathematics methods course that the first author of this article designed and taught.

Analysis

First, the three authors created a list of preliminary codes reported in our experience through the research project. This list was extensive and detailed (Saldaña, 2015). The three authors met to discuss the initial code list and we generated a common code list (Saldaña, 2015). After this coding process, we meet to compare the codes and carry out a triangulation process (Vallejo & de Franco, 2009).

Results and Discussion

The interactions between the teachers in training and the 5th grade students showed important aspects in the use of language in the teaching-learning process. The following represent a few reflections on the discoveries made during this analysis process.

Language: A Hidden Resource

The interactions between Aurora and her student, Yerina, allowed us to notice the fluidity with which each one changed from one language to another. For our study, we have defined fluency as the ability to move from one language to another. In the following transcript of Aurora interacting with Yerina, it can be seen that they both use English and Spanish, while Aurora makes sure that Yerina understands the context of the problem:

Aurora: *Carlos tiene una caja* [Yerina repeats with Aurora] *de comida de gato* [Carlos has a box of cat food]

Yerina: *Él le da a su gato un* [Aurora helps Yerina to read one-fourth] *un cuarto* [He feeds his cat one fourth]

Aurora: *One-fourth*

Yerina: *…de la caja para la comida. ¿Cuánto le queda de la caja?* […of the box. How much of the box is left?]

Aurora: [Aurora repeats the problem] *Entonces Carlos tiene una caja de comida de gato. Él le da a su gato un cuarto de la caja de la comida. ¿Cuánto queda de la caja?* [Carlos has a box of cat food. He feeds his cat ¼ of the box. How much of the box is left?]

This type of interaction is an example of what García and Sylvan (2011) has described as translanguaging. In the context of our study, this practice is particularly important for two reasons: 1) while Aurora made sure that Yerina understood the problem statement, she was able to see that any difficulty that might appear in relation to understanding the context of the problem, is not related with the ability to understand one language or another, that is, it is not a linguistic barrier, 2) providing the space to express mathematical ideas, as provided to Yerina, facilitated the communication of mathematical ideas and the interaction between Aurora and Yerina.

Language: Not an Isolated Resource

The following example shows how in the initial interactions between Aurora and Yerina, Aurora focuses on guiding Yerina towards a specific strategy. In this way Yerina's strategy becomes more like Aurora's strategy and Yerina's mathematical thinking is no longer the main source of instruction at that time.

Aurora: *So what number do we see? ¿Qué números ves?* [What numbers do you see?]
Yerina: *One fourth.*
Aurora: *One fourth, okay. Un cuarto ¿de qué?* [A fourth of what?]
Yerina: *¿De la caja?* [Of the box?]
Aurora: *Entonces ¿cuántas cajas tiene Carlos?* [Then, how many boxes does Carlos have?]
Yerina: *Una.* [One]
Aurora: *Una, nada más tiene una. ¿Podemos dibujar la caja? ¿Sí? …. y ¿es comida para Carlos? O ¿para quién?* [One, he only has one. Can we draw the box? Yes? … and is it food for Carlos?]
Yerina: *Gatos* [Cats]
Aurora: *Aja, para el gato. Y dice … él le da a su gato un cuarto de la caja para la comida. So, he gives one fourth.* [Aha, for the cat. And it says … he feeds his cat a fourth of the box. So,…]
Yerina: *So he gives her like this much.*
Aurora: *So, it would be, okay kind of like in the middle. So, let's draw one in the middle. Let's see.*
[Yerina draws a line]
Aurora: *Okay, ¿entonces cuántas partes tenemos ahí?* [Okay, then how many parts do we have there?]
Yerina: *Uno…dos.* [One … two]

In our experience as mathematics educators we have seen that teacher educators tend to guide the student towards a specific strategy. Typically, they suggest the use of a drawing, as we saw in this sample. At the same time, Jacobs and Empson (2016) have documented the same experience working with in-service teachers. Although, language seemed to help in the interaction between Yerina and Aurora, we noticed language was not the only resource needed in this instance to help Yerina express her mathematical thinking. Aurora, could have used this opportunity to uncover how Yerina was thinking, instead of suggesting that she draw the box.

Conclusions

We found that the availability of the two languages made it easier for BPSTs 1) to reach the same understanding of the context of the problems, ruling out the use of one language or another as an obstacle to reaching a common agreement, and 2) that the teacher also needs to develop the ability to respond instantly to the student's mathematical ideas. This practice in the case of bilingual interactions does not necessarily depend on the use of a specific language. Once language has been ruled out as a non-influencing factor in solving the problem, the teacher needs to have the ability to create the space for the student to freely express their mathematical ideas. In our study we investigated the language factors in one-to-one interactions between BPSTs and a bilingual student in the context of approximations, but we need more studies that investigate the representation and decomposition of practice.

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LENGUAJE: UN RECURSO OCULTO EN LA FORMACIÓN DE MAESTROS BILINGUES

LANGUAGE: A HIDDEN RESOURCE IN PREPARING BILINGUAL PRE-SERVICE TEACHERS

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En este estudio se identifica de que manera la disponibilidad de dos idiomas, español-inglés, facilita la enseñanza de las matemáticas cuando el pensamiento matemático de los estudiantes es la base para la instrucción. Identificamos dos aspectos intrínsecos del lenguaje en la enseñanza de las matemáticas en el aula bilingüe: (1) El lenguaje es un recurso oculto que poseen los maestros bilingües y (2) el lenguaje no debe ser estudiado como un recurso aislado en la enseñanza de las matemáticas. Argumentamos que estos dos aspectos están situados e integrados en la práctica de la enseñanza y de esta manera deben formar parte en la preparación de maestros bilingües.

Palabras clave: Educación Primaria, Preparación de Maestros en Formación, Educadores de Docentes

Nuestro artículo presenta un estudio que realizamos teniendo como base el entender más a fondo cómo apoyar la preparación de maestros bilingües en formación, particularmente aquellos en español e inglés. El estudio se realizó en el contexto de un programa extracurricular con estudiantes bilingües de 5to grado de primaria (español-inglés). Este programa extracurricular se enfocó en la enseñanza de fracciones. Nuestro enfoque en fracciones tiene dos fundamentos teóricos: (1) el pensamiento matemático de los niños (Empson & Levi, 2011; Carpenter et al., 2014; Jacobs & Empson, 2016), y (2) cómo el pensamiento matemático de los estudiantes apoya la preparación de maestros en formación (Hunt et al., 2019; Krause & Maldonado, 2019).

Nuestro trabajo se enfoca en entender qué factores influyen en la negociación lingüística entre el maestro y el alumno durante la enseñanza de las matemáticas. Específicamente nuestra investigación se centra en identificar estos factores cuando maestros en formación y estudiantes de primaria interactuaban durante la enseñanza y aprendizaje de fracciones. En este contexto especifico, la presente investigación responde la siguiente pregunta: ¿De qué manera la disponibilidad de dos idiomas, español e inglés, facilita la enseñanza de las matemáticas cuando el pensamiento matemático de los estudiantes es usado como base para la instrucción de fracciones?

Antecedentes Teóricos

Recientes reformas propuestas en el campo de la educación de matemáticas promueven entornos que fomentan la participación de los estudiantes en el razonamiento matemático, la resolución de problemas y el uso del sentido común al aprender matemáticas (National Council of Teachers of Mathematics, 2014). Todas estas reformas tienen un fundamento de investigación que proviene de aulas monolingües. Hasta la fecha, existe poca evidencia que sugiera que el éxito de estas reformas en el contexto monolingüe, podría trasladarse sin ningún cambio al aula bilingüe. Por ejemplo, décadas de investigación confirman que la instrucción basada en el pensamiento matemático de los estudiantes mejora el aprendizaje de los estudiantes (Fennema et al., 1996; Simon & Schifter, 1993; Saxe et al., 1999; Jacobs et al., 2007). Sin embargo, estas investigaciones no mencionan, o tienen en cuenta, la importancia e influencia del idioma en la manera cómo los estudiantes expresan sus ideas matemáticas. Por ejemplo, sólo lo que implica usar el pensamiento matemático de los estudiantes en la práctica, requiere que acomodemos estos pensamientos para nuestro propio entendimiento en el mismo momento en que recibimos las respuestas y explicaciones de los estudiantes sobre lo que piensan y entienden de una idea matemática. Además, como maestros bilingües muchas veces debemos adaptar el uso del lenguaje. Esta práctica en particular agrega un nivel de dificultad en la enseñanza debido a la idiosincrasia de cómo los diferentes hablantes aprenden idiomas en contextos multiculturales (Urciuoli, 1985).

Marco Teórico

A continuación, se abordan los conceptos que fueron considerados como fundamentos teóricos para el estudio que se presenta en esta propuesta.

Entrevistas Uno-a-Uno Como Aproximación de la Práctica Docente

Grossman et al. (2009) presentaron un marco teórico para la práctica docente que incluye tres componentes principales: aproximación, representación y descomposición. Nuestro trabajo se enfoca en una de estas prácticas centrales: aproximación de la práctica docente. La aproximación de la práctica docente está definida por Grossman et al. (2009) como una forma de proporcionar oportunidades para la representación y experimentación a través de simulaciones de aspectos de la práctica docente. Grossman et al. (2009) también exponen que participar en la aproximación de la práctica permite al aprendiz participar en la repetición. La repetición permite a los maestros en formación ganar facilidad y fluidez con movimientos de enseñanza comunes, para que puedan poner mayor atención en más matices y aspectos individualizados de la práctica.

En nuestro estudio, diseñamos una aproximación de la práctica educativa mediante el desarrollo de un programa extracurricular en el que los maestros en formación trabajaron en entrevistas individuales (Ginsburg, 1997) con un estudiante bilingüe de 5to grado de primaria.

TransLenguaje Como Medio Comunicación en el Aula Bilingüe

Además de trabajar en espacios donde aproximaciones de la práctica educativa están disponibles para los maestros en formación, también centramos nuestro estudio alrededor de el teórico de *TransLenguaje.* García et al. (2017) definen un aula en dónde se puede observar el uso *TransLenguaje* como un "espacio construido en colaboración" por docentes y alumnos, donde cada uno tiene sus propias prácticas lingüísticas, y que tiene como objetivo el de enseñar y aprender de manera "profundamente creativa y crítica" (pág. 2). Maldonado et al. (2018) han señalado la falta de estudios que se centran en esta práctica cuando se enseñan o aprenden matemáticas. Ellas argumentan que es necesario que los educadores de matemáticas cultiven *TransLenguaje* y construyan aulas en las que los maestros y los estudiantes trabajen y enriquezcan la práctica y la cultura del lenguaje, involucren a las familias y las comunidades como fuentes de conocimiento y creen un aula democrática que co-crea conocimiento (Maldonado et al., 2018). Usamos este lente para comprender las prácticas lingüísticas de los maestros en formación y sus estudiantes para comprender el desarrollo de las prácticas de instrucción de los maestros en formación.

Contestando en el Momento a las Ideas Matemáticas de los Estudiantes

Para ilustrar cómo se ubica *TransLenguaje* durante la instrucción de matemáticas y cómo los maestros en formación se involucraron en la práctica de *TransLenguaje* al hacer matemáticas, usamos el marco teórico de Jacobs y Empson (2016), enseñanza receptiva*.* Ellas, conceptualizan enseñanza receptiva como un "tipo de enseñanza en la que las decisiones de instrucción de los maestros sobre qué idea seguir y cómo seguirla se ajustan continuamente durante la instrucción en respuesta al pensamiento específico de los estudiantes, en lugar de determinarse de antemano" (pág. 1). Su trabajo establece un marco de preguntas que identifica 4 categorías de instrucción: (a) asegurar que el estudiante entienda el contexto del problema, (b) explorar detalles de la estrategia que el estudiante usa para resolver el problema, (c) animar al estudiante a considerar otras estrategias, y (d) conectar el pensamiento matemático del estudiante con notación simbólica.

Estos tres principios proporcionaron la base teórica en el diseño del programa extracurricular en nuestro estudio y en el análisis de datos.

Método de Análisis

Para el análisis seguimos los parámetros de un estudio de casos exploratorios (Yin, 1984).

Recopilación de datos

Los datos de este estudio provienen de un programa extracurricular para estudiantes bilingües de 5to grado. Los maestros en formación trabajaron una vez por semana por aproximadamente 30 minutos a la vez. Se realizaron un total de 20 sesiones durante el transcurso de un semestre.

También realizamos tres sesiones de entrevistas con los maestros en formación. La primera entrevista se realizo al inicio del programa extracurricular, la segunda durante la mitad del semestre y la tercera al final del programa extracurricular.

Participantes

En este estudio participaron siete maestros bilingües en formación, una maestra Latina bilingüe con mas de 12 años de experiencia y 7 estudiantes de 5to grado. Todos los maestros se identificaron como Latinx. Los siete maestros en formación habían tomado (el semestre anterior) el curso de métodos de enseñanza de matemáticas bilingües que el primer autor de este artículo diseñó y enseñó.

Análisis

Primero, los tres autores creamos una lista de códigos preliminares informados en nuestra experiencia a través del proyecto de investigación. Esta lista fue extensiva y detallada (Saldaña, 2015). Los tres autores nos reunimos para discutir la lista de códigos iniciales y generamos una lista común de códigos (Saldaña, 2015). Después de este proceso de codificación, nos reunimos para comparar los códigos y realizar un proceso de triangulación (Vallejo & de Franco, 2009).

Resultados y Discusión

Las interacciones entre los maestros en formación y los alumnos de 5to grado mostraron aspectos importantes en el uso del lenguaje en el proceso de enseñanza-aprendizaje. A continuación, se muestran algunas reflexiones sobre los descubrimientos que se obtuvieron durante el este proceso de análisis.

Idioma: Un Recurso Oculto

Las interacciones entre Aurora y su estudiante, Yerina, nos permitió notar la fluidez con la que cada una, cambiaba de una lengua a otra. Para nuestro estudio, hemos definido fluidez como la capacidad de moverse de un idioma a otro. En la siguiente transcripción de Aurora interactuando con Yerina, se puede notar que ambas usan el inglés y el español, mientras que Aurora se asegura que Yerina entienda el contexto del problema:

Aurora*: Carlos tiene una caja* [Yerina repite con Aurora] *de comida de gato*
Yerina: *Él le da a su gato un* [Aurora ayuda a Yerina a decir la palabra] *un cuarto*
Aurora: *One-fourth*
Yerina: *De la caja para la comida, ¿cuánto le queda de la caja?*
Aurora: [Aurora repite la pregunta] *Entonces Carlos tiene una caja de comida de gato. El le da a su gato un cuarto de la caja de la comida. ¿Cuánto queda de la caja?* [Aurora espera unos cuatro segundos]

Este tipo de interacción es un ejemplo de lo que García y Sylvan (2011) ha descrito como *TransLenguaje*. En el contexto de nuestro estudio esta práctica es particularmente importante por dos razones: 1) mientras Aurora se aseguraba que Yerina entendía el enunciado del problema, se pudo percatar que cualquier dificultad que pudiera aparecer con relación a comprender el contexto del problema, no está relacionada con la capacidad de comprender una lengua u otra, es decir no es una barrera lingüística. 2) El proveer el espacio para expresar las ideas matemáticas, tal como le fue facilitado a Yerina facilitó la comunicación de las ideas matemáticas y la interacción entre Aurora y Yerina.

Idioma: No es un Recurso Aislado

El siguiente ejemplo muestra como en las interacciones iniciales entre Aurora y Yerina, Aurora se enfoca en guiar a Yerina hacia una estrategia especifica. De esta manera la estrategia de Yerina se convierte mas en la estrategia de Aurora y el pensamiento matemático de Yerina, ya no es la fuente principal de la instrucción en ese momento.

Aurora: *So, what number do we see? ¿Qué números ves?*
Yerina: *One fourth.*
Aurora: *One fourth, okay. Un cuarto ¿de qué?*
Yerina: *¿De la caja?*
Aurora: *Entonces ¿cuántas cajas tiene Carlos?*
Yerina: *Una.*

Aurora: *Una, nada más tiene una. ¿Podemos dibujar la caja? ¿Sí? …. y ¿es comida para Carlos? O ¿para quién?*
Yerina: *Gatos*
Aurora: *A ha, para gato. Y dice … él le da a su gato un cuarto de la caja para la comida. So, he gives one fourth.*
Yerina: *So, he gives her like this much.*
Aurora: *So, it would be, okay kind of like in the middle. So, let's draw one in the middle let's see.* [Yerina dibuja una línea]
Aurora: *Okay, ¿entonces cuántas partes tenemos ahí?*
Yerina: *Uno…dos.*

En nuestra experiencia como educadores de matemáticas hemos visto que los maestros en formación tienden a guiar al estudiante hacia una estrategia especifica. Típicamente, sugieren el uso de un dibujo, como lo vimos en esta muestra. Al mismo tiempo, Jacobs y Empson (2016) han documentado la misma experiencia trabajando con maestros en servicio. En este instante notamos que el lenguaje no era el único recurso necesario para ayudar a Yerina a expresar su pensamiento matemático. Aurora podría haber aprovechado esta oportunidad para descubrir cómo estaba pensando Yerina, en lugar de sugerirle que dibujara la caja.

Conclusiones

Encontramos que la disponibilidad de los dos idiomas facilitó a los maestros en formación 1) llegar a un mismo entendimiento del contexto de los problemas descartando el uso de un idioma u otro como obstáculo para llegar a un acuerdo común, y 2) que el maestro necesita también desarrollar la capacidad de responder en el momento a las ideas matemáticas del estudiante. Esta práctica en el caso de interacciones bilingües no depende necesariamente del uso de una lengua especifica. Una vez que el lenguaje ha sido descartado como un factor que no influye en solucionar el problema, el maestro necesita tener la capacidad de crear el espacio para que el estudiante pueda libremente expresar sus ideas matemáticas. En nuestro estudio investigamos los factores del lenguaje en las interacciones individuales entre BPST y un estudiante bilingüe en el contexto de aproximaciones en la práctica. Sin embargo, necesitamos más estudios que investiguen las representaciones y la descomposición de la práctica docente.

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INTRODUCING FRACTION MULTIPLICATION. A STUDY ON TEACHER'S PEDAGOGICAL KNOWLEDGE

INTRODUCIR LA MULTIPLICACIÓN DE FRACCIONES. UN ESTUDIO SOBRE EL CONOCIMIENTO DIDÁCTICO DEL PROFESOR

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The aim of this article is to describe the characteristics of the pedagogical knowledge used by a teacher to introduce the multiplication of fractions in elementary education. In order to achieve this aim, a sixth-grade teacher was observed while she was teaching the multiplication of fractions according to the Mexican curriculum. Results show that the teacher defines the multiplication of fractions as a repetitive sum and she uses this notion to guide students in carrying out multiplications; however, this strategy creates difficulties since the curriculum presents the fraction also as a multiplicative operator.

Keywords: Culturally Relevant Pedagogy; Teacher Knowledge; Rational Numbers; Elementary School Education.

Background

The multiplication of fractional numbers is taught in basic education. Teaching this topic is aimed at helping children in developing their mathematical reasoning, and it is considered an essential topic for understanding further contents and for applying it in their daily lives (NCTM, 2013, 2014; Lamon, 2012; SEP, 2011). It is expected that children understand the fraction as a multiplicative operator, in terms of calculating a part *of* a whole (Son, 2012). However, the different meanings of the fraction and the relationship between the factors pose difficulties for students' understanding (De Castro, 2008; García, 2014; Lamon, 2012). These difficulties might be derived also from teaching practices focused on the repetition of the fractional numbers multiplication algorithm and from the students' belief that fractions and natural numbers share the same properties and laws (De Castro, 2008). Researchers like Isiksal and Cakiroglu (2011) point out that, in the classroom, the multiplication of fractions is reduced to a routine and mechanized procedure, instead of understanding its meaning and functionality.

In addition to the conditions mentioned above, teachers might hold misconceptions regarding this content; for example, believing that the product is bigger than the factors, understanding this multiplication as a repetitive sum (Isiksal & Cakiroglu, 2011; Rifandi, 2014; Thompson & Saldanha, 2003; Valdemoros, 2010), and dealing with it as a routine problem (Chinnappan & Desplant, 2012). Yasoda (2009) states that teacher knowledge to teach the multiplication of fractions is, mainly, of algorithmic nature, which makes difficult to students to comprehend its meaning and the relationship between the factors (Son, 2012). A didactic barrier is to take as a reference the natural numbers as a way of understanding the multiplication of fractions, specifically, the rule that the product is bigger than the factors (Prediger, 2008). Related to what has been stated before, the aim of this research is to describe the didactic knowledge related to the practice of a teacher who introduces and teaches the multiplication of fractions to children in basic education.

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Reference Framework

To teach mathematics, the teacher requires, besides knowing mathematics, a didactic knowledge (Ball et al., 2008; Carrillo, Escudero & Flores, 2014; Carrillo, Climent, Contreras, & Muñoz, 2013; Shulman 1986). This knowledge refers to a group of strategies that the teacher has to represent ideas, analogies, examples, illustrations and explanations related to a mathematical content (Chick, Baker, Pham & Cheng, 2006). The teacher is expected "to hold clear concepts, images, structures and basic approaches related to a topic…, as well as knowing how to identify in his students the difficulties and conceptual errors that they could face (problems related to the derivation rules, for example the rules related to the product, quotient **or the chain**), as well as what this means in their learning. This knowledge also requires teachers to use activities or methodological strategies so that students can identify and build new understandings based on their previous ideas" (García, 2009, p. 42).

In addition to the relevance of understanding the didactic knowledge in mathematics, it is necessary to specify the concept of multiplication of fractions when using whole numbers. Son (2012) considers that this type of multiplication makes reference to the *part-part* or *part-whole*. When the fraction involves whole numbers as a first factor (*n out of a/b*), the multiplication points out a repeated sum, where the whole number is the number of times the fraction is repeated; on the other hand, if the fraction is the first factor (*a/b out of n*) the fraction is an operator and it refers to the part that will be taken from the whole number.

Methodology

The study is qualitative, and it was designed as a case study. Data was collected through non-participant observation in order to grasp the natural context where the mathematical teaching-learning process occurs. A sixth grade teacher participated in the research; she works in a rural school in Mexico and we have named her, Elena. Lessons related to the *Multiplication of fractions* were audio and video recorded, according to the current study plan (SEP, 2011). In total, two lessons were video recorded, each lasting three hours approximately. Recordings were carried out under the informed consent of the teacher, respecting the dates and timing established by the teacher, in order to avoid affecting de natural and cultural context. In addition to the recordings, notes were taken in a fieldwork journal, to support in triangulating information.

Video-recordings were transcribed and fragmented into units of analysis. The analysis was based on Miles and Huberman (2007) framework, this allowed to identify aspects related to the didactic knowledge that Elena used when teaching the multiplication of fractions. The syllabus (SEP, 2011) posits that the student should use the fraction as a multiplicative operator through problems type *a/b out of n.* The results of class observation are shown in the following section.

Analysis

To achieve the learning objective stated in the syllabus (SEP, 2011), Elena used specific and centered strategies to help students in constructing their knowledge through the interaction with their peers, working individually and under her guidance. Elena first taught the *part-whole* concept of the fraction and implemented an activity in which she presented the multiplication of fractions. The activity is described next.

To introduce the *part-whole* concept of the fraction, Elena gave the half of a sheet paper (1/2) to 18 out of the 24 students of the group, and asked them to identify which part of the group (3/4) had a piece of paper:

Elena: How many are in total?
Students: 24!

Elena: If you are 24 and I want to give half of a paper sheet to $\frac{3}{4}$ of the class. How many of you are going to get a piece of paper?
Students: 18!

In the construction of this interpretation of the fraction, Elena induces the answer that she wants to get from students by asking directly "I want to give half of a paper sheet to $\frac{3}{4}$ of the class. How many of you are going to get a piece of paper?" Students' responses evidence their use of their previous knowledge on fractions (SEP, 2011), since they first recognized the denominator, called "whole number", then they determine that $\frac{3}{4}$ from the total of students is the same as 18. With respect to Elena's knowledge, it was noticed that she does not only expects an answer but its validation:

Elena: If I wanted to know how many halves of paper sheets I used with $\frac{3}{4}$ of you, what do I have to do? Check it out, there you have the sheets of paper… stand up and count how many [children have $\frac{1}{2}$ of a piece of paper].

This validation leads students to work on the multiplication as a repeated sum ($8\times\frac{1}{2}$) where the fraction as an operator is left out. Elena gave suggestions, such as separating into two groups, one group with the students who had a piece of paper and the other group without it, she told them: "get together or count each other". Getting students into groups allowed Elena to construct the multiplication of fractions concept, however, she formalized and institutionalized the demonstration $\frac{3}{4}$ *out of the group = 18 students* (Chevallard, 1998), and she raised the meaning of the multiplication as a repetitive sum: "What have we been doing? Counting each… a half plus a half, plus a half, plus a half, plus a half, plus a half until I completed $\frac{18}{2}$. Therefore, in the same way as in the sum, [in the multiplication] I was adding a half, plus a half…" In her discourse, it is possible to see that the multiplication is 18 students per $\frac{1}{2}$ of a paper sheet.

Elena generalized the fraction multiplication as a repetitive sum, even though she did not point out the relationship between $\frac{18}{2}$ (total halves of paper sheets) and $\frac{3}{4}$ of the class. Based on this meaning, Elena introduced new activities; for example, she asked children to solve $4\times\frac{2}{3}$ by using paper strips:

Elena: How can I multiply $4\times\frac{2}{3}$? To make this simpler, we are always going to try to write the whole numbers first, ok? In a multiplication, it is the same if I write the numbers before [as a first factor] or if I write them after [as a second factor]. But now, to make it easier, I am going to write the whole number first. Then, what do I need? Out of this paper sheet, we are going to get whole number [she gives the paper sheets to the students]. The problem says that I need four strips… because we need four whole numbers.
Students: [They measure and cut the paper sheets to obtain the strips that represent the whole numbers, as it is shown in Figure 1].

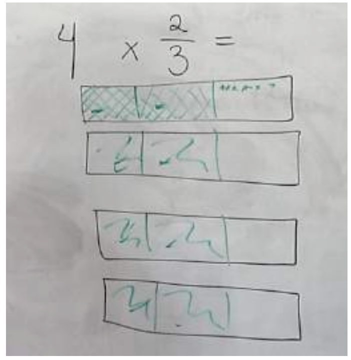

Figure 1: Graphic Representation to calculate $4\times\frac{2}{3}$.

In this activity, Elena allows students to discover the algorithm by themselves, which consists in $a\times\frac{b}{c}=\frac{a}{1}\times\frac{b}{c}=\frac{a\times b}{1\times c}$. In Elena's explanation, a limitation regarding content knowledge was identified, which could influence students' comprehension on the topic, we refer to the position of the whole numbers in the multiplication of fractions (commutation property). Although the product is the same, the meaning of the factors changes; when the whole number is the first factor ($4\times\frac{2}{3}$) the multiplication represents an abbreviated sum, but when the whole number is the second factor ($\frac{2}{3}\times 4$) it corresponds to a multiplication of fractions as an operator (Son, 2012). Elena's interpretation could be based on the commutation law.

It is possible to argue that for Elena there is no difference when using the whole number as a first or second factor. Elena's argument is focused on the product, as she tells their students when institutionalizing the algorithm "To make this simpler, we are always going to try to write the whole numbers first, ok? In a multiplication, it is the same if I write the numbers before [as a first factor] or if I write them after [as a second factor]. But now, to make it easier, I am going to write the whole number first". In doing so, the difference between a reduced sum and a fraction as a multiplicative operator is not recognized.

Conclusions

Results show that as part of her didactic knowledge (SEP, 2011), Elena uses strategies and tools according to the sixth graders academic level and to the syllabus requirements, which is identified in the examples to represent graphically and to work the multiplication algorithm with rational numbers. The use of paper, as didactic tool, allowed students to represent the algorithm and the multiplication of fractions product, as Elena does in class. It is evident that through students' interaction in the classroom, they can construct and validate this content knowledge. However, in Elena's practice, we identified that the meaning of this multiplication differs from the curricular objectives, because the fraction is considered only as a repetitive sum, which can, in turn, generate comprehension problems to students. Therefore, didactic and mathematical knowledge is fundamental to comprehend the difference in meaning of *a/b out of n* and *n out of a/b.*

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INTRODUCIR LA MULTIPLICACIÓN DE FRACCIONES. UN ESTUDIO SOBRE EL CONOCIMIENTO DIDÁCTICO DEL PROFESOR

INTRODUCING FRACTION MULTIPLICATION. A STUDY ON TEACHER'S PEDAGOGICAL KNOWLEDGE

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El artículo tiene como objetivo describir el conocimiento didáctico que una profesora utiliza para introducir la multiplicación de fracciones en educación básica. Para ello se observó a una profesora de sexto grado de educación primaria (México), enseñando este contenido de acuerdo con el Plan de estudios vigente durante la toma de datos. Los resultados muestran que la docente recupera el *concepto de fracción para definir y trabajar la multiplicación de fracciones como una suma reiterativa, pero esta estrategia genera dificultades dado que el Plan de estudios plantea la fracción como operador multiplicativo.*

Palabras clave: Pedagogía culturalmente relevante; Conocimiento del Profesor; Números Racionales; Educación Primaria.

Antecedentes

La multiplicación de números fraccionarios se enseña en la educación básica, con la finalidad de que los niños desarrollen un razonamiento matemático para comprender otros contenidos de mayor complejidad y para su vida cotidiana (NCTM, 2013, 2014; Lamon, 2012; SEP, 2011). Se espera que los niños entiendan la fracción como un operador multiplicativo, en términos de calcular una parte *de* un conjunto (Son, 2012). Sin embargo, para los alumnos es difícil comprender la multiplicación de fracciones debido a los diferentes significados de la fracción y de la relación entre los factores (De Castro, 2008; García, 2014; Lamon, 2012). Tal dificultad, en ocasiones, es producto de una enseñanza centrada en la mecanización y memorización del algoritmo de la multiplicación con números fraccionarios, y debido a que los niños creen que las fracciones presentan las mismas propiedades y leyes que los números naturales (De Castro, 2008). Investigadores como Isiksal y Cakiroglu (2011) apuntan que, en el aula, la multiplicación de fracciones se reduce a un procedimiento rutinario y mecanizado, en lugar de entender su significado y su funcionalidad.

Aunque existe el compromiso de enseñar la multiplicación de fracciones, el profesor llega a tener conceptos erróneos de este contenido, por ejemplo, que el producto es mayor que los factores o generalizar su definición como una suma reiterada (Isiksal & Cakiroglu, 2011; Rifandi, 2014; Thompson & Saldanha, 2003; Valdemoros, 2010), así como abordar problemas rutinarios (Chinnappan & Desplant, 2012). Yasoda (2009) afirma que el conocimiento del profesor para enseñar la multiplicación de fracciones es, principalmente, de naturaleza algorítmica, lo cual le dificulta a los estudiantes comprender su significado y la relación entre los factores (Son, 2012). Un obstáculo didáctico es tomar como referente los números naturales para entender la multiplicación de fracciones, en específico, la regla de que el producto es mayor que los factores (Prediger, 2008). En relación con lo anterior, la presente investigación tiene como objetivo describir el conocimiento didáctico inmerso en la práctica de una profesora que introduce y enseña la multiplicación de fracciones a niños de educación básica.

Marco de referencia

Para enseñar matemáticas el profesor requiere, además de saber matemáticas, un conocimiento didáctico (Ball et al., 2008; Carrillo, Escudero & Flores, 2014; Carrillo, Climent, Contreras, & Muñoz, 2013; Shulman 1986). Este conocimiento se refiere al conjunto de estrategias que el docente dispone para representar ideas, analogías, ejemplos, ilustraciones y explicaciones en torno a un contenido matemático (Chick, Baker, Pham & Cheng, 2006). Se espera que el profesor "tenga claro los conceptos, imágenes, estructuras y planteamientos básicos vinculados a un tema…, además sepa identificar en sus estudiantes las dificultades y errores conceptuales que enfrentarán estos (problemas con las reglas de derivación, como por ejemplo las del producto, cociente o de la cadena), así como lo que esto signifique en su aprendizaje. Este conocimiento también reclama al profesor que, mediante actividades o estrategias metodológicas, el estudiante pueda identificar y discernir sobre sus ideas previas" (García, 2009, p. 42).

Además de la relevancia de entender el conocimiento didáctico en matemáticas, es indispensable precisar el concepto de multiplicación de fracciones cuando se tiene números enteros. Son (2012) considera que este tipo multiplicación hace referencia a la *parte-parte* o *parte-todo*. Cuando involucra números enteros como primer factor (*n de a/b*), la multiplicación indica una suma reiterada, donde el entero es la cantidad *de veces* que se repite la fracción; en cambio, si la fracción es el primer factor (*a/b de n*) la multiplicación indica que la fracción es un operador y refiere *la parte que se tomará* del entero.

Metodología

El estudio es de corte cualitativo y está centrado en el estudio de casos. Se recurrió a la observación no participante con la finalidad de tener un acercamiento al contexto natural donde ocurre el proceso de enseñanza y aprendizaje de las matemáticas. En la investigación participó una profesora de sexto grado de educación primaria, quien labora en una escuela rural en México y a quien hemos llamado Elena. El acopio de datos comprendió la video y audiograbación de las clases en las que Elena trató el contenido *Multiplicación de fracciones* de acuerdo con el Plan de estudios vigente en ese momento (SEP, 2011). En total se videgrabraron dos sesiones de clases, totalizando 3 horas aproximadamente. Las videograbaciones se realizaron previo consentimiento de la docente, respetando las fechas y tiempos programados por ella, de tal manera que no afectaran el escenario natural y cultural del salón de clases. Además de las grabaciones en audio y video, en una bitácora se registraron aspectos puntales de la práctica de la profesora, esto facilitó la triangulación de la información.

Las videograbaciones fueron transcritas y fragmentadas en unidades de análisis, para ello se tomó como referencia la propuesta de análisis de Miles y Huberman (2007), lo cual permitió identificar aspectos relacionados con el conocimiento didáctico que Elena pone en juego al enseñar la multiplicación de fracciones. El Plan de estudios (SEP, 2011) apunta que el alumno debe usar la fracción como *operador multiplicativo* mediante problemas de tipo *a/b de n*. En la siguiente sección se muestran los resultados de la observación en aula.

Análisis

Para lograr el objetivo de aprendizaje dado en el Plan de estudios (SEP, 2011), Elena recurrió a estrategias específicas y centradas en que los alumnos construyeran sus conocimientos a partir de interactuar con sus compañeros, trabajar de manera individual y bajo la guía de la profesora. Para ello, Elena partió del concepto de fracción como *parte-todo* y de una actividad en la cual presentó la multiplicación de fracciones. A continuación, se describe esta actividad.

Para introducir el concepto de fracción como parte-todo, Elena proporcionó media hoja de papel (1/2) a 18 de los 24 estudiantes que conforman el grupo, para que posteriormente todos determinaran qué parte del grupo (3/4) tiene papel:

> Elena: ¿Cuántos somos en total en el grupo?
> Alumnos: ¡24!
> Elena: Si son 24 y yo les quiero dar a $\frac{3}{4}$ de ustedes media hoja de papel. ¿A cuántos les voy a dar?
> Alumnos: ¡18!

En la construcción de esta interpretación de fracción, Elena induce la respuesta que espera obtener al plantearles directamente "les quiero dar a $\frac{3}{4}$ de ustedes media hoja. ¿A cuántos [estudiantes] les voy a dar?" Las respuestas de los alumnos evidencian que partieron de sus conocimientos sobre fracciones (SEP, 2011), pues primero reconocen el denominador, llamado "entero" por los alumnos, y posteriormente determinan que $\frac{3}{4}$ del total equivalen a 18 alumnos. Como características del conocimiento de Elena, se observa que involucra no sólo obtener el resultado sino que el alumno lo sustente:

> Elena: Si yo quisiera saber cuántos medios de hoja me gasté en $\frac{3}{4}$ de mi salón, de mis alumnos, ¿cómo le puedo hacer? Compruébenmelo, ahí están las hojas… párense y cuenten cuántos [niños con $\frac{1}{2}$ de hoja de papel].

Esta validación le permite a Elena llevar a los estudiantes a trabajar la multiplicación como una suma reiterada ($8\times\frac{1}{2}$), donde deja de lado la fracción como operador. La profesora da sugerencias a los estudiantes, tales como hacer dos grupos, aquellos que tienen hoja de papel y quienes no la tienen, pues ella les dice: “júntense o cuéntense”. Agrupar a los estudiantes permite a Elena construir el concepto de multiplicación de fracción, sin embargo, ella es quien formaliza e institucionaliza la demostración $\frac{3}{4}$ *del grupo = 18 alumnos* (Chevallard, 1998), y plantea el significado de la multiplicación como una suma reiterativa: “¿[qué] tuvimos que ir haciendo? Contando cada… medio más medio, más medio, más medio, más medio, más medio hasta que completé los $\frac{18}{2}$. Entonces al igual que en la suma, [en la multiplicación] yo tendría que estar sumando medio, medio más…” En su discurso se evidencia que la multiplicación queda como 18 estudiantes por $\frac{1}{2}$ de hoja papel.

Elena generaliza la multiplicación de fracción como la suma reiterada, aunque no muestra la relación entre $\frac{18}{2}$ (total de medios de hojas) y $\frac{3}{4}$ de grupo. Con base en este significado de la multiplicación de fracciones, Elena introdujo nuevas actividades orientadas a que los estudiantes lo pongan en práctica; por ejemplo, les pidió resolver $4\times\frac{2}{3}$ mediante el uso de tiras de papel:

> Elena: ¿Cómo puedo multiplicar cuatro enteros por $\frac{2}{3}$? Para que se nos haga más fácil vamos a tratar siempre de poner primero los enteros, ¿sí? En una multiplicación es lo mismo si lo pongo acá [como primer factor] que lo ponga acá [como segundo factor]. Pero por lo pronto, para que se nos haga más fácil, voy a poner primero los enteros. ¿Qué es lo que necesito entonces? En esta hoja vamos a ir sacando enteros [reparte hojas de papel a los alumnos]. Dice que necesito cuatro enteros, yo necesito cuatro tiras… porque son cuatro enteros.
>
> Alumnos: [Miden y recortan las hojas de papel para obtener las tiras que representen los enteros, como se muestra en la Figura 1].

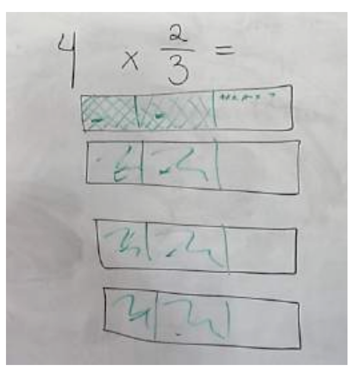

Figura 1: Representación gráfica para calcular $4\times\frac{2}{3}$.

Esta actividad Elena deja que los alumnos descubran el algoritmo de la multiplicación de fracciones por ellos mismos, el cual consiste $a\times\frac{b}{c}=\frac{a}{1}\times\frac{b}{c}=\frac{a\times b}{1\times c}$. En la explicación de Elena se evidencia una limitación en el conocimiento del contenido que podría afectar la comprensión del niño acerca del tema, nos referimos a la posición de los enteros en la multiplicación de fracciones (propiedad de la conmutación). Aunque el producto es mismo, el significado de los factores cambia; cuando el entero es el primer factor ($4\times\frac{2}{3}$) la multiplicación representa una suma abreviada, pero cuando es el segundo factor ($\frac{2}{3}\times 4$) corresponde a multiplicación de fracciones como operador (Son, 2012). La interpretación de Elena podría deberse a la prevalencia de la ley de conmutación.

Es notorio que para ella no hay una diferencia de significado en cuanto al entero como primer o segundo factor. El argumento de Elena está centrado en el producto, pues como ella les dices a los estudiantes al momento de institucionalizar el algoritmo: "Para que se nos haga más fácil vamos a tratar siempre de poner primero los enteros, ¿sí? En una multiplicación es lo mismo si lo pongo acá [como primer factor] que lo ponga acá [como segundo factor]. Pero por lo pronto para que se nos haga más fácil voy a poner primero los enteros". En este sentido no se reconoce la diferencia entre una suma reducida y la fracción como operador multiplicativo.

Conclusiones

Los resultandos muestran que, como parte de su conocimiento didáctico, Elena usa estrategias y recursos acorde con el nivel educativo de los estudiantes de sexto grado de educación primaria y a las exigencias del Plan de estudios (SEP, 2011), lo cual se refleja en los ejemplos para representar gráficamente y trabajar el algoritmo de la multiplicación con números racionales. El uso de papel, como recurso didáctico, le permite al alumno representar el algoritmo y el producto de la multiplicación de fracciones, tal como Elena lo hace en la clase. Es evidente que a través de la interacción que tienen los alumnos en el salón de clases se construye y se validan los conocimientos en torno a este contenido. Sin embargo, en la práctica de Elena se evidencia cómo el significado de la multiplicación difiere con los objetivos curriculares al considerarla sólo como una suma reiterada, los cual puede generar obstáculos de compresión en el estudiante. En este sentido es fundamental un conocimiento didáctico y matemático para comprender la diferencia en los significados *a/b de n* y *n de a/b.*

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MATHEMATICS TEACHERS' PERCEPTION OF INDUCTIVE REASONING AND ITS TEACHING

PERCEPCIÓN DE PROFESORES DE MATEMÁTICAS DEL RAZONAMIENTO INDUCTIVO Y SU ENSEÑANZA

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This paper presents the perception that middle school mathematics teachers attribute to inductive reasoning and its teaching from working with the concept of quadratic equation. The data was obtained from a questionnaire given to 16 teachers and from their expanded responses in a group conversation. Through the thematic analysis method, it was found that most teachers perceived this type of reasoning as a process to move from the particular to the general and as a way to guide teaching a concept through questioning. However, they encountered difficulties in using inductive processes to teach the concept and attach it to an inductive logic.

Keywords: Teacher Knowledge, Reasoning and Proof, Middle School Education.

Introduction

Inductive reasoning for learning mathematics in basic and middle school education is important for two reasons. On one hand, it is a means for the development of concepts and the resolution of mathematics problems (Molnár, Greiff, & Csapó, 2013; Papageorgiou, 2009; Sosa, Cabañas y Aparicio, 2019; Sriraman & Adrian, 2004) that assists the abstraction and generalization of the invariant characteristics of particular objects or situations. Klauer (1996) claims that it leads to "detecting regularities, be it classes of objects represented by generic concepts, be it common structures among different objects, or be it schemata enabling the learners to identify the same basic idea within various contexts" (p. 53). On the other hand, it supports processes to speculate, argue and generalize in mathematics (Cañadas et al., 2007; Cañadas, Castro and Castro, 2008; Conner et al., 2014; Martinez & Pedemonte, 2014).

This implies that middle school teachers should develop and interpret the inductive reasoning of students (AMTE, 2017; NCTM, 2000). NCTM (2000) establishes that this form of reasoning must progress in students throughout each grade and education level so that they can become more proficient in the formulation of conjectures and generalizations from specific cases. In this sense, it is desirable that teachers have clarity about inductive reasoning and the phases that go along with the transition from particular instances to the general. On the contrary, they may have difficulties incorporating it into their practice. Therefore, the goal of this study is to examine and describe the perception that middle school teachers show about the inductive reasoning in relation to the teaching of the quadratic equation concept.

Literature review

Much of the research on inductive reasoning and professional development of mathematics teachers has been conducted with preservice teachers and most of them focused on issues associated to the teacher cognition, such as ways of recognizing similarities by induction from numerical and figural representations (Rivera & Becker, 2003), levels of deepening understanding and strategies used to solve a generalization problem (Manfreda, Slapar, & Hodnik, 2012), the role of induction and abduction in making generalizations of classes of abstract objects (Rivera & Becker, 2007), and the relationship between inductive and deductive reasoning with learning styles (Arslan, Göcmencelebi, & Tapan, 2009). Results indicate that future teachers tend to induce numerically over strategies used based on the use of figures. Difficulties are also reported in generalizing quadratic patterns, even when a numerical pattern was identified. However, Sosa, Aparicio and Cabañas (2019) show that secondary school teachers who achieve generalization in these kinds of patterns are those who managed to connect inductive processes; they also identified difficulties to establish and abstract a pattern. This reinforces the need to study how teachers perceive inductive reasoning and its teaching.

Conceptual framework

In this study, inductive reasoning is understood as a means to produce generalizations from particular cases, be they ideas, qualities, objects, facts, phenomena or situations. This understanding is consistent with those who refer to it as a mental process oriented to infer laws or general conclusions through observation and connection of particular instances of a class of objects or situations (Glaser & Pellegrino, 1982; Haverty et al., 2000, Polya, 1957).

The works of Reid and Knipping (2010), Polya (1967) and Sosa et al. (2019) are examples of this understanding. Reid and Knipping (2010) identify three characteristics of inductive reasoning: it comes from specific cases to conclude general rules, uses what is known to conclude something unknown and, it is only likely but not true. Polya (1967) proposes the following four phases of such reasoning to discover properties, principles and general cases in mathematics: observing particular cases, formulating a conjecture, generalizing and verifying conjecture. More recently, from a cognitive approach, Sosa et al. (2019) report that the connection of the following three processes is necessary to achieve generalization inductively: observation of regularities, establishment of a pattern and formulation of a generalization.

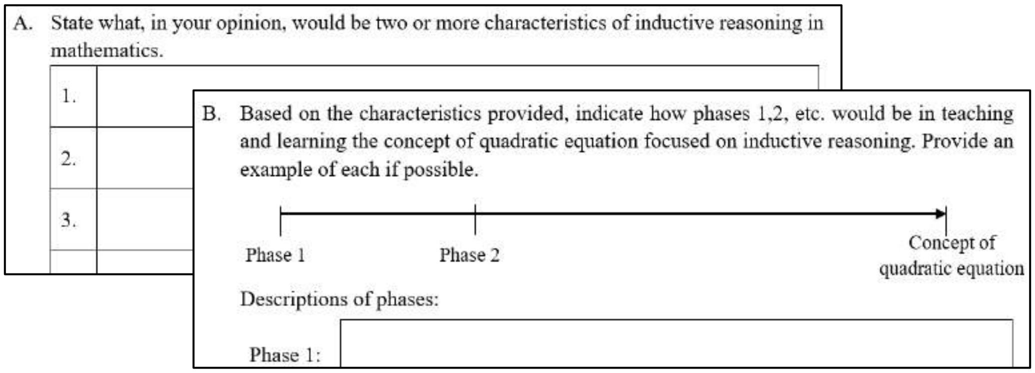
Methodology

Context and participants

This study is part of a professional teacher development program in mathematics, in which 16 secondary school teachers (10 women and 6 men) participated. The data was collected in the first of the five sessions that make up the program. Due to the relationship between inductive reasoning and generalization, as well as the difficulties of teachers to obtain a generalization of quadratic patterns as reported in the literature, the selection criteria for their participation was that they had at least one year of teaching experience in the third year of secondary school. This criterion is explained by the fact that, in the Mexican curriculum, "patterns and equations" is a topic associated with generalization, and the quadratic structure is studied in that education level.

Data collection

Data collection was conducted with a written questionnaire and audio recordings. The questionnaire had two items A and B (Figure 1). Item A asked for the enunciation of at least two characteristics of inductive reasoning in mathematics, and item B requested the description of the phases to be followed in order to teach some aspect of the quadratic equation in an inductive way. The replies were recorded in writing, and individually, and were subsequently communicated orally to the group for further information or clarification.


A. State what, in your opinion, would be two or more characteristics of inductive reasoning in mathematics.

1.	
2.	
3.	

B. Based on the characteristics provided, indicate how phases 1,2, etc. would be in teaching and learning the concept of quadratic equation focused on inductive reasoning. Provide an example of each if possible.

Descriptions of phases:

Phase 1:

Figure 1: Questionnaire for data collection

Data analysis

A thematic analysis was conducted to describe the perception of teachers considering the written and oral answers to item A. Then, the responses given to item B were associated to the categories of perception previously generated and contrasted with the conceptual framework in order to identify how teachers interpret inductive reasoning in teaching the concept of quadratic equation.

Thematic Analysis. This method consists of identifying, analyzing, organizing and systematically obtaining patterns (themes) in a data set by detecting and giving sense to the experiences and meanings shared in a group (Braun & Clarke, 2006; 2012). This helped to identify patterns of meanings in the common characteristics that teachers attribute to inductive reasoning and to form categories of their perception. To do this, the six phases of thematic analysis were followed: familiarize with the data, search for topics, review those that have potential, define and name themes, and produce a report (Braun & Clarke, 2012).

Results

Inductive reasoning perception categories

Five categories were identified on the perception of inductive reasoning, among them were *as a guide for mathematical knowledge* and *as a cognitive process.*

- **Category 1:** Inductive reasoning as a way to guide mathematical knowledge. This category consists in the fact that the students can be guided from their previous knowledge to new knowledge through questions. An example of this category is shown in the following excerpts of responses:

Teacher L: Give students an exercise and based on their previous knowledge draw their own knowledge. Create a brainstorm to learn what students know.

Teacher M: One of the characteristics is to begin asking key questions for the exercises and introducing students to the topic. Students begin to reason about the topic through questions and are able to visualize the previous knowledge. Guide questions. During the class, doubts may emerge [...] and questions may be asked [...], students can achieve the appropriation of concepts.

- **Category 2:** Inductive reasoning as a cognitive process. This category consists of perceiving it as a process to move from particular instances (ideas, particular cases or situations) to the inference of a general conclusion or result. For example:

Teacher E: It goes from the particular to the general...

Teacher N: It is a type of reasoning that consist of moving from particular to general ideas. Starting from concrete ideas to ideas in general. Generalize based on experiences of the given results.

Interpretation of inductive reasoning in teaching: logic and phases

Four different ways of interpreting teaching a concept based on inductive reasoning were identified. Eight teachers interpreted it as a guide for knowledge, such was the case of teacher M (Table 1). Five followed a deductive logic rather than inductive logic, for example teacher O; this means that they begin with the approach of general formulas or definitions of quadratic equations and conclude with a particular example. The deductive or inductive logic was not identified in the phases described by two teachers, they focused on "iconic" treatments based on the association of a quadratic property with the area of a square figure or the product of a number with itself. Strictly speaking, only the phases described by one teacher could be considered as an inductive logic. Overall, inductive processes were found to be absent in the phases proposed by the teachers for teaching quadratic equation, except for those described by teacher B (Table 1).

Table 1: Transcription of the phases proposed by two teachers

Phase	Teacher M	Teacher B
1	Previous knowledge: Introductory questions about algebraic expression, algebraic language, power, law of exponents.	Specific cases or situations which can be quantified, manipulated, or visualized are provided.
2	Application of the concept of "basic" shapes areas (with square shapes).	Different cases that meet the observed characteristic or property are asked.
3	Delete data and replace it with literals. Start with formulas.	It is required a prediction that this characteristic or property is fulfilled for other cases that are not tangible or directly observable.
4		A rule or formula that covers all possible cases is obtained; that is, a generalization.

Conclusions

Little clarity was identified in teachers about what inductive reasoning is. Most perceive or interpret it as a way of guiding mathematical knowledge in a teaching situation. However, this perception differs from the idea of inductive reasoning as a means for the construction of concepts; that is, to abstract and generalize the key characteristics of an object in specific situations (Sosa, Cabañas y Aparicio, 2019; Sriraman & Adrian, 2004; Klauer, 1996). It was also identified that few teachers perceive induction as a means to promote processes of generalization and resolution of problems. While reference is made to the transition from the particular to the general as a feature of this reasoning, the responses reveal a lack of clarity about the underlying processes because there is an inadequate interpretation when describing the phases to teach this mathematical concept; some of them even used a deductive logic. Therefore, it is necessary to compare and broaden teachers' knowledge of inductive reasoning through learning experiences in which they recognize and articulate inductive processes in contexts of mathematical generalization.

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DEFINING KEY DEVELOPMENTAL UNDERSTANDINGS IN CONGRUENCE PROOFS FROM A TRANSFORMATION APPROACH

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Previous work by the authors (St. Goar et al., 2019) identified two potential key developmental understandings (KDUs) (Simon, 2006) in the construction of congruence proofs from a transformation perspective for pre-service secondary teachers in an undergraduate geometry course. We hypothesized the independence of the potential KDUs in previous work, meaning that students may have one potential KDU but not the other, and vice versa. We tested this hypothesis with analysis of an expanded data set and found that this hypothesis did not hold in general. We report on the expanded analysis and discuss implications for the scope and limitation of the potential KDUs.

Keywords: Teacher Knowledge, Geometry and Geometrical and Spatial Thinking, Reasoning and Proofs

A change has come to K-12 geometry instruction, and as a result changes to preparation of future teachers must follow. Many guidelines (Catalyzing Change in High School Mathematics: Initiating Critical Conversations [NCTM], 2018) and standards (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010) now recommend or require the teaching of geometry from a transformation perspective instead of the more traditional approach originating from Euclid's *Elements* (Sinclair, 2008). The concepts and proofs involving congruence and similarity now appeal to rigid motions: reflections, rotations, and translations. That is, two figures are said to be congruent if and only if there exists a sequence of rigid motions carrying one figure onto another. This definition is notably different from those in Euclid's *Elements,* where the criteria for congruence differs for each type of shape. Thus, the reader will note that the differences in mathematical structure between the transformation and *Elements* contexts are substantial.

The resulting danger is that some future teachers may lack the content knowledge to handle the new approach. Without sufficient content knowledge, they may struggle to know what can be proved in this new context and how these proofs may be structured. This lack may affect how they write lesson plans and course materials, adapt or modify materials for the context of their class, and evaluate student thinking and alternate approaches. Future teachers may need support in the transformation context to allow them to thrive in the teaching of geometry.

Relationship to Prior Literature

To answer the call, some undergraduate instructors are beginning to incorporate transformation geometry into their geometry courses for future teachers. Because transformation geometry is becoming a more prominent feature of geometry in post-secondary contexts, research on how pre-service teachers learn these topics are particularly salient. However, at this point research on how pre-service teachers learn transformation geometry is just beginning. Jones and Tzekaki (2016) noted the "limited research explicitly on the topics of congruency and similarity, and little on transformation geometry" (p. 139).

Some key results informing our work are the following. Edwards (2003) explained that students in middle school through undergraduate contexts tend to view transformations from a motion view, as opposed to a map view of transformations. A motion view is characterized by conceptualizing transformations as physical movements, such as picking up a figure and shoving it to where it needs to go. A map view is characterized in terms of inputs and outputs of transformation, and

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distinguishes the preimage from the image. For instance, a person with a motion view may think of an image and preimage of a figure as being the same object, simply with a different location. But a person with a map view can hold the idea that the image and preimage as different objects, and hence compare them. Research conducted after Edwards' (2003) study with middle school students corroborate her results, even for high school teachers (Hegg et al., 2018; Portnoy et al., 2006; Yanik, 2011). These results also note the difficulty that a motion view may present to generating congruence proofs from a transformation approach.

Based on analysis of future teachers' work on two congruence proofs on a midterm examination, we previously highlighted the importance of supporting pre-service teachers in understanding *both* directions of the "if and only if" in the definition of congruence. Further, we identified two potential key developmental understandings (KDUs; Simon, 2006), stated below:

"Potential KDU 1: Understanding that applying the definition of congruence to prove congruence of two figures means establishing a sequence of rigid motions mapping one *entire* figure to the other *entire* figure" (St. Goar et al., 2019).

"Potential KDU 2: Understanding that using a sequence of transformations to prove that two figures are congruent means justifying deductively that the image of one figure under the sequence of transformations is exactly the other figure" (St. Goar et al., 2019).

As the results by St. Goar et al. (2019) were based on analysis of teachers' work from a single, timed assessment, more work is needed to interrogate the accuracy of these potential KDUs.

Further, *we previously hypothesized the independence of these potential KDUs, meaning that teachers might hold KDU 1 but not KDU 2, or hold KDU 2 but not KDU 1.* We generated this hypothesis empirically from examples of teachers' work in our previous analysis. In considering the literature, we might also support and refine this hypothesis as follows. First, potential KDU 1 pertains to constructing a sequence of rigid motions, and not explicit deductive reasoning about images and preimages, which is the scope of potential KDU 2. Second, constructing a sequence of rigid motions can be consistent with either a motion view or a map view. However, deductive reasoning as needed for congruence proofs might require distinguishing between overlapping figures. Although this could be done under a motion view, it seemed plausible to us that conceiving transformations as maps was more likely to support a teacher in careful work with images and preimages – particularly if the figure is disconnected. It seemed plausible that it is more difficult to conceive of "moving" a disconnected figure than "moving" a connected one. In lieu of the literature, although it is possible for these potential KDUs to be independently held, *the following is a better hypothesis: Teachers hold neither potential KDU (if neither motion or map view is developed), potential KDU 1 but not potential KDU 2 (representing a motion view), or both potential KDUs (representing a map view).*

Objectives

Hence, we proceeded with the following research questions, with the same teachers' work on different congruence proofs than previously analyzed: (1) Do we continue to see evidence of the previously identified potential KDUs? (2) What are the scope and limitations, including the independence, of these potential KDUs?

Conceptual Perspective

Based on Usiskin and Coxford (1972), a *transformation approach* assumes without proof that rigid motions (e.g., reflections, rotations, and translations) are bijections of the plane that preserve both distance and angle measure. Additionally, under such an approach, two subsets of the plane are considered to be congruent if and only if there exist a sequence of rigid motions mapping one subset to the other. Similarity is treated analogously, incorporating dilations.

Key developmental understandings (KDUs) are described by Simon (2006). A key developmental understanding has two primary aspects: (1) Achieving a KDU represents a conceptual advance by the student. A conceptual advance is "a change in a students' ability to think about and/or perceive particular mathematical relationships" (Simon, 2006, p. 362) and (2) Acquiring KDUs does not tend to happen "as the result of an explanation or demonstration. That is, the transition requires a building up of the understanding through students' activity and reflection and usually comes about over multiple experiences" (Simon, 2006, p. 362).

As Simon noted, KDUs generally cannot be found by a mathematician examining their own understanding of a topic, but rather through observing students' mathematical work. As a result, our first steps in identifying these potential KDUs have been through the analysis of future teachers' work. Simon noted also that KDUs may be identified with varying amounts of detail and that "the level of detail specified for a key developmental understanding is adequate if it serves to guide the effort for which it is needed (e.g. curriculum design, further research)" (Simon, 2006 p. 364). Hence our analysis here is meant to achieve this necessary detail so that the potential KDUs can be used to improve undergraduate geometry curricula and research.

We use the term "potential KDU" rather than "KDU" because we see our understanding of teachers' understanding as a work in progress that is only based on analysis of written work as opposed to cognitive interviews, which would be ideal and needed to substantiate a claim of being a KDU. We return to this critical piece in the discussion and questions to the audience.

Methods

We collected the coursework of twenty teachers in an undergraduate geometry course taught by Lai. We examined homework assignments and midterm exams from throughout the semester for tasks where teachers specifically worked on congruence proofs. Here we report analysis of four tasks. This resulted in 69 total proof submissions included in the analysis.

We coded teachers' work on tasks based on evidence of potential KDU 1 and KDU 2. During the course of this analysis, if some criteria had to be changed, then codes were reworked to reflect these updated criteria, consistent with constant comparison (Strauss & Corbin, 1994).

Results

Addressing the first research question, the basic statements of the potential KDUs remained intact after analysis of teachers' work on additional tasks. Addressing the second research question, this analysis provided possible disconfirming evidence for the independence of the potential KDUs. We begin this section by reviewing the scope and limitations of the potential KDUs, and then compare evidence of each potential KDU.

Scope and Limitations of Potential KDUs

Potential KDU 1 is primarily focused on the *construction* of the sequences of rigid motions. That is, in order to have this potential KDU, teachers must construct a sequence of rigid motions from one entire figure to another entire figure. This means that aside from the creation of the rigid motions themselves, the rest of the deductive logic in a transformation proof is not a part of this potential KDU.

Potential KDU 2 focuses on the *deductive reasoning* used in the proof. Specifically, teachers need to attempt to deductively show that their transformation extends to the entire figure. Note that a teachers' work need not show entirely correct logic in order to show evidence of this potential KDU so long as they are attempting to extend arguments about the image of a transformation to entire figures and are using deductive logic to do so.

(Non) Independence of Potential KDUs

We hypothesized previously the independence of potential KDU 1 and potential KDU 2, meaning that, teachers' capacity to engage in deductive reasoning about the correctness of a proof may not depend on their capacity to construct sequences of rigid motions. We refined our view in lieu of the literature to hypothesize that it is most likely that teachers may hold neither potential KDU 1 nor potential KDU 2, hold potential KDU 1 and not potential KDU 2, or hold both. Our analysis suggests that our initial hypothesis is not well-supported, but our new hypothesis is. For brevity, we limit discussion of this to a visual summary of the results of this analysis, shown in Figure 1.

Homework Task 1	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	12	2
No Evidence of KDU 2	2	0

Homework Task 2	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	9	0
No Evidence of KDU 2	1	6

Midterm Task 1	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	4	0
No Evidence of KDU 2	12	1

Midterm Task 2	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	7	3
No Evidence of KDU 2	5	5

Across all Tasks	Evidence of KDU 1	No Evidence of KDU 1
Evidence of KDU 2	32	5
No Evidence of KDU 2	20	12

Figure 1: The above is a summary of evidence of potential KDU 1 and potential KDU 2 across two homework tasks and two midterm examination tasks.

Discussion and Conclusion

In this report, we expanded on the research by St. Goar et al. (2019) by analyzing future teachers' work on transformation congruence from an undergraduate geometry course. The results confirm the viability of potential KDU 1 and potential KDU 2 as codes for teachers' written work on congruence proofs from a transformation approach. Moreover, the results do corroborate the authors' revised hypothesis that that teachers may hold neither potential KDU 1 nor potential KDU 2, hold potential KDU 1 and not potential KDU 2, or hold both. In other words, the *least likely scenario* is that teachers hold potential KDU 2 but not potential KDU 1. Indeed, across the tasks, there are only 5 out of 69 instances (7%) where teachers' work shows evidence of potential KDU 2 but not potential KDU 1.

While our work was able to corroborate part of our revised hypothesis described above, the revised hypothesis was based on the construct of map view and motion view. We were not able to deduce from the available written work which type of view a teacher might hold, and as a result further research is needed to investigate this possible role of motion view and map view.

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USING TASK DESIGN METHODOLOGY TO UNPACK TEACHERS' (MIS)CONCEPTIONS ABOUT PROCEDURAL AND CONCEPTUAL KNOWLEDGE

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Star (2005) argues that "current characterizations of the terms procedural knowledge and conceptual knowledge are limiting and are in fact impediments to careful investigation of these constructs" (p. 405). Addressing this argument, we examined secondary mathematics teachers' understanding of procedural and conceptual knowledge through the design of mathematical tasks. We asked 55 secondary mathematics teachers to design a procedural and a conceptual task on a given topic and explain why they think that the task they designed is a procedural and/ or conceptual task. The study results showed that 78% of teachers were able to design and correctly explain procedural tasks. However, only 5.5% of teachers were able to correctly design conceptual tasks. Teachers' narratives were examined to categorize emerging characteristics of procedural and conceptual tasks as well as to address teachers' (mis)conceptions about procedural and conceptual knowledge.

Keywords: procedural knowledge, conceptual knowledge, task design, secondary mathematics teachers.

Objective

Mathematics education reform calls for building students' and teachers' mathematical proficiency that, among other strands, include conceptual understanding and procedural fluency (Kilpatrick, Swafford, & Findell, 2001). Star (2005) suggests that the widespread use of the terms conceptual and procedural in learning and teaching mathematics can be attributed to Hiebert (1986) who defined procedural knowledge as knowledge of procedures (e.g., syntax, steps, conventions, rules) and conceptual knowledge as knowledge of relationships (e.g., connected web of knowledge, a network of linked information). However, there are different interpretations of the conceptual/procedural framework (Star & Stylianides, 2013). Therefore, teachers may agree that reform-oriented mathematics teaching and learning should focus on conceptual knowledge, but it could be difficult to implement "if teachers do not have a common understanding on what conceptual knowledge is" (Star & Stylianides, 2013, p. 5). Considering this challenge, the purpose of this study is to utilize task design methodology as a way to explore secondary mathematics teachers' understanding of procedural and conceptual knowledge. This study addresses the following research question: how do secondary mathematics teachers' operationalize the distinction between procedural and conceptual knowledge through task design methodology?

Perspectives

Procedural vs. Conceptual Knowledge

There is a vast existing literature on the differences between procedural and conceptual knowledge. Star (2005) argues that procedural and conceptual knowledge can have a superficial and/or a deep quality. Deep procedural knowledge is "knowledge of procedures that is associated with comprehension, flexibility, and critical judgment and that is distinct from (but possibly related to) knowledge of concepts" (Star, 2005, p. 408) while deep conceptual knowledge is about knowledge of concepts with rich connections.

In this study, we examine different ways secondary mathematics teachers express their procedural and conceptual knowledge by evaluating mathematical tasks that they designed. We view the superficial and deep quality of the two types of knowledge as an intersection between procedural and conceptual knowledge, as shown in figure 1. The arrows at the intersection show how the procedural knowledge can deepen into conceptual knowledge and how conceptual knowledge can be surfaced into procedural knowledge.

Task Design

We employed task design as a methodology to unpack teachers' understanding of procedural and conceptual tasks. Research on task design has been common to study teachers' content knowledge (Gellert et al., 2012). Several studies have focused on pre-service teachers' designing tasks as part of their training (Chinnappan & Forrester, 2014; Hannigan et al., 2013; Rayner et al., 2009). Some studies have focused on designing tasks aligned with technology (Gueudet et al., 2016; Hansen et al., 2016; Misfeldt & Zacho, 2016). Additionally, researchers have created tasks to be used by teachers (Jung & Brady, 2016; Tempier, 2016; Wake et al., 2016) while other researchers discussed the design of tasks with teachers (Coles & Brown, 2016; Johnson et al., 2016; Thanheiser et al., 2016). However, when examining mathematical tasks with in-service teachers, most studies have examined how in-service teachers choose mathematical tasks (Cartier et al., 2013; Roth McDuffie & Mather, 2006). The field lacks studies that use task design as a methodology to address teachers' misconceptions. By asking teachers to design their own tasks we can analyze further their reasoning (Cartier et al., 2013).

Methods of Data Collection

Context

This study was part of a larger project that took place during four years from 2013 to 2016. The larger project was a series of professional development workshops focused on mathematics content. This study took place at a university located on the U.S.-Mexico border. The vast majority of people in this area identify themselves as Hispanics (80%). Many of them are recent immigrants from Mexico. The population of the main school districts reflects the demographics of the city. The workshop was aimed to support the training and retention of secondary school mathematics teachers.

Participants

Workshop participants (N=55) were selected from local secondary schools. Teachers that attended the professional development workshop were from five different school districts across the region. Most of the teachers were female (62%). Also, the majority of the teachers reported their race/ethnicity as Hispanics (81%), 17% reported their race as White, Non-Hispanic, and 2% as African American. Years of teaching experience varied from half a year to 15 years.

Data Sources

All 55 teachers that participated in the study answered a survey that required them to design a procedural and conceptual task and explain their reasoning. The purpose of this survey was to examine the teachers' understanding of procedural and conceptual knowledge. Two topics were used for the survey: area and proportion. In addition, we conducted semi-structured interviews where teachers were asked to talk about their understanding of procedural and conceptual tasks.

Data Analysis

Once the survey data was graded, both researchers analyzed the tasks that the teachers designed and looked for patterns. We were interested in examining closely the types of tasks the teachers designed. The tasks were graded on whether they were surface or deep procedural or surface or deep conceptual. We also looked for patterns on their explanations. The interviews were coded to look for instances in which the teachers talked about the design of mathematical tasks. Emergent codes were extracted using linguistic analysis and meaning coding techniques (Kvale & Brinkmann, 2009). After the authors coded the data separately, the two researchers held meetings to reach a consensus on the codes to separate them into final categories.

Results

Teachers created a wide array of tasks as procedural and conceptual tasks. Table 1 shows the percentage of the types of tasks teachers designed. The table clearly shows that teachers were able to correctly design procedural tasks (78%) while the majority designed a procedural task when they intended to design a conceptual task (80%). There is also a percentage of teachers that created tasks that were ill-designed or provided no answer (i.e., 22% for procedural tasks and 14.5% for conceptual tasks). When the explanations were analyzed along with the tasks that teachers designed, we found different patterns. Tables 2 and 3 show the different codes that were created based on teachers' explanations and the types of tasks. The majority of teachers argue that their task is procedural because it includes a procedure (35%) or because it requires to substitute or "plug-in" values in a formula (29%). When they were designing the conceptual task they argued that their task is conceptual because: it is about finding a relationship (26%), it is a multi-step problem (26%), it is a word problem (23%) or it has a real-world connection (21%).

Besides designing a task teachers had to provide a solution as well as an explanation of why they think the task is either procedural or conceptual. For example, a teacher designed the following task: "solve the following, $\frac{x}{3} = \frac{7}{20}$" as a procedural task. This teacher wrote the following explanation: "Must find x using cross multiplication, then division, very procedural, no connection." For this teacher, this problem is procedural because is about just solving for x. Using Star's (2005) classification, we rated it as a superficially procedural task, The following task was intended as a conceptual task:

Laura types 168 words in 25 minutes, if she continues typing at this rate, how much time will she spend typing a 1500 word paper?

The explanation for this task written by the teacher was: "Because students need to apply what they learned on proportions by solving real-world problems in order to make connections." Based on this teacher's explanation, there is some understanding about conceptual knowledge by using words and phrases like "real-world problem" and "connections." However, upon further examination of this task, we can see that this task requires just procedural knowledge since, after setting up the equation, the solution would look very similar to the previous one. The main difference is that this is a word problem, which would require a student to read the problem, determine if this is a proportional situation, and set up the equation. Therefore, we rated it as a deep procedural task using Star's (2005) classification.

Another teacher designed the following task: "What is the maximum area of a rectangle if the perimeter is 20?" with an explanation that said, "it requires to use prior knowledge of area and perimeter". The use of "prior knowledge" in the explanation might imply that the teacher was thinking about how the student would have to make connections between fixed perimeter and changing area. This task was one of the few that was rated as a deep conceptual task following Star's (2005) classification. During interviews, teachers expressed the desire to design more conceptual tasks but said they need help. For instance, a teacher said about conceptual tasks, "to get them (students) to apply it to the real world and forces them to kind of make the connections, so it's something I think I am improving on, I don't think I am quite at the area but I am improving on it…"

Discussion and Conclusions

Teachers have an understanding of procedural knowledge related to the steps that require solving a mathematical task. While some teachers used language related to conceptual knowledge in their explanations, they face challenges in designing conceptual tasks. For the majority, the actual tasks that they designed illustrate some misconceptions about procedural and conceptual knowledge. This study adds to the growing literature about procedural and conceptual knowledge (Hannigan et al., 2013; Rayner et al., 2009; Rittle-Johnson et al., 2015; Star & Stylianides, 2013) by utilizing task design as a methodology. Based on teachers' (mis)conceptions of procedural and conceptual tasks, more studies need to be conducted to aid teachers not only in selecting tasks but in designing them as well.

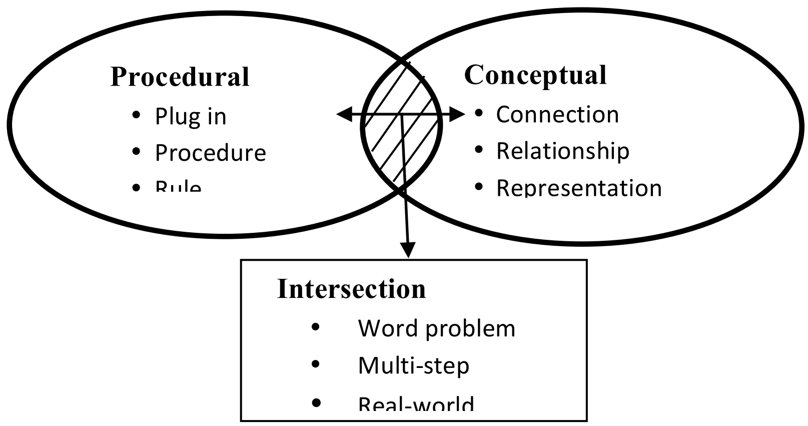

Figure 1: Relationship between Conceptual and Procedural Tasks

Table 1: Teacher designed tasks rated by experts as procedural and/or conceptual

	Procedural	Conceptual
Rated as Procedural	43 (78%)	44 (80%)
Rated as Conceptual	0 (0.0%)	3 (5.5%)
Ill-designed/no answer	12 (22%)	8 (14.5%)
Total	55 (100%)	55 (100%)

Table 2: Number and Percentage of Teacher Explanations for the Procedural Task

Procedure	Word-Problem	Substitute	Multi-step	No real world connections	No explanation	Total codes
18 (35%)	8 (16%)	15 (29%)	4 (8%)	6 (12%)	0 (0%)	51 (100%)

Table 3: Number and Percentage of Teacher Explanations for the Conceptual Task

Procedure	Word-Problem	Substitute	Multi-step	Real world connection	No explanation	Total codes
12 (26%)	11 (23%)	1 (2%)	12 (26%)	10 (21%)	1 (2%)	47 (100%)

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MATHEMATICAL KNOWLEDGE FOR TEACHING:

POSTER PRESENTATIONS

A STUDY ON THE RELATIONSHIP BETWEEN TUTOR'S CONTENT KNOWLEDGE AND THEIR TUTORING DECISIONS

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Keywords: Mathematical Knowledge for Teaching, Informal Education, University Mathematics

When studying mathematics education and student success, most research tends to study the in-classroom teaching aspect. Another important aspect of mathematics education occurs outside the traditional classroom with tutors. While it has been shown that tutoring leads to student success (Xu, Hartman, Uribe, & Mencke, 2001), research has not necessarily focused on what tutoring is or what makes it effective. In recent years, efforts have been made to expand research in this field. Two major themes are the study of the types of knowledge necessary for effective tutoring and the interplay between these domains of knowledge to better understand the tutoring process.

Mathematical Knowledge for Tutoring

Burks and James (2019) began to create a theoretical framework for what constitutes "Mathematical Knowledge for Tutoring (MKTu)" (Burks & James, 2019) derived from Mathematical Knowledge for Teaching (MKT; Ball, Thames, & Phelps, 2008) model. What they determined is that a MKTu model would differ slightly from MKT in that the MKTu would include two overarching domains of *affect* and *self-regulation*. Additionally, certain domains shared by MKT and MKTu may not necessarily be implemented in the same manner. For example, while a classroom teacher is typically expected to be a master of their subject, a tutor is not, and thus, their common content knowledge tends to be more general, with a focus on *solving* problems rather than conceptual understanding. This new framework prompts a number of new avenues for research.

The Study

One such avenue is research into the relationship between a tutor's content knowledge and the pedagogical decisions they make while tutoring. In this poster, we present the results of a study in which we develop and facilitate mock-tutoring scenarios for tutors at a generalist-model tutoring center, and analyze their interactions with an actor-student (Jose Saul Barbosa) through the lens of MKTu, with consideration given to the dimensions for tutoring centers laid out by Byerly et al. (2019). In a generalist model, tutors are not experts in a single content area, rather they have a more general knowledge on a variety of subjects (Byerly et al. 2019). This variation provides an excellent opportunity to study how one's content knowledge interacts with the other domains of MKTu. In addition, we present the results of analyzing brief content assessments associated with the scenarios to draw comparisons between a tutor's content knowledge and the choices they make while tutoring. This study has implications not only for understanding the ways in which a tutor's content knowledge informs their tutoring, but also the ways it interacts with the other domains of MKTu. In studying this, we hope to contribute to future research into determining what factors and decisions can lead to effective tutoring.

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NOVICE TEACHERS INTERPRETATION OF FRACTIONS

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Keywords: Teacher Knowledge, Teacher Education

Past research suggests that teacher's intentions to root mathematics instruction in conceptual understanding is impeded by their content knowledge and pedagogical beliefs (Borko et.al., 1992; Fuller, 1996). A significant source of this knowledge and beliefs may be attributed to their own experiences as mathematical learners (Chicoine, 2004). Implementation of the Common Core Mathematical Standards and Practices, launched in 2009, have emphasized the importance of instruction and learning focused on developing both mathematical procedural knowledge and conceptual understanding. Novice teachers have experienced the implementation of the Common Core during their own k-12 experience, as well as within their teacher preparation program. It may be that the renewed emphasis on productive mathematical dispositions and conceptual understanding present in the Common Core era may be realized in the content and pedagogical knowledge of these novice teachers.

In an effort to better understand novice teacher's mathematics knowledge and the potential influence of development of mathematical understanding under the Common Core, we have designed a study to examine one specific content area- the interpretation of fraction concepts. Fraction were chosen specifically due to the challenge they present to both pre-service and in-service teachers (Ma, 1999). Our data collection initially focused on how novice teachers across the k-12 grade level apply their own mathematical understanding to interpreting fractions and applying fraction operation. To address this area, artifacts of teacher work comparing, adding, and division of fractions was collected. We then compared the solution strategies utilized by the teachers in their own work was compared to the instructional strategies chosen by these teachers in their classrooms. Within this comparison, we questioned teachers to understand the impetus for these instructional decisions.

Results of the study suggest diverse solution strategies to the fraction problems within and across certification levels (e.g. elementary compared to high school). When choosing instructional strategies to apply for the instruction of fraction concepts the school curriculum was not often cites as a common resource. Instead, many of the teachers focused on their own backgrounds and experiences to determine their instructional approaches. This finding is of interest because of the potential for contradictions between teacher held beliefs on appropriate methods of instruction and instruction that exists in school and state curriculums.

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ADVANCING REASONING AND PROOF IN SECONDARY MATHEMATICS CLASSROOMS: INSTRUCTIONAL MODULES FOR SUPPORTING TEACHERS

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Bringing to life a vision of mathematics teaching that emphasizes reasoning and proof across topics and grade levels can be a challenging task for teachers. To support and enhance teachers' content and pedagogical knowledge for teaching reasoning and proof, we developed and systematically studied a capstone course for preservice teachers (PSTs) *Mathematical Reasoning and Proving for Secondary Teachers.* The course emphasized integration of reasoning and proof with teaching secondary mathematics across *any* curricular topic, not just high-school geometry.

During our design-based research project, we developed and tested four instructional modules, each focusing on a particular proof theme: (1) direct reasoning and argument evaluation, (2) conditional statements, (3) quantification and the role of examples in proving, and (4) indirect reasoning. Each module comprised three types of activities: *crystalize, connect* and *apply*. The *crystalize* activities aim to help PSTs refresh their memory of a particular proof theme, within secondary school content. The PSTs enhance their knowledge by solving problems, as well as discussing and clarifying questions or misconceptions, e.g., the difference between proof by contradiction and by contrapositive. In the *connect* activities, the PSTs have an opportunity to connect their mathematical knowledge with knowledge of students' conceptions. The PSTs read cases or watch video or animations of classroom scenarios depicting students working on problems in a particular proof theme. For example, in the *connect* activity of the Quantification and the Role of Examples in Proving module the PSTs analyze a dialog between three students contemplating how to interpret a statement "There exist three consecutive even numbers whose sum is a multiple of four" and what kinds of examples can prove or disprove (if possible) this statement. Next, the PSTs envision possible pedagogical moves to support student thinking in the scenario. The *apply* tasks invite PSTs to identify opportunities in secondary curriculum to make proof themes explicit to students, and to develop a lesson plan that achieves that goal. In our course, we had PSTs implement these lessons in actual middle and high school classrooms, videotape themselves, and reflect of their teaching (Buchbinder & McCrone, 2018).

All four course modules were designed to be independent from each other and applicable for individual use in courses for PSTs or teacher workshops. The design of the instructional modules is grounded in the literature on best practices for teacher learning and professional development, such as focusing on deepening both content and pedagogical knowledge, engaging teachers in active learning experiences, and making direct connections to teachers' classroom practices (American Federation of Teachers 2002; Boston & Smith, 2009, 2011; Copur-Gencturk & Papakonstantinou, 2015). In our poster, we will show the four modules, explicate design features underlying their development, and provide evidence for the effectiveness of the modules.

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TECHNOLOGICAL KNOWLEDGE OF MATHEMATICS PRE-SERVICE TEACHERS AT THE BEGINNING OF THEIR METHODOLOGY COURSES

CONOCIMIENTO TECNOLÓGICO DE LOS FUTUROS MAESTROS DE MATEMÁTICAS AL INICIAR SUS CURSOS DE MÉTODOLOGÍA

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Mathematics pre-service teachers must learn how to use tools like scientific calculators, Computer Algebra System (CAS), text processors and dynamic mathematical environments. These tools allow users to work with mathematical objects, perform specialized tasks, respond in a defined mathematical way, and transmit mathematical knowledge (Dick & Hollebrands, 2011). To achieve the integration of technology in Mathematics Education, the teacher's role is very important, since their beliefs and knowledge will dictate how they use technology in the classroom (Julie et al., 2010).

The goal of this research is to determine the beliefs and knowledge about technology and its integration into the teaching of mathematics by a group of pre-service teachers at the beginning of their first course of methodology in the teaching of mathematics at the secondary level (N=11). Interviews were conducted, and a questionnaire was administered to determine the profile participants use of technology at their schools and universities.

The results show that participants have used scientific calculators, content management software and online platforms. However, they have little experience with programs that allow them to work with mathematical objects (Dynamic Geometry or CAS). The participants have had a low level approach to technology, since both, they and their teachers, use the computer mainly for presentations, and calculators to corroborate results obtained with pencil and paper (Sacristán, 2017). 98% of the pre-service teachers that participate have used word processors, spreadsheets, presentations, emails, and cloud storage for academic purposes. Likewise, 59% of the participants indicated that they learned to use these technologies on their own. In the case their beliefs, participants indicated that the technology oriented to mathematical learning improves the quality of education, increases student participation, cooperative work experiences, individualized learning, and the motivation of students.

The participants have a positive perspective towards the use of technology in the mathematics classroom. However, their experiences and knowledge are not enough to do mathematical work or to teach mathematics. Therefore, it is necessary to expose pre-service teachers to experiences that allow them an integration that facilitates the learning of their students. (Julie et al. 2010; Önal, 2016; Sacristán 2017). We propose that, in the preparation of future teachers, a systematic and theoretical framework of the Technological Pedagogical Content Knowledge (TPACK), which is necessary to teach mathematics, is used (Harris et al., 2009; Rosenberg & Koehler, 2015).

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

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CONOCIMIENTO TECNOLÓGICO DE LOS FUTUROS MAESTROS DE MATEMÁTICAS AL INICIAR SUS CURSOS DE MÉTODOLOGÍA

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Palabras clave: Conocimiento Matemático para la Enseñanza, Tecnología, Educadores de Docentes, Conocimiento del Profesor

Los maestros en formación deben aprender a utilizar herramientas como las calculadoras científicas, los sistemas algebraicos computadorizados, los procesadores de textos y los entornos matemáticos dinámicos. Estas herramientas permiten a los usuarios trabajar con objetos matemáticos, realizar tareas especializadas, responder de una forma matemática definida y trasmitir conocimientos matemáticos (Dick & Hollebrands, 2011). Para lograr la integración de la tecnología en la educación matemática el rol del maestro es muy importante ya que de sus creencias y conocimientos dependerá el uso que haga de ellas en el salón de clases (Julie et al., 2010).

El objetivo de este trabajo es determinar las creencias, el conocimiento sobre la tecnología y su integración en la enseñanza de las matemáticas de un grupo de maestros en formación al inicio de su primer curso de metodología de enseñanza de las matemáticas en el nivel secundario (N=11). Se realizaron entrevistas y se administró un cuestionario para determinar el perfil de uso de la tecnología en la escuela y en la universidad de los participantes.

Los resultados demuestran que los participantes han utilizado calculadoras científicas, programas gestores de contenido y plataformas en línea. Sin embargo, tienen poca experiencia en programas que les permitan trabajar con objetos matemáticos (Geometría Dinámica o "Computer Algebra Systems"). Los participantes de este estudio han tenido un acercamiento con la tecnología de bajo nivel ya que tanto ellos como sus maestros utilizaron la computadora principalmente para hacer presentaciones y las calculadoras para corroborar resultados obtenidos con lápiz y papel (Sacristán, 2017). El 98% de los maestros en formación ha utilizado para fines académicos procesadores de palabras, hojas de cálculo, presentaciones, correos electrónicos y nubes de almacenamiento. Igualmente, un 59% indica que han aprendido ha utilizar estas tecnologías por su propia cuenta. En relación con las creencias, los participantes indican que la tecnología orientada al aprendizaje matemático mejora la calidad de la educación, la participación, el trabajo cooperativo, el aprendizaje individualizado y la motivación en los estudiantes.

Los participantes tienen una orientación positiva hacia el uso de la tecnología en las matemáticas. Sin embargo, sus experiencias y conocimientos no son suficientes para hacer trabajo matemático o para enseñar matemáticas. Por ello, es necesario exponer a los futuros maestros a experiencias que les permita una integración que facilite el aprendizaje de sus estudiantes (Julie et al. 2010; Önal, 2016; Sacristán 2017). Proponemos que en la formación de maestros se utilice sistemáticamente un marco teórico del conocimiento pedagógico, tecnológico y de contenido necesario para enseñar matemáticas (Harris et al., 2009; Hollebrands, 2017; Rosenberg & Koehler, 2015).

Reconocimientos

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EXPLORING MATHEMATICS TEACHER EDUCATORS' AVENUES FOR PROFESSIONAL GROWTH: A REVIEW OF THE RESEARCH LITERATURE

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There is a growing consensus that mathematics teacher educators (MTEs) need a robust knowledge base in order to prepare prospective teachers (e.g., Chavout, 2008; 2009). However, research has shown that the majority of MTEs are not provided the support or training necessary to develop this knowledge (Masingila et al., 2012). Professional organizations have called for improved preparation of MTEs (AMTE, 2017; CBMS, 2012), but the ways in which *MTEs develop Mathematical Knowledge for Teaching Teachers (MKTT)* it is not yet understood.

For the past three years, our research group has been studying the literature on MKTT (Castro Superfine et al., in preparation; Olanoff et al., 2018; Welder et al., 2017). As part of this work, we have conducted an extensive literature search of research on the knowledge MTEs use in their work with prospective and practicing teachers and how this knowledge is developed. Thus far, we have organized the extant literature into five overall themes capturing the ways in which MTEs have developed MKTT. In this poster, we will offer the five themes, summarize pertinent research for each, and discuss implications this work has on the preparation and professional development of MTEs. Below we offer a sampling of our findings.

1. ***Learning through reflective self-study***. Many MTEs who are also researchers have conducted reflective self-studies of their teacher education practices (e.g. Alderton, 2008; Allen et al., 2018; Chavout, 2009; Marin, 2014; Muir et al., 2017; Taylan & da Ponte, 2016). These studies vary (for example, individual vs. collaborative efforts), including whether or not the researchers identified professional growth as a finding of their self-study.
2. ***Learning through communities of practice***. Some groups of MTEs (e.g., Applegate et al., 2020; Jaworski, 2003; Olanoff et al., in press) have formed communities of practice to improve the MKTT of all of the group members and/or initiate novice MTEs into the field.
3. ***Learning through graduate preparation and/or professional development activities***. Few studies describe how MTEs have taken graduate-level courses (Flores et al., 2017), or participated in formal professional development opportunities, as part of their own learning (Hauk et al., 2017; Castro Superfine & Li, 2014). Flores and colleagues (2017) suggest that taking graduate courses or participating in other forms of professional development can provide meaningful opportunities for MTEs to reflect on and build upon their own knowledge.
4. ***Learning through research***. Some authors (e.g., Chauvot, 2008, 2009; Chen et al., 2008) suggest that MTEs develop MKTT by being *Mathematics Teacher Educators/Researchers*. This work can involve studying one's own practices of teaching teachers as well as studying the literature involving the teaching and learning of MKT.
5. ***Learning through doing the work of teaching teachers***. Ball and her colleagues (2008) suggest that MKT is the knowledge required to perform the mathematical tasks of teaching. Building on this, some researchers (e.g., Zopf, 2010) define MKTT as the knowledge required to perform the mathematical tasks of teaching teachers. Zopf and others, such as Jankvist and colleagues (2019) suggest that MTEs develop MKTT on the job by performing the work of teaching teachers.

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1/2, 1/3 OR 2/4? INTERPRETATION OF ELEMENTARY SCHOOL TEACHERS' ANSWERS TO AN EVALUATION

¿1/2, 1/3 O 2/4? INTERPRETACIÓN DE RESPUESTAS DE DOCENTES DE NIVEL BÁSICO A UNA EVALUACIÓN

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Keywords: Teacher knowledge, Concept of number and operations; Rational Numbers

The need to strengthen teachers' mathematical knowledge for teaching in Mexico has been identified as a recurrent topic in the literature (Eugenio & Zaldívar, 2019; García, et al., 2019; Juárez Eugenio & Arredondo López, 2017; OCDE, 2012). Addressing this need, we present results from a diagnostic study administrated to 355 elementary teachers in Mexico enrolled in a course focused on rational numbers. The results are contrasted with both a similar diagnostic administrated to 360 university students enrolled in other programs and the way in which this topic is addressed in textbooks in Mexico and Canada.

We focus our analysis on the distinct representations of mathematical objects, as well as the diverse images, metaphors and analogies related to rational numbers (Lakoff y Núñez, 2000; Núñez y Marghetis, 2014). For the case of rational numbers, we considered the division of objects in equal parts (Confrey et al., 2009). However, our review of textbooks from Canada and the official student resource in Mexico (Comisión Nacional Libro de Texto Gratuito, 2019) reveled that images and descriptions for fractions make emphasis on dividing a shape in congruent parts. However, the area model for fraction requires parts with the same measure of surface area, without the need of being congruent. Figure 1 shows two of the questions in the diagnostic test.

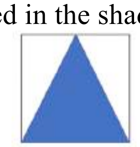
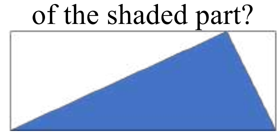
Question 1: Which fraction is represented in the shaded shape?	Question 2: If the area of the rectangle is 12, what is the area of the shaded part?
	

Figure 1: Diagnostic test questions related to rational numbers and the are model.

The results of the diagnostic test administrated to teachers suggest that many of them attended to the shape of the shaded part, instead of its area. The diagnostic to other undergraduate students, which included written justifications of their answers, was consistent with the suggested results from the diagnostic to teachers. In this sense, the answers to Question 1 were 1/2, 1/3 and 2/4. Although 1/2 and 2/4 are equivalent rational numbers, the justifications for 2/4 included the division of the shaded triangle in two equal parts, dividing the square in four congruent triangles. Many people who answered 2/4 did not answer Question 2 correctly, which suggests that they focused on the shape of the parts rather than the surface area.

These results plead for the need to clarify the meaning of "equal" in the definition of fractions, both in teachers' knowledge and textbooks. We consider that this analysis based on representations and metaphors can be extended to other school content.

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¿1/2, 1/3 O 2/4? INTERPRETACIÓN DE RESPUESTAS DE DOCENTES DE NIVEL BÁSICO A UNA EVALUACIÓN

1/2, 1/3 OR 2/4? INTERPRETATION OF ELEMENTARY SCHOOL TEACHERS' ANSWERS TO AN EVALUATION

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Palabras clave: Conocimiento del profesor, Conceptos de Números y Operaciones, Números Racionales

La necesidad de fortalecer el conocimiento matemático para la enseñanza de los profesores en México ha sido identificada como tema recurrente en la literatura (Eugenio & Zaldívar, 2019; García, et al., 2019; Juárez Eugenio & Arredondo López, 2017; OCDE, 2012). Atendiendo esta necesidad, presentamos resultados de un estudio diagnóstico a 355 profesores de educación básica en México en el contexto de un curso enfocado en números racionales. Los resultados se comparan con un diagnóstico similar a 360 estudiantes universitarios en otras carreras, así como la forma en que se aborda el tema en libros de texto de educación básica en México y Canadá.

Enfocamos nuestro análisis en las distintas representaciones de objetos matemáticos, así como las diversas imágenes, metáforas y analogías relacionadas con los números (Lakoff y Núñez, 2000; Núñez y Marghetis, 2014). Para el caso de números racionales se considera la división de objetos en partes iguales (Confrey et al., 2009). Sin embargo, nuestra revisión de libros de texto de Canadá y del recurso oficial en México (Comisión Nacional Libro de Texto Gratuito, 2019), reveló que las imágenes y descripciones de fracción hacen énfasis en la división de una figura en partes congruentes. Sin embargo, el modelo de área para fracciones requiere que las partes compartan la misma medida de su área, sin ser necesario que sean congruentes. La Figura 1 muestra dos de las preguntas del diagnóstico.

Pregunta 1: ¿Qué fracción representa la figura?	Pregunta 2: Si el área del rectángulo es 12, ¿cuál es el área de la parte sombreada?
	

Figura 1: Preguntas correspondientes a números racionales usando el modelo de área

Los resultados de la evaluación diagnóstica a maestros sugieren que muchos pusieron atención a la forma de las partes en lugar del área. El diagnóstico a estudiantes universitarios, que incluyó respuestas escritas con justificación, fue consistente con los resultados sugeridos en el diagnóstico a profesores. En este sentido, las respuestas a la Pregunta 1 fueron 1/2, 1/3 o 2/4. Si bien 1/2 y 2/4 son equivalentes como números racionales, las justificaciones para 2/4 incluyeron la división del triángulo sombreado en partes iguales, dividiendo al cuadrado en cuatro triángulos congruentes. Muchas personas que respondieron 2/4 no respondieron correctamente la Pregunta 2, lo que sugiere que se enfocaron en la figura, no en el área.

Estos resultados dan cuenta de la necesidad de aclarar qué se entiende por “igual”, tanto en el conocimiento del profesor como en los libros de texto. Consideramos que este análisis basado en representaciones y metáforas se puede extender a otros contenidos escolares.

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THE IDEAS OF GIORDANO BRUNO AS A TEACHING ALTERNATIVE, IN ORDER TO UNDERSTAND THE PRINCIPLE OF RELATIVITY

LAS IDEAS DE GIORDANO BRUNO, ALTERNATIVA DE ENSEÑANZA PARA LA COMPRENSIÓN DEL PRINCIPIO DE RELATIVIDAD

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Keywords: Giordano Bruno, Secondary Education, Relativity Principle.

We report what took place in our qualitative research in mathematics education and science teaching, on the Relativity Principle (RP) proposed by Giordano Bruno (GB), which, up until today, has been adjudicated to Galileo. De Angelis and Espirito (2015) report on that situation.

We considered the 1993, 2006, 2011 and 2017 curricula used in Mexico for science teaching, with emphasis in Physics, particularly of the free fall topic, which ask to contrast what was carried out by Aristotle and Galileo Galilei. Bruno's innovation of the relativity principle (1984/1972) is highlighted as follows: "TEO: With Earth move therefore, everything what is on it. Because of that, if, from one place outside Earth an object is launched toward it, it could lose the straightness due to its movement" (p.61).

In the Concept Models (CM) theory, invented by teachers, researchers, engineers, architects in order to facilitate the comprehension or teaching of physical systems, Mental Models (MM) are

> models that people build to represent physical states. They do not need to be technically accurate but they must be functional. They evolve naturally. They are limited by factors such as knowledge, previous experience and the own structure of information processing of human been. (Moreira, 1997, p. 45).

The following problem was given to 117 students of seventh grade in a public school of a metropolitan area near Mexico City:

> Imagine a ship that moves in a constant velocity. From the tallest mast a rock is released. Do you believe that the rock will fall exactly at the mast´s base where the rock was released? Justify your answer. The proposed problem is based on one of Bruno (1584/1972, p. 161).

The following is the most frequent Mental Model from the studied population:

> MM: "No, because, while it falls through the air, the ship moves and the rock doesn't fall in the same place"

Figure 1. Image retrieved from Matías & Gallardo (2019, p. 98).

The studied problem shows the need of analyzing important concepts for the comprehension of relativity principle, making it possible to use Giordano Bruno's proposal as a teaching alternative.

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LAS IDEAS DE GIORDANO BRUNO, ALTERNATIVA DE ENSEÑANZA PARA LA COMPRENSIÓN DEL PRINCIPIO DE RELATIVIDAD

THE IDEAS OF GIORDANO BRUNO AS A TEACHING ALTERNATIVE, IN ORDER TO UNDERSTAND THE PRINCIPLE OF RELATIVITY

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Se reporta lo acontecido en una investigación cualitativa en la enseñanza de la matemática y las ciencias, usando el Principio de Relatividad (PR) propuesto por Giordano Bruno (GB) hasta hoy adjudicado a Galileo. De Angelis, A. y Espirito, C. (2015) también reportan y reclaman dicha situación.

Se contemplaron los Planes y Programas de los años 1993, 2006, 2011 y 2017 usado en México respecto a la enseñanza de las ciencias con énfasis en física particularmente sobre la caída libre, los cuales piden contrastar lo realizado por Aristóteles y Galileo Galilei. Se resalta la primicia del PR en Bruno (1584/1972) como sigue: "TEO: Con la tierra se mueven por tanto, todas las cosas que se encuentran en ella. Por consiguiente, si desde un lugar fuera de la tierra se arrojara algún objeto hacia ella, perdería la rectitud debido al movimiento de ésta." (p. 61)

Se usó la teoría de los Modelos conceptuales (MC): "Aquellos inventados por los profesores, investigadores, ingenieros, arquitectos, para facilitar la comprensión o la enseñanza de sistemas físicos…" En ella, los Modelos Mentales (MM) son

> modelos que las personas construyen para representar estados físicos. No requieren ser técnicamente precisos, sino que deben ser funcionales. Evolucionan naturalmente. Están limitados por factores tales como conocimiento, experiencia previa y la propia estructura del sistema de procesamiento de información del ser humano. (Moreira, 1997, p. 45).

Se aplicó a 117 estudiantes de segundo grado de una escuela pública ubicada en el Estado México el problema siguiente:

> Imagina un barco que se desplaza a velocidad constante, desde el mástil más grande, se deja caer una piedra. ¿Crees que la piedra caerá justo en la base del mástil donde se dejó caer? Justifica tu respuesta. Problema propuesto con base en (Bruno, 1584/1972, p. 161).

A continuación se muestran los MM del grueso de la población analizada.

> **"MM.** No, porque mientras cae sobre el aire, el barco se mueve y la piedra no cae en el mismo lugar"

Figura 1. Imagen recuperada de (Matías y Gallardo, 2019, p. 98)

El problema usado, despierta la necesidad del análisis de conceptos importantes en la comprensión del PR pudiendo usar lo propuesto por Giordano Bruno como alternativa de enseñanza.

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MATHEMATICAL PROCESSES AND MODELING

RESEARCH REPORTS

EXPLORING SECONDARY STUDENTS' PROVING COMPETENCIES THROUGH CLINICAL INTERVIEWS WITH SMARTPENS

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Twenty students who earned A or B course-grades in the proof unit(s) of a secondary course that addressed proof in geometry were asked to work on two proof tasks while sharing their thinking aloud and using smartpens. Students were classified into two categories: those who were successful with both proofs and those who were unsuccessful with both proofs. Large differences were observed in how often students in the two groups exhibited certain competencies. The largest gaps occurred in the ways that students: attended to the proof assumptions; used warrants in their proofs; and demonstrated logical reasoning.

Keywords: reasoning and proof, geometry and geometrical and spatial thinking

Introduction

Proving is an important aspect of mathematical competence, and geometry is the site which has historically been considered to be a good starting point to teach and learn mathematical argumentation and proving in secondary mathematics (Reiss, Hellmich, & Reiss, 2002). Hoyles (2002) suggested that proving involves a range of non-trivial "habits of mind" such as looking for structures and invariants; identifying assumptions; and organizing logical arguments. These processes, she noted, must be coordinated with visual or empirical evidence and mathematical results and facts. The need to coordinate the linguistic and visual registers is a particularly distinguishable feature of proof in geometry (Sinclair, Cirillo, & de Villiers, 2017). Yet, a number of studies conducted in the context of secondary geometry provide evidence that the teaching and learning of proof in secondary geometry is a challenging endeavor (Balacheff, 1988; Cirillo, 2011; Healy & Hoyles, 1998; Senk, 1985). To make progress on this challenge, we sought to gain insight into the competencies and behaviors displayed by students who were successful and unsuccessful with two proof tasks.

Theoretical Framework

A number of studies have documented students' difficulties with proof in geometry. For example, Senk (1985) conducted a study wherein she administered six proof-writing tasks to 1520 students in the U.S. After scoring the tasks on a scale of 0-4 where scoring >3 deemed a student to be "successful" on an item, Senk concluded the following: only about 30% of students in a full-year geometry course reached a 75% mastery of proof, and approximately 29% of students could not write even one valid proof. In another study, in this case, a study of high-attaining 14-15 year-old students in England and Wales, Healy and Hoyles (1998) found that only 19% of students were able to construct a proof of a familiar geometry statement, and fewer than 5% of students could construct a proof of an unfamiliar geometry statement. Last, in their study of 81 German upper secondary students, Reiss, Klieme, and Heinze (2001) found that only 20% of students were able to construct correct Euclidean geometry proofs. Various recommendations and student difficulties were noted in these studies.

Based on study results, Senk recommended that we must look for more effective ways to teach proof in geometry, noting, for example, that we must find ways to support students to start a chain of reasoning. In the studies conducted by Healy and Hoyles (1998) and Reiss and colleagues (2001), the researchers found that students were better at judging others' proofs than they were at constructing their own proofs. Also, in Reiss et al.'s (2001) study, only 10% of students were able to provide a definition of the central concept of "congruence" and name a mathematical theorem related to congruence (i.e., a triangle congruence theorem). Furthermore, this study provided evidence that geometric competence with respect to proving is dependent on a combination of metacognition, spatial reasoning, methodological knowledge, and declarative knowledge (e.g., the notion of congruence).

Some researchers have elaborated on why proof is so challenging for students. For example, Healy and Hoyles (1998) explained that the process of building a valid proof is complex in that it involves sorting out what is "Given" from what can be deduced, and then organizing the conclusions that can be drawn from the "Given" into a coherent and complete argument that meets the proof goal. When considering the large number of possible inferences that could be made from the "Given(s)" in a typical school geometry proof problem, Koedinger and Anderson (1990) noted: "Geometry proof problem solving is hard" (p. 512). To better understand this "hard" situation, they observed geometry "experts" and found that, prior to writing up the details of their proofs, experts tended to quickly and accurately develop an abstract proof plan that skips many of the steps required in the full proof. In other words, they first applied global thinking (i.e., considered the "big picture") rather than local thinking (i.e., worked on one step at a time) at the start of the process. This conclusion is consistent with Cai's (1994) finding in a study of problem solving in geometry that the more-experienced participants spent the majority of their time on orientation and organization, while less-experienced students spent the majority of their time on execution (i.e., doing rather than thinking or planning).

In Battista's (2007) review of school geometry research, he posed several unanswered questions related to students' learning of proof, including: Why do students have so much difficulty with proof? What components of proof are difficult for students and why? and How can proof skills best be developed in students? (pp. 887-888). Ten years later, speaking back to these questions in their research review, Sinclair et al. (2017) concluded that while some researchers have attempted to address these questions, the reported studies tended to focus on only one or a few teachers or did not provide evidence of effectiveness at scale. They also suggested that more research is needed on students' development of geometric proof skills and their understanding of the nature of proof. The lack of research in this area led us to pursue this topic. Thus, this study addresses the following research questions: (1) What proving competencies and behaviors are observed in students who were successful with solving geometric proofs? and (2) How do these competencies and behaviors compare to those of students who were unsuccessful with the proofs?

Methods

The study reported on in this paper is part of a larger research project titled: *Proof in Secondary Classrooms: Decomposing a Central Mathematical Practice* (*PISC;* PI: Cirillo).The goal of the *PISC* project is to better understand the difficulties involved in the teaching and learning proof in secondary geometry and to develop a new and improved intervention to address these challenges. Students who earned high marks (grades of A or B) in the geometry proof unit(s) were selected for individual clinical interviews for this sub-study. The rationale for interviewing students with high marks was to understand what high-performing students were taking away from the proof unit(s). Because past studies have shown that even high-attaining students struggle with non-routine as well as routine proof tasks in geometry (see, e.g., Healy & Hoyles, 1998; Cirillo, 2018), two proof tasks that, in theory, should have been somewhat familiar to the students, were selected for data collection and analysis. We chose triangle congruence proof as a topic for exploration because it is considered to be a central concept in school geometry and because limited time to conduct the interviews in the school setting did not permit us to take up large amounts of students' time with longer problems.

The criteria for participant selection for this sub-study were as follows: (1) students were enrolled in a secondary course that addressed proof in geometry; (2) students earned an A or a B in the proof unit(s); (3) students were identified by their teacher as students who would be willing to share their thinking aloud during the interview; and (4) students completed the full interview protocol in the allotted time; (5) there were no technology glitches during the data collection, and (6) students' success results on each of the two proof tasks were the same (i.e., successful or unsuccessful on both proofs). This selection process reduced the sample size from 31 students interviewed to 20 selected. Participating students spanned Grades 8-11 (ages 13-17).

The first author conducted all student interviews. The goal was to spend about 35 minutes with each student; the mean interview length was about 32 minutes. The full interview protocol consisted of seven items. The first item was a simple "warm-up" task about geometric notation. The next four tasks were selected or adapted from Cirillo and Herbst (2011). The last two tasks, which were the ones selected for this analysis, were full-proof tasks (see Figure 1). Students spent an average time of 7.28 minutes on Task 6 and 5.95 minutes on Task 7. Students were asked to read each task aloud to ease them into talking through the task and guarantee that they had read the "Given" information for each task. Smartpen technology (i.e., Livescribe pens) was used to audio-record the students' explanations of their thinking and capture their pen strokes as they worked through the proofs. This methodology allowed us to capture students' thinking in the form of verbal explanations and simultaneous diagram markings and other written work.

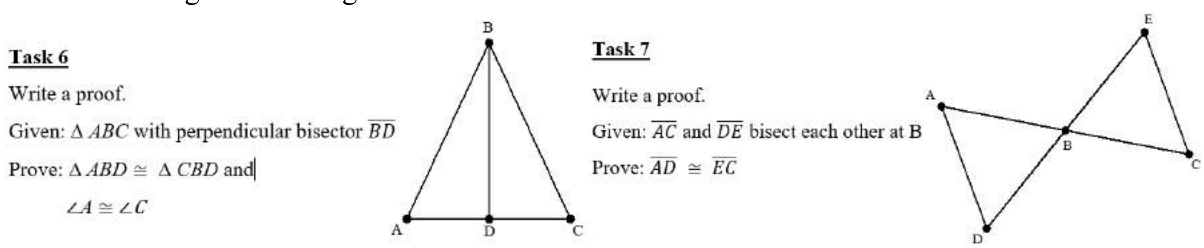

Figure 1: The two full-proof interview tasks analyzed for this study

Smartpen data were digitized to create a "pencast," or video, that simultaneously replays each student's handwriting and audio-recording (Livescribe, 2012). Prior to analyzing the smartpen data, students' final proofs were quantitatively scored from the paper hardcopies in ways that followed Senk's (1985) methods. Specifically, we adapted the rubrics for Senk's full-proof tasks so that every proof was scored on a scale of 0-4. Following Senk's approach, if students scored a 3 or a 4 on a proofs, they were considered to be Successful with the Proof tasks (abbreviated as SP; n=7). Students who scored less than 3 were considered Not successful with the Proof tasks (abbreviated as NP; n=13). The resulting data set consisted of two Proof Task Interviews (PTIs) from 20 students. The units of analysis are the individual proof tasks, resulting in 40 units of analysis. Tables 1 and 2 include age, grade level, course grade (A or B), and task scores (0-4) for each participant in the NP and SP groups.

We used constant comparative analysis (Boeije, 2002) to develop a coding dictionary. To develop the codes, the research team watched the PTI pencasts for six participants for both Tasks 6 and 7. Codes were developed for observed problem-solving behaviors and competencies that were exhibited through spoken and written work. This iterative process resulted in 45 possible codes for both NPs and SPs. After garnering an 86.59% interrater reliability and reconciling incongruent decisions, the second author coded the remaining data. The final phase of analysis involved looking across the coding results for patterns and themes.

Table 1: Study participants who were Not successful with both Proof tasks (NPs)

Participant Number	Age	Grade Level	Course Grade	Task 6 Score	Task 7 Score
P1	14	8	A	0	0
P2	16	10	B	0	0
P3	15	10	B	0	0
P4	15	10	B	0	0
P5	14	8	A	0	1
P6	14	8	A	1	1
P7	17	9	B	1	1
P8	17	10	B	1	1
P9	16	10	B	1	1
P10	15	9	A	1	1
P11	13	8	A	2	1
P12	17	11	B	2	2
P13	17	11	A	2	2

Table 2: Study participants who were Successful with both Proof tasks (NPs)

Participant Number	Age	Grade Level	Course Grade	Task 6 Score	Task 7 Score
P14	14	8	A	3	3
P15	14	8	A	3	4
P16	13	8	A	4	4
P17	13	8	A	4	4
P18	14	8	A	4	4
P19	14	8	A	4	4
P20	13	8	A	4	4

Findings

In reporting the findings, we first describe the most prevalent competencies and behaviors among the students who were **S**uccessful with both **P**roof tasks (SPs; n = 7). Because there were multiple competencies and behaviors that were exhibited by a high percentage of the SPs, we chose a threshold of 60% for the occurrences that would be reported. That is, we report on behaviors and competencies observed at least 60% of time for the SPs. This decision yielded five findings across the data set and includes behaviors related to working with the "Givens," marking the diagram, demonstrating logical thinking, and so forth.

Regarding the second research question about the competencies and behaviors of the students who were **N**ot successful with either **P**roof task (NPs; n = 13), the data were more inconsistent. We first compare NPs with SPs by reporting on the frequencies (as a percentage of the total occurrences for each PTI) with respect to the six findings from the SP group. Table 3 provides a summary of these behaviors and the frequency of occurrence. We then share three additional findings that relate to behaviors observed at least 20% of the time in the NP group.

Competencies and Behaviors of Students Who Were Successful with the Proofs

All seven SPs made productive and explicit use of the "Given" information for both tasks (i.e., 100% of the time). They did so either as they were planning their proofs or as they began to work on a proof. They explicitly identified the relevant mathematical objects from the assumptions (i.e., the "Given"). Here is an example of P14 thinking aloud about Task 7 (P14-T7; from here forward, we will use the notation PX-TY to denote each participant and task): "First thing we know is that ABC and DE bisect each other at B. Why? Because it's the Given. Next. Well you know that B, B is the midpoint. Why? Because definition of line segment bisector…"

Table 3: Frequencies of observed competencies and behaviors for both groups

Observed Competencies & Behaviors (rounded to nearest whole percentage)	**SPs (%)**	**NPs (%)**
***Students productively attended to the "Given" information**	**100**	**23**
Students correctly identified bisectors	100	12
Students indicated what object was being bisected	93	12
Students used the diagram as a resource	**100**	**47**
Students marked the diagram	100	65
Students used the diagram as a check list or planning tool	100	46
Students made valid claims supported by assumptions about the diagram	100	31
***Students identified warrants as postulates, axioms, definitions, or theorems**	**86**	**8**
Students clearly connected claims to definitions	86	12
Students stated or explained a definition	79	4
Students articulated a definition of congruent triangles	100	8
***Students demonstrated that they were thinking in a logical manner**	**93**	**8**
Students attended to important details while working through their proofs	**68**	**13**
Students articulated a plan for the proof prior to writing the proof	64	15
Students attended to rigor in sub-arguments	100	4
Students attended to triangle congruence criteria	79	15
Students attended to the "Prove" statement in explicit ways	64	23

* Indicates the main findings with the largest percentage gap between SPs and NPs (> 75%)

All SPs accurately marked the diagrams for both proofs (i.e., 100% frequency). The smartpen technology enabled us to see noticeable differences in the ways this occurred. For Task 6, SPs tended to mark the diagram in two distinct ways. Either they worked through the details of the proof, marking off congruent parts as they made their inferences, or they marked the congruent parts, using the diagram as a checklist to show that they had proven the triangles congruent. For Task 7, three SPs seemed to immediately recognize how to solve the proof, so they explained a plan for the proof *and* marked the congruent parts prior to beginning the proof (see Figure 2). Each proof task included some type of bisector in the "Given." SPs were clear about what type of bisector they were working with 100% of the time (e.g., line segment bisector). They explicitly indicated what was being bisected 92.9% of the time.

When SPs wrote or articulated their warrants (i.e., reasons for their statements), they typically indicated the typology in explicit ways 85.7% of the time. SPs appropriately connected claims to definitions 85.7% of the time, sometimes even stating the exact definition (78.6%). For both proofs, the concept of congruent triangles was critical toward developing a valid proof. All SPs wrote CPCTC as the warrant for their triangle congruence statements. When asked what it meant or stood for, 100% of SPs were able to articulate what CPCTC stood for or explain what it meant (i.e., Corresponding Parts of Congruent Triangles are Congruent).

As they were thinking aloud, SPs' explanations contained logical connectives, such as "next," "and then," and "we can conclude," in 92.9% of the PTIs. The P18-T6 excerpt above is a good example of this. Also, P20-T6 used logical connectives "then" and "so" in various ways:

> <u>So, we have</u> this is congruent to that and we have that this is perpendicular to that <u>so I guess we could use</u> the right angles theorem to prove that these are congruent <u>and then we could prove</u> that this is congruent by the reflexive property of congruence. <u>And then we can get</u> the angles congruent by C-P-C-T-C.

Although, in this explanation, P20 seemed to skip over the step of stating that the triangles were congruent, it was included in the written proof.

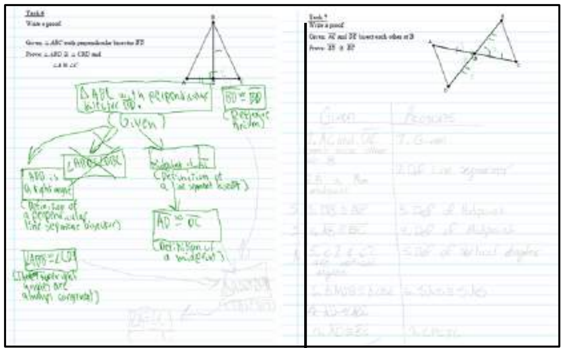

Figure 2: P15-T6 Used Diagram as a Checklist & P20-T7 Used Diagram as a Planning Tool

Arrow 1	Arrow 2	Arrow 3 and Triangle Congruence Statement
		

Figure 3: P16-T6 attends to sub-arguments and triangle congruence criteria

SPs attended to the details of their proofs in multiple ways. They articulated a plan for the proof before writing the proof 64.3% of the time. SPs attended to rigor in their sub-arguments 100% of the time. They attended to triangle congruence criteria in explicit ways 78.6% of the time. And they explicitly attended to the "Prove" statement 64.3% of the time. P16-T6's work provides evidence of attending to sub-arguments (i.e., branches of a proof claim and consequence) and triangle congruence criteria. His written work (see Figure 3) is very methodical in that he established three congruent parts prior to drawing in the arrows in his flow proof to connect the three congruent statements to the triangle congruence statement:

> Ok so then we have our three parts [draws three arrows]. So, we know that these are congruent and then we can say that triangle ABD is going to be congruent to triangle C, CBE. [Pause] I had to take a moment there to see which point was corresponding with point A. So, then we have our two triangles. And we can say this, because of S-A-S theorem. And after this, we can use my favorite theorem again to say that line segment AD is congruent to line segment EC because of C-P-C-T-C. So yeah.

The smartpen allowed us to see how the student worked out the three congruence statements prior to writing and then drawing arrows to the triangle congruence statement. We can see from the combination of transcript and smartpen images that he attended to triangle congruence criteria when he said: "we have our three parts," paused to accurately write the triangle congruence statement in a way that matched up the corresponding parts, and drew the three arrows before writing the triangle congruence statement.

Competencies and Behaviors of Students Who Were Unsuccessful with the Proofs

Large discrepancies between SPs' and NPs' behaviors were noted in the data. Table 4 includes frequencies from the PTIs (as percentages) for both groups for each finding and sub-finding. The differences in occurrences of the six main findings range from 35-92% with a gap of more than 80%

for 3 of the 6 main findings (as indicated in the table by *). For the second research question, it is not so productive to go through each finding one-by-one because there is little to say about the absence of something. Instead, we address the kinds of behaviors observed in NPs given the absence of many of the competencies exhibited by high percentages of SPs. Because the percentages of occurrences of common behaviors were much lower in this data set, we chose a lower threshold of >20% for the codes that we discuss in this section.

First, there were multiple issues noted with the ways in which students dealt with the "Given" information. In particular, 23.1% of the time, NPs incorrectly stated the "Given" when they read it aloud. The most common error for both tasks was saying "line" rather than "line segment." Second, 46.2% of the time, NPs omitted notation or information when they wrote the "Given" statement in the first line of their proof.

NPs' warrants were vague 61.5% of the time. They often did not identify warrants as postulates, definitions, or theorems, and they frequently did not seem to know definitions of relevant concepts. For example, P12-T6 wrote "Definition of bisect" as a reason for Line 3. Yet, in order for the corresponding statement, ∠ABD ≅ ∠CBD, to be true, $\overline{BD}$ would have had to have been an *angle* bisector rather than a *perpendicular* bisector. In the next line of the proof, the student's warrant references the particular diagram, rather than a definition.

NPs displayed a lack of confidence in 34.6% of the PTIs. They sometimes would say that they could not remember definitions or reasons for their statements. For example, when asked about a reason for one of the steps in her proof, one NP said, "I don't have one." This statement also indicates a lack of using logical reasoning. Other comments made by NPs as they shared their thinking included: "I don't know how to do this one" [P1]; "I don't know how to explain it" [P4]; and "I remember having a lot of trouble on this because I didn't understand" [P11].

Discussion and Conclusions

Making use of smartpen technology, we explored the competencies and behaviors of high-attaining students with the expectation that even high-attaining students would have gaps in their abilities to prove. Clear differences were noted in the approaches taken by students who were successful with the two proof tasks, compared to students who were unsuccessful with those tasks. These findings contribute to the research on the teaching and learning of proof in geometry, specifically, the first part of Battista's (2007) question: What components of proof are difficult for students and why? There are several important take-aways from these findings.

First, when we compare the frequency percentages of competencies and behaviors observed in the two student groups in Table 4, the differences are relatively large. For example, the difference between how often SPs were observed productively attending to "Given" information compared to how often NPs did so was 77%, with SPs doing so 100% of the time and NPs only 23% of the time. This finding indicates that more work is needed to support students to productively attend to proof assumptions.

With respect to geometric diagrams, 100% of SPs marked their diagrams, made valid claims supported by assumptions about diagrams, and used the diagrams as a check list or a planning tool, particularly when solving a proof at the "global" level (Koedinger & Anderson, 1990; Cai, 1994). In contrast, NPs did not always mark their diagrams; they sometimes used the diagrams to plan; and they rarely made appropriate assumptions about the diagrams. Rather than using the "Given" information to draw valid conclusions, NPs put forth inappropriate inferences which often seemed to come from what the diagram "looked like." Doing so follows the perceptual proof scheme described by Harel & Sowder (2007).

There was also ample evidence to suggest that SPs had strong understandings related to the typology of warrants and that they understood how to use them to develop proof arguments. In contrast, this competency was rarely observed in the PTIs of NPs. Instead, NPs were sometimes observed justifying statements using reasons related to particular diagrams (e.g., B is the midpoint). Additionally, at times, NPs simply wrote, as their warrants, some mathematical object (e.g., bisector) without writing anything further, such as clearly identifying the type of bisector or explicitly stating whether they were thinking about a theorem or definition. NPs explicitly connected claims to definitions, stated definitions, and were able to articulate a definition of congruent triangles less than 15% of the time. These issues connect to Reiss and colleagues' (2001) claims about the importance of coordinating declarative knowledge with higher order skills. Regarding higher order skills, evidence that students were reasoning through their proofs in a logical manner was observed 93% of the time for SPs, compared to only 8% of NPs.

The likelihood of SPs attending to important details in their proofs was also much greater than the likelihood of NPs doing so. The area of greatest difference was with respect to sub-arguments. SPs attended to sub-arguments in rigorous ways 100% of the time; NPs did so only 4% of the time. This finding relates to logical reasoning in that a sub-argument is a chain of reasoning that begins with an assumption and involves more than one deduction (Cirillo, Murtha, McCall, & Walters, 2017). SPs were also observed attending to other kinds of details in ways that were not observed in the NP data, such as: developing a plan for their proof (i.e., thinking at the global rather than local level); attending to triangle congruence criteria; and attending to the "Prove" statement. SPs were observed doing these things most of the time; while NPs were not. These findings are consistent with Healy and Hoyle's (1998) descriptions of what makes proving so complex. In particular, they argued that proving involves: sorting out what is "Given" from what can be deduced and then organizing the conclusions that can be drawn from the "Given" into a coherent and complete argument that meets the proof goal.

The use of smartpen technology allowed us to "see" and analyze the data in ways that we would not have been able to see or do without it. For example, the active ink feature in the pencast videos allowed us to track the ways in which students marked their diagrams and how they shifted back and forth between diagram and proof. We conjecture that this kind of analysis is only the tip of the methodological iceberg in terms of what is possible to do with this and other tracking technologies. What kinds of insights into student thinking might be gained from more studies like this one, is an open but exciting question.

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DEVELOPING A FRAMEWORK FOR CHARACTERIZING STUDENT ANALOGICAL ACTIVITY IN MATHEMATICS

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This report proposes a framework for describing student analogical reasoning activities in abstract algebra that moves beyond the traditional literature-based treatment of analogical mapping. The Analogical Reasoning in Mathematics (ARM) framework captures the activities that students engage in when anticipating, creating, and reasoning from mathematical analogies. This considers activities along several dimensions including: inter/intra domain activity, foregrounded/backgrounded domain, and attention to similarity/difference. These dimensions are integrated with Gentner's (1983) analogical mapping framework to characterize student activity when they are presented with tasks where reasoning by analogy can productively support their mathematical investigations. By characterizing these activities, we can better develop tasks to support students in productively analogizing between mathematical domains.

Keywords: Advanced Mathematical Thinking, Algebra, Analogical Reasoning

Throughout history, analogies have played a vital role in the development of key mathematical concepts and connections such as Descartes' recognition of the analogical similarities between algebra and geometry (Crippa, 2017). In modern mathematics instruction, analogies have been argued to be useful in developing student conceptual understanding by assisting students in utilizing prior knowledge to make sense of new contexts and develop conceptual understanding rich in connections across mathematical domains. However, unguided analogical reasoning may result in unproductive mathematical reasoning (Sidney & Alibali, 2015). By investigating the nature of students' mathematical reasoning as they develop and reason from analogies in mathematical contexts, we can begin developing support for students to productively reason by analogy.

Outside of mathematics education, Gentner (1983) introduced the Structure-Map Theory (SMT) to describe analogical reasoning as mappings across domains. In particular, this notion of mapping requires attention to similarity across domains. This type of focus is consistent with much of the theories that followed (e.g. Holyoak & Thagard, 1989). Attending to similarity is crucial for generating analogies. Surface similarities provide an access point for the generation of analogies (Holyoak & Koh, 1989). These components of analogical reasoning are shown in Figure 1 below.

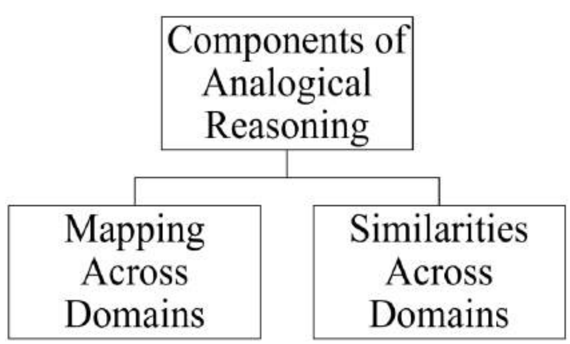

Figure 1: Components of Analogical Reasoning

Although the framework above is useful for categorizing a particular analogical mapping, there exist nuances of analogical reasoning within mathematical thinking that are not explicitly captured by

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this framework. In this paper, I present the Analogical Reasoning in Mathematics (ARM) framework for characterizing the mathematical activity of students as they anticipate, create, and reason from analogies between mathematical domains. In particular, this framework expands upon the ubiquitous components of mapping across domains and attending to similarities, and introduces a new component of foregrounding a domain. This interpretive qualitative study seeks to contribute to answering the following research question:

What are the mathematical activities of students as they reason by analogy about structures between group theory and ring theory?

Theoretical Framing

I adopt Gentner's SMT as a foundation for developing a conceptual framework for identifying and describing analogical reasoning in mathematics. In particular, I borrow the concept of *domains* and the process of *mapping across domains* as a basis for identifying and describing analogical reasoning in mathematics. I define analogical reasoning as the act of identifying or conjecturing about a perceived correspondence between two (or more) domains. In addition, I define *analogical activity* as mathematical activity occurring within and around analogical reasoning.

A domain is a collection of knowledge held about a mathematical concept or situation. For example, one could reason specifically about the domain of two-digit addition problems, or more generally about the domain of binary operations. A key aspect of reasoning by analogy is to utilize knowledge in one domain in order to develop knowledge within another. This process occurs through mapping across domains. The domain from which knowledge originates is known as the *source domain*, while the domain to which source knowledge is being applied is the *target domain*.

I also adapt the content of mappings described by Gentner to this conceptual framework. Gentner proposed that three categories of content may be mapped between domains: objects, attributes and relations. Objects can be a single entity (e.g., a triangle), component parts of a larger object (e.g., the angles of a triangle), or coherent combinations of smaller objects (e.g., all equilateral triangles) An attribute is defined as a property or description of an object. This could include a definitional property of an object or a non-definitional descriptor of an object. Relations are properties that relate two or more objects, attributes or other relations together. A visual of these aspects in the context of an analogy between a ruler and a number line can be seen in Figure 2.

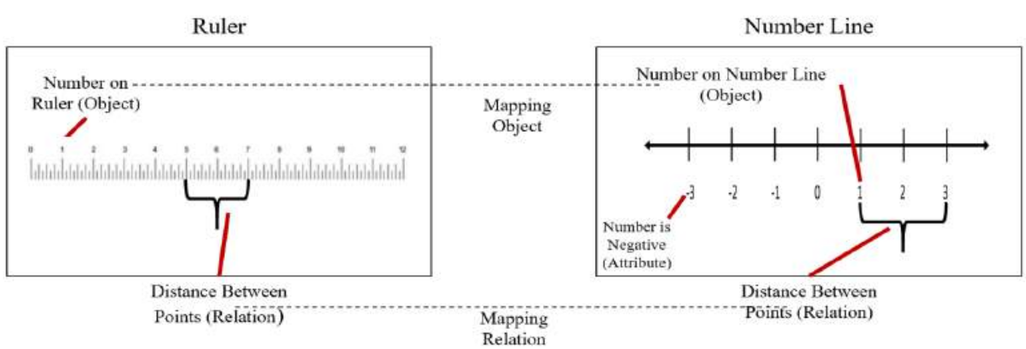

Figure 2: Mapping Between a Ruler and a Number Line

Methodology

Data Collection

The context of abstract algebra was chosen because of the existence of several naturally occurring structural similarities between group theory and ring theory. I conducted an initial pilot study with two mathematics students, one an undergraduate student in pure mathematics, and the other a graduate student pursuing a PhD in mathematics education. The participants in the present study included two undergraduate mathematics students who had previously taken a course in modern algebra emphasizing the theory of groups. Five 60-90-minute-long task-based interviews were conducted with the each participant. The initial interview helped to assess the participant's content knowledge of group theory before beginning to explore topics in ring theory. The ring interview provided the participants with the definition of ring and various tasks designed to acclimate the student to working with rings. The three subsequent interviews provided students with the analogizing task focused on reconstructing one of subrings, ring homomorphisms, or quotient rings by analogy with a structure in group theory. The interview tasks were constructed around three basic types: (1) Explicit elicitation of analogy generation, (2) example generation and checking (i.e., "give an example of a subring."), and (3) proof-writing (i.e., "Is the homomorphic image of a commutative ring commutative?") An example of a task meant to elicit explicit analogies is the following: *Make a conjecture for a structure in ring theory that is analogous to subgroups in group theory.*

Data Analysis

I used techniques outlined by Corbin and Strauss (2015) to analyze the data. First, the transcripts were segmented by identifying shifts in a particular mathematical idea or focus. Segments were identified by two criteria: (1) presence of analogical activity, and (2) shifts in mathematical focus. Each of the six interviews in which the analogizing task was given was coded for mapping activity and attending to similarity, as well as open coded for other aspects of analogical activity. Microanalysis was intermittently performed on segments when the nature of the analogical activity was unclear within a segment. Diagramming was incorporated to aid in making sense of how concepts fit together with one another. As a part of ongoing analysis, I wrote research memos to aid in explicating my thinking about concepts and generating new hypotheses. Results of microanalysis, diagramming, and memoing were regularly shared with colleagues to assist in ensuring the viability of my interpretations. Finally, the interviews focused on group theory and rings were used to help triangulate interpretations of the students' activity when possible.

From this process of coding and subsequent axial coding, two dimensions of activity related to analogical reasoning were identified in addition to the activity of mapping and attending to similarity, and one component of analogical reasoning was identified. These are each discussed at length in the following section.

Results

In this section, I share an overview of the Analogical Reasoning in Mathematics (ARM) framework with extended attention on categorizations that did not exist in the literature on analogy: intra-domain activity, and attending to differences. I then describe the component of foregrounding a domain which allows for deeper description of student mathematical activity while reasoning by analogy. Finally, I exhibit examples of mathematical activity during analogical reasoning that are characterized with the aid of the expanded framework.

The Analogical Reasoning in Mathematics (ARM) Framework

As shown in Figure 1, the literature on analogy is heavily focused upon mapping activity and attention to similarity. Within Gentner's (1983) framework, the heart of analogical reasoning is the process of mapping between domains. I refer to activity in which mapping occurs as inter-domain activity due to the nature of the activity as necessarily involving two or more domains along with activity occurring across those domains. However, there are times when a student may not engage with mapping activity. Instead, the student engages with activity that lies completely within a single domain. This category of activity is referred to as *intra-domain* activity. Intra-domain activities are those activities that operate within a single domain, either by focusing specifically on one domain, or by blending domains together. In addition, students may also attend to differences during the process of reasoning by analogy. Figure 3 exhibits the expanded framework for analogical reasoning including three components of analogical activity along with the dimensions of each component. Those marked with a '*' represent new aspects identified within the present study.

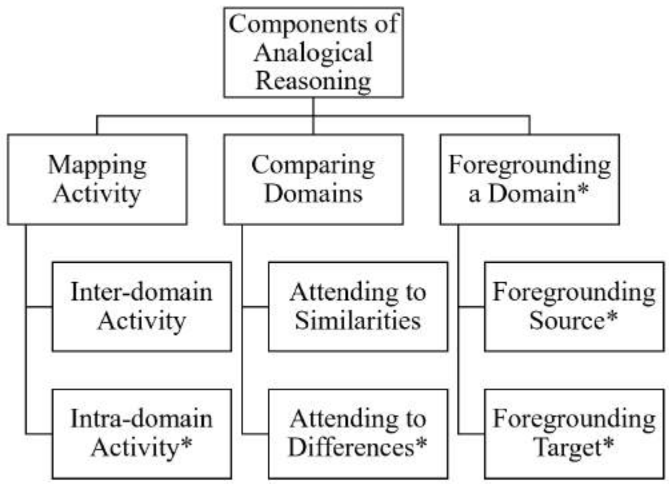

Figure 3: The ARM Framework

Foregrounding a Domain. The participants in this study were seen to place emphasis on different domains throughout the process of reasoning by analogy. This component of activity is referred to as *foregrounding a domain*. Consider the following quote from a student providing a rationale for their definition of what they call "normal subrings":

> Well I used the normal subgroup definition to apply it to normal subrings just because that act of using an operation I guess to apply it to your subring and making sure it's still and the ring itself is kind of the point of a normal subgroup so it's got to be important to use that for subrings or for normal subrings.

Within this first example, the student is justifying her definition of 'normal subring' by specifically pointing to the definition of normal subgroup and claiming that she "applied it to normal subring." The student is emphasizing the source domain as the motivation for her definition as is thus foregrounding the source domain of groups.

In contrast, consider the following exchange in which another student is reasoning about a proposed idea of normality in rings:

> Student B: I don't know. We didn't talk about normal rings, so I didn't even think about that. I don't know. Well, with rings, you do have a lot more conditions than you do with groups, so I don't think I would need this normal on my ring because that's already included in my ring. So, I don't think I would have to mess with it.
>
> Interviewer: Okay. Could you explain what you mean by that? Like you say, "It's already included in the ring."
>
> Student B: Groups have four conditions or something to be called a group, and then rings have seven of them. So, since rings have more conditions, I think being normal is already one of those conditions, in a sense.

Within the second example, the focus of the student's reasoning is on the concept of 'normal rings' to which he then draws comparisons to normal subgroups. Thus, in this second example, the student is foregrounding the *target* domain of rings.

Characterizing Activity Using the ARM Framework

Through reintegration with the widely identified dimensions of mapping activity and similarity, the dimensions of intra-domain activity and attending to differences, and the component of foregrounding domains offer insights into detailing ways in which student mathematical activity during analogical reasoning can be characterized and interpreted. In this section, I use the ARM framework to characterize several identified mathematical activities during analogical reasoning. Table 1 provides a brief overview of the activities characterized and described in this section.

Table 1: Overview of Identified Activities

Activity	Description of Activity	Dimensions of Activity
Exporting	Projecting known aspects of the source domain into the target domain.	• Inter-domain • Attending to similarities • Foregrounding source
Importing	Selectively pulling aspects of the source domain into the target domain.	• Inter-domain • Attending to similarities • Foregrounding target
Recalling	Recalling information that one possesses about the source domain.	• Intra-domain • Neither similarities or differences • Foregrounding source
Distinguishing	Recognizing an anomaly between the source and target domain.	• Inter-domain • Attending to differences • Foregrounding source
Adapting	Modifying the target to accommodate a distinction between the source and target.	• Inter-domain • Attending to differences • Foregrounding target

Exporting and Importing Across Domains. The activity of exporting across domains occurs when a student projects an aspect of the source domain over to the target domain. In contrast, importing occurs when a student selectively pulls aspects over from the source domain into the target domain. The terms "exporting" and "importing" are chosen purposefully to be analogous to the meaning of the words in the context of international trade in the sense that you export outward from the country in which you reside, but import into the country in which you reside.

Consider the following example of a student exporting attributes from the source domain of groups to the target domain of rings:

> So, normal subgroup… First condition is that H is a subgroup of G, and the second condition is that gHg^{-1} is a part of H, and then you can say, therefore, H is normal to G. So, now we're going to call this normal subring. We give this one a name. First condition is that S is a subring of R. Second condition, I don't know. Maybe we say rSr^{-1} is in S, just to copy it.

A visual of this student's work is seen in Figure 4 below. In this example, the student is constructing a definition for what they call "normal subrings." The student constructs a "normal subring" by copying over known aspects of normal subgroups into the context of ring theory.

To contrast with the activity of exporting, consider the following example from the pilot study in which a student is making a conjecture about what comes next in the study of ring theory after having developed the concept of subring:

> Like Abelian rings, or like... giving them that type of thing where you give them special names… Special types of rings, "This is the golden ring." So these, you gave me these properties on the last page. But I'm sure if you have all these properties, it's probably a special type of ring.

In this example, the student is importing the objects of “special” groups, such as Abelian groups, from the source domain into the target domain. The emphasis is on the target which is evident by the fact that the student does not immediately assume that there are Abelian rings just as there are Abelian groups. Rather, this student has discriminately chosen which aspect of group theory they wished to pull over into the target domain of rings.

Characterizing the activities of exporting and importing with respect to the components of mapping and comparing, it is clear that each of these activities are examples of inter-domain activity focused on attending to similarity. They are indistinguishable by examining these two components alone. However, a distinguishing feature can be determined by examining which domain is being foregrounded while engaging with the activity.

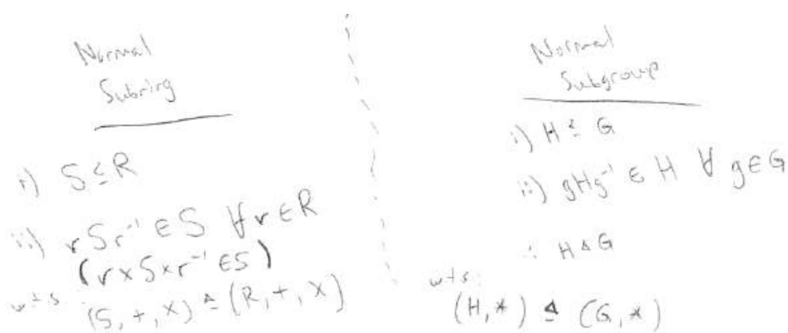

Figure 4: Student Exporting Properties from Source to Target

Recalling Source Knowledge. By expanding the component of mapping across to domains to include intra-domain activity, a greater range of mathematical activities during analogical reasoning become observable. One such activity is that of *recalling source knowledge*. Consider the following example of a student recalling attributes after being asked to make a conjecture for a structure in ring theory analogous to group homomorphisms:

> So, let me just try to recall that… So, group homomorphism. So, there exists a phi that maps from *A* to *B*. So, *A* ... or, I guess maybe it'd be easier to say phi maps from $(A,*)$ to $(B,*)$. So, phi of *a* equals some *b*.

In this case, the student is foregrounding the source and is neither attending to similarity or difference. Although the student is not explicitly engaging in reasoning by analogy in this example, the student is recalling information about the source domain with the intent of utilizing the information for the purpose of analogical reasoning. This is evidenced in the following quote from the same student in which the student exports from the source:

> So, now let's talk about ring homomorphism. Homomorphism. I don't know, I feel like it would be the same thing. There exists, let's just say psi or something from one ring to another ring such that psi is one-to-one. Or, I don't know what you say, onto.

Distinguishing and Adapting. Just as expanding the component of mapping to include intra-domain activity revealed new perspectives on analogical activity, so too does adding the dimension of attending to differences along with similarities provide new insight. One such activity is *distinguishing* between domains. Distinguishing occurs when a student recognizes an anomaly between the source and target domain. Consider the following example in which a student is making observations about the definition of ring:

> So it has ... what I noticed immediately is that, it has two binary operations addition and multiplication. Whereas with group theory we're only dealing with one binary operation at a time.

It is clear from this example that the student is attending to differences and mapping objects between the source and target domains. During distinguishing activity, the student is foregrounding the source. This is due to the nature of anomalies being detected within the target only by a direct comparison to what is already known about the source.

The attention to differences in examples such as this impacted the student's reasoning by analogy later on in the interview process through the activity of *adapting the target domain*. Adapting occurs when a student modifies the target to accommodate a distinction between the source and target. Consider the following statement from the student as they conjectured about the definition of ring homomorphism:

> What would be one for.... We have two [operations] here. Start with phi going from G to H. There's two operations here so I'm like, I don't exactly know if it should just be one of them, or both of them, or how I would do that here. Could I do like three elements, like *a*, *b*, and *c*, and then have like the addition and multiplication?

A visual of this student's work is seen in Figure 5. In this instance, the student is keying in on the difference she identified between the domains two interviews prior and attempting to adapt the homomorphism property she learned in group theory to the context of rings. Adapting activity is characterized by inter-domain activity and attending to differences. Unlike distinguishing activity, adapting activity foregrounds the target since the focus is on constructing new information within the target.

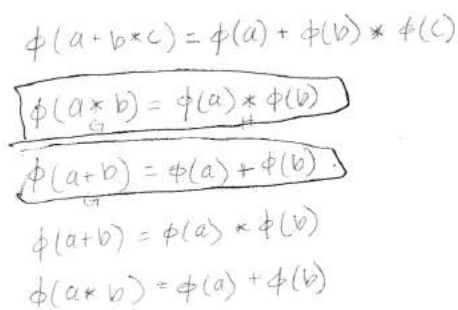

Figure 5: Student Adapting a Structure to the Target

Discussion

An Application to a K-12 Context

Although the ARM framework was developed in the context of abstract algebra, I argue the framework has utility across mathematical contexts. I use an example of students' reasoning about integer operations to illustrate the utility of the framework to capture analogical activity in other contexts. Consider the following statement in which a child is solving the problem -5 – (-3) = ☐ (Bishop et al, 2014).

> Five minus 3 is an easy fact for me. So, um, using negatives it will probably be the same thing like using normal numbers. It will probably be the same thing, but with negatives it'll probably be negative 2.

The child is mapping between the source domain of "normal" numbers and the target domain of negative integers. Because they are emphasizing their knowledge about how subtraction of positives work, they are foregrounding the source. In addition, it is clear that the child is attending to similarities (i.e., "…it will probably be the same thing…") This characterization of the child's activity allows us to see that the child is engaging in the activity of *exporting* from the source domain. To summarize this brief analysis, this child appears to be relying heavily upon their knowledge about "normal numbers" in order to understand a subtraction problem with negative numbers.

Conclusions and Implications

Analogical reasoning in mathematics provides students with the opportunity to develop their conceptual understanding in mathematics that is rich in connections across mathematical contexts (Sidney & Alibali, 2015). Within this study, students were provided the opportunity to leverage analogy and analogical reasoning to make comparisons across domains and reinvent mathematical structures by analogy. By characterizing mathematical analogizing activity, we can support students in coming to create connections across mathematical domains by providing a tool for carefully analyzing how students engage with analogical reasoning in mathematics.

For the purposes of research, the ARM framework provides a foundation for analyzing student analogical activity specific to mathematics. Thus, the framework can be used for generating insight into student's thinking involving analogical reasoning. The results in this paper have only shown a snapshot of students' analogical reasoning in an interview setting. Further research can focus on how student's leverage analogical reasoning over an extended period of time. Finally, the ARM framework and the analogical activities identified in this paper can aid instructors in developing tasks which leverage analogical reasoning by attending to the possible activities of their students while working the tasks. Future research should address ways in which to foster more productive reasoning by analogy in the context of instruction.

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HOW MATHEMATICAL MODELING ENABLES LEARNING?

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In this theoretical paper we compare the Piagetian perspective on knowledge construction to mathematical model construction, with the aim to understand how mathematical modeling enables learning of mathematics and learning of science, as is often claimed. We do this by examining data through two lenses:(i) examining the role of cognitive conflict as it arises during validation of a model and (ii) viewing model validation as a reflection on activity-effect relationship. We explain why we chose to look deeply into model validation specifically, present examples for each lens, and consider implications.

Keywords: Learning Theory, Modeling, Cognition

There has been much interest over the past few decades in the teaching and learning of mathematical modeling. Typically, investigations seek to understand the process of model construction. However, research has also looked into how learning of curricular mathematics beyond modeling may occur as students generate and validate their mathematical models (Zbiek & Connor, 2006). Taking on a Neo-Vygotskian, socio-cultural perspective, Zbiek and Connor elaborated on the cognitive processes that constitute modeling as to situate thinking about how learning takes place during mathematical modeling. In addition, empirical studies have also shown how a modeling approach to instruction may have an impact on student achievement (e.g. Czocher, 2017; Schukajlow et al, 2012). At the same time, two lines of inquiry have used mathematical modeling as an instructional paradigm to guide students' construction of mathematical knowledge. The first uses mathematical modeling tasks to teach mathematical concepts (Lesh et al., 2000) and the second uses the term model to capture the evolution of conceptual models through mathematical activity (Gravemeijer, 1999; Lesh, Doer, Carmona, & Hjlmarson, 2003). Both lines of inquiry agree that mathematics can be learned through constructing models. However, for one to know how mathematical modeling can best be leveraged to learn mathematical concepts, one first needs to understand how mathematical modeling may enable learning. In this paper, we illuminate data drawn from cognitive modeling task-based interviews using two theoretical lenses on mathematical modeling in order to elaborate how learning may enabled through mathematical modeling.

Perspective on Learning and Knowledge Construction

In order to understand how learning is occasioned through modeling we take on a Piagetian view on learning and knowledge construction. In this view, learning is considered as a process of transforming one's way of knowing and acting. According to Piaget, all construction consists of activity and all activity is goal-directed. In this sense, all construction (of cognitive structure) is goal directed (von Glasersfeld, 1983). Hence, we begin from the position that mathematical modeling is a goal-directed activity and the modeler is working towards an anticipated model as a goal. Two theories have been highlighted in the constructivist perspective as ways of learning to occur: the theory of equilibration and reflective abstraction. To support our view of modeling as a process of construction, we adapt both these views to mathematical modeling and compare their merits.

Theory of equilibration

One tenet highlighted in constructivist theory is that conceptual transformation is induced by a perturbing experience. *Perturbation* is experienced when the cognizing subject is met with a constraint or clash in the externalized world and therefore goes through adaptation to regain

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equilibrium (absence of clashes). According to Piaget, disequilibria is stimulated by conflict, either between an individual's action schemes and external realities or among different schemes within an individual. The cognitive structure undergoes assimilation and accommodation repeatedly until it seems "fit" in the externalized world. A *scheme* is an intellectual structure that organize events as they are perceived and classified according to common characteristics. *Assimilation* is the cognitive process by which a person integrates new matter into existing schemata or patterns of behavior. Assimilation does not result in a change of schemata, but it does affect the growth and its part of the development. *Accommodation* modifies the cognitive structure (scheme) to make it "fit" the external world. According to Piaget, accommodation can happen in two ways: one can create a new scheme in which to place the new stimulus or modify an existing schema so that the stimulus fits into. Both forms of accommodation result in change in the configuration. Piaget refers to the process of assimilation and accommodation as *adaptation* (Wadsworth, 2004).

Scholars have since explored the contours of disequilibria and cognitive conflict in different ways. Limon (2001) defined cognitive conflict as something that occurs when a students' mental balance is disturbed by experiences that do not fit their current understanding. Zazkis & Chernoff (2007) stated cognitive conflict is "invoked when a learner is faced with a contradiction or inconsistency of his or her ideas" (p. 196). Berlyne (1970) elaborated cognitive conflict as "a condition in which mutually interfering processes occur simultaneously and in which selection of a motor response from a set of competing alternatives is therefore hampered" (p. 968), which is more amenable to empirical work seeking to understand it in the context of mathematics teaching and learning. Zazlavasky (2015) argued that perplexity, confusion and doubt are often associated with and evoked by cognitive conflict, suggesting that they may be used as proxies for identifying instances of cognitive conflict. Within the literature on mathematical modeling, Lesh et al (2003) identified three kinds of cognitive conflicts arise as models are constructed: within-model mismatches, model-reality mismatches, and between-model mismatches. Researchers have studied how cognitive conflict influences or changes a students' conceptual understanding (Chan, Burtis, & Bereiter, 1997; Ernest, 1996). At the same time, there is also a body of research questioning the role of cognitive conflict in the learning of a concept with evidence that cognitive conflict is only one of the many important factors contributing to learning a concept (Kang, et al., 2004; Zimmerman & Bloom, 1983).

Theory of Reflective Abstraction

The theory of equilibration only considers how a conceptual change is established when there is a presence of clashes between the cognizing subject and the stimuli. However, it is incapable of explaining how we learn during the absence of clashes. Reflective abstraction addresses this issue. Piaget's (2001) reflective abstraction is a process by which higher level mental structures could be developed from lower level structures. This is done in two phases. In the first phase, the structure at the lower developmental level is projected onto a higher level and in the second phase these structures are reorganized (Campbell,2001). Piaget (2001) acknowledged that reflective abstraction is not necessarily a conscious process.

Reflective abstraction was a significant contribution to addressing the learning paradox (Pascual-Leone, 1976) because it allows for knowledge to be constructed from already-existing knowledge. Simon and colleagues elaborated on reflective abstraction, offering a new explanation for conceptual learning in mathematics that not only addresses the learning paradox but also can contribute to the basis for the design of mathematics instruction (Simon, Tzur, Heinz, & Kinzel, 2004). The mechanism, Reflection on Activity-Effect Relationship (Ref*AER) builds on von Glaserfeld's (1995) tripartite model of a scheme: (1) recognition of a certain situation (S), (2) specific activity associated with that situation (A), and (3) the expectation that the activity produces a certain, previously experienced result or the anticipated the activity-effect relationship (A/E) (Tzur & Simon, 2004). According to Simon and colleagues, an occasion that can result in learning is present when a learner sets a goal (G). The goal is then assimilated into situations (S) that are part of the learner's existing conceptions. From the set of conceptions related to S, activities (A) are called upon to work towards the goal to which the learner anticipates the effect of these activities (A/E). While carrying out these activities, the learners' mental systems engage in continual monitoring, including distinguishing effects of the activity that advance the goals from effects that do not advance them. During the reflection, the learner identifies patterns in the outcomes and abstracts a relationship between the activity and the effect it had on reaching the goal. This abstraction results in a new activity-effect relationship. Here, activities refer to mental activities, the learners' goal are not necessarily conscious, and the effects are the assimilatory conceptions that the learner brings to the situation.

Perspective on Mathematical Model Construction

We view mathematical modeling as a goal-directed activity. To elaborate the modeling process, we appeal to the cognitive perspective on modeling (Kaiser, 2017) where a mathematical model is considered to be a cyclic process that transforms a real-world problem into a mathematical problem. From this perspective it is common to represent model construction through a mathematical modeling cycle (MMC) such as Blum & Leiß's (2007) characterization. Empirical studies have described dimensions along which a model can change as it is constructed (Czocher & Hardison, 2019) and different ways a modeler can validate her model (Czocher, 2018). Validation is a crucial part of mathematical modeling, because non-viable models are of little use for solving real-world problems. In many mathematical modeling cycles, validating occurs at the end of the process (e.g. Blum & Leiß's, 2007). However, Czocher (2018) argued that validating not only occurs when one checks the final results against the real-world phenomena, she attempted to model but in different ways throughout model construction. When a student attempts to validate her model, she holds two models in her mind: the model she is constructing and the model she anticipates constructing. As a consequence of this comparison, the modeler chooses to accept, revise, or reject the model she is constructing. In this way, validating is responsible for the iterative nature of modeling as well as ongoing monitoring (Czocher, 2018). Therefore, we conclude that since (a) the outcomes of validating lead to modifications of the model, and (b) modelers validate both their final products and monitor their evolving models, validating has a significant contribution in model construction.

For these reasons, we argue that looking deeply into model validation will lead us to understand how learning happens through modeling. To move the field forward, the paper focuses on what happens during validation that leads to the acceptance, rejection or revision of the model, specifically by looking at model validation through two related but different lenses: (1) cognitive conflicts during model validation and (2) viewing validation as a ref*AER. Informed by the review of the relevant constructs, we conceptualize cognitive conflict that arises during validating the model as a discomfort the modeler experiences due to a perceived discrepancy between the model under construction and the model she anticipates constructing. At the same time, validating can be seen as a reflection on Activity-Effect relationship. When a student engages in a modeling task, she is working towards a goal(G) of modeling a real-world situation. To reach this goal, she calls upon activities or activity sequences (A), which she had previously abstracted as having certain effects (A/E), that will help her to map her understanding of the real-world situation to a mathematical structure. While executing these activities, she then monitors the effects of these activities through the interpretation of her constructed model. Then, validation is the reflection that compares the anticipated effect to the constructed effect. As Simon and colleagues stated, "the ability to set the goal subsumes the ability judge the results" (2004, p.318).

We make the case that if cognitive conflicts and reflective abstraction contribute to the construction of knowledge, then in the mathematical modeling context, it is through model validation that cognitive conflict and ref*AER enable learning. This paper first presents an analysis using the first lens to investigate the decisions made during validation, addresses the constraints, and then presents the second lens that could address the limitations of the first.

Methods

Data for this study were drawn from a larger study of one-on-one modeling task-based interviews with undergraduate STEM majors at a large university in the United States. The students were enrolled in a semester course on differential equations. The overarching goal of the interviews were to explore and document students' mathematical reasoning during modeling. We present examples, to illustrate our case, from one student Jayden, working on the falling body problem.

> The falling body problem: On November 20, 2011, Willie Harris, 42, a man living on the west side of Austin TX died from injuries sustained after jumping from a second-floor window to escape a fire at his home. What was his impact speed?

Jayden was purposefully selected to look deeply into the mechanisms of model validation, because he employed multiple strategies to model the scenario and exhibited observable modeling mechanisms that helped us in explaining our lenses on model validation. Our primary research goal was to build second-order models (Steffe & Thompson, 2000) of his mental activities to explain the factors that shaped his decisions about revising his mathematical model (or not) as an outcome of his engagement in model validation. Since we did not have direct access to Jayden's mental activities, the second-order models are what we inferred from Jayden's observable activities including his language, verbal descriptions and discourse, written work, and on occasion gestures, when they were salient.

For our retrospective analysis of Jayden's engagement with the falling body task, we carried out five rounds of data analysis to arrive at examples that could serve for theory-building. First, we coded the interview for instances of validating, using the method of constant comparison and according to the operationalization in Czocher (2018). Next, we surveyed the validating instances for any identifiable cognitive conflicts and these instances were isolated. Third, we selected examples illustrating cognitive conflict to seek evidence of learning. Fourth, we catalogued instances of validation that failed to be instances of cognitive conflict. Fifth, we applied ref*AER to explain the failed examples. Below, we share illustrations of the third and fifth steps.

Findings

Lens 1 - Cognitive Conflicts during Model Validation

We offer two illustrations of when cognitive conflict arose for Jayden during model validation. We leverage the illustrations to explain how Jayden modified the model under construction to accommodate the anticipated model or otherwise left the conflict unresolved.

Jayden began from kinematics equations and successfully modeled the falling body situation without accounting for air resistance. He justified his choice, asserting that air resistance would be negligible "when there is either no air or no fluid to fall through, or you were infinitely close to the ground." The interviewer challenged Jayden to take air resistance into consideration. In response, he constructed a first order, linear, homogeneous equation to model the falling body. He wrote $\frac{dQ(t)}{dt} + \beta Q = 0$, where Q represented the position of the body and $\frac{dQ}{dt}$ represented its velocity. He then wrote the generic solution $Q(t) = Ce^{-\beta t}$. Jayden wrote $Q(t) = Ce^{-\lambda t}$ with the intention of figuring out "what λ has to equal". Jayden modeled the situation with the initial condition for position as $Q(0) = 0$. Later, Jayden indicated that he was not sure if the model he constructed was correct. Jayden stated, "I'm not sure if that's right, I'm not sure if there should be some sort of constant increase as you get faster". He drew two graphs showing an increasing relationship between velocity and the air resistance (figure 1). However, he was unsure which representation best matched the situation. He indicated that the linear relationship or the exponential relationship will determine if $\frac{dQ(t)}{dt} + \beta Q$ would equal zero or would equal a forcing term, respectively.

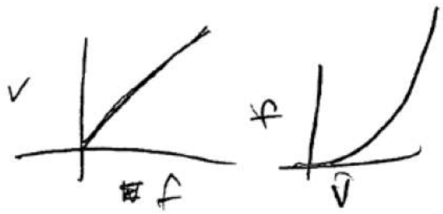

Figure 1: Students' representation of the relationships between drag force and velocity

He continued to solve the differential equation assuming the initial positions and initial velocity to be zero. He substituted the general solution $Q(t) = Ce^{-\beta t}$ in $\frac{dQ(t)}{dt} + \beta Q = 0$ and obtained the expression $\lambda Ce^0 + \beta Ce^0 = 0$ which resulted in$\lambda = -\beta$ (figure 2). He then engaged in validating the model he presented by commenting on the reasonableness of it by stating the following:

> it doesn't really tell me a whole lot because I don't know what the graph should look like. I feel like it probably equal some sort of forcing term…because I don't think that the solution would end up being…as he increases in position, I don't think it's going to be $Ce^{-\beta t}$ ... I don't think that this correctly models it.

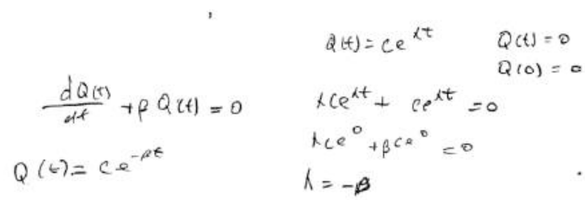

Figure 2: First order linear differential equation with initial conditions

Jayden engaged in model validation when he commented on the reasonableness of the model. Here, the model under construction is the mathematical expression based on the assumption that the velocity and force change linearly and the anticipated model is the mathematical expression based on his assumption that "as the velocity gets larger, the force might get greater and greater and greater". Jayden was experiencing a conflict between the model he constructed and the model he idealized, hence anticipated.

Jayden was able to resolve the conflict when he realized that "the wind is always just an opposing force [so] it could be treated like the force of friction." He then rejected his mathematical model by attempting a different solution that used Newton's laws of motion because they incorporated the surface area of the body and air resistance. In this episode, Jayden attended to the model under construction by modifying the assumptions that the model was based on in order to accommodate the anticipated model. We inferred, based on his sketches, that his anticipated model was his idealization (based on his real-world knowledge) that as the velocity increases the force due to air resistance should increase nonlinearly.

Next is an example where Jayden left the conflict unresolved. Assuming the presence of air resistance, Jayden modelled the falling body using Newton's laws of motion, taking into the consideration the surface area of the falling body and a coefficient to capture the influence of air resistance. He introduced the downward force that the body would experience as $F = ma,$ the air resistance as $F_{wr} = \mu_w \cdot s_a$, and the net force the body would experience as the addition of the two forces. Here μ_w is was the coefficient of air resistance and s_a was the surface area (Figure 2). However, he mentioned that the velocity should be somewhere in these equations as well. This was evidenced by the following statement he made:

> I just kind of thought of something. His velocity should be somewhere in here also. Because the faster you go the more the force will be…but I have no clue how to put that in.

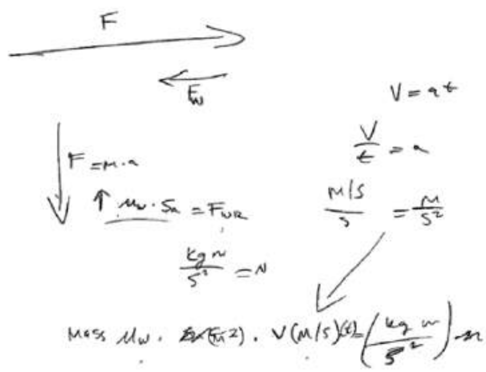

Figure 3: Student's model of the falling body including air resistance and surface area.

In order to incorporate velocity in his model Jayden performed a dimensional analysis to balance both sides of the equation in terms of units. He equated $1N = \frac{kg \cdot m}{s^2}$, to the units of $\mu_w \cdot s_a$. While performing the dimensional analysis, he decided that the surface area should not be there. He scratched out the symbol for surface area and instead added the "change in velocity for a time" of the body to the expression (Figure 3). He equated the mass of the body times the coefficient of air resistance times the "change in velocity for a time" of the body to net the force that the body would experience due to air resistance. After arriving at the aforementioned model, Jayden explained:

> Intuitively I don't think I trust that…I mean that's the answer that I reached, but I really think that has something to do with the surface area. Because this pencil will drop faster [drops his pencil from his hands] than a big piece of paper weighing the same amount…so I don't know.

In this instance, Jayden validated the model by commenting on the reasonableness of it, appealing to his lived experiences. Jayden's statement that the model was not trustworthy indicates that he experienced a cognitive conflict. In this case the model under construction was the mathematical expression he produced (without surface area) and the anticipated model was his idealized view of the world, where an object's surface areas affects its velocity through air resistance. Jayden indicated that he did not know how to rectify the dispute and therefore presented this as the final expression for wind resistance. He then discussed how he would set the force equal to $\frac{dQ(t)}{dt} + \beta Q$, obtained from earlier work, in order to find the falling man's impact speed. In this scenario, Jayden accepted his model. However, the conflict was left unresolved.

While analyzing cognitive conflicts during model validation was a useful way to look at what happened during model validation that led to the acceptance, rejection, and revisions of the model, there were limitations to it. First, taking this perspective assumes that learning during mathematical modeling only occurs during the rejection or/and revision of the model. This is not necessarily true. Learning could also happen when one is satisfied with the model and accepts it because accepting the model may also have transformed the modelers way of knowing and acting about the model. This perspective ignores this case. Second, not all validating instances coincide with instances of cognitive conflict. Therefore, it is necessary to explain such instances where model validation is present, but conflict is not. The second lens of looking at validating was drawn upon to address some of these limitations.

Lens 2 – Validation as a Ref*AER

The following is an example of model validation which could not be explained through the first lens, can now be explained by viewing validating as a Ref*AER. Recall the scenario where Jayden modelled the falling body with air resistance with the expression $\frac{dQ(t)}{dt} + \beta Q = 0$ and initial values $Q(0) = 0$. While considering the initial conditions to solve the differential equation, Jayden stated:

> I'm just trying to think what initial conditions I need to use. I guess I'll have to just say… $Q(0) = 0$ because his position is 0. But I guess it will be better if I said that this was [pause] let's see [long pause] … I guess this is fine [pointing at $Q(0) = 0$].

Jayden validated his model through evaluating the reasonableness of the initial condition $Q(0) = 0$. However, he was not experiencing a conflict because there was no evidence for a discrepancy between the model under construction and the anticipated model. When Jayden stated "I'm just trying to think what initial conditions I need to use" we take that as an indication of him recalling the activities that would lead him to the desired effect and filtering the ones that would not. Here the goal is to solve the differential equation (G), the activity is drawing on the appropriate initial condition (A), and the effect is what comes out of solving the differential equation using the selected initial condition (E). Jayden first considered the initial condition $Q(0) = 0$, and next he considered whether they would advance him toward his desired goal. This is evident when he said, "But I guess it will be better if I said that this was…" Through reflecting, Jayden ultimately conformed to his initial choice $Q(0) = 0$, and therefore accepted his model. In this instance, Jayden was continuously monitoring and reflecting on the effect of selecting $Q(0) = 0$ as the initial condition would have towards reaching his ultimate goal.

The following is an example where Jayden rejected his model, which can also be explained using the Ref*AER lens. Jayden's initial approach was to draw from the equations of motion from mechanics. To find the impact speed of the falling body, Jayden wrote the equation $s = ut + \frac{1}{2}at^2$, where s is the distance the body travelled, u is the initial velocity, a is the acceleration due to gravity, and t is the time it took to travel a distance s. As soon as he realized that the equation contains the time of fall t, Jayden scratched out the expression and resorted to $v^2 - u^2 = 2as$. The reason being the first expression required the time of fall, which was not given in the task. This was an instance of validation because he scratched out the first expression and attempted a different solution. However, there was no evidence of conflict. In this instance, the goal for Jayden was to find the impact speed without using the time of fall (G). He stated, "I could find the time of fall, but it's not necessary". His activity (A) was selecting $v^2 - u^2 = 2as$ over $s = ut + \frac{1}{2}at^2$ through cataloguing existing equations and reflecting on the effect they had in reaching the desired outcome (E). As a result of validating, he rejected his initial expression and selected another one to meet his desired effect.

Discussion & Conclusions

This study investigated the mechanisms of model validation through two lenses: (i) looking at cognitive conflicts that arise due to the discrepancy between the model under construction and the anticipated model, and (ii) viewing model validation as a reflection on activity-effect relationship. Our analysis offers insight into potential mechanisms for model construction and suggests a strong link between model construction and Piagetian explanations of knowledge construction. Studying the nature of cognitive conflicts students experience while engaging in mathematical modeling and viewing model validation as Ref*AER may be an avenue towards elaborating how learning occurs through mathematical modeling because it may inform us about how students make decisions about the viability of their models.

Given the preceding analysis, we close with two considerations: limitations and future directions. This study only informs us how learning may be enabled through mathematical modeling and is not capable to inform us on what was learned. At the same time, the paper does not discuss the explicit treatment of the two lenses and how they can be leveraged to analyze the mechanism of model construction, yet. Future analysis will investigate this. In order to understand what was learned through modeling, instances of validating will be analyzed closely, using the lenses presented in this paper, to see the following: why do modelers chose to accept, revise, and reject the models? how do they do so? and in what ways? However, this theoretical paper outlines the extent to which these learning theories are applicable to mathematical modeling. This we believe is a significant contribution as it sets us open to understanding what is it that is being learned through mathematical modeling. These mechanisms can then be leveraged to develop instructional theory that fosters mathematical conceptual learning through mathematical modeling.

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ESTABLISHING A CARTESIAN COORDINATION IN THE ANT FARM TASK: THE CASE OF GINNY

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In this report, we present how one prospective elementary teacher (PT) engaged in the Ant Farm Task, which we designed to investigate PTs' reasoning about coordinate systems. We highlight the cognitive resources the PT drew upon in solving the task via the establishment of a Cartesian coordination and consider educational implications.

Keywords: Geometry and Geometrical and Spatial Thinking, Representations and Visualization, Modeling, Coordinate System

From fifth grade onward, students and their teachers are often expected to represent and reason about various mathematical concepts (e.g., geometrical shapes, functions, etc.) using the Cartesian plane. However, coordinate systems are often taken for granted; meaning, students are assumed to develop proficiency in using this representational tool in relatively unproblematic ways, and teachers are assumed to have developed understandings of coordinate systems capable of supporting their students' mathematical activity. Additionally, textbooks and curricular standards (e.g., CCSSM) describe how to draw and use a Cartesian plane but rarely provide motivation for establishing a Cartesian coordination. Generally, the rules of "generating" a Cartesian plane are given with minimal explanation for *why* we construct it in such a way or *why* using an ordered pair of numbers locates a point. In this report, we present how one prospective elementary teacher (PT) engaged in the Ant Farm Task (AFT), a task we found helpful in motivating a Cartesian coordination. We highlight the cognitive resources the PT drew upon in solving the task via establishing a Cartesian coordination and consider educational implications of the task and our findings.

Theoretical Framing

By *coordinate system* we mean a representational space in which an individual systematically coordinates quantities (Thompson, 2011) to organize some phenomenon. A coordinate system does not represent by itself; it must be created and interpreted by a cognizing subject (cf., von Glasersfeld, 1987). Put differently, we consider coordinate systems to be constructed by an individual in goal-directed activity. Relatedly, we have distinguished between two types of coordinate systems depending on the goal they serve: spatial and quantitative (Lee, Hardison, & Paoletti, 2020).

Spatial coordinate systems are used to quantitatively organize a space in which a phenomenon is situated. Constructing a spatial coordinate system involves (mentally) overlaying a coordinate system onto some physical or imagined space being represented and tagging objects within that space with coordinates. For example, consider a Cartesian plane overlaid onto an amusement park from a bird's eye view where the axes are aligned with two streets in the park. On the other hand, quantitative coordinate systems are used to coordinate sets of quantities by constructing a geometrical representation of the product of measure spaces. Constructing a quantitative coordinate system involves an individual extracting quantities from the space in which a phenomenon occurs and projecting them onto a new space, different from the space in which the quantities were originally conceived. For example, imagine a Cartesian plane with one axis representing the amount of wait time for a ride in the amusement park and the other axis representing the number of people in line for that ride.

Both coordinate system types involve coordinating quantities. Our use of *spatial* and *quantitative* as modifiers is intended to foreground the different mental actions involved in establishing a coordination of quantities in each type of coordinate system. We use the Cartesian plane as an example to illustrate this distinction. When constructing a *spatial* Cartesian plane, an individual starts with a space they want to organize. In order to quantitatively describe the location of objects or points within this space, the individual can establish a reference point and orthogonal lines through the reference point and use these frames of reference to describe each point's location in terms of its horizontal and vertical distance from the reference point. In other words, the individual can establish a Cartesian coordination via decomposing the location of a point along two orthogonal lines in relation to a reference point (see Figure 1a). In this case, the point's location is conceived of as a logical multiplication (Piaget et al., 1960) of the horizontal and vertical displacement from the reference point.

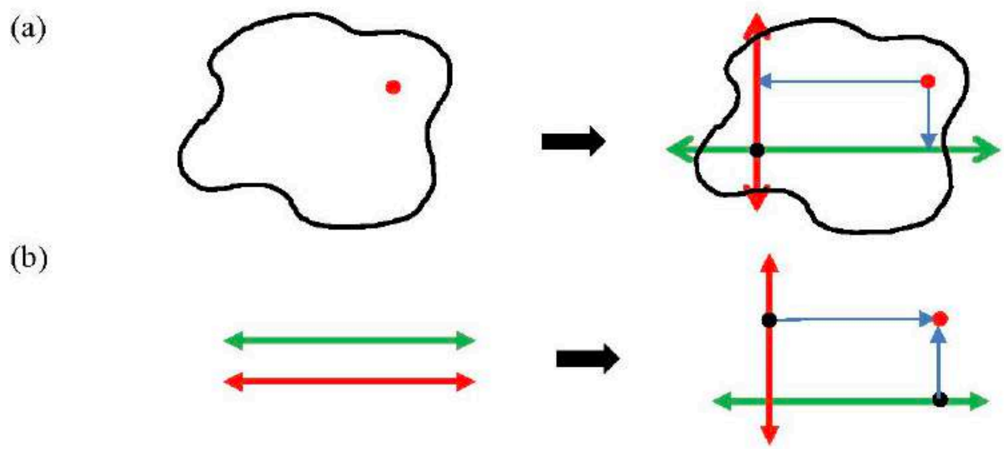

Figure 1: Model of the operations involved in establishing (a) a spatial Cartesian coordination and (b) a quantitative Cartesian coordination

In contrast, when constructing a *quantitative* Cartesian plane, an individual starts with two quantities they have disembedded (Steffe & Olive, 2010) from the space in which a phenomenon occurs (i.e., the individual has extracted them from the situation while maintaining an awareness of the quantities within the situation). Overlaying the quantities onto two number lines, and arranging the number lines orthogonally, the individual can produce a two-dimensional space, different from where the quantities were originally conceived. Finally, a point is constructed as the intersection of the perpendicular projections from each point/quantity on each number line (see Figure 1b) and a quantitative Cartesian coordination is established. In this case, the point is conceived of as a multiplicative object holding both quantities' values simultaneously (Saldanha & Thompson, 1998).

Students' Cartesian Coordinations in Literature

Several studies provide insights into students' Cartesian-like coordinations described above in graphing contexts. For example, Nemirovsky and Tierney (2004) presented a situation to Rose, an eight-year-old, in which blocks were added to or taken away from a paper bag over time. Rose was asked to show how the number of blocks in the paper bag changed over time using a line marked with Start and End. Rose produced a curve (Figure 2a) to represent the change in quantity over time.

This example shows a child spontaneously using the space above the line to show change in quantities over time. DiSessa et al. (1991) and Sherin (2000) presented a scenario describing the motion of a motorist over time to middle and high school students and asked them to produce a picture describing the motorist's motion. In solving this task, students tended to use a horizontal line representing the road and made marks such as dots and line segments to represent quantities (e.g., speed). Gradually, the horizontal line transformed into a number line representing time and some students produced a graph such as that in Figure 2b to describe the motorist's speed over time. This example demonstrates how middle and high school students utilized the vertical distance from a horizontal line to represent change in speed over time. Collectively, these examples illustrate how students, starting with a horizontal line, can use a vertical dimension to represent change in quantities over time and hence construct Cartesian-like systems. However, they do not explain how students might establish a Cartesian coordination starting with two (number) lines.

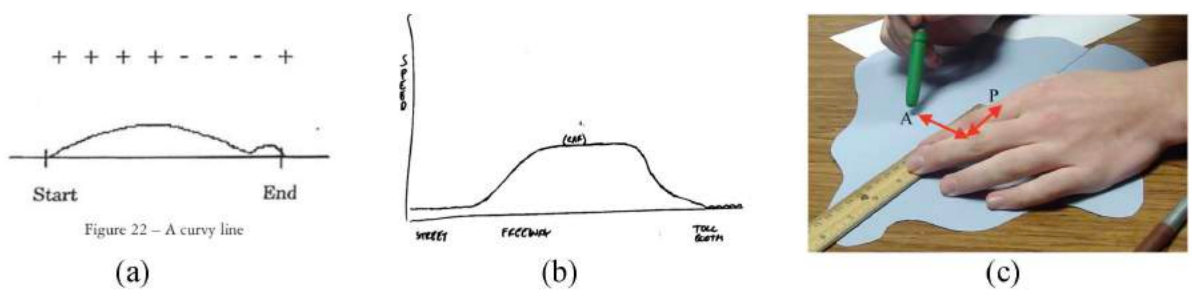

Figure 2: Two examples: (a) Rose's "time line" representation in Nemirovsky & Tierney (2004), p. 42 (b) Carl's representation in Sherin (2000), p. 432

In her previous work with four ninth-graders, Lee (2017) examined students' constructions of spatial coordinate systems and observed students establishing a spatial Cartesian coordination. For example, when asked to locate a missing person (point *A*) in reference to a rescuer (*P*) on a map, one ninth-grader described the location of *A* in relation to *P* by decomposing the straight motion from *P* to *A* into a horizontal and vertical movement from P to A (see Figure 2c). After observing students' constructions of spatial coordinate systems, we were motivated to explore how students might leverage their ways of coordinating spatially in order to coordinate quantitatively via the AFT (Lee & Hardison, 2016; 2017). The findings we present in this report extend the literature base by (a) examining PTs' constructions of coordinate systems which can inform educational support for PTs, which are scarcely documented and (b) identifying the cognitive resources that may be leveraged to engender a spontaneous Cartesian coordination starting with two (number) lines.

The Ant Farm Task

In previous work, we have hypothesized that spatial coordinations necessarily precede quantitative coordinations (Lee & Hardison, 2017). We designed the AFT as a possible way of bridging these two types of coordinations. In contrast to the above tasks, we designed the AFT to (a) be entirely situated in a spatial context in which PTs could leverage their spatial coordination, (b) start with two given lines, and (c) have the potential for engendering some of the mental actions involved in establishing a quantitative coordination. In the AFT, PTs were provided with two transparent tubes representing two ant farms (Figure 3a) and asked to imagine that each contained exactly one giant ant moving around haphazardly. Additionally, we provided a model of this situation in a dynamic geometry environment (DGE; Figure 3b); the DGE sketch contained two long, thin rectangles (ant farms), each containing a point (ant) moving haphazardly. The points' movement could be paused/activated by action buttons, and the rectangles could be moved or rotated within the DGE. Given this scenario and the DGE sketch, we asked PTs, “Can you make a single point to show the locations of both ants at any moment in time?” Following this prompt was an explanation that if we were to hide both ants, they should be able to use their new point to determine the location of the two hidden ants. We presented the spatial situation without referencing any quantities explicitly (e.g., an ant’s distance from the end of the tube). One possible solution to the AFT involves establishing a conventional Cartesian coordination. Through the AFT, we investigated (a) how PTs might modify their ways of coordinating spatially to coordinate quantitatively and (b) what cognitive resources PTs draw upon when constructing a two-dimensional coordinate system from two one-dimensional lines.

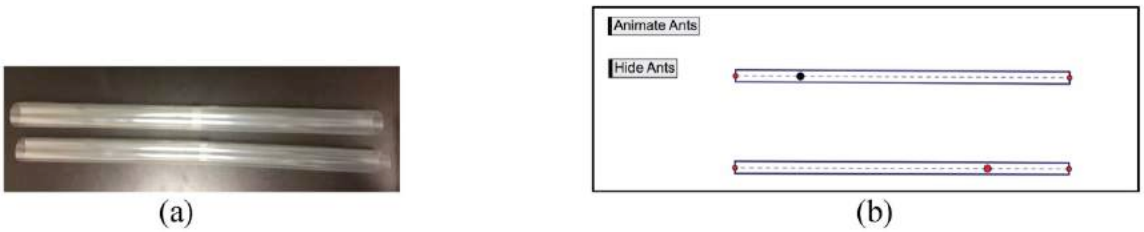

Figure 3: Ant Farm Task (a) plastic tubes representing the ant farms and (b) accompanying dynamic geometry environment sketch.

Methods

We draw on data from a teaching experiment (Steffe & Thompson, 2000) with four PTs. The overarching goal of the teaching experiment was to investigate how PTs construct and reason about coordinate systems, as well as how PTs’ ways of thinking changed throughout the teaching experiment. All four PTs were enrolled in an elementary or middle grades teacher preparation program at a university in the southern U.S. PTs participated in eight 60-minute long teaching sessions, which were conducted individually or in pairs. In this report, we present and analyze data from one teaching session wherein one PT, Ginny, and her partner, Hermione, solved the AFT. We focus our analyses on Ginny with occasional remarks regarding Hermione as appropriate. We focus specifically on Ginny because her solution to the AFT contained features common to other PTs’ activities on the task and a unique feature—introducing number lines.

Data Sources and Analysis Methods

Investigating PTs’ mathematical thinking, which is not directly accessible, requires making inferences from PTs’ observable activities. Therefore, the models of Ginny’s thinking we build are second-order models (Steffe & Thompson, 2000) of what we infer from her visual illustrations, verbal descriptions, and physical gestures. For each teaching episode, we collected video recordings of PTs’ actions, a screen recording of PTs’ activities in the DGE, and digitized written work.

We conducted both on-going and retrospective analyses and modeled PTs’ constructive activities (Steffe & Thompson, 2000). On-going analyses involved testing and formulating hypotheses during the teaching experiment based on ways PTs engaged in each teaching episode. We inferred, from PTs’ engagement, instances that corroborated or contraindicated our hypotheses. After the completion of the teaching experiment, we re-visited the data corpus to do an in-depth retrospective analysis. The retrospective analysis involved four main activities that collectively refined the initial explanatory models developed during the teaching experiment. The four activities were (a) watching the entire video set or subsets of the video holistically without interruption to observe recurring patterns in PTs’ activities or shifts in their reasoning, (b) identifying instances that offered insights in building working models of the recurring patterns or shifts in their reasoning, (c) constructing annotated transcripts of such instances with rich descriptions of PTs’ actions, and (d) constructing/refining explanatory models of PTs' constructions of coordinate systems. Specific to the AFT, we analyzed the cognitive resources PTs drew upon when constructing a two-dimensional system from two one-dimensional lines.

Findings

We present our findings regarding Ginny's reasoning about the AFT in four phases. We highlight Ginny's activities in each phase and analyze the cognitive resources that supported her in establishing a Cartesian coordination.

Attending to Variability and Locational Simultaneity by Superimposition

When first posed with the prompt in the AFT, Ginny reiterated the problem, asking, "So, we need to make one point *while they're moving* to show where they are while *they're moving*? *At any point in time*?" to which her partner Hermione added, "I think it's hard because they're moving around a lot." We took these initial comments to indicate that both PTs were attending to, and perturbed by, two things: variability in the ants' locations and locational simultaneity required in the desired single point.

Ginny and Hermione proposed some ideas to address these elements. For instance, Hermione suggested using the mid-points of each tube, since "the ants always pass the middle." However, both PTs noticed that the ants were not always at the middle of each tube. Instead, Ginny superimposed the tubes in various ways to observe where the ants crossed each other. Specifically, Ginny aligned the two ant farm rectangles on top of each other in the DGE so that one rectangle was perfectly overlaid onto another and animated the ants. Next, she observed instances where the ant points occupied the same location at the same time. She identified three such points and claimed that one of those could be used as the desired point. However, Ginny acknowledged that these three points do not capture all of the possible ants' locations.

Seeking alternative solutions, Ginny joined the ends of the two plastic tubes, making the two tubes into one long tube, and explained, "You know both of the ants are on the same path. In the direct center, I guess that would be the one point to describe where they both are," as she drew a circle and point in the circle's center. As such, Ginny assumed the ants were moving in the center of each tube and, taking a cross-sectional view, identified one point as a projection of both ants in each tube.

From her activities, we infer that superimposing and attending to the intersection of the two points was a way for Ginny to account for locational simultaneity. When she superimposed one tube onto another, she accounted for locational simultaneity for three different instances in time; when she superimposed the tubes by joining their ends, she accounted for simultaneity for all instances in time, but from a different perspective. Although these points did not indicate where the ants were in each tube, Ginny demonstrated flexibility in taking different perspectives in viewing the ants and tubes. Collectively, Ginny's activities indicated that she viewed the tubes as objects she could manipulate and rearrange to serve her goal; they were not fixed objects, which we viewed as a critical cognitive resource in her thinking.

Establishing the Single Point as Being Dynamic

Approximately 20 minutes into the session, Ginny placed the tubes in the DGE perpendicularly (see Figure 4a) and explained she wanted to see if the ants met in the middle, where the tubes intersected. With the ants animated in the DGE, Ginny and Hermione observed that the ants crossed each other at the middle, but Ginny was unsure how to proceed: "I still don't think [we can come up with just one point] because they're never, they're not always in the same place at the same time." Wondering if the PTs conceived of the desired point as being static, the teacher-researcher (TR) asked, "So, what if somehow that point might move appropriately with the ants?" To which Hermione expressed, "So, you're saying the point can move now?" Hermione's response suggested she had interpreted the initial prompt as requesting a static desired point and she was now considering whether the desired point could be dynamic. Although this interaction was occasioned by the TR, we view this interaction as critical because the PTs established the desired point as potentially being dynamic.

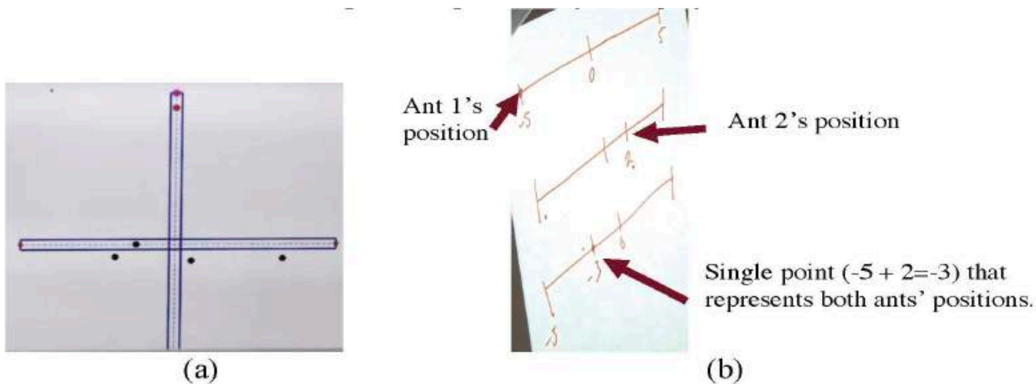

Figure 4: (a) Tubes arranged perpendicularly in the DGE and (b) Ginny's three number lines.

Using Number Lines

Approximately 30 minutes into the session, Ginny proposed a new idea: "What if we made the edges of the tubes a number line and the middle of the tube zero. So this [*pointing to the left side of a tube*] would be the negative side and this [*pointing to the right side of the same tube*] would be the positive side and this [*pointing to the center of the tube*] would be zero." Next, Ginny drew three number lines as shown in Figure 4b, explaining that the first and second number line each represented the ant's location in Tube 1 and Tube 2, respectively. Picking two locations as examples, she explained that if Ant 1 is at –5 and Ant 2 is at 2, then the single point –3 on the third number line, obtained by adding –5 and 2, should represent both ants' positions. When asked what the third number line was, Ginny explained, "[It is] corresponding to where the point is on, it would be I guess both of the tubes." As such, instead of superimposing one ant farm onto the other, Ginny created a third object to account for locational simultaneity.

Up to this point, we hypothesized that Ginny overlaid a number line onto each tube, and thus constructed a one-dimensional spatial coordinate system. Relatedly, there were two hypotheses to be tested: (a) whether Ginny's number lines were viewed as number lines superimposed onto the spatial situation, and (b) whether the numbers on her number lines were conceived of as distances (e.g., from the edge of the tube) or numerical values labeling each ants' location.

In Ginny's three number line representation (Figure 4(b)), although simultaneity was accounted for, the variability of the ants' positions yielded a non-unique point on the third number line. Relatedly, the TR asked, "what if Ant 1 is at –4 and Ant 2 is at 1?" Almost immediately, Ginny drew two perpendicular lines and explained, "So, if we had a graph and we do... negative five and then two, this point right here [*plotting the point in Figure 5a*] would describe where they are. So, instead of doing the adding them together you would graph it on a graph." She further explained that the point she plotted showed that the black ant is at 2 and the red ant is at –5. At this point, the two tubes in the DGE placed in front of the PTs were positioned perpendicularly (like in Figure 4a). We account for the sudden shift in her thinking to (a) Ginny's attention to locational simultaneity and attempting to capture both ants' locations and (b) recalling her previous graphing experience from the image on the DGE.

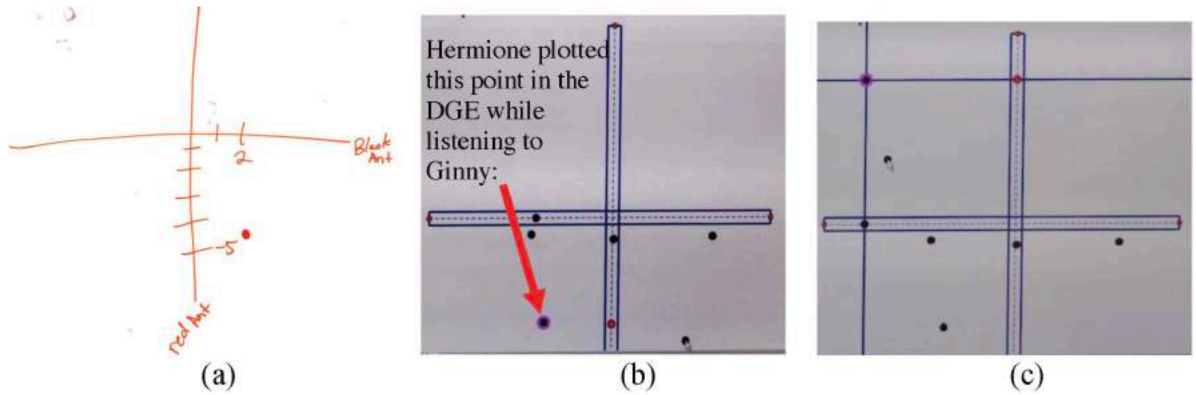

Figure 5: (a) Ginny's drawing of number lines arranged perpendicularly; (b) Hermione's point in the DGE; (c) Ginny's construction of the single point.

While Ginny was showing her new solution for finding the desired point, Hermione plotted the point shown in Figure 5b in the DGE and asked Ginny if the point she plotted would be the single point. Ginny explained "So, this [*pointing to her drawing in Figure 5a*] would be a totally different graph... so you'd get information from where the ants are on the number line on the tube. Then you would graph it on a different paper. So, it's not like a point on these [*pointing to the plastic tubes*] it's like a point on this [*pointing to her paper*].

Ginny's explanation in response to Hermione's question indicated Ginny has constructed number lines, disembedded from the Ant Farm situation, and arranged them to produce a different space from the original Ant Farm space. Thus, we inferred she constructed a quantitative coordinate system with her number lines in the sense that she viewed her "graph" as a space different from the space in which the number lines were initially conceived. Also, by the way she explained "getting information from where the ants are on the number line on the tube," we inferred her number lines consisted of numerical values indicating location in reference to the middle of the tube and that her number lines were not yet spatial coordinate systems superimposed onto the spatial situation.

Establishing the Cartesian Coordination

Noticing Ginny's differentiation between her "graph" and the ant farm space, approximately 35 minutes into the session, the TR commented, "I'm curious, so it sounds like you see this as a different space from that space, is what it sounds like." Ginny responded, "Yeah, I mean, I guess you can make it the same space but..." and sat in thought looking back and forth at the DGE screen and her sketch in Figure 5a for approximately 7 seconds. She then continued, "I guess you *could* make it the same space because...if put a number line on this," pointing to the ant farm tubes in the DGE. Pointing to each ant farm tube in the DGE, Ginny further explained, "So this [*referring to the horizontally placed tube*] would be your x and your y [*pointing to the vertically arranged tube*] of your graph and your dots [*referring to the ants*] would be your values and you just need to connect them to make your point [*moving her fingers in the air consistent with the blue arrows in Figure 1b*]."

Finally, Ginny constructed a line through Ant 1, perpendicular to Tube 1, and a line through Ant 2, perpendicular to Tube 2 and indicated that the intersection of the two lines would show the location of both ants in each tube (Figure 5c). After the TR hid the two perpendicular lines, both PTs verified their new method worked by considering several positions of ants by hiding the ants, animating the single point, guessing where the ants should be within each tube, and then checking the ants' locations. When the TR asked why they think this method works in general, Ginny explained that by placing the tubes perpendicularly, a horizontal line can be used to describe the red ant moving vertically and a vertical line to describe the black ant moving horizontally. In conclusion, Ginny established a Cartesian coordination leveraging her spatial coordinations.

Conclusion and Discussion

From Ginny's activities in the AFT, we highlight three cognitive resources we see as critical for establishing a Cartesian coordination given two lines that would enable holding a sustained image of two locations simultaneously for arbitrary positions of points on each line. First, Ginny's *attention to variability in the ants' locations* coupled with *imagining the single point as moving along with the two ants* was a critical development during the teaching session. Second, Ginny *recognized the tubes (or number lines) as objects that could be manipulated and rearranged* which supported her to arrange them in a particular way (e.g., perpendicularly) so that the locations of each ant could be accounted for simultaneously. Third, *drawing from her spatial coordinations*, Ginny utilized the two-dimensional space outside of the one-dimensional tubes spaces to construct a point outside of the tubes (or number lines). Specifically, she projected vertically from the horizontal tube in which the ant moved horizontally and projected horizontally from the vertical tube in which the ant moved vertically. Supported by these cognitive activities, Ginny successfully constructed a single point that simultaneously captured the location of both ants.

For Ginny and Hermione, devising a system to coordinate the location of two points using a single point appeared novel, meaning that establishing a Cartesian coordination by rearranging lines orthogonally and projecting from the two lines was non-trivial despite their prior school experiences. Recall that although Ginny eventually constructed a system she called a "graph," she initially viewed it as different from the ant farm space. We suspect this was due to her past experiences with coordinate systems focused predominantly on quantitative coordinate systems. Thus, viewing the Ant Farm space as a space analogous to her coordinated number line space was an additional critical realization for Ginny in solving the AFT.

Given the preceding findings, we close with two considerations: one limitation and one direction for further study. First, the wording of the initial AFT prompt may have hindered PTs from viewing the single point as being dynamic; therefore, variations of the prompt might be considered in future implementations. Second, the cognitive resources presented here that foster the establishment of a Cartesian coordination may be specific to Ginny and the AFT, which require further research with more PTs. Future research can also look into what engendered Ginny's transition from one phase to another (e.g., TR moves) and how these can be leveraged to support PTs and consequently their future students' constructions of coordinate systems.

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PROSPECTIVE K-8 TEACHERS' PROBLEM POSING: INTERPRETATIONS OF TASKS THAT PROMOTE MATHEMATICAL ARGUMENTATION

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This study examines pre-service teachers' (PSTs') views of tasks that engage students in mathematical argumentation. Data were collected in two different mathematics courses for elementary school education majors (n = 51 total PSTs). Analyzed were (a) written journals in which PSTs defined tasks that promote student engagement in argumentation, (b) tasks PSTs posed to engage students in mathematical argumentation, and (c) accompanying explanations in which PSTs motivated tasks they posed. The analysis revealed that PSTs interpret tasks that foster argumentation in terms of activities of argumentation that a task elicits and space for argumentation that the task provides. Several features that PSTs associated with each of the two major task characteristics were identified. While posing tasks to engage students in argumentation, PSTs did not place equal emphasis on all of the identified features.

Keywords: Reasoning and Proof, Mathematical Knowledge for Teaching, Teacher Education-Pre-service.

Background

Curricular standards in mathematics recognize mathematical argumentation as an essential disciplinary practice with which all students should engage and become proficient (e.g., National Governors Association Center for Best Practices and Council of Chief State School Officers [CCSSM], 2010; National Council of Teachers of Mathematics [NCTM], 2000). Engaging students in constructing viable arguments and critiquing the reasoning of others is the desired instructional goal (CCSSM, 2010). Past research on argumentation shows, however, that contrary to curricular recommendations and recognized importance of argumentation in student learning, teaching mathematics with a focus on argumentation is still far from a common practice (e.g., Bieda 2010; Staples, Bartlo, Thanhaiser, 2012).

Past research with practicing (and pre-service teachers, PSTs) documented challenges that teachers face while facilitating argumentation in their classrooms (e.g., Bieda 2010; Kosko, Rougee, & Herbst, 2014), explored the role of the teacher in promoting argumentation (e.g., Aylon & Even, 2016; Conner, Singlertary, Smith, Wagner, Francisco, 2014; Graham & Lesseig, 2018), and explored how teachers interpret argumentation in the context of mathematics classrooms (e.g., Park & Magiera, 2019). While overall, the research interest in argumentation is growing, research attention to teachers' views of tasks that promote student engagement in mathematical argumentation has been limited.

Researchers agree that tasks play an essential role in how students experience mathematics (Krainer, 1993; Simon & Tzur, 2004; Zaslavsky, 2008). However, research on curricular materials reveals that school mathematics textbooks, even textbooks designed to support mathematics curriculum reforms, offer limited collections of tasks that, by their inherent design, provide opportunities for engaging students in argumentation (e.g., Bieda, Ji, Drwencke, & Pickard, 2013; Dolev & Even; 2015; Stacey & Vincent, 2009). Understanding how teachers interpret tasks that engage students in argumentation could help gauge students' opportunities for experiencing argumentation in mathematics classrooms. Research-based information about teachers' views of tasks that engage students in mathematical argumentation can also aid the efforts of helping teachers develop a more comprehensive knowledge of argumentation. Thus, in a bid to address these existing gaps, this study uses problem-posing as a context for exploring elementary PSTs' views of mathematical tasks that engage students in the practice of argumentation. This research was guided by the following question: *What characteristics of tasks that promote mathematical argumentation emerge from the analysis of problems PSTs' pose to build students' capacities in mathematical argumentation, and PSTs' descriptions of problems that engage students in argumentation?*

Conceptual Framework

Problem Posing

Problem-posing, which includes designing new and modifying existing tasks, is recognized as an essential element of mathematical activity (Silver, 1994). Teachers' ability to design and pose mathematical tasks is one of the central aspects of mathematics teaching (Krainer, 1993; NCTM, 2000). Classroom problems provide students with the opportunity for thinking and learning (Smith & Stein, 1998). Thus problem-posing is perceived as integral to teaching a "high leverage" practice, a gateway to understanding that serves as a learning and instructional tool (Ball & Forzani, 2009). While posing problems, teachers go beyond thinking about problem-solution, they need to consider the overall goal of the task, think about what and how students can make sense of the mathematics they learn, and what understandings, skills, and attitudes they develop (Crespo, 2015; Lavy & Shriki, 2007; NCTM, 1991). Researchers recognize that the activity of problem-posing can provide a window into an understanding of teachers' mathematical and pedagogical content knowledge (Ellerton 2015; Lee, Capraro, & Capraro, 2018).

Mathematical Argumentation

Toulmin, Rieke, and Janik (1984) described argumentation broadly as "the whole activity of making claims, challenging them, backing them up by producing reasons, criticizing those reasons, rebutting those criticisms, and so on" (p. 14). This description is consistent with the notion of argumentation presented in the Standard for Mathematical Practice #3 (CCSSM, 2010). Mathematics education researchers generally agree that in school mathematics, argumentation involves a wide range of activities. These activities include constructing, validating, or refuting mathematical claims, producing and criticizing justifications, formulating conjectures, generalizing, representing mathematical ideas, constructing counterexamples, or communicating reasons, to name some. (e.g., Lakatos, 1976; Knudsen, Lara-Meloy, Stevens, & Rutstein, 2014; Krummheuer, 1995; Ramsey & Langrall, 2016). The existing frameworks that guide the examination of textbook tasks for their affordances of engaging students in argumentation (e.g., Bieda et al., 2014; Stylianides, 2009) classify the kinds of argumentation-related activities elicited by the task. Given that the focus of this research was on PSTs' interpretations of tasks that engage students in argumentation, not on the implementation of classroom tasks to engage students in argumentation, Toulmin et al. (1984) broad description of argumentation together with frameworks proposed to classify the types of argumentation-related activities elicited by written tasks provided an attractive guide for this study. They allowed negotiating a wide range of meanings that PSTs bring while thinking about tasks that engage students in argumentation and to place argumentation within the individual and social space a task might create for student engagement in argumentation.

Method

Participants and Study Context

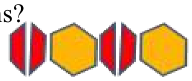
The study was conducted in the Midwestern university in the U.S. Participants were 51 PSTs preparing to teach grades 1-8 mathematics enrolled in two mathematics content and concurrent pedagogy with field experience set of courses for elementary education majors. The two pairs of courses were *Number Systems and Operations for Elementary School Teachers* and *Teaching Elementary School Mathematics* (n = 23) and *Algebra and Geometry for Teachers* and *Teaching Middle School Mathematics* (n = 28). Curricula of each set of courses were coordinated. In the context of their mathematics courses, PSTs studied concepts fundamental to the K-8 mathematics and engaged in mathematical argumentation as learners. In their corresponding pedagogy with field experience courses, they focused on teaching strategies that support students learning of K-8 mathematics and support students' mathematical reasoning skills. They conducted focused observations in their field placement classrooms to identify teacher moves, instructional strategies, and classroom interactions that supported student reasoning. In the context of their education and field experience work PSTs also prepared and conducted two problem-based interviews with students for the purpose of engaging students in the practice of mathematical argumentation and learning about student mathematical thinking.

Data and Data Analysis

Collected in the Number Systems and Operations for Elementary School Teachers course were (a) written journals in which each PST described tasks that engage students in argumentation, (b) two tasks each PST posed (one at a time) in preparation for their interviews of elementary school students, and (c) explanations in which each PST described why task they posed creates an environment for student engagement in argumentation.

Collected in the Algebra and Geometry for Teachers course were: (a) written journals in which each PSTs described tasks that engage students in argumentation, (b) PSTs' analyses and critiques of three instructor-provided tasks (completed one at a time) in which they discussed each task's potential to engage students in argumentation, and (c) revisions of instructor-provided tasks PSTs' proposed to enhance each task's potential to engage students in argumentation and PSTs' explanations for each revision. Instructor-provided tasks that provided context for PSTs' problem-posing activity are shown in Figure 1.

Task 1: How many red pattern blocks will be used if the pattern of figures is extended until there is a total of 13 polygons?



Task 2: Kay made summer lemonade from a mix using l2 tablespoons of lemonade mix and 20 cups of water. How many tablespoons of lemonade mix will she need if she plans to use 30 cups of water to make lemonade that tastes just the same?

Task 3: Below is a growing sequence of figures.

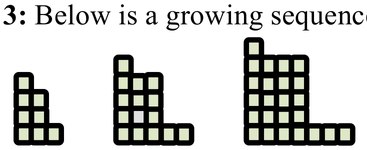


Fig 2 Fig 3 Fig 4

a) Draw the 1st, 5th, and 6th figures
b) How is the pattern changing?
c) What would the 100th figure look like? How many tiles it has? How can you justify your prediction?

Figure 1: Instructor-provided Tasks that PSTs Analyzed and Revised

Qualitative analytical-inductive methods were used for data analysis. In the first round of the analysis, a large subset of all data (about 25%) that consisted of all types of artifacts was first carefully annotated to discern PSTs' perceptions of tasks that engage students in argumentation. The goal was to create a code-book that could reliably capture task characteristics identified across PSTs' definitions, tasks they posed (designed or revised), journals in which they described why tasks they posed could engage students in argumentation, or interviews during which they discussed their tasks. To illustrate the coding process, consider included in Figure 2 task that PST A44 posed together with explanation PST A44 provided about her task. Following the initial round of the analysis, the annotations were further compared and contrasted, and their descriptions revised to formulate a final set of codes that were then applied to the entire corpus of data. If any new task characteristics were identified during this process, new codes were introduced, and the code-book was augmented. The final list of codes was further compared and contrasted until no new task characteristics were identified. In the final stage of analysis, all codes were again compared and contrasted, leading to the identification of two major task characteristics. Task features identified across each PST's responses were then tabulated to identify the overall patters in PSTs' interpretations of tasks that facilitate student engagement in mathematical argumentation.

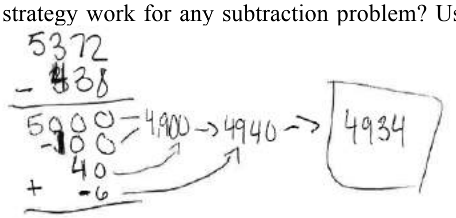
PST A44's Task 1 (task posed)	Annotations
Kim has 5,372 songs on iTunes. She deletes 438 songs. How many songs does she have on iTunes now? A student solves this problem by subtracting the thousands, hundreds, tens, and one's from 5,372 by 438. Then she adds the sum of each number place together to get the final answer of 4,934 songs. Does this strategy work? Why or why not? Can this strategy work for any subtraction problem? Use examples to explain your reasoning. 	Call for justification of a strategy (Does this strategy work? Why or why not?) Call for evaluating reasoning of others Call for generalization (Can this strategy work for any subtraction problem?) Call for communicating thinking, reasoning (Use examples to explain your reasoning)
PST A44's Explanation about Task 1	Annotations
This task takes a simple subtraction problem and turns it into an engaging mathematical problem for the student. The student must understand how this subtraction problem was solved by someone else using what I assume will be a method that is different from how the student traditionally solves a subtraction problem. The task asks several questions of the student to clarify that they understand how the problem was solved and help them build upon their argumentation of how they got the solution and how it works. The task can be applied to other example problems to help the student with their understanding of the subtraction strategy and to help them explain their mathematical reasoning. I think the most likely problem that could occur is that the student may not fully grasp or understand the technique given to solve the subtraction problem. They may not get the idea of subtracting the place value or may struggle with the negative numbers that show up in this problem because they are not used to working with negatives in a standard subtraction problem. Lastly, it might be the wording of the problem, that stumps the student. However, with a little help, I do think that most sixth graders can absolutely understand this technique and argue how it works. The main math skill required to understand this task is place value and sixth graders should certainly have a strong understanding of this concept.	Task is engaging Task engages in evaluating the reasoning of others Task engages in generalizing Task is non-routine, requires deeper thinking, challenging Task builds on student existing knowledge, understanding

Figure 2: Example of Task Posed and Task Explanation (PST A44, Task 1)

Results

Table 1 summarizes the characteristics of tasks identified across PSTs' responses. The analysis revealed that while posing tasks to engage students in argumentation, PSTs considered (a) activities of argumentation in which students could engage given their task and (b) space for argumentation that their task provides. Overall across the analyzed tasks, task explanations, and PSTs' definitions of tasks that engage students in argumentation, individual PSTs included between 2 to 11 different task features.

Table 1: PSTs' views about tasks that promote mathematical argumentation

Major Task Characteristics	Task Features	(*n*, %*)
Activities of Argumentation	Task promotes providing justifications	48 (94%)
	Task encourages making generalizations	34 (67%)
	Task elicits explorations, generating conjectures, evidence, and claims	31 (61%)
	Task promotes evaluating arguments or reasoning of others	28 (55%)
Space for Argumentation	Task elicits communicating thinking and reasoning	47 (92%)
	Task enables the use of multiple solution strategies and ways of thinking	36 (71 %)
	Task draws on students' existing knowledge and allows them to make connections	34 (67%)
	Task engages in deeper thinking, is complex	33 (65%)
	Task supports the use of multiple representations, manipulative materials to guide thinking	31 (61%)
	Task requires that students reflect and make sense of their results	20 (39%)
	Task fosters the development of concepts	14 (27%)

* Rounded to the nearest %.

Activities of argumentation. As summarized in Table 1, PSTs associated argumentation with a broad range of activities in which a student could engage in the context of posed tasks. These activities, categorized as task features, were emphasized across the analyzed data to a different degree. For example, while almost all PSTs (94%), in at least one of their tasks, included an explicit call for justifying a result, claim or strategy, only about half of the participants (55%) designed tasks that would engage students in evaluating arguments or reasoning of others. About two-thirds of PSTs (67%) formulated tasks that engaged students in generalizing, and a little less than two-thirds of PSTs (61%) formulated tasks that encouraged explorations, generating conjectures, evidence, and claims.

Consider the presented earlier task posed by PST A44 (Figure 2). By its design, this task engages a student in evaluating the validity of a given strategy. The task statement requires that a student justifies his or her strategy assessment. The task also elicits thinking about the strategy generality by prompting the student to reason about whether or not the presented strategy can be applied to other subtraction problems. Consider also the following task which PST A15 posed modifying the instructor-provided Task 1 (Figure 1): "How many red pattern blocks will be used if the pattern is extended until there are 200 polygons? Justify your response." PST A15's task also includes an explicit call for justification. The intention to engage a student in generalizing was evident is explanation PST A15 included. PST A15 described her thinking about this task modification and her desire to engage a student in thinking about pattern generalization and exploring and developing a general conjecture about the pattern sharing:

> I increased the number of polygons in order to prevent the student from merely counting the blocks. Increasing the number requires that the student finds a general rule or equation. They can investigate the relationship between the blocks. I also asked the student to justify it. (PST A15, Task 1)

Space for argumentation. This category of task characteristics describes ways in which the PSTs thought about and designed their tasks to create an environment for student engagement in argumentation. As summarized in Table1, PSTs varied in approaches they used to generate a context for student engagement in argumentation within their tasks. For example, almost all PSTs (92%) designed tasks, so the task elicited communicating thinking and reasoning. A large group of PSTs (71%) described and proposed tasks that were open to diverse ways of thinking and solution strategies. The focus on the latter task feature is illustrated with the excerpts below:

> Mathematical tasks that foster argumentation must be challenging., directional, often have more than one specific answer and can be represented in multiple ways (PST A44)

> Flexibility is also very important in fostering mathematical argumentation. Understanding that there are multiple ways to view a problem or multiple solutions that could be found is important because it allows students to challenge ideas and use evidence to prove why their answer is efficient. (PST B17)

About a two-thirds of PSTs, (67%), thought about opportunities their task might give students for drawing on students' existing knowledge and making connections, for engaging in deeper thinking (65%), or for supporting student thinking by encouraging them to use multiple representations or manipulative materials (61%). For example, PST A23 shared:

> Using manipulatives in tasks helps students reason and make claims. When students are able to visualize and make structure of their work, they better understand the problem and are able to make and justify claims. (PST A23)

In contrast, only 39% of PSTs considered tasks that require students to reflect and make sense of their results as one that can engage students in argumentation, and only 27% of PSTs envisioned that tasks that facilitate concept development might engage students in argumentation. An excerpt from PST A18's journal presented below illustrates the former task feature:

> Tasks should encourage students to go back to their own work. A student should see if they used evidence or showed enough work to support their explanation. Are the equations and tables labeled? Can everything be proved? Tasks [that encourage reflection] can help and improve students' skills in making, justifying, and evaluating mathematical claims. (PST A18)

Presented below task posed by PST A2, together with accompanying task explanation, serve as an illustration of PSTs' thinking about how a task that provides space for concept development can engage students in argumentation. PST A2 shared:

> The mathematical task I designed is a word problem that will require students to think about multiplication- this task is designed for a third-grade student. Example task: Sue invited 8 friends to her birthday party. She was making goodie bags for each of her friends. If she puts 5 pieces of candy in each bag how many total pieces of candy does she need? Justify your answer.

While motivating her task PST A2 wrote:

> Mathematical argumentation requires a student to not only explain how they arrived at their answer but to think about the mathematical ideas, concepts, theories, and reasoning that are used in the problem. This task presents students with a fundamental multiplication property – equal-sized groups and repeated addition. Multiplication can be viewed as repeated addition of equal-sized groups. With this task, the student will be exposed to this idea because the task is asking them to find the total pieces of candy when there are 8 groups (bags of candy) with 5 pieces each (candy pieces). […] One potential error I could see with this problem is a student potentially grouping 8 pieces of candy 5 times. If this were to happen, this would still result in the correct solution of 40 and would be a great opportunity to talk about the commutative property of multiplication.

Summary and Discussion

This study contributes to the research on argumentation in school mathematics in two ways. First, it extends the existing analytical models for engagement in mathematical argumentation, which emphasize the structural or cognitive aspects of argumentation (e.g. types of arguments being generated, or degree of justification) and provide a framework that can serve as a guide for the design of tasks that support student engagement in argumentation. Second, it offers insights into PSTs' understanding of mathematical argumentation by describing the kinds of opportunities for argumentation that PSTs envision as they pose tasks to engage students in argumentation.

The analysis revealed that while posing written tasks and thinking about task affordances for engaging students in argumentation, PST considered (a) activities of argumentation in which a student might be involved while working on the task and (b) the space for argumentation that the task generates. PSTs viewed both of these task characteristics as contributing to the overall potential of the task for building students' capacities in argumentation. This finding extends previous conceptual frameworks for analyzing the potential of written tasks for engaging students in argumentation, which exclusively focused on the types of activities of argumentation that task elicits (Bieda et al., 2014; Stacey & Vincent, 2009; Stylianides, 2009).

The results also document task features related to the two identified major task characteristics and show that PSTs do not equally emphasize these features while designing tasks to engage students in argumentation. For example, concerning the activities of argumentation, almost all PSTs in this study posed tasks that elicited justifying. A large proportion of PSTs formulated tasks that promoted conjecturing and generalizing, but tasks that engaged students in evaluating arguments and reasoning of others were posed less frequently. PSTs' choices of task features identified as representative of the space of argumentation posed tasks afforded also varied. For example, the results suggest that PSTs might be more likely to associate opportunities for mathematical argumentation with tasks that elicit communicating thinking and reasoning, or tasks that allow for divergent ways of thinking and solution strategies. About two-thirds of PSTs in this study considered also task complexity, the extent to which task allows students to build on their prior knowledge and make connections, or facilitates the use of multiple representations as a viable task environment that offers space for engaging students in argumentation. Less frequently, PSTs envisioned that tasks that promote concept development or elicit students' reflections on their thinking might provide space for argumentation.

The results of this study provide important insights for mathematics teacher educators about supporting PSTs' visions of argumentation in mathematics classrooms. For example, it is likely that without intentional efforts focused on heightening PSTs' awareness of tasks that engage students in analyzing and critiquing the reasoning of others, PSTs might limit students' opportunities for experiencing this aspect of argumentation. Particularly, because, as reported by Bieda and colleges (2014) in their review of several elementary school textbooks in the U.S., tasks designed to engage students in evaluating claims were rarely present within the elementary school textbooks. This study did not examine how PSTs' envision classroom implementation of tasks for the purpose of engaging students in argumentation. To generate a more robust picture of PSTs' knowledge in the area of mathematical argumentation future research should investigate PSTs' interpretations of tasks that engage students in argumentation and the nature of opportunities for engaging students in argumentation PSTs see in classroom tasks, with concurrent attention to PSTs' visions of task implementation.

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MATHEMATICAL KNOWLEDGE AND SKILLS OF UNIVERSITY STUDENTS WHEN SOLVING A MEA

CONOCIMIENTOS Y HABILIDADES MATEMÁTICAS DE ESTUDIANTES UNIVERSITARIOS AL REALIZAR UNA MEA

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In this paper we present the results of an investigation related to the developing of mathematical knowledge and skills by first semester university students when solving a Model Eliciting Activity [MEA] which involves quadratic function knowledge. This was a qualitative research. The theoretical framework was Models and Modeling Perspective [MMP]. The results show that the students used their mathematical knowledge and skills related to linear and quadratic functions to describe the situation; they moved from a quantitative cycle of understanding (associated with linear and quadratic behaviors), to an algebraic cycle of understanding (associated with quadratic behaviors).

Keywords: Modeling, problem solving, university level.

Learning the concept of function has been studied by several researchers (Gutiérrez and Prieto, 2015; Hernández, 2013; Oviedo, 2013). Why is so important to learn the concept of function? López, Navarro and Fuchs (2018) answer that it is a necessary knowledge to model events and phenomena in different professional areas. Villarraga (2012) describes, for example, the type of situations that can be modelled using the quadratic function such as: optimization problems related to cost, demand, and areas, or physical problems. Despite the importance of understanding the quadratic function, difficulties have been identified in its learning, such as the poor articulation between algebraic and graphic representations (Díaz, Haye, Montenegro and Córdoba, 2013) needed to describe situations and phenomena.

Several authors (Gutiérrez and Prieto, 2015; Hernández, 2013; Oviedo, 2013) have created proposals to address the learning of the quadratic function. However, many of the proposals are reduced to the manipulation of parameters of algebraic expressions and the study of the transformations in the corresponding graph. Little research addresses the development of mathematical knowledge and skills associated with the quadratic function in the context of problem solving or modeling situations close to real life. One of the studies carried out in this direction was that of Aliprantis and Carmona (2003). They used the Models and Modelling Perspective framework (Lesh, 2010) to design and implement an activity to promote the development of knowledge of the quadratic function and associated concepts, such as variables, the relationship among them (quadratic and linear), the product of linear relationships, and the maximization; as well as to encourage students to develop skills for modeling and problem solving, such as conjecture, argument, description, and explanation. The participants in this study were high school students.

The research described in this paper has been carried out with students from the first university semester. The goal was to promote the development of mathematical knowledge and skills associated with the quadratic function during the process of solving situations close to real life. The research question is: What knowledge and skills do first-semester college students exhibit when performing an MEA in which the concept of quadratic function underlies?

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Theoretical Framework

Learning mathematics, according to MMP (Lesh and Doerr, 2003), is based on the construction of models, which

> are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently. (Lesh and Doerr, 2003, p. 10).

These models can be internal and external, that is, they inhabit both the thinking of students and the equations, schemes, computer applications or other representational resources used by science experts or schoolchildren (Lesh and Doerr, 2003). Models can be created by carrying out Model Eliciting Activities (MEAs), which are simulations of "real life" situations. In carrying out MEAs, students go through iterative sequences where they express, test, and revise their own ways of thinking (Lesh and Caylor, 2007). During the knowledge development process students build and modify their models through the phases of differentiation and refinement of the conceptual systems they construct (Lesh and Doerr, 2003).

MEAs "involve sharable, manipulatable, modifiable, and reusable conceptual tools (e.g., models) for constructing, describing, explaining, manipulating, predicting, or controlling mathematically significant systems" (Lesh and Doerr, 2003, p. 3). Researchers mention that such descriptions, explanations, and constructions should not be considered as simple processes that students create to get 'the answer', but they are key elements in the learning process. Thus, the process is the product.

MEAs "involve mathematizing—by quantifying, dimensionalizing, coordinatizing, categorizing, algebratizing, and systematizing relevant objects, relationships, actions, patterns, and regularities" (Lesh and Doerr, 2003, p. 5). One feature that distinguishes MEAs from other problem-solving activities is the writing of a letter. In the letter, students must explain the method they used to find the solution to the problem; this method can be used by a client to solve other problems with similar characteristics.

Methodology

The methodology was qualitative. The MEA (Figure 1) whose results are described in this paper, is part of a didactical sequence (Lesh, Cramer, Doerr, Post and Zawojewski, 2003) designed during the research project. It was implemented in a two-hour session with a group of 12 undergraduate students, who were 18 years old. Students were organized in four teams of three members each. The participants had a laptop and Excel and GeoGebra software. The process of solving the problem was carried out in four phases. 1) students read an informative article related to the context of the problem (warm-up activity according to the MMP). 2) they read the problem (Figure 1) and worked as a team to solve it. 3) They presented their results in a plenary discussion. 4) Students solved the problem individually at home. In this paper, the results of the phase 2 are presented, based on the discussions generated during the plenary (phase 3). The data were obtained from the worksheets, Word, Excel and GeoGebra files, and from video recordings of the face-to-face session. The concepts associated with this activity are those mentioned before (Aliprantis and Carmona, 2003).

Students in the last semester of Mechanical Electrical Engineering at the University of Guadalajara's Centro Universitario de Ciencias Exactas e Ingenierías (CUCEI) want to organize a trip to the Mazamitla Cabins to raise funds to celebrate their graduation.

For this purpose, the University will lend them a bus with capacity for 49 passengers, so that the cost of the round trip from CUCEI (diesel and driver's fees) and the tours inside the magic town will be covered. In addition, Carlos, a partner of the students, has taken on the task of searching on the Internet for accommodation and activities in order to organize a tourist package and be able to offer a complete experience to the travelers who attend the tour.

The tourist package includes: accommodation for three days and two nights in a cozy cabin located in La Cañada; a tour of the Sierra del Tigre, with a guide; ice cream factory and tasting of typical products; a ride on a four-wheel-drive vehicle through the most striking places in the sierra. Also, for those who like adrenaline, the zipline, climbing wall, hanging bridges and gotcha are included.

Carlos budgeted the cost of $1300.00 for each tour package per traveler. But, in order to make a profit, he will offer it at a price of $3650.00. To encourage potential travelers, Carlos proposes to make a discount of $50 for each person who goes, except if only one goes. That is, if two people go, they get a $50 discount; if there are three, there is a $100 discount per passenger.

Help Carlos. Write him a letter explaining whether and how much organizing the trip will allow him to make a profit. Support your statements with clear and valid arguments.

Figure 1: MEA

Data analysis was based on two criteria: quantitative and algebraic cycles of understanding (Vargas, Reyes & Cristóbal, 2016). The quantitative cycle of understanding is one in which students are able to describe the variables and relations among them in a numerical way. The information and relationships can be organized in tables and graphs. In the Algebraic cycle of understanding, students exhibit some mastery of the language of algebra to solve the problem. Those who reach this last cycle have gone through different stages of differentiation, integration, and refinement of their conceptual systems and have a deeper domain of the representations. During each of the cycles, verbal representation was present to justify the conjectures and explanations.

Results

Students went through two cycles of understanding (Table 1). The first cycle was quantitative and the second one was algebraic. At the beginning, four teams (100% of the total) revealed a way of thinking related to linear variation (column 2, Table 1) during the quantitative cycle. Then, students from teams 1, 2 and 3 (75% of teams) moved from their procedures characterized by linear to a quadratic variation (column 3, Table 1). Only the members of the teams 1 and 2 extended their ways of thinking into an algebraic cycle of understanding. A detailed description is shown below.

Table 1: Student’s cycles of understanding to solve the MEA

	Cycle of understanding			Result	
	Quantitative		Algebraic	Correct	Incorrect
	Linear	Quadratic	Quadratic		
E1	✓	✓	✓		✓
E2	✓	✓	✓	✓	
E3	✓	✓		✓	
E4	✓				✓

First Cycle of Understanding: Quantitative

All the students identified the data: capacity of the bus, cost of the journey, initial price, and the discount. They also realized that they needed to write a letter that included the procedure for solving the problem. Two ways of addressing the situation in this cycle were distinguished: linear and quadratic behavior.

Linear behaviour. All the students calculated the value of the profit corresponding to the maximum number of passengers, i.e. 49 persons. Students' conjecture was that the more passengers, the greater profit, which denotes linear thinking. Two procedures were distinguished, one in which the profit per passenger was obtained, exhibited by teams 1, 2 and 3, and another in which the profit per group was obtained, realized by team 4. The procedures are discussed below.

Procedure to obtain the profit when 49 passengers travel (maximum capacity of the bus)

Procedure: profit per passenger. The procedure of the teams 1, 2 and 3 (75% of teams) was to subtract the corresponding discount for 49 passengers from the initial price, i.e. they carried out the operation: 3650-2400. The result (1250) was reduced by the cost of the trip, i.e. 1250-1300. Students interpreted this quantity (-50) as the profit per passenger. However, because the amount was negative, they expressed that it was a loss of $50 per person. Thus, $2450 would be the loss corresponding to the trip of 49 passengers, as shown in Figure 2.

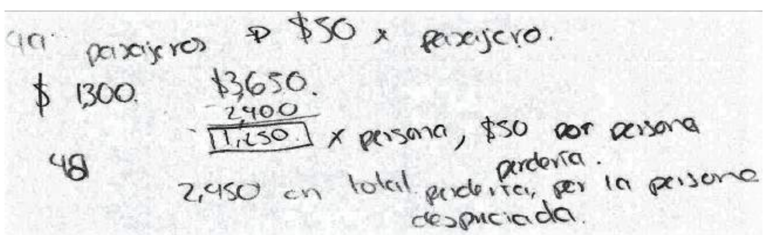

Figure 2: Operations carried out by Team 1 to obtain the Profit related to 49 Passengers. Procedure: Profit per passenger

Procedure to obtain the profit corresponding to 49 passengers (maximum capacity of the bus). Procedure: profit per group. Students of team 4 used the spreadsheet to operate with the cost of the trip, the maximum passenger capacity and the initial price, i.e., $1300, 49, and $3650, respectively (cells D6, E6 and D10 in Figure 3). They related these data using formulas (Table 2) to calculate the discount, the expenditure, the income related to the initial price, the income related to the discounted price, and the profit corresponding 49 passengers.

	A	B	C	D	E	F	G	H
1	2	50						
2	3	100		SUMATORIA DE LOS DESCUENTOS				
3	4	150			117600			
4	5	200						
5	6	250		PRECIO TOTAL DEL VIAJE				
6	7	300		$1,300	49			
7	8	350			63700			
8	9	400						
9	10	450		COBRO TOTAL		178850		
10	11	500		$3,650	61250			
11	12	550						
12	13	600						
13	14	650		GANANCIAS TOTALES				
14	15	700			-2450	COMO PODEMOS OBSERVAR ES UNA		
15	16	750				CANTIDAD NEGATIVA, POR TANTO SE		
16	17	800				PERDERIA ESE DINERO		

Figure 3: Procedure to obtain the profit corresponding to 49 Passengers by Team 4. Procedure: profit per group

Table 2: Formulas Used by Team 4

Cell	Formula
E3	=(B48*A48)
E7	=(D6*E6)
E10	=(D10*E6)-E3
E14	=(E10-E7)
F9	=(D10*E6)

Quadratic behavior. The members of teams 1, 2 and 3 (75% of the teams) made several operations using different amounts of passengers and analyzed how the results varied. Students from teams 1 and 2 (50%) used trial and error procedure, in a disorganized way. Students from team 3 (25%) performed a systematized procedure. Team 4 (25%) was the only team that did not perform many operations.

Trial and error procedure. Students from teams 1 and 2 (50% of the total) performed operations with different quantities of passengers. This allowed the students from team 1 to identify how income varied and to find out for what quantity of passengers a maximum income was produced (Figure 4). In turn, the students from team 2 identified in a more organized way how the profit varied and found the quantity of passengers that corresponded to the maximum profit.

Figure 4 shows the procedure developed by the team 1, as a representative example of the teams 1 and 2 procedures. Students calculated the income corresponding to 24, 28, 37, 36 and 38 passengers. However, they believed that they were finding the profit. They noted that there was a dependency relationship involving the number of passengers; they expressed that "the profit depends on the passengers".

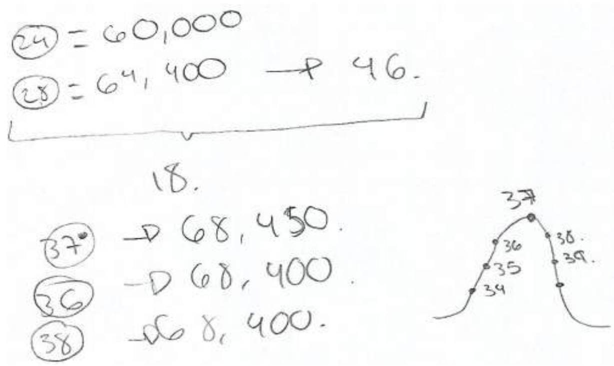

Figure 4: Procedure to obtain the Income and to analyze the Variation for Different Amounts of Passengers by the Team 1

Figure 4 shows how students made a graph that describes the way in which the income varies. Students noticed that an "income maximum value" or "peak" was obtained when they made operations with 37 passengers. They did not identify an interval where the function was increasing or decreasing. They pointed out that, from the maximum value (37), the income corresponding to 36 and 38 passengers "goes down the same", in the same way they mentioned that "35 and 39 have the same [corresponding income value]".

Exploration of results by systematized test. Students from team 3 constructed a table (Figure 5) with the labels: "Passengers", "Price per passenger", "Discount per passenger" and "Total Profit". They related the amounts of each row in a horizontal way, and obtained the profit, according to the quantity of passengers. They identified that the maximum profit, $28800, is obtained when 24 passengers are traveling. Figure 5 shows part of the table created by team 3.

Relación de clientes/ganancias

Pasajeros	Precio p/p	Descuento p/p	Ganancia total
1	$3650	0	2350
2	$3600	50	4600
3	$3550	100	6750
4	$3500	150	8800
5	$3450	200	10750
6	$3400	250	12600
7	$3350	300	14350
8	$3300	350	16000
9	$3250	400	17550
10	$3200	450	19000

Figure 5: Team 3 Procedure (passengers, price per person, discount per person, total profit)

Algebraic Cycle of Understanding

Students of teams 1 and 2 (50% of teams) generalized patterns. Students from team 3 (25% of the total) constructed syncopated expressions to perform the calculation. Students from team 4 (25%) did not generalize.

Generalization of patterns through algebraic expression. Figure 6 shows the expression obtained by the students of team 1 to calculate the profit. It is not identified as a function by the students, but as a formula. The quantities 3650 and 50 represent the initial price and the discount per person, respectively. The value 63700 is the result of multiplying 1300*49, that is, the cost per person multiplied by 49. Therefore, 63700 is the expenditure when attending the maximum passenger capacity.

The corresponding expenditure of n passengers would be $1300n$. The correct expression was $(3650-(n-1)*50)n-1300n)$. The algebraic model that the students had to construct to calculate the profit was $f(x)=-50x^2+2400x$, in its simplest form, where x represents the number of assistants.

(3650- (n-1)*50)n-63700

*Sea "n" el número de personas que asistan

Figure 6: Algebraic Expression of Team 1 for Calculating the profit

These students used the GeoGebra software to identify the number of passengers corresponding to the maximum profit. They found that 37 passengers were needed to produce a maximum value of 4750 (Figure 7). However, according to the problem data, the correct values were 24 passengers and $28800.

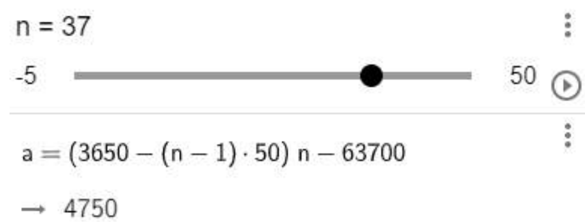

Figure 7: Team 1 procedure to obtain the maximum profit.

Generalization of patterns in a syncopated way. Students from team 3 (25% in the group) generalized their procedure relationships through natural language and mathematical symbols (Figure 8).

Número de personas -1 x 50 = Descuento por persona

Precio base(3650) - Descuento - Costo(1300) = Ganancia por persona

Ganancia por persona x N de personas = Ganancia total

Figure 8: Generalization of Patterns to obtain the profit by Team 3.

Plenary Discussion

During the plenary session the students presented their letters and discussed their results. Students from team 4, based on their letter, showed how they calculated the profit. They were challenged by the rest of the teams with questions such as "What if there were 10", "How many would have to attend so that I could earn a lot of money". Students from teams 2 and 3 mentioned that 24 passengers were needed to get a maximum profit of $28800. In the letters they wrote this result, however, they did not explain the method they used to find the solution so they did not develop a shareable and reusable model (Lesh and Doerr, 2003). Students from team 1 told their classmates that they "discovered a formula by trial and error". They wrote in the letter the quantities corresponding to the maximum profit (according to their expression), as well as the expression itself. In other words, they not only mentioned how many passengers were needed to obtain a maximum profit, but also presented the client with a shareable and reusable tool with which they could know the profit for any number of passengers. However, they also did not explain how they found their procedure.

Conclusions

What knowledge and skills do first-semester college students exhibit when performing an MEA in which the concept of quadratic function underlies? Students exhibited the knowledge mentioned by Aliprantis and Carmona (2003): variable recognition, variation, linear relation and quadratic relation, maximum. They were able to identify the data of the problem and relate them to obtain new data. The relationships were expressed verbally and in writing, through operations on paper and formulas in Excel. Regarding mathematical skills, they used trial and error procedures, and built tables and graphs to analyze the variation of quantities. They identified a maximum value denoted as "peak", "mountain" or "bell curve" in the graphical form as the maximum profit. Students were able to generate conjectures (associate profit with linear behavior), describe and explain the situation, and finally, evaluate their conjectures. The results showed how the students were able to identify patterns, generalize and express them in a rhetorical and symbolic way, and use the GeoGebra CAS system to find answers.

One aspect that is emphasized in the MMP is letter and, therefore, model building. Although students obtained solutions, it was difficult for them to describe the procedures they used to arrive at their answers, as well as to develop general procedures that would be useful for similar situations. Considering the process as the product was not easy, as it involved giving importance to the process of mathematization. Students are used to giving unique and accurate answers, and this is what happened when they carried out the MEA.

One thing that stands out in this study is that before students associated quadratic behavior to the situation, they associated linear behavior. In other words, the activity presented in this paper has the potential to give students elements to characterize each type of function and, based on the context, discuss the differences between linear and quadratic behaviors.

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CONOCIMIENTOS Y HABILIDADES MATEMÁTICAS DE ESTUDIANTES UNIVERSITARIOS AL REALIZAR UNA MEA

MATHEMATICAL KNOWLEDGE AND SKILLS OF UNIVERSITY STUDENTS WHEN SOLVING A MEA

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En este documento se presentan los resultados de una investigación relacionada con el desarrollo de conocimientos y habilidades matemáticas por estudiantes universitarios de primer semestre al resolver Actividades Provocadoras de Modelos [MEA] que implican conocimiento acerca de la función cuadrática. La investigación fue de tipo cualitativa. El marco teórico fue la Perspectiva de Modelos y Modelación [PMM]. Los resultados muestran que los estudiantes utilizaron sus conocimientos y habilidades matemáticas relacionadas con la función lineal y cuadrática para describir la situación; transitaron de un ciclo de entendimiento caracterizado por procedimientos cuantitativos (en el que asociaron comportamientos lineales y cuadráticos a la situación), a un ciclo de entendimiento algebraico (en el que asociaron comportamientos únicamente cuadráticos).

Palabras clave: modelación, resolución de problemas, matemáticas de nivel universitario.

El aprendizaje del concepto de función ha sido objeto de estudio en diversas investigaciones (Gutiérrez y Prieto, 2015; Hernández, 2013; Oviedo, 2013). ¿Por qué es importante aprender el concepto de función? López, Navarro y Fuchs (2018) responden que es un conocimiento necesario para modelar sucesos y fenómenos en distintas áreas profesionales. Villarraga (2012) da cuenta, por ejemplo, del tipo de situaciones que pueden ser modeladas mediante la función cuadrática como: circunstancias de optimización relacionadas con costo, demanda y áreas, o en problemas físicos como intensidad de iluminación sobre una superficie. Pese a la importancia de comprender la función cuadrática, se ha identificado que existen dificultades en su aprendizaje, como la escasa articulación entre las representaciones algebraica y gráfica (Díaz, Haye, Montenegro y Córdoba, 2013) necesaria para describir situaciones y fenómenos.

Diversos autores (Gutiérrez y Prieto, 2015; Hernández, 2013 y Oviedo, 2013) han creado propuestas para atender el aprendizaje de la función cuadrática. Sin embargo, muchas se reducen a la manipulación de parámetros de la expresión algebraica para estudiar transformaciones en la gráfica correspondiente. Son escasas las investigaciones en las que se atiende el desarrollo de conocimientos y habilidades matemáticas asociadas a la función cuadrática en el contexto de la resolución de problemas o modelación de situaciones cercanas a la vida real. Una de las investigaciones realizadas en esta dirección fue la de Aliprantis y Carmona (2003), quienes en el marco de la Perspectiva de Modelos y Modelación (Lesh, 2010) diseñaron e implementaron una actividad para promover el desarrollo del conocimiento de la función cuadrática y conceptos asociados como el reconocimiento de variables, relación entre ellas (cuadráticas y lineales), producto de relaciones lineales y maximización, así como para propiciar en los estudiantes el desarrollo de habilidades para la modelación y solución de problemas, como conjeturar, argumentar, describir y explicar. Los participantes en estudio fueron estudiantes de secundaria.

La investigación descrita en este artículo se llevó a cabo con estudiantes del primer semestre universitario. El objetivo fue propiciar el desarrollo de conocimiento y habilidades matemáticas asociadas a la función cuadrática durante el proceso de resolución de problemas cercanos a la vida real. La pregunta de investigación es: ¿Qué conocimientos y habilidades exhiben estudiantes universitarios de primer semestre al resolver un problema en el que subyace el concepto de función cuadrática?

Marco Teórico

Aprender matemáticas, de acuerdo con la PMM (Lesh y Doerr, 2003), se basa en la construcción de modelos, los cuales

> son sistemas conceptuales (que consisten de elementos, relaciones, operaciones y reglas que rigen interacciones) que son expresados al usar sistemas de notación externa y que son utilizados para construir, describir o explicar los comportamientos de otros sistemas –quizá de manera que el otro sistema pueda ser manipulado o predicho inteligentemente. (Lesh y Doerr, 2003, p. 10).

Estos modelos pueden ser internos y externos, es decir, habitan tanto en el pensamiento de los estudiantes como en las ecuaciones, esquemas, aplicaciones computacionales u otros recursos de representación que utilizan expertos en ciencia o bien escolares (Lesh y Doerr, 2003). Los modelos pueden ser creados al realizar Actividades Provocadoras de Modelos (MEA), las cuales son simulaciones de situaciones de la “vida real”. Al realizar las MEA los estudiantes pasan por secuencias iterativas donde expresan, prueban y revisan sus propias formas de pensamiento (Lesh y Caylor, 2007). Durante el proceso de desarrollo de conocimiento, los estudiantes construyen y modifican sus modelos mediante las fases de diferenciación y refinamiento de los sistemas conceptuales que construyen (Lesh y Doerr, 2003).

Las MEA implican el uso de “herramientas conceptuales que son compartibles, manipulables, modificables y reutilizables (por ejemplo, modelos) para construir, describir, explicar, manipular, predecir, o controlar matemáticamente sistemas significativos” (Lesh y Doerr, 2003, p. 3). Los investigadores mencionan que tales descripciones, explicaciones y construcciones no deben ser consideradas como simples procesos que los estudiantes crean para conseguir “la respuesta”, sino que son elementos clave en el proceso de aprendizaje. De manera que, el proceso es el producto.

Las MEA “usualmente involucran la matematización, es decir, cuantificar, dimensionar, coordinar, categorizar, algebrizar y sistematizar objetos relevantes, relaciones, acciones, patrones y regularidades” (Lesh y Doerr, 2003, p. 5). Una característica que distingue a las MEA de otras actividades de resolución de problemas es la escritura de una carta. En ella los estudiantes deben explicar el método que utilizaron para encontrar la solución del problema; este método puede ser utilizado por un cliente para resolver otros problemas de características similares.

Metodología

La metodología que se siguió en esta investigación fue de tipo cualitativa. La MEA (Figura 1) cuyos resultados se describen en este documento, forma parte de una secuencia didáctica (Lesh, Cramer, Doerr, Post y Zawojewski, 2003) diseñada durante el proyecto de investigación. Fue implementada con un grupo de 12 estudiantes universitarios de nuevo ingreso, de aproximadamente 18 años, en una sesión de dos horas. Los estudiantes trabajaron en cuatro equipos de tres integrantes cada uno. Cada participante tenía una laptop con la MEA, Excel y GeoGebra. El proceso de resolución del problema se llevó a cabo en cuatro fases. 1) los estudiantes leyeron un artículo informativo relacionado al contexto del problema (actividad de calentamiento de acuerdo con la PMM). 2) leyeron el problema (ver Figura 1) y trabajaron en equipo en la resolución de éste. 3) expusieron sus soluciones en una discusión plenaria. 4) los estudiantes resolvieron el problema de manera individual en casa. En este documento solo se presentan los resultados de la fase 2, sustentados en las discusiones generadas durante la plenaria (fase 3). Los datos del estudio se obtuvieron del trabajo hecho por los estudiantes (hojas escritas); archivos de Word, Excel o GeoGebra; y de videograbaciones de la sesión presencial. Los conceptos asociados a esta actividad son los mencionados en la introducción (Aliprantis y Carmona, 2003).

Los estudiantes del último semestre de Ingeniería Mecánica Eléctrica del Centro Universitario de Ciencias Exactas e Ingenierías (CUCEI) de la Universidad de Guadalajara, quieren organizar un paseo a las cabañas de Mazamitla con el fin de recaudar fondos para celebrar su graduación.

Para tal propósito, la Universidad ha puesto a su disposición un autobús con capacidad para 49 pasajeros, de manera que quede cubierto el costo del recorrido ida y vuelta desde las instalaciones del CUCEI (diesel y pago al chofer), así como los traslados dentro del pueblo mágico. Además, Carlos, compañero de los estudiantes, se ha dado a la tarea de buscar en internet alojamiento y actividades para formar un paquete y poder ofrecer una experiencia completa a los viajeros que asistan al paseo.

El paquete turístico incluye el hospedaje para tres días y dos noches en una acogedora cabaña situada en La Cañada; un tour por la Sierra del Tigre, con guía, hielera y degustación de productos típicos; un paseo en cuatrimoto en el que se recorren los lugares más llamativos de la sierra. También, para aquellos que gustan de la adrenalina, se incluye tirolesa, muro de escalar, puentes colgantes y gotcha.

Carlos presupuestó el costo de $1 300.00 para cada paquete turístico por viajero, pero con el fin de obtener una ganancia, lo ofrecerá a un precio de $3650.00. Para animar a los posibles viajeros, Carlos propone hacer un descuento de $50 por cada persona que vaya, excepto si sólo va una. Es decir, si van dos personas, éstas reciben un descuento de $50; si son 3, se hace un descuento de $100 a cada una de ellas.

Ayuda a Carlos. Escríbele una carta en la que le expliques si su propuesta de organizar la excursión les permitirá obtener ganancias, y de qué monto. Sustenta tus afirmaciones con argumentos claros y válidos.

Figura 1: MEA

El análisis de datos se realizó con base en los ciclos de entendimiento cuantitativo y algebraico (Vargas, Reyes & Cristóbal, 2016). El ciclo de entendimiento cuantitativo es aquel en el que los estudiantes son capaces de describir de manera numérica las variables involucradas en el problema.

La información y las relaciones pueden ser organizadas en tablas y gráficas. En el ciclo de entendimiento algebraico los estudiantes exhiben cierto dominio del lenguaje del álgebra para solucionar el problema. Quienes alcanzan este ciclo han transitado por distintas etapas de diferenciación, integración y refinamiento de sus distintos sistemas conceptuales y tienen un dominio superior en el manejo de las representaciones. Durante cada uno de los ciclos, la representación verbal estuvo presente para justificar las conjeturas y las explicaciones de los estudiantes.

Resultados

Los estudiantes transitaran por dos ciclos de entendimiento (Tabla 1). El primero fue cuantitativo y el segundo algebraico. Inicialmente, los cuatro equipos (100% del total) revelaron una forma de pensar relacionada con la variación lineal (columna 2, Tabla 1) durante el ciclo cuantitativo. Enseguida, los alumnos de los equipos 1, 2 y 3 (75% de equipos) transitaron de sus procedimientos caracterizados por la variación lineal a una de tipo cuadrática (columna 3, Tabla 1); sin embargo, el equipo 4 no lo logró. Solamente los integrantes de los equipos 1 y 2 extendieron sus ideas a un ciclo de entendimiento algebraico. Una descripción en detalle se muestra enseguida.

Tabla 1: Ciclos de entendimiento de los estudiantes al resolver la MEA

	Ciclos de entendimiento			Respuesta	
	Cuantitativo		Algebraico	Correcta	Incorrecta
	Lineal	Cuadrático	Cuadrático		
E1	✓	✓	✓		✓
E2	✓	✓	✓	✓	
E3	✓	✓		✓	
E4	✓				✓

Primer Ciclo de Entendimiento: Cuantitativo

Todos los estudiantes identificaron los datos del problema: capacidad del camión, costo del viaje, precio inicial y el descuento. También, se percataron que tenían que elaborar una carta en la que desarrollaran un procedimiento para solucionar el problema. Se distinguieron dos formas de abordar la situación en este ciclo, porque algunos equipos la asociaron a un comportamiento lineal y otros a uno cuadrático.

Comportamiento lineal. Los integrantes de los cuatro equipos (100% de los equipos) calcularon el valor de la ganancia correspondiente a la cantidad máxima de pasajeros, es decir, 49 personas. La conjetura de los estudiantes fue que, a mayor cantidad de pasajeros, mayor ganancia, lo que denota un pensamiento lineal. Se distinguieron dos procedimientos, uno en el que se obtuvo la ganancia por pasajero, exhibido por los equipos 1, 2 y 3; y otro en el que se obtuvo la ganancia por grupo, elaborado por el equipo 4. Enseguida se discuten los procedimientos.

Procedimiento para obtener la ganancia para 49 pasajeros (capacidad máxima del camión). Método: ganancia por pasajero. El procedimiento de los equipos 1, 2 y 3 (75% de equipos) consistió en sustraer al precio inicial el descuento correspondiente por 49 pasajeros, es decir, efectuaron la operación: 3650-2400. Al resultado (1250) le restaron el costo del viaje, o sea, 1250-1300. Los estudiantes interpretaron esta cantidad (-50) como la ganancia por pasajero. No obstante, debido a la naturaleza negativa, expresaron que se trataba de una pérdida, en este caso de $50 por persona. De esta manera, la pérdida por 49 pasajeros era de $2450, como se muestra en la Figura 2.

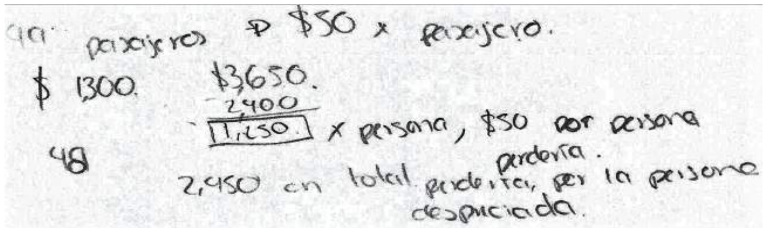


Figura 2: Operaciones llevadas a cabo por el equipo 1 para obtener la Ganancia relacionada a 49 Pasajeros por el Equipo 1. Método: Ganancia por Pasajero

Procedimiento para obtener la ganancia correspondiente a 49 pasajeros (capacidad máxima del camión). Método: ganancia por grupo. Los estudiantes del equipo 4 utilizaron la hoja de cálculo para operar con el costo del viaje por pasajero, la capacidad máxima de pasajeros y el precio inicial del paquete, es decir, $1300, 49 y $3650, respectivamente (celdas D6, E6 y D10 en la Figura 3). Relacionaron estos datos mediante fórmulas (Tabla 2) para calcular el descuento, el egreso, el ingreso relacionado con el precio inicial, el ingreso relacionado con el precio con descuento y la ganancia correspondiente a 49 pasajeros.

	A	B	C	D	E	F	G	H
1	2	50						
2	3	100		SUMATORIA DE LOS DESCUENTOS				
3	4	150			117600			
4	5	200						
5	6	250		PRECIO TOTAL DEL VIAJE				
6	7	300		$1,300	49			
7	8	350			63700			
8	9	400						
9	10	450		COBRO TOTAL		178850		
10	11	500		$3,650	61250			
11	12	550						
12	13	600						
13	14	650		GANANCIAS TOTALES				
14	15	700			-2450	COMO PODEMOS OBSERVAR ES UNA		
15	16	750				CANTIDAD NEGATIVA, POR TANTO SE		
16	17	800				PERDERIA ESE DINERO		

Figura 3: Procedimiento para obtener la Ganancia correspondiente a 49 Pasajeros por el Equipo 4. Método: Ganancia por Grupo

Tabla 2: Fórmulas Utilizadas por el Equipo 4

Celda	Fórmula
E3	=(B48*A48)
E7	=(D6*E6)
E10	=(D10*E6)-E3
E14	=(E10-E7)
F9	=(D10*E6)

Comportamiento cuadrático. Los integrantes de los equipos 1, 2 y 3 (75% de los equipos) hicieron varias operaciones, usaron diferentes cantidades de pasajeros y analizaron cómo variaban los resultados. Los estudiantes de los equipos 1 y 2 (50%) recurrieron al ensayo y error, de manera desordenada. Los estudiantes del equipo 3 (25%) exhibieron un procedimiento sistematizado. El equipo 4 (25%) fue el único que no realizó varias operaciones.

Procedimiento de ensayo y error. Los estudiantes de los equipos 1 y 2 (50% del total) realizaron operaciones con cantidades distintas de pasajeros. Esto permitió que los integrantes del equipo 1 identificaran la forma en cómo variaba el ingreso y encontraran para qué cantidad de pasajeros se producía un ingreso máximo (Figura 4). Por su parte, los alumnos del equipo 2 identificaron de manera más organizada cómo variaba la ganancia y encontraron la cantidad de pasajeros que correspondía a la ganancia máxima.

En la Figura 4 se muestra el procedimiento elaborado por el equipo 1, como ejemplo representativo del trabajo de los equipos 1 y 2. Los estudiantes calcularon el ingreso que se generaba si viajaban 24, 28, 37, 36 y 38 pasajeros. No obstante, creían que habían obtenido la ganancia. Observaron que existía una relación de dependencia en la que estaba involucrada la cantidad de pasajeros; expresaron que "la ganancia depende de las personas que vayan [pasajeros]".

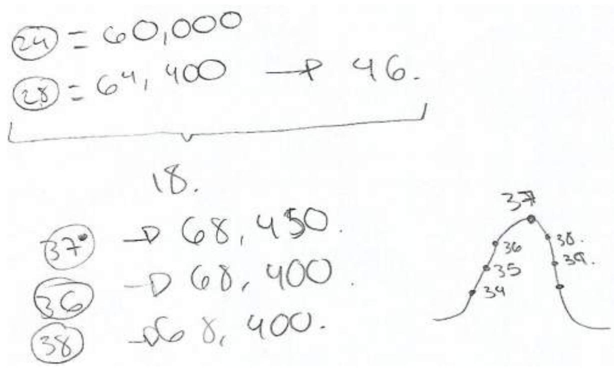
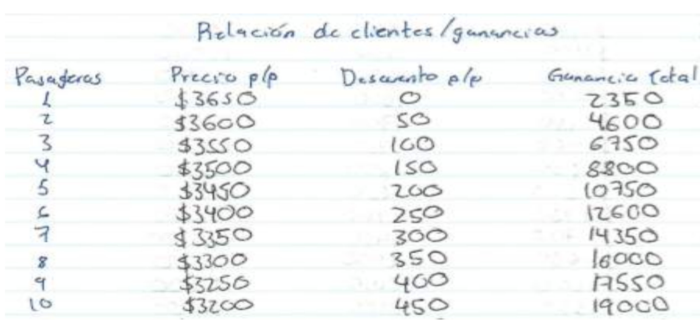


Figura 4: Procedimiento para obtener el Ingreso y para analizar la Variación para Distintas Cantidades de Pasajeros por el Equipo 1

En la misma Figura 4 se observa el bosquejo de una gráfica que describe la forma en cómo varía el ingreso. Los estudiantes notaron que se obtenía un ingreso "máximo" o "cúspide" para 37 pasajeros. No identificaron un intervalo de crecimiento o decrecimiento. Señalaron que, a partir del valor máximo, los ingresos correspondientes a 36 y 38 pasajeros "van bajando igual", del mismo modo mencionaron que "35 y 39 tienen el mismo [correspondiente valor de ingreso]".

Exploración de resultados por ensayo sistematizado. Los estudiantes del equipo 3 construyeron una tabla (Figura 5) con los rótulos: "Pasajeros", "Precio p/p", "Descuento p/p" y "Ganancia Total". Relacionaron las cantidades de cada fila de manera horizontal, y obtuvieron la ganancia, según la cantidad de pasajeros. Identificaron que la ganancia máxima, $28800, se obtiene cuando viajan 24 pasajeros. En la Figura 5 se muestra parte de la tabla creada por el equipo 3.



Relación de clientes/ganancias

Pasajeros	Precio p/p	Descuento p/p	Ganancia Total
1	$3650	0	2350
2	$3600	50	4600
3	$3550	100	6750
4	$3500	150	8800
5	$3450	200	10750
6	$3400	250	12600
7	$3350	300	14350
8	$3300	350	16000
9	$3250	400	17550
10	$3200	450	19000

Figura 5: Procedimiento del Equipo 3

Ciclo de Entendimiento Algebraico

Los estudiantes de los equipos 1 y 2 (50% de equipos) generalizaron patrones. Por su parte, los estudiantes del equipo 3 (25% del total) construyeron expresiones sincopadas para realizar el cálculo. Los estudiantes del equipo 4 (25%) no presentaron generalizaciones.

Generalización de patrones mediante expresión algebraica. En la Figura 6 se muestra la expresión obtenida por los estudiantes del equipo 1 para calcular la ganancia. No es identificada como función por los estudiantes, sino como una fórmula. Las cantidades 3650 y 50 representan el precio inicial del paquete y el descuento por pasajero, respectivamente. El valor 63700 es el resultado de multiplicar 1300*49, es decir, el costo del paquete por pasajero por 49 pasajeros. De manera que 63700 es el egreso cuando asiste la capacidad máxima de viajeros.

El egreso correspondiente a n pasajeros sería $1300n$. La expresión correcta era $(3650 - (n-1) * 50)n - 1300n$. El modelo algebraico que los estudiantes debían construir para calcular la ganancia era $f(x) = -50x^2 + 2400x$ en su forma simplificada, donde x representa la cantidad de asistentes.

(3650- (n-1)*50)n-63700

*Sea "n" el número de personas que asistan

Figura 6: Expresión Algebraica del Equipo 1 para Calcular la Ganancia

Estos estudiantes emplearon el software GeoGebra para identificar la cantidad de pasajeros con la que se producía la ganancia máxima. Encontraron que eran necesarios 37 pasajeros para producir un valor máximo de 4750 (Figura 7). Sin embargo, de acuerdo con los datos del problema, los valores correctos eran 24 pasajeros y $28 800.

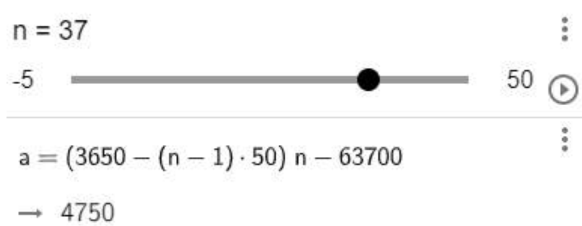


Figura 7: Procedimiento del equipo 1 para obtener la Ganancia Máxima

Generalización de patrones de manera sincopada. Los estudiantes del equipo 3 (25% en el grupo) generalizaron su procedimiento mediante lenguaje natural y símbolos matemáticos (Figura 8).

Número de personas-1 x50 = Descuento por persona

Precio base(3650) - Descuento - Costo(1300) = Ganancia por persona

Ganancia por persona x N de personas = Ganancia total

Figura 8: Generalización de Patrones para el Cálculo de la ganancia por el Equipo 3.

Discusión Plenaria

Durante la plenaria los estudiantes expusieron sus cartas y discutieron sus resultados. Los integrantes del equipo 4, con base en su carta, mostraron cómo calcularon la ganancia. Fueron cuestionados por el resto de los equipos con preguntas como "¿Qué pasa si van 10?", "¿Cuántos tendrían que asistir para que pudiera ganar mucho dinero?". Los alumnos de los equipos 2 y 3 mencionaron que eran necesarias 24 personas para obtener una ganancia máxima igual a $28800. En sus cartas escribieron este resultado, sin embargo, no explicaron el método que utilizaron para encontrar la solución por lo que no elaboraron un modelo compartible y reutilizable (Lesh y Doerr, 2003). Los estudiantes del equipo 1 comunicaron a sus compañeros que "descubrieron una fórmula al tanteo". Mostraron en su carta las cantidades correspondientes a la ganancia máxima (de acuerdo con su expresión), así como la expresión misma. Es decir, no sólo mencionaron qué cantidad de pasajeros eran necesarios para obtener una ganancia máxima, sino que presentaron al usuario una herramienta compartible y reutilizable con la que podrían conocer las ganancias para cualquier cantidad de pasajeros. No obstante, tampoco explicaron cómo encontraron su procedimiento.

Conclusiones

¿Qué conocimientos y habilidades exhiben estudiantes universitarios de primer semestre al resolver un problema en el que subyace el concepto de función cuadrática? Los estudiantes exhibieron los conocimientos mencionados por Aliprantis y Carmona (2003): reconocimiento de variables, variación, relación lineal y relación cuadrática, máximo. Fueron capaces de identificar los datos del problema y relacionarlos para obtener nuevos datos. Las relaciones las expresaron de manera verbal y escrita, mediante operaciones en papel y fórmulas en Excel. Respecto a las habilidades matemáticas, emplearon procedimientos de ensayo y error, y construyeron tablas y gráficas para analizar la variación de las cantidades. Identificaron un valor máximo denotado como "cúspide", "montaña" o "campana de Gauss" en la forma gráfica, el cual relacionaron con la ganancia máxima. Los alumnos fueron capaces de generar conjeturas (asociar a la ganancia un comportamiento lineal), describir y explicar la situación, y finalmente, evaluar sus conjeturas. En los resultados se observó cómo los estudiantes lograron identificar patrones, generalizarlos y expresarlos de manera retórica y simbólica, y utilizar el sistema CAS de GeoGebra para encontrar respuestas.

Un aspecto que se enfatiza en la PMM es la carta y, por lo tanto, la construcción de modelos. Si bien los estudiantes obtuvieron soluciones, les fue difícil describir los procedimientos que utilizaron para llegar a sus respuestas, así como desarrollar procedimientos generales que fueran útiles para situaciones similares. Considerar que el proceso es el producto no fue sencillo, ya que implicaba darle importancia al proceso de matematización. Los estudiantes están acostumbrados a dar respuestas únicas y exactas, y ello fue lo que ocurrió cuando realizaron la MEA.

Algo destacable en este estudio, es que antes de que los estudiantes asociaran a la situación un comportamiento cuadrático, asociaron un comportamiento lineal. Es decir, la actividad que se presentó en este documento tiene el potencial de dar elementos para que los estudiantes caractericen cada tipo de función y, con base en el contexto, discutan las diferencias entre comportamientos lineales y cuadráticos.

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MATHEMATICAL PROCESSES AND MODELING:

BRIEF RESEARCH REPORTS

MODEL ELICITING ACTIVITY FOR HYPOTHESIS TESTING WITH ENGINEERING STUDENTS

ACTIVIDAD PROVOCADORA DE MODELOS PARA PRUEBA DE HIPÓTESIS CON ESTUDIANTES DE INGENIERIA

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This report describes the models that students from the Electronics Engineering Division of the University of Guadalajara built with the implementation of a model eliciting activity (MEA) called Nanomaterials. The purpose was to document and analyze the underlying ideas and relationships that the students exhibited when solving a near-real-life problem in which various concepts of statistical inference emerge. The theoretical framework used was the Models and Modeling Perspective (MMP). The results indicate that the implementation of the activity generated hypothesis testing models, which integrated various means of representation, decision making and the use of concepts such as: null hypothesis, alternative hypothesis, confidence level, mean, sample, among others associated with statistical inference.

Keywords: Experiment design, Modeling, Problem solving, STEM.

Introduction

Nowdays there is a lot of information in print and digital formats. This demands the mastery of tools to analyze and understand that information, and use it to make decisions for the benefit of society (Lesh & Doerr, 2003). People who understand the information around them in their different economic, social, political or cultural contexts can make sense of it and reap its benefits.

Hypothesis testing is considered to support data interpretation, decision making and statistical inference, which "is where the power of statistics lies" (Makar & Rubin, 2018, p. 264). However, there are conceptual complications associated with statistical inference, especially in hypothesis testing, such as: a) confusion in the logic of hypothesis testing, b) the way in which the null hypothesis and the alternative are combined, c) the construction of a statistical hypothesis (Inzunsa & Jiménez, 2013; Lesh, 2010; Alvarado, Estrella, Retamal & Galindo, 2018; López, Batanero & Gea, 2018). These complications illustrate the complexity of the logic and fundamental concepts of hypothesis testing (Makar & Rubin, p. 269).

This study presents the results obtained in the implementation of a Model Eliciting Activity (MEA). It was developed under the Models and Modeling Perspective (MMP) (Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly & Post, 2000). The activity simulated a problem close to real life where students could build mathematical interpretations and manipulate information to make decisions. With it, an alternative is sought to generate better access to statistical inference.

The questions that guided this research were: 1) what are the models that engineering students create to give meaning and solution to a real life problem situation, which is related to hypothesis testing, 2) what mathematical concepts emerge during the implementation of the activity associated with statistical inference?

Theoretical framework

In the MMP "models are expected to be among the most important kinds of knowledge that students" (Lesh, 2010, p.17). According to Lesh & Doerr models are considered as:

> conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)—perhaps so that the other system can be manipulated or predicted intelligently. (Lesh & Doerr, 2003, p. 10)

Furthermore, "models are assumed to be highly situated, continually adapting, richly distributed, and socially shaped human constructs " (Lesh, 2010, p. 19), where modeling, as the process of building and developing models, includes a "series of parallel, interactive sequences of interactive cycles in which current forms of thought are repeatedly expressed, tested and revised" (Lesh, 2010, p. 17). In the framework of MMP, MEAs are proposed in order to motivate students to solve real-life problems through the construction of mathematical models that allow them to generate solutions, and they can "repeatedly revealing, testing and refining or extending their ways of thinking " (Lesh et al., 2000, p. 597).

Methodology

The MEA, named Nanomaterials, developed from Ramirez, Yu, Xu & Chen (2015), poses a situation of uncertainty related to a production process. The activity consists of four pages, the first two of which present a newspaper article describing the manufacturing process (printing) of nano circuits for biosensors. The third page contains a section of warm-up questions that introduce the student to the production process and uses of biosensors in health. The fourth page introduces the students to the problem, which refers to the discrepancy that exists between the molds and the magnitude of the printed circuits, and asks them to support the production process technicians to determine with certainty and with the information they have, the type of discrepancy that exists between the molds and the printed circuits.

The activity was refined with five previous implementations with different students (April-October 2019), where several aspects were adjusted. It is until the sixth implementation that we consider that we achieved an activity that meets the attributes of the MPA.

The implementation of the activity was developed according to Lesh & Doerr's (2003) suggestions which are: 1) reading a newspaper article that introduces the student to the context (production of biosensors), 2) construction of interpretations developed in teams, and 3) exposition and discussion of constructed models. The teacher performs the roles of facilitator and observer.

The six principles for the design of MEAs were used in the construction and refinement of the activity. 1) Reality: the situation is likely to occur in the students' real life; 2) Model construction: the activity generates in students the need to build, modify or refine a model; 3) Documentation of the model: the students' constructions explicitly reveal how they are thinking about the situation and its resolution (initial, intermediate and final interpretations); 4) Self-evaluation: the students are clear how to evaluate if their constructions are useful or good enough; 5) Generalization of the model: the model built by the students can be used for other situations and shared with other people; 6) Simple prototype: the activity solution provides a useful prototype for interpreting other structurally similar situations (Lesh et al. 2000; Lesh & Doerr, 2003). However, for the analysis of results only the first four design principles could be evaluated, due to the availability of resources.

The initial, intermediate and final ideas outlined by Lesh and Doerr (2003) were integrated into the principle of documentation of the model, and these were characterized in the MEA as follows. a) The initial ideas were proposals about what data are important, how to address the problem, and what steps for solution are most useful, and may or may not be associated with the use of statistics. b)

Intermediate ideas were those focused on the identification and explanation of patterns, relations or specific behavior of the data of the situation to be solved; it includes the use measures of central tendency and dispersion, construction of descriptive graphs or simple operations such as obtaining differences between the data, among others. c) Final ideas were the formal approaches of null and alternative hypothesis, confidence levels to decide, making the decision to reject or accept the null hypothesis, and includes possible suggestions to improve the production process according to the results obtained.

The study was of a descriptive qualitative nature. The means of collecting information were: videos, audios, and the written (physical and digital) of seven students from the Electronic and Computer Engineering Division of the University of Guadalajara, who were finishing the Probability and Statistics course (second semester). The final information was obtained in a 60-90 minute session, where the implementation of the activity took place. Three work teams were organized, Team 1 with two members (1A, 1B), Team 2 with three (2A, 2B, 2C) and Team 3 with two students (3A, 3B).

Results

The main results of the study are summarized below. They have been grouped according to the six principles for the design of MEAs, which were outlined above.

Reality principle. The reading and discussion of the newspaper article on biosensor production allowed the students to engage with the context of the activity, and revealed the knowledge they had about biosensors. This was done by relating the activity to some biosensors that they already knew about: "Apple Watch, exercise band, cardio exercise machines, heart rate sensor, temperature sensor and step counter".

Model construction principle. Team 1 made a graph to visualize the behavior of the data (mold measurements and impressions), which helped them to understand and interpret the situation visually. Team 2 proposed procedures such as: obtaining differences between the measurements of the shapes and impressions, determining an average of each of the measurements, calculating the standard deviation and variance of the data in order to analyze the information.

Model documentation (initial, intermediate, and final ideas). The students during the implementation developed a series of interpretations that went through several modifications and refinements, which helped them to orient their work towards what they considered to be the best answer.

Initial interpretations. The teams in their first working dialogues mentioned ideas of how to analyze the information of the problem, but without arguing why they were useful: 1A: "What if we make a graph", 2B: "What if we take out the differences between the mold and the print", 2B: "Let's make a bell graph", 2A: "Let's make a scatter graph". It was identified that students initially resort to data analysis with graphs and manipulate them with simple operations (differences); with the idea of making sense of the information of the problem.

Intermediate interpretations. Students identified patterns and relationships after graphing the data set and calculating some statistics (mean, variance, standard deviation).For example, Team 1 looked at their line graph and noted a dependency relationship between the mold and print measurements; 1B: "the mold is larger than the print," 1B: "look at the means are different," but without formally making a correlation or hypothesis to prove the differences, they only made some informal statements that could lead them to make the null and void hypothesis; 1A: "in fact the measurements should be the same," 1B: "the mold is larger than the print".

Final interpretations. The final model in all teams was a hypothesis testing procedure, which integrated graphic, tabular, algebraic, written and verbal representations. Here they posed the relationship between the measurements of the molds and the prints, for example: 1A: "the null hypothesis is that the measurements of the molds are equal", posed by the student that defines the type of contrast to be developed. 1B: "the other hypothesis is that the moulds are larger than the impressions". The approach of this alternative hypothesis complemented and guided the construction of his model, in addition to using other concepts such as: sample and confidence level to refine and define their model.

Principle of self-evaluation. Team 1, modified their decision to use graphs and linear regression to explain their proposed model. This was because, after a second analysis of the activity specifications, they felt that their proposal was not sufficiently useful. This redirected their work towards building a hypothesis test, a decision that they felt allowed them to test with greater certainty whether their guess was acceptable or not. In other words, the students self-evaluated the usefulness of their answer and modified it in search of a better one.

Conclusions

The MEA Nanomaterials contributed to the study of MMP, specifically in topic of hypothesis testing of inferential statistics, being an original design for electronic engineering students.

According to the research questions, the MEA Nanomaterials promoted the construction of models to perform the hypothesis testing procedure; which integrated different mathematical concepts such as: sample, mean, variance, null and alternative hypothesis, confidence level, scatter plot, among others. In addition, it made evident that the hypothesis test is not a trivial issue, since students had to develop a series of interpretations (initial, intermediate and final) (Lesh et al. 2000). They used various means of representation (graphical, tabular, written, algebraic, spoken language) (Lesh & Doerr, 2003) to refine, modify and extend their ways of thinking about this relevant procedure within statistical inference. In addition, access to the web and the use of Excel made it easier for students to build visual, manipulable and dynamic representations (Lesh, 2010).

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ACTIVIDAD PROVOCADORA DE MODELOS PARA PRUEBA DE HIPÓTESIS CON ESTUDIANTES DE INGENIERIA

MODEL ELICITING ACTIVITY FOR HYPOTHESIS TESTING WITH ENGINEERING STUDENTS

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En este reporte se describen los modelos que estudiantes de la División de Ingeniería en Electrónica y Computación de la Universidad de Guadalajara, construyeron con la implementación de una Actividad Provocadora de Modelos (APM) llamada Nanomateriales. El propósito fue documentar y análizar las ideas y relaciones subyacentes que los alumnos exhibieron al resolver una problemática cercana a la vida real en la que emergen diversos conceptos de inferencia estadística. El marco teórico utilizado fue la Perspectiva de Modelos y Modelación (PMM). Los resultados indican que la implementación de la actividad generó modelos de prueba de hipótesis, que integraron diversos medios de representación, la toma de decisiones y el uso de conceptos como: hipótesis nula, alternativa, nivel de confianza, media, muestra, entre otros asociados con la inferencia estadística formal.

Palabras clave: Diseño de experimentos, Modelación, Resolución de problemas, STEM.

Introducción

La gran cantidad de información en formatos impresos y digitales en la era actual demanda el dominio de herramientas de interpretación, análisis y toma de decisiones para poder comprender esa información y utilizarla en beneficio de la sociedad (Lesh & Doerr, 2003). Las personas que comprenden la información que les rodea en sus diferentes contextos económicos, sociales, políticos o culturales pueden darle sentido y aprovechar sus beneficios (Lesh, 2010, p.27).

Se considera que la prueba de hipótesis apoya la interpretación de datos, la toma de decisiones y la inferencia estadística, que "es donde reside el poder de la estadística" (Makar & Rubin, 2018, p. 264). Sin embargo, se han identificado varias complicaciones conceptuales asociadas con la inferencia estadística, especialmente en prueba de hipótesis, tales como: a) confusiones en la lógica del contraste de hipótesis, b) la forma en la que se combinan la hipótesis nula y la alternativa, c) la construcción de una hipótesis estadística (Inzunsa & Jiménez, 2013; Lesh, 2010; Alvarado, Estrella, Retamal & Galindo, 2018; López, Batanero & Gea, 2018). Estas complicaciones ilustran aún más la complejidad de la lógica y los conceptos fundamentales de las pruebas de hipótesis (Makar & Rubin, 2018; p. 269).

El presente estudio expone los resultados obtenidos en la implementación de una Actividad Provocadora de Modelos (MEA por su nombre en inglés: Model Eliciting Activities). Misma que fue desarrollada bajo la Perspectiva de Modelos y Modelación (PMM) (Lesh & Doerr, 2003; Lesh, Hoover, Hole, Kelly & Post, 2000). La actividad simuló una problemática cercana a la vida real donde los alumnos pudieron construir interpretaciones matemáticas y manipular información para tomar decisiones. Con ella se busca una alternativa para generar un mejor acceso a la inferencia estadística.

Las preguntas que guiaron esta investigación fueron: 1) ¿cuáles son los modelos que los estudiantes de ingeniería crean para dar sentido y solución a una situación problemática de la vida real, que está relacionada con la prueba de hipótesis?, 2) ¿qué conceptos matemáticos emergen durante la implementación de la actividad asociados con la inferencia estadística?

Marco teórico

En la PMM, se "espera que los modelos se encuentren entre los tipos de conocimiento más importantes que desarrollan los estudiantes" (Lesh, 2010, p.17), de acuerdo con Lesh & Doerr los modelos son considerados como:

> Sistemas conceptuales (consisten en elementos, relaciones y reglas que gobiernan las interacciones) expresados mediante el uso de sistemas de notación externa, y utilizados para construir, describir, o explicar los comportamientos de otros sistemas de tal forma que el otro sistema pueda ser manipulado o predicho de manera inteligente (Lesh & Doerr, 20003, p. 10)

Además, "se asume que los modelos están altamente situados, adaptándose continuamente, ricamente distribuidos y con construcciones con formas sociales" (Lesh, 2010, p. 20), donde la modelación, como el proceso de construcción y desarrollo de modelos, incluye una "serie de secuencias paralelas e interactivas de ciclos interactivos en los que las formas actuales de pensamiento se expresan, prueban y revisan repetidamente" (Lesh, 2010, p.17). La PMM, propone MEAs, para que los estudiantes se motiven a resolver problemas de la vida real a través de la construcción de modelos matemáticos que les permitan generar soluciones y puedan "revelar, probar y refinar repetidamente o ampliar sus formas de pensamiento "(Lesh et al., 2000, p. 597).

Metodología

La APM, nombrada Nanomateriales, desarrollada a partir de Ramirez, Yu, Xu & Chen (2015), plantea una situación de incertidumbre relacionada con un proceso de producción. La actividad consta de cuatro páginas, en las dos primeras se presenta un artículo de periódico que describe el proceso de fabricación (impresión) de nano circuitos para biosensores. La tercera página contiene una sección de preguntas de calentamiento que introducen al estudiante con el proceso de producción y usos de los biosensores en la salud. En la cuarta página se les presenta a los estudiantes la problemática, que se refiere a la discrepancia que existe entre los moldes y la magnitud de los circuitos impresos, y se les solicita que apoyen a los técnicos del proceso de producción para determinar con certeza y con la información que tienen, el tipo de discrepancia que existe entre los moldes y los circuitos impresos.

La actividad fue afinada con cinco implementaciones previas con diferentes alumnos (abril-octubre de 2019), donde se ajustaron varios aspectos. Es hasta la sexta implementación en la que consideramos logramos obtener una actividad que reúne los atributos propios de las APM.

La implementación de la actividad se desarrolló de acuerdo con las sugerencias de Lesh & Doerr (2003) que son; 1) lectura de un artículo de periódico que introduce al estudiante en el contexto (producción de biosensores), 2) construcción de interpretaciones desarrolladas en equipo y 3) exposición y discusión de los modelos construidos. El docente asume los roles de facilitador y observador.

Los seis principios de diseño de APM se utilizaron en la construcción y refinación de la actividad: 1) Realidad: la situación es posible que ocurra en la vida real de los alumnos; 2) Construcción del modelo: la actividad genera en los estudiantes la necesidad de construir, modificar o refinar un modelo; 3) Documentación del modelo: las construcciones de los alumnos revelan explícitamente cómo están pensando en la situación y su resolución (interpretaciones iniciales, intermedias y finales); 4) Autoevaluación: los alumnos tienen claro cómo evaluar si sus construcciones son útiles o suficientemente buenas; 5) Generalización del modelo: el modelo construido por los alumnos puede utilizarse para otras situaciones y compartirse con otras personas; 6) Prototipo simple: la solución para la actividad proporciona un prototipo útil para interpretar otras situaciones estructuralmente similares. (Lesh et al. 2000; Lesh & Doerr, 2003). Sin embargo, en el análisis de resultados sólo se pudieron evaluar los primeros cuatro principios de diseño, debido a la disponibilidad de los recursos.

Las ideas iniciales, intermedias y finales que señalan Lesh y Doerr (2003) se integraron en el principio de documentación del modelo, y se caracterizaron en la APM como sigue. a) Ideas iniciales, fueron las propuestas acerca de qué datos son importantes, cómo abordar el problema y qué pasos para la solución son más útiles, pudiendo o no estar asociadas con el uso de la estadística. b) Ideas intermedias, son aquellas centradas en la identificación y explicación de patrones, relaciones o comportamiento específico de los datos de la situación a resolver; incluye el uso de medidas de tendencia central y de dispersión, construcción de gráficas descriptivas u operaciones sencillas como obtener diferencias entre los datos, entre otras. c) Ideas finales, son los planteamientos formales de hipótesis nula y alternativa, niveles de confianza en la decisión, tomar la decisión de rechazar o aceptar la hipótesis nula, e incluye posibles sugerencias para mejorar el proceso de producción en función de los resultados obtenidos.

El estudio fue de carácter cualitativo descriptivo. Los medios de recolección de información fueron: vídeos, audios, y los escritos (físicos y digitales) de siete estudiantes de la División de Ingeniería en Electrónica y Computación de la Universidad de Guadalajara, que estaban terminando el curso de Probabilidad y Estadística del segundo semestre. La información final se obtuvo en una sesión de 60-90 minutos, donde se llevó a cabo la implementación de la actividad. Se organizaron tres equipos de trabajo, el Equipo 1 con dos integrantes (1A, 1B), el Equipo 2 con tres (2A, 2B, 2C) y el Equipo 3 con dos alumnos (3A, 3B).

Resultados

A continuación, se resumen los principales resultados del estudio. Se han agrupado de acuerdo a los seis principios para el diseño de APMs, que ya antes se señalaron.

Principio de realidad. La lectura y discusión del artículo de periódico (sobre producción de biosensores) permitió a los estudiantes involucrarse con el contexto de la actividad, y reveló el conocimiento que tenían sobre biosensores al relacionar la actividad con algunos que ellos ya conocían como: "Apple Watch, banda de ejercicio, máquinas de cardio para el ejercicio, sensor de ritmo cardiaco, sensor de temperatura y contador de pasos".

Principio de construcción de modelo. El Equipo 1 hizo un gráfico para visualizar el comportamiento de los datos (medidas de molde e impresiones), que les ayudó a entender e interpretar la situación visualmente. El Equipo 2 propuso procedimientos tales como: obtener las diferencias entre las medidas de las formas e impresiones, determinar un promedio de cada una de las medidas, calcular la desviación estándar y la varianza de los datos para analizar la información.

Documentación del modelo (ideas iniciales, intermedias y finales). Los estudiantes durante la implementación desarrollaron una serie de interpretaciones que transitaron por varias modificaciones y refinamiento, que les ayudó a orientar su trabajo hacia lo que consideraban como mejor respuesta.

Interpretaciones iniciales. Los equipos en sus primeros diálogos de trabajo mencionaron ideas del cómo analizar la información del problema, pero sin argumentar el por qué eran útiles: 1A: "¿Y si hacemos una gráfica?", 2B: "Y si sacamos las diferencias entre el molde y la impresión", 2B: "hagamos una gráfica de campana", 2A: "hagamos una de dispersión". Se identificó que los alumnos recurren inicialmente al análisis de datos con gráficas y a la manipulación de éstos con operaciones sencillas (diferencias); con la idea de dar sentido a la información del problema.

Interpretaciones intermedias. Los alumnos identificaron patrones y relaciones después de graficar los datos y calcular algunos estadísticos (media, varianza, desviación estándar). Por ejemplo el Equipo 1 observó su gráfica de líneas y señaló una relación de dependencia entre las medidas de moldes e impresiones; 1B: "el molde es más grande que la impresión", 1B: "mira las medias son diferentes", pero sin llegar a plantear de manera formal una correlación o el planteamiento de hipótesis para probar las diferencias, solo realizaron algunas afirmaciones informales que pudieran llevarlos al planteamiento de la hipótesis nula y alternativa; 1A: "de hecho las medidas deberían ser iguales", 1B: "el molde es más grande que la impresión",

Interpretaciones finales. El modelo final en todos los equipos fue un procedimiento de prueba de hipótesis, que integró representaciones gráficas, tabulares, algebraicas, escritas y verbales. Aquí ellos plantearon la relación entre las medidas de los moldes y las impresiones, por ejemplo: 1A: "la hipótesis nula es que las medidas de los moldes son iguales", planteamiento por el alumno que define el tipo de contraste a desarrollar. 1B: "la otra hipótesis es que los moldes son más grandes que las impresiones". El planteamiento de esta hipótesis alternativa complementó y orientó la construcción de su modelo, además de utilizar otros conceptos como: muestra y nivel de confianza para refinar y definir su modelo.

Principio de autoevaluación. El Equipo 1, modificó su decisión de usar gráficas y regresión lineal para explicar su modelo propuesto. Esto debido a que, después de analizar por segunda vez las especificaciones de la actividad, consideraron que su propuesta no era lo suficientemente útil. Esto reorientó su trabajo hacia la construcción de una prueba de hipótesis, decisión que, según ellos, les permitía probar con mayor certeza si su conjetura era aceptable o no. En otras palabras, los alumnos autoevaluaron la utilidad de su respuesta y la modificaron en búsqueda de una mejor.

Conclusiones

La APM Nanomateriales contribuyó al estudio de la PMM, específicamente en el tema de prueba de hipótesis de la estadística inferencial, al ser un diseño original para los estudiantes de ingeniería electrónica. De acuerdo con las preguntas de investigación la APM Nanomateriales propició la construcción de modelos tipo procedimiento de prueba de hipótesis que integraron diferentes conceptos matemáticos como; muestra, media, varianza, hipótesis nula y alternativa, nivel de confianza, gráfica de dispersión, entre otros. Además, dejó en evidencia que la prueba de hipótesis no es una idea trivial, ya que los alumnos tuvieron que desarrollar una serie de interpretaciones (iniciales, intermedias y finales) (Lesh et al. 2000) donde usaron varios medios de representación (gráfico, tabular, escrito, algebraico, lenguaje hablado) (Lesh & Doerr, 2003) para refinar, modificar y ampliar sus formas de pensar acerca de este procedimiento relevante dentro de la inferencia estadística. En adición, el acceso a la red y uso de Excel les facilitó a los alumnos la construcción de representaciones visuales, manipulables y dinámicas (Lesh, 2010).

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AN ANALYSIS OF STUDENTS' MATHEMATICAL MODELS FOR MUSIC

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This paper describes a modeling task designed to improve students' understanding of music and related unit structures (e.g., whole note, half note). Fourteen upper elementary students were asked to build models of melodies using Cuisenaire rods and make arguments about how their models represented what they heard. Our analysis of students' models suggested four categories of models. Students exhibited one- or two-dimensional reasoning with either (or both) height and length correspondence that varied in terms of duration and/or pitch features.

Keywords: Modeling, Representations and Visualization, Measurement

Background and Literature Review

Mathematical modeling focuses the relevance of mathematics through the use of authentic contexts where students use their mathematics to solve relevant problems (COMAP & SIAM, 2016). There is a growing emphasis on the inclusion of mathematical modeling in school mathematics (e.g., National Council of Teachers of Mathematics, 2000; National Governor's Association Center [NGAC] & Council of Chief State School Officers [CCSSO], 2010). While the phrase *mathematical modeling* has been used in many ways, we consider the description of mathematical modeling from the Common Core State Standards, which describes modeling as "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (NGAC & CCSSO, 2010). In this description, the main focus of mathematical modeling is learning to make decisions and assumptions when interpreting a real-world scenario using a mathematical lens. These scenarios are often posed using open-ended tasks where students have the freedom and flexibility to create their own non-prescribed models (COMAP & SIAM, 2016). Because mathematical modeling requires creativity and allows for varied solution strategies, modeling tasks inherently provide multiple entry points and differentiation opportunities (Cirillo et al., 2016).

Prior research studies showed that mathematical modeling tasks were helpful in revealing student thinking and that modeling tasks enable students of differing performance levels to interpret, invent, and find solutions (e.g., Aguilar Battista, 2017; Carmona & Greenstein, 2007; Koellner-Clark & Lesh, 2003; Mousoulides, Pittalis, Christou, & Sriraman, 2010). Despite the existing literature on mathematical modeling, there is a need for further research in the elementary grade levels. An analysis of 29 articles (published between the years 1991-2015) that focused on elementary mathematical modeling (ages 10 and below) revealed that more research (as well as teacher training) related to mathematical modeling in the elementary grades is needed (Stohlman & Albarracin, 2016).

In the modeling task that we share in this report, students are expected to use "the language of mathematics to quantify real-world phenomena and analyze behaviors" (COMAP & SIAM, 2016, p. 8). The real-word phenomena is the representation of musical notes. We chose to develop a modeling task for music because musical notes are inherently mathematical due to the proportional relationship of their size (i.e., duration of each note). Additionally, integrating music and mathematics appears to be a particularly effective intervention for students to improve students' conceptual understanding of

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

fractions, especially for high needs students (Courey et al., 2012). In order to contribute to the understanding of framing instruction with modeling tasks in earlier grades, we focused on the following research questions in our study: *What were the mathematical assumptions and decisions students made when creating physical models to represent musical melodies? What were the underlying mathematical characteristics of their models and were there any similarities and/or differences between models?*

When learning a mathematical concept, children use actions. While these actions can initially be physical or mental, ultimately, the actions are mental that may or may not have been derived from physical actions or words (Sarama & Clements, 2009). When creating our own models during the design phase of the task, we determined that our own mental actions included unitizing: defining a unit and a sub-unit (i.e., whole and half notes). Unitizing is defined as "the process of constructing chunks in terms of which to think about a given commodity" (Lamon, 2012, p. 104). Because unitizing is a subjective process, encouraging flexibility and highlighting the relationship between unitizing and understanding fractions and equivalence is important (Lamon, 2012). We focused on students' unitizing mental actions while analyzing their models.

Methodology

The motivation for the *Modeling Music* task was to utilize the multiple ways in which music can be represented to emphasize the proportional relationship of musical notes. To show the different representations of music as well as how these different representations are related, we developed a framework which had the components of song, sound wave, sheet music, and physical tools. This particular modeling task attended to the bi-directional relationships between melody, sheet music, and physical tools representations.

Four melodies (Melodies A, B, C, and D) were created and then purposefully sequenced to highlight differences in the length of the notes (Figure 2). The first two melodies (A and B) were solely comprised of either whole or half notes. The third melody (C) was a combination of whole and half notes and the fourth melody (D) was a combination of whole, half, and quarter notes.

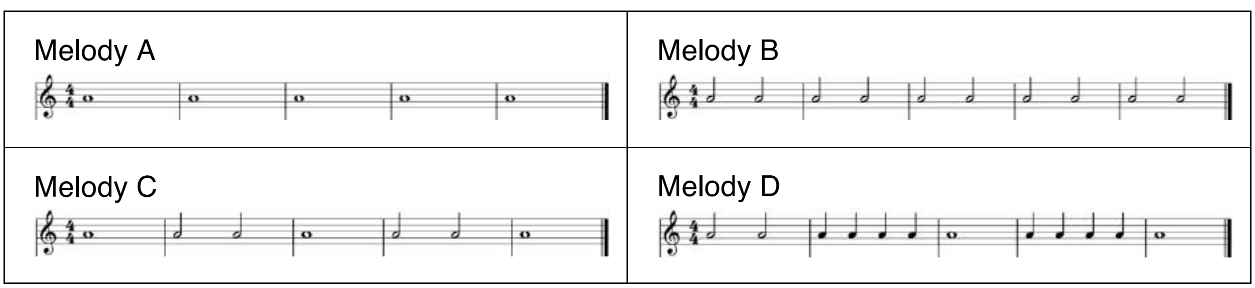

Figure 1. Sheet music for Melodies A, B, C, and D

Participants and Implementation

Fourteen upper-elementary (fourth and fifth grade) students participated in the *Modeling Music* task during a summer ice skating camp in July 2019. The daily schedule of the camp limited the time allotted for the *Modeling Music* task to 45 minutes and as a result, students were only able to create models for the first three melodies. The activity sequence for the *Modeling Music* task consisted of three parts: (1) listening to the melody, (2) recording and sharing notices and wonders about the melody, and (3) building the model using Cuisenaire rods. Students were not provided with any guidance or direction when building their models, which required them to make their own assumptions and decisions during the modeling process, as well as identify the underlying mathematical relationships in their models.

Data Collection and Analysis

In order to better understand students' modeling strategies, the data we collected during task implementation included students' individual written responses to the notices and wonder prompts for each melody, a written record of students' verbal descriptions of each melody, and photographs of the Cuisenaire rod models students created for each melody. The students' models and their written descriptions were analyzed using comparative analysis (Merriam, 1998). The similar models were first categorized into similar chunks (e.g., models using one-dimensional reasoning). In the next revision, this classification was elaborated into more defined categories and we looked for the unitizing structures involved in the models. We used measurement ideas to analyze the multiple representations of proportional relationships and we used basic principles of measurement (e.g., relating size and units) to explore how these relationships were connected within the context of music.

Results and Discussion

Students' notices and wonders for each of the melodies highlighted several common themes. Some of these themes revealed the underlying mathematics students observed (e.g., distance between notes, length of notes). Other themes revealed students' perceptions of the sound (e.g., pitch, tempo). Students' Cuisenaire rod models of the melodies revealed their modeling strategies, including the assumptions and decisions they made for mathematizing the melodies.

Modeling Single Note Melodies (Melody A and B)

When modeling single note melodies, students built either a single rod model or a collection of rods model to represent one note (see Table 1). The main difference between these models was how students decided to represent one unit. With the single rod model, students decided to define one note with one rod, whereas with the collection of rods model, students decided to define one note with a collection of rods in a staircase shape. With both the single rod model and collection of rod models, students assumed that the notes in the melody were identical and chose to iterate their unit to reflect this assumption.

Table 1: Student Models of Single Note Melodies (Melody B)

Single Rod Model	Collection of Rods Model
	

Modeling Two-Note Melodies (Melody C)

When modeling the two-note melody, students had to decide how to represent both whole and half notes in a single model. Students' models were categorized based on which characteristics of the rods they attended to when representing the different notes as summarized in Table 2.

Table 2: Categories for Two-Note Melodies (Melody C)

Category and Sample Model	Defining Characteristics
(1) 1-D: Length Correspondence (Duration) 	Attended to rod length to represent each note. Length of half note (red) corresponded to length of whole note (purple).
(2a) 2-D: Height Correspondence (Duration) 	Attended to horizontal length (number of rods) and height (length of rods) to represent each note. Length of starting rod of half note (yellow) corresponded to length of starting rod of whole note (orange).
(2b) 2-D: Length Correspondence (Duration) 	Attended to horizontal length (number of rods) and height (length of rods) to represent each note. Number of rods representing each note had a 4:2 proportion.
(3) 2-D: Height, Length Correspondence (Duration) 	Attended to horizontal length (number of rods) and height (length of rods) to represent each note. Length half note (yellow) corresponded to length of starting rod of whole note (orange) and number of rods representing each note had a 2:1 proportion.
(4) 2-D: Height, Length Correspondence (Duration and Pitch) 	Attended to horizontal length (number of rods), height (length of rods), and pitch (starting rod) to represent each note. Number of rods representing each note had a 4:2 proportion. Used same starting rod for both whole and half notes.

Conclusion

The *Modeling Music* task clearly provided students with multiple entry and exit points as evidenced by the sheer variety in students' models. In addition, unpacking students' mental actions when building their models revealed commonalities in students' thinking related to unitizing and proportional reasoning (e.g., half/whole note relationships). Our analysis provided a method of categorizing students' models based on their defining characteristics, which brought to light the assumptions and decisions made by students during the modeling process.

Research related to students' mathematical modeling strategies provides opportunities for rich descriptions of student thinking. Our findings are promising in terms of further study of modeling tasks and the value of using modeling tasks to explore students' reasoning and strategies, including application of prior knowledge, when solving open-ended problems. The *Modeling Music* task also suggests a framework for task design and model categorization that can allow for further mathematical modeling research in the elementary grades.

Our findings can also inform instructional decisions. Having a framework for model categorization (in terms of underlying mental actions) allows us to anticipate student thinking, which can help educators better prepare instruction related to both mathematical modeling and the development of measurement concepts.

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BEYOND PATTERNS: MAKING SENSE OF PATTERNS-BASED GENERALIZATIONS THROUGH EMPIRICAL RE-CONCEPTUALIZATION

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Identifying patterns is an important part of mathematical reasoning, but many students struggle to justify pattern-based generalizations. Some researchers argue for a de-emphasis on patterning activities, but empirical investigation has also been shown to support discovery and insight into problem structures. We introduce a phenomenon we call empirical re-conceptualization, which is the development of a generalization based on an empirical pattern that is subsequently re-interpreted from a structural perspective. We define and elaborate empirical re-conceptualization by drawing on data from secondary and undergraduate students, and identify three major affordances: Empirical re-conceptualization can serve as (a) a source of verification, (b) a means of justification, and (c) a vehicle for generating insight.

Keywords: Reasoning and Proof, Cognition, Algebra and Algebraic Thinking

Objective: Leveraging the Power of Pattern-Based Generalizations

Recognizing and developing patterns is a critical aspect of mathematical reasoning. Many students are adept at recognizing and formalizing patterns (Pytlak, 2014), but they can also struggle to understand, explain, and justify those very patterns they develop (Čadež & Kolar, 2014). One source of students' difficulties may rest with the empirical nature of those generalizations. Students can become overly reliant on examples and infer that a universal statement is true based on a few confirming cases (Knuth, Choppin, & Bieda, 2009). One potential solution is to help students understand the limitations of empirical evidence and thus recognize the need for deductive arguments (e.g., Stylianides & Stylianides, 2009). These approaches have shown some success in helping students see the limitations of examples, but they also frame empirical reasoning strategies as stumbling blocks to overcome.

In contrast, we have identified a phenomenon that we call *empirical re-conceptualization*, in which students identify a pattern, form an associated generalization, and then re-interpret their findings structurally. From this perspective, students can bootstrap their pattern-based generalizations into mathematically meaningful insights and arguments. In this paper, we describe and elaborate the construct of empirical re-conceptualization and address the following questions: (a) What characterizes students' abilities to leverage pattern-based generalizations in order to develop mathematical insights? (b) What are the conceptual affordances of empirical re-conceptualization? We offer a secondary example, discuss the affordances experienced, and consider ways in which instruction can support the practice of empirical re-conceptualization.

The Drawbacks and Opportunities of Empirical Reasoning

While an emphasis on patterning that lacks meaning can promote the learning of routine procedures without understanding (Fou-Lai Lin et al., 2004), there are also a number of affordances that can arise from empirical investigation. The act of developing empirically-based generalizations can foster the discovery of insight into a problem's structure, which could consequently support proof development (de Villiers, 2010). The degree to which pattern generalization is an effective route to proof is an open question, but there is evidence that students can and do engage in a dynamic interplay between empirical patterning and deductive argumentation (e.g., Schoenfeld, 1986).

Students lack sufficient experience with developing meaning from patterns. Curricular materials emphasize patterning activities that end with a generalization, typically an algebraic rule; developing an associated justification is seldom emphasized in standard classroom tasks. In fact, students typically receive little, if any, explicit instruction on how to strategically analyze examples in developing, exploring, and proving generalizations (Cooper et al., 2011). We propose that empirical re-conceptualization can be one way to provide opportunities to develop mathematical insight and deductive argumentation from pattern-based generalizing activities.

Theoretical Perspectives: Structural Reasoning

Harel and Soto (2017) identified five major categories of structural reasoning: (a) pattern generalization, (b) reduction of an unfamiliar structure into a familiar one, (c) recognizing and operating with structure in thought, (d) epistemological justification, and (e) reasoning in terms of general structures. The first category further distinguishes between result pattern generalization (RPG) and process pattern generalization (PPG) (Harel, 2001). RPG is a way of thinking in which one attends solely to regularities in the result. The example Harel gave is observing that 2 is an upper bound for the sequence $\sqrt{2}$,$\sqrt{2 + \sqrt{2}}$, $\sqrt{2 + \sqrt{2 + \sqrt{2}}}$, … because the value checks for the first several terms. When we refer to empirical re-conceptualization and the identification of a pattern based on empirical evidence, we are referring to RPG. In contrast, PPG entails attending to regularity in the process. Harel discussed how one might engage in PPG to determine that there is an invariant relationship between any two consecutive terms of the sequence, $a_{n+1} = \sqrt{a_n + 2}$, and therefore reason that all of the terms of the sequence are bounded by 2 because $\sqrt{2} < 2$.

We define empirical re-conceptualization as the process of re-interpreting a generalization based on RPG from a structural perspective. By structural perspective, we mean engaging in any of the following activities: (a) shifting from RPG to PPG; (b) reducing an unfamiliar structure into a familiar one; (c) carrying out operations in thought without performing calculations; (d) forming and reasoning with a new conceptual entity; or (e) shifting from figurative to operative activity. In short, re-interpreting a generalization from a structural perspective entails the ability to recognize, act upon, and reason with general structures.

Methods

Barney (a 7th-grade student) and Homer (a 9th-grade student) participated in a paired teaching experiment (Steffe & Thompson, 2000), which took place across five sessions averaging 75 minutes each. An aim of the teaching experiment was to investigate the students' generalizations about the areas and volumes of growing figures, and then to study their development of combinatorial reasoning by exploring the growing volumes of hypercubes and other objects in 4 dimensions and beyond.

All teaching sessions were videoed and transcribed. We first drew on Ellis et al.'s (2017) RFE Framework to identify generalizations, and then used open coding to infer categories of generalizing activity based on the participants' talk, gestures, and task responses. We then identified an emergent set of relationships between the participants' patterning activities and the types of generalizations they formed; this yielded the category of empirical re-conceptualization. In a final round we re-visited the data corpus in order to identify all instances of empirical re-conceptualization, the generalizations that led to each instance, and the subsequent explanation or justification. In this manner we were able track the changes in students' activity after engaging in re-conceptualizing, which led to the identification of the affordances detailed below.

Results

We found three major affordances of engagement in empirical re-conceptualization. Namely, empirical re-conceptualization can serve as (1) a source of *verification*, (2) a means of *justification*, and (3) a vehicle for *generating insight*. Within the third category, we identified three types of insight: (3a) re-interpretation within a different context or representational register, (3b) the creation of a new generalization, and (3b) the establishment of a new piece of knowledge. In order to characterize the phenomenon of empirical re-conceptualization and its associated affordances, we present an exemplar case.

Secondary Case: Growing Volumes in Three Dimensions and Beyond

Barney and Homer explored the added volumes of three-dimensional, four-dimensional, and other n-dimensional "cubes" that grew uniform amounts in every direction. They began by determining the added volume of an n by n by n cube that grew 1 cm in height, width, and length. The students worked with physical cubes to consider the component pieces and determined that the added volume would be $3n^2 + 3n + 1$. When they then investigated the added volume of a cube that grew x cm in each direction, the students simply generalized from their prior result. Homer wrote "$(3x)n^2 + (3x)n + x^2$", replacing the 3 in the first two terms of his original expression with a $3x$, and replacing the 1 in the last term, which he had conceived as 1^2, with an x^2. Unsure about the correctness of this expression, Barney said, "let me model on the cube", which he used to verify that the first term, $3xn^2$, was correct because it represented three additional rectangular prisms, each with a volume of xn^2. Both students then realized errors in the next two terms. Barney explained that the second term should actually be $3x^2n$ "because you're adding 3 of x by x by n." Both students also realized the final term would have to be x^3.

The students' original generalization was based on the *result* of their prior activity in building up additional volume components, rather than attending to the *process* by which they grew the cube's volume. However, Barney then experienced a need to verify Homer's result, which led to re-conceptualizing the generalization within the context of volume. He took the algebraic structure and made sense of it geometrically, in the process coordinating his mental activity of constructing component volumes and translating those quantities to algebraic representations.

The students eventually went on to determine expressions of added volume for the 2nd, 3rd, and 4th dimensions, which the teacher-researcher wrote in Figure 1. Homer then saw a pattern in the expressions, exclaiming, "Oh, I know what's happening!":

> Homer: It is simple, as 2 – sorry I'm writing on it. [Begins to draw the blue lines.] Two plus 1 is 3, and 2 plus 1 is 3, 3 plus 3 is 6, 3 plus 1 is 4, 1 plus 3 is 4. [Writes the red numbers.]
> TR: Whoa. Huh.
> Barney: Wow. It's just that one triangle, Pascal's triangle, right?

Homer recognized the pattern in which each coefficient could be determined by adding the sum of the coefficients of the prior consecutive terms. Pascal's triangle then became a mechanism for determining the additional volume of a 5th-dimensional solid, which the students wrote as "$5n^4 + 10n^3 + 10n^2 + 5n^1 + 1^5$". They then decided to check their answer by listing the arrangements of three ns and two 1s (the $10n^3$) case, which served to verify that the coefficient was indeed 10. Barney then realized that given that they had verified the $10n^3$ case, they did not need to check the $10n^2$ case: "We can basically just take this and switch all the ns to 1s and 1s to ns." This explanation of symmetry caused Homer to then extend that finding to new cases: "Oh, and you know what? You can do the same for these (pointing to the $5n^4$ and the $5n^1$ terms)…you can just replace these 1s for ns."

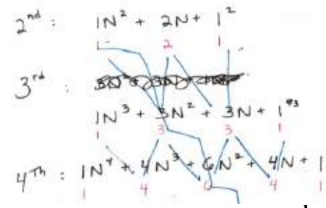

Figure 1: Expressions for added volume in the 2nd, 3rd, and 4th dimensions

Homer and Barney initially developed a generalization based on Pascal's triangle, which allowed them to determine the expression for added volume. Their subsequent listing activity enabled the students to re-interpret that expression combinatorially. That pattern allowed the students to engage in a verification process and subsequently reason about outcomes to develop a new insight, that there must be symmetry in the coefficients. Barney was able to reflect on his operations in listing the ten outcomes and realize that there was nothing special about the characters n and 1, and that they could simply be reversed in the case of determining the combinations of two ns and three 1s. This then supported Homer's new generalization.

Discussion

Empirical re-conceptualization can serve as a source of *verification*, such as when Barney checked the algebraic expression for adding x cm to a cube by appealing to the notion of volume. It can also serve as a source of *justification*, which we saw when Barney justified Homer's pattern of xs in the expression $3xn^2 + 3x^2n + n^3$. We also saw the students developing *insight*. They developed new knowledge and understanding, such as when Barney generated the idea that the coefficient of n^3 must be identical to the coefficient of n^2, which then supported Homer's ability to establish a new generalization that could be extended to the other terms, $5n^4$ and $5n$.

These affordances suggest that empirical re-conceptualization can serve as a vehicle to transform empirical patterns into meaningful sources of verification, justification, and insight. Certainly, students may also identify and generalize patterns that they do not understand or cannot justify. A danger is that students will engage in empirical investigation but then not seek to re-conceive their findings structurally. We find it useful to explore the conditions that can best support students' transition to the productive next step, that of empirical re-conceptualization. Our data suggest that directing students back towards the contextual genesis of the patterns they generalize may be an effective strategy for supporting empirical re-conceptualization. With the support of concrete contexts for meaning making, the activity of generalizing empirical patterns can serve as a bridge to more generative and productive mathematical activity.

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CREATIVITY-IN-PROGRESS RUBRIC ON PROBLEM SOLVING AT THE POST-SECONDARY LEVEL

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Promoting students' mathematical creativity while problem solving is critical to prepare students for future learning and careers. In this paper, we introduce the Creativity-in-Progress Rubric (CPR) on Problem Solving as a tool to enhance mathematical creativity while cultivating problem-solving heuristics and fostering metacognition. With its two categories, Making Connections and Taking Risks, the CPR aims to develop mathematical discourse centered around aspects of creativity involving fluency, elaboration, flexibility, and originality.

Keywords: Advanced Mathematical Thinking, Assessment and Evaluation, Metacognition, Problem Solving

Mathematical creativity and problem solving are two interrelated research constructs in that "[t]rue problems need the extra-logical processes of creativity, insight, and illumination, in order to produce solutions" (Liljedahl, Santos-Trigo, Malaspina & Bruder, 2016, p.19). Numerous research studies and curriculum documents have emphasized the importance of mathematical creativity in mathematics and mathematics courses (e.g., Borwein, Liljedahl, & Zhai, 2014; CUPM, 2015; Leikin, 2009; Silver, 1997; Sriraman, 2009). Similarly, many research studies (e.g., Carlson & Bloom, 2005; Pólya, 1957; Schoenfeld, 2013) have emphasized the importance of problem-solving practices and identified a need to foster skills (e.g., metacognition, creativity) beyond accumulation of facts or procedural steps during problem solving. It seems that exploring mathematical creativity and problem solving together at the tertiary level in mathematics courses is rare (e.g., Zazkis & Holton, 2009). As a first step towards understanding ways to foster and enhance students' mathematical creativity at tertiary level, our research team designed a formative assessment tool, the Creativity-in-Progress Rubric (CPR) on Problem Solving that capitalizes on interactions between creativity and problem-solving constructs. In this paper, we introduce the CPR on Problem Solving and its development. We provide empirical examples from undergraduate Calculus 1 student interviews to illustrate potential benefits of using CPR.

Theoretical Background

In our work, we view mathematical creativity as a process of offering new solutions or insights that are unexpected for the student with respect to their mathematics background or the problems they have seen before (Liljedahl & Sriraman, 2006; Savić et al., 2017). In contrast to examining final products of those processes, this definition is process-oriented, providing a dynamic view of creativity rather than a static one. This definition also encompasses creativity relative to the student versus creativity relative to the field of mathematics (Leikin, 2009).

Our conception and development of the Creativity-in-Progress Rubrics (CPR) was guided by this operational definition of mathematical creativity and situated within two theoretical perspectives: *Developmental*, and *Problem Solving and Expertise-Based* (Kozbelt, Beghetto, & Runco, 2010). The primary assertion of the *Developmental* theory is that creativity develops over time, and the main

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focus of investigation is a person's process of creativity. This perspective also emphasizes the role of environment, in which interaction takes place, to enhance the creativity. The *Problem Solving and Expertise-Based* theory with the emphasis on the role of an individual's problem-solving process brings forth key concepts such as problems and heuristics.

In our work, we adopted Schoenfeld's (1983) formulation of a problem as a task that the problem solvers "don't know how to go about solving it" (p. 41). Thus, problem solving becomes a process in which the problem solver tries to attain some outcomes without having an immediate access to known methods (to that particular individual) (Schoenfeld, 2013). This description of problem solving aligns with our mathematical creativity definition as both of them focus on a process relative to the individual.

Creativity-in-Progress Rubric

In our previous research studies (see Creativity Research Group, n.d.), we explored the ways in which mathematical creativity could be explicitly valued and fostered in tertiary level proof-based mathematics courses. The CPR on Proving was rigorously constructed through triangulating research-based rubrics, mathematicians' and students' views on mathematical creativity, and students' proving attempts (Karakok et al., 2015; Savić et al., 2017; Tang et al., 2015). Following the development, the CPR on Proving was implemented as a formative assessment tool in several proof-based courses. Some instructors used it to facilitate in-class discussions on proof construction and evaluation of this process (El Turkey et al., 2018) whilst others gave it to students to be used on homework problems and write-ups of solutions (Omar et al., 2019). For example, one instructor, in an elective proof-based combinatorics course asked students to reflect on their proving process of assigned problems using the CPR. One of the students of this course, when asked to discuss the use of the CPR, stated "The reflection process – the rubric itself helped kind of outline where you should go if you were lost, in a very general sense." Another student said, "I think it's helped me …reflect on the sort of creative process that I have and it's kind of helped me understand the ways that I can be mathematically creative."

We have expanded our research program by modifying the CPR on Proving to problem solving by utilizing existing studies in problem solving. This effort allowed us to include more tertiary mathematics courses and student populations in our exploration of creativity. The CPR on Problem Solving has two categories: *Making Connections* (Figure 1) and *Taking Risks* (Figure 2). These categories are divided into subcategories that are reflective of the different aspects of creativity found in prior research. The rubric provides three general levels: Beginning, Developing, and Advancing, each of which serves as a marker along the continuum of a student's progress in that subcategory. This continuum among levels of the rubric communicates the possible states of growth, aligning with the theoretical constructs of the Developmental perspective.

Making Connections Category

The category of *Making Connections* is defined as a process of connecting the problem with definitions, formulas, theorems, representations, and examples from the current or prior courses and connecting the attempted problem solutions to each other. Various researchers (e.g., Schoenfeld, 2013; Silver, 1982) have highlighted the importance of prior knowledge in problem-solving processes acknowledging that such knowledge helps the problem solver to understand the problem and influences the choices of approaches and tools to be used (e.g., examples, representations). The subcategories in Making Connections communicate these ideas to the problem solver and encourage them to push their processes in these areas forward along the continuum. Furthermore, the Between Solutions subcategory encourages the solver to examine their different solution attempts, connect them, and generalize them for thorough understanding.

MAKING CONNECTIONS

	Beginning	Developing	Advancing
Between Definitions/Formulas/ Theorems	Recognizes some relevant definitions/formulas/theorems from the course with no attempts to connect them in a solution	Recognizes some relevant definitions/formulas/theorems from the course and attempts to connect them in a solution	Uses relevant definitions/formulas/theorems from the course or other resources outside the course in a solution
NA ☐			
Between Representations[1]	Provides a representation with no attempts to connect it to another representation	Provides multiple representations and recognizes connections between representations	Provides multiple representations and uses connections between different representations
NA ☐			
Between Examples	Generates one or two specific examples with no attempt to connect them	Generates one or two specific examples and recognizes a connection between them	Generates several specific examples and uses the key idea synthesized from those examples
NA ☐			
Between Solutions	Attempts to connect multiple solutions to each other	Connects multiple solutions to each other	Connects multiple solutions to each other and generalizes common properties
NA ☐			

Figure 1: Making Connections Category of the CPR on Problem Solving

This category encompasses the *fluency* and *elaboration* components of Torrance's definition of creativity (Leikin, 2009). As *fluency* describes flow of associations and use of basic knowledge, with its subcategories of between definitions, formulas, theorems, between representations, and between examples and continuum levels, Making Connections provides opportunities to enhance fluency. As *elaboration* relates to generalization of ideas, moving in rubric's the continuum toward advancing levels of each subcategory provides opportunities for generalization.

Taking Risks Category

The category of Taking Risks in our rubric is defined as a process of actively attempting a solution, demonstrating flexibility in using multiple solution paths, posing questions about reasoning within solutions, and evaluating solution attempts or solutions. The subcategories of Flexibility, Posing Questions, and Evaluation of Solution Attempt align with Pólya's (1957) problem-solving heuristic. In the third step of this heuristic, Pólya discusses the process of carrying out a plan and in the fourth step, the solver examines the reasoning and results of their solution attempt and tries to solve the problem in different ways. In addition, the continuum levels of the Posing Questions subcategory provide ways for the solver to move from the state of being stuck to less stuck by explicitly asking various types of questions.

TAKING RISKS

	Beginning	Developing	Advancing
Tools and Tricks[2]	Uses a tool or trick that is usual for the course or the student	Uses a tool or trick that is partly unusual[3] for the course or the student	Creates a tool or trick that is unusual for the course or the student
NA ☐			
Flexibility[4]	Introduces one solution path	Introduces more than one solution path	Uses more than one solution path
NA ☐			
Posing Questions	Recognizes there should be a question asked, but does not pose a question[5]	Poses questions clarifying a step within a solution	Poses questions about reasoning within a solution
NA ☐			
Evaluation of Solution Attempt	Checks surface-level[6] features of a solution attempt	Checks an entire solution attempt for reasoning	Revises or validates an entire solution attempt for reasoning
NA ☐			

Figure 2: Taking Risks Category of the CPR on Problem Solving

We note that the Tools and Tricks and Flexibility subcategories directly relate to the *originality* and *flexibility* components of Torrance's definition of creativity (Leikin, 2009), respectively. Torrance describes *originality* as a unique way of thinking, which could be evident in the process of using a trick (e.g., adding one and subtracting one) or introducing a mathematical object (e.g., defining a new function) that is unconventional for a student or a course that the student is in. Torrance defined *flexibility* as approaching a problem in multiple ways and producing multiple solutions, which is captured in our Flexibility subcategory. Within the Taking Risks category, we claim that the process of moving forward in the continuum of levels towards the advancing level requires a problem solver to take an intellectual risk in their problem-solving process.

Discussion

In our research project, instructors of Calculus 1 at several different institutions were asked to use the CPR on Problem Solving with tasks that we designed (El Turkey et al., in press). Each instructor decided how to implement these tasks and the CPR, where some used them as part of assignments and others had in-class sessions. We conducted interviews with students from these courses. In our preliminary analysis, we noted that students' experience and the usage of the CPR align with four themes of a problem-solving activity that Schoenfeld (2013) claimed to be necessary and sufficient for the analysis of the success of a problem solver's problem-solving attempt: a) The individual's knowledge; b) The individual's use of problem solving strategies, known as heuristic strategies; c) The individual's monitoring and self-regulation (an aspect of metacognition); and d) The individual's belief systems (about him- or herself, about mathematics, about problem solving) and their origins in the students' mathematical experiences.

We claim that the first two themes (a & b) directly relate to the CPR. When students utilize the CPR during their problem-solving attempt, they demonstrate their knowledge and use of problem-solving strategies. For example, one Calculus 1 student stated that the rubric prompted her to think about class work during problem solving. Discussing her required use of the CPR on an assignment during an interview, she said, "I was trying to think about the definitions we used in class and like drawing pictures with that" and continued by discussing that the flexibility and evaluation subcategories guided her problem-solving approach.

We believe the third theme (c) was encompassed by the usage of the rubric as a reflection tool as the problem solver tried to move forward on the continuum. The CPR connects to the fourth theme (d) as it may increase students' awareness and shift in their perception about their own creative processes (Cilli-Turner et al., 2019). For example, a student from another Calculus 1 course at a different institution stated that, "So, I feel like [the rubric has] definitely improved my creativity the way that …made me think a little bit more about what I'm actually writing down instead of just doing the problem." Our preliminary analysis seem to indicate that as a reflective tool, the CPR can help facilitate discussions on students' attempts and provide guidance on how to enhance students' mathematical reasoning and creative potentials. Ultimately, it may serve to make the link between problem solving and mathematical creativity more salient and accessible in any classroom context.

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A QUANTITATIVE REASONING STUDY OF STUDENT-REPORTED DIFFICULTIES WHEN SOLVING RELATED RATES PROBLEMS

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This paper extends work in the area of quantitative reasoning at the undergraduate level. Task-based interviews were used to examine 16 calculus students' difficulties when solving three related rates problems. Analysis of students' verbal responses and written work revealed several difficulties, including dealing with several time-dependent quantities. The paper concludes with a recommendation for the teaching of related rates problems at the undergraduate level.

Keywords: quantitative reasoning, related rates problems, derivatives, problem solving.

Related rates problems involve at least two rate quantities (i.e., instantaneous rates of change) that can be related algebraically by an equation, function, or formula. Although related rates problems constitute an essential part of any first-semester calculus course in the United States, several researchers have argued that there is a shortage of research that has examined students' thinking about related rates problems at the undergraduate level (e.g., Engelke, 2007; Mkhatshwa, 2020; Speer & King, 2016). Of the few studies involving related rates problems, Mkhatshwa (2020) reported on students who exhibited poor calculational knowledge of the product and quotient rules of differentiation, something that limited their success in a non-routine related rates problem they were asked to solve. Engelke (2007) described beneficial components of a successful solution to a related rates problem, including drawing a diagram, determining a functional relationship (algebraic equation), and checking the answer for reasonability. Other studies have found that mathematizing (Freudenthal, 1993) related rates problems is problematic for students (Martin, 2000; White & Mitchelmore, 1996).

While these studies have provided useful information about how students set up and solve related rates problems, there is still much to be explored about what different modes of reasoning, such as quantitative reasoning (Thompson, 1993, 1994b, 2011) might reveal about students' difficulties with solving related rates problems that have real-world contexts such as kinematics. Quantitative reasoning seems a particularly important lens for studying students' understanding of related rates problems since they inherently deal with quantities. In addition, students' difficulties with solving related rates problems in these studies have all been reported from a researcher's perspective (i.e., observed difficulties), and have not considered a student perspective (i.e., student-reported difficulties). Thus, in order to build on these studies, the present study investigated students' difficulties with solving related rates problems from a student perspective. The research question we investigated is: What do calculus students identify as difficulties when engaged in reasoning quantitatively about solving related rates problems?

Related Literature

Evidence from studies that have examined students' reasoning about geometric related rates problems (Mkhatshwa, 2020) shows that students who are able to visualize and perform physical enactments of situations described in related rates problems tend to be successful in solving these problems (Carlson, 1998; Carlson, Jacobs, Coe, Larson, & Hsu, 2002; Monk, 1992). Several researchers have identified lack of facility with implicit differentiation as a major cause for students' failure to solve related rates problems successfully (Clark et al., 1997; Engelke, 2004; Mkhatshwa, 2020; Piccolo & Code, 2013). Piccolo and Code (2013) argued that students' difficulties with solving

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related rates problems stem from a weak understanding of implicit differentiation, rather than a misunderstanding of the physical context of such problems. Hare and Phillippy (2004) posited that "implicit differentiation is a difficult concept for many students to understand because the level of difficulty of the concept is higher than the level of difficulty of explicit functions" (p. 7). Conflicting findings have been reported on calculus students' ability to mathematize related rates problems (cf., Martin, 2000; Mkhatshwa, 2020; White & Mitchelmore, 1996). Analysis of students' written responses to geometric related rates problems by Martin (2000) revealed that overall performance was poor, and that "the poorest performance was on steps linked to conceptual understanding, specifically steps involving the translation of prose to geometric and symbolic representations" (p. 74). Findings of a recent study (Mkhatshwa, 2020) on students' thinking about related rates problems in real-world contexts indicated that mathematizing routine related rates problems is straightforward for students.

Theoretical Perspective

This study draws on the theory of quantitative reasoning (Thompson, 1993, 1994b, 2011). Quantitative reasoning is the act of analyzing a problem in terms of the quantities and relationships between the quantities involved in the problem (Thompson, 1993). In this study, quantitative reasoning refers to how students interpreted rate quantities (i.e., instantaneous rates of change) when solving related rates problems, and how they reasoned about quantities and relationships between quantities when engaged in talking about difficulties they had with solving these problems. What is important in quantitative reasoning is making sense of quantities and relationships between quantities (Smith III & Thompson, 2007; Thompson, 1993). Thompson (2011) described three tenets that are central to the theory of quantitative reasoning, namely a quantity, a quantitative operation, and quantification. A quantity is a measurable attribute of an object (Thompson, 1994b). Examples of quantities in this study include the speed of an airplane, the area of a puddle, and the volume of a balloon. A quantitative operation is the process of forming a new quantity from other quantities (Thompson, 1994b). We designed three tasks (Task 1, Task 2, and Task 3 in the methods section) that provided opportunities for students to perform quantitative operations by creating new quantities through the process of implicit differentiation. Quantification is the process of assigning numerical values to quantities (Thompson, 1994b). The three tasks used in this study provided opportunities for students to engage in quantification.

Methods

Task-based interviews (Goldin, 2000) were used to investigate calculus students' quantitative reasoning while solving related rates problems. The interviews covered three tasks:

> **Task 1 [motion context]:** Two small planes approach an airport, one flying due west at a speed of 100 miles per hour and the other flying due north at a speed of 120 miles per hour. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 miles from the airport and the northbound plane is 200 miles from the airport?
>
> **Task 2 [non-motion context]:** A leak from the sink is creating a puddle that can be approximated by a circle, which is increasing at a rate of $12\ cm^2$ per second. How fast is the radius growing at the instant when the radius of the puddle equals 8 cm?
>
> **Task 3 [non-motion context]:** For the next problem, let me give you a little background on a formula that we will use. Suppose a gas is inside a container. Many gases under normal conditions follow the "ideal gas law," $PV = kT$, where P is the pressure the gas exerts on the container, V is the volume of the container, T is the temperature of the gas, and k is a constant. P is measured in "atmospheres," V is measured in cubic meters, and T is measured

> in Kelvins. Kelvins is a lot like Celsius, except that it is scaled so that 0 means absolute zero (lowest possible temperature), which makes water's freezing point to be $273\ °K$. Do you have any question(s) about this formula, or any of the quantities [like temperature in Kelvins] before we proceed?
>
> In a laboratory, an experiment is being done on a gas inside a large flexible rubber balloon. For the experiment, the temperature of the gas is being heated at a rate of 8 degrees Kelvin per second. At one point, when the temperature of the gas is $300\ °K$, the pressure is 1.5 atmospheres, the volume of the gas is one cubic meter, and the volume of the gas is increasing at a rate of 0.01 m^3 per second. At that moment, is the pressure in the balloon increasing or decreasing? What is the rate of that increase/decrease?

After students concluded their work on each task, the interviewer asked the following questions: (i) What does your answer [derivative] tell you in the context of this task? (ii) How would you answer the question posed in this task? (iii) What was the easy part for you when solving this task? and (iv) What was the difficult part for you when solving this task? With these questions, our goal was two-fold. First, we wanted to examine students' interpretations of derivatives in motion and non-motion contexts (questions i. and ii.). Second, we wanted to gain an insight on what is straightforward and what is difficult about solving related rates problems from a student perspective (questions iii. and iv.).

Setting, Participants, Data Collection, and Data Analysis

The study participants were 16 undergraduate students at a research university who were enrolled in five different sections of a calculus I course taught by three different professors. Details about the participants, including opportunities they had to learn about related rates problems during classroom instruction are provided in Mkhatshwa (2020). Data for the study consisted of transcriptions of video-recordings of the task-based interviews and work written by the students during each interview session. On average, each interview session lasted for about 65 minutes. The data was analyzed in two stages. In the first stage, we used two emergent codes i.e., student actions that evolved from the data. These codes are: (1) the difficulty of dealing with several time-dependent variables (quantities), and (2) the difficulty of finding the value of the constant k in Task 3. In the second stage of the analysis, we tallied the number of students in each of codes found in the first stage of the analysis.

Results, Discussion, and Conclusions

Since Task 3 was the only non-routine task, and one that most of the students were least successful in solving, we limit our discussion of student-reported difficulties with solving related rates problems to this task. There are three findings from this study. First, when asked about the difficult part about solving the task, six students stated that everything about the task was hard. Amos' reasoning about the difficulty of solving Task 3 is representative of the six students.

> Researcher: What was the easy part for you when solving this task [Task 3]?
> Amos: None of the problem was easy for me.
> Researcher: What was the difficult part for you when solving this task?
> Amos: Reading the task, differentiating the given equation [$PV = kT$], and figuring out where to plug in the given values [quantities in the task, e.g., the temperature of the gas given as $300°K$] in order to solve the problem.

In response to the first question (i.e., the easy part), Amos stated that "none of the problem was easy" for him. When asked about the difficult part, he noted reading the task, differentiating the equation given in the task, and using all the given information in the task to solve the problem posed in the task. When probed about the type of differentiation he would use in this task, Amos stated that he would "have to use implicit differentiation." Two other students identified implicit differentiation as the challenging part when solving the task. Students' difficulties with implicit differentiation (i.e., performing quantitative operations) when solving related rates problems have been reported by other researchers (cf., Clark et al., 1997; Mkhatshwa, 2020; Piccolo & Code, 2013).

Second, four other students stated that there were several variables (quantities) to keep track of, and that this was the main challenge for them when solving the task. The following excerpt illustrates how Felix, whose reasoning is representative of these students, commented about the difficult part when solving the task.

Researcher: What was the difficult part for you when solving this task?
Felix: There are more than two variables [quantities], P, V, and T in the same equation [$PV = kT$].

Felix remarked that having several quantities, namely pressure (P), volume (V), and temperature (T) in the same problem was problematic for him when solving the problem posed in the task. He, however, did not elaborate on this. We argue that the unfamiliar context may have been the challenge for Felix more than having several variables. This is because in Task 1 (a familiar task to Felix and one that has several variables as well), Felix did not claim that having several variables in the problem was the difficult part. Instead, he said the difficult part was finding an equation that relates the quantities in the task, that is, mathematizing the problem. On the contrary, 10 students claimed that Task 2 (a routine task) was easy to solve because it had fewer variables compared to Task 1 and Task 3. When asked about the easiest part about solving Task 2, one of the 10 students, James, commented, "we only had one variable to track and that's the radius, so it was fairly easy to solve." When asked about the difficult part when solving Task 2, he said, "I don't think there were any challenges." We argue that the number of variables play a huge role in students' ability to solve related rates problems successfully.

Third, four students stated that finding the value of the constant k was the difficult part for them when solving Task 3. We note that although only four students identified solving for the constant k as the problematic part in Task 3, half of the 16 students in this study were unsuccessful in finding the value of k. Since finding the value of the constant k entails engaging in the process of quantification i.e., substituting the given values of the quantities of P, V, and T in the equation $PV = kT$ and then solving for k, we argue that substituting known quantities and solving for an unknown quantity in an equation is perhaps not only problematic for secondary school students, but also for undergraduate students. Based on the student-reported difficulties when solving related rates problems in this study, we recommend that calculus instruction should provide more opportunities for students to make sense of, and to solve non-routine related rates problems that have several quantities. The interested reader is referred to Mkhatshwa (2020) for observed (i.e., researcher-reported) difficulties that were exhibited by the students when solving the three tasks used in the present study.

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SECONDARY TEACHERS' DIFFERING VIEWS ON WHO SHOULD LEARN PROVING AND WHY

DIFERENTES OPINIONES DE MAESTROS DE PREPARATORIA SOBRE QUIÉN DEBE APRENDER DEMOSTRACIÓN Y POR QUÉ

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Reasoning-and-proving is viewed by many scholars to be a crucial part of students' mathematical experiences in secondary school. There is scholarly debate, however, about the necessity of formal proving. In this study, we investigated the notion of "proof for all" from the perspective of secondary mathematics teachers and we analyzed, using the framework of practical rationality, the justifications they gave for whether or not all students should learn proof. Based on interviews with twenty-one secondary teachers from a socioeconomically-diverse set of schools, we found that teachers do not share the same opinion on who should learn proving but they expressed obligations toward individual student learning as justifications both for teaching proving to all students and for not teaching proving to some students.

Keywords: Reasoning and Proof; Teacher Beliefs; High School Education; Equity and Diversity.

Reasoning-and-proving, the broad mathematical practice of conjecturing, justifying, critiquing arguments, constructing proofs and more (Stylianides, 2008), is central to the discipline of mathematics and can also be a powerful process through which students learn mathematics (de Villiers, 1995; Stylianides et al., 2017). Policymakers (National Governors Association & Council of Chief State School Officers, 2010; Secretaría de Educación Pública, 2014) and scholars (e.g., Mariotti, 2006) alike have called for reasoning-and-proving to be a part of all students' learning experiences in school. But there are also critiques of this general framing of learning "for all" such as Martin (2003) who pointed out that "for all" often comes as impositions on underserved groups, and Battey (2019) pointed out that "for all" can gloss over learners' individuality, proposing "for each and every" as a replacement framing. With regard to formal proof in particular, Weber (2015) noted that it may be unnecessary at the secondary level to explicitly develop "proving" and that it may be sufficient to push for clear explanations and valid justifications and that doing so may more easily integrate with students' mathematical experiences prior to secondary school.

Where do mathematics teachers, as the ones directly responsible for enacting curricular recommendations, stand on this issue of "proof for all"? How are teachers thinking about the scope and appropriateness of proof for students? Past studies have examined teachers' views of proof (e.g., Ko, 2010) or their views on mathematical processes including proof (e.g., Sanchez et al., 2015) but the question of who they think should learn proof is fundamental. In this study, we interviewed 21 secondary mathematics teachers from an economically-diverse set of schools in Cape Town, South Africa. Although outside North America, it has similarities to North American contexts in terms of mathematics teaching being heavily influenced by European colonization and having typical instruction that is procedural in nature (Webb & Roberts, 2017). Moreover, the question of who should experience proof is one with worldwide relevance as we consider broadly students' mathematical experiences.

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Personal and Theoretical Perspectives

Because this study involves our analysis of teachers' perspectives on proving, it is important that we reveal salient information about our own perspectives for the sake of transparency. Samuel is an American white man of Western European descent who attended rural public schools and then public universities where he earned degrees in both mathematics and education. Mitchelle is an African woman who attended the Kenyan elementary, secondary, and undergraduate education system, and is currently in the U.S. pursuing a doctoral degree. Rajendran, an Indian born in South Africa, attended urban primary and secondary public schools during the apartheid era and then proceeded to study at public universities where he earned degrees in both mathematics and education. Although from diverse backgrounds, we all share a view that proving—in the sense of constructing reasonably complete and logically valid arguments for mathematical claims—is important for all students in the general education system as well as most students in the special education system. Although this is our opinion, we value hearing the voices of teachers and taking seriously their perceptions of what is possible and why.

In terms of our approach to teachers' perceptions, we see teachers as participants in a cultural practice of teaching governed by norms (i.e., tacitly expected behaviors or unquestioned historical practices) and obligations (i.e., requirements perceived as inherent to their role as a mathematics teacher) (Herbst & Chazan, 2011). These norms and obligations influence the choices that teachers make in their own teaching (e.g., Webel & Platt, 2015). Obligations, in particular, can be used to categorize the justifications that teachers provide for their instructional choices. For example, a teacher may decide to present a proof to students rather than have them construct the proof independently because she feels an obligation to complete the lesson in a single class period and stay "on pace." Or a teacher may decide to emphasize formal terminology in a proof because he feels an obligation to the mathematics discipline to maintain "rigor."

We have two central research questions. RQ1) According to secondary mathematics teachers, who should learn proving in their formal mathematics education? RQ2) What justifications do secondary mathematics teachers provide for their answer about who should learn proving?

Method

The study was conducted in the Cape Town metropolitan area of South Africa, which is a port city on the southwest coast. South Africa, since its democratization in the 1990s, has pursued curricular reforms centered on universal education (Webb & Roberts, 2017). Its official standards call for elements of reasoning-and-proving to be taught to all learners. The 21 teachers participating in this study varied in their professional preparation and experience (from 1 year to 15 years teaching) but they all were mainly involved in teaching mathematics to grades 10–12 leaners. Their five schools were in drastically different socio-economic neighborhoods.

The first author, sometimes with the third author, conducted two types of semi-structured interviews. All 21 teachers participated in focus group interviews (approx. 20–40 minutes), organized by school, focused on the purposes of mathematics education and curricular issues related to proving. Ten of the 21 teachers also participated in individual interviews (approx. 10– 30 minutes) focusing on proving tasks and their experiences with proof learners. The analysis reported here specifically addresses the question of "who should learn proof in school?"

The interviews were transcribed and coding was in two phases. Phase 1 involved reading the transcripts and applying broad codes to any segments that related to the overarching research questions. We noted the groups of learners that teachers identified as who should learn proving. Phase 2 involved qualitative coding based on the practical rationality framework (Herbst & Chazan, 2011), particularly the professional obligations. We briefly describe these codes here:

- *Disciplinary*: teachers' perceived obligations related to mathematics as a subject area (e.g., proving is the "core" of mathematics)
- *Institutional*: teachers' perceived obligations related to the educational system, school policies, or administrators (e.g., proving is included on official assessments)
- *Individual*: teachers' perceived obligations related meeting the needs and expectations of specific learners (e.g., proving can help learners gain deeper understanding)
- *Interpersonal*: teachers' perceived obligations to balance the needs of a diverse class of learners and managing productive interactions (e.g., proving promotes respectful critique)
- *Worldly obligations*: teachers' perceived obligations related to the real-world usefulness of what is being taught (e.g., proving will help learners use logic beyond mathematics)

The *worldly obligation* code emerged from our own data set. Overall, multiple authors coded the obligations and met regularly to clarify (e.g., code two obligations within the same justification statement) and reconcile any discrepancies in the coding.

Findings

Several of the teachers expressed the opinion that some students should be exempted from the opportunity to learn proving (see below), but more than twice as many teachers expressed that all students should have an opportunity to learn proving. Others (i.e., some who only participated in the focus group interviews) did not express an opinion on this question, but the sections below provide brief findings with regard to the rationality that the teachers exhibited.

Teachers' Rationality for All Students Learning Proving

Teachers who stated that all students should learn proving provided a variety of justifications for that position. The most common justification related to the teachers' obligation toward *individual* student learning. Teachers explained that proving can help students to understand mathematical content in deeper or more inter-connected ways. For example, Panyanga said:

> All of [the students] should know how we get to things, not just the application… I'll just make an example with Pythagoras' theorem, you find that they know how to use it but they don't really understand it properly [without proving it].

An implicit obligation here is for the teacher to support students in understanding mathematics "properly," not just execute applications, and proving is something that promotes an understanding of "how we get to things." A similar point was raised by another teacher, Rhyan, who said about proving opportunities, "You have to give people space to experience the idea" because this helps them to move beyond knowing just "that a parallelogram has opposite sides equal" to understanding how that result connects with other pieces of knowledge in geometry.

Other teachers, in justifying that all students should learn proving, looked beyond the classroom. Specifically, teachers expressed a *worldly obligation* by connecting mathematical proving to students' current or future lives beyond mathematics. For example, Portia said that

> …to prove something is not just a mathematical skill, it's a skill in logic, it's a skill in trying to figure out and to validate your arguments. And that is not confined to only mathematics… I think it is a valuable skill and I think that everybody has the ability to do it. You don't need to say, 'Okay, this is exclusively for those who score high marks [in mathematics].'

Shabeer made a similar point that "being able to prove something in geometry, it helps you even being able to prove it in something unrelated to geometry." We also viewed references to general "critical thinking" as part of this worldly obligation.

Beyond individual and worldly obligations, a few teachers cited disciplinary or institutional obligations. We turn now, however, to those who had different opinions altogether.

Teachers' Rationality for Not All Students Learning Proving

Although one teacher commented that, were it in her power, she would remove proof from the official curriculum, we focus in this section on the several teachers who stated that only certain groups of students should learn proof. The most frequent justification that teachers provided for this position had to do with their *individual obligation* to support or cater to students' needs. For example, when asked who should learn proof, Shannon said the following:

> I think our students must learn proofs but not all the students. Because … with a particular class I can prove a theorem or I can do a proof. But with another particular class I will see that if I want to prove this theorem, I will do more harm than good… For those learners whom I can say they may be average to above average, yes. If we say let us prove it and let us use it, they will acutely enjoy the proving and the using of the proof.

For Shannon, she wants to teach proof when the result, as she perceives it, is an enjoyment of learning and students who see the usefulness of the proof. She has identified these benefits as being attached to some "but not all" students. For the other students, she refers to proving as doing "more harm than good," which we interpret to refer to confusion and struggles that can occur when she teaches proving. Other teachers expressed a similar obligation to help students avoid struggle. One said that proofs can cause students to become "discouraged" and another mentioned that those who "shouldn't learn the proofs are those learners [who] at the beginning are struggling… it's not going to be worth it to learn the proofs," whereas "those learners who excel should be focusing on the proofs because it'll help them understand actually most of the math much better." In this excerpt, avoiding proof and teaching proof were both rooted in individual obligations to either help students avoid struggle or help them achieve understanding.

Another justification that teachers mentioned had to do not with students' struggles but with their future plans after secondary school. For example, Shabeer said that "there is a certain group of students [who] should learn proofs and it depends maybe on what that student plans to do when he's done with school." This life-after-school idea connects with the *worldly obligation* in our framework. A similar point, but with different underpinnings, came from Trevelyan who said that proof should be "for the other students who are going to do something with the mathematics… are going to go and study further with the mathematics… because that's the core of what mathematics is." In this instance, a *disciplinary obligation* is evident as proof is an essential part of mathematics and thus is relevant for those who will pursue higher mathematics.

Discussion

Our goal was to recognize teachers' rationality because this is important as we attempt to work collectively to improve proof learning. As scholars, it is not safe to presume that all teachers necessarily view proving as worthwhile for all students. Yet, our findings suggest that teachers on both sides of the issue were deeply attuned to individual obligations. Thus framing proof as a way to support individual learning, and in particular addressing the idea (not expressed by the teachers) that proving can be fruitful for "struggling" students may be a way to find common ground. Conversely, appeals to the disciplinary obligation of proving may inadvertently send the message that proving is only for students who are on a pathway to higher mathematics. The worldly obligation may be an opportunity to promote more universal opportunities for proof learning. Or in listening to some teachers we may, like Weber (2015), have to take seriously the notion that *formal* proof may not be the most productive approach to reasoning for all students.

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THE ROLE OF THE RESEARCHER'S QUESTIONS IN A CLINICAL INTERVIEW ON STUDENTS' PERCEIVED PROBLEM SOLVING

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In this work we considered interactions between researcher/interviewer and two case subjects in the context of two task-based interviews to isolate ways that subjects' problem-solving performance was influenced by the choice of interventions that researcher opted to employ. In order to capture shifts in the students' actions that could be attributed to specific interventions a problem-solving map was constructed for each individual. Shifts and transitions in actions were then corresponded to the nature of exchanged taking place prior to the shift. In the case of Tuna, probing guidance lead her struggle to a productive end. For Sam, suggesting the use of a two-way table as scaffolding shifted his struggle in a productive manner.

Keywords: Problem Solving, Intervention, Clinical Task-Based Interview

The goal of the research reported here was twofold. First, we examined problem solving processes of two 5th grade students when working on a task that entailed algebraic thinking. The goal was to identify junctions where students' progress on the task was constrained. The second goal was to identify ways in which the researchers' modes of interventions seemingly impacted the problem solvers' performance, particularly at constrained junctions. Our research was motivated by the desire to better unpack ways in which researchers' choices of questions might influence results concerning children's problem-solving abilities and performance, an area rarely explored in mathematics education. With increased interest in using clinical interviews and teaching experiments that rely on direct interactions between subjects and researchers such an exploration is both timely and needed.

Literature Review

Mathematical problem solving is defined as "an activity that relies heavily on the problem solvers' in-the-moment decision making and improvising and the type of insights that they may develop in the course of their actions" according to Manouchehri & Zhang (2013, p.68). What remains unknown is how students decide what strategies to use and what might contribute to shifts in their approaches. Much of the literature concerning the mathematical problem-solving performance of learners relies on task-based interviews, either in structured or semi-structed setting. Rarely has the interviewer's role and their comments in the course of problem-solving process has been scrutinized, linking learners' performance to potential impact of the probing questions the interviewer might have asked. This considers even in occasions where questions may consist of eliciting the learners' own thinking (i.e. explain what you were thinking, why did you do this, etc.) All these comments are forms of intervention that force reflection, either implicitly or explicitly and elucidate cognitive reactions resulting to some kind of mathematical outcome.

Methodology

Videotapes of two task-based interviews (Litchman, 2012) with two 5th grade students were used as data sources to carefully unpack researcher/participant interactions as one task was used towards capturing students' problem-solving performance. During these interviews, the researcher had used a common task focused on capturing students' algebraic thinking (see Figure 1). Each interview lasted approximately 20 minutes.

Analysis consisted of two phases. First, Using Schoenfeld's (1985) problem solving path as a platform we traced the learners' actions throughout their encounters with the problem with the goal of launching a solution. Phases included reading the problem, analyzing the problem, exploring, planning, implementing and verifying the solution method. Of particular interest was determining stages at which progress seemed constrained.

Task 1: Where am I?

The seats in the auditorium of Joyful Elementary School with a capacity of 300 people are labeled as follows and the rest of the seats follow the same pattern.

Front

Row 1	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Row 2	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Row 3	**21**	**22**	**23**	**24**	**25**					

Figure 1

1. Chris is in seat 48? What row is he in?
2. Tyler is two rows behind and four seats to the right of Seat 42. What is his seat number?

During the second phase of data analysis we considered the influence of the researcher/interviewer's comments on the problem solvers' transitions along the problem-solving map. Note that during the interviews, as a protocol, the researcher was allowed to ask "why" and "how" questions to understand the reasoning of students' problem solving strategies. Further, common to task-based interview tradition, students were asked whether their answer made sense and to explain their reasons. Additionally, questions regarding how students assessed their own progress, what may have caused them to be stuck or what they found confusing were considered ways that the research could gain a better understanding of sources that contributed to the problem solvers' choices. Indeed, we examined how these seemingly "unobtrusive comments" impacted the mathematical work that problem solvers produced.

Preliminary Findings

Our findings will be grounded in illustrations of shifts in two students' problem-solving practices in the presence of interactions with the interviewer. Due to the limitation of space, we consider only two examples to highlight ways that the task-based interactions influenced the learners' problem-solving actions and progress.

Sam and Tuna both encountered the same impasse during the episodes of exploring and planning, but their resolutions were different. Sam and Tuna both struggled as they worked on the task but there were significant differences in the nature of their struggle. Sam directly implemented his solution method after analyzing the problem based on his understanding from the task. On the other hand, Tuna asked analytical questions to the interviewer to clarify her understanding from the question and then she went back and forth between either correcting or recalling her prior knowledge and creating new knowledge in the exploring phase. While Sam struggled making sense of the 1st question, Tuna's constraint was misinterpreting her prior knowledge of multiplication for the 2nd question.

Initial actions

Figure 2 illustrates problem solving maps of the two interviewed subjects (Sam and Tuna). Figure 2(left column) depicts a map of Sam's problem-solving process. He first reads the problem and analyze the pattern going by ten then implement his strategy of counting by 10 backwards from 48 until he reaches 8. Then, he verifies his solution by his multiplication fact. (6x8=48). Once he was asked how he had arrived the number 6, he we went back to planning phase and adjusted his solution method. Then, he verified his new solution with another multiplication fact. (8x5=40). Lack of reliance on an organizational scheme hindered his ability to move forward in generalizing his answers. However, once Sam was provided with two-way table, he analyzed the problem again, relied on his initial interpretation of the task and solved it. The transcript below shows Sam’s shift in his struggle with multiplication facts to a productive end. Using two-way table reminded him his first interpretation of the task and obtained the final result. However, once he was suggested to use a two-column table to organize his data, he managed to successfully launch a solution.

S: So then, if you are like 6 rows over, it will probably be 48 because if he is if he is wait, if it says he was 48?

I: Yeah! Chris is in seat 48. It is asking for uhm what row is he in? **What we can do is I can give you this table Ok? Did you use two-way table before?**

S: Yeah

I: Ok. Just go with that.

(C is filling in the table as Row: 5 Seat Number: 41,42,43, 44…)

S: (While he is writing the number 44) AHA!

I: What happened?

S: It is row 5.

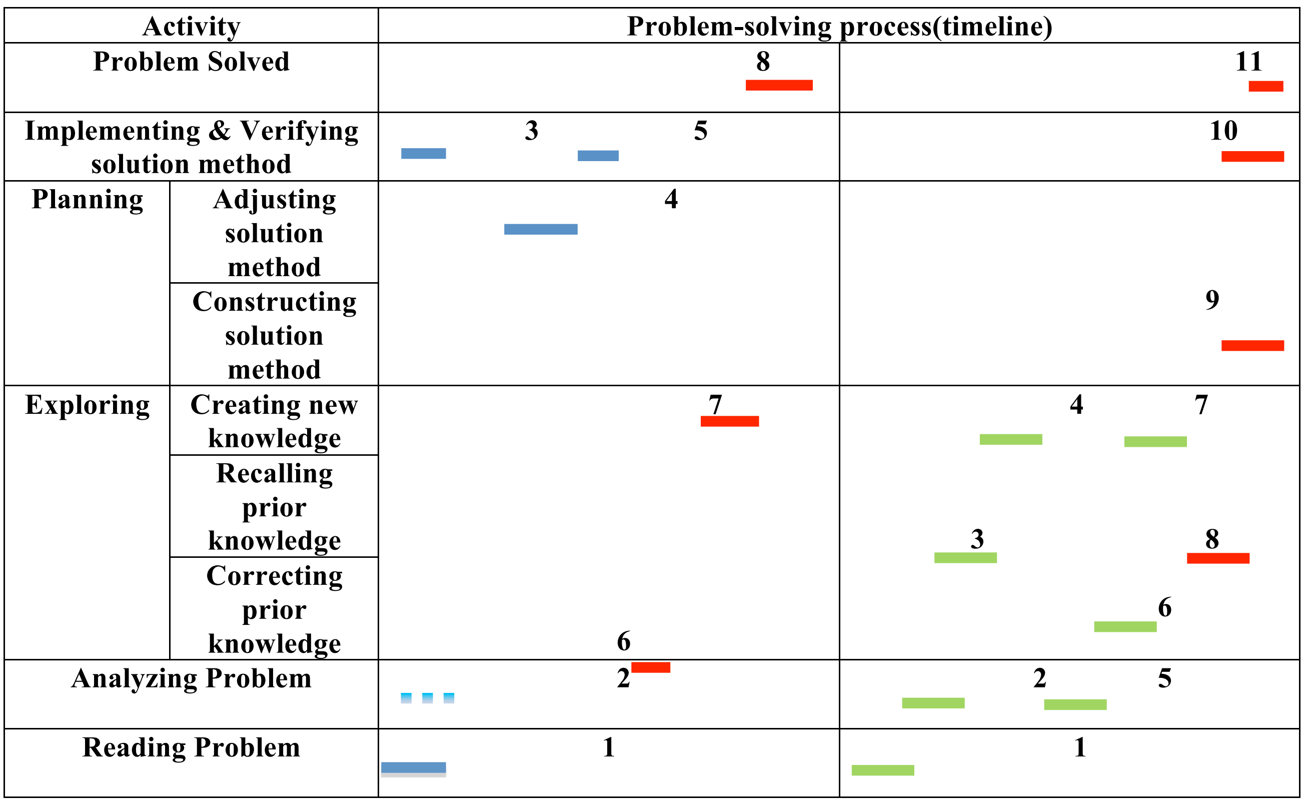

Figure 2: Tracking of Sam and Tuna’ s problem solving (Red lines represent intervention)
Sam: Blue lines 1-5 & red lines 6-8
Tuna: Green lines 1-7&red lines 8-11

I: Uhm how do you know that? Why did you use 6 times 7 is equal to 42? How did you decide to use that?

T: Uhm. Uhm yeah, uhm wait oh wait never mind. I will just solve it never mind. Because I was thinking like going like that uhm wait. I think it will be row 6 because like seat 42. Because in each row, one of the multiples are in 7 are (inaudible) in it. So, 42 will be in the row 6.

I: So, in each row, there are multiples of what?

T:7 and uhm because I know. Wait no! You are actually in row 5. Because in row 4, last seat will be 40 and row 5, seat 42 will be in it needs to keep going through and you find in in row 5.

Transcript above shows Tuna's transition from her previous strategy to the new solution due to the interactions with the interviewer. Immediately after the interviewer re-voiced her claim as in each row there are multiples of seven, Tuna recalled her prior knowledge of last seat in each row is 10 times the row number and she constructed new solution method based on her first interpretation of the task. She then implemented her solution method as adding 2 rows to row 5 and claimed that Tylor will be in row 7 and she said that "since it will be in row 7 last seat in row 7 will be 70 but the seat it is in row 7 so I know that his seat would be the 6^{th} seat in row. If 70 is the last seat you have to take away 4 will get you to 66.". She finally obtained the correct response after going back and forth between problem solving phases.

Conclusion

Our results highlight several important theoretical considerations. While clinical interviews are widely used as vehicles to learn about what students know and how they work mathematically around selected tasks with the desire to identify gaps and strengths in their approaches, little attention has been paid to how these actions may have been influenced by the researchers' choices of questions. It is commonly assumed that interviewer's role is to be an objective observer that asks why and how questions without a careful examination of how such interventions might have influenced the work that learners produce in the course of interactions. In our analysis we offered how students' problem-solving pathways were influenced by the interviewer's comments. As such, we problematize we perceived notions of students' problem solving competencies reported in the literature without a careful examination of interactions in the course of interviews that could have impacted participates' work either through provoking reflection or prompting schemes that may not have been recalled during their work on tasks.

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PROPORTIONAL THINKING: A COMPARATIVE STUDY AMONG FOURTH AND FIFTH GRADE CHILDREN

EL PENSAMIENTO PROPORCIONAL: UN ESTUDIO COMPARATIVO ENTRE NIÑOS DE CUARTO Y QUINTO DE PRIMARIA

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The theme of proportionality begins formally in the last years of primary school, and involves the mastery of notions such as comparison of ratios, fractions, and percentages. In this research, a study was carried out with 18 students of fourth and fifth grades of primary education between the ages of 9 and 13, with the objective of assessing their performance in solving problems that involved comparing ratios and having relative thinking. Both types of problems involve proportional thinking. The purpose is to know what is the difference between the two groups, as well as to identify if there is a difference between performance in the two types of problems. It was found that students in both grades have more difficulty in solving relative thinking type problems and that there is no substantial difference between the two groups that participated

Keywords: Problem solving, Reasoning and Proof.

In research aimed at studying proportional reasoning in children who attend primary school, the difficulty they face in solving problems that involve operating with notions such as ratio, fraction and developing relative thinking is recognized.

This report presents the results of a study carried out with children in 4th and 5th grade of primary school using computer software in which two types of problems were presented that required the use of reasoning and relative thinking for their resolution.

Background and theoretical framework

Among the studies that focus on the comparison of ratios, Fishbein, Pampu, and Manzat, (1970) investigated the influence of the total number of marbles, in any set, when comparing ratios between sets, noting the possibility of estimating using W1/B1 and W2/B2 type ratios. Sing (2001), based on studies with sixth graders students on the understanding of concepts of proportion and ratio when solving problems such as buying sweets, delivering pizzas and a situation of enlarging and reducing a rectangle, emphasizes that it is necessary to have built multiplicative structures and iteration schemes for proportional thinking. Boyer, Levine, and Huttenlocher (2008), conducted two studies using computer programs, based on the application of a proportional equivalence task. They examined where students make mistakes in processing proportions involving discrete quantities.

The interpretation of the results of this study is made within the framework of the following theoretical approaches. Vergnaud (1991), mentions that the first acquisitions of numerical structures are made during the first years of primary school: additive structures and multiplicative structures. Addition is associated with grouping situations, which make it possible through directed situations to know the rules and procedures of the additive structures, necessary for the acquisition of the multiplicative structures. With respect to multiplication, it establishes that the first great form of a multiplicative relation, implies establishing a relation between two quantities of the same type with other two of the same type, that is, there are four quantities put in relation.

Multiplication requires an understanding of functions that assume the quantities involved, notions such as scalar operator, fractional operator, unit value, etc., Vergnaud (1991). According to this approach it is possible to write relationships and representations between quantities in the form of proportions and arrive at the notion of fraction, ratio and consequently the comparison of ratios. On the other hand, Lamon (1999) defines two types of thinking in the field of proportionality: relative thinking, in which the comparison between two quantities is made from one with respect to the other and not independently, which is what characterizes additive thinking. Noelting (1980), based on his experiment of comparing mixtures of orange juice and water, established a hierarchical differentiation of stages of development of proportional reasoning, based on the comparison of ratios.

Research questions, objectives and method

General objective

To assess whether there is a difference between fourth and fifth grade students in terms of their performance on two types of problems that involve recognizing relationships of proportionality and having relative thinking.

Specific objectives

Recognize how the results of reason comparison problems differ from relative thinking problems.

Observe what difference there is between the results of the fourth grade of primary school students with respect to the fifth grade of primary school students.

Research questions

What is the difference between the results of ratios comparison problems and relative thinking problems?

What is the difference between the results of the fourth grade of primary school students with respect to the fifth grade of primary school students?

Hypothesis

H0= There is no difference in results between ratios comparison problems and relative thinking problems.

H1= There is a difference in the results between the problems of comparison of ratios and the problems of relative thinking.

H0= There is no difference in the answers of 4th grade students compared to 5th grade students in primary education.

H1= There is a difference in the answers of 4th grade students compared to 5th grade students.

A total of 18 students participated in this research, 10 fourth graders and 8 fifth graders from a private school in Mexico City. The participants were between the ages of 9 and 13, with a mean of 10.17 years and a standard deviation of 0.89.

A program was designed in the PsychoPy software version v.3 .2.3. The program that was carried out in PsychoPy, is the instrument that allowed obtaining the data in the present investigation, it was made up of three general parts, the first one presented the instructions of the activity to the students, in the following part of the instrument eight problems of the type comparison of reasons were presented to the students, the activities carried out by Noelting (1980) were taken as a model, in the third part of the instrument eight problems in the category relative thought were proposed, some activities carried out by Lamon (1999), referring to relative and absolute thought were taken into account and others were adapted. For the implementation the institution provided access to the computer room and the test was done twice to each student, when the children arrived at the room they were informed that it was an investigation that was being done for a university and that it was not to qualify them.

Results and discution.

The procedure implemented to evaluate the answers was to calculate the percentage of correct answers per student in the first moment, then to calculate the percentage of correct answers per student in the second moment and finally to calculate the average of moments one and two.

With the mean of the moments one and two of the students, an analysis of variance (ANOVA) of repeated measures (grade five vs. grade four) X2 (Types of problems: Comparison of ratios vs. relative thinking) was performed in the statistical program called JASP. The school grade is defined as the inter-subject variable and the problem type variable is defined as the intra- subject variable.

For the main effect with respect to the types of problems, within subjects, a large statistic of $F(49,201)$, $(0.001<p)$ is reported, it can be said that without considering the group that responded to the instrument there are significant differences with respect to the two types of problems presented, the problems of comparison of reasons with respect to the problems of relative thinking. That is to say, the students of both courses respond better to the problems of comparison of reasons than to those of relative thinking, in this way the null hypothesis is rejected and the alternative hypothesis is accepted which mentions that if there is a difference between both types of problems.

On the other hand, another effect within subjects in relation to the two school grades is reported as a statistic $F(0.274)$, $(p= 0.608)$ this indicates that there is no difference between the two groups that answered the instrument. The ANOVA test suggests that in this case, regardless of the grade level of the students, both grades show a higher percentage of correct answers in one type of problem than in the other, thus accepting the null hypothesis.

On the other hand, as an effect of the test between subjects, there was no effect between the factors degree and types of problems $F(.411)$, $p=0.53$. That is, the grade does not affect performance on either type of problem. The analysis of variance does not reveal any significant interaction.

Figure 1 shows the lack of interaction between problem type and grade, both perform better in answering problems of the comparison of reasons type and lower in solving problems of the relative thinking type, fourth graders are slightly above correct answers relative to fifth graders.

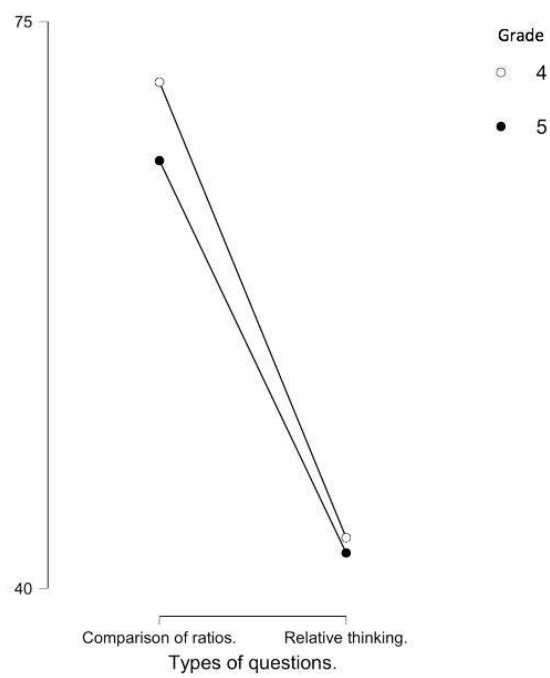

Figure 1: Comparative chart between the results of two types of problems in two school grades.

The results of the statistical test indicated that if there are differences between students' responses to the two types of problems, both problems show very large differences in student responses, even though proportional reasoning is at play in both.

It is believed that problems of the Relative Thinking kind present a greater level of difficulty to students because, as reported by Lamon (1993), students are more familiar with *using* additive structures and when facing situations that involve the use of multiplicative structures, they cannot perceive the multiplicative nature of situations involving ratio and proportion. Unlike problems of the comparison of ratios, it is not necessary to give meaning to any quantity, "in these problems the notion of ratio is the very object of the question, by means of a certain quality: which orange tastes more like orange? Which vehicle goes faster?" (Block, Mendoza and Ramírez, 2010, p 67).

It is necessary to provide students with opportunities to develop relative thinking, to create contexts in classrooms to encourage multiplicative thinking while giving meaning to the notion of ratio, and to find the value of ratio where necessary.

Future research should seek to link students' relative thinking and their understanding of ratio and proportion. It is essential that bridges be built between these two issues related to proportional thinking, so that it can contribute to the creation of new educational programs that favor understanding in proportionality. Work must also be done on the training of teachers, who face various challenges in teaching proportional reasoning (Hilton and Hilton, 2018).

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EL PENSAMIENTO PROPORCIONAL: UN ESTUDIO COMPARATIVO ENTRE NIÑOS DE CUARTO Y QUINTO DE PRIMARIA

PROPORTIONAL THINKING: A COMPARATIVE STUDY AMONG FOURTH AND FIFTH GRADE CHILDREN

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El tema de la proporcionalidad se inicia formalmente en los últimos años de la escuela primaria, implica el dominio de nociones, como la comparación de razones, fracciones y porcentajes. En esta investigación se realizó un estudio con 18 alumnos de cuarto y quinto grado de educación primaria con una edad entre los 9 a 13 años, con el objetivo de evaluar su desempeño al resolver problemas que implicaron comparar razones y tener un pensamiento relativo. Ambos tipos de problemas conllevan tener un pensamiento proporcional. El propósito es conocer cuál es la diferencia entre los dos grupos, así como identificar si existe diferencia entre la ejecución en los dos tipos de problemas. Se encontró que los alumnos de ambos grados presentan más dificultad al resolver los problemas del tipo pensamiento relativo y que no hay una diferencia sustancial entre los dos grupos que participaron.

Palabras clave: Resolución de problemas, razonamiento y demostraciones

En las investigaciones dirigidas a estudiar el razonamiento proporcional en los niños que asisten a la primaria, se reconoce la dificultad que enfrentan ante la resolución de problemas que implican operar con nociones como razón, fracción y el desarrollo del pensamiento relativo.

En este reporte se presentan los resultados de un estudio realizado con niños de 4° y 5° grado de primaria a través de un software informático en el que se plantearon dos tipos de problemas que para su resolución requería el uso de comparación de razones y del pensamiento relativo.

Antecedentes y marco teórico

Entre los estudios que focalizan la comparación de razones, Fishbein, Pampu, y Manzat, (1970) investigaron acerca de la influencia del número total de canicas, en cualquier conjunto, al comparar razones entre conjuntos, señalando la posibilidad de estimar utilizando razones del tipo W_1/B_1 y W_2/B_2 . Sing (2001), con base en estudios con alumnos de sexto grado sobre la comprensión de conceptos de proporción y razón al resolver problemas como la compra de dulces, reparto de pizzas y una situación de ampliar y reducir un rectángulo, enfatiza que se necesita tener construidas estructuras multiplicativas y esquemas de iteración para el pensamiento proporcional. Boyer, Levine, y Huttenlocher (2008), realizaron dos estudios mediante el uso de programas de computadora, basados en la aplicación de una tarea de equivalencia proporcional. Examinaron dónde se equivocan los estudiantes al procesar proporciones que implican cantidades discretas.

La interpretación de los resultados de este estudio, se realiza en el marco de los siguientes planteamientos teóricos. Vergnaud (1991), menciona que en el curso de los primeros años de la escuela primaria se hacen las primeras adquisiciones de las estructuras numéricas: las estructuras aditivas y las estructuras multiplicativas. La adición está asociada con situaciones de agrupamiento, que posibilitan a través de situaciones dirigidas conocer las reglas y procedimientos de las estructuras aditivas, necesarias para la adquisición de las estructuras multiplicativas. Respecto a la multiplicación, establece que la primera gran forma de una relación multiplicativa, implica establecer una relación entre dos cantidades del mismo tipo con otras dos del mismo tipo, es decir, hay cuatro cantidades puestas en relación.

En la multiplicación se requiere de la comprensión de funciones que asumen las cantidades implicadas, nociones tales como operador escalar, operador fraccionario, valor unitario, etc., Vergnaud (1991). De acuerdo a este planteamiento se da la posibilidad de que las relaciones y representaciones entre cantidades se escriban en forma de proporciones y se arribe a la noción de fracción, de razón y en consecuencia a la comparación de razones. Por otra parte, Lamon (1999), define dos tipos de pensamiento en el campo de la proporcionalidad: pensamiento relativo, en el que la comparación entre dos cantidades se hace de una respecto a la otra y no de manera independiente que es lo que caracteriza al pensamiento aditivo. Noelting (1980), con base en su experimento de comparar mezclas de jugo de naranja y agua, estableció una diferenciación jerárquica de etapas de desarrollo del razonamiento proporcional, basándose en la comparación de razones.

Preguntas de investigación, objetivos y método

Objetivo General:

Evaluar si existe diferencia entre los alumnos de cuarto y quinto grado de primaria en cuanto a su desempeño en dos tipos de problemas que implican reconocer las relaciones de proporcionalidad y tener un pensamiento relativo.

Objetivos específicos:

Reconocer qué diferencia existe entre los resultados de los problemas de comparación de razones con respecto a los problemas de pensamiento relativo.

Observar qué diferencia hay entre los resultados de los alumnos de cuarto grado de primaria con respecto a los alumnos de quinto grado de primaria.

Preguntas de investigación:

¿Qué diferencia existe entre los resultados de los problemas de comparación de razones con respecto a los problemas de pensamiento relativo?

¿Qué diferencia existe entre los resultados de los alumnos de cuarto grado de primaria con respecto a los alumnos de quinto grado de primaria?

Hipótesis

H0= No hay diferencia en los resultados entre los problemas de comparación de razones y los problemas de pensamiento relativo.

H1= Existe diferencia en los resultados entre los problemas de comparación de razones y los problemas de pensamiento relativo.

H0= No hay diferencia en las respuestas de los alumnos de 4º grado respecto a las de los alumnos de 5º grado.

H1= Existe diferencia en las respuestas de los alumnos de 4º grado respecto a las de los alumnos de 5º grado.

Un total de 18 alumnos participaron en esta investigación, 10 niños de cuarto grado y 8 niños de quinto grado de educación primaria de una escuela privada de la Ciudad de México.. Los participantes cuentan con una edad comprendida entre los 9 a 13 años de edad, con una media de 10.17 años y una desviación estándar de 0.89.

Se diseñó un programa en el software PsychoPy versión v.3.2.3. El programa que se realizó en PsychoPy, es el instrumento que permitió obtener los datos en la presente investigación, se conformó de tres partes generales, la primera de ellas presentó a los estudiantes las instrucciones de la actividad, en la siguiente parte del instrumento se presentaron a los estudiantes ocho problemas del tipo comparación de razones, se tomaron como modelo las actividades realizadas por Noelting (1980), en la tercera parte del instrumento se proponen ocho problemas en la categoría pensamiento relativo, se tuvieron en cuenta algunas actividades realizadas por Lamon (1999), referentes al pensamiento relativo y absoluto y otros se adaptaron. Para la implementación la institución brindó el acceso al aula de cómputo y la prueba se hizo dos veces a cada estudiante, cuando los niños llegaban al aula se les informaba que era una investigación que se estaba realizando para una universidad y que no era para calificarlos.

Resultados y discusión.

El procedimiento implementado para evaluar las respuestas fue calcular el porcentaje de respuestas correctas por alumno del primer momento, posteriormente calcular el porcentaje de respuestas correctas por alumno en el segundo momento para finalmente calcular la media del momento uno y dos.

Con la media de los momentos uno y dos de los estudiantes se realizó un análisis de varianza (ANOVA) de medidas repetidas (grado escolar: quinto de primaria vs. cuarto de primaria) X2 (Tipos de problemas: Comparación de razones vs Pensamiento relativo) en el programa estadístico llamado Jasp. El grado escolar se define como la variable inter-sujeto y la variable tipo de problema se define como la variable intra-sujeto.

Para el efecto principal con respecto a los tipos de problemas, dentro de sujetos, se reporta un estadístico grande de $F(49.201)$, $(0.001<p)$, se puede decir que sin considerar el grupo que respondió el instrumento hay diferencias significativas respecto a los dos tipos de problemas presentados, los problemas de comparación de razones con respecto a los problemas de pensamiento relativo. Es decir, los alumnos de ambos cursos responden mejor los problemas de comparación de razones que los de pensamiento relativo, de esta manera se rechaza la hipótesis nula y se acepta la hipótesis alternativa que menciona que si existe diferencia entre ambos tipos de problemas.

Por otra parte, otro efecto dentro de sujetos en relación con los dos grados escolares se reporta un estadístico $F(0.274)$, $(p= 0.608)$ esto indica que no existe diferencia entre los dos grupos que respondieron el instrumento. La prueba ANOVA sugiere que en este caso sin tomar en cuenta el grado escolar que cursen los alumnos, ambos grados muestran tener un porcentaje mayor de respuestas correctas en un tipo de problema que en el otro, aceptando así la hipótesis nula.

Por otro lado, como efecto de la prueba entre sujetos, no hubo efectos entre los factores grado y tipos de problemas $F(.411)$, $p=0.53$. Es decir, el grado escolar no afecta el rendimiento en ninguno de los dos tipos de problemas. El análisis de varianza no revela alguna interacción significativa.

La figura 1 muestra la falta de interacción entre el tipo de problema y el grado, ambos tienen un mejor rendimiento al responder problemas del tipo comparación de razones y un rendimiento más bajo al resolver problemas del tipo de pensamiento relativo, los alumnos de cuarto grado están ligeramente por encima con respecto a las respuestas correctas en relación con los alumnos de quinto grado.

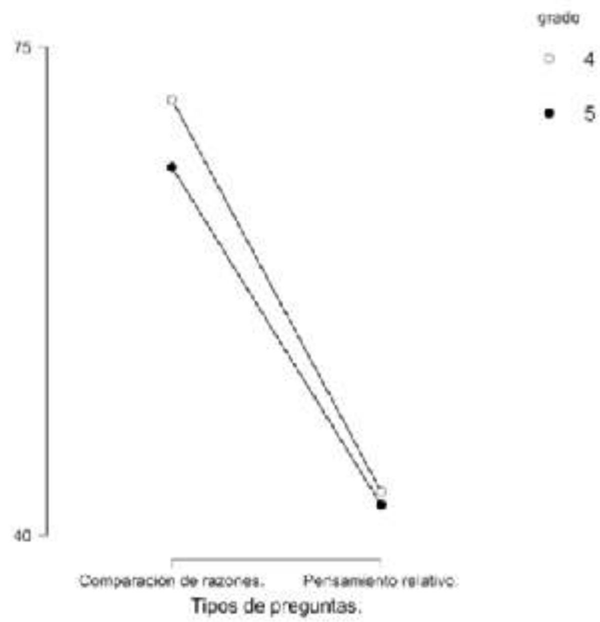

Figure 1: Gráfica comparativa entre los resultados de dos tipos de problemas en dos grados escolares.

Los resultados de la prueba estadística realizada indicaron que si hay diferencias entre las respuestas de los alumnos de los dos tipos de problemas, ambos problemas muestran diferencias muy grandes en las respuestas de los estudiantes, aunque el razonamiento proporcional está en juego en ambos.

Se cree que los problemas del tipo Pensamiento Relativo presentan un nivel de dificultad mayor a los estudiantes debido a que como reporta Lamon (1993), los estudiantes están más familiarizados con utilizar estructuras aditivas y al enfrentar situaciones que implican el uso de estructuras multiplicativas, ellos no pueden percibir la naturaleza multiplicativa de las situaciones que implican la razón y proporción, a diferencia de los problemas de tipo comparación de razones, en los que no es necesario dar significado a ninguna cantidad, "en estos problemas la noción de razón es el objeto mismo de la pregunta, mediante determinada cualidad: ¿qué naranjada sabe más a naranja? ¿Qué vehículo va más rápido?" (Block, Mendoza y Ramírez, 2010, p 67).

Es necesario brindar a los estudiantes oportunidades para desarrollar un pensamiento relativo, crear contextos en las aulas de clase para incitar el pensamiento multiplicativo a la par de dar significado a la noción de razón y encontrar el valor de la razón en los casos en los que sea necesario.

Futuras investigaciones deben buscar vincular el pensamiento relativo de los estudiantes y su comprensión de la razón y la proporción. Es indispensable que se construyan puentes entre estos dos temas relacionados al pensamiento proporcional, de modo que se pueda contribuir a la creación de nuevos programas educativos que favorezcan la comprensión en la proporcionalidad. También se debe trabajar en la capacitación de los profesores, los cuales enfrentan varios desafíos en la enseñanza del razonamiento proporcional (Hilton y Hilton, 2018).

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STAGES IN USING PROOF TECHNIQUES: STUDENT DEVELOPMENT IN THE TRANSITION TO PROOF

ETAPAS EN EL USO DE TÉCNICAS DE PRUEBA: DESARROLLO DE LOS ESTUDIANTES EN LA TRANSICIÓN A LA PRUEBA

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The transition to learning how to prove is difficult for undergraduates. We are aware of the many varied struggles students have, but we know less about their development as they are learning. This is vital as development is more than just an accumulation of competencies. To examine these developments, a series of four task-based interviews across a semester were conducted with (N=11) undergraduate students enrolled in a transition to proof course. Video of students constructing proofs was analyzed qualitatively; changes in how students chose what proof technique to use were common. Stages in students' rationales are illustrated using two students as cases. The results show students' decision-making in starting a proof and remind us that such judgement takes time to grow. Instructors and curriculum developers may use these results in designing tasks and supports for the transition-to-proof.

Keywords: Reasoning and Proof, Post-Secondary Education

The transition-to-proof is difficult for undergraduate students (Moore, 1994; Selden & Selden, 1987). Students struggle with learning how to prove (Iannone & Inglis, 2010; Selden & Selden, 2013). The transition-to-proof is a shift in the "game" of mathematics, from answering "exercises" that are largely procedural (Schoenfeld, 1992) to now writing arguments and justifications. Researchers have identified the types of errors students make (Selden & Selden, 1987) and their struggles (Selden & Selden, 2003): use of examples, notation and symbols, quantifiers, and general logic (Epp, 2003; Selden & Selden, 1987). Students also struggle with larger issues, such as providing empirical rather than deductive arguments (Harel & Sowder, 2007) and having difficulty writing formal arguments (Alcock & Weber, 2010). Another strand of research has focused on students' strategies and approaches to the proving process (Karunakaran, 2014; Savic, 2012).

We know then students' struggles and strategies while proving at singular points in time, but few have looked at how students develop, at how their strategies change over the course of the learning process. Development is not necessarily about accumulating competencies: "For some psychologists, development is reduced to a series of specific learned items, and development is thus the sum, the culmination of this series of specific items. I think this is an atomistic view which deforms the real state of things" (Piaget, 1964, p. 38). Thinking about proving as the sum of skills and assessing whether or not students have those skills is not enough for us to understand students' learning process. We do not yet know *how* students put the pieces together while they are learning how to prove nor the *order* in which they occur. We lack models of students' cognitive development in this domain.

In response to this gap, the research question guiding this work is: How does undergraduate students' proving develop over the duration of a transition to proof class? The purpose of this study is to understand how students come to learn how to prove. In this paper, I examine one prevalent development that occurred and illustrate it through two participants.

Conceptual Framework

Proving as Problem Solving

While there are multiple ways to think about proving as an activity, I take the conceptualization of proving as a form of problem solving (Stylianides et al., 2017). I further take problem solving to be the activity a person engages in when stuck, reaching an impasse (Savic, 2012). Under this definition, a task can elicit problem solving in one student but not another, depending on whether or not they become stuck at any point. There are a lack of robust frameworks for characterizing a student's proving (Savic, 2012), but by considering proving as problem solving, we can look to work on problem solving. I focused on the components of strategies (heuristics) and monitoring and judgement of problem solving (Schoenfeld, 1985b; 1992). Moreover, the focus here was on proving as a *process* (Karunakaran, 2014), rather than on the product, the correct proof.

Development at its most base level may refer to change over time. Development does not happen in a vacuum; it is undoubtedly informed by instruction. A common way to consider development is in terms of stages, in which a person passes through each stage on their way to full mastery (e.g. Lo, Grant & Flowers, 2008; van Hiele, 1959). I conceptualize development simply as taking a "snapshot" - a characterization of some construct at a point in time - and looking across these at multiple timestamps for change (Figure 1).

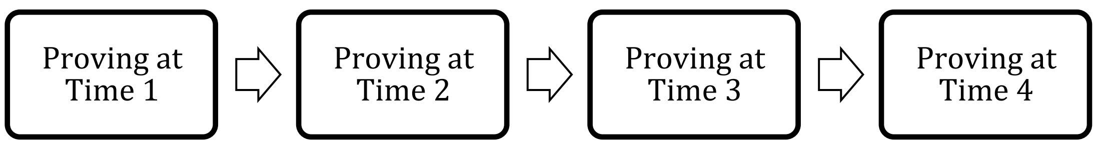

Figure 1: Conceptualization of development, by capturing snapshots of student's proving and comparing over time.

The purpose in taking this simple view of development is to provide as much description as possible and look for natural change, which may then inform the creation of potential frameworks and models for how students develop in a transition to proof course.

Method

A series of four semi-structured interviews were conducted with N=11 undergraduate students in a transition to proof mathematics course at a large Midwestern university. Their ages were 18 and up. This transition to proof course was designed to ease the transition from calculus-based to upper-level math courses that involved writing proofs. This course was a prerequisite for Linear Algebra, so a variety of STEM (science, technology, mathematics, and engineering) majors were enrolled in this course as well. The first half of the course focused on logic, including direct proof, proof by contradiction, proof by contrapositive, and proof by cases. The second half introduced basic concepts in real analysis, linear algebra, and number theory.

Data Collection

The four interviews were spread across a semester. Each interview was also task-based, consisting of two proof construction tasks. Participants worked for no more than 15-20 minutes on each proof construction and debriefed their thought process after with the interviewer. All eight tasks were from one content area, basic number theory. Tasks were selected by the researcher to not be heavily dependent on content knowledge nor a singular specific proof technique. Interviews were audio- and video-recorded, and interview notes and student work were collected.

In order to capture their strategies and reasons for using certain strategies, I used a think-aloud protocol (Ericsson & Simon, 1980; Schoenfeld, 1985a), where participants voice their thoughts aloud about a task. Based on the affordances and constraints of asking probing questions (Schoenfeld, 1985a), I minimized interviewer intervention during task performance. Because the phenomenon of interest was the proving *process* itself, keeping the process intact without interruption as much as possible was of the utmost importance.

Analysis

Qualitative analysis was conducted on video data of participants working on the proof construction tasks. First, video was analyzed for moments when students became stuck. Then, in those moments, I recorded students' strategies, termed proof-specific intentions (Satyam, 2018). Students' strategies were refined using open coding and constant comparison. Lastly, I looked for change in each student's strategies over the eight tasks spanning the semester.

Indicators of an impasse. Through watching videos of students' attempts to prove, certain observable behaviors contributed to my judgment of when a student was stuck. A list of these include: silence, no writing, staring at paper, holding paper closer to one's face, sitting back from the paper to look at it as if from a distance, tapping/playing with their pen/pencil, and touching face with hand or pencil. These behaviors were not exhaustive and individuals exhibited different behaviors specific to themselves, but they cover much of what we see when a person is stuck.

Results: Shifts in How Students Chose A Proof Technique to Use

A common development that occurred across participants was change in how they chose what proof technique to pursue, when trying to construct a proof. By *proof technique*, I mean tools such as direct proof, cases, etc. Proof by contradiction may be referred to here as just contradiction and proof by contrapositive as contrapositive for brevity sake. Eight of eleven participants (pseudonyms used) showed this development, based on interview notes and across all tasks (see Table 1). I discuss two participants here, to illustrate this development.

Table 1: Select Developments in Proving by Participant

	Rationale for a proof technique	Harness awareness of solution attempt	Check examples w/ other strategies	Explore and monitor
Amy				X
Charlie	X		X	
Dustin	X			
Granger		X	X	X
Gabriella	X	X		
Joel		X		
Jordan	X			
Leonhard	X			
Stephanie	X			
Shelby	X			
Timothy	X	X	X	X

Case: Favoring One Proof Technique

Stephanie was chosen here to illustrate the early stages, of where a student uses one proof technique predominantly. From the beginning, Stephanie favored proof by contradiction over all other techniques when constructing a proof. In Interview 1 – Task 1, she jumped to trying proof by contradiction: She immediately identified the assumption as “A,” the conclusion of the statement as “B,” and wrote the negation.

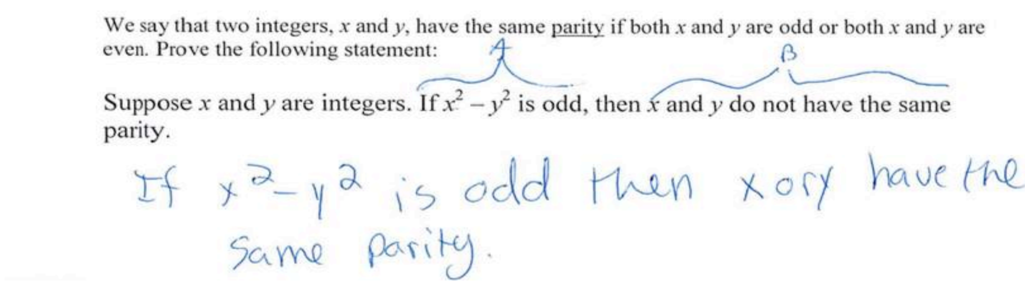

Figure 2: Beginning of Stephanie’s work on Interview 1 – Task 1

She explained that she used contradiction because the statement was an implication, having an “if-then” structure: “When I see the if-then statement, I immediately think I can do this by contradiction.” She explained further that she felt comfortable using this technique. Note that Stephanie technically wrote the negation incorrectly; the correct negation is “A and not B,” i.e. “$x^2 - y^2$ is odd and x or y have the same parity.” Instead, she wrote the negation as an implication, a common error. However, this error did not affect the rest of her proof and her reasoning for picking contradiction was unaffected by her execution.

In the next interview, Stephanie’s go-to method was still contradiction. Upon starting the second task of Interview 2, she said, "I can see that this is an if-then statement, so automatically I'm going to try to use contradiction, but I don't know if it will work or not." She explained that “When I read an if-then statement, I'm most comfortable using negation or a contradiction. So then I just try that, even though I know it doesn't always work, but I just try it." The use of contradiction was automatic for her, saying outright she does not always know if proof by contradiction will lead to a correct solution. The general structure – of a statement having “if” and “then” clauses – was enough to determine that she could use her favored technique, but she did not make use of the statement in any further way to guide her choice of technique.

Stephanie did get stuck on her proof by contradiction, so she switched to contrapositive. She explained during the debrief, “I'll try contrapositive and then I felt a little better after I tried contrapositive just because I thought [out of] both of them, probably one of them was gonna be right." Stephanie did not give a rationale for why, just that it was another technique.

Summary. Stephanie’s articulations and work during Interviews 1 and 2 show how a student can favor one proof technique and use it whenever they can. Stephanie did have a condition for when to use proof by contradiction, whenever she saw an if-then statement, but this applied to nearly all statements to be proven in the course. Stephanie becomes less dependent on proof by contradiction and her rationale did become more sophisticated over time, but her work was unfortunately incorrect on all four tasks on Interviews 3 and 4 so they are not presented here.

Case: Recognizing When Best to Use A Certain Proof Technique

We turn now to a different student, Timothy, to see how rationales shift over time. Timothy was similar to Stephanie in having favored proof techniques in the beginning, but his rationales became more sophisticated and based on the statement itself as his interviews progressed, in addition to producing correct or partially correct proofs.

Timothy began his interviews similar to Stephanie in terms of his rationales. Figure 3 shows Timothy’s attempt in Interview 1 - Task 1 (same as Stephanie’s task). When stuck in the beginning, he re-read the question and wrote what was known. At this point he switched from his direct proof attempt to proof by contrapositive.

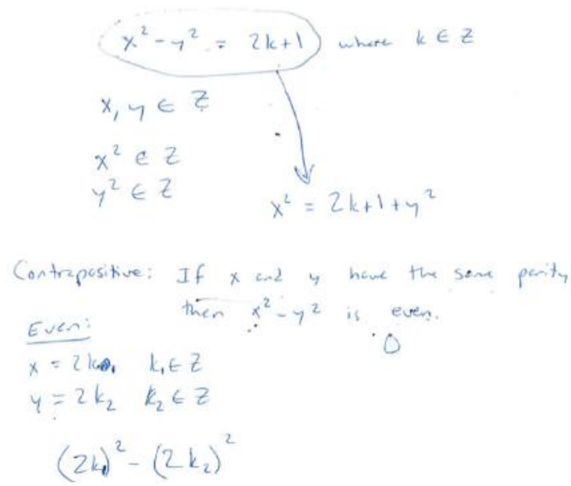

Figure 3: Beginning of Timothy's work on Interview 1 - Task 1

When asked why he selected contrapositive, he explained it was a method from class but also that it was a tool logically equivalent to direct proof that he could use:

> Timothy: It was confusing me when I'd try to think of it the normal way so I knew the contrapositive is true, it's basically the equivalent, logical equivalent.
>
> …
>
> Interviewer: So actually, so how did you come up with contrapositive?
>
> Timothy: Looking at it straightforward didn't…it wasn't working for me so I know we learned in class that the contrapositive is basically not B implies not A. I knew we said that was logically equivalent, so if I could prove the contrapositive was true, then I could prove the original statement was true was kinda my thinking with that.

He explained that direct proof was not helpful for generating a proof, but he gave no specific rationale for choosing contrapositive over other proof techniques. His explanation implied that contrapositive was a legitimate tool from class, so why not use it? While it is possible he may have had some internal reason for using contrapositive, he neither mentioned this on his own nor articulated any further reasons when questioned.

Later in this interview, he talked more about contradiction being one of his "go-to" methods and why:

> Timothy: I always go about it with either contradiction or induction or straight up [direct proof] so I kinda knew that I might be able to contradict this never equaling that, so I wrote out the contradiction...I guess contradiction is a little easier for me to think about. You just say the first part of the implication is true and the second part is false. So it's just easier in my head, I guess, just to think about rather than switching around the implication, negating both parts.
>
> Interviewer: Okay.
>
> Timothy: So I guess that's why I go to that first.

Timothy expressed here that contradiction was easier for him than contrapositive, which involves switching and negating both the assumption and conclusion. He did have some rationale for why he might use contradiction, but it was couched in terms of ease of use, first and foremost.

The notion of "ease of use" as determining choice of proof techniques showed up in latter interviews. In his work for Interview 2 - Task 1 (Figure 4), Timothy started by defining x and y using the definition of consecutive numbers but in calculating xy, he became stuck over what to do next. He then switched to contrapositive because "sometimes that's an easier way for me to look at it." He knew that contrapositive was easier on some level for him but not for any reasons specific to the statement and did not further articulate why. Ultimately, his contrapositive proof was not to his liking and also not correct.

If x and y are consecutive integers, then xy is even.

$x = k$ where $k \in \mathbb{Z}$

$y = k+1$ where $k \in \mathbb{Z}$

Then $xy = k(k+1)$ Goal: $2(m)$ where $m \in \mathbb{Z}$

$= k^2+k.$

Contrapositive: If xy is odd, then x and y are not consecutive integers.

Figure 4: Timothy's work for Interview 2 – Task 1

By the end of the interviews, however, Timothy showed sophisticated rationale when considering which proof techniques to use. In Interview 4 - Task 1 (Figure 5), Timothy became stuck after computing the goal, a+b, directly.

If a and b are odd perfect squares, then their sum a + b is never equal to a perfect square.

$a = n_1^2$ where $n_1 = 2k_1+1$ for some $k_1 \in \mathbb{Z}$

$a = (2k_1+1)^2$ for some $k_1 \in \mathbb{Z}$

$b = (2k_2+1)^2$ for some $k_2 \in \mathbb{Z}$

Then $a+b = (2k_1+1)^2 + (2k_2+1)^2$

$= (4k_1^2+4k_1+1) + (4k_2^2+4k_2+1)$

$= 4(k_1^2+k_1+k_2^2+k_2) + 2$

Let $k_1^2+k_1+k_2^2+k_2 = m \in \mathbb{Z}$ since $k_1, k_2 \in \mathbb{Z}$

Contradiction: a and b are odd perfect squares and $a+b$ is a perfect square.

Then $4m+2 = n^2$ for $m \in \mathbb{Z}$

$4m = n^2-2$

$m = \frac{n^2}{4} - \frac{1}{2}$

Figure 5: Timothy's work on Interview 4 – Task 1

He explained that he used contradiction because "it's easier when I know something like is equal to something or is something." He then gave this further rationale for why contradiction:

> I was trying to prove that it's not equal to a perfect square and I know from past experiences, *it's easier when I know something is equal to something or is something.* So I tried to use contradiction because *I knew I could say then it is a perfect square.*

His argument was that he wanted to be able to work with an equality. Timothy also gave a reason for why he did not use another method, contrapositive:

> I thought about contrapositive, too, but then it would say that A and B are not perfect squares and that's again, like something's *not* so I mean, it's easier for me to work when I know like a straight definition of something. So if I could keep this, I knew if I could keep this, like they are perfect squares and say this is a perfect square, then it'd be easier to work with.

This explanation was similar to his prior one about equality of objects being easier, i.e. knowing things are not equal is not as helpful. His sub goal then was to find a proof technique that would give him a+b *is* a perfect square. This task is also notable for drawing out Timothy's observations on contradiction:

> I never really thought about it this way but I realized when you use the contradiction, you don't really have the assumption and conclusion anymore…*you can actually pick any part of that statement you want and work with it. Rather than with an if/then statement, you start with the assumption and try to work to the conclusion.* So you're not as limited, I guess.

Timothy gave a high-level explanation of the nature of proof by contradiction. He found proof by contradiction freer than other techniques, due to being able to work with all parts of the statement. This is in contrast to starting with the assumption and trying to prove the conclusion, as is done in direct proof but also to an extent proof by contrapositive. Note that this revelation came during this interview context, based on his "I never really thought about it this way but..." clause. The interview served as a vehicle for reflection on proof techniques for Timothy.

Summary. Timothy went from picking a proof technique because (1) it existed as a tool, to (2) having a general sense that certain ones would be easier, to (3) explaining how the content of the statement can drive the approach, to (4) articulating understanding at the meta-level of how a technique functions as logical tools. His later interviews revealed insight on when to use contradiction that did not depend on statement content but instead meta-level structure.

Discussion

Both Stephanie and Timothy showed similar growth in how they chose a proof technique to pursue through most of their interviews. Both discussed liking and being drawn to certain techniques, as their go-to method. Timothy's latter interviews especially illustrated weighing the utility of different techniques, to think about which would be *better*, whether it be a cleaner proof or just easier. He noticed that being able to set things equal provided more to work with and often preferred proof by contradiction for this reason.

The difference between these two cases lies in where they ended: by the end of the interviews, Timothy articulated a general insight for when contradiction was useful. Across all the students, a general trajectory for how students grew in how they chose which techniques to use emerged. Conceptualizing this specific development as a series of stages, Figure 6 illustrates the stages students tended to step through.

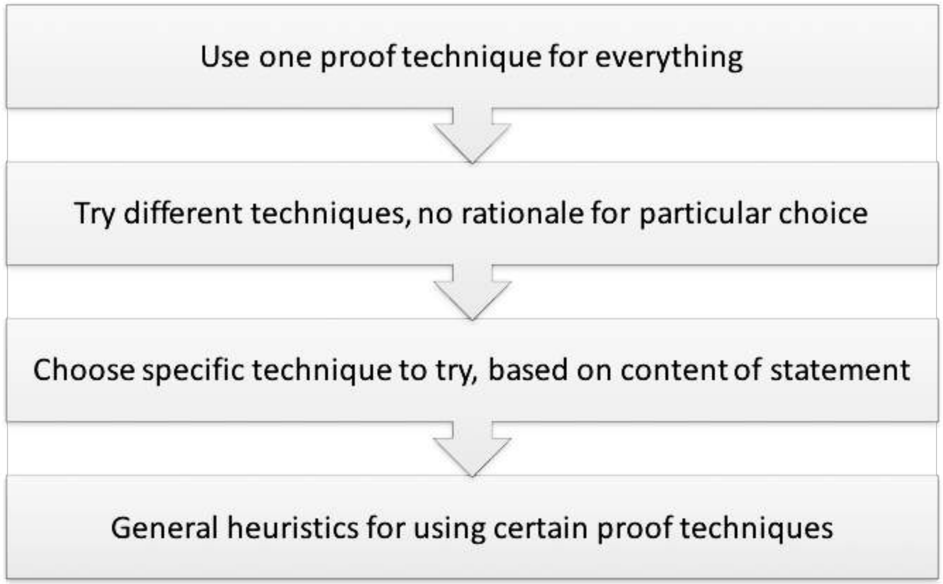

Figure 6: Stages of development in how students choose proof techniques to pursue

This development reveals the amount of decision-making that can go into writing even the first line of a proof. Timothy took the content of the statement to be proven into account when deciding how to begin a proof. This suggests a revisiting of the distinction between formal-rhetorical and problem-centered aspects of proving (Selden & Selden, 2007). Acts that we expect to be formal-rhetorical, such as writing the first line of a proof, may retain some of the problem-solving aspects too for students new to proving, as they consider the content as well. It is important that the interplay between these two aspects – formal-rhetorical and problem-centered parts of proving – not be lost when teaching students.

This development is significant because it shows that students do over time grow in their sense of when certain proof techniques are best suited for a problem and that there are general stages. One limitation is that becoming better at using tools is not necessarily reflective of deeper mathematical understanding, as Guin and Trouche (1999) noted about students using calculators as tools. But the cases here shows it is natural for even this kind of judgment to take a while to develop; noticing what proof technique works best for a given statement does not happen instantly but also it must be nurtured. This means that as instructors, we cannot expect students to have this reasoning immediately. Development is of course informed by instruction, so this may be an area that can be supported via instruction, by designing tasks that probe students to consider the strengths and weaknesses of each proof technique. Through this, we can better help students understand and learn how to prove, as a mathematical activity that makes sense.

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ATTEND TO STRUCTURE AND THE DEVELOPMENT OF MATHEMATICAL GENERALIZATIONS IN A DYNAMIC GEOMETRY ENVIRONMENT

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Central to mathematical generalization is the development of structural thinking. By examining the relationship between structural thinking and mathematical generalization, this study found that learners' attention to different elements of a problem can result in different mathematical generalizations and structural generalization occurs only when learners reason based on identified properties. These findings imply that learners should be cultivated to attend to mathematical structures and to generalize beyond numerical patterns.

Keywords: Advanced mathematical thinking

Generalizing involves transportation of a mathematical relation from a given set to a new set for which the original set is a subset, perhaps adjusting the relation to accommodate the larger set. It has been argued that generalizing should be at the heart of mathematics activity in school (e.g., Mason, Johnston-Wilder, &Graham, 2005). Within the past a few decades researchers have differentiated different forms of mathematical generalizations (Dörfler, 1991; Yerushalmy,1993; Mason, Burton, & Stacey, 2010), among which are *empirical* and *structural* generalizations. Empirical generalization is the process of forming a conjecture about what might be true from numerous instances. It occurs when a learner looks at several, sometimes many, cases or instances and identifies the sameness among these cases as a general property. Structural generalization arises when a learner recognizes a relationship from one or very few cases by attending to the underlying structure within these cases and perceives this relationship as a general property. The distinction implies the need for learners to move from empirical to structural generalization. Central to this advancement is the development of structure thinking, which can be defined as a disposition to use, explicate, and connect mathematical properties in one's mathematical thinking (Mason, Stephens, & Watson, 2009). However, most studies on generalizing were conducted in the context of pattern recognition. More importantly, by providing the first few terms of a pattern, the tasks used in these studies tend to promote generalization that does not necessarily demand structural thinking (Küchemann, 2010). To extend the study of mathematical generalization beyond the context of pattern recognition and to bring structure thinking to the forefront of the development of mathematical generalization, this study aimed to examine the relationship between structure thinking and mathematical generalization in a dynamic geometry environment (DGE). It was guided by one research question: How does learners' structure thinking evolve and influence their generalizing activity when working on a carefully designed sequence of tasks in DGE?

Theoretical Framework

Mason et al. (2009) described mathematical structure as the identification of general properties that are instantiated in a particular situation as relationships between elements and differentiated five states of learner's attention to mathematical structure. *Holding wholes* involves a certain way of looking at a whole situation that produces a global image that will undergo further analysis. In this awareness state, a learner attends to an object as a whole without explicit regard to its components. *Discerning details* shift the learner's attention toward further analysis and deep description, in which parts begin to be discerned and described in detail based on what the learner finds meaningful to inspect. The attention can focus on parts that either change or remain invariant. *Recognizing* *relationships* occurs when changing or invariant relationships are detected and analyzed critically. In this awareness state, the learner attends to relationships between parts or between part and whole. *Perceiving properties* occurs when the learner perceives the discerned relationships as instantiations of general properties which can apply in many different situations. It involves the transition from seeing something in its particularity to seeing it as representative of a general class. This state enables a further categorization of different (classes of) objects. The separation of stages three and four indicates a subtle but vital difference between recognizing relationships in particular situations and perceiving relationships as instantiations of general properties which can apply in many different situations. *Reasoning on the basis of the identified properties* is the critical phase in which inductive and abductive reasoning about specific objects transforms into deductive reasoning by examining what other objects may belong to the perceived structure. In this awareness state, the learner attends to properties as abstracted from and independent of any particular objects and forms axioms from which deductions can be made. This model provides a useful tool to examine the development of structural thinking.

Methodology

The data for this study was collected from a series of task-based interviews that were a part of a larger research project aimed to investigate preservice secondary mathematics teachers as learners and teachers of mathematical generalizations in a technology-intensive learning environment. The task-based interview was chosen to gain knowledge about individual preservice teacher's processes to generalize mathematical ideas and the mathematical knowledge resulting from it. Each task in this study consisted of a sequence of closely related problems that aimed to promote learners to generalize a mathematical idea to a broader domain. These tasks were design to engage learners in not only empirical but also structural generalizations.

The participants were 8 undergraduate preservice secondary mathematics teachers enrolled in a course that focused on teaching mathematics with various types of mathematical action technologies. The course took a problem-solving approach and engaged the preservice teachers in the processes of representing, conjecturing, generalizing, and justifying by solving and extending mathematically rich problems in technology-rich learning environments. Outside the class each participant participated in four task-based interviews, each of which was about 2 hours. During each interview, a participant would solve one or two mathematical tasks with the technologies they had learned in class. Participants' interactions with technology were screen-recorded. During each session, the interviewer frequently asked the participant to articulate his/her thinking process and to make general statement based on his exploration. Those interactions between the interviewer and the participant were recorded with a camera.

Data analysis consisted of three stages. First, the generalizations a participant constructed while solving each mathematical task were identified and categorized into empirical and theoretical generalizations. A generalization was coded as empirical if it was constructed on the basis of perception or numerical pattern by comparing numerous instances; it was coded as structural if it was constructed based on the generality of the inferred ideas, methods, or processes. Second, Mason et al. (2009)'s model was used to analyzed a participant's evolution of the state of attention to mathematical structure when constructing each mathematical generalization. The final stage involved coordinating the analysis in the first two stages to look for patterns about the evolution of structural thinking and the development of generalization.

Results and Discussion

Results from data analysis indicated a close relationship between the state of attention to mathematical structure and the forms of mathematical generalization that can potentially emerge.

More specifically, the study found that (1) learners' attention to different elements of a problem can result in different mathematical generalizations and (2) structural generalization occurs only when learners can reason on the basis of identified properties. I will use participants' work on task to illustrate the findings from this study. In the task, participants were asked to decide the conditions under which the area of the square created from the largest side of a triangle is equal to the sum of the areas of the squares created from the other two sides of the triangle (Part 1) and to further extend this relationship to quadrilateral (Part 2) and other polygons (Part 3).

When solving Part 1 of the task, Although Joe quickly connected it with the Pythagorean theorem, he focused his attention on the relationship of square from a right leg and the square from the hypotenuse, conjectured that the areas of the two squares grew proportionally and the vertex A shared by the two squares moved along a line, and then validated his conjecture by perception and measurement (Figure 1a). When solving Part 2 of the task, Joe made one interior angle of the quadrilateral a right angle by dragging and then dragged the vertex opposite to the right angle such that the area of the largest square was equal to the sum of the areas of the other three squares. After creating multiple instances of the desired diagram through dragging, informed by his knowledge gained from earlier exploration, he conjectured that the vertex D opposite to the right interior angle moved along a line and the areas of the two squares that share the vertex A grew proportionally (Figure 1b). Here, Joe attended to the relationship between the areas of the two squares and generalized this relationship from triangle to quadrilateral.

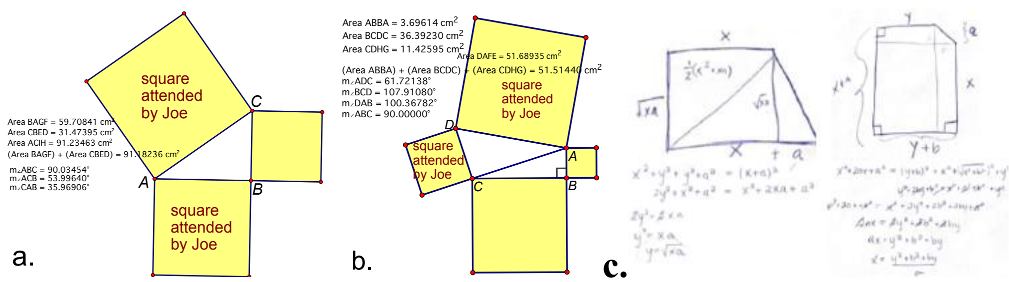

Figure 1: Snapshots of participants' work

In contrast, when exploring Part 2 of the task, Jen considered a right isosceles trapezoid, made two right triangles inside the trapezoid, and labeled the shorter base as x, the longer base as $x + a$, and a lateral side as y (see Figure 1c). By using the fact that the area of the largest square should be equal to the sum of the areas of the three squares and the Pythagorean theorem, she created an equation $x^2 + y^2 + y^2 + a^2 = (x + a)^2$ and concluded that $y = \sqrt{ax}$ after symbolic manipulations. When asked how to further extend the relationship to other polygons, Jen drew a pentagon with three right interior angles as shown in Figure 3 and labeled $x, y, x + a, y + b$ as the length of its four sides. By using the fact that the area of the largest square should be equal to the sum of the areas of the four squares and the Pythagorean theorem, he created an equation $x^2 + 2ax + a^2 = (y + b)^2 + x^2 + (\sqrt{a^2 + b^2})^2 + y^2$ and concluded that $x = \frac{y^2+b^2+by}{a}$ after symbolic manipulation. Moreover, Jen noticed that there was one right angle in the case of triangle, two right angles in the quadrilateral, three right angles in the pentagon and concluded that there would be $n - 2$ right angles by extending the perceived numerical pattern. Here, Jen attended to the desired symbolic relationship between the sides of a polygon and the algebraic identity expressed in the Pythagorean theorem to search for a class of polygons that would satisfy the problem condition. What was generalized was a symbolic relationship rather than the underlying structure expressed in the Pythagorean theorem.

Different from both Joe and Jen, Jack made generalizations by attending the underlying structure of the Pythagorean theorem. The following excerpt shows his generalization of the Pythagorean theorem when solving Part 2 and Part 3 of the problem.

> Interviewer: Now let's think a little bit of what we have done here. What if it is a nonagon, decagon, or an n-sided polygon, how can you create the polygon such that the area of the largest area is equal to the sum of the areas of the other squares drawing from each side of the polygon?
>
> Jack: From one of the vertices of the octagon, the vertex on the largest square, I need the side of each square and the line connecting A to each vertex of the nonagon or the n-gon to form a 90-degree angle. So, you need to make $n - 2$ right angles because the only ones that aren't are the two vertices from the largest square.

The above excerpt provides evidence that Jack extended the Pythagorean theorem to any polygon and generalized that the area of the largest square is equal to the sum of the areas of the n-1 squares created from each side of an n-sided polygon when the polygon is created by sequentially drawing n-2 right angles from a vertex of the polygon to the sides of the polygon.

This study found a close relationship between the elements that the participants attended to and the possible mathematical generalizations they might develop. As shown in the above examples, when solving the task, Joe focused on the covariation of the areas of the two squares, Jen attended to the algebraic identity expressed in the Pythagorean theorem, and Jack focused on the structure underlying the Pythagorean theorem. As a result, Joe generalized the proportionality of the areas of the two squares from triangle to quadrilateral, Jen applied the algebraic identity to deduce algebraic equations that specify a given set of quadrilaterals and pentagons that satisfy the problem condition, and Jack extended Pythagorean theorem and used it to decide the particular shape of an n-sided polygon that satisfies the problem condition. One productive way of helping learners to identify mathematical useful relations is to engage them to examine the generalizability of the perceived mathematics relations and the structures behind them.

Pattern generalization is a typical generalization activity in school mathematics, in which a figurative, numerical, or tabular pattern is usually presented in the form a systematic sequence of elements, and learners are expected to generate a systematic set of ordered pairs from which an empirical relationship can be induced. This approach allows learners to identify and express a numerical relationship without necessarily seeing the mathematical structure that produces it. This study found that although the inductive nature of the dynamic geometry environment made it relatively easy for the participants to observe, conjecture, validate, and generalize mathematical relations based on perception and numerical patterns, identifying structure underlying these relations and generalizing them to broader contexts proved to be challenging. For instance, when solving the above task, the participants produced various generalizations relying on measurement and dragging, but only two of them were able to generalize the Pythagorean theorem from triangle to other polygons. A similar result was found in other tasks given to the participants. Therefore, engagement in pattern generalization does not necessarily support learners' development of structural thinking. One plausible reason that many participants in this study were not able to generalize on the basis of mathematics structure is that they were not provided sufficient opportunities to engage in this way of thinking in their own mathematics learning experience. In order to develop learners' ability to make structural generalization, they should be provided opportunities to initiate into structural thinking.

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MATHEMATICAL PROCESSES AND MODELING:

POSTER PRESENTATIONS

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HOW TO POSE IT: DEVELOPING A PROBLEM-POSING FRAMEWORK

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Posing problems is an important mathematical activity. In fact, Halmos (1980) views problems as the essence of mathematics and he, as well as other mathematicians (e.g., Polya, 1973) and mathematics educators (e.g., Brown & Walter, 1983, 1993; Silver 1994), argue that we should prepare our students to become better problem posers. It is not surprising then that professional organizations (e.g., Australian Education Council, 1991; National Council of Teacher of Mathematics [NCTM], 1989, 1991, 2000) have called for increased attention for students to be given opportunities to "create new problems by modifying the conditions of a given problem" (NCTM, 1991, p. 95). The research community still continues to investigate the different aspects of teaching and learning how to pose mathematical problems (Felmer, Pehkonen, & Kilpatrick, 2016; Silver, 2013; Singer, Ellerton, & Cai, 2013, 2015).

To help my students and I to become better problem posers within dynamic geometry environments, I developed a problem-posing framework. The problem-posing framework includes the following systematic strategies to pose new problems by modifying the conditions of a given problem: reversing, proving, specializing, generalizing, extending, and further extending. The problem-posing framework has been a powerful tool that has helped both my students and I to create new problems related to a given problem within dynamic geometry environments. The initial problem from which we created new problems was the following: What type of quadrilateral has as vertices the points of intersection of the angle bisectors of the angles of a parallelogram? I will refer to this problem as the base problem.

During the poster presentation, I will display the problem-posing framework and illustrate its usefulness with some of the problems that I and my students have generated by systematically varying the attributes of the base problem. Examples of posed problems include the following:

Problem 1: The vertices of quadrilateral EFGH are the points of intersection of the angle bisectors of a quadrilateral ABCD. If EFGH is a rectangle, what sort of quadrilateral is ABCD? (Converse or reverse problem)

Problem 2: Let E, F, G, and H be the points of intersection of the angle bisectors of the angles of a rectangle. Prove that EFGH is a square or a point. (Special and proof problem)

Problem 3: What kind of quadrilateral has as vertices the points of intersection of the angle bisectors of the consecutive angles of a quadrilateral? (General problem)

Problem 4: Prove that the angle bisectors of the angles of a kite are concurrent. (Extended and proof problem)

Problem 5: The vertices of quadrilateral EFGH are the points of intersection of the consecutive exterior angles of an isosceles trapezoid ABCD. Characterize EFGH. (Further extended problem)

Problem 6: Let E, F, G, and H be the points of intersection of adjacent angle trisectors of the interior angles of a parallelogram. What type of quadrilateral is EFGH? (Further extended problem)

The author will also provide solutions to problems that are supported by proofs in some cases or conjectures supported by empirical evidence in other cases (such as geometric diagrams created with Dynamic Geometry Software or numerical examples).

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DOMAIN APPROPRIATENESS AND SKEPTICISM IN VIABLE ARGUMENTATION

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Keywords: Reasoning and Proof, Empiricism, Argument

Several recent studies have focused on helping students understand the limitations of empirical arguments (e.g., Stylianides, G. J. & Stylianides, A. J., 2009, Brown, 2014). One view is that students use empirical argumentation because they hold empirical proof schemes—they are convinced a general claim is true by checking a few cases (Harel & Sowder, 1998). Some researchers have sought to unseat students' empirical proof schemes by developing students' *skepticism*, their uncertainty about the truth of a general claim in the face of confirming (but not exhaustive) evidence (e.g., Brown, 2014; Stylianides, G. J. & Stylianides, A. J., 2009). With sufficient skepticism, students would seek more secure, non-empirical arguments to convince themselves that a general claim is true. We take a different perspective, seeking to develop students' awareness of *domain appropriateness* (DA), whether the argument type is appropriate to the domain of the claim. In particular, DA entails understanding that an empirical check of a proper subset of cases in a claim's domain does not (i) guarantee the claim is true and does not (ii) provide an argument that is acceptable in the mathematical or classroom community, although checking *all* cases does both (i) and (ii). DA is distinct from skepticism; it is not concerned with students' confidence about the truth of a general claim.

We studied how ten 8th graders developed DA through classroom experiences that were part of a broader project focused on developing viable argumentation. One important classroom task in the project was the Circle-and-Spots problem (Stylianides, G. J. & Stylianides, A. J., 2009, Brown, 2014), which was meant to develop DA and to provide a rationale for why empirical arguments are not considered viable. Semi-structured interviews were conducted, in which we provided students with the claim "For every whole number value of n, if you compute $7n - 1$ you will not get a perfect square," and "Thomas'" empirical argument that checked the first seven cases in the claim's domain. Students were asked questions such as if they thought the claim was true, whether the argument was viable, and what they would have to do to make the argument viable. Thematic analysis was used to develop themes among the student responses (*Braun & Clarke, 2006).*

After collapsing themes, we found that five of the ten students displayed robust understanding of DA. They said that Thomas' argument was not viable because it did not account for all cases in the claim's domain. All of them suggested both of the following ways to make Thomas' argument viable: (a) restrict the domain to just the seven cases that were checked or (b) find some sort of "equation," "pattern," or "relationship" to show why the claim was always true. Two students nonetheless expressed confidence that the claim was true, supporting our view that DA is distinct from skepticism. Two other students showed partial understanding of DA; the other three displayed empirical reasoning. The results provide evidence of how DA can develop in middle grades, and raise the question of how robustly DA can develop without students having significant prior experience with viable general arguments.

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EXAMINING JUSTIFICATION OF THIRD-GRADE CHILDREN WHEN THEY ENGAGED IN EQUAL-SHARING PROBLEMS

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Keywords: Elementary School Education, Fractions, Justification, Problem Solving

Justification is a crucial practice that involves conjecturing and justifying mathematical claims. Researchers have shown that elementary children are able to provide sophisticated arguments for their conjectures during generalizing activities (e.g., Ball, 1993; Reid, 2002). To understand the justification of children in the context of fractions tasks, this study examined four third graders' discourse when they justified their claims in solving equal-sharing problems.

Four third-grade children in the Midwestern United States participated in this study. Through prior interviews, they showed only limited understanding of fraction concepts, particularly part-whole relationships, as is typical for this age. Provided with a tablet, pencils, and paper, the participants worked in pairs to solve equal-sharing story problems (e.g., Mary, Sam, and you are sharing eight pizzas; how do you share the pizzas so that each of you gets the same amount?). During the videotaped sessions of each pair, I facilitated the participants' interactions and observed how they used their intuitive knowledge for justification. Here presents an example that illustrates the arguments of two children when they engaged with the aforementioned task:

> Amy: …3 wholes, 3 wholes, 3 wholes [repeats her answer with confidence]. **[claim]**
> Betty: 1, 2, …2 wholes…and 2 halves [counts the wholes and slices of pizzas]. **[claim]**
> Amy: Therefore the 2 halves equals 1 whole.
> Betty: But the 2 halves do not make 1 whole. One half is here, and it takes two more (halves) to make 1 whole [explains while pointing to Amy's drawing]
> Amy: [Writes down numerical values inside the wholes and parts of the pizza figure]. The 2 slices should be counted as 1 whole. Here one slice is 5, and there one slice is 5 [points to graphics]. 5 added by 5 is 10, so together is 10. Since each person has 2 wholes already, then (the total is) 10, 10, and 10. Therefore, it's 3 (wholes) (for each person).
> Betty: Mine is kind of different. I would put 15 and 15 [writes down 15 for each circle that represents a whole pizza]. Everyone wants 3 slices to make 15, but the two slices are not sufficient (to make a whole). Since each person would take 2 slices. Those make a 10.…Each person gets 15, 15, and 10.

These results demonstrate informal strategies the children employed for justification. Particularly, whole number magnitudes were used to reason and describe part-whole relationships. Further analysis of Betty's discourse found that she seemingly followed a reasoning pattern similar to a proving technique, namely proof by contradiction. This technique first assumes the opposite of a claim and then uses the established facts to invalidate the claim.

This study reveals that some third-grade children created distinct strategies to justify or refute their claims for fractions problems. The finding of Betty's reasoning approach resonates the findings of Reid (2002) that specified logical patterns the fifth graders implicitly used for generating and testing conjectures. Since many U.S. elementary textbooks were found lacking written justification tasks (Bieda, Ji, Drwencke, & Picard, 2014), it is imperative for elementary teachers to consider enacting such practice in typical problem-solving fractions tasks.

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A CONTENT FOCUSED TASK SCHEMATIZATION AROUND MATHEMATICAL MODELING PROBLEMS: QUANTITIES

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Keywords: Number Concepts and Operations, Modeling, High School Education

Researchers have developed design categories or classification schemes for modeling problems (Bock, Bracke, & Kreckler, 2015; Czocher, 2017; Maaß, 2010) to make appropriate task choices for the target group of students and mathematical objectives. The task design around mathematical modeling needed to be carefully studied to evoke students' modeling process so that those processes could be traced systematically (Albarracin, Arleback, Civil, & Gorgorio, 2019). The goal of this study aimed to provide a task classification system to examine secondary students' modeling behaviors and decision-making processes while they are engaged in mathematical modeling tasks that draw on the content of Quantities.

The choice of mathematical content for this work was deliberate for two reasons: (1) the content of Quantities under the Number and Quantity section (CCSSM, 2010) plays a foundational role in the development of advanced mathematical domains (e.g., algebra, functions, vectors); and (2) the content naturally implicates to study modeling since it requires choosing, interpreting quantity units, and defining appropriate quantities to create descriptive models while coordinating both mathematical and non-mathematical knowledge to solve problems.

National and international research resources were reviewed to compile a list of modeling tasks (85 tasks). Two task design heuristics were followed in this project. First, by adopting Maaß's (2010) modeling task design framework, I examined modeling tasks under the five categories: the scope of modeling (whole process or sub-process), the amount of data provided (superfluous, inconsistent, missing, matching), the nature of the task's relationship to reality (level of authenticity or artificiality), the contextual situation (personal, occupational, public, scientific), and the type of model used (descriptive or normative). I used these categories along with the target mathematics content. Second, following Czocher's (2017) task selection method, the modeling cycle (Blum & Leiß, 2007) was utilized to filter the tasks from the list in order to map each task with the anticipated stages and transitions of the modeling expected to be evoked.

The final list of modeling tasks (14 tasks) was evaluated and critiqued by a panel of mathematics educators and field-tested by two researchers in high school classrooms. The tasks ranged from targeting specific steps to whole steps of the modeling cycle (Blum & Leiß, 2007) to study closely how secondary students might move between mathematical modeling stages and how their cognitive resources might influence their problem-solving process. This task scheme can be used for identifying modeling problems for the use of one-on-one clinical interviews or implemented in classrooms for tracking the kinds of mathematical thinking among high schoolers.

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EXPLORING HIGH SCHOOL STUDENTS' VALIDATION METHODS IN THE MATHEMATICAL MODELING PROCESS

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Keywords: Modeling, Cognition, Advanced Mathematical Thinking, High School Education

Calls have been made for the need to understand and advance students' mathematical modeling behaviors (Cai et al., 2014). Validation is a crucial step of the modeling process that occurs when modelers compare their mathematical results to a real-world situation that they attempt to understand (Blum & Leiß, 2007), and helps them to decide whether the model needs revisions or fulfills the need of the problem (Zawojewski, 2013). In this study, I investigated how high school students evaluate and validate their models in the mathematical modeling process.

The framework for this study stems from the two integrated theoretical stances—embodied cognition perspective (Lakoff & Núñez, 2000) and cognitive mathematical modeling perspective (Kaiser, 2017). Students' model-based problem-solving is influenced by their internal resources (i.e., mathematical knowledge and beliefs) (i.e., Stillman, 2011) and external relationships with the environment and other individuals (Lesh & Doerr, 2003). Prior experiences might be difficult to communicate at times for students, but linguistic tools can be rich with representational elements (Kövecses & Benczes, 2010) that can be turned into a validation method in the mathematical modeling process (Czocher, 2018). As a result, students embody experiences, intuitions, and means to support transfer through language, thought, and action while engaging cognitive steps of the mathematical modeling process (Manouchehri & Lewis, 2017).

This research was a qualitative, descriptive account of the validation ways employed in the mathematical modeling problem-solving process by eight high school students from different grade levels (2 ninth-graders, 2 tenth-graders, 2 eleventh-graders, and 2 twelfth-graders). Each participant completed 4 interviews lasting approximately 1 hour each based on one-on-one think-aloud tasks (Ericsson & Simon, 1998) at a public university in a Midwestern state. The selection of the participants was deliberate, targeting variability in mathematical backgrounds and self-efficacy toward solving mathematical problems. The common requirement of the interview tasks was choosing and interpreting quantity units and defining appropriate quantities to create descriptive models while using both mathematical knowledge (i.e., estimation) and non-mathematical knowledge to solve problems. Each interview was audiotaped and transcribed. The ongoing data analysis focused on the categorization of the forms of reasoning employed by the students while they were evaluating and validating their models, and then deciding whether their mathematical models needed more revisions or not. A content analysis of the transcripts (Patton, 2002) was used to detect themes in the students' validation methods.

The preliminary findings provided evidence in support of two of the themes identified by Ferri (2006)—*knowledge-based validation* and *intuitive-based validation*—and described three additional validation elements of implementation that appear to support those identified two themes: *formalistic validating* (the focus is on abstraction, formulas, mathematical correctness), *realistic validating* (the focus is on references to real situations that are enriched with verbalized or visual representations), and *formalistic–realistic validating* (the focus is on a balance between formal mathematical aspects and reality-based aspects of the problem).

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PROMPTING QUANTITATIVE REASONING PATTERNS WITH MATHEMATICAL MODELING PROBLEMS

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Keywords: Problem-Solving, Number and Operation, Modeling, High School Education

Contributing to the call for improving secondary instruction, the National Council of Teachers of Mathematics (NCTM, 2000) emphasized that students should gain quantitative reasoning abilities that lead to work with quantitative information and be able to use formulas and graphs to represent how quantities in real-life phenomena are related to one another and change together. Ellis (2007) suggested that one way to support students' quantitative reasoning is by engaging them in problem-solving activities that require (a) exploring how changing initial quantities will affect the emergent quantities, (b) determining how to adjust the initial quantities while keeping the emergent quantities constant, and (c) determining how to adjust the emergent quantities with the initial quantities (p. 475). In that sense, mathematical modeling problems naturally provide an environment for fostering and nurturing quantitative reasoning skills (Carlson, Larsen, & Lesh, 2003; Thompson, 2011).

In this poster presentation, a partial report of a larger study, we examined two tenth-graders'—Carlos and Ahmad (pseudonyms)—quantitative reasoning patterns and quantification processes while they solved mathematical modeling problems. Each student was interviewed one-on-one and given four modeling problems. Each interview was approximately 60 minutes long, and the students were encouraged to explain their reasoning processes (Ericsson & Simon, 1998). In the data analysis, we adopted the quantitative reasoning in context (QRC) framework (Mayes, Peterson, & Bonilla, 2013), which has four elements: (a) the quantification act (QA), the ability to identify the mathematical objects and their unit measures; (b) quantitative literacy (QL), the ability to identify, compare, manipulate, and draw conclusions from variables; (c) quantitative interpretation (QI), the ability to discover patterns and trends; and (d) quantitative modeling (QM), the ability to create representations to explain the problem and to revise them based on their fit into reality (p. 130).

The initial findings indicate that both the students were comfortable when identifying variables and their unit measures. Both recognized that they had assigned numbers as assumptions. Two distinct patterns emerged when comparing and manipulating the unit measures throughout the four modeling problems: (a) when Carlos assigned numbers, he primarily used the smallest unit as a measure and made calculations from the part to the whole *(inductive thinking approach)* (Simon,1996), whereas Ahmad always simplified the whole unit to reach the smallest unit at the end and made calculations from the whole to the part (*deductive thinking approach*) (Simon, 1996), (b) those thinking approaches impacted their quantitative interpretations on the mathematical models they had created (i.e., tables and graphs). For example, while Carlos explained the patterns on his graphs in a *descriptive modeling way* (Maaß, 2010) as explaining or forecasting the real-life situation, Ahmad's explanations were solely focused on the generalized mathematical results and mathematical accuracy in a *normative modeling way* (Maaß, 2010). In the presentation, the excerpts will be shared under each reasoning pattern, and possible instructional implications will be discussed.

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MAKING SENSE OF SENSELESS THINGS: AN ENACTIVIST ANALYSIS OF HARMONY AND DISSONANCE IN PROBLEM SOLVING

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Keywords: Learning Theory, Learning Tools, Problem Solving, Embodiment and Gesture

Operating from an enactivist theory of cognition, this work seeks to understand the emergent nature of mathematical activity mediated by manipulatives. Enactivism takes a biological approach and theorizes that perception consists in perceptually guided action enabled by cognitive structures that emerge from recurrent sensorimotor patterns (Proulx, 2013; Varela, Rosch, & Thompson, 1992). These processes are non-linear, unfolding, and ongoing events where meanings emerge and transform in interactions, not inside of minds and bodies (Malafouris, 2013; Proulx, 2019). Further, one's way of knowing is driven by an evolutionary imperative to act in an adequate, fitting, and harmonious way with one's environment (Maheux & Proulx, 2015). This search for harmony leads to a *structural coupling* between the individual and their environment with the individual's history of recursive interactions playing a crucial role in *structurally determining* this course of evolution (Proulx, 2013).

This work seeks to elucidate the nature of emergent mathematical activity mediated by manipulatives by addressing the question, *"What role might manipulatives play in the emergent processes of sense making?"* To do so, we analyzed the activity of "Dolly" and "Lyle" as they aimed to make sense of the flip-and-multiply algorithm for fraction division in a problem-solving interview using a manipulative Dolly created for engagement with fraction concepts. The data comes from a larger study that is exploring how an open-ended and iterative design experience centered in Making (Halverson & Sheridan, 2014) might inform prospective mathematics teachers' (PMTs') pedagogy. We took a revelatory case study approach to analyze and transcribe the video data (Yin, 2014), and focused our analysis on the particular interactions aiming to coordinate meanings of fraction division in the manipulative and in the algorithm that presumably substantiates those meanings (Malafouris, 2013).

Our analysis illuminates the role manipulatives can play in establishing a notion of sense making that is grounded in embodied understandings. For example, although Dolly and Lyle arrived at the correct answer with the manipulative early in their problem solving, they were dissatisfied because it did not seem to *fit* with the answer they derived from the algorithm. Eventually, this dissonance gave way as they established harmony between the two, thereby revealing the compelling power that embodied tool use can have for altering a space of possible actions and consequently on sense-making activity. Our analysis also reveals what might be problematic about a pedagogical practice where a procedure is adequate and *sense making* is not the criteria for fit. The enactment of the algorithm was disrupted through use of a tool, ultimately leading to an authentic understanding of what it means to *do* fraction division. These findings further substantiate extant arguments for engaging mathematics learners in embodied, tool-mediated problem-solving activity in conjunction with the learning of procedures.

Acknowledgments

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TALKING IN MATHEMATICS – DO WE KNOW HOW?

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Keywords: Communication; Classroom Discourse; Teacher Education - Inservice / Professional Development; Mathematical Knowledge for Teaching

With communication being highlighted as a process standard in The Curriculum and Evaluation Standards for School Mathematics by the National Council of Teachers of Mathematics (NCTM) in 1989 (NCTM, n.d.), the notion of communicating mathematically (Pimm, 1991) has since become an increasing important yet demanding task both for students and for teachers. Moreover, the role of spoken communication and how it effectively helps in the development of mathematical thinking did not seem clear. On the one hand, it was often assumed that students know how and what to communicate in the mathematics classroom, i.e. mathematical communicative competence was assumed to be a given (Adler, 2002; Pimm, 1987; Sfard et al., 1998) when it is the exact opposite. On the other, teachers face the challenge of orchestrating and facilitating meaningful mathematical conversations with and for their students, as Sfard et al. (1998) argued that it is "an extremely demanding and intricate task" (p.51) for conversations (either orchestrated or spontaneous) to be meaningful or productive in the mathematics classroom.

As part of this poster presentation, the author has attempted to explore the value and process of spoken communication in the mathematics classroom; and surfaced some corresponding implications on the teaching (and learning) of mathematics. In particular, on the value of spoken communication in the mathematics classroom, Pimm (1991) has suggested how spoken communication can be considered as the pathway to written communication if used purposefully with the intent of acquiring the mathematics register with regard to the notion of communicating mathematically. As for the form of spoken communication in the mathematics classroom, Barnes' (1976) studies on classroom talk can be a possible source of reference in providing a frame to understand classroom talk which contributes to learning. Based on these ideas, a preliminary framework (Figure 1) is proposed with the intent of explaining why and how spoken communication (or mathematical talk) can contribute to the teaching (and thus learning) of mathematics. While it may not fully explicate the value and process of spoken communication in the mathematics classroom, this idea can be further explored and refined through future research, e.g. the use of the framework as a possible structure for teacher professional development activities, focusing on developing the necessary mathematical knowledge for teaching (Ball et al., 2008) to orchestrate and facilitate mathematics talk.

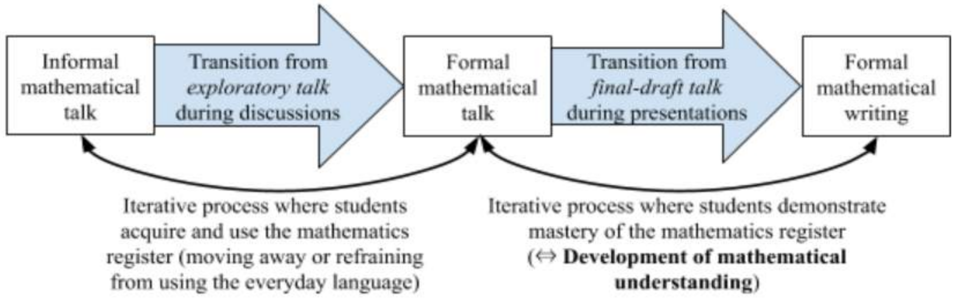

Figure 1: Spoken Communication as a Process in Mathematics Classrooms

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

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DEVELOPMENT OF THE MATHEMATIC MODELING PROCESS IN MATHEMATICS UNDERGRADUATE STUDENTS

DESARROLLO DEL PROCESO DE MODELACIÓN MATEMÁTICA EN ESTUDIANTES DE LICENCIATURA EN MATEMÁTICAS

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Posing of the Problem and Justification

In light of the distinct changes that modern society faces and the new "normal" it is necessary that individuals relate mathematics to their environment. Mathematical modeling is a process that allows one to determine a real-world problem, which is subsequently subject to observation and experimentation in order to obtain data and conclusions on said phenomenon (Villa-Ochoa y Ruiz, 2009). However, at the time of implementing the modeling in the classroom, it becomes evident that there are difficulties that do not permit its full development. Since modeling tends to be presented as a mathematical application (Villa-Ochoa et al., 2009) and therefore aspects such as observation and experimentation are left out of this process (Berrio, Peña y Torrenegra, 2018).

Methodology and Results

For this research an experimental activity was developed in the classroom, which consisted of rolling100 dice and removing those dice that landed on the number 5.

With the activity defined, we proposed an a priori analysis on how this process should have been developed (figure 1). The results obtained are shown in figure 2.

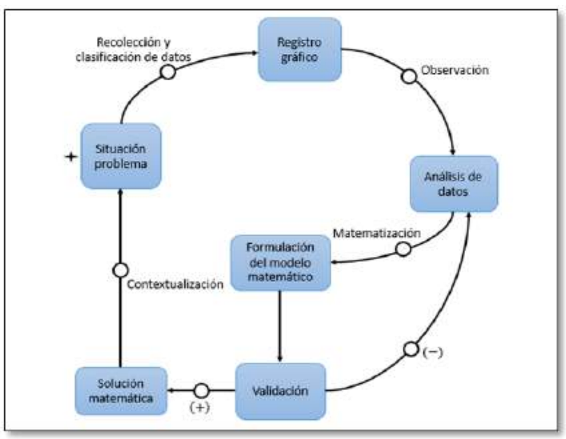

Figure 1: Berrio et al (2019)

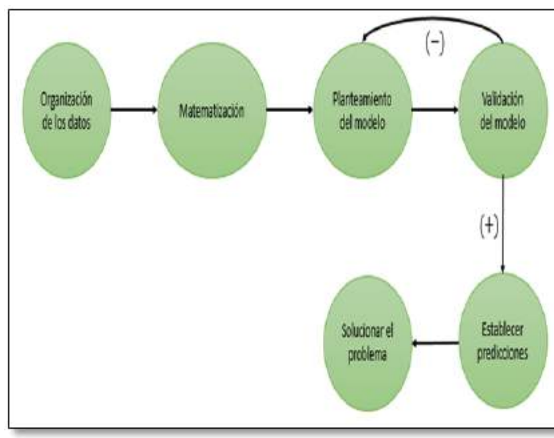

Figure 2: Berrio et al (2019)

Conclusions

- Experimental activities in the classroom give students a more concrete sense of reality, allowing inferences to be made outside the initial domain, which allows them to identify the limitations of the proposed model.

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- There are difficulties in solving problems of mathematical modeling on the part of students, because the notion of mathematical modeling that they have consists of a mathematical application or problem.

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DESARROLLO DEL PROCESO DE MODELACIÓN MATEMÁTICA EN ESTUDIANTES DE LICENCIATURA EN MATEMÁTICAS

DEVELOPMENT OF THE MATHEMATIC MODELING PROCESS IN MATHEMATICS UNDERGRADUATE STUDENTS

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Planteamiento del problema y justificación

Ante los distintos cambios que enfrenta la sociedad actual y la nueva “normalidad” se requiere de individuos que relacionen las matemáticas con su entorno. La modelación matemática es un proceso que permite determinar un problema de la realidad, que posteriormente es sometido a la observación, y a la experimentación con el fin de obtener datos y conclusiones sobre dicho fenómeno (Villa-Ochoa y Ruiz, 2009). Sin embargo, al momento de implementar la modelación en el aula, se evidencian dificultades que no permiten su completo desarrollo. Pues se tiende a presentar la modelación como una aplicación matemática (Villa-Ochoa et al., 2009) y por consiguiente aspectos como la observación y experimentación quedan por fuera de este proceso (Berrio, Peña y Torrenegra, 2018).

Metodología y resultados

Para esta investigación se desarrolló una actividad experimental en el aula que consistió en el lanzamiento de 100 dados y aquellos que quedaban con la cara superior en el número 5 se retiraban.

Con la actividad definida, planteamos un análisis a priori sobre cómo se debía desarrollar el proceso (figura 1) los resultados obtenidos se muestran en la figura 2.

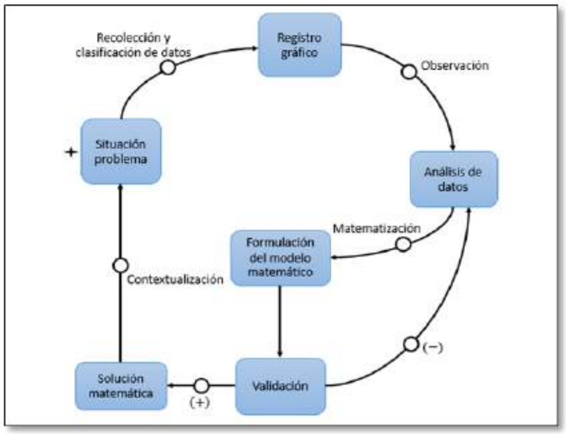

Figura 1: Berrio et al (2019)

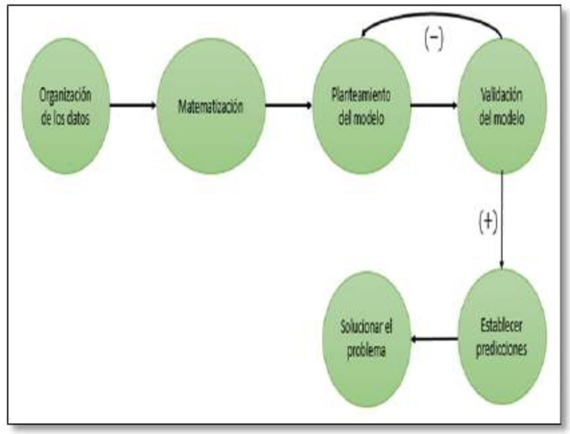

Figura 2: Berrio et al (2019)

Conclusiones

- Las actividades de experimentación en el aula, les da a los estudiantes una noción de realidad más concreta, lo que permite realizar inferencias fuera del dominio inicial, lo que les permite identificar las limitaciones del modelo planteado.
- Existen dificultades en la resolución de problemas de modelación matemática por parte de los estudiantes, pues la noción de modelación matemática que tienen consiste en una aplicación o problema matemático.

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DIDACTIC SEQUENCE FOR LEARNING PROBLEM SOLVING CONCERNING THE USE OF RELATIVE EXTREMA

SECUENCIA DIDÁCTICA: EL APRENDIZAJE DE RESOLUCIÓN DE PROBLEMAS REFERENTES AL USO DE EXTREMOS RELATIVOS

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Theoretical Approach

In Schoenfeld's (1985) Mathematical Problem-Solving (MPS) theory it is key to illustrate ideas, reflections of the use of diagrams and representations, heuristic strategies, and discussion of all the possible and distinct solution methods. For Santos (2014), the learning process is categorized into resources, prior mathematical knowledge; heuristic strategies, methods to reach a solution; metacognitive strategies, monitoring and regulation of the process itself for decision making; and beliefs, ideas about mathematics and how to solve problems. To verify and evaluate this process, Polya's four-phases of his Problem-Solving theory (1945) consisting of comprehension of the problem, design of a plan, carrying out the plan, and looking back has been followed; these particular questions provide information on the learning process.

Research Questions and Design

It was necessary to analyze the effects of a didactic sequence based on MPS theory applied on first-semester students of *Teoría del Cálculo I* from the LIMA programme at UdeG, for the learning of resources, strategies, beliefs and competences in mathematical Problem Solving. In this course, students are expected, in particular, to model the volume function applying the concept of Relative Extrema in a realistic scenario. The sequence consisted of a non-routine problem and working sheets with questions relating to Polya's phases. A group of 12 students were asked to work in triads to design a prototype of a container in the shape of a trapezoid using a 50cm x 65cm cardboard sheet; it had to guarantee the maximum possible volume taking into consideration that the trapezoidal face must be an isosceles trapezium containing the assigned acute angles (University of Colorado, s.f.)

Data Collection and Analysis

Since this is a qualitative phenomenological research, focus groups' activities were monitored and recorded; additionally, the answers to demi-structured interviews were analyzed through worksheets provided to the students. Altogether, they were triangulated with a matrix of categories and indicators based on the analysis of the data and the dimensions of Schoenfeld's theory.

Summary

According to data triangulation, students were able to follow Polya's solving problem phases and showed a satisfactory performance according to MPS theory. The most relevant effect of the sequence was the fact that the contextualization of the problematic proved beneficial to the development of the metacognitive process by giving students guidance in the decision making of the procedure to follow in the operations to perform, and in the direction to reach a solution. In general,

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

the sequence of a mathematical non-routine problem influenced positively in the learning of the concept of Relative Extrema, in the development of metacognitive strategies, and in modelling skills.

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SECUENCIA DIDÁCTICA: EL APRENDIZAJE DE RESOLUCIÓN DE PROBLEMAS REFERENTES AL USO DE EXTREMOS RELATIVOS

DIDACTIC SEQUENCE FOR LEARNING PROBLEM SOLVING CONCERNING THE USE OF RELATIVE EXTREMA

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Perspectiva Teórica

En la teoría de resolución de problemas matemáticos (RPM) de Schoenfeld (1985) es clave ilustrar ideas, reflexiones sobre el uso de diagramas y representaciones, estrategias heurísticas y discutir todos los métodos de solución posibles. Para Santos (2014), el proceso de aprendizaje se clasifica en recursos, conocimiento matemático previo; estrategias heurísticas, métodos para llegar a una solución; estrategias metacognitivas, monitoreo y regulación del proceso mismo para la toma de decisiones; y creencias, ideas sobre las matemáticas y sobre cómo resolver problemas. Para verificar y evaluar este proceso, se han seguido las cuatro fases de Polya de su teoría de Resolución de Problemas (1945) que consiste en la comprensión del problema, el diseño de un plan, la ejecución del plan y la visión retrospectiva; sus preguntas características proporcionan información sobre el proceso de aprendizaje.

Preguntas de Investigación y Diseño

Se requiere analizar los efectos de una secuencia didáctica basada en la teoría RPM aplicada en estudiantes de primer semestre de *Teoría del Cálculo I* del programa de la LIMA en UdeG, para el aprendizaje de recursos, estrategias, creencias y competencias en resolución matemática de problemas. En particular, se espera que los estudiantes modelen la función de volumen aplicando el concepto de Extremos Relativos en un escenario realista. La secuencia consistió en un problema no rutinario y hojas de trabajo con preguntas relacionadas con las fases de Polya. Se pidió a un grupo de 12 estudiantes que trabajaran en tríadas para diseñar un prototipo de un contenedor en forma de trapecio usando una lámina de cartón de 50 cm x 65 cm; tenía que garantizar el máximo volumen posible teniendo en cuenta que la cara trapezoidal debe ser un trapecio isósceles que contenga los ángulos agudos asignados (Universidad de Colorado).

Técnicas y Análisis de Recopilación de Datos

Al tratarse de una investigación cualitativa fenomenológica, se monitorearon y registraron las actividades de los grupos focales. Las respuestas a las entrevistas semiestructuradas se analizaron a través de las hojas de trabajo proporcionadas a los estudiantes. Además, la información recopilada se trianguló con una matriz de categorías e indicadores basados en el análisis de los datos y en las dimensiones de la teoría de Schoenfeld.

Resumen de Hallazgos

Según la triangulación de datos, los estudiantes lograron seguir las fases de resolución de problemas de Polya y mostraron un desempeño satisfactorio de acuerdo con la teoría RPM. El efecto más relevante de la secuencia fue que la contextualización de la problemática benefició el desarrollo del proceso metacognitivo al brindar orientación a los estudiantes en la toma de decisiones del procedimiento a seguir, operaciones a realizar y la dirección a la solución. En general, la secuencia de un problema matemático no monótono influyó positivamente en el aprendizaje del concepto de Extremos Relativos, en el desarrollo de estrategias metacognitivas y en las habilidades de modelado.

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SECONDARY TEACHERS' PRACTICAL RATIONALITY OF MATHEMATICAL MODELING

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Keywords: Mathematical modeling, practical rationality, norms, obligations

Various policy leaders and educational scholars have advocated for teachers to implement mathematical modeling in their classrooms (e.g., Common Core State Standards in Mathematics, 2010). In spite of arguments in favor of modeling, the implementation of modeling is still relatively rare in teaching practice in most countries including the United States (Blum, 2015). What makes modeling not viable or desirable from the perspective of secondary mathematics teachers? In what ways do teachers' day-to-day routines or views of their professional responsibilities align or not with modeling? In this study, we employed *practical rationality* as a framework for examining how teacher decisions are rationalized at the level of the instructional situation and to further understand the potential challenges of enacting modeling in classrooms.

The *practical rationality* approach suggests that instructional norms and professional obligations come into play in teachers' instructional decisions (Herbst & Chazan, 2012). Teachers view their role as entwined with obligations to different stakeholders corresponding to four sources of obligations: *disciplinary obligation*, *institutional obligation*, *individual obligation*, and *interpersonal obligation*. This study presents an analysis of the norms that are perceived by secondary teachers in relation to modeling and the professional obligations that they use to justify their departure from or alignment to the associated norm through the use of a scenario-based survey.

Secondary mathematics teachers (n=176) from the Midwestern United States participated in the study, varying in terms of their experience of teaching different courses and experience of enacting modeling tasks. They were randomly assigned to one of two groups. Each group includes four narrative sets in the situation of modeling. Within each narrative set, teachers were asked to choose what they would do next, presenting three options that included a hypothesized normative instructional action (e.g., close off the opportunities for students to use their everyday life knowledge) and two less typical actions (e.g., encourage students to bring in their background experiences). These hypothesized actions are based on prior studies of the enactment of modeling tasks (e.g., Leiß, 2007), non-traditional tasks (e.g., Herbst, 2003), and word problems (e.g., Chazan, Sela, & Herbst, 2012), and classroom observations in the United States.

Our findings show that while 68.2% of teachers chose to give clear directions for factor selection for their students and 94.3% of teachers expected students to find a symbolic representation (e.g., functions) as their final product of modeling, these teachers felt strongly obligated to teach the underlying mathematical concepts and the properties. In addition, only 34.1% of teachers were amenable to emphasize the social justice aspect of the task. However, for those who chose not to attend to social justice issues, they felt strongly obligated to interpersonal obligations (e.g., maintain a classroom environment that is conceived to learning) and disciplinary obligations (e.g., teach a valid representation of the mathematical knowledge and practices). The findings imply that the environmental impacts (e.g., disciplinary obligations) on the instructional practice of mathematical modeling should be taken into account. In the poster session, we will further illustrate how practical rationality can be used to better understand how teacher decisions are rationalized in mathematical modeling.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

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MISCELLANEOUS TOPICS

RESEARCH REPORTS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

NUMERICAL ESTIMATION SKILLS, EPISTEMIC COGNITION, AND CLIMATE CHANGE: MATHEMATICAL SKILLS AND DISPOSITIONS THAT CAN SUPPORT SCIENCE LEARNING

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Texts presenting novel statistics can shift learners attitudes and conceptions about controversial science topics. However, not a lot is known about the mechanisms underlying this conceptual change. The purpose of this study was to investigate two potential mechanisms that underlie learning from novel statistics: numerical estimation skills and epistemic cognition. This research investigated two treatments—a numerical estimation and epistemic cognition intervention—that were expected to enhance people's ability to make sense of key numbers about climate change. Results indicated that undergraduate students (N = 516) who were given instruction on numerical estimation strategies before shown novel climate change statistics had fewer misconceptions when compared with people who did not. Findings provide emerging evidence that supporting mathematical reasoning skills can enhance conceptual change in science.

Keywords: Numerical Estimation, Epistemic Cognition, Conceptual Change, Plausibility Judgments, Mathematics For Sustainability

Now more than ever, people need to be skeptical of the information that they encounter online. Inaccurate, self-authored misinformation is being created and circulated at an alarming rate (see, e.g., Allcott, Gentzkow, & Yu, 2019; Kata, 2012). Internet searches for controversial science topics like climate change, genetically modified foods, and vaccinations reveal millions of articles, much of which include scientifically incorrect information (e.g., Kortum, Edwards, & Richard-Kortum, 2008; Scheufele, & Krause, 2019); and much of this misleading information relies on misleading data.

Numerical data (e.g., statistics) found in the news can be a powerful tool for conceptual change, whether that change is for better or for worse. On the one hand, prompting people to estimate just a handful of statistics about climate change and then presenting them with the actual value can shift their attitudes, beliefs, and misconceptions to be more aligned with scientists (Ranney & Clark, 2016). On the other hand, presenting people with misleading statistics can shift their scientifically correct conceptions and attitudes to be less aligned with those of scientists (Ranney & Clark, 2016). Taken as a whole, this research suggests that statistics can be used as a catalyst for conceptual change. However, the mechanisms that underlie this change process remain understudied.

The purpose of this study was to examine mechanisms that underlie the learning that occurs when people encounter novel statistical information. Namely, I draw from theory on conceptual change (Dole & Sinatra, 1998, Lombardi, Nussbaum, & Sinatra, 2016), and epistemic cognition (the active reflection on whether information is true or justified; Chinn, Rinehart, & Buckland, 2014) to examine the impact of two mechanisms of conceptual change when learning from real-world numbers—numerical estimation skills and epistemic cognition.

Theoretical Framework

Conceptual Change

When individuals encounter statistics in the news or online that conflict with their prior conceptions, conceptual change may occur. Conceptual change represents a particular kind of learning that occurs when new information conflicts with a learners' background knowledge, leading

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to a restructuring of conceptual knowledge (Dole & Sinatra 1998; Murphy & Mason, 2006). Conceptual change researchers tend to describe concepts as either consistent or inconsistent with the understanding of experts and many define conceptual change as a correction of scientifically inaccurate conceptions, or *misconceptions*. For example, if a person holds the misconception that scientists believe that humans are not responsible for climate change and reads a statement that "97% of scientists agree that climate change is caused by humans," then there may be potential for the learner to question their idea and shift them to be more consistent with scientists. In this way, a single number has the potential to instigate conceptual change. Of course, there are many contributing factors and processes left unexplained in in this simplistic example, as conceptual change can be viewed as a process that is contingent upon people's motivation, emotion, and attitudes—factors that are often called *warm constructs* (see Dole & Sinatra, 1998; Pintrich, Marx, & Boyle, 1993; Sinatra, 2005; Sinatra & Seyranian, 2016). As such, the extent to which people engage with and learn from numerical data may be influenced by motivational factors such as their beliefs about their ability to succeed in mathematics (*self-efficacy*; e.g., Bandura, 1997), or emotional factors such as their trait-level anxiety associated with engaging in mathematics (*mathematics anxiety*; e.g., Ramirez, Shaw, & Maloney, 2018).

Plausibility judgments for conceptual change. When individuals encounter a novel statistic, they may implicitly or explicitly judge whether that information is plausible and then shift their conceptions accordingly. Research on plausibility judgments for conceptual change offers a useful frame for investigating these shifts in understanding. The Plausibility Judgments for Conceptual Change model (PJCC), posits that novel information (like novel statistics) can incite conceptual change because they prompt learners to appraise or reappraise the plausibility of their existing beliefs (Lombardi et al., 2016). When people encounter a novel explanation like a statistical figure, they first pre-process the information (e.g., by employing numerical estimation skills to judge the reasonableness of the number), and then make a judgment of the plausibility of the conception supported by the new information. Plausibility judgments can be either implicit or explicit. The extent to which people explicitly evaluate the plausibility of a conception depends, in part, on their views about knowledge (epistemic motives and dispositions); more explicit plausibility evaluations are thought to lead to greater potential for conceptual change—but only if the learner finds the new conception to be more plausible than their previous conception. That is, learners process statistical information and then appraise the plausibility of their initial conceptions based on this information; learners that find a novel conception more plausible than prior conceptions have higher potential for conceptual change.

Numerical Estimation

One way that learners process numbers is by estimating whether they are reasonable (e.g., Reys & Reys, 2004). Research on measurement estimation concerns the explicit estimation of real-world measures (Bright, 1976; Sowder & Wheeler, 1989) and is useful for understanding factors that help people judge whether real-world quantities are reasonable. Findings suggest that peoples' estimation accuracy and judgments of reasonableness improve when they use measurement estimation strategies, such as the benchmark strategy—the use of given standards and facts that can be applied by the learner through mental iteration and proportional reasoning to better estimate and judge the plausibility of real-world quantities (e.g., Brown & Siegler, 2001; Joram et al., 1998). For example, a person's estimate of the number of jellybeans in a container is likely to be more accurate and they will be a better judge of reasonableness of other peoples' guesses if they are first told the number of jellybeans in a different container. Measurement estimation strategies may therefore support people's comprehension and evaluation of given real-world quantities.

Epistemic Cognition

Epistemic cognition is the thinking that people do about knowledge and knowing (Chinn, et al., 2014; Sandoval, Greene, Braten, 2016) and is hypothesized to predict the extent to which learners evaluate the plausibility of a claim in light of new information (Lombardi et al., 2016). There are multiple models of epistemic cognition (for a review, see Sandoval et al., 2016), but for the purpose of this study, I draw from the AIR model of epistemic cognition (Chinn et al., 2014). According to this model, epistemic cognition is considered to be a situated process that relies on individuals' Aims (goals and associated values of goals), Ideals (espoused standards for achieving epistemic aims), and Reliable processes for knowing (schema for producing true, justified beliefs; Chinn et al., 2014).

An Existing Learning Intervention: EPIC

Prior classroom and laboratory studies have demonstrated the impact of presenting people with surprising numbers about controversial topics on their understanding of social issues (for reviews, see Ranney et al., 2019; Yarnall & Ranney, 2017). Many of these studies are grounded in a paradigm called "Numerically Driven Inferencing" (NDI, Ranney, Cheng, Garcia de Osuna & Nelson, 2001; Ranney & Thagard, 1988), which assumes that individuals' understanding of numerical information is connected to their knowledge, attitudes, and beliefs about larger issues. One of the central techniques from this perspective is called EPIC, an acronym for an intervention which introduces novel numerical information by prompting learners to Estimate quantities, state a Preference for what they would like the quantity to be, Incorporate the answer, and then Change their preferences afterward (e.g., Ranney & Clark, 2016; Rinne et al., 2006). Studies that use EPIC often operationalize conceptual change in terms of shifts in the preferences that individuals state for given numbers (i.e., differences between the "P" and the "C" in EPIC).

In sum, I contend that in order for learners to select high quality content from which to learn, they must develop skills to evaluate epistemic aspects of new information and also develop estimation skills necessary to accurately evaluate the statistics that they encounter along the way. That is, they must learn epistemic cognition and numerical estimation skills. Currently, there is little to no empirical research that investigates the role of estimation skills and epistemic cognition in conceptual change processes. My research is therefore guided by five questions:

1. To what extent does estimation of and exposure to novel statistics regarding climate change (i.e., an adapted EPIC intervention) shift learners' knowledge of climate change?
2. To what extent does enhancing this intervention with instruction on estimation strategies change learners' knowledge of climate change?
3. To what extent does enhancing this intervention with prompts to activate epistemic aims change learners' knowledge of climate change?
4. Is there an interaction between estimation skills and epistemic thinking on conceptual change?
5. To what extent do warm constructs (i.e., mathematics anxiety, mathematics self-efficacy, epistemic dispositions, and reported surprise from reading statistical information) mediate relations between pre- and post-intervention knowledge?

Methods

To answer my research questions, I formed a nationally representative Qualtrics panel of 516 undergraduate students to participate in an experimental online survey. Participants' median reported age was 20years, and 81% identified as Female, 64% White, 11% African American, 9% Asian, 9% Hispanic, and 43% as either Liberal or Very Liberal. All participants (a) completed a pretest to measure their misconceptions about climate change, mathematics self-efficacy and anxiety, and prior epistemic dispositions, (b) were randomly assigned to one of five conditions created by a control group and combinations of two interventions (see below), and (c) completed an identical post-test of knowledge and a demographics questionnaire.

Outcome Measure

Knowledge. Knowledge of human-induced climate change was a primary outcome in this study and was measured using seven items from the 28-item human induced climate change knowledge questionnaire (HICCK; Lombardi, Sinatra, & Nussbaum, 2013). Construct and content validity of the abbreviated scale was established through pilot studies and cognitive interviews (see Thacker, 2020). The knowledge questionnaire was given to participants just prior to and immediately after instruction and was intended to measure participants' conceptions about the consensus on human-induced climate change and were selected to align with information presented in the EPIC intervention. For example, participants rated their agreement with statements such as, "greenhouse gas levels are increasing in the atmosphere" on a scale from 1 (*strongly disagree*) to 5 (*strongly agree*). The measure at pre and posttest was reliable at conventional levels (Cronbach's alpha = .85 pre, .88 post).

Covariates

Mathematics Self-Efficacy and Anxiety Questionnaire (MSEAQ). Participants mathematics-specific self-efficacy and anxiety were measured using the Mathematics Self-Efficacy and Anxiety Questionnaire (MSEAQ; May, 2009). The MSEAQ consists of 28 items that can be divided into two subscales, mathematics self-efficacy (13 items) and mathematics anxiety (15 items). Construct validity was established in a prior study using factor analytic methods with an online sample and by establishing strong correlations with a classic measures of mathematics anxiety (s-MARS) and mathematics self-efficacy (see May, 2009). The instrument was shown to be reliable overall (Cronbach's Alpha = .96), as were the two subscales for mathematics self-efficacy (Cronbach's Alpha = .94) and mathematics anxiety (Cronbach's Alpha =.93). Average scores for the two subscales were computed and used in mediation analyses.

Epistemic dispositions. Baseline epistemic dispositions were measured using the Actively Open-Minded Thinking scale (AOT; Stanovich & West, 1997). The AOT is a measure of epistemic dispositions toward knowledge that consists of seven items. Participants reported their agreement with five statements (e.g., "Changing your mind is a sign of weakness") on a scale from 1 (*completely disagree*) to 7 (*completely agree*). The Chronbach's alpha was found to be .70 with the main analytic sample. The AOT was included in mediation analyses to observe whether epistemic dispositions mediate conceptual change outcomes, as inferred from the Plausibility Judgments for Conceptual Change model (Lombardi et al., 2016).

Surprise. Participants in the main analytic sample who were assigned to estimate quantities about climate change by way of the EPIC intervention were also prompted to report their sense of surprise after being shown the true values. Namely, participants were asked to "Rate how surprised you are by this number" on a scale from 1 (*not at all*) to 7 (*extremely surprised*). Surprise ratings had Cronbach's alpha = .82. Similar to prior research (e.g., Munnich et al., 2007), I expected that participants' sense of surprise from exposure to novel statistics would correspond with change in climate change beliefs. Participants in the control group did not estimate climate change numbers and therefore were not prompted to report surprise.

Interventions and Experimental Conditions

Participants were randomly assigned to one of five conditions: (1) a control group in which participants were presented with an 817 word expository text about the greenhouse effect (2) the EPIC task; (3) the EPIC task accompanied with an estimation skills modification that presents learners with strategies for using the given "hints," (4) an EPIC task accompanied with an epistemic cognition modification, or (5) an EPIC task accompanied by both estimation and epistemic cognition modifications. These interventions and modifications are described below.

The EPIC task required learners to estimate 12 climate change-related quantities before being presented with the scientifically accepted answer. Six of these items were taken from Ranney & Clark (2016) and asked participants to estimate unitless proportions. The remaining six were created by Thacker (2020) to be more mathematically challenging, requiring participants to estimate raw units of length, area, volume, mass, and temperature and included a "hint" that might be rescaled to better estimate the unknown quantity (see Table 1 for sample items).

The estimation skills modification consisted of a 132-word text that provided direct instruction on how to use the "hints" embedded in half of the EPIC items to more accurately estimate unknown numbers followed by two interactive examples (see Table 1 for an excerpt). The epistemic cognition modification was intended to activate epistemic aims and consisted of an open answer text-box that appeared after each of the twelve number estimates, prompting participants to "...reflect on the differences between your estimate and the true value. How does the true value change what you know about climate change or the way you think about climate change? Explain." This prompt was intended to activate epistemic aims.

Table 1. Sample Items from the EPIC Intervention and Modifications to the Intervention.

Sample EPIC Items			
Source	# of items	Sample item	Correct Answer
Ranney & Clark (2016)	6	What is the change in percentage of the world's ocean ice cover since the 1960s? (units in %)	40% Decrease
Thacker (2020)	6	What was the average Arctic Sea ice thickness in 2008? *Hint: Arctic ice thickness was 3.64 meters in 1980*	1.89 meters
Excerpt from Numerical Estimation Strategies Modification			
Numbers that you already know can help you estimate numbers that you do not know. For example, if you know that about 300 pennies fit in a small, 8oz milk carton, you can use this information to estimate the number of pennies that fit in a larger container… When using benchmarks, you may want to round values to make mental computation easier. For example...			
Excerpt from Epistemic Cognition Instruction Modification			
...Please reflect on the differences between your estimate and the true value. How does the true value change what you know about climate change or the way you think about climate change? Explain.			

Results

Preliminary analyses revealed no significant differences in pre-intervention knowledge between conditions (F = 1.54, p = .187). Skew ranged from -.78 to -.34 and kurtosis ranged from .01 to .36 for the revised knowledge measure though both failed the Shapiro-Wilk normality test (p < .001 for both pre- and post-knowledge), as such, both classic and robust analyses are presented. An initial omnibus test revealed significant differences between the five conditions when the seven-item knowledge score at post-test was used as the main outcome (F = 3.126, p = .0147). This finding was corroborated with nonparametric ANOVA analyses using a Kruskal-Wallis rank sum test (Kruskal-Wallis Chi-squared = 17.18, df = 4, p = .001). Raw means and standard deviations by condition and overall for all variables are shown in Table 2.

Table 2. Descriptives by Condition for the Main Analytic Sample of N = 516 Undergraduate Students.

	Min, Max	Alpha	Full Sample (n=516)		Control (n=103)		EPIC (n=103)		EPIC+EC (n=103)		EPIC+EST (n=104)		EPIC+EC+EST (n=103)	
			Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
Knowledge (Pre)	1, 5	.85	3.88	0.60	3.82	0.58	3.98	0.56	3.84	0.66	3.93	0.53	3.81	0.66
Knowledge (Post)	1, 5	.88	4.08	0.75	3.88	0.66	4.20	0.72	4.06	0.78	4.19	0.68	4.06	0.80
Knowledge Gain (Post − Pre)	-4, 4	na	0.20	0.57	0.06	0.42	0.22	0.50	0.22	0.73	0.26	0.49	0.25	0.65
Active Open Mindedness	5, 5	.70	4.84	0.86	4.74	0.81	5.03	0.89	4.83	0.83	4.83	0.89	4.78	0.85
Mathematics Self-Efficacy	1, 5	.94	3.27	0.87	3.31	0.97	3.20	0.81	3.33	0.83	3.24	0.86	3.29	0.87
Mathematics Anxiety	1, 5	.93	2.98	0.86	2.98	0.90	2.95	0.84	2.98	0.90	2.95	0.81	3.03	0.87
Surprise (in Reaction to EPIC Items)	1, 5	.82	2.83	0.73	NA	NA	2.79	0.67	2.83	0.75	2.85	0.81	2.87	0.71

Control versus all other conditions (RQ1). To address my first research question, I used contrasts to assess the knowledge of the control group compared with the combined average of the remaining four groups. A Welch's two sample t-test revealed significant differences in mean post-intervention knowledge between control (M = 3.88) and EPIC conditions (M = 4.12, t = 3.23, p = .001, Cohen's d = .33), as did Yuen's method of trimmed means, bootstrapped T, and bootstrapped medians (all $p <$ .009). In other words, students assigned to the EPIC conditions performed about one third of a standard deviation better on the seven-item knowledge posttest when compared with the control.

Estimation intervention versus no estimation intervention (RQ2). To address my second research question, I first dropped the control from analysis to consider only the four EPIC conditions, and then used planned contrasts to compare those who were given estimation instruction with those who were not. A Welch's two sample t-test revealed a marginally significant and positive impact of the estimation intervention on post-intervention knowledge (b = .09, SE = .05, p = .086). After adjusting for prior knowledge, nonparametric ANCOVA methods using a Thiel-Sen estimator revealed significant differences in post-intervention knowledge scores for those at the upper third (*Difference* = .31, *95% CI* = 0.04-0.58) and fourth (*Difference* = .17, *95% CI* = 0.08-0.27) of five evenly spaced points along the range of prior knowledge, a range that includes 67% of the analytic sample. In other words, the estimation intervention appeared to be effective in shifting knowledge for participants on the upper end of the prior knowledge range[1].

Epistemic cognition intervention versus no epistemic cognition intervention (RQ3). To answer my third research question, I again used contrasts to compare those who were given epistemic cognition prompts with those who were not after dropping the control from analysis. Contrasts

[1] Pairwise comparisons using the Benjimani-Hochberg method revealed significant differences between post-intervention knowledge scores when comparing the control and unmodified EPIC intervention (p = .022, Cohen's d = .46) and when comparing the control and EPIC supplemented with estimation strategy instruction (p = .026, Cohen's d = .46).

revealed no significant differences on the revised knowledge scale at post-test, even after adjusting for prior knowledge.

Tests for interactions (RQ4). To answer my fourth research question, I tested for main effects and interactions of the two modifications to the EPIC intervention. I first ran classic two-way ANOVAs followed by robust two-way ANOVAs using Johansen's heteroscedastic method for trimmed means (see Wilcox, 2017, Chapter 10). Both sets of tests revealed no significant main effects or interactions when post-intervention knowledge was the outcome.

Mediating role of warm constructs (RQ5). To explore relations between prior knowledge, warm constructs, and post-intervention knowledge, I tested a hypothesized model inferred from Lombardi and his colleagues (2016; presented in Figure 1a.) using maximum likelihood estimation with robust (Huber-White) standard errors and a scaled Yuan-Bentler test statistic in R using Lavaan 0.6-3 (Rosseel, 2012). The model resulted in acceptable fit at conventional levels (CFI = .99, TLI = .93, $RMSEA$ = .077; Hu & Bentler, 1999).

As expected, results revealed that warm constructs mediated relationships between prior- and post-intervention learning outcomes (see Figure 1b for all coefficients). Notably, I found indirect effects of prior knowledge on post-test knowledge through active open-minded thinking (*indirect effect* = .059, $p < .01$).

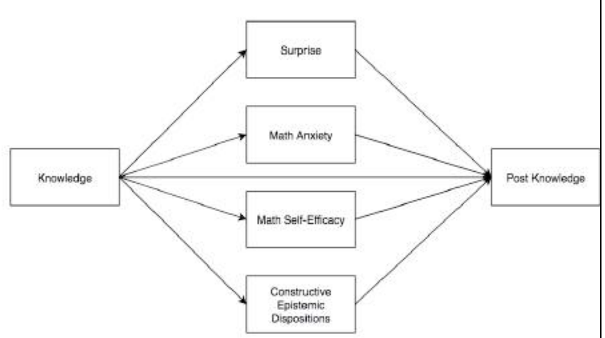

Figure 1a. Hypothesized model to address RQ5. Intercorrelations between warm constructs (surprise, mathematics anxiety, mathematics self-efficacy, and constructive epistemic dispositions) are included in the model but not shown in this figure.

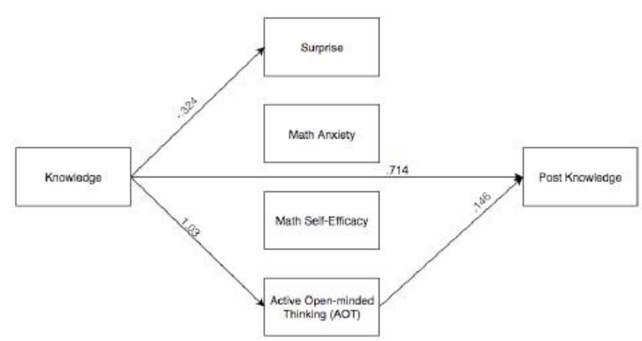

Figure 1b. Full path model. Only significant paths at the .05 level are shown. Intercorrelations between warm constructs were included in the model but not shown in this figure. Standard errors range between .010 and .132.

Significance

I sought to investigate whether the learning that occurs when people encounter novel statistics was enhanced with additional instruction on estimation strategies or prompts to activate epistemic aims. I found that students who learned from novel statistics performed about a third of a standard deviation better than a control group on a post-test of climate change knowledge, which is consistent with prior findings demonstrating the effectiveness of EPIC for climate change learning (e.g., Ranney & Clark, 2016; Ranney et al., 2019).

I also found that enhancing this intervention with numerical estimation instruction had a small but positive impact on students' science learning; an effect that was concentrated among students in the upper range of the prior knowledge distribution. These findings provide emerging evidence that numerical estimation skills can be leveraged for improved scientific learning. Future research support students' numerical estimation skills as applied to additional policy-relevant topics.

Findings also revealed that prompts to activate epistemic aims had no detectable effect on undergraduate students learning. To date, efforts to design micro-interventions intended to shift epistemic dispositions are only emerging. Only longer interventions spanning the duration of several weeks have yielded impacts on patterns of epistemic thinking (e.g., Lombardi et al., 2013; Chinn & Buckland, 2012). More research is needed to explore whether such an intervention is possible. Related to this, I found no significant interactions between intervention conditions, likely due to the very small and insignificant effects of the epistemic cognition intervention. With improved intervention design, future research might explore whether such an interaction might exist.

Though the brief online intervention created for this study was not found to *shift* learners' epistemic dispositions, learners' *baseline* epistemic dispositions were shown to be important mediators of conceptual change processes. Namely, a path model revealed that epistemic, motivational, and affective constructs were important predictors of conceptual change outcomes, as predicted by the Plausibility Judgments for Conceptual Change model (Lombardi et al., 2016), and that epistemic dispositions significantly mediated relationships between pre-intervention knowledge and post-intervention knowledge.

Conclusions

Findings from this study contribute to better understanding the extent to which individuals shift their conceptions about climate change based on just a handful of novel statistics and illuminate mechanisms that underlie such conceptual changes. Evidence that epistemic cognition, estimation skills, motivational, and emotional factors play a role in conceptual change provide empirical support for the Plausibility Judgments for Conceptual Change model (Lombardi et al., 2016). Findings also provide emerging evidence that mathematical knowledge can be leveraged for conceptual change regarding scientific topics. By creating and testing instructional interventions, this study also provides both mathematics and science instructors and those concerned with public understanding of science with a collection of strategies for better preparing people with skills to navigate the minefield of deceptive statistics found in today's online news landscape.

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MISCELLANEOUS TOPICS:

BRIEF RESEARCH REPORTS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

UTILIZING MATHEMATICS TO EXAMINE SEA LEVEL RISE AS AN ENVIRONMENTAL AND A SOCIAL ISSUE

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The phenomenon of the sea level rise is a pressing environmental and social issue of the present age. Starting with the assumption that mathematics can be utilized to help students explore this phenomenon, we designed a simulation in NetLogo, in which students investigated the relationships between the quantities of temperature rise, height of future sea level, and total land area. In this paper, we present the analysis of a whole-class design experiment in a sixth-grade classroom and discuss how our design helped students to examine sea level rise as both an environmental and a social issue.

Keywords: Social Justice, Interdisciplinary Studies, Technology, Mathematics for Sustainability

In 2001, The Intergovernmental Panel on Climate Change (IPCC) projected that the global sea level would rise up to .88 meters by 2100, which was only .09 meters in 1900 (Raleigh, Jordan, & Salehyan, 2008). Sea level rise would not only cause inundation and displacement of wetlands and lowlands, coastal erosion, and flooding (Nicholls & Mimura, 1998), it would also bear a severe impact on people residing in low-lying coastal areas (Rowley, Kostelnick, Braaten, Li, & Meisel, 2007) as these would be the first people to experience flooding. Further, damage of properties, loss of lives, and injuries caused due to increased sea level would disproportionately impact the poorer section of the society, who, despite being the least contributor to sea level rise, would be most vulnerable to its impact (Dodman & Satterthwaite, 2008). Lack of preparedness and financial limitation would make poor people more susceptible to the effects of sea level rise (Walker & Burningham, 2011). Hence, like any other climatic issue, sea level rise also qualifies as an issue of social injustice.

Climate Issues and School Curriculum

Research shows that the introduction of climatic issues in the school curriculum would help students as the future citizens to develop an awareness about and cultivate sensitivity towards the climate (Shepardson, Niyogi, Choi, & Charusombat, 2009). Mathematics education inarguably plays a significant role in the process. Mathematics literacy is not only necessary to identify the different traits that indicate climatic disruptions, but it also helps students to predict the future impacts of climate change (Barwell, 2013). Although school mathematics has traditionally modified itself and accommodated issues that marked the needs of the time, climatic phenomena have seldom been incorporated in mathematics textbooks or tasks (Renert, 2011). When Abtahi et. al. (2017) investigated Norwegian and Canadian mathematics teachers' opinion regarding inclusion of climatic issues in mathematics classroom, they found that even though the teachers acknowledge their moral obligation towards educating students about climate, they indicate that the complexity of climatic issues, the lack of students' mathematical and technical knowledge, and the lack of resources and time are some of the roadblocks towards implementation of climatic issues in mathematics classrooms. The study reported in this paper aimed to address those challenges by designing an interactive simulation and accompanying tasks and questioning that would help students explore the causes and consequences of sea level rise in a way that would make this complex phenomenon accessible to sixth grade students. Specifically, we report on how we assisted students to reason

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

covariationally about the quantities involved and how this reasoning helped students understand how their own city can be affected by the sea level rise.

Design and Methods

Most of the information about sea level rise in the news and public media is in the form of data and graphs. To support students' interpretation of data and graphs, this study focused on students' covariational reasoning about the quantities underlying the phenomenon. Covariational reasoning involves coordinating two quantities as the values of those quantities change (Confrey & Smith, 1994). A student reasons covariationally when she envisions two quantities varying simultaneously (Thompson & Carlson, 2017). For instance, as air temperature increases, the height of sea level also increases. To support students' understanding of the sea level rise, we designed an interactive simulation and a set of integrated activities that asked students to reason about the relationships of the quantities.

We designed the Sea Level Rise simulation using NetLogo (Wilensky, 1999), a multi-agent programmable modeling environment. We hoped that the dynamic environment of NetLogo, together with its animated outputs and result plots, would provide students with a self-exploratory space to change and reverse change the values of different quantities, which is not always practical with physical manipulations. Four cities familiar to the students were selected and arranged vertically according to their elevations from the sea level (Figure 1). The user can drag the temperature rise slider to the left and right, manipulate its value, and observe the impact of the change on the height of sea level and total land area. The simulation was accompanied by a set of activities and discussion questions that we hoped would prompt students to reason about different covarying quantities and identify the environmental and social aspects of sea level rise. For example, questions such as "What would happen to Manhattan if height of future sea level doubles?" not only required students to focus on the covariational relationship between height of future sea level and elevation of Manhattan, but also to identify the consequences of sea level rise on lives of people living at lower elevation, such as Manhattan.

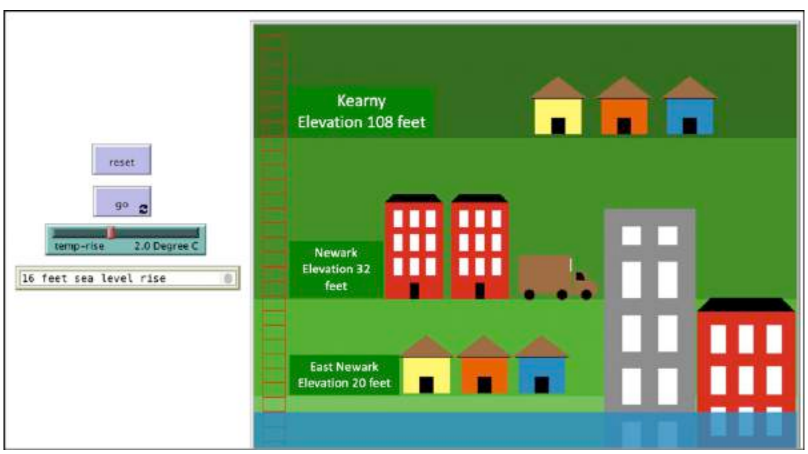

Figure 1: Sea level rise simulation

Our goal was to explore the ways that our design, which included engineering learning opportunities for students to reason covariationally, helped students to reason about sea level rise as an environmental and a social issue. More specifically, we examined the research question: *How did our design help students develop an understanding of sea level rise as an environmental and a social issue?*

This study took place in a public elementary school located in the North-Eastern part of the United States. We conducted a week-long design experiment (Cobb et. al., 2003) in a sixth-grade classroom containing 17 students. The teacher conducted the whole-class instruction and a research team member interacted with a small group of students. All the sessions were video recorded, transcribed, and coded using the software program Quirkos. In this paper, we focus on our interaction with a student named Ani to illustrate how our design helped students explore the phenomenon as an environmental and a social issue.

Findings

The Sea Level Rise simulation provided the students with a dynamic environment to drag the temperature rise slider and observe its impact on the height of sea level (Figure 2). For instance, when Ani was asked "What happens if I lower the temperature?," he dragged the temperature rise slider to the left and said, "the lower the height of sea level."

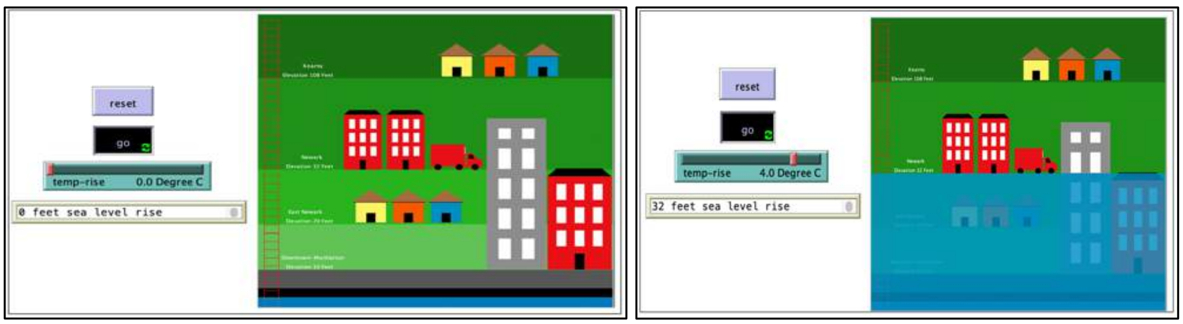

Figure 2: Temperature rise increases, height of sea level increases, total land area decreases

To prompt the students to reason numerically between the two covarying quantities, we asked them to graph the relationship between temperature rise and the height of future sea level. Students used the simulation to find the height of future sea level for different values of temperature rise and plotted the ordered pairs on a graph. When Ani was asked to explain the graph, he stated that the graph was "rising like super straight line" because "when temperature rises 0.5, it rises by 4 feet every time." From his response it seems that Ani attributed the "straight" shape of the graph to the constant increase of height of future sea level for a uniform change of temperature rise.

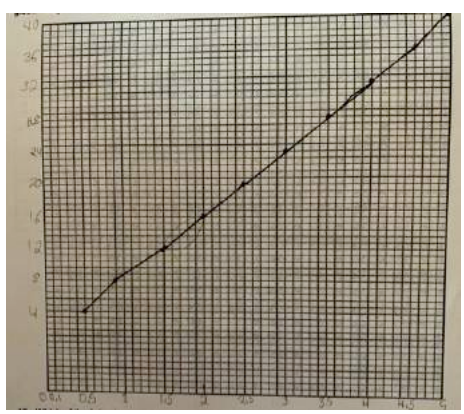

Figure 3: Ani's graph showing the relationship between global temperature (horizontal axis?) and height of future sea level (vertical axis)

To help students identify the consequences of sea level rise in their own lives, we encouraged them to think about the impact of sea level rise on total land area. When we asked Ani to state what would happen to the total land area if the sea level rises, he responded, "the less land, the total land area is going to be less." He further justified, "because the more higher the sea level is, it takes over land. So, instead of land over water, it will be under water." Through his reasoning, Ani identified the direction of change between the height of sea level and total land area. The graphics of the simulation (Figure 2) were powerful in helping Ani coordinate the direction of change of the two quantities. We further prompted Ani to think and explain why an increasing temperature results in a rise of sea level and a reduction of land area. Ani thought briefly and said, "The higher the global temperature, the higher the sea level. Rising the global temperature, the ice caps in the Antarctica will melt which makes more water to go into the water and sea level rises, which means less land area." Ani not only explicitly described the relationship between the three quantities but also identified melting ice caps in Antarctica as a consequence of increased temperature and a cause of the rising sea level.

In the Sea Level Rise simulation, the inclusion of the names of places familiar to the students helped them identify the consequences of sea level rise in connection to their own lives. Students were relieved to find themselves located at a higher sea level, compared to their neighboring towns of Newark and Manhattan. They identified that if sea level rises, then that will "cause places like…low elevation like Newark go under water." Students also expressed their anxiety about the lack of economic affluence of people to endure the impact of displacement caused by flooding. For example, during the small group conversation when we asked the students, "What is going to happen to our home (if sea level rises)?", Ani replied, "It is gonna be destroyed, and we cannot rebuild it." Further, he added that the situation would be different for rich people, since their homes would also be "Destroyed, but they can rebuild it." Ani resonated the argument of Dodman and Satterthwaite (2008) that climatic threats, such as sea level rise and flooding are issues of social injustice since they bear down a disproportionate impact on the people belonging to different socio-economic strata. The students' articulations "they can rebuild it" and "we cannot rebuild it" indicate that students recognized how low socioeconomic conditions of certain people limit their access to resources and opportunities to fight the impact of climatic disruption.

Conclusion

Consistent with Barwell's (2013) assertion, this study illustrates that students' mathematical reasoning provided them a platform to engage in a meaningful discussion around sea level rise. Students not only reasoned covariationally between rising air temperature, height of future sea level, and total land area, and examined the environmental aspect of sea level rise, they also explored the social aspect of the climatic phenomena. Students identified that economic disparity makes poor people more vulnerable to the risk associated with sea level rise (Dodman & Satterthwaite, 2008), while wealthy people possess both resources and financial stability to escape its impact. So, through this study we convey that incorporating climatic issues in mathematics classroom is complex, but it is high time that mathematics educators and researchers acknowledge their roles and responsibilities in empowering students mathematically and helping future citizens to become more sensitive towards the climate.

Acknowledgements

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SOURCES OF EVOLUTION OF UNIVERSITY STUDENTS' VIEWS ON MATHEMATICAL CREATIVITY

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In a world that is in increasing demand for creativity, mathematics courses and programs need to shift from more routine and computational to more creative and problem-solving focused. We present preliminary results of a qualitative research study in which we examined students' perceptions of mathematical creativity in an introduction-to-proofs course. We conducted interviews with students as well as collected their reflection assignments at the end of the semester. Using a definition of creativity from a relativistic perspective, we analyzed interview data to describe how students' perspectives of mathematical creativity evolved throughout the semester and the sources of those shifts. Students shifted from previously not seeing themselves, others, or mathematics as creative, to believing they are creative. The sources found in the data are related to content and course design.

Keywords: University Mathematics, Creativity, Affect, Emotion, Beliefs, and Attitudes

Introduction

Curriculum-standard documents, both in the United States and internationally, mention creativity as an important skill when learning mathematics (Askew, 2013). Additionally, creativity has become one of the most sought-after skills for academia and industry employers (World Economic Forum, 2016). While the mathematical creativity literature at the K-12 level is well-developed, there remain few studies at the undergraduate level and fewer still that investigate students' beliefs about creativity and its role in mathematics. In this qualitative study, we explored students' perceptions of mathematical creativity and how they evolved over the semester of an introduction-to-proofs course. Furthermore, we examine the sources of these shifts as evidenced by the students' own words.

Theoretical Perspective

As with many of our research projects on mathematical creativity (Tang et al., 2015; Savic, Karakok, Tang, El Turkey, & Naccarato, 2017), this study uses a *developmental* perspective of creativity (Kozbelt, Beghetto & Runco, 2010). This theoretical lens contends that creativity develops over time and emphasizes the role of the environment in the development of creativity. Such an environment should provide students authentic mathematical tasks and opportunities to interact with others (Sriraman, 2005).

We operationalize mathematical creativity as "a process of offering new solutions or insights that are unexpected for the student, with respect to their mathematical background or the problems [they've] seen before" (Savić et al., 2017; p.1419). This definition focuses on the *process* (Pelczer & Rodriguez, 2011) of creation, rather than the *product* that is created at the end of a process (Runco & Jaeger, 2012).

Background Literature

Moore-Russo and Demler (2018) examined the perceptions of U.S. faculty and staff participants from gifted mathematics programs and found that, through counts of coding using several creativity frameworks, mathematical creativity in education was more of a process than "a subjective experience" (p.23). This particular orientation allows us to keep a dynamic view rather than a static one to capture nuances in the individual's thinking. Furthermore, our definition takes a *relativistic* perspective—creativity relative to the student—in contrast to absolute creativity for the field of mathematics (Leikin, 2009). For example, Levenson (2013), using a similar viewpoint, focused on the discussion of ideas put forth by individual students and how these ideas helped in developing a product of collective mathematical creativity in fifth- and sixth-grade mathematics classrooms. Levenson also emphasized the teachers' roles in facilitating these discussions.

While there is literature on mathematicians' and mathematics instructors' perceptions on mathematical creativity (Borwein, Liljedahl & Zhai, 2014; Sriraman, 2009), research on students' perceptions on mathematical creativity as well as classrooms that impact these perceptions has received less attention. In one of our earlier studies, we examined university students' and mathematicians' definitions of mathematical creativity using three process categories: taking risks, making connections, and creating ideas (Tang, El Turkey, Savić, & Karakok, 2015). We found that students rarely associated making connections using different mathematical content with creativity compared to mathematicians (9% of students' responses compared to 38% of mathematicians' responses). This study alerted us to think about explicitly valuing and discussing the processes that are deemed to be important in developing mathematical creativity (El Turkey et al., 2018). In this paper, we explore the following research question: In what ways do students' views on creativity evolve in an introduction-to-proofs course which explicitly valued mathematical creativity?

Methods

Data were collected in an introduction-to-proofs course at a small liberal arts college in the Southwestern United States. This course was taught using an inquiry-based learning (IBL) pedagogy (Laursen et al., 2014), where students often worked on proofs in small groups and gave presentations to the class on proofs constructed both in class and for homework. The instructor explicitly valued creativity by making use of the Creativity-in-Progress Rubric (CPR) on Proving (Savić et al., 2017; El Turkey et al., 2018), a formative assessment tool developed by the authors that students can use to persevere in proving and encourage creative processes. The rubric has two main categories: making connections and taking risks (see Author, 2017 for a more detailed discussion of the CPR on Proving). The instructor gave assignments and exam questions where students had to use the rubric to assess their own or other's work.

At the end of the semester, 4 female and 3 male students agreed to be interviewed and participated in 60 to 90-minute semi-structured interviews. During the interview, students were asked to describe the course, discuss their views on creativity, and discuss the use of the CPR in the course. As part of a larger study, interviews were coded using hypothesis coding (Saldaña, 2013) with five categories, one of which being creativity. This is the coding category we focus on for this report. Three of the seven participants' transcripts were coded separately by the first and second author with 97% agreement. Because of this high degree of inter-rater reliability, the remaining transcripts were coded by only the first author.

Results

From three of the students interviewed (all of whom identified as female), an explicit shift in the way they thought about creativity or how they viewed themselves as creative people was reported. The students that reported an evolution in perspective on creativity were able to ascribe this to one of two sources: mathematical content and course design. In what follows, we show a sampling of student quotes where they indicate a shift in perspective and ascribe a reason to this change.

For instance, Stephanie (all names reported are self-chosen pseudonyms) spoke about content with respect to *learning new tools* to work with. That is, she feels that having a larger mathematical toolbox allows one to be more creative when proving or problem solving.

> I think I started to look at creativity a little bit different through this course...Prior to this it's been all very applied mathematics...So before, just using the trig equations to solve geometry was creative for me. Whereas now, this has just opened up a whole new door of opportunities for it because I can solve a proof using a contradiction, while somebody else used a contrapositive and somebody else used a direct proof and somebody else used induction, and we all do it completely different.

Whereas, Olivia attributed her shift to the social structure of the course. As the course included collaboration and presentation, Olivia reported that the environment was conducive for growth and students were able to see each other's creativity and began to feel more creative as the semester progressed.

> We kind of all went in with kind of not really feeling confident in our abilities to be creative, so it was really interesting to see students that were quiet, reserved early on like show their work later in the semester and they had done something like totally cool and amazing...So, I feel you know their ability, like their confidence levels went up and I could say that's true of me as well. So, I wanna say that it's, you know it wasn't that like all the creative people took this course because I didn't consider myself creative and I took the course, and I would say that that's probably true of other students as well.

In a later part of her interview, Stephanie echoed Olivia's comment almost exactly with her assessment of the course culture and its contribution to everyone's creativity.

> At the beginning of the semester, I think a lot of people in that class were very shy and quiet, and so it was kind of hard to judge where their creativity was because they weren't sharing it as much. Um, by the end of the course you had everybody speaking, you had everybody giving their opinions and how to work on things together, and you saw everyone grow. You saw everyone coming up with their own tools and tricks. And everyone was posing questions, not just the few of us that were outspoken to begin with. So, you definitely saw growth in the class, um not only with the shyness but with the creativity and coming up with their own ideas to change things and make them better.

The IBL practices of the course required students to present their work to each other. The instructor also especially encouraged multiple presentations on the same problem if different students approached the problem using different methods. Two of the interviewees spoke directly to this aspect of the course design as contributing to their own creativity. That is, this shift seems to be a result of *seeing others' work as creative and reflecting it back on themselves*. For instance, Peyton said:

> I really, I really did not feel like I was being creative at all throughout the course. It really was just things in my head, it makes sense that led to a conclusion that made sense. But, considering that I thought other people were exceptionally creative, I kind of thought that maybe they though that about me too.

In fact, Peyton had perhaps the starkest change in her beliefs on mathematical creativity and in seeing herself as a creative person. The following excerpt shows that Peyton started the semester believing that mathematics was not a creative subject and ended with a completely opposite viewpoint.

Interviewer: And in your reflections you said…'I think I am on the spectrum that generally believes that, believes there is no need for creativity in mathematics. That's been a key reason why I enjoy math. I know, I know if I get the answer then I have done it correct. There is a set process and if I learn the process then…I'll be successful'. So, do you wanna comment on that part?

Peyton: I…should have made that more in the past tense, because I believed that prior to taking this course…There has been, you can figure out problems and it's creative in the sense that you can figure out how, where you wanna start with the problem. But I like being able to know that if I am doing it correctly, the process correctly, then I will get to the answer… I enjoy knowing when I'm gonna do something correctly as opposed to just spending a lot of time and then not even knowing if it's gonna yield good results. But this course changed that quite a bit, because there really was no assurance that anything would be correct, but it still… required me to use different thought processes to get to a result hoping for the best, which was stressful to say the least, but still, it was fun.

Discussion

These three females explicitly acknowledged that their previous perceptions of not seeing themselves, others or mathematics as creative shifted to thinking they or mathematics are creative. We found two main sources of these shifts a) *content* - having more mathematical tools to work with, b) *course design* - developing a mathematical community that allows students to see each other's creative work with opportunities to reflect and connect back to their own work. Thus, for these students, content and course design seem to be important sources in shifting students' perceptions of themselves, others, or mathematics as creative.

Furthermore, although Stephanie does not explicitly mention the CPR on Proving, she mentions two of the subcategories "Tools and Tricks" and "Posing Questions". By using the CPR on Proving, it is evident that this particular instructor's course design and teacher actions aimed to explicitly value and foster students' mathematical creativity. This facilitated the evolution of students' perspectives on mathematical creativity. The connection between course design, teachers' actions, and changing students' perspectives on mathematical creativity requires additional exploration and our future work aims to examine this connection in detail and catalog specific *creativity-fostering teacher actions*. In particular, we wish to determine not only which teacher actions are more fruitful to afford such changes, but also what other course design features can contribute to shifts in student appreciation of mathematical creativity and fostering of creative behavior in the classroom.

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LEARNING TO POSE PROBLEMS WITHIN DYNAMIC GEOMETRY ENVIRONMENTS: A SELF STUDY INVOLVING VARIGNON'S PROBLEM

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This paper reports my second experience on my trajectory to learn how to pose mathematical problems within Dynamic Geometry Environments. I used The Geometer's Sketchpad and mathematical reasoning as tools to verify the plausibility and reasonability of each new problem situation. Using a problem-posing framework that I had developed during my first problem-posing experience within dynamic geometry environments, and subsequently refined and enriched with subsequent tasks, I was able to generate a diversity of problems by modifying the attributes of Varignon's problem. Among the problems generated were special problems, general problems, extended problems, further extended problems, converse problems, and proof problems. Examples of each of these types of problems are provided.

Keywords: Problem posing, problem solving, teacher educators, technology

Engaging in problem-posing tasks is recognized by mathematicians (e.g., Halmos, 1980; Polya, 1945/1973), mathematics educators (Brown & Walter, 1983, 1993; Kilpatrick, 1987; Silver, 1994, 2013), and professional organizations (Australian Education Council, 1991; National Council of Teacher of Mathematics [NCTM], 1989, 1991, 2000) as a worthwhile mathematical activity. According to Halmos (1980), the heart, the essence, of mathematics consists of problems. NCTM (1991), on the other hand, calls for all students to "be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem" (p. 95).

Purpose of the Study

Problem posing continues to receive increased attention from curricular, pedagogical, and research perspectives as attested by the recent publications of two books: *Mathematical problem posing: From research to effective practice* (Singer, Ellerton, & Cai, 2015) and *Posing and solving mathematical problems: Advances and new perspectives* (Felmer, Pehkonen, & Kilpatrick (2016). Initially, most research focused on understanding and documenting students' abilities to pose mathematical problems (Ellerton, 1986a, 1986b, 1988; English, 1996, 1997, 1998, 2003; Silver & Cai, 1996). If teachers and prospective teachers are to engage their students in problem-posing activities, it is important that they have experiences in problem generation. To help students to enhance their problem-posing abilities, research also examined teachers' approaches to pose mathematical problems (Author, 1998; Crespo, 2003; Ellerton, 2013; Engström & Lingefjärd, 2007; Lavy & Shriki, 2010; Silver et al. 1996). However, as noticed by Beswick and Goos (2018) and Castro Superfine and Li (2014), mathematics teacher educator knowledge has received limited attention.

While numerous studies on problem posing have investigated both students and teachers' abilities to pose problems, little research has been done on mathematics teachers educators' abilities to pose mathematical problems. I extend this research on problem posing by focusing on myself as teacher educator, a teacher of teachers. As noted by Suazo-Flores et al. (2019), qualitative methodologies such as narrative inquiry, self-study, and autoethnography have increasingly becoming modes of inquiry in mathematics teacher education research.

The purpose of this paper is to describe the types of problem that I have generated by modifying the conditions of Varignon's problem. To understand how I came to pose the problems, I present a brief story of my experiences with a problem-posing framework and how it enhanced my abilities to pose mathematics problems with the support of The Geometer's Sketchpad (GSP).

Perspectives on Mathematical Problem Posing

Problem posing tasks involve both the generation of new problems aimed at exploring and examining a given situation, as well as the reformulation of given problems (Silver, 1994). As noted by Silver (1994), problem posing can occur before, during, and after solving a given problem.

When we are trying to solve a challenging problem, a strategy is to reformulate the problem into an equivalent problem to make it more accessible. For example, we could reformulate a geometric problem in terms of algebra. A second way to reformulate a problem is to "think of a related, more accessible problem" (Polya, 1945/1973).

Problem posing can also occur before and after problem solving. It can occur before problem solving when the goal of the task is not to solve a mathematical problem, but to simply create new mathematical problems. It can occur after solving a problem as we examine the problem and pose follow-up questions or problems, a stage in the problem-solving process coined "looking back" by Polya. Brown and Walter (1983, 1993, 2004) have reported extensively about this type of problem posing by applying what they call the "What-if?" and "What-if-not" strategies in which problem conditions and constrains are changed.

While solving problem is recognized almost universally as an important mathematical, curricular, and pedagogical activity, problem posing is not, as evidenced by research examining opportunities to pose problems afforded by textbooks (Cai & Jiang, 2016; Cai, Jiang, Hwang, Nie, & Hu, 2016).

Methods of Inquiry

As stated by Pinnegar (1998), self-study is a "methodology for studying professional practice settings" (p. 33). LaBoskey (2004) adds that 'the aim for teacher educators engaged in self-study is to better understand, facilitate, and articulate the teaching-learning process" (p. 857). To illuminate the process of learning to pose mathematical problems, I decided to conduct a self-study research of how I came to learn to pose mathematical problems within dynamic geometry environments.

My Background

I was a high school mathematics teacher for 7 years at a state University in Mexico. After completing a bachelor's degree in Mathematics with a minor in mathematics teaching, I came to the USA and completed a Master's degree and a Ph. D degree in mathematics education. I have about 24 years of teaching experience at the University level. Currently, I teach content and methods courses at the undergraduate and graduate levels, mostly for prospective and practicing teachers.

First encounter with the concept of mathematical problem as the essence of mathematics. As undergraduate, I did not realize the importance of problems for mathematics. I conceived mathematics mainly as a well-integrated body of knowledge involving concepts and procedures connected through theorems whose proofs revealed explicitly the connections. As part of an assignment in one on my methods courses, I read Halmos's (1980) article *The Heart of Mathematics* where he argues that "the heart of mathematics consists of problems". Halmos concludes his article with a call to all instructors that they should "train our students to be better problem-posers" (p. 524). However, I did not interiorize nor appreciate the importance of the idea of learning how to pose problems.

First explicit encounter with the concept of posing problems. As a graduate student, I was one day perusing some books at the library when I encountered by chance Brown & Walter's (1983) *The art of problem posing*. The title of the book intrigued and intimidated me. It intrigued me because it seemed like a book from which I could learn how to pose problems. It intimidated me because learning how to pose problems seemed more like an art, and I did not see myself as a creative person. I left the book where it was and I did not think for a longtime of learning how to enhance my abilities to pose mathematical problems.

First experience on posing problems within dynamic geometric environments. The first problem-posing experience within dynamic geometry environments that I had was with the following problem: Prove that the angle bisectors of the angles of a parallelogram form a rectangle (Landaverde, 1970, p. 85). As a result of this experience and other experiences posing problems without the use of technology, I developed the problem-posing framework displayed in Figure 1 (Contreras & Martínez-Cruz, 2003). Notice that the base problem is the initial given problem whose attributes are to be modified to pose new related problems.

The base problem. I used as base problem the well-known Varignon problem. Typically, the Varignon problem is stated as a theorem (The midpoints of a quadrilateral are the vertices of a parallelogram). I consider this theorem as a mathematical situation within an implicit problem that we can reformulate as a proof problem or as a more open-ended problem. My version of Varignon' problem is as follows: Let E, F, G, and H be the midpoints of the consecutive sides of a parallelogram ABCD. What type of quadrilateral is EFGH?

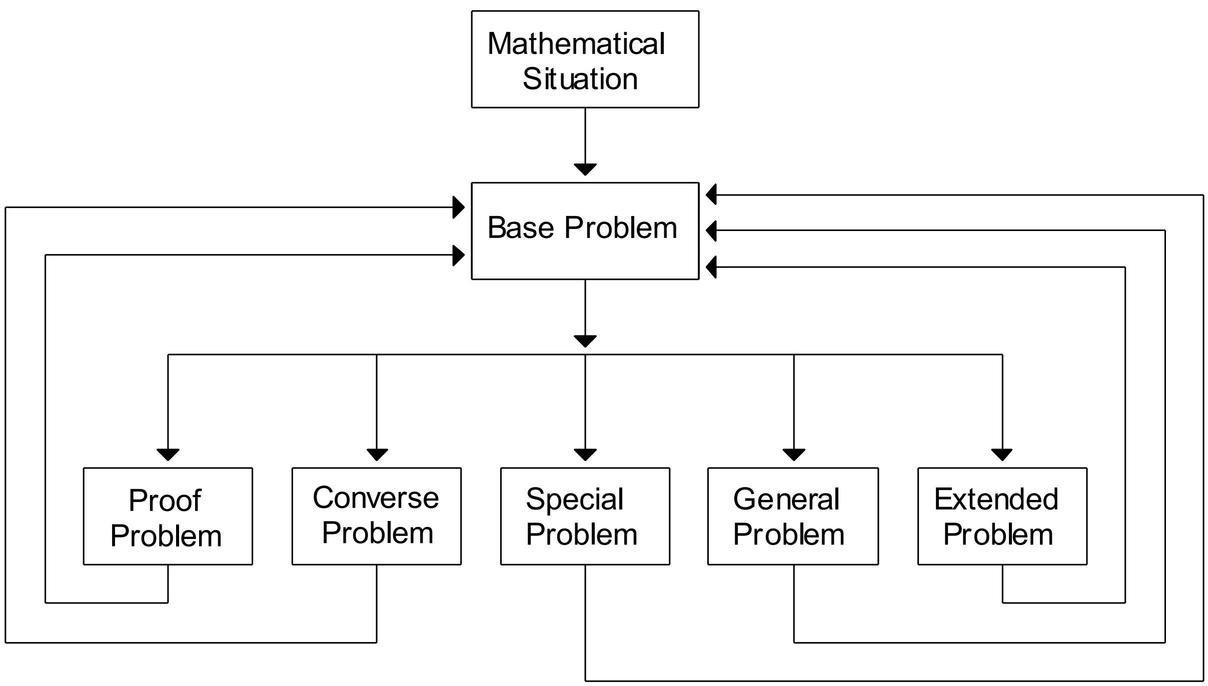

Figure 1: A Problem-Posing Framework

Analysis and Results

Using the problem-posing framework, I posed a diversity of problems that after analysis I classified as special problems, converse problems, extended problems, prove problems, and further extended problems. Typical problems of each of these types are displayed in Table 1.

Table 1: Examples of problems generated using the problem-posing framework

Type of problem	Problem
Special and proof problem	If E, F, G, and H are the midpoints of the consecutive sides of a rhombus ABCD, prove that EFGH is a rectangle.

Converse problem of a special problem	E, F, G, and H are the midpoints of the consecutive sides of a quadrilateral ABCD. If EFGH is a rectangle, what type of quadrilateral is ABCD?
Converse problem of a general problem	If E, F, G, and H are the midpoints of the consecutive sides of a quadrilateral ABCD. If EFGH is a parallelogram, what sort of quadrilateral is EFGH?
Extended problem	ABC is a triangle. Characterize quadrilateral BDEF where D, E, and F are the midpoints of the sides BC, CA, and AB, respectively. (Extended problem to a triangle, which is a degenerate case of a quadrilateral)
Extended and proof problem	Prove that the medial quadrilateral of a kite is a rectangle.
Further extended and proof problem	Prove that the points of intersection of the angle bisectors of the consecutive interior angles of a parallelogram ABCD are the vertices of a rectangle.
Further extended problem	I, J, K, and L are the points of intersection of the sides of a parallelogram ABCD with the interior angle bisectors. What sort of quadrilateral is IJKL?

Conclusion

Researchers (e.g., Crespo, 2003; Crespo & Sinclair, 2008; Nicol, 1999; Silver at al., 1996) report that students, teachers, and prospective teachers typically generate problems that are "predictable, undemanding, ill-formulated, and unsolvable" (Crespo & Sinclair, 2008). While there is some degree of predictability on the types of problems suggested by the problem-posing framework, I used a diversity of language to make them more interesting. I believe that I created a diversity of well-posed problems, each of which is a good and interesting problem because each one opens the mathematics involved or required by the problem (Crespo & Sinclair, 2008). In addition, I used mathematical reasoning and conceptual understanding to generate each problem. The plausibility of each problem was supported with GSP, but I went beyond exploring each problem with GSP and I provide a mathematical solution. In summary, I was actively engaged in the authentic process of doing mathematics. I have made public my second experience in posing mathematical problems within dynamic geometry environments to challenge other mathematics educators to test the problem-posing framework in other appropriate mathematical contexts.

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A LARGE-SCALE STUDY ON TEACHER NOTICING

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Teachers' noticing of key aspects of instruction is an important skill for learning from and improving their teaching because noticing enables the opportunity for change. We investigated what teachers notice in short video clips of a real classroom teacher's interaction with students around a mathematics problem by conducting the largest survey study on teacher noticing to date. According to our analysis of data collected from 496 fourth- and fifth-grade teachers from 48 states, the key issues that were vital to improving teaching and students' learning caught the attention of only 13.7% of teachers. However, 67.5% of the teachers focused on interpreting issues around content-specific teaching and learning, and 17.7% paid attention to general issues, such as the classroom climate.

Keywords: Mathematical Knowledge for Teaching, Teacher Knowledge

Teacher noticing, or the act of observing and interpreting classroom events (e.g., Sherin & van Es 2009), influences the likelihood for desirable teacher actions such as responding to student mistakes and making a variety of other pedagogical choices. Yet, the act of noticing classroom events that are pedagogically relevant for improving teaching and advancing students' thinking is not a simple skill. During any second of classroom instruction, teachers are inundated with numerous inputs (e.g., each individual student's attention, students' reaction to given tasks, or the impact of his/her choice of sequencing of activities on students' thinking), requiring teachers to be selective in their noticing. Research demonstrates that teachers' noticing of classroom events is widespread; aspects of the classroom that capture teachers' attention include classroom climate, students' math thinking, and the organization of the classroom (e.g., Sherin & van Es, 2009). Furthermore, when teachers attend to one particular classroom occurrence, they are consciously or unconsciously missing other events occurring in the classroom. Effective teaching thus relies in part on noticing and attending to the most pedagogically relevant aspects of the class and filtering out other aspects (Sherin, Russ, & Colestock, 2011).

Researchers acknowledge the importance of noticing for teaching expertise and have conducted explorations of teacher noticing skills that made significant contributions to the field. Yet studies on teacher noticing to date also have certain limitations, namely that they are conducted with limited numbers of teachers, with teachers who were attending a professional development program targeting their noticing skills, or with teachers from only certain school districts (e.g., Jacobs, Lamb, & Phillip, 2010; van Es, 2011). Our current understanding of mathematics teacher noticing is thus informed in large part by research settings that involved prompts and facilitators guiding teachers to attend to certain aspects of classroom events and samples that limit generalizability. We argue that an analysis investigating what teachers across the United States notice independent of a professional development setting or teacher education program seeking to improve their noticing skills is needed to better understand the overall trend in teachers' noticing skills. This analysis will advance not only our general understanding of teachers' noticing, but also our preparedness to help teachers improve their noticing skills to develop more effective teaching.

Objectives

The present study is the first large-scale analysis of mathematics teachers' noticing. Fourth- and fifth-grade teachers (N = 496) watched four short videos of classroom mathematics instruction that targeted fraction concepts. The instruction was aligned with the upper elementary mathematics standards (National Governors Association and Council of Chief School Officers, 2010). We consider certain types of noticing to be more beneficial for teachers to make changes to their content-specific pedagogical practices. As such, we intentionally selected videos that showed instructionally problematic teaching moments and/or students' confusion around the targeted mathematical content. Our aim was to investigate whether these key moments would attract teachers' attention compared with other generic issues. Although we did not direct teachers' attention to these specific particular issues, we specifically asked them to report what they noticed around the mathematical content. Again, our rationale was that teachers cannot make content-specific pedagogical decisions if they cannot notice these issues. Building on prior work on teachers' noticing (van Es & Sherin, 2008), we aimed to explore the following research questions:

1. What overall topics of the classroom instruction presented in the video clips caught the teachers' attention? What levels of analysis did teachers' noticing entail?
2. What subtopics of the classroom instruction at each level of analysis did teachers notice?

Methods

This study used data from 496 fourth- and fifth-grade teachers. The teachers completed an online mathematics teaching survey that included four videos of classroom instruction from Kersting and colleagues developed to capture teachers' useable knowledge (2008, 2010, 2012). For each of the four videos, teachers were asked, "Please list the three most significant things that you notice regarding how the teacher and the students in the clip interacted around the targeted mathematical content." The videos were presented in a random order for each participant. For our analysis, we include teachers who provided responses to at least one of the videos.

Analysis

We developed a 4-point rubric to evaluate the depth and topics of teachers' responses. Our goals were to differentiate between teachers' surface-level noticing and more sophisticated noticing, and also to identify responses focused on *content-specific teaching and learning-related issues* that limited students' understanding of the concepts in the videos. Thus, we created our rubric to distinguish among purely descriptive responses, analytical responses, and responses that focused on the problematic content-specific issues in each video. We also coded a subsample of responses to ensure that our rubric captured qualitative differences in teachers' responses. An important distinction between our rubric and those used in prior studies is that we consider both content and depth of analysis within single codes, whereas other rubrics use separate codes to capture content and stance of analysis (e.g., Sherin & van Es, 2009)

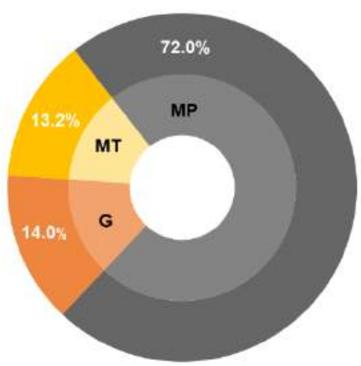
In our rubric, Level 1 responses did not include mathematics-specific events (e.g., describing seating arrangements, describing the teacher's tone of voice); Level 2 responses focused on content-specific aspects that were either purely descriptive or that contained a binary judgment (e.g., restating the problem, stating that the lesson was good); Level 3 responses analyzed some aspect of students' mathematical thinking or the teacher's mathematics pedagogy (e.g., noticing that the students were confused, interpreting why the teacher chose to use a strategy); Level 4 responses included responses that focused on the problematic issues related to students' mathematical understanding or teachers' mathematical instructional practices.

Our rubric also captured the topics of teachers' noticing responses. Adapting the methods of van Es and Sherin (2008), we differentiated among responses related to three topic categories: the mathematics pedagogy code identified responses that focused on teaching actions and strategies, such as the use of manipulatives or questioning techniques; the mathematical thinking code identified responses that focused on students' thinking and ideas; and the general code identified responses that were not related to the mathematical content.

After we finalized our rubric and gained confidence in using the rubric to reliably code responses, we coded the remaining responses by rotating the order and combination of the responses to the videos. Interrater reliability as measured by exact agreement was 95.3% for Noticing Levels and 95.9% for the Noticing topics. Furthermore, coefficient kappa was .927 and .909 for noticing levels and noticing topics. Teachers' responses were scored with a high degree of consistency.

Results

On the basis of our analysis of 5,382 responses, the vast majority of responses (72%) were related to pedagogical content, such as the actions, choices, or strategies the teacher used during the lesson (see Figure 1). 13.2% of the responses were focused on students' mathematical thinking, and 14% of the responses were about the general classroom climate and environment not specific to mathematics.[1] In terms of the depth of teachers' analysis of classroom events, we found that 14.8% of the responses had no focus on mathematics and 33% were purely descriptive (i.e., they did not indicate any analysis or interpretation; see Figure 2). Nearly half of all the responses (47.79%) included some level of analytical thinking about students' learning and teachers' pedagogical choices around the mathematics content but did not identify the key mathematical ideas related to the problematic mathematics content. Only about 6% of responses included an analysis of key mathematical issues around either the teachers' pedagogical choices, students' understanding, or both.



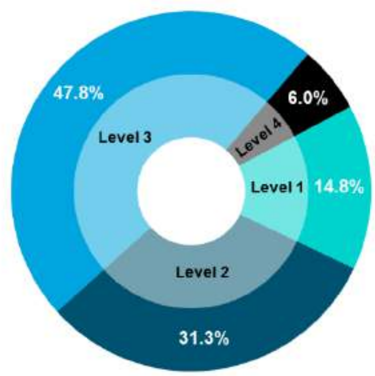


Figure 1. Percentages of overall noticing topics across 5,382 responses. MP = mathematics pedagogy; MT = mathematical thinking of students; G = general. Figure 2. Percentages of noticing depth levels across 5,382 teacher responses.

Aspects of Mathematics Classroom Events Teachers Noticed at Level 1 and Level 2

Level 1. Level 1 of our rubric contained responses that did mention mathematics but merely in a descriptive or evaluative way (e.g., "good" or "important" or "difficult"), without any analysis. The vast majority of Level 1 responses (91.4%) focused on teaching-related issues, such as the instructional tools and questioning strategies the teacher used. Among those responses focusing on content-specific pedagogical issues, almost one-third of the responses (30%) mentioned the use of manipulatives, visuals, or hands-on materials (e.g., "The students were working with manipulatives;" "I like that they used the pie fraction pieces").

Level 2. Level 2 responses included analytical or interpretive statements about the mathematics content in the video, but they did not identify the problematic mathematics content. At Level 2,

[1]Recall that teachers were asked to list three things they noticed. Some listed two and gave no answer for the third one. We assigned the "no answer" responses a score of 0, and we include these responses in the "general" noticing category.

79.8% of responses focused on content-specific pedagogical issues, whereas 20.2% focused on students' mathematical thinking.

Among the issues related to students' mathematical thinking, the majority of responses (74.6%) focused on what students seemed to understand or were struggling to understand ("Students are manipulating the pieces, but I can't tell if they are truly understanding the concept;" "The kids don't seem to have an understanding of parts to whole"). In relation to how students engaged with the problem ("The student used trial and error to find the correct fraction pieces to use;" "I noticed students were engaged in the lesson with the chips and did not seem to give up in understanding in solving the problem"), 8.5% of the responses focused on students' readiness to deal with the given concept or problem ("Her work with one student seemed effective, but I don't think the whole class was ready to tackle this problem;" "The students obviously had background knowledge on how to solve these problems").

Discussion

The concept of teacher noticing has important implications for student learning, research and teacher education because teachers do not address events that do not catch their attention. The majority of prior work on mathematics teacher noticing has been conducted with teachers in a program aiming to improve teachers' noticing skills; thus, the present study is unique by investigating trends in what a national sample of fourth- and fifth-grade mathematics teachers noticed independent of professional development or teacher education programs.

In alignment with prior work, our study indicated that pedagogical topics caught teachers' attention more often than any other topic (e.g., Sherin & van Es, 2009). In fact, more than two-thirds of teachers noticed content-specific pedagogical topics in each video; however, one-third of the teachers did not report anything on students' mathematical thinking.

Our study contributed to the current understanding of noticing in that teachers analyzed pedagogical strategies in greater depth, and their analyses targeted a wide range of pedagogical strategies. In contrast, teachers' analysis of students' mathematical thinking seemed limited. As scholars in several studies have noted, attending to and interpreting students' thinking is an important aspect of quality teaching (e.g., Ball & Cohen, 1999; Jacobs et al., 2010) and one that contributes to students' learning (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Thus, teachers' lack of attention to students' mathematical thinking may indicate a lack of attention to their own students' thinking. Therefore, teachers may need more targeted interventions to learn to focus on students and how they analyze students' thinking.

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REASONING ABOUT THE RATE OF CHANGE WHILE LINKING CO_2 POLLUTION TO GLOBAL WARMING

PENSANDO EN LA RAZÓN DE CAMBIO AL VINCULAR CONTAMINACIÓN POR CO_2 CON EL CALENTAMIENTO GLOBAL

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This paper characterizes the way three preservice mathematics teachers (PSTs) understand and quantify the rate of change as they model the link between carbon dioxide (CO_2) pollution and global warming. I also discuss what PSTs learned about the concept of forcing by CO_2, a key metric of global warming. The PSTs completed a mathematical task during an individual, task-based interview. The study revealed three levels of understanding of the rate of change in relation to quantitative operations (comparison versus coordination), graphing activity (pointwise versus smooth and continuous), and concavity (discovering versus anticipating). Depending on their level of understanding, PSTs could imagine the rate of change changing discretely or continuously with respect to an independent variable. PSTs also learn four central ideas regarding the forcing by CO_2 as a result of working on the task.

Keywords: Cognition, Modeling, STEM/STEAM, Teacher Education - Preservice

Introduction

Climate change is a pressing issue for this century with potentially irreversible and disastrous consequences for social and natural systems (Intergovernmental Panel on Climate Change [IPCC], 2013). The *United Nations* has called for incorporating climate change education in schools (Anderson, 2012; Global Education Monitoring [GEM], 2016). Since students have different interests and learning abilities, teachers from all disciplines can contribute to climate change education (McKeown & Hopkins, 2010). Mathematics teachers can play a central role in this endeavor since mathematical modeling represents a promising approach for connecting mathematical learning and climate change education (González, 2018, 2019; Barwell & Suurtamm, 2011; Barwell, 2013a, 2013b). Teachers, however, need to be prepared for the challenge, which requires teacher education programs to prepare preservice mathematics teachers (PSTs) for incorporating climate change into their instruction.

Lambert and Bleicher (2013) have identified two key concepts from climate sciences that preservice science teachers need to learn about in order to understand climate change: (a) the Earth's energy balance, and (b) the link between carbon dioxide (CO_2) pollution and global warming. It is reasonable to extend this premise to PSTs since they are less familiar with concepts from climate science than preservice science teachers. Therefore, a starting point may involve studying the energy balance and the link between CO_2 and global warming as dynamic situations where two (or more) variables change together (covariation). In this paper, I characterize, from a covariational reasoning perspective, the way three PSTs think about the rate of change as they model the link between CO_2 pollution and global warming. I also discuss what PSTs learned about the concept of *Forcing by CO_2*, a key metric for assessing the impact of CO_2 pollution on global warming.

Conceptual Framework

Covariational reasoning refers to "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson,

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Jacobs, Coe, Larsen, & Hsu, 2002, p. 354). Johnson (2015) distinguished two categories of quantitative operations that students use when reasoning about covariation and rate of change: operations of *comparison* (QO-Comp) and operations of *coordination* (QO-Coord). *QO-Comp* involves conceiving a quantity's variation in chunks and produces associations of amounts of change between covarying quantities. The amounts of change in *y* are compared for (not necessarily equal) amounts of change in *x* in order to make viable claims about the rate of change. *QO-Coord* involved conceiving a quantity's variation smoothly and produces relationships between covarying quantities. The relationships are coordinated through division to create a new quantity measuring *degrees of change* that supports accurate claims about the rate of change. Carlson and colleagues' concept of covariational reasoning and Johnson's (2015) QO-Comp and QO-Coord informed the discussion about the ways PSTs understood and quantified the rate of change.

Methodology

This paper is part of a larger study that investigated how PSTs make sense of simple mathematical models of climate change. Three secondary PSTs —hereafter Jodi, Pam, and Kris— enrolled in a mathematics education program at a large Southeastern university in the United States participated in that larger study. Here, I focus on their responses to one task of the larger study: the *Forcing by* CO_2 *Task*.

The Forcing by CO_2 Task

The Erath's energy balance accounts for all heat flows (in Joules per second per square meters, or $Js^{-1}m^{-2}$) that there exit in the continuous heat exchange between the sun, the planet's surface, and the atmosphere (Figure 1a). The sun warms up the planet's surface at an approximately constant heat flow *S*. As the surface heats up, it radiates heat to the atmosphere (R). A small fraction of it escapes to space (L), but the majority (B) is absorbed by *greenhouse gases* (GHG) in the atmosphere. The atmosphere then re-radiates a fraction of the absorbed heat back to the surface (A), further increasing its temperature. The heat flow A represents the magnitude of the *greenhouse effect*, which enhances the planet's mean surface temperature. The energy balance shows that changes in the concentration of GHG result in changes in the planet's mean surface temperature. The Fording by CO_2 Task (Figure 1b) focuses on carbon dioxide (CO_2) because it is a key driver of global warming, as human activity produces large amounts of it by burning fossil fuels (IPCC, 2013).

The task defines the forcing by CO_2 as $F = (S + A) - R$, which is a measure of the warming effect over the planet's surface produced by an instantaneous increase in the atmospheric CO_2 concentration, C, (in parts per million, or ppm). If C increases, then the atmosphere can absorb more heat and, consequently, can radiate more heat towards the surface (A increases). Thus, as C increases, so does F, but $lim_{C\to\infty} F(C) = 45$ since S and R remain constant, which puts a cap on the growth of A and, consequently, on the growth of F. This suggests that F increases asymptotically towards 45 $Js^{-1}m^{-2}$ as C increases, producing an increasing, concave-downward graph.

Data Collection

Each PST completed the task during an 80-minute long, individual, task-based interview (Goldin, 2000). The interview followed a *semi-structured* format and was video recorded and transcribed for analysis. I started the interview by showing each PST a 7-minute long video introducing the concepts of energy balance and greenhouse effect. After the video, the PST and I had a Q&A session in which I summarized the central ideas regarding the energy balance and the greenhouse effect and clarified any questions they may have had about those ideas. The video and Q&A session were meant to provide PSTs with a basic knowledge regarding the energy balance and the greenhouse effect so that they could start working on the task.

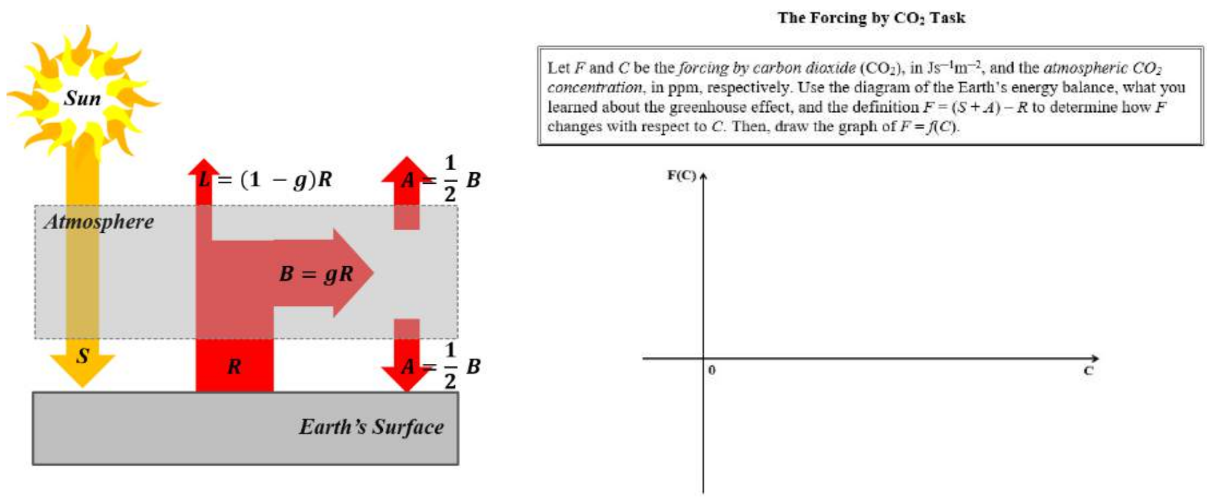

Figure 1. (a) the Earth Energy Balance (Left) and (b) the Forcing by CO_2 Task (Right).

Once the Q&A session ended, PSTs were given the Forcing by CO_2 Task along with a diagram of the energy balance (Figure 1a). The interview had four distinct parts. First, PSTs were asked to think about how F changes as C increases by examining the diagram of the energy balance. The diagram had no values for the heat flows to encourage PSTs to imagine changes happening dynamically. When PSTs experienced difficulties, I gave them initial values for the heat flows so that they could find F-values by using the given definition $F = (S + A) - R$. Second, PSTs had to think about two theoretical scenarios *Scenario 1* described a completely transparent atmosphere (an atmosphere that absorbs no surface heat) and was assumed to happen for $C = 0$ ppm. Scenario 1 corresponded to the minimum forcing (F-value) for the given initial values of the heat flows. *Scenario 2* described a completely opaque atmosphere (an atmosphere that absorbs all surface heat) and was assumed to happen for $C = 1{,}000{,}000$ ppm (highest concentration possible). Scenario 2 corresponded to the maximum forcing (F-value). The PSTs were expected to imagine how F increased from Scenario 1 to Scenario 2 and anticipate the graph's concavity. Third, I introduced the *Excel Simulation*, a spreadsheet that allowed PSTs to enter C-values and obtain the corresponding F-values. The Excel Simulation assisted PSTs in examining and quantifying changes in F for corresponding changes in C and evaluating the accuracy of their graphs. Finally, I asked PSTs to draw the graph of the *Sensitivity of F to C*, or the rate of change of F with respect to C. Here, I examined the PSTs' ability to conceive the rate of change as a measure of sensitivity and as a quantity in and of itself that covaried with C.

Data Analysis

Interview videos and transcripts were analyzed through the *Framework Analysis* (FA) method (Ward, Furber, Tierney, & Swallow, 2013). I watched all videos and divided them into smaller episodes. For each episode, I took notes regarding PSTs' views of forcing, covariational reasoning, and understandings of rate of change. I used the notes to develop an analytic framework, which included six codes about forcing, eight codes regarding covariation, and five codes about rate of change. The analytic framework was applied back to the data to code all episodes. Next, I looked for patterns across the participants' responses and categorized codes into themes. The patterns and themes helped me characterize the way PSTs understand the forcing and the rate of change.

Results

The Direction of Change of the Forcing

During the first part of the interview, PSTs looked at the diagram (Figure 1a) and identified the heat flows that changed when *C* increased and those that remained constant (i.e., unaffected by changes in *C*). In doing so, PSTs thought about how an increment in *C* influences the atmosphere's capacity to absorb and radiate heat (changes in *B* and *A*, respectively), which represents a foundational idea to understand the forcing by CO_2. PSTs inferred the direction in which *F* was changing by utilizing the given definition $F = (S + A) - R$. They noticed that an increase in *C* resulted in an increase in *A*, while the heat flows *S* and *R* remained constant, which meant that *F* increased when *C* increased. For instance, Kris stated "if *B* increases, then *A* is going to increase, and *S* and *R* stay the same [*pauses*]. So, [*F*] is going to be positive".

During the second part of the interview, PSTs thought about *Scenario 1* and *Scenario 2*. They realized that the scenarios represented the minimum and maximum forcing, respectively. For instance, Jodi described Scenario 2 as follows:

A would be 390 over 2, which is going to be [*uses calculator*]. So, A is 195, and we would need, we would want $_S$ to equal A [*writes S = A*]. But, since 240 is greater than 195, we would need to add [*F*] [*writes 240 = 195 + F*]. And, that would make F = 45. In the case we add more CO_2 to the atmosphere and L no longer is emitted

The PSTs assumed Scenario 1 occurred for $C = 0$ and found that $F = (240 + 0) - 390 = -150$ $Js^{-1}m^{-2}$. For Scenario 2, they assumed it occurred for $C = C_M$ and had $F = (240 + 195) - 390 = 45$ $Js^{-1}m^{-2}$. They represented these scenarios in the coordinate plane by the points $(0, -150)$ and $(C_M, 45)$, respectively. Then, Pam and Jodi drew a line incident to both points as the graph of *F* (Figure 2), while Kris could not decide whether the graph should be an increasing, concave-downward curve or an increasing line. She stated that a line "would imply that it is like a constant rate of change with *C* and [*F*]." Kris's understanding of rate of change appeared more advanced than Jodi and Pam's since it involved the realization that the shape of a graph is related to the variation in the rate of change.

The Rate of Change of the Forcing

During the third part of the interview, the *Excel Simulation* was introduced. Here, the PSTs also learned that *F* follows the rule "*F* increases by 4 $Js^{-1}m^{-2}$ every time *C* doubles[1]" which is widely accepted among the experts (Huang & Shahabadi, 2014; IPCC, 2013).

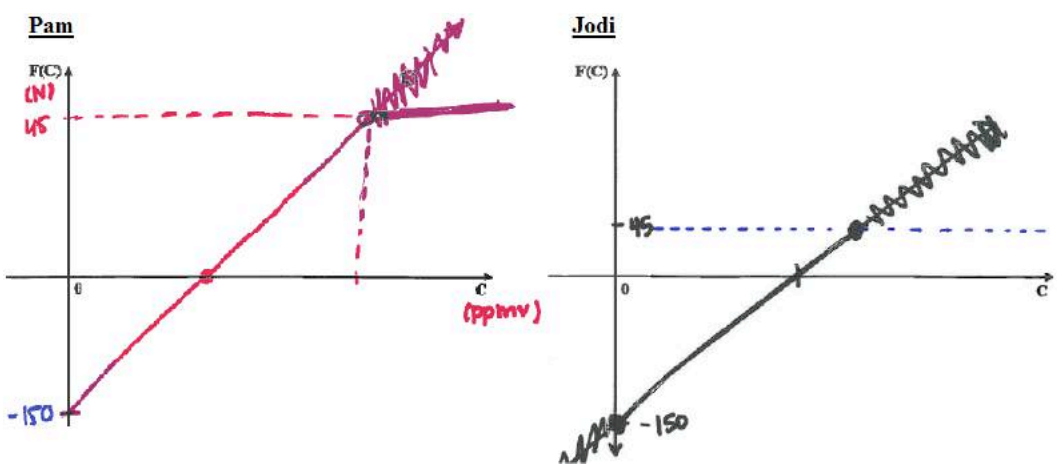

Figure 2. (a) Pam's linear graph of F (left) and (b) Jodi's linear graph of F (right).

[1] A more real estimate is about 3.7 $Js^{-1}m^{-2}$ (IPCC, 2013), but I rounded it to 4 $Js^{-1}m^{-2}$ for simplicity.

PSTs were asked to find out whether *F* was a linear or a nonlinear function of *C*. All three PSTs determined *F*-values corresponding to equally spaced *C*-values. Then, they compared the differences $\Delta_i F$ and noticed they were decreasing, discarding the linear model. After that, PSTs demonstrated three different ways of quantifying the rate of change and understanding its connection to the concavity of the graph of *F*, as they drew new versions of that graph. Pam did not anticipate the concavity of the graph from interpreting the decreasing increments $\Delta_i F$, suggesting she did not see them as an indicator of concavity or a measure of the variation in the *degree of change* of *F* with respect to *C*. Instead, Pam used the rule "*F* increases by 4 $Js^{-1}m^{-2}$ when *C* doubles" to coordinate *C*-values with *F*-values, creating a discrete collection of pairs (*C* , *F*) and drawing the graph of *F* using a pointwise approach (Figure 3a). When finished, Pam said "Oh! This looks like a logarithmic thing I hate", suggesting she did not anticipate the concavity of her graph as much as she *discovered* it.

In contrast, Jodi anticipated the concavity of *F* by interpreting the decreasing increments $\Delta_i F$ as indicating that *F* increased less and less as *C* increased.

So, the relationship is not linear because the change in $_y$ over the change in $_x$ is not equal between two points. But, I see that, as we increase [$_C$], the change in $_F$ is less. So, we may end up getting a function that looks like that *[draws a tiny, increasing, concave-down curve]*

Although Jodi anticipated the concavity, she still used the Excel Simulation to create a discrete collection of pairs (*C* , *F*). She then used a pointwise approach to draw her final version of the graph of *F*. This is an interesting behavior because it suggests that she did not have complete confidence on her interpretation of the differences $\Delta_i F$ in terms of concavity. A possible explanation is that her understanding of those differences as an indicator of concavity and a measure of the degree of change of *F* may have been still stabilizing in her mind.

Finally, Kris anticipated the concavity of *F* by interpreting the decreasing average rate of change of *F*. Her interpretation confirmed her previous suspicion that the graph was an increasing, concave-downward curve.

K: That is really weird, how like, if you look at the change from [*C* = 0] to [*C* = 1] [*pauses*]
I: There is a big jump
K: Yeah, like over a hundred (*F* increases more than 100 $Js^{-1}m^{-2}$). And then you get from [*C* = 10] to [*C* = 20] and it is only like four (*F* increases by approximately 4 $Js^{-1}m^{-2}$). So like, for every change [of] 2.5 [in *C*], [*F*] changes like one-ish. So that is what I was thinking about when I said that [the graph] may look like this [*draws an increasing, concave-downward curve*]

Kris's way of quantifying the average rate of change of *F* supported both anticipating concavity *and* drawing the graph of *F* in a smooth and continuous way (Figure 3b). Also, Kris's use of ratios represents a step forward in the formalization of the concept of rate of change in relation to the comparison of the differences $\Delta_i F$ for equal increments ΔC.

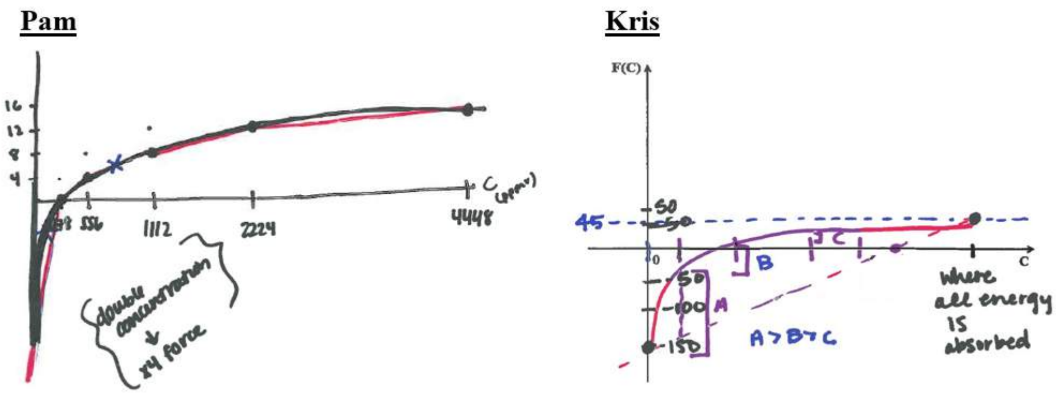

Figure 3. (a) Pam's final graph of F (Left) and (b) Kris's final graph of F (Right).

The Sensitivity of F to C

Pam and Kris participated in the fourth part of the interview involving the *Sensitivity of F to C* (i.e., the rate of change of *F* with respect to *C*). Unfortunately, Jodi did not have time to participate during that last part. The analysis of Pam and Kris's responses suggest two different ways of quantifying the *sensitivity* of *F* to *C* and two different ways of conceiving that sensitivity *covarying* with *C*.

Pam quantified the sensitivity by the *steepness* of the graph of *F* corresponding to unequally-spaced values of *C* (each interval was twice as long as the previous one). She then attended to the variation in the steepness as she moved from one interval of *C* to the next. She translated that variation into degrees of sensitivity (e.g., more or less sensitive).

P: So, it is not super sensitive here [*uses two fingers to indicate the steepness of the graph of F for* $2224 \leq C \leq 4448$]

I: Could you tell me a little bit more about how you figured that out by looking at this graph [*point at her graph of F*]?

P: Here [*points at the interval [0 , 278]*], [*C*] increased a little bit, and the force [*sic*] went crazy [*moves her index finger up quickly to indicate a large increase in F*], I mean compare to everything else, it went higher. Here [*points at the interval [278 , 556]*], [*C*] increased a little bit more, and the sensitivity didn't increase that much. So, [*F*] is not as sensitive when there is more concentration [*moves her fingers to the right to indicate the increase in C*]

The transcript above shows how Pam imagined the steepness decreasing as she moved from one interval of *C* to the next. This helped her identify the direction of change of the *sensitivity* (i.e., it decreases as *C* increases). However, she did not notice that the decline in steepness slowed down as *C* increased, hence she could not anticipate the concavity of the graph of the sensitivity. In order to draw the graph, Pam first found four values of the average rate of change of *F*: 4/278, 4/556, 4/1112, and 4/2224, corresponding to the intervals [278 , 556], [556 , 1112], [1112 , 2224], and [2224 , 4448], respectively. Then, Pam notice that "my concentration increases by double, and my sensitivity goes down by half [*writes 'concentration × 2, sensitivity ÷ 2'*]". She used that rule to create the discrete collection of pairs (278 , 1/2 *F'*(0)), (556 , 1/4 *F'*(0)), (1112 , 1/8 *F'*(0)), and (2224 , 1/16 *F'*(0)), where *F'* represents the sensitivity of *F* to *C*. Then, Pam drew the graph of the sensitivity with a pointwise approach (Figure 4a). This suggests she discovered the concavity of the graph of the sensitivity rather than anticipating it.

In contrast, Kris's ways of quantifying the sensitivity involved thinking in terms of *how resistant F* was to changes in *C*, as indicated by the graph of *F*. When I asked her how the sensitivity changes as *C* increases, Kris replied "sensitivity decreases because [*F*] is more resistant to a change in *C*". She then drew a decreasing, concave-upward graph of the sensitivity in a smooth and continuous way

(Figure 4b). Since she did not justify the concavity of her graph, I asked her to elaborate on how she figured the concavity out, to which she replied:

As we increase C by equal amounts each time *[uses two fingers to indicate equal increments in C], F is increasing by smaller, and smaller, and smaller amounts [uses two fingers to indicate decreasing increments in F]*. So, it is becoming less sensitive to the changes in C. Because it takes a bigger change in C to equal the equal change in F.

Because one must double *C* to create the same increment in *F*, she claimed that "the sensitivity decreases at a decreasing rate". Kris's quantification of the sensitivity allowed her to anticipate concavity, draw an accurate graph, *and* make viable claims about the rate of change of the sensitivity. This suggests that Kris not only reasoned about the rate of change of *F*, but also about the rate of change of the rate of change of *F*, which is foundational to understand second derivative in Calculus.

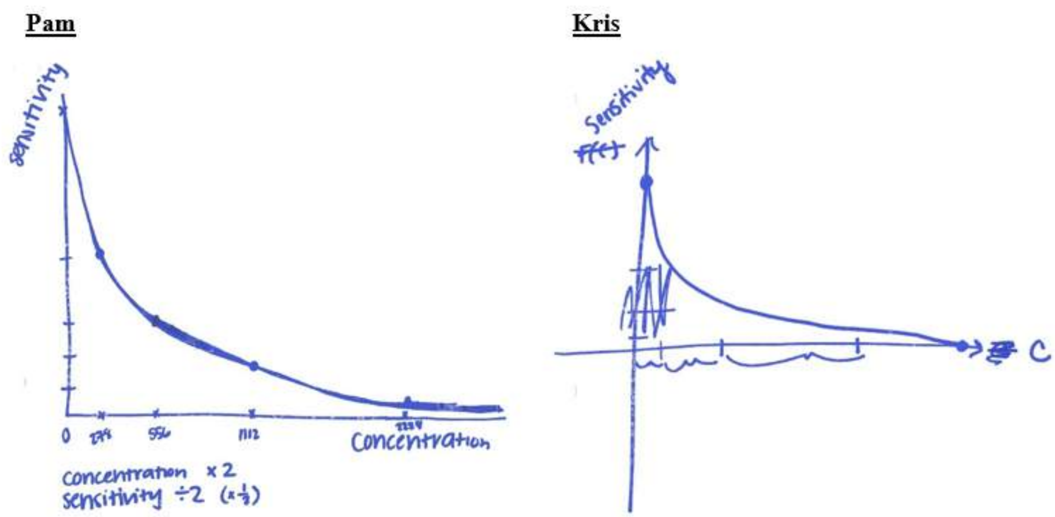

Figure 4. (a) Pam's graph of sensitivity (Left) and (b) Kris's graph of sensitivity (Right).

Finally, by thinking about the sensitivity of *F* to *C*, Pam and Kris learned that the forcing by CO_2 becomes less sensitive to changes in *C* as *C* increases. This is another characteristic of the forcing widely accepted among the experts (Huang & Shahabadi, 2014; IPCC, 2013).

Conclusions

The study revealed three different levels of understanding of the rate of change among the PSTs. *Level 1* is represented by Pam; she did not demonstrate quantitative operations related to reasoning about the rate of change *F*. She created a discrete collection of pairs (*C* , *F*) and used a pointwise approach to draw the graph of *F*. The concavity was discovered after finishing the graph and no viable claims about the rate of change were made. *Level 2* is represented by Jodi; she associated changes $\Delta_i F$ with equal changes ΔC and compared those associations (QO-Comp) to anticipate concavity and make viable claims about the rate of change. She, however, created a discrete collection of pairs (*C* , *F*) and used a pointwise approach to draw the graph of *F*. This suggests that her understanding of the relationship between a graph's shape and the rate of change was not completely stable in her mind. *Level 3* is represented by Kris; she coordinated changes $\Delta_i F$ with changes ΔC through division (QO-Coord) to create a single quantity that allowed her to anticipate concavity, make viable claims about the rate of change, and draw an accurate graph of *F*.

The analysis of Pam and Kris's responses suggest two different ways of quantifying the *sensitivity* of *F* to *C* and two different ways of conceiving covariation between the sensitivity and *C*. Pam quantified the sensitivity by the *steepness* of the graph of *F* corresponding to an interval of *C*. This allowed her to identify the direction of change of the sensitivity (i.e., it decreases when *C* increases). Then, she *compared* (QO-Comp) the values of the average rate of change of *F* for consecutive, unequally-long intervals of *C* (each interval was twice as long as the previous one) in order to define a correspondence rule between values of sensitivity and values of *C*: the sensitivity halves every time *C* doubles. Pam's QO-Comp allowed her to draw an accurate graph but did not support the ability to make claims about the rate of change of the sensitivity or anticipate concavity. In contrast, Kris quantified the sensitivity as the *resistance* of *F* to changes in *C*, as defined by the graph of *F*. This allowed her to identify the direction of change of the sensitivity (i.e., it decreases when *C* increases). Then, she *coordinated* (QO-Coord) changes in resistance with changes in *C* in order to draw an accurate graph of the sensitivity in a smooth and continuous way, make claims about the rate of change of the sensitivity, and anticipate concavity. Most interestingly, Kris's QO-Coord supported reasoning about the rate of change of the rate of change of *F*, a key idea to understand the second derivative in Calculus (Johnson, 2012).

Finally, the study also shows that PSTs learned four important aspects about the *forcing by* CO_2: (1) an increase in atmospheric CO_2 concentration enhances the atmosphere's capacity to absorb and radiate heat, which further warms the planet's surface; (2) the forcing has a theoretical minimum value, when the atmosphere absorbs no surface heat, and a theoretical maximum value, when the atmosphere absorbs all surface heat; (3) the doubling CO_2 rule for the forcing (*F* increases by 4 $Js^{-1}m^{-2}$ every time *C* doubles); and (4) the forcing by CO_2 becomes less sensitive to changes in *C* as *C* increases.

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MISCELLANEOUS TOPICS:

POSTER PRESENTATIONS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

POSING PROBLEMS ABOUT GEOMETRIC SITUATIONS: A STUDY OF PROSPECTIVE SECONDARY MATHEMATICS TEACHERS

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Mathematics has developed into an extensive body of knowledge because there is, and has been, a continuous search for finding solutions to problems posed by someone. Therefore, problem posing is a fundamental activity of doing mathematics (Brown & Walter, 1983, 1993; Halmos, 1980; Contreras, 2019, 2020; Kilpatrick, 1987; Polya, 1973; Silver, 1994). Even though some researchers (Crespo, 2003; Crespo & Sinclair, 2008; Ellerton, 1986a, 1986b, 1988; English, 1996, 1997, 1998, 2003; Lavy & Shriki, 2010; Silver & Cai, 1996; Silver et al., 1996) have provided some insights about issues pertaining to this line of investigation, we do not know enough about the extent to which preservice teachers, who themselves are students, are able to pose problems by modifying the conditions of a given problem. In fact, the research community continues to investigate the different aspects of teaching and learning how to pose mathematical problems (Ellerton, 2013; Felmer, Pehkonen, & Kilpatrick, 2016; Silver, 2013; Singer, Ellerton, & Cai, 2013, 2015) including analysis of textbooks (Cai & Jiang, 2016; Cai et al., 2016).

From a mathematical point of view, generalizing, proving general statements, and generating problems by considering converse-type problems, are important mathematical activities. Generalizing mathematical patterns is one of the most important processes that contributes to the development of mathematics. According to Sawyer (1982), generalizing “is probably the easiest and most obvious way of enlarging mathematics knowledge” (p. 55). Proving these general statements is one of the central activities of doing mathematics. When creating or discovering a theorem, it is often worthwhile to investigate whether the converse of the theorem holds or what additional conditions or restrictions must be added for having a converse-type theorem.

In this study, 17 prospective secondary mathematics teachers were asked to pose problems related to each of four geometric situations. The four problem situations were chosen as to allow for posing general problems, problems about proving general formulas, and converse-type problems. The students generated a total of 225 responses (199 mathematical problems or questions, 4 nonmathematical problems or questions and 22 statements). The 199 mathematical problem were categorized as well-posed problems (168) and ill-posed problems (31). I used Author’s (1998) and Moses, Bjork, & Goldenberg’s (1990) frameworks for analyzing the strategies that the students used to pose the problems. The framework includes mainly the following seven strategies to pose the problems: variation of unknowns, variations of knowns or givens, variations of restrictions, reversing knows and unknowns (converse-type problems), generalizing, thinking of patterns, and proving (Contreras, 2003; Moses et al., 1990).

The most common strategies used by the students to pose the problems, and number of problems, were: generalization (38), variation of knowns (25), variation of unknowns (21), and a combination of strategies (12). Even though students generated a diversity of problems, only 10 students posed general problems and only two students posed at least one general problem for each geometric situation. In addition, the students rarely posed converse-type problems and proving problems. Given that most of the students were majoring in mathematics, the findings are not very encouraging. Thus, appears to be a need for prospective mathematics secondary mathematics teachers to learn how to pose these types of problems.

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COMPREHENSION OF COMPLEX MATHEMATICAL TASKS WITHIN THE SCHOLARIZED CULTURAL ENVIRONMENT IN THIRD-GRADE

ENTENDIMIENTO DE TAREAS MATEMÁTICAS COMPLEJAS DENTRO DEL ENTORNO CULTURAL ESCOLARIZADO EN TERCER GRADO DE PRIMARIA

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Keywords: Cognition, Algebraic thinking, Order of operations, Problem solving

The understanding of complex quantitative relationships requires analyzing the cognitive processes of mathematical representations in students (Vygotsky, 1988), within school cultural environments. The development of concepts at an early age represents an opportunity in the forming of habits of abstract thinking for students of basic education (Carpenter et al, 2005; Carraher & Schliemann, 2007). The understanding of the quantitative relationships of mathematical concepts requires several areas: 1) the analysis of cognitive processes, through their mathematical representations and their discourse; 2) contemplate the school cultural environment in which children learn 3) the possibility of understanding complex concepts such as the order of operations or algebraic thinking.

For the present study, the understanding of quantitative relationships of 30 third-grade primary school students was analyzed, in complex mathematical tasks of the order of operations and algebraic thinking, in their schooled cultural environment (public primary). A teaching experiment was implemented based on tests of mathematical competence, concrete manipulative tasks based on part-whole relation (Davydov, 1962), and a content analysis of the students' discourse. The assessment of understanding was based on reactive tests of the order of operations and algebraic thinking. These tests were complementary with interviews with each participant. The tasks in the three stages of the teaching experiment were correlated with each other (RhO Spearman) to assess the internal consistency of the assessment.

The results indicate that 27% of the third-grade students expressed at least a potential understanding of the algebraic thinking tasks. The meaning of the variable and the unknown was linked to unknown or hidden quantities. 47% also expressed this understanding of the order of operations tasks. The justification for using the order of operations was the union between quantities that are multiplied compared to quantities that are added. The average scores of the children during the 10 sessions of the sequence had a high correlation with the average scores of the students in the tasks of algebraic thinking (0.733 ($p < 0.000$)) and the order of operations (0.769 ($p < 0.000$)). The results are compatible with the findings of multiple investigations of the order of operations or PEMDAS (Glidden, 2008; Gunnarsson et al., 2016; Linchevski & Livneh, 1999; Papadopoulos & Gunnarsson, 2020; Taff, 2017; Zorin & Carver, 2015).

It is possible, thanks to the potential of the formation of complex thinking habits, to understand algebraic thinking tasks and the order of operations schoolchildren at an early age, from concrete and significant experiences; When a child comes to understand and see the significance of mathematical tasks, he becomes enthusiastic about learning mathematics.

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TRANSFORMATIONS OF FIGURES IN O'DAM EMBROIDERY´S

TRANSFORMACIONES DE FIGURAS EN MORRALES O'DAM

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Keywords: Ethnomathematics, Geometry and spatial thinking, Secondary Education

A current of Ethnomathematics ("... set of knowledge produced or assimilated by an indigenous sociocultural group: counting, measuring, organizing space and time, designing, estimating and inferring, valid in their own context" (Villavicencio p.173 2001, cited by Avila p. 22 2014) is defined as the study of mathematics from other cultures, as indigenous peoples in various parts of the world (Suarez , Acevedo and Huerta s , 2009). Since each indigenous culture has its own language, representations of numbers, symbols, and number systems, so its instruments, units of measurement, and local forms of measurement make up a unique mosaic such as the O'dam culture that has a history ancestral in embroidery.

From the above, it is important then to prioritize meaningful learning in the classroom and support of learning strategies that enable students that are facing real problem situations, relevant and link with their environment, for this , the pedagogical approach that Challenge-Based Learning has meets the necessary conditions to achieve meaningful learning in students, discarding the idea that the teacher must teach a mathematical area or content so that learning can arise in students.

So, how to potentiate the learning of geometric concepts about transformations of figures in the plane, taking into account the cultural context of O'dam students? Thus, the challenge was proposed to the students to develop an embroidered backpack (bhai'mkar) or napkin, typical of their culture, with designs that included figure transformations such as translations, rotations and symmetries. The research was based on the ACODESA methodology: collaborative learning, scientific debate and self-reflection (Hitt and Cortés, 2009). Data was collected through observation, photographs, videos and products made by students (Figure 1). The proposed activities allowed students to explore different sources of available queries, interviews and search for information in textbooks. Through exhibitions, it was observed that the students were able to describe the type and characteristics of the transformations used to design their embroidery, thus helping to naturally identify movements in the plane in embroidery's in their community.

The O'dam indigenous cultural group uses a naturalistic and geometric cut in their handicrafts, so even though the students did not explicitly know the types of transformations used in their embroidery's; applied, demonstrated and communicated learning based on figure transformations.

Figure 1. O'dam embroidery designs based on figure transformations

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TRANSFORMACIONES DE FIGURAS EN MORRALES O'DAM

TRANSFORMATIONS OF FIGURES IN O'DAM EMBROIDERY´S

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Palabras clave: Etnomatemática, Geometría y pensamiento espacial, Educación Secundaria.

Una corriente de la Etnomatemática ("…conjunto de los saberes producidos o asimilados por un grupo sociocultural autóctono: contar, medir, organizar el espacio y el tiempo, diseñar, estimar e inferir, vigentes a su propio contexto" (Villavicencio, p. 173, 2001, citado por Ávila p. 22 2014,) está definida como el estudio de las matemáticas desde otras culturas, como pueblos indígenas de diversas partes del mundo (Suárez, Acevedo y Huertas, 2009). Dado que cada cultura indígena tiene su propio lenguaje, representaciones de números, símbolos, y sistemas numéricos, por lo que sus instrumentos, unidades de medida y formas de medición local conforman un mosaico único como lo es la cultura O'dam que tiene un historial ancestral en bordados.

A partir de lo anterior es importante priorizar el aprendizaje significativo en el aula, y apoyarse de estrategias de aprendizaje que permitan a los estudiantes se enfrenten a situaciones problemáticas reales, relevantes y de vinculación con su entorno, para esto el Aprendizaje Basado en Retos cumple con las condiciones necesarias para logar el aprendizaje significativo en los estudiantes para que pueda surgir el aprendizaje en los estudiantes con su propia vivencia.

Entonces, ¿Cómo potencializar el aprendizaje de conceptos geométricos sobre transformaciones de figuras en el plano, tomando en cuenta el contexto cultural de alumnos O'dam de educación secundaria? Así, se propuso el reto a los estudiantes de elaborar un morral bordado (*bhai´mkar)* o servilleta, propios de su cultura, con diseños que incluyeran transformaciones de figuras como traslaciones, rotaciones y simetrías. La investigación se apoyó en la metodología ACODESA: aprendizaje en colaboración, debate científico y auto reflexión (Hitt y Cortés, 2009). Se recolectaron datos a través de la observación, fotografías, videos y productos elaborados de los estudiantes (Figura 1). Las actividades propuestas permitieron a los estudiantes indagar en diferentes fuentes de consultas disponibles, entrevistas y búsqueda de información en libros de texto. Por medio de exposiciones se observó que los estudiantes lograban describir el tipo y características de las transformaciones utilizadas para el diseño de su bordado, ayudando así a identificar de forma natural los movimientos en el plano en tejidos elaborados en su comunidad.

El grupo cultural indígena O'dam usa en sus trabajos manuales un corte naturalista y geométrico, por lo que aun cuando los estudiantes explícitamente no sabían los tipos de transformaciones que se usan en sus bordados; aplicaron, demostraron y comunicaron el aprendizaje con base en las transformaciones de figuras.

Figura 1. Diseños de bordados O'dam basados en transformaciones de figuras.

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YOU SAY BRUTAL, I SAY THURSDAY: ISN'T IT OBVIOUS?[1]

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Keywords: Systemic Change, STEM/STEAM, Equity and Diversity

Programmatic collaborations involving mathematicians and educators in the U.S. have been valuable but complex (e.g., Heaton & Lewis, 2011; Bass, 2005; Bass & Ball, 2014). Sultan & Artzt (2005) offer necessary conditions (p.53) including trust and helpfulness. Articles in Fried & Dreyfus (2014) and Bay-Williams (2012) describe outcomes from similarly collaborative efforts; however, there is a gap in the literature in attending to how race and gender intersect with issues of professional status, culture, and standards of practice. Arbaugh, McGraw and Peterson (2020) contend that "the fields of mathematics education and mathematics need to learn how to learn from each other - to come together to build a whole that is greater than the sum of its parts" (p. 155). Further, they posit that the two must "learn to honor and draw upon expertise related to both similarities and differences" across disciplines, or cultures. We argue that in order to do this, we must also take into account race, gender, language. For example, words like *trust* or *helpfulness* can read very differently when viewed from personal and professional culture, gender, or racial lenses.

This poster shares personal vignettes from the perspective of three collaborators – one black male mathematician, one white female mathematics educator, and one white woman who was trained as a mathematician but works as a mathematics educator - illustrating some of the complexity of collaboration. The title of the poster comes from a moment in conversation among the authors. One of the women, recalling a conversation with the mathematician, said, "oh, that conversation was brutal," without acknowledging or considering the history and potential painful ramifications of the word "brutal" when used about an interaction between a white woman and a black man - ramifications that could create barriers to collaboration. The mathematician returned with, "you say brutal, I say Thursday," meaning that the conversation was totally within the norms of a conversation between mathematicians. The vignettes in the poster serve as a contribution toward an eventual framework for studying and discussing intersectionality in collaborations in education.

We come to this work embodying a "humanistic perspective of mathematics as a discipline that drives and is driven by human endeavor" (PME-NA equity statement). Mathematics is deeply connected to the stories and histories of the people *doing* the mathematics. There has been a long history of positioning mathematics educators and mathematicians in problematic ways. Even more troubling is the way in which marginalized groups are positioned with respect to mathematics. Not attending to these critical identities falls short when we try to understand the goals, outcomes, and effects of collaborations between mathematicians and mathematics educators.

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KINDERGARTEN STUDENTS' SPATIAL THINKING: PRACTICES ON DEBUGGING OF BUILDING BLOCKS

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Keywords: Spatial Thinking, Kindergartener, Debugging, Computational Thinking

Spatial thinking is a process to mentally represent and transform objects and to interpret the relationship among these objects (Clement. 1998). Abstraction is a core component supporting faster and more effective spatial thinking by removing unnecessary details and producing a simpler representation (e.g., in mental rotation; Lovett & Schultheis, 2014). Computational thinking (CT) is a concept rooted in computer science, which refers to a set of problem-solving methods to solve complex problems (Wing, 2006). CT has suggested abstraction as one of the main concepts (Wing, 2006). Indeed, CT and spatial thinking interrelate with each other (Román-González, Pérez-González, & Jiménez-Fernández, 2017). This study focuses on the interrelation between spatial thinking and CT.

Giving more attention to abstraction as a concept of CT could support students' mathematics learning (Rich & Yadav, 2020). Likewise, spatial thinking is essential for students to develop some mathematical ideas (e.g., number sense) in early ages (Cheng & Mix, 2014; Geary & Burlingham-Dubree, 1989; Gunderson et al., 2012; Stieff & Uttal, 2015; Verdine et al., 2014). Further, some researchers argued that playing blocks is associated with improving children's spatial thinking (e.g., Caldera et al., 1999; Connor & Serbin, 1977; Verdine et al., 2014). Although many studies have concentrated on constructing blocks as an intervention to promote children's spatial thinking, far fewer have examined children's computational thinking in the constructing and deconstructing process. To help fill this knowledge gap, this study investigated the possible impact of debugging (finding and fixing mistakes) –an approach of CT (Angeli et al., 2016) – on kindergartener's spatial thinking. Further, debugging might also be helpful for their mathematics skills by enhancing spatial thinking.

The design of the study includes a paper guideline that serves as algorithms and an authentic model. The guideline is a series of solid figures showing detailed steps. Each step includes the shape of a building block being used and the outcome for this step while the model has mistakes when matching with the guideline (e.g., the incorrect orientation of piece on correct place, missing piece). The students need to deconstruct the given model by following the guidelines to detect the mistakes. Once they find and fix a mistake, they need to reconstruct the model with the guideline. In addition, pre- and post-tests will be used to measure students' spatial thinking and mathematics. We hypothesize that CT-inspired-heuristic such as debugging an authentic model of building blocks will have a positive impact on spatial thinking and mathematics in early ages. The findings will contribute to the theory and practice of developing students' spatial thinking in early ages. The study will also expand and enrich the discussion on the interplaying relations between computer science and mathematics.

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PRECALCULUS, CALCULUS, OR HIGHER MATHEMATICS

RESEARCH REPORTS

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TWO PROSPECTIVE MIDDLE SCHOOL TEACHERS REINVENT COMBINATORIAL FORMULAS: PERMUTATIONS AND ARRANGEMENTS

DOS FUTUROS MAESTROS DE ESCUELA INTERMEDIA REINVENTAN FÓRMULAS COMBINATORIAS: PERMUTACIONES Y ARREGLOS

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We report on findings from two one-on-one teaching experiments with prospective middle school teachers (PTs). The focus of each teaching experiment was on identifying and explicating the mental processes and types of intermediate, supporting reasoning that each PT used in their development of combinatorial reasoning. The teaching experiments were designed and facilitated to guide each PT toward reinventing multiple combinatorial formulas. Drawing on a subset of this data, we describe the development of the PTs' mental processes and reasoning as they came to construct formulas for counting permutations and arrangements without repetition, and we analyze our findings through a psychological constructivist framework.

Keywords: Advanced Mathematical Thinking; Cognition; Teacher Education–Preservice

Enumerative combinatorics is a mathematical discipline concerned with the activity of *counting*. More specifically, by "counting," we mean finding the cardinality of particular set, either by exhaustive listing or by using a more sophisticated technique. Researchers have taken an increased look into students' combinatorial reasoning (e.g., Batanero et al., 1997; English, 1991; Fischbein & Gazit, 1988; Lockwood, 2011; Maher et al., 2010; Tillema, 2013), but relatively little research has investigated the combinatorial reasoning of *teachers*, either current or prospective (exceptions to this include McGalliard & Wilson, 2017, and Speiser et al., 2007). Research on teachers' development of combinatorial reasoning can be an important component of studying the development of the specialized mathematics content knowledge (Hill et al., 2008) needed by teachers.

The aim of the present study was to investigate the nature of prospective middle school teachers' (PTs') combinatorial reasoning. The study involved one-on-one teaching experiments with two PTs, DC and NK. In these teaching experiments, DC and NK were guided to develop increasingly sophisticated conceptualizations and ways of reasoning needed to solve increasingly complex tasks. Ultimately, both PTs constructed multiple generalized counting formulas and ways of making sense of those formulas. We present a subset of data in which the PTs developed formulas for counting permutations and in which DC developed a formula for counting arrangements without repetition. The two tasks outlined below, as well as numerous extensions to these tasks, were instrumental in guiding the PTs' development.

- *3-Cube Towers with 3 Colors.* Using three different colors of cubes, how many different towers 3-cubes-high can be made without repeating colors? [Answer: 3!, or 6]
- *3-Cube Towers with 5 Colors.* Using five different colors of cubes, how many different towers 3-cubes-high can be made without repeating colors? [Answer: $5\times4\times3$, or 60]

The following research question guided this study: How do two PTs' conceptualizations and forms of reasoning develop as they are given increasingly complex tasks pertaining to permutations and arrangements with repetition, particularly tasks within the context of constructing block towers?

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Literature Review

Research on combinatorial thinking has consistently shown that students tend to struggle with solving counting problems. This includes, for instance, choosing an appropriate combinatorial operation for a given situation (e.g., Batanero et al., 1997; Fischbein & Gazit, 1988). Researchers have identified certain practices that help students develop in their combinatorial reasoning, such as productive listing (Lockwood & Gibson, 2016) and reflecting on the outcomes being counted (Lockwood, 2014). Tasks within the context of enumerating block towers have been shown to be particularly useful for developing combinatorial reasoning, both with children (Maher et al., 2010; Maher & Speiser, 1997) and with prospective elementary teachers (McGalliard & Wilson, 2017; Speiser et al., 2007).

One study investigated the processes by which two undergraduates (former integral calculus students) came to reinvent four combinatorial formulas (Lockwood et al., 2015), including those pertaining to this study. Framing their analysis using Lockwood's (2013) model of combinatorial thinking (sets of outcomes, counting processes, and formulas/expressions), the authors conjectured that a reinvention of counting formulas would require reasoning about counting processes—in particular, about the Multiplication Principle. Instead, the authors found their participants relied heavily on empirical patterning to reinvent certain formulas. We situate our study within the literature, guiding two PTs to construct generalized combinatorial formulas and develop in their combinatorial reasoning along the way.

Theoretical Perspectives

This study used a teaching experiment methodology (Steffe & Thompson, 2000) which was most appropriate given that the primary purpose of the study was to investigate the development of NK's and DC's conceptualizations and ways of reasoning as they were posed with increasingly complex combinatorial tasks. Ultimately, these tasks led to the construction of several combinatorial formulas and ways of using and making sense of those formulas; in this sense, our study is consistent with the principle of guided reinvention within the theory of Realistic Mathematics Education (Freudenthal, 1973; Gravemeijer, 1999). We view this work as contributing toward the overarching goal of developing a hypothetical learning trajectory (Simon, 1995) for permutations and arrangements without repetition, tracing the development of the PTs' conceptualizations and ways of reasoning as needed to make sense of increasingly complex tasks and—including the construction of generalized counting formulas.

Mathematically, we focus on the development of two combinatorial structures: permutations and arrangements without repetition. A *permutation* is a particular ordering of a set of n (distinct) objects. The number of permutations of n objects can be expressed as $n\times(n-1)\times(n-2)\times\ldots\times2\times1$, or $n!$. An *arrangement without repetition* is an ordering of a subset of k objects from a set of n distinct objects, the number of which can be expressed as $\frac{n!}{(n-k)!}$.

Reformulating the elaboration of Piaget's theory of abstraction by von Glasersfeld (1995) and Steffe (Steffe, 1998; Steffe & Cobb, 1988), Battista (1999, 2007) proposed *levels* of abstraction of sensory objects and motor activities (collectively called mental items). At the most basic *perceptual* level, a person has abstracted an item from their experiential flow and can perceive the item as a coherent unit. At this level, the item cannot be re-presented (visualized) without the presence of relevant sensory input. At the *internalized* level, a person can either re-present sensory objects in their mind in the absence of perceptual material or reenact a motor activity in the absence of kinesthetic signals from physical movement. However, the internalized level is limited in that a person cannot yet reflect upon an item's re-presentation or analyze an item's composition and structure. Only upon reaching the *interiorized* level of abstraction can a person reflect upon and analyze internalized items.

Interiorization "leads to the isolation of structure (form), pattern (coordination), and operations (actions)" (Steffe & Cobb, 1988, p. 337). Procedurally, a student's focus shifts from performing a sequence of actions to analyzing the meanings and results of those actions, treating the procedure as an object of reflection. Upon reaching the *second level of interiorization*, a person can perform *operations* on mental items without re-presenting or generating the material, and they can use symbols as "pointers" to the abstracted material, substituting the material with these symbols. Procedurally, second level interiorization allows a person to mentally operate on a procedure's components without actualizing the procedure using numbers. Symbols can refer to abstracted spatial components (e.g., "positions" in a generic 3-cube tower) or numerical components (e.g., a numerical procedure from reasoning about positions). At the *third level of interiorization*, a person can meaningfully represent the arithmetic/algebraic structuring of a generalized computational procedure with algebraic notation.

Findings and Analysis

In this section, we present, make inferences about, and analyze selected key events in the development of the PTs' conceptualizations and forms of reasoning.

Permutations

Episode 1. Both teaching experiments began with the 3-Cube Towers with 3 Colors task, and in both cases the PT was shown an example. NK constructed all six 3-cube towers using a single blue, green, and red cube, deconstructing the previous tower in order to make the next one.

> NK: The easiest way to think about it would be to start with a color. So, I would start with blue at the top. I always start at the top and go down. So like blue, green, red would be one, and then blue, red, green would be two.

NK then constructed the two 3-cube towers with a green top cube and the two towers with a red top cube. Notably, NK's count did not match her intuition (that there would be 3^2 towers), so she used the cubes to construct all towers in order to verify her count. Given the first follow-up task—counting 4-cube towers with 4 colors without repetition—NK responded,

> NK: OK, so same thing. Let's start with blue. So we have blue, black, green, red. I'm not gonna sit here and make all of these because, if you start with blue, you know that there's, if we have 3, then there's 6 combinations that can have blue at the top. … So blue times 6, 6 times 4, 24.

After asking for further elaboration, NK continued her explanation.

> NK: So, however many times you can arrange these three [bottom three cubes] is gonna be however many times you can arrange this whole thing, because blue will constantly be at the top. So you can just kind of omit it [the top cube] out of your thinking and see how many times this [bottom three cubes] can be combined, and then throw the blue at the top and that's the tower of four.
>
> NK then indicated the same number of towers could be made with each choice of top-cube color, motivating her multiplication of 6×4.

DC, on the 3-Cube Towers with 3 Colors task, constructed all six towers (so that all towers were present on the work-table) using a strategy similar to NK's, except his construction process was anchored by the color of the bottom cube (which he called the "base") instead of the color of the top cube. Given the 4-cube tower follow-up task (with black cubes added as the fourth color), DC deconstructed each of his original six towers; he then placed three green cubes in a line on the work table and was going to place three red cubes next to them, but he shifted his approach and instead constructed six 4-cube towers each with a green cube as base. He reasoned,

> DC: Each color has two possible towers if that color is the base [of a 3-cube tower]. But I *rose* it a level. So this is still the red base [*pointing to the two 4-cube towers with red cube second from the bottom*], but it's on top of the green base.

To clarify, DC had constructed six 4-cube towers, all with a green base and two with a red cube second-from-the-bottom. When he said he "rose it a level," he was referring to his action of taking six 3-cube towers and adding a green cube to each. After this, DC constructed all six 4-cube towers with a red base. He then predicted there would be the same number of towers with blue and black bases, concluding there would be $6 \times 4 = 24$ total 4-cube towers. Later, DC further explained, similar to NK, that 4-cube towers with green bases can be made by taking the composite of six 3-cube towers and appending a green cube to the bottom of each tower.

Inferences. We infer that NK conceptualized towers spatially as composites consisting of a single cube appended to an (n-1)-subtower, while DC had a similar spatial structuring but with the appending cube in the bottom position rather than the top. This spatial structuring led the PTs to organize their processes of tower construction by using the appended cube as an anchor, similar to what English (1993) calls a "major constant item." When transitioning to the 4-cube-tower follow-up task, both PTs used a recursive strategy, now reconceptualizing each 3-cube tower as a composite unit and operating on the composite of six 3-cube towers (mentally, in NK's case, or perceptually, in DC's case) by appending to each tower an additional cube. We also infer that, initially, DC planned to construct each 4-cube tower systematically, but he realized the composite of 4-cube towers could be constructed by building on the composite of 3-cube towers.

Analysis. Both NK and DC had interiorized the process of constructing 3-cube towers, indicated by the fact that their constructions were coordinated by a spatial structuring. Further, we interpret NK's reasoning on the 4-cube-towers task as reasoning about "symbolic" positions (of a generic 4-cube tower) rather than about specific instances of towers, indicating second level interiorization of the process of constructing 4-cube towers. DC's reasoning was also coordinated by a spatial structuring, but his reasoning relied on the perceptual material available on the worktable and did not appear to draw on the structure of a generic 4-cube tower. Thus, we interpret that DC interiorized the process of constructing 4-cube towers.

Episode 2. *NK.* Extending to counting 5-cube towers with 5 colors of cubes, and without constructing any towers perceptually, NK reasoned,

> NK: So you could do the same thing where you start with a different color, but how many different combinations you can make with 4 is going to be 24. …Let's pretend [the fifth color] is white. So you could start with white, and you would have 24 different combinations of these [gestured to other four colors of cubes] that would go under the white. So it would be 24 times 5, because each one of these colors would get their chance to be at the top.

Extending to counting 6-cube towers with 6 colors, NK described her strategy more generally.

> NK: So there's a pattern. So it's, whatever 5 is, which we found that 5 is 120, it would be 120 times 6. … Sorry, I keep thinking with the mindset of, just like, you can omit the top cube. So you can start with like a constant, your constant would be like whatever color [is at the top].

The next follow-up task jumped to counting 9-cube towers with 9 colors. She did not have an immediate answer, but she took colored pencils and paper and started reflecting on her previous enumerations, searching for a pattern.

> NK: Oh! Okay, okay. So, if you were to start with two colors, there are two combinations. Then you go to three, we found that there are six. Then you go to four, and find whatever 6 times 4 is, which is 24. Then you take 24 and multiply it by 5. 'Cause we have 5 things, so I multiplied it by 5 and got 120. And we came here, and we multiplied 120 by 6, which is… 720. And then you can

> take that and multiply it by 7… which is 5,040. And then you can do that and multiply it by 8, which is the number of combinations you get if you had 8 colors, which is 40,320. And then you can multiply it by 9 to finally find how many it would be, which would be 362,880.

Inferences. As with her spatial structuring of 4-cube towers from 3-cube towers, NK visualized appending a fifth-color cube to the top of each 4-cube subtower, then reasoning multiplicatively with the number of available colors. Similar to her reasoning in Episode 1, NK counted 5-cube towers by reasoning about positions—that is, her reasoning drew on an abstraction of the spatial structure of a generic 5-cube tower. With this spatial structuring guiding her reasoning, NK used a recursive strategy once more, multiplying the number of top-cube color possibilities by the number of ways to arrange the cubes in the (n-1)-cube subtower.

Analysis. NK used an interiorized spatial structuring to guide her reasoning about the number of towers of any height, although at this point her reasoning was recursive in the sense that she relied on knowing the number of (n-1)-cube-towers to compute the number of n-cube-towers. As with the previous episode, NK's reasoning is based on positions, indicating second level interiorization of her recursive scheme for counting n-cube towers.

Episode 2. *DC.* Counting 5-cube towers with 5 colors, DC searched for a pattern in the number of towers that are present with a given color base cube.

> DC: I'm going to take a guess. So, we went from two of each to six of each, which means we multiply by 3. So when we multiply by 3 again… oh no, I should multiply by 4. I'm gonna multiply that [pointed to the six constructed 4-cube towers, each with a green base] by 4 because there are four colors, excluding the new color. I don't know if that's right…. So, 6 times 4, 24 combinations. No, 24 is what we have now if we did just have the 4 of them.

He placed the six 4-cube towers with green bases in one group, and adjacent to those he placed the six 4-cube towers with red bases. He then lined up six blue cubes and six black cubes to symbolize the 4-cube towers with blue and black bases, respectively. With this, he could see there would be 24 4-cube towers in total. Although he still expressed uncertainty, DC predicted there would be 24×5 5-cube towers that could be made with 5 colors.

To help DC resolve his uncertainty, he was given five different colors of cubes arranged in a line on the table and was asked how many different ways the five cubes could be rearranged by interchanging cubes. DC exhaustively enumerated 24 of the possible 5-cube orderings, keeping the right-most cube (which he still called the "base") constant and multiplying, at the end, by 5. His enumeration was systematic although not a complete odometer pattern (English, 1991). Moving to counting 6-cube orderings in a row, DC reasoned,

> DC: I know that in the 5-color scheme, there are 120 combinations per base color. Now, with a *super-base* sixth color, there are six different options for the super-base, and 120 options for the regular base. And so 6 times 120 is… 720.

Of note, DC's verbal comments indicated he thought about constructing towers and enumerating orderings of cubes interchangeably. Extending to counting 9-color orderings in a row, DC said,

> DC: I would go through if we had seven colors and then eight colors first. All I would do is take that 720… multiply it by 7, to get 5,040. That's how many combinations we have for seven colors. Then multiply that by 8… 40,320. And then for nine, we would multiply that value by 9.

Inferences. DC had constructed 12 of the 24 4-cube towers with 4 colors, and he placed six blue and six black cubes on the table to represent the remaining 12 towers. With this visual aid, DC predicted there would be 24×5 5-cube towers, but he did not seem to visualize "raising" this 24 5-cube-tower composite with the added sixth color of cubes (as he did to construct 4-cube towers from

3-cube towers). After moving to counting 5-cube orderings, DC's counting was systematic, and he reflected on this process to enumerate orderings of larger numbers of cubes.

Analysis. At first, DC seemed to have internalized, but not interiorized, the process of constructing 5-cube towers. After perceptually finding all 5-cube orderings with a fixed right-most cube (and reflecting on those actions), DC reasoned about positions—specifically, multiplying the number of 5-cube orderings with a fixed right-most cube (24) by the number of possible right-most cubes (5)—which indicated second-level interiorization of the process of constructing 5-cube towers. With subsequent tasks, his reasoning extended to counting increasingly larger orderings, indicating second level interiorization for a more general (recursive) scheme for counting permutations, although it became less apparent that DC's reasoning was guided by his spatial structuring.

Episode 3. Extending to counting 20-cube towers with 20 colors, NK began to search for a function that would produce the number of towers of a given height (and number of colors), n. Thinking the function would be exponential, but unable to predict a specific formula, she reflected on her previous strategies.

> NK: So you would start at 20, so it would be 20 times 19 times 18 times 17 times 16 times 15 times 14 times 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1. The whole time I've been multiplying *this* direction [motioned from left to right, indicating starting at 1 and ending with 20], but I just started multiplying by the number of colors we have.

On follow-up tasks, NK then predicted the number of 100-cube towers with 100 colors to be $100\times 99\times 98\times \ldots\times 1$, and she wrote a general expression for the number of n-cube towers with n colors to be $n\times(n-1)\times(n-2)\times(n-3)\times \ldots\times 1$.

DC predicted the number of 20-cube towers with 20 colors by multiplying the number of 9-cube towers, 362,880, by 10, then by 11, by 12, etc., and finally by 20. He searched for an "equation" that would calculate the number of towers, but he was not able to find one. Extending to counting 100-cube towers, DC described that he would continue multiplying the product found in the 20-cube-tower task by 21, by 22, by 23, etc., until finally multiplying by 100. After leading DC through a quick review of his work so far, DC said,

> DC: *Oh wait*! Would this work, where we have 2 there, and that's the 2 combinations we can get with 2 colors…. Then we multiply that by 3, because then we added a third color. And then we would multiply that by 4, because we added a fourth color. Then we multiplied that by 6, which is what I'm doing, but then you could just go up to 100.

He wrote the following to express what he had verbalized: $(((((2)3)4)5)6\ldots)100$. He initially kept the parentheses so as to maintain the order of multiplication, but he later cited the associative property of multiplication to reason that he could remove the parentheses. Finally, extending to counting n-cube towers with n colors, DC said there would be $2\times 3\times 4\times \ldots\times n$ different towers.

Inferences. NK's calculations, now reversed from their original order, no longer seem to be guided by a mental process of recursively operating on towers of height n-1 to form towers of height n. For DC, after guiding him through a review of his previous enumerations, he reconceptualized 362,880 as the product of consecutive integers $(2\times 3\times 4\times \ldots\times 9)$. He then extended and generalized this strategy, although at this point his reasoning still seemed recursive.

Analysis. Both NK and DC reached the third level of interiorization for counting permutations, indicated by both PTs' generalized algebraic expressions. Even still, NK's reasoning seemed more sophisticated than DC's as she realized she could reverse her multiplicative process and achieve the same result, which we suspect is important for counting and reasoning about arrangements.

Overall, both PTs exhibited remarkable progress in their conceptualizations and ways of reasoning about permutations. NK and DC were able to enumerate block tower composites successfully and efficiently with cubes, either mentally or with perceptual material, and use this to structure their numerical reasoning.

Arrangements—DC

Due to space constraints, only a subset of DC's progress toward thinking about arrangements without repetition is presented.

Episode 4. DC first considered the task of counting 3-cube towers with 4 possible colors of cubes. He systematically constructed the six 3-cube towers with a red base, then reasoned:

> DC: If we use red as our first bottom color, our first base, after red only 3 other colors can be the middle color in our tower of 3. Then, for each of those, there are only 2 colors remaining. And since it's a tower of 3, it's just those 2 colors on top. … So 6 total for red, times 4.

He clarified "times 4" was to account for the number of possible base-cube colors. Given the 3-Cube Towers with 5 Colors task, he constructed the remaining 3-cube towers with a red base, including those with the newly-added fifth color, culminating in 12 3-cube towers with a red base on the worktable.

> DC: So now there are 12 different combinations for each color as base, times 5—60. There are 60 possible combinations.

Given the next follow-up task, counting 3-cube towers with 6 colors, DC initially tried to find a pattern, first noticing a doubling pattern in which the number of towers "per base" went from 6 (using 4 different colors) to 12 (using 5 different colors). DC predicted there would be 24 towers per base in the 6-color task, but he was not certain. He resorted to systematically constructing all 3-cube towers with a red base using 6 colors, finding 20 towers in total. He reflected on the number of towers that were added when transitioning from towers with 5 colors to towers with 6 colors, but this did not lead to a new insight. We returned to the task of counting 3-cube towers with 4 colors, and it was in this task that DC made a new insight.

> Int: With the first problem with 4 colors making towers 3-cubes-high, I remember you saying in this original problem that you thought about it as 3 times 2…. So, you saw the 3 times 2 in that problem. Could we see something similar when we move up to 5 colors?
>
> DC: I mean, it's 4 times 3. 'Cause there are 4 possible options for the second tier, and then on top of each of those there are 3 possibilities for what can be left…. That's 12 per base, and then 12 times the possible 5 bases is 60 combinations. With 6 colors, if we look at it in the same way we just did, there are 5 combinations for what can come on top of each base, times 4 possibilities for what can come on top of that second tier… 5 times 4 is 20, times the possible 6 bases, which is 120 possible combinations. So, if we were dealing with 4 colors—*Oh!* Here it is, I think I might have just done it. We have n times $n-1$ times $n-2$ equals that value…. This gives us the number of possible combinations above each base, times the number of bases, gives us the total number.

Inferences. DC's reasoning was guided by a specific spatial structuring: He conceptualized composites of 3-cube towers by focusing on a subset with a particular base-cube color appended to 2-cube towers, then generated the entire composite of 3-cube towers by multiplying by the number of possible colors for the base. When the teacher-researcher led DC to revisit and reflect on his numerical procedures, he realized a numerical pattern connected to his spatial structuring: the number of color possibilities "on top of each base" multiplied by the number of color possibilities "on top of that second tier."

Analysis. DC's process of counting 3-cube towers reached second-level interiorization, indicated by his reasoning about positions within 3-cube towers, and quickly progressed to third-level interiorization as he generalized his numerical procedure to counting 3-cube towers with any number of colors.

Episode 5. DC was asked to count k-cube towers with n different colors, without repetition.

> DC: Let's just say k=5 in this instance. Then the equation would be $n(n-1)(n-2)(n-3)(n-4)$. And… the integer that you're subtracting is k minus—like, goes up to k-1, but doesn't quite get to k. And so, it's like the reverse factorial here. Like, this is k-1 [pointed to 4 in n-4], k-2 [pointed to 3 in n-3], k-3 [pointed to 2 in n-2], k-4 [pointed to 1 in n-1].

When asked to generalize from k=5 to a general k, DC used this same "reverse factorial" idea and wrote $n(n-k-(k-1))(n-k-(k-2))(n-k-(k-3))\ldots$. He was asked if $(n-k-(k-1))$ might be able to be simplified; after some algebraic manipulation, he realized that, for any value of k, the number of towers of height k with n colors of cubes is $n(n-1)(n-2)(n-3)\ldots(n-(k-1))$.

Inferences. DC thought that a generalized expression for counting k-cube towers with n colors would require an expression involving both n and k, leading to his conceptualization of the "reverse factorial."

Analysis. DC's process for counting arrangements had reached third-level interiorization.

Conclusions

The PTs' spatial structuring was instrumental toward guiding the development of their reasoning about permutations and arrangements. Both PTs developed recursive strategies but progressed past recursion through processes of action, reflection, and abstraction. This ultimately led their processes of enumerating permutations and arrangements to third-level interiorization.

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REPORT OF A CLASSROOM EXPERIENCE FOR THE DEVELOPMENT OF DISTRIBUTION MODELS

EXPERIENCIA EN EL AULA PARA EL DESARROLLO DE MODELOS PARA EL REPARTO

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This paper presents the design and analysis of a Model-Eliciting Activity (MEA), which aims to support the refinement of the conceptual system associated with directly proportional distribution problems (example: linear function, percentage, proportion, variation, etc.). The situation explores a context that is close to the student, encouraging the construction of a generalizable mathematical model that can be transferred to diverse contexts. For the design, the principles of the Models and Modeling perspective (Lesh and Doerr, 2003) were considered. The target population were students enrolled in the first semesters of a university degree in Accounting. For the analysis, we considered the construction of models (performed in teams); the mathematical representations within the process observed (such as diagrams and tabular organizations); as well as the individual solutions of the students.

Keywords: Model-Eliciting Activities, Models and modeling, Distribution problems

Objectives and purpose of the study

Distribution problems may represent an opportunity for college students to delve into mathematical notions related to the multiplicative conceptual system. Researchers such as Martínez-Juste, Muñoz-Escolano and Oller-Marcén (2019) consider that distribution problems have gained relevance in recent years because they can explore a diversity of contexts close to the reality of the students.

The multiplicative conceptual system associated with distribution problems elicit fundamental ideas of mathematics, that´s why it is present in the curricula from elementary education to high school. In Mexico, these kind of problems are introduced in the third grade of elementary school (SEP, 2017) and, in subsequent years, variants of these problems are developed where notions such as: proportionality (direct and inverse), linear function, percentage, ratio, linear equation, rational number, proportion, variation and arithmetic progressions. According to Sánchez-Ordoñez (2013), distribution problems allow on the one hand, to shape reason, proportion and proportionality as mathematical objects that contribute to the understanding and mastery of the multiplicative conceptual field by students, and on the other hand, to identify the way in which ratios, proportions and proportionality are recognized and manipulated by students in classroom situations (p. 71,72).

Based on this, the intention is to answer the question: Which are the cycles of understanding that emerge in undergraduate students when they are solving a distribution MEA? In order to develop a possible answer, we present the analysis of the solutions for a situation which was solved by a group of students enrolled in the first semester of a degree in Accounting at the School of Economics, Accounting and Administration of our institution (UJED), with the intention of eliciting the notion of proportionality and to develop integrated knowledge in order to connect and refine the associated conceptual system. It should be noted that the activity was led by the teacher who is also the author of this article.

The Models and Modeling perspective (Lesh and Doerr, 2003) was fundamental for the theoretical framework. The proposed activity was designed as a MEA. Thus, the proposed situation aims to promote the construction of a reusable and generalizable model that provokes mathematical understanding in students in order to achieve the objective of refining and eliciting concepts

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associated with the multiplicative conceptual system; such as those associated with algebra (ratio, linear variation, function, etc.) as the ones associated with statistics (percentage, frequency tables, etc.). This activity is part of a sequence of activities which were aimed at achieving the course objectives, however, in this paper only the scope of the distribution MEA will be mentioned.

The analysis presented includes the solution interpretation of the student's productions that emerged during the problem solving process where the student used mathematical resources available in order to build a model that can be generalized to solve similar but varied situations.

The Models and Modeling Perspective (PMM) as a theoretical framework

This perspective was developed by Lesh and Doerr (2003) and proposes the resolution of problematic situations endowed with a real context, which can be approached from the particular, taking, as a starting point, the previous knowledge that each participant may have. Through the socialization of the solution proposals, the refinement of the solution model that satisfies the requirements of the approach is reached. The MMP considers learning as a process for developing conceptual systems (models), which emerge when students share and analyze situations that have more than one answer, so that the solution is not a number or word. The problems addressed should encourage students to describe, argue and explain the solution processes used. Doerr (2016) explains that learning mathematical content is a process of developing an adequate and productive model that could be used and reused in a certain range of contexts.

The MMP considers that learning mathematics is a process that includes progressive construction of understanding cycles, modification, extension and refinement of ways of thinking where the subjects manage to elicit a mathematical concept or construct at different levels by relating data, goals and the possible routes of solution they are exposed to when facing a problematic situation (Lesh and Doerr, 2003). Therefore, the product of learning is the process in which the fluidity of mathematical representations allows for the understanding of most of the mathematical constructions (Lesh and Doerr, 2003).

In MMP, the ADM or MEA are proposed as a way to generate products that go beyond short answers to specific questions. MEA allow students to get involved in an interactive and iterative process where they express, test and redefine their ways of thinking about significant problem situations (Doerr, 2016). Therefore,,the meaningfulness of the context provokes new mathematical understanding in the students, which allows them to express, in their procedures, their current knowledge that can come from their experience and/or their previous mathematical knowledge.

MEA must comply with the six design principles established by Lesh, Hoover, Hole, Kelly, and Post (2000) which are: reality or personal meaning, construction of the model, self-evaluation, model externalization or model documentation. simple prototype and model generalization.

The characterization of understanding cycles exposed by Vargas-Alejo, Reyes-Rodríguez and Cristóbal-Escalante (2016) was used for the analysis of the model construction process. The researchers propose a characterization based on their interpretation of the MMP in which they identify cycles (qualitative, quantitative and algebraic) that emerge when individuals approach contextualized situations. Understanding cycles is also associated with eliciting a mathematical construct or concept, which is why, for example, in a *qualitative understanding cycle* the individual gives meaning to the situation where the problem develops and can identify the variables involved, as well as the possible relationship between them. However, in order to express this relationship the individual only elaborates verbal descriptions, diagrams, analogies or metaphors. In contrast, in a *quantitative understanding cycle* the individual makes assumptions, discards useless information and may establish a quantitative meaning of linguistic expressions by making numerical comparisons, for this, the student may use tabular and graphical representations which leads to a more elaborate interpretation of the situation. When the individual seeks to describe and interpret a phenomenon through the construction, use, transit and coordination of different representations (tabular, graphic, verbal, etc.) where there is fluency between them and algebraic symbols can be manipulated to solve a situation, then he has reached the *algebraic understanding cycle*. Researchers agree that, in order to reach the last cycle, it is possible for the individual to pass through different interpretations of the situation and, previously, has managed to differentiate, integrate and refine different models.

Design studies

Within design research there are multiple methodologies and theoretical perspectives in education and other fields. Some researchers consider that process and tangible products (didactic sequences, associated conceptual systems, etc.) are involved in the design (Hjalmarson and Lesh, 2008). However, intangible products formed during the process are also relevant, since the objective of this methodology is to improve the design processes and, consequently, to obtain different results both within the process and the product.

The design of this experiment is composed of three phases: preparing for the experiment, experimenting in the classroom and conducting retrospective analyzes of the data generated during the implementation with the students (Cobb & Gravemeijer, 2008). This type of study aims to develop particular forms of learning while studying the learning obtained in the designed environments.

Investigation method

The target population, the context and the data collection techniques, as well as the theoretical and methodological tools for its analysis, are described below.

Study population

The activity presented was implemented in a group of 36 students enrolled in the first semester of studies for a degree in accounting and whose ages ranged between 18 and 20 years of age. In previous sessions, the members had solved a modeling activity, so they already had experience in dealing with this type of situation. The implementation of the activity required 4 sessions of one hour each.. During the implementation, data was collected through photographs, video, and student logs.

The participants were organized into nine teams made up of four members each. The activity was solved as a team and, later, the solution reached was socialized in a group discussion. In a final production, the students submitted an individual solution to the problem. The role of the teacher was key throughout the process, asking key questions to the students aimed at deepening the solution model reached.

Description of activities

The design and characterization of the models developed during the implementation were based on the MMP theoretical framework. To familiarize the students with the context of the situation, the "warm-up" activity is presented (Figure 1), which consisted of reading a newspaper article and a subsequent group discussion to clarify possible doubts regarding the understanding of the context. The MEA (Figure 2) was then presented, which addressed a problem related to the context of the reading. These activities are described below, accompanied by the image of the worksheets.

Warm-up activity (Figure 1). It had the aim of introducing the context from real and current data to sensitize students to the problem and motivate them to resolve the profit sharing situation in the ADM (Figure 2).

Model Eliciting Activity: Income Distribution. In this activity (Figure 2), the situation shows a sister and brother (Alondra and Paco) who decide to start a business together where the tasks they would perform are divided equally. However, the time that Alondra spends doing her tasks represents eight times more than what her brother Paco dedicated to his tasks. The situation also takes into consideration that Paco contributed a little more than half of the initial investment. This has given rise to the dilemma of how to distribute the income; therefore, the brother and sister have decided to ask for help so that the students have to propose solutions for the distribution of the profits.

Figure 1: Warm-up activity


PROBLEMA 3

Para comenzar su negocio de pasteles, Alondra y Paco habían invertido $800.00 pesos, de los cuales Paco había aportado $455.00 y Alondra $345.00 y, desde el principio, ambos hermanos habían comenzado trabajando tanto en la elaboración como en la venta. Sin embargo, negocio de pasteles había crecido, por lo que consideraron necesario dividir el trabajo, Paco se dedicó a calendarizar los pedidos, cobrar y entregar los pasteles, mientras que Alondra se encargó de comprar los ingredientes, hacer los pasteles y decorarlos. Paco tenía varios conocidos que habían probado los pasteles elaborados por Alondra, por lo que no tardó en comenzar a calendarizar los pedidos de pasteles y pastelitos. Los pedidos quedaron así:

Pedidos

- Para el Lunes debía entregar dos pasteles para el cumpleaños de Jorge
- Para el martes debía entregar seis pasteles para la cafetería "El encuentro"
- Para el miércoles debía entregar un pastel y una docena de pastelitos o cupcakes para el cumpleaños de Regina
- Para el jueves debía entregar 20 pastelitos o cupcakes para el babyshower de Sofi
- Para el viernes debía entregar 2 pasteles y 62 cupcakes para el cumpleaños del jefe de Raúl

Pedidos semanales

Al finalizar la semana, Alondra estaba muy cansada pues, por cada cuarto de hora que Paco había destinado a la venta, ella había dedicado 2 horas al proceso de preparación de los pasteles.

Ayuda a los hermanos a llegar a una forma de repartir las ganancias entre ambos. Puedes tomas como base la información que se te presenta y analizar las posibles maneras de dividir las ganancias entre ambos, también puedes buscar maneras alternativas de repartir las ganancias, sin embargo debes sugerir la que consideres mejor y justificar el por qué. Considera que Alondra y Paco continuarán con su negocio de pasteles, por lo que las condiciones pueden variar, por ejemplo, puede que haya veces que Paco trabaje más horas que Alondra o que Alondra decida invertir más dinero en la empresa.

Redacta una carta donde expliques cual es la manera que sugieres para la repartición de las ganancias, justifica las razones que te llevaron a concluir esto y realiza una comparación de los beneficios que conlleva esta repartición en comparación de las otras.

Figure 2: MEA income distribution

Results

In this section, students' productions are analyzed during the resolution of the proposed activities, first in a team and then in a group discussion. The characterization of the understanding cycles of Vargas-Alejo, Reyes-Rodríguez and Cristóbal-Escalante (2016) was used for the analysis of the construction process of a model that responds to the situation presented in the MEA exhibited by the students.

Understanding cycles

The warm-up activity (Figure 1) introduced the context. It was during the implementation of the MEA (Figure 2) were the students put into evidence the understanding cycles they went through, as well as the concepts and representations that emerged from their previous individual knowledge. The interaction with their teammates allowed to observe, in the written procedures, the solution model for income distribution. The observations made are briefly explained below.

Qualitative understanding cycle. This cycle took place after the students read the problem. In it, they selected the data that they considered important for the resolution of the ADM and discarded the information that they considered irrelevant. For example, within the data shown in student procedures is the division of labor. This data was part of the writing of the ADM; however, the information does not follow the arrangement in the document that allows for the visual comparison of the quantities. Therefore, the order where a visual comparison can be made of the data selected as relevant to the situation was notable in the students' productions. This allowed us to identify a qualitative approach to solving the problem.

Quantitative understanding cycle. After the students made a qualitative comparison of the selected information, they decided to make different mathematical representations in the construction of a solution model. Among the procedures exhibited, the following was observed:

Circular diagrams. The implicit use of percentages is shown to designate the proportion of the graph that corresponds to a category as a percentage of investment or time dedicated to a task.

Arithmetic representations. For the calculation of the fourth proportional, the students used the algorithm known as the “rule of three”. For example, this procedure allowed them to find the percentage proportion corresponding to the initial contribution of Paco and Alondra.

Tabular representation. The evidence shows the tabular organization where the linear variation was related, the arithmetic progression and the search for the unit value. This representation was used by teams 1 and 2. It shows the use of rational numbers and the designation of time units (hours); the teams used this to be able to identify and compare the time that each member dedicated to their tasks.

Algebraic understanding cycle. The evidence written by the members shows algebraic representations of linear functions.

Solution models formalization prior to group discussion

Students were asked to give a short presentation regarding their solution. All the teams presented their solutions. The solutions presented show: the distribution of the income taking into account each partner’s investment proportion; the time spent on their tasks; or a mix of both elements to decide the fairest distribution, taking into account some other characteristics such as the difficulty of the tasks.

Group discussion of solutions achieved by the teams

It was during the group discussion of the results by the team that a refinement of the model was observed. The group presentation and discussion allowed the students to appeal to mathematical representations that emerged during the understanding cycles and allowed, not only to support and argue about the solution model reacheds, but also to construct other representations that had not emerged in the models achieved by the teams.

Table 2 summarizes some of the procedures shown, the mathematical concepts that emerged, as well as their representations.

Table 1 Results of the implementation of the distribution activity

Cycles	Representation	Procedures	Mathematical concepts
Qualitative	Verbal		Comparison
Quantitative	Arithmetic	Fourth proportional calculation Use of the circular diagram to represent quantities (investment and time worked).	Fraction Reason Proportion Percentage Proportionality
	Diagrams	Arithmetical progression Frequency	
	Tabular	Use of algebraic symbolism	Constants and variables Value unit Dimensional Unit
Algebraic	Algebraic	Use of algebraic symbolism	Linear function

Source: Our own elaboration based on Vargas-Alejo, Reyes-Rodríguez and Cristóbal-Escalante (2016)

Discussion and conclusion

As a design study, a first conclusion is that the activity complies with the design of a MEA by verifying that it satisfies the design principles established by the researchers Lesh, Hoover, Hole, Kelly, and Post (2000). The topics addressed in the modeling tasks documented in this article have interested students who have been involved in developing a solution that contrasts with the situation raised to validate it and, later were able to refine it in more than one cycle of understanding. The transition between the qualitative, quantitative, as well as the algebraic cycles was observed in the procedures carried out by the participants during the implementation of the activity. However, the researchers Vargas-Alejo, Reyes-Rodríguez and Cristóbal-Escalante (2016) mention: "conceptual understanding or understanding is not achieved through a linear process [...] there may be intermediate understanding cycles between the previous cycles, which indicate a transition between cycles and involve an incomplete development of the differentiation, integration and refinement phases" (p. 70). Figure 3 shows an example regarding the transit between cycles of understanding, in this case, students are between the cycle of quantitative and algebraic understanding since they identify a "formula" to generalize the possible solutions for the distribution of profits, however, the expression used is not algebraic.

◦ Los 2 Recursos que en este caso se invierten en el negocio de la Repostería son: El Tiempo y El Dinero.

Como sabemos 800$ es la inversión total, Alondra invirtió 345$ ósea un 43.125% y Paco 455 un 56.875%

Cuando alondra trabaja 2 horas, Paco apenas trabaja 15 minutos, en términos de porcentajes

Alondra hace 87.5% del trabajo mientras que Paco un 12.5%

Y si se siguiera una de las 2 escalas, la repartición de la ganancia seria muy desigual, porque al pagarle según su inversión, Paco recibiría mas dinero solo por haber invertido mas y no trabajar casi nada en comparacion, mientras que si se hiciera la repartición en cuestión del tiempo invertido Alondra recibiría mucho mas que Paco e incluso Paco terminaría perdiendo dinero.

Por lo tanto la mejor opción es realizar un promedio entre ambos parámetros con la siguiente formula

(%de aportación + %de Tiempo)/ 2 Para así realizar una proporción Ideal

Figure 3: Evidence of transition from the quantitative to algebraic understanding cycle

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EXPERIENCIA EN EL AULA PARA EL DESARROLLO DE MODELOS PARA EL REPARTO

REPORT OF A CLASSROOM EXPERIENCE FOR THE DEVELOPMENT OF DISTRIBUTION MODELS

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Este escrito presenta el diseño y análisis de una Actividad Detonadora de Modelos (ADM), la cual pretende apoyar el refinamiento del sistema conceptual asociado a los problemas de reparto directamente proporcional (ejemplo: función lineal, porcentaje, proporción, variación, etc.). La situación explora un contexto cercano al estudiante alentando a la construcción de un modelo matemático generalizable y transferible a contextos diversos. Para el diseño se consideraron los principios de la perspectiva de Modelos y Modelación (Lesh y Doerr, 2003). La población objetivo fueron estudiantes inscritos en los primeros semestres de la carrera de Contador Público. En el análisis se consideraron la construcción de modelos (realizada en equipos); las representaciones matemáticas dentro del proceso observados (como diagramas y organizaciones tabulares); al igual que las soluciones individuales de los alumnos.

Objetivos y propósitos del estudio

Los problemas de reparto pueden representar una oportunidad para que los estudiantes universitarios profundicen en nociones matemáticas vinculadas con el sistema conceptual multiplicativo. Investigadores como Martínez-Juste, Muñoz-Escolano y Oller-Marcén (2019) consideran que los problemas que plantean una situación de reparto han tomado relevancia en los últimos años debido a que pueden explorar una diversidad de contextos cercanos a la realidad de los estudiantes.

El sistema conceptual multiplicativo asociado con los problemas de reparto vincula ideas fundamentales de las matemáticas, es por ello que está presente en la currícula desde la educación inicial hasta la educación media superior. En México, este tipo de problemas son introducidos a partir del tercer año de educación básica (SEP, 2017) y, en los años subsecuentes, se desarrollan variantes de dichos problemas donde se abordan nociones como: proporcionalidad (directa e inversa), función lineal, porcentaje, razón, ecuación lineal, número racional, proporción, variación y progresiones aritméticas, entre otros. Según Sánchez-Ordoñez (2013) los problemas de reparto permiten «por un lado, dar forma a la razón, la proporción y la proporcionalidad como objetos matemáticos que contribuyen al entendimiento y dominio del campo conceptual multiplicativo por parte de los estudiantes, y por el otro, identificar la manera como son reconocidas y manipuladas las razones, las proporciones y la proporcionalidad por los estudiantes en situaciones de aula» (p. 71,72).

Dado lo anterior, se pretende dar respuesta a la pregunta: ¿Cuáles son los ciclos de entendimientos que emergen en los estudiantes de nivel superior al resolver la ADM de reparto? En aras de elaborar una posible respuesta, se presentará el análisis de las soluciones para la situación planteada y, la cual, fue resuelta por un grupo de estudiantes inscritos en los primeros semestres de la carrera de Contador Público en la la UJED con la intención de profundizar en la noción de proporcionalidad y desarrollar conocimientos que se integren para conectar y refinar el sistema conceptual asociado. Cabe señalar que la actividad fue dirigida por el docente quien también es el autor de este artículo.

La perspectiva de Modelos y Modelización (Lesh y Doerr, 2003) fue fundamental para el marco teórico, dado que la actividad de reparto propuesta, fue diseñada como una ADM. Así, la situación planteada pretende promover la construcción de un modelo reusable y generalizable que, a su vez, provoque entendimiento matemático en los estudiantes para alcanzar el objetivo de refinar y profundizar en conceptos asociados al sistema conceptual multiplicativo; tanto aquellos asociados al álgebra (razón, variación lineal, función, etc.) como a la estadística (porcentaje, tablas de frecuencia, etc.). Esta actividad forma parte de una secuencia de actividades las cuales estaban encaminadas a lograr los objetivos del curso, sin embargo, en este escrito sólo se mencionarán los alcances de la ADM de reparto.

El análisis que se presentará comprende la interpretación de los procedimientos de solución que surgen entre los estudiantes durante la resolución del problema al hacer uso de los recursos matemáticos dispuestos a su alcance para construir un modelo que pueda generalizarse para resolver situaciones similares pero variadas.

La Perspectiva de Modelos y Modelación (PMM) como marco teórico

Esta perspectiva fue elaborada por Lesh y Doerr (2003) y propone la resolución de situaciones problemáticas dotadas de un contexto real, las cuales, pueden ser abordadas desde lo particular tomando, como punto de partida, los conocimientos previos que cada participante pueda tener. El refinamiento se lleva a cabo mediante la socialización de las propuestas de solución donde se alcanza un modelo de solución que satisfaga los requerimientos del planteamiento. La PMM considera al aprendizaje como un proceso de desarrollo de sistemas conceptuales (modelos), los cuales emergen cuando los estudiantes comparten y analizan situaciones que tienen más de una respuesta, por lo que la solución no es un número o palabra. Las problemáticas abordadas deben alentar a los estudiantes a describir, argumentar y explicar los procesos de solución empleados. Doerr (2016) explica que el aprendizaje de un contenido matemático surge durante el proceso de desarrollo de un modelo adecuado y productivo el cual puede ser usado y reusado en cierto rango de contextos.

La PMM considera que aprender matemáticas es un proceso que comprende ciclos progresivos de construcción de entendimiento, modificación, extensión y refinamiento de formas de pensar donde los sujetos logran profundizar en un concepto o constructo matemático a distintos niveles al relacionar datos, metas y posibles rutas de solución expuestos al enfrentar una situación problemática (Lesh y Doerr, 2003). Por lo tanto, el producto del aprendizaje es el proceso durante el cual la fluidez de las representaciones permite entender la mayoría de las construcciones matemáticas (Lesh y Doerr, 2003).

Dentro de la PMM se proponen a las ADM o MEA como una manera de generar productos que vayan más allá de respuestas cortas a preguntas específicas. Las ADM permiten a los estudiantes involucrarse en un proceso interactivo e iterativo donde expresan, prueban y redefinen sus maneras de pensar respecto a situaciones problemáticas significativas (Doerr, 2016), de esta manera, al estar en un contexto significativo, provocan entendimiento matemático nuevo en los estudiantes permitiéndoles expresar en sus procedimientos su conocimiento actual el cual puede provenir tanto de su experiencia como de su conocimiento matemático previo.

Las ADM deben cumplir con los seis principios del diseño establecidos por Lesh, Hoover, Hole, Kelly, y Post (2000) los cuales son: significado de la realidad o personal, construcción del modelo, auto-evaluación, externalización del modelo o de documentación del modelo. prototipo simple y generalización de modelos.

La caracterización de los ciclos de entendimiento expuesta por Vargas-Alejo, Reyes-Rodríguez y Cristóbal-Escalante (2016) fue utilizada para el análisis del proceso de construcción de modelos. Los investigadores proponen una caracterización a partir de su interpretación de la PMM en la cual identifican ciclos (cualitativo, cuantitativo y algebraico) que surgen cuando los individuos abordan situaciones contextualizadas. Los ciclos de entendimiento también están asociados con la profundización de un constructo matemático o concepto, es por ello que, por ejemplo, en un *ciclo de entendimiento cualitativo* el individuo da sentido a la situación donde se desarrolla el problema y puede llegar a identificar las variables involucradas, así como la posible relación entre ellas, sin embargo, para expresar dicha relación sólo elabora descripciones verbales, diagramas analogías o metáforas. En contraste, en un *ciclo de entendimiento cuantitativo* el individuo elabora supuestos, descarta información que considera inútil y puede llegar a establecer cuantitativamente el significado de expresiones lingüísticas haciendo comparaciones numéricas, para ello, puede llegar a utilizar representaciones tabulares y gráficas lo que conlleva a una interpretación más elaborada de la situación. Cuando el individuo busca describir e interpretar un fenómeno a través de la construcción, utilización, tránsito y coordinación de distintas representaciones (tabulares, gráficas, verbales, etc.) fluidez entre ellas y puede manipular símbolos algebraicos para resolver una situación, entonces ha alcanzado el *ciclo de entendimiento algebraico*. Los investigadores coinciden que, para poder alcanzar el último ciclo, es posible que el individuo haya pasado por diferentes interpretaciones de la situación y, previamente, haya logrado diferenciar, integrar y refinar distintos modelos.

Estudios de diseño

Dentro de la investigación del diseño se encuentran múltiples metodologías y perspectivas teóricas en la educación y otros campos. Algunos investigadores consideran que, tanto el proceso como los productos tangibles (secuencias didácticas, sistemas conceptuales asociados, etc) están involucrados en el diseño (Hjalmarson y Lesh, 2008). Sin embargo, también son relevantes los productos intangibles formados durante el proceso, puesto que el objetivo de esta metodología es mejorar los procesos de diseño y, en consecuencia, a resultados diferentes tanto dentro del proceso como en el producto.

Es importante considerar en el experimento de diseño las tres fases: prepararse para el experimento, experimentar en el aula y realizar análisis retrospectivos de los datos generados durante la implementación con los estudiantes (Cobb & Gravemeijer, 2008). Este tipo de estudios pretende desarrollar formas particulares de aprendizaje mientras se estudia el aprendizaje provocado en los ambientes diseñados.

Método de investigación

Enseguida se describe la población objetivo, el contexto y las técnicas de recolección de datos, así como las herramientas teórico metodológicas para su análisis.

Población de estudio

La actividad presentada fue implementada en un grupo de 36 estudiante inscritos en los primeros semestres en la carrera de Contador Público y cuyas edades oscilaban entre los 18 y 20 años de edad. En sesiones anteriores, los integrantes habían resuelto una actividad de modelación, por lo que ya contaban con experiencia para abordar este tipo de situaciones. La implementación de la actividad requirió de 4 sesiones con duración de una hora cada uno. Durante la implementación, los datos fueron recolectados mediante fotografías, video y bitácoras de los estudiantes.

Los participantes fueron organizados en nueve equipos compuestos por cuatro integrantes cada uno. La actividad fue resuelta en equipo y, posteriormente, la solución alcanzada se socializó en una discusión grupal. En una última producción, los estudiantes entregaron una solución individual al problema. El rol del docente fue clave durante todo el proceso realizando preguntas clave a los estudiantes encaminadas a profundizar en el modelo de solución alcanzado.

Descripción de las actividades

El diseño y la caracterización de los modelos desarrollados durante la implementación se sustentaron en el marco teórico de la PMM. Para familiarizar a los estudiantes con el contexto de la situación planteada se presenta la actividad calentamiento (Figura 1) que consistía en la lectura de un artículo de periódico y, su posterior discusión grupal para esclarecer posibles dudas respecto a la comprensión del contexto. Acto seguido se presentó la ADM (Figura 2) la cual abordaba una problemática relacionada con el contexto de la lectura. A continuación, se describen dichas actividades acompañadas de la imagen de las hojas de trabajo.

MÉXICO ES EL PAÍS DE LA OCDE QUE MÁS HORAS LABORA

Unas 2,255 horas al año es el promedio, mientras que Alemania es la nación con la cifra más baja entre los miembros de la organización, pero una de las más productivas.

La cultura laboral actual sostiene que cuántas más horas se trabaja, mejor será el sueldo percibido. Este no es el caso de México, uno de los países miembros de la OCDE donde los profesionales dedican más horas de trabajo semanales y reciben a cambio sueldos muy por debajo del promedio. Es por esto que cada día más trabajadores buscan mejorar sus condiciones salariales a través de nuevos empleos o emprendimientos propios.

Esta situación, sumada a la cantidad de trabajadores informales que existen en nuestro país (que alcanzan los 30 millones), genera inestabilidad y disconformidad en los profesionales mexicanos. Para cambiarlo, es necesario un control de la normativa laboral existente y la generación de nuevas políticas que alienten a las empresas y compañías a generar una mayor retribución por productividad y brindar beneficios de acuerdo a los intereses de cada trabajador.

Las estadísticas muestran que México es el país donde más horas se trabaja de la OCDE y el de peor sueldo promedio (los profesionales cobran, en promedio, 1 dólar por hora)

Mientras el ingreso salarial de una familia de cualquier otro país de la OCDE es más del doble

De acuerdo con el Foro Económico Mundial (WEF, por sus siglas en inglés), "las diferentes actitudes culturales y los factores socioeconómicos juegan un papel clave en la cantidad de horas que los empleadores esperan de los trabajadores".

Figura 1:Actividad de calentamiento

PROBLEMA 3

Para comenzar su negocio de pasteles, Alondra y Paco habían invertido $800.00 pesos, de los cuales Paco había aportado $455.00 y Alondra $345.00 y, desde el principio, ambos hermanos habían comenzado trabajando tanto en la elaboración como en la venta. Sin embargo, negocio de pasteles había crecido, por lo que consideraron necesario dividir el trabajo, Paco se dedicó a calendarizar los pedidos, cobrar y entregar los pasteles, mientras que Alondra se encargó de comprar los ingredientes, hacer los pasteles y decorarlos. Paco tenía varios conocidos que habían probado los pasteles elaborados por Alondra, por lo que no tardó en comenzar a calendarizar los pedidos de pasteles y pastelitos. Los pedidos quedaron así:

Pedidos

- Para el Lunes debía entregar dos pasteles para el cumpleaños de Jorge
- Para el martes debía entregar seis pasteles para la cafetería "El encuentro"
- Para el miércoles debía entregar un pastel y una docena de pastelitos o cupcakes para el cumpleaños de Regina
- Para el jueves debía entregar 20 pastelitos o cupcakes para el babyshower de Sofi
- Para el viernes debía entregar 2 pasteles y 62 cupcakes para el cumpleaños del jefe de Raúl

Pedidos semanales

Lunes, Martes, Miércoles, Jueves, Viernes, Sábado, Domingo, Notas

Al finalizar la semana, Alondra estaba muy cansada pues, por cada cuarto de hora que Paco había destinado a la venta, ella había dedicado 2 horas al proceso de preparación de los pasteles.

Ayuda a los hermanos a llegar a una forma de repartir las ganancias entre ambos. Puedes tomas como base la información que se te presenta y analizar las posibles maneras de dividir las ganancias entre ambos, también puedes buscar maneras alternativas de repartir las ganancias, sin embargo debes sugerir la que consideres mejor y justificar el por qué. Considera que Alondra y Paco continuarán con su negocio de pasteles, por lo que las condiciones pueden variar, por ejemplo, puede que haya veces que Paco trabaje más horas que Alondra o que Alondra decida invertir más dinero en la empresa.

Redacta una carta donde expliques cual es la manera que sugieres para la repartición de las ganancias, justifica las razones que te llevaron a concluir esto y realiza una comparación de los beneficios que conlleva esta repartición en comparación de las otras.

Figura 2: ADM Reparto de ganancias

Actividad de calentamiento. Expuesta en la Figura 1 con el objetivo de introducir el contexto desde datos reales y actuales para sensibilizar a los estudiantes respecto a la problemática y motivarlos a resolver la situación de reparto de ganancias en la ADM (Figura 2).

Actividad Detonadora de Modelos: Reparto de ganancias En esta actividad (Figura 2) se plantea una situación en la cual dos hermanos (Alondra y Paco) quienes deciden iniciar juntos un negocio donde las tareas que realizarían están dividas a la por igual, sin embargo, el tiempo que Alondra le dedicada a las tareas que realiza representa ocho veces más del dedicado por su hermano Paco. La situación también contempla que, al momento de realizar la inversión inicial, Paco aportó un poco más de la mitad de dicha inversión. Lo anterior ha dado pie a la disyuntiva de cómo repartir las ganancias, por ello, los hermanos han decidido pedir ayuda para que los estudiantes propongan soluciones de reparto de las ganancias.

Resultados

En este apartado se analizan las producciones de los estudiantes durante la resolución de las actividades propuestas, primero en equipo y, posteriormente, la discusión de las mismas de manera grupal. Para el análisis del proceso de construcción de un modelo que dé respuesta a la situación presentada en la ADM exhibido por los estudiantes, se utilizó la caracterización de los ciclos de entendimiento de Vargas-Alejo, Reyes-Rodríguez y Cristóbal-Escalante (2016)

Ciclos de entendimiento

La introducción del contexto se llevó a cabo con la actividad de calentamiento (Figura 1). Fue durante la implementación de la ADM (Figura 2) cuando los estudiantes pusieron en evidencia tanto los ciclos de entendimiento por los que atravesaron, como los conceptos y representaciones que surgieron desde su conocimiento individual previo. La interacción con sus compañeros de equipo permitió observar, en los procedimientos escritos, el modelo de solución para distribuir las ganancias. A continuación, se explican, brevemente, las observaciones realizadas.

Ciclo de entendimiento cualitativo. Este ciclo tuvo lugar después de que los estudiantes leyeron el problema. En él seleccionaron los datos que consideraron importantes para la resolución de la ADM y desecharon la información que consideraron irrelevante. Por ejemplo, dentro de los datos que muestran en los procedimientos de los estudiantes está la división de trabajo. Estos datos formaban parte de la redacción de la ADM, sin embargo, la información no sigue la disposición en el documento que permita la comparación visual de las cantidades, por ello, fue notable en las producciones de los estudiantes el orden donde puede hacerse una comparación visual de los datos seleccionados como relevantes en la situación. Lo anterior, permitió identificar una aproximación cualitativa a la solución del problema.

Ciclo de entendimiento cuantitativo. Después de que los estudiantes realizaron una comparación cualitativa de la información seleccionada, decidieron realizar distintas representaciones matemáticas en la construcción de un modelo de solución. Dentro de los procedimientos exhibidos se observaron:

Diagramas circulares. se exhibe el uso implícito de porcentajes para designar la proporción de la gráfica que corresponde a una categoría como porcentaje de inversión ó tiempo dedicado a una tarea.

Representaciones aritméticas. Para el cálculo del cuarto proporcional los estudiantes utilizaron el algoritmo conocido como “regla de tres”. Por ejemplo, este procedimiento les permitió encontrar la proporción porcentual correspondiente a la aportación inicial de Paco y Alondra.

Representación tabular. La evidencia muestra la organización tabular donde la variación lineal fue relacionada, la progresión aritmética y la búsqueda del valor unitario. Esta representación fue utilizada por los equipos 1 y 2. En ella se muestra el uso de números racionales y la designación de unidades de tiempo (horas), los equipos la utilizaron para poder identificar y comparar el tiempo que cada integrante le dedicaba a sus tareas.

Ciclo de entendimiento algebraico. Las evidencias escritas por los integrantes muestran representaciones algebraicas de funciones lineales.

Formalización de los modelos de solución previa a la socialización en grupo

Se les pidió a los estudiantes que realizaran una presentación corta respecto a su solución. Todos los equipos expusieron sus soluciones, entre ellas se encontraron: repartir la ganancia tomando en cuenta la proporción de la inversión entregada por cada socio; el tiempo dedicado a sus tareas; o bien, una mezcla entre ambos factores para decidir la repartición más justa tomando en cuenta algunas otras características como dificultad de las tareas.

Socialización de las soluciones alcanzadas por equipos

Fue durante la socialización de los resultados por equipo donde se observó un refinamiento del modelo. La exposición y discusión grupal permitió que los estudiantes recurrieran a representaciones matemáticas que surgieron durante los ciclos de entendimiento y permitieron, no sólo, sustentar y argumentar respecto el modelo de solución alcanzado en el equipo, también dio paso a la posibilidad de construir otras representaciones que no habían surgido en los modelos alcanzados por equipo.

En la Tabla 2 se resumen algunos de los procedimientos mostrados, los conceptos matemáticos que emergieron, así como las representaciones.

Tabla 2 Resultados de la implementación de la actividad de reparto

Ciclos	Representación	Procedimientos	Conceptos matemáticos
Cualitativo	Verbal		Comparación
	Aritmética	Cálculo del cuarto proporcional	Fracción
Cuantitativo	Diagramas	Uso del diagrama de circular para representar cantidades (inversión y tiempo laborado).	Razón Proporción Porcentaje Proporcionalidad
	Tabular	Progresión aritmética Frecuencial	Constante y Variables Valor unitario Unidad dimensional
Algebraico	Algebraica	Uso de simbolismo algebraico	Función lineal

Fuente: Elaboración propia con base en Vargas-Alejo, Reyes-Rodríguez y Cristóbal-Escalante (2016).

Discusión y conclusión

Al ser un estudio de diseño una primera conclusión es que la actividad cumple con el diseño de una ADM al verificarse que satisface los principios de diseño establecidos por los investigadores Lesh, Hoover, Hole, Kelly, y Post, (2000). La temática abordada en las tareas de modelación documentadas en este artículo han interesado a los estudiantes y se han involucrado para desarrollar una solución que contrastan con la situación planteada para validarla y, posteriormente, la retoman para refinarla en más de un ciclo de entendimiento. Se observó, en los procedimientos llevados a cabo por los participantes durante la implementación de la actividad, el tránsito entre los ciclos cualitativo, cuantitativo, así como algebraico. Sin embargo, aquí se encuentra también lo que Vargas-Alejo, Reyes-Rodríguez y Cristóbal-Escalante (2016) mencionan: «la comprensión o entendimiento conceptual no se logra por medio de un proceso lineal […] pueden existir ciclos de entendimiento intermedios entre los ciclos anteriores, los cuales señalan un tránsito entre ciclos e involucran un desarrollo incompleto de las fases de diferenciación, integración y refinamiento». (p. 70). La Figura 3 muestra un ejemplo respecto al tránsito entre ciclos de entendimiento, en este caso, los estudiantes se encuentran entre el ciclo de entendimiento cuantitativo y el algebraico puesto que identifican una "formula" para generalizar las posibles soluciones de reparto de ganancias, sin embargo, la expresión utilizada no es algebraica.

- Los 2 Recursos que en este caso se invierten en el negocio de la Repostería son: El Tiempo y El Dinero.

Como sabemos 800$ es la inversión total, Alondra invirtió 345$ ósea un 43.125% y Paco 455 un 56.875%

Cuando alondra trabaja 2 horas, Paco apenas trabaja 15 minutos, en términos de porcentajes

Alondra hace 87.5% del trabajo mientras que Paco un 12.5%

Y si se siguiera una de las 2 escalas, la repartición de la ganancia seria muy desigual, porque al pagarle según su inversión, Paco recibiría mas dinero solo por haber invertido mas y no trabajar casi nada en comparacion, mientras que si se hiciera la repartición en cuestión del tiempo invertido Alondra recibiría mucho mas que Paco e incluso Paco terminaría perdiendo dinero.

Por lo tanto la mejor opción es realizar un promedio entre ambos parámetros con la siguiente formula

(%de aportación + %de Tiempo)/ 2 Para así realizar una proporción Ideal

Figura 3: Evidencia de tránsito del ciclo de entendimiento cuantitativo al algebraico

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CONSTRASTING SOCIAL AND SOCIOMATHEMATICAL NORMS OF TWO GROUPS OF STUDENTS IN A POSTSECONDARY PRECALCULUS CLASS

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This paper characterizes the engagement of two groups of students in a Precalculus course at a four-year public university. A set of "Multiple Solutions Activities" was designed for the course to expose groups of students to alternative solution methods, allowing instructors to explicitly negotiate productive norms to foster students' flexible knowledge. Over the duration of the semester, the groups developed contrasting social and sociomathematical norms. One group's norms seem to be particularly influenced by students' experience taking the same course the prior semester in a more traditional format.

Keywords: Post-Secondary Education, Precalculus, Instructional Activities and Practices

Purpose

Many developmental mathematics courses, like College Algebra or Precalculus, tend to emphasize remedial content coverage and practicing procedures (Cox, 2015; Grubb, 2013; Mesa et al., 2011). This may not require students to change their mathematical practices and habits that contributed towards some students' need for further mathematical background development (Carlson et al., 2010; Goudas & Boylan, 2013). Consequently, some researchers have suggested focusing on developing students' argumentation skills and reasoning strategies (Chiaravalloti, 2009; Partanen & Kaasila, 2014).

One way to do this is to provide opportunities for students to compare, reflect, and discuss multiple solution methods (Rittle-Johnson & Star, 2007). This has been shown to help develop flexible knowledge, which Star and Rittle-Johnson (2008) characterize as the awareness of multiple problem-solving strategies and when to use them. However, students with underdeveloped mathematical skills often prefer a dependent learning style focused on mastering algorithms, making it necessary for instructors of developmental courses to negotiate productive norms, and promote mathematical practices that can help students develop flexible knowledge.

We conducted a teaching experiment in a Precalculus class at a four-year public university, in which the course instructor (first author of this paper) negotiated such productive norms and practices with the students. Specifically, the instructors attempted to aid the development of students' flexible knowledge by negotiating the social norm that it is important to understand others' work and the sociomathematical norm that an acceptable solution is one that follows any mathematically valid approach. In this paper, we analyze two groups' in-class engagement to answer the following research question: What social and sociomathematical norms developed in these groups over the semester?

Framework

Our study is framed within the emergent perspective, which views psychological and social factors as necessary to characterizing classroom activity. Continual student and teacher interactions formulate mutually established and regulated activity, which constitute norms (Cobb et al., 2001). Social norms portray the classroom participation structure, whereas sociomathematical norms are those specific to mathematical aspects of students' activity (Yackel & Cobb, 1996). Social constructs are reflexively related to psychological constructs (See Table 1). For example, as students develop sociomathematical norms they reorganize their mathematical values and beliefs, while productive social norms support students' positive perspectives of communal mathematical activity.

Table 1: Modified Interpretive Framework (Yackel & Cobb, 1996)

Social Perspective		Psychological Perspective
Classroom Social Norms	$\leftrightarrow$	Beliefs about one's own role, others' roles, and the general nature of mathematical activity in school
Sooicomathematical Norms	$\leftrightarrow$	Mathematical values and beliefs

Data Sources and Methods

The data were collected in a post-secondary Precalculus class of about eighty students at a four-year public university. This is the only developmental mathematics course offered by the university, and is largely populated by students who intend to major in engineering or the physical sciences, but who did not meet the pre-requisite for enrolling in Calculus. In the semester of our study, the majority of the students enrolled in the course were retaking it because they did not attain the necessary score to advance to Calculus. The instructor and teaching assistant for the course were Mathematics Education Ph.D. candidates, who previously taught this course multiple times, but not in the semester preceding this study.

Multiple Solutions Activities were designed to expose students to a variety of solution strategies, while creating opportunities for students to critique and analyze mathematical arguments. The activities were intended for groups of three or four students. Each activity had three phases. First, students solved a mathematics problem and cooperatively constructed a grading key for it. Second, students used their grading key to evaluate three fictitious students' solutions to the same problem. These solutions contained errors and/or used different approaches than those previously discussed in class (see Fig. 1-2). After analyzing these sample solutions, students were given reflection questions to compare and contrast the solutions. The last phase was a class discussion facilitated by the instructors, which helped them respond to students' concerns and bring attention to various aspects of the solutions. This allowed the instructors to model practices and explicitly negotiate norms such as: an acceptable mathematical solution may follow any mathematically valid approach and that solutions must contain explanations. Four such activities were implemented throughout the semester, and served as a key data source for the study.

Additionally, students participated in a pre- and post-course survey, which asked questions about their mathematical and role beliefs. Each item used a four point Likert scale to assess student's agreement with (1 – Disagree, to 4 – Agree) or importance of (1 – Not Important, to 4 – Very Important) given claims. For example: "The most valid ways of solving a problem are the ones discussed in class," and, "To receive full credit, my solutions must use the same methods used in class." These items aimed to assess students' openness towards other approaches, a theme the instructors advocated for to support the development of students' flexible knowledge.

Another data source was weekly writing prompts, in which students were asked to reflect on various topics such as their individual mathematical beliefs and practices. Several students were interviewed during and/or at the end of the semester to expound on their written responses.

Analysis

We analyzed video data by classifying students' utterances and activity into categories within the interpretive framework (Table 1) and coded videos in conjunction with students' written original solutions, grading keys, and evaluations of the sample solutions. Particular attention was given to characterizing the sociomathematical norm of what constitutes an acceptable mathematical solution and the social norm of interpreting others' solutions. Similarly, students' individual responses to writing prompts were partitioned into meaning units (Tesch, 1990) and classified into categories within the interpretive framework.

In this paper, we focus on the analysis of two groups of students. Group 1 included Albert, Dwayne, Gordon, and Harry, and Group 2 was composed of Molly, Steve, Peter, and Chad (all names are pseudonyms). One of the primary reasons for choosing these two groups was that three of four students in Group 1 were taking the course for the first time whereas all of the students in Group 2 were taking the course for the second time.

The survey data were analyzed by item, using a paired t-test (JMP refers to this as a Matched Pairs test). The pre- and post-survey were paired using a non-identifying code, which students created when completing the surveys. Pre-course surveys' that did not have a matching code in the post-course survey pool, and vice versa, were not included in the analysis. In total, we analyzed 42 students' surveys, of which 26 reported taking the course in a previous semester.

Results

The quantitative analysis revealed that over the duration of the semester, in general, students developed beliefs that were supportive of developing flexible knowledge. But the qualitative analysis revealed major variations in students' perceptions, which could be seen in the norms developed in various groups of students. This was particularly evident in Group 1 and Group 2's interactions with Multiple Solutions Activities, as we will show below.

Survey Results on Flexibility

Table 2 shows the results of two survey items that assessed students' beliefs associated with flexible knowledge, both of which demonstrate a statistically significant change.

Table 2: Flexibility Survey Items and Paired T-test Results, n = 42 (1- Disagree, 4-Agree)

Question	*Pre-Mean*	*Post-Mean*	*Prob < t*
The most valid ways of solving a problem are the ones discussed in class.	2.88095	2.54762	0.0058
To receive full credit, my solution must use the same methods used in class.	2.14634	1.8297	0.0178

The decrease in mean scores suggest that students came to assign less value to following specific procedures, and view it as having less influence on receiving full credit for their work. This suggests improved openness to learning about multiple solution approaches. This change was not homogenous across all students in the class, as the next sections show.

Norms developed in Group 1

Social Norms. One of the most evident social norms that developed within this group during the Multiple Solutions Activities, was the importance of interpreting and understanding others' solutions. As the semester progressed, the students spent increasing effort to analyze the provided solutions to understand and evaluate novel approaches and to find errors in them. Even when the group to initially criticized novel approaches, this did not detract from their efforts to interpret a new method.

Another social norm that developed in this group is the importance of all group members' participation in collaboratively discussing each solution and their evaluation of it. The group exhibited a shared responsibility group to explain what they understood about each solution and to help clarify confusion to each other when possible. When analyzing novel solutions, group members would verbally share their confusion with one another. Naturally, not all group members were uniformly vocal. To accommodate Albert's introverted demeanor, the group would often ask for his opinion on the solutions to integrate him into the group discussions. The group demonstrated that they valued each other's concerns, questions, and suggestions.

Sociomathematical Norms. We provide one vignette of Group 1's typical work that depicts the characterization of their sociomathematical norm of what constitutes an acceptable solution. During the last Multiple Solutions Activity of the semester, on the topic of inverse trigonometry, as Group 1 formed their grading rubric, they explicitly expressed awareness that there are different ways to solve the problem besides their chosen method. Harry described reluctance to form a rubric that would be limited to only one familiar way of solving:

Harry: I don't know if there is another way to solve it, so I don't want to write [grading] rules.

As they looked at the sample solutions, the group was initially dismissive of "Jennifer's" solution (Figure 2-b), which utilized right triangle trigonometry with the angle $u = sin^{-1}\left(\frac{1}{2}\right)$. This represented a novel approach that the group was unfamiliar with.

Gordon: This person is doing some weird math.
Dwayne: What did you do here? What kind of [stuff] is this? How the [heck] did you get to that?

Their lack of familiarity with her solution was obviously discomforting to them. But, despite these initial reactions, the group continued to investigate.

Gordon: [Jennifer] didn't find the inverse sine, so. They never even solved for u.
Harry: She's saying this is sine of u, this triangle, so then tangent would be opposite over adjacent, so one over one. That's what she's saying … she just didn't do it right.
Gordon: Right, because this should be one half, square root of three over two, and one (pointing to the triangle, and referring to a common right triangle).

Gordon's remark suggested that when using trigonometry, the triangle must have a hypotenuse of one. Gordon did not seem to understand how Jennifer formed her triangle. But, as Dwayne asked questions about Jennifer's approach, he was able to clarify Gordon's misconception.

Dwayne: "a" squared plus "b" squared" is "c" squared. How did [she] get two? (Pointing to the hypotenuse). Oh! [She] did one over two. That's correct though. That's just a different proportion. That is right.

This insight helped Gordon, who eventually located the exponent mistake in Jennifer's solution. After he explained the mistake to the group, he noted:

Gordon: If she did her math right, she actually would have got it, because "a" would have come out as square root of three.
Dwayne: So her process is right … but she just made one mistake. And technically her tangent work is correct for the work.

This particular example demonstrates how the social norm of collaborative analysis of an unknown solution mediated the development of sociomathematical norms within the group. This example demonstrates the group's openness to unfamiliar solutions and the sustainment of their sociomathematical norm of what constitutes an acceptable solution: a solution is acceptable if it follows any mathematically valid method. Conversely, this sociomathematical norm may have influenced the social norm of understanding other's solutions.

Norms developed in Group 2

Social Norms. All students in Group 2 had taken the course the semester prior. As the instructor tried to negotiate productive classroom social norms, this group of students developed their own set of norms that reflected a more traditional mathematics class. One social norm that quickly developed within this group was a rejection of engagement with alternative solutions. This norm was sustained throughout the semester, as the group tried to avoid analyzing others' arguments or investigating different solutions. Instead, the group members tried to finish the activities as soon as possible, did not seek input from or ignored quiet group members. Once finished responding to the reflection questions, the group would often disengage for the rest of the class-period, including whole class discussions, which may have been particularly detrimental to the instructors' efforts of promoting students' flexible knowledge (Rittle-Johnson & Star, 2007). Since student participation in these discussions was not included in the assessment structure of the course, this may have implicitly negotiated less importance or value than other aspects of the course.

Sociomathematical Norms. The instructors attempted to negotiate the sociomathematical norm that an acceptable solution is one that follows any mathematically valid approach, not just a familiar one. However, the group was conflicted with the instructors' negotiations, instead developing an alternative norm: An acceptable solution to a problem is one that uses a familiar approach or leads to the correct answer. The following illustrates this norm.

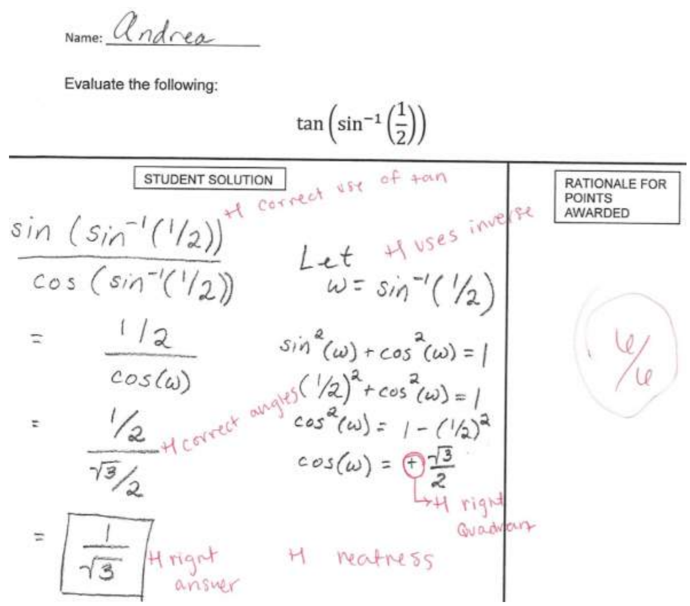

Figure 1: Molly's Grading of Andrea's Solution

During one activity, the group had to evaluate three sample solutions by the fictitious students "Andrea," "Dan," and "Jennifer," and then to compare and contrast these solutions. Andrea's solution used an unfamiliar approach but resulted in the correct answer, Dan's solution followed a method shown in class but had a wrong answer because of an intentionally included error, and Jennifer's solution was both unfamiliar and also yielded an incorrect answer.

The group favored Andrea's solution (Fig. 1), which yielded a correct answer, although it used an unfamiliar method. The group concluded that Andrea's solution was "interesting" and viable, since it "got them the right answer." The students relied on the authority of the answer to determine whether or not the approach was valid, but without thoughtful investigation.

The group was also receptive towards Dan's solution (Fig. 2-a), but for a different reason. Dan's solution resembled the approach the instructor modeled for similar problems. The approach was familiar to the group members, and eventually both Molly and Steve concluded that, "He has everything right except the answer." When students were familiar with a procedure, they were able to recognize patterns and locate errors, unlike in novel solutions like Jennifer's.

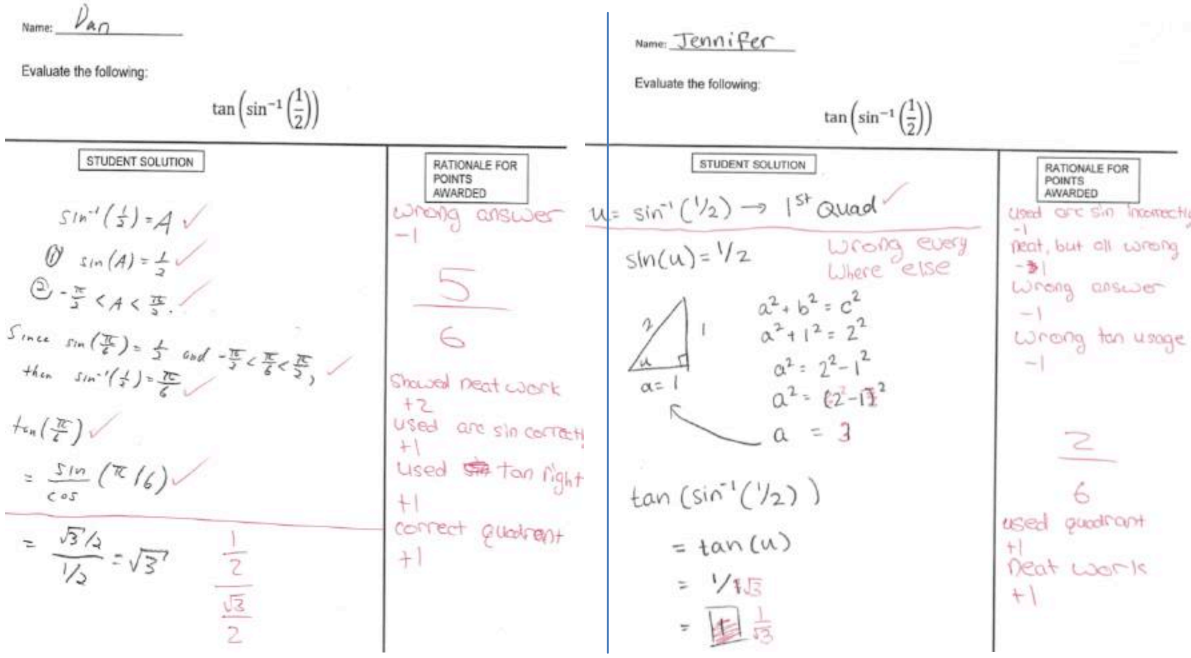

Figure 2 a & b: Steve's Grading of Dan (a) and Jennifer's (b) Solutions

Jennifer's solution (Fig. 2-b) used an unfamiliar approach and resulted in an incorrect answer. The group had a scathing first response towards the solution:

> Steve: Oh God, this already looks bad. Oh yeah, this is real bad. 0 out of 6 … I hope this is not a real student, I really hope.

The only discussion in the group was to determine if Jennifer should earn points for neatness or for "getting the quadrant right." The group did not notice the arithmetic mistakes until the instructor pointed it out to them.

In general, this group did not develop the sociomathematical norms that the instructors advocated and negotiated for. Instead, they chose to focus on the correct answer, as in Andrea's solution (Fig. 1), or a familiar procedure, as in Dan's solution (Fig. 2-a). The former is indicative of intellectual hegemony, relying on authority to determine that an approach is mathematically valid, which hinders the development of students' autonomy. The group's affinity towards familiar approaches coincides with their adopted social norm of aversion to exploring novel solutions. Consequently, they were eager to discredit novel solutions. Without a source of authority or the familiarity of an approach, the group was unable to determine its mathematical validity and vehemently rejected the solution, such as Jennifer's (Fig. 2-b).

Further Differences between the Groups

The two groups described above had varying perspectives, beliefs, and practices, which may help further explain some of the differences in the norms that developed amongst them. Below are some students' written responses to the question whether they found it helpful to learn about different approaches (asked near the end of the semester).

> Dwayne (Group 1):I think it is very helpful to me … I think multiple ways of solving a problem gives me an overall better [perspective] on the problem itself and gives me a better understanding of how it is broken down.
>
> Albert (Group 1): It is also pretty helpful to try different things to prove it in different ways, because this will increase understanding of different methods of proving things which you may find useful in other problems.

Harry (Group 1): I find it very helpful to learn about multiple ways of solving problems. Sometimes when I see a problem approached from a different mindset I can create a mental connection between various concepts or strengthen my knowledge of how a concept works. I also like seeing how you can use seemingly unrelated math concepts to find the solution to a problem.

Molly (Group 2): No, I like to learn one set way to do the problem. The more sets and procedures there are to a problem, the more confusing it can get.

Peter and Chad (group 2) expressed that they liked learning different methods but with the intention of finding a method that was easiest for them to replicate. In general, a common theme in Group 2's responses was their preference to learn and practice through repetition, as was evident in their responses to writing prompts:

Steve: When I am learning math, I heavily rely on seeing something done out in front of me and then having myself try the example myself and try and get the same answer as the example. I will then try more examples that relate towards that problem, I just need to know the answer in the end.

Peter: I also learn from observing, and repetition … I locked myself in a study lounge and kept doing complete the square problems until it came like second nature for me, just kept repeating the steps and applying them to different problems.

Molly: Given problems to do on our own with some way, either discussion or an answer key, we are given a way to check that we are doing it right. I personally like this way because it's repetitive and that's how I learn best in math.

The differences between the two groups manifested themselves in students' attitudes towards instructor's attempts to negotiate productive norms in the course, specifically, the importance of flexible knowledge of mathematics. Group 2 expressed their frustration with the instructor's approach towards teaching the class, which differed drastically from the previous semester:

Steve: Last semester they constantly drilled in our head that there was only one way to do it.
Molly: Yeah. So that's why I feel like a lot of us, or at least personally why I'm struggling.
Steve: It's a lot different.
Molly: I don't have a set rule to follow.

These comments may represent role beliefs that reflect the expectation that the instructor is responsible for abiding by the norms of a traditional mathematics class. Despite the instructor's efforts, most of the Group 2 members' beliefs and practices remained unaffected and staunchly sustained throughout the whole semester. At the end of the semester, Steve reflected:

Steve: Throughout the semester my studying habits have not changed, I have [continued] the same strategy that I used since the beginning, but my grade has started get worse and worse, but I do not believe that it due on my part.

This comment represented a deep-rooted conflict that may explain the nature of the norms that developed in Group 2 in contrast to the instructors' negotiations and expectations.

Discussion

This study examined student engagement in a post-secondary Precalculus course, in which an instructor implemented novel instructional activities and pedagogical strategies. The course and its activities aimed towards negotiating productive social and sociomathematical norms, which were intended to support students' flexible mathematical knowledge.

Our data analysis showed that the intervention produced mixed results. At a large scale, we saw some improvement in students' placing less emphasis on a single solution strategy, and possibly more openness towards multiple solutions (Table 2). Closer examination of student engagement revealed the variability between groups of students, which was evident in social and sociomathematical norms developed in different groups. Group 1 developed a social norm of interpreting and understanding others' solutions, which coincided with development of the sociomathematical norm that an acceptable solution may follow any mathematically valid approach. Meanwhile, Group 2's social norm of aversion to interpret or understand another's work developed concurrently with the sociomathematical norm that an acceptable solution is one with a correct answer or follows a familiar procedure. These data, along with the analysis of the differences between the two groups, suggest that there are several processes in place.

One is the interconnectedness and co-development of social and sociomathematical norms. The emergent perspective (Yackel & Cobb, 1996) emphasizes the reflexive relationships between psychological and social factors (see Table 1). In addition to these connections, our data suggest that social and sociomathematical norms may mutually influence the development of one another. For example, Group 2's sociomathematical norm of an acceptable solution as one following a familiar procedure may have influenced, and may be influenced by, the development of the social norm of avoiding engagement with non-familiar solutions. Thus, we assert that the interpretive framework can be enriched by incorporating this new dimension of reflexivity.

Second, our study demonstrates the lingering effects of detrimental classroom practices and norms. The emergent perspective describes a reflexive relationship between classroom norms and students' beliefs. Although classroom practices and norms dissolve after the conclusion of a course, the norms developed in one course affect students' individual beliefs and practices, which our study shows can persist and act as barriers to the negotiation of different norms and classroom practices in another course. This was particularly evident in Group 2's preference for repeated practice of a single procedure. Although there is value in developing procedural competences, unreflective repetitive practice may result in an illusion of competence. In our study, students seemed to hold onto inefficient practices that constrained their growth in the past, which continued to disservice them in the present. The case of Group 2 shows that changing these beliefs and negotiating productive norms, especially in developmental mathematics courses, is a gradual process. However, the case of Group 1 demonstrates the importance and positive effects of a constructive participation structure to the development of productive sociomathematical norms and improved learning outcomes.

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STUDENTS' REASONING ABOUT MULTIVARIATIONAL STRUCTURES

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Covariation and covariational reasoning are key themes in mathematics education research. Recently, these ideas have been expanded to include cases where more than two variables relate to each other, in what is termed multivariation. Building on the theoretical work that has identified different types of multivariation structures, this study explores students' reasoning about these structures. Our initial assumption that multivariational reasoning would be built on covariational reasoning appeared validated, and there were also several other aspects of reasoning employed in making sense of these structures. There were important similarities in reasoning about the different types of multivariation, as well as some nuances between them.

Keywords: Multivariation, Covariation, Student reasoning, Mental actions

Covariation and the cognitive activities involved in reasoning about it have become important themes in mathematics education research (e.g., Johnson, 2012; Moore, Paoletti, & Musgrave, 2013; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994; Thompson & Carlson, 2017). Yet, work on *co*-variational reasoning has essentially been limited to examining two variables changing in tandem with each other. Mathematical and scientific contexts often include more than two variables that are potentially related to each other. For example, the quantities pressure, volume, and temperature of a fix amount of gas inside of a flexible container are given by $PV = kT$, where P, V, and T could all be changing simultaneously. Mathematical functions of more than one variable, $z = f(x, y)$, also contain this feature. Note that we use "variable" in this paper to generically mean any potentially varying value, including values of real-world quantities as well as mathematical function inputs and outputs.

Recently, Jones (2018) used the term "multivariation" to theoretically describe situations where more than two variables relate to and change with one another. However, we do not yet have empirical data on the reasoning students might employ in making sense of these situations. This study was intended to explore, open-endedly, types of reasoning students might use when asked to think about multivariation structures. We went into this study with the assumption that multivariational reasoning would be related to covariational reasoning. Thus, our guiding research question was as follows: When analyzed through the lens of previous work on covariational reasoning, what reasoning mental actions are observed in students as they are asked to discuss different multivariational structures?

Background Research on Covariation

Because of our assumption that multivariational reasoning would be closely connected to covariational reasoning, we briefly review here some research work on covariation (see Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Castillo-Garsow, 2012; Johnson, 2015; Thompson & Carlson, 2017). The central theme to this work is that covariation consists of imagining "two quantities [i.e. variables] changing together" (Castillo-Garsow, 2012, p. 55) in which "they are changing simultaneously and interdependently" (Johnson, 2012, p. 315). The work of Carlson et al. (2002) provided the field with a framework of covariational reasoning mental actions, and then the more recent work of Thompson and Carlson (2017) heavily revised this into a new framework. For our purposes, we use the newer framework by Thompson and Carlson, though we draw on one major aspect of Carlson et al.'s original work. In particular, in Carlson et al.'s (2002) original framework,

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the first mental action of covariational reasoning was (1) coordinating one variable with changes in another. We believe this to essentially mean that students *recognize the dependence* between two variables, in that they perceive a change in one to correspond to a change in another (see also Oehrtman et al., 2008). This first mental action did not find its way into the revised framework by Thompson and Carlson, but it is important to our study because of the connections it has to some of our results.

Beyond this first mental action, we then used the revised framework (Thompson & Carlson, 2017) for the remaining mental actions. These subsequent mental actions are: (2) imagining related, but asynchronous changes in variables (*precoordination*), (3) imagining generic increases/decreases between the variables (*gross coordination*), (4) coordinating the variables' values (*coordination of values*), (5) coordinating changes in variables' values in "chunks" (*chunky continuous*) and (6) imagining changes in the variables' values happening smoothly (*smooth continuous*). These six mental actions were used in our study as baseline codes to categorize and organize student statements, as described in the methods section. We also extended this work by identifying new mental actions pertinent to *multi*variational reasoning.

Multivariation

Multivariation consists of situations where more than two variables are related to and possibly changing in conjunction with each other (Jones, 2018). Conceptual analysis has revealed different possible types of multivariation, which we recap in this section.

Independent Multivariation

Jones (2018) described *independent multivariation* as situations where certain variables can be held constant while others vary. For example, in $F = GmM/r^2$, one can change the distance (*r*) to produce a different amount of force (*F*), while keeping mass (m) constant. Multivariable functions, $z = f(x, y)$, typically also behave in this way. Yet, in independent multivariation, what is held constant and what can change can be switched. In $F = GmM/r^2$ one can keep the distance (*r*) the same to see how *F* and *m* might covary with each other. It is critical to note, though, that independent multivariation is more than simply the covariation of two variables while holding the others constant. Rather, one can imagine *multiple* variables changing at the same time. For example, in $F = GmM/r^2$, r and m could both be changing simultaneously, each impacting how F changes. In $z = f(x, y)$, one could trace a path in $\mathbb{R}^2$ in which both *x* and *y* change at the same time, with *z* changing as one traces along that path (see also Martinez-Planell, Trigueros-Gaisman, & McGee, 2015). Finally, another aspect of this multivariation is that it can include as many variables as desired, such as $z = f(x_1, x_2, \ldots, x_n)$ having *n*+1 variables.

Dependent Mulvariation

Next, Jones (2018) described *dependent multivariation* as situations in which it is *not* possible to hold some variables constant while changing others. A change in any one variable in this situation will produce simultaneous changes in *all* other variables. Some real-world contexts exhibit this behavior (Bucy, Thompson, & Mountcastle, 2007; Roundy et al., 2015), such as $PV = kT$. If the gas inside the flexible container is heated up, the increase in temperature (*T*) would cause simultaneous changes in both the internal pressure (*P*) and volume (*V*). As another example, in free-market economics, if one changes the price of a commodity, both demand and supply will react simultaneously. It may not be realistic or even possible to hold "demand" constant to observe only changes in supply. Similarly, for parametric functions in mathematics, $\langle x(t), y(t)\rangle$, if one changes *t*, then *x* and *y* both change simultaneously.

Nested Multivariation

Third, Jones (2018) described *nested multivariation* as situations where the variables are related in a function composition structure. In the structure $z(y(x))$, as one imagines a change in x, there is a corresponding change in y. That change in y then automatically corresponds to a change in z. It may be necessary sometimes to *perceive* the intermediary variable if it is not explicitly labelled, such as $y = sin^2(x)$ consisting of the quantities x, sin(x), and y. As x changes, sin(x) changes, which in turn makes y change. Of course, it is possible to conceptualize two-variable covariation between x and y directly in this example. However, a complete understanding of their relationship would require interpreting the intermediary sin(x) value (see also Breidenbach, Dubinsky, Hawks, & Nichols, 1992). Otherwise, for instance, if x decreases into negatives, the values of y might not be accurately tracked. Real-world quantities can also have this nested structure. For example, in the theory of relativity, as an object's velocity changes, the object's mass changes, given by $= m_o/\sqrt{1 - v^2/c^2}$. As the velocity (v) changes, the ratio between it and the speed of light (v/c) changes, which in turn changes the Lorenz factor $1/\sqrt{1 - (ratio)^2}$, which in turn changes the mass (m) (see also Jones, 2015).

Study Methods

To document students' multivariational reasoning, we recruited 10 undergraduate students to participate in interviews, referred to as Students A–J. Students E, G, and J were female and the others male. Because our study was exploratory in terms of the types of reasoning students might use, we decided to recruit students who were more advanced in their mathematical studies, to better ensure that they had had exposure to and experience with multivariational contexts. Similar to how Carlson et al. (2002) recruited second-semester calculus students to investigate covariational reasoning, we recruited students in multivariable calculus (from two different classes) to investigate multivariational reasoning. In the interview, the students were given two contexts for each type of multivariation (Table 1). These contexts were chosen for their connection to the conceptual analysis that helped define multivariation (Jones, 2018). For each context, the students were allowed to clarify the context first, and then were asked, "What does this equation/formula mean? What does it say about the variables in it?" The students open-endedly discussed the context, but were also asked several scripted questions, including how the variables related to each other, how changes in one variable impacted the others, whether multiple variables could change at the same time or whether variables could remain unchanged, and what impact increases or decreases in certain variables might imply.

Table 1. Contexts Given to the Students in the Interviews

Multivariation	Context 1	Context 2
Independent	Let $z = x^2 - y^2$ be a function of two variables. [The function's graph was also given to the student.]	The formula $F = \frac{GMm}{r^2}$ relates gravitational force (F) with mass (m) and distance (r). M (Earth's mass) and G are constants.
Dependent	For a certain amount of gas in a flexible balloon, $PV = kT$ relates pressure (P), volume (V), and temperature (T).	The price (p) of a specific book is related to the number of books people want to buy (d for demand) and number of books the publisher is willing to print (s for supply).
Nested	Let $y = \sin(x)$ and $z = y^2$. In other words, $z = \sin^2(x)$.	$m = \frac{m_o}{\sqrt{1-(v^2/c^2)}}$ relates an object's mass (m) to its velocity (v). Note that m_o is the "resting mass" and c is the speed of light.

Our analysis was based on our assumption that multivariational reasoning would be related to covariational reasoning, though also extended beyond it as well. Thus, our initial analysis consisted of using the covariation mental actions described previously as starting codes. We marked any place in the data where a student exhibited reasoning behaviors similar to one of the covariational reasoning mental actions. While we did, we also used open coding to mark any reasoning instances that did not align with one of the covariational reasoning mental actions. After doing so, we examined these "other" reasoning instances in order to identify themes among them. This led to the emergence of new codes that were not a part of the covariational reasoning mental actions. Once we had our final set of codes, we recoded the entire data set again, using our completed coding scheme. Then, within each of independent, dependent, and nested multivariation contexts, we compared the reasoning used across the 10 students. We looked for trends in how the students tended to reason about each multivariation type. We also compared the reasoning used in one type of multivariation with reasoning used in another to identify if certain kinds of reasoning were distinctive to one type of multivariation or common across them.

Results

Our first main result was that the students did, in fact, employ much covariational reasoning within these multivariation contexts, supplying evidence for our assumption that multivariational reasoning is rooted in covariational reasoning. We observed all of the mental actions from Thompson and Carlson (2017) in our students. In fact, as seen subsequently, imagining only two variables at a time was a common and useful action that students did. Space constraints do not permit a full treatment of how each aspect of covariational reasoning was observed, and we instead focus the remainder of our results on reasoning mental actions specific to multivariation.

Students' Independent Multivariational Reasoning

In the independent multivariation contexts, all 10 of our students engaged in reasoning that was related to Carlson et al.'s (2002) first *recognize dependence* mental action. Yet, a slightly different aspect of that reasoning in these contexts was a similar mental action we call *recognize independence*. In this, the students decided which variable was to be treated as constant and which were to vary. For example, one of the first things Student B said when shown $z = x^2 - y^2$ was, "x and y are variables, independent variables. Which basically means as one changes the other doesn't necessarily have to change." In $F = GmM/r^2$, Student C stated, "As I am getting farther from the earth with a bowling ball, I'm not changing the mass of the bowling ball." Recognizing independence then permitted the students to use another new mental action that we call *decompose into isolated covariations*. For instance, when discussing $z = x^2 - y^2$, Student D early on stated, "Whether x is increasing or decreasing… it is going to be increasing the z either way." Then a few statements later, Student D described, "Let's just pretend that the x^2 doesn't exist and we're only playing with the y^2… We see the parabola for y, which is negative, it starts at the origin and then curves down in both directions." Here, Student D simplified the context to two variables at a time in order to understand the covariational relationships between x-z and y-z. After using this mental action to reduce the multivariation to covariation, covariational reasoning mental actions were then used to analyze that relationship between those two particular variables.

In conjunction with recognizing independence and decomposing into covariations, a third new mental action we observed was that students could *switch constants/variables*, in which they shifted their conceptualization of which variables were held as constant and which were allowed to vary. For example, a little after the previous excerpt, Student B explained, "If we follow this path $x = y$, our z stays constant." Similarly, Student C later stated, "Increasing the mass, getting a pebble compared to a rock and having them the same distance from the earth, I have increased the mass but the distance is still the same." These students demonstrated they were able to reason from different perspectives within the same context in terms of what changes or stays the same.

The students were also able to perform mental actions regarding the variables all changing at the same time. We call one such mental action *imagining simultaneous changes in inputs*. In this mental action, two variables were considered the "inputs" and their changes were imagined as linked together *before* then coordinating this with the variable considered to be the output. To illustrate, in discussing $z = x^2 - y^2$, Student H explained, "I could move simultaneously in both an x direction and a y direction. That's going to determine how my z direction is changing… If I'm changing my x and my y at the same time, then z can potentially change as well."

Building on imagining simultaneous changes, another mental action for independent multivariation was what we call *coordinate these multiple simultaneous changes*. For example, in the force context, after Student G had first decomposing into isolated covariations and then subsequently imagined simultaneous changes, she explained, "Say the mass is increasing and the distance is also, r is decreasing, then the force would definitely be increasing." This mental action consists of aligning the results of the isolated covariations together into an overall image of all the variables' changes. Student G also considered the possibility of m and r both increasing or both decreasing. She explained, "If the mass and the distance are both increasing or both decreasing, then it gets a little bit iffy. It depends on which one has a greater impact. [Pause] If m is increasing, at a rate that's greater than the rate at which r^2, the distance squared, is increasing, then the force will still increase." Here we can see Student G comparing the covariations between m and F and r and F. She decided that if m changes by more than r^2, then the positive covariation between m and F will overcompensate for the negative covariation between r and F.

In Student G's explanation, we also see another important mental action. Here, she did a mental action close to what Thompson and Carlson (2017) call *coordinating values*, but she did so without ever using specific numeric values. Thus, from our study, we decided that covariational and multivariational reasoning research would benefit from separating out what we call *qualitative amounts of change* versus *numeric amounts of change*. In other words, Student G was able to *qualitatively* image that the increase in F due to a large increase in m would be larger than the decrease in F due to a smaller increase in r^2. She could have done this by using specific numeric values, but her coordination at the qualitative level was productive for what she wanted to accomplish. We see this mental action as applying to both covariation and multivariation.

Lastly, another new mental action we saw was students attempting to *articulate the type of relationship* present between two or more variables. It appeared helpful for students to determine well-known relationships present between the variables. For example, for $z = x^2 - y^2$ Student I explained, "You maybe pick some value of x and keep it there and then you just basically have $z = c - y^2$. So it's just an upside parabola." Visualizing a parabola helped him think of how z and y would covary with each other. Students used other well-known relationships to assist imagining the situation, such as thinking of F and m as *proportional*, and F and r^2 as *inversely proportional*.

Students' Dependent Multivariational Reasoning

Recall it is not possible to hold some variables constant in dependent multivariation. Thus, an important mental action students used was, again, *recognize independence/dependence*. Yet, the way this mental action was carried out varied from student to student. For instance, in the $PV = kT$ context, Student H explained, "So, if my temperature were increasing… I can think of both my pressure and my volume increasing. The balloon is getting bigger and the pressure inside it is increasing." Here, Student H envisions a dependent relationship between all three variables simultaneously. However, when Student A was asked if T could change so that only P changes, without V changing, he explained, "If you just keep, I mean, is this according to the equation?

According to the equation, then I would say yeah [V can be kept constant], because if V is constant and P increases, then T would increase." Student A did accept that all three could be changing simultaneously, but also observed that in the mathematical equation, one can leave one as constant. These examples illustrate two things. First, whether a context is independent or dependent multivariation consists, at least in part, in how the person *conceptualizes* it. Regardless of how things behave in the "real world," if a student perceives that it is possible to hold some variables constant, then that is the type of situation they cognitively work with. Second, whether the students chose to conceptualize it as independent or dependent multivariation seemed connected with whether they believed they should operate mentally in a "math" world of the symbolic equations, or the way quantities behave in the "real" world.

When students determined that they were in a dependent multivariation situation, they also often decomposed into isolated covariations. But, then, the mental action *coordinate simultaneous changes* became important. For example, after decomposing $PV = kT$ into covariations, Student D stated, "If the balloon is being heated up, then its volume will greatly increase and its pressure will increase a little bit, depending on the capacity of the balloon to contract." He put the individual relationships together into a coherent whole. However, an important difference in dependent multivariation is that the simultaneous nature of the changes is *required*, where it is not required in independent multivariation. This excerpt again shows *qualitative amounts of change*, because Student D imagined relative changes without using exact numeric values. As another example, Student J stated, "If the increasing amount of T is greater than either [the change in] P or V, that means both would be increasing." She realized that if P and V both increase, they could not each increase relatively as much as T does by itself. Of course, students did engage in *quantitative amounts of change*, too, such as Student E examining what possibilities he could get for changes in P and V if temperature changed from "6 to 10."

Like for independent multivariation, in these contexts the students also spent time trying to *articulate the type of relationship* between the variables. For example, several students discussed "proportionality" between V and T, or "inverse proportionality" between P and V. In the supply and demand context, students also used ideas of proportionality and inverse proportionality. Some students tried to create a rough symbolic formula to relate them, which took the forms of $p \propto \frac{d}{s}$ (Student A), $p = k\frac{d}{s}$ (Student B), $p = d - s$ (Student F), and $d = \frac{p}{s}$ (Student H). Note that different relationships can be seen depending on which quantity was seen as changing first. For example, if price is seen to increase first, that might signal a decrease in demand. Alternatively, if demand increases first, that might signal an increase in price. Other students drew graphs of p versus d and p versus s that matched these equations. *Articulating these relationships* seemed to help students organize their thinking about relative changes between the variables. In particular, Student J drew the familiar supply and demand curves (Figure 1) and used them to help her organize her thinking of relative changes. She first imagined that where the decreasing demand curve intersected the increasing supply curve defined the price. An increase in demand was represented as a shift upward in the demand curve, which resulted in an intersection point at a higher supply and higher price (left image). Similarly, an increase in price (right image) resulted in a point lower along the demand curve but higher along the supply curve. This kind of reasoning seems to suggest another possible mental action, *identifying order of effect between variables*.

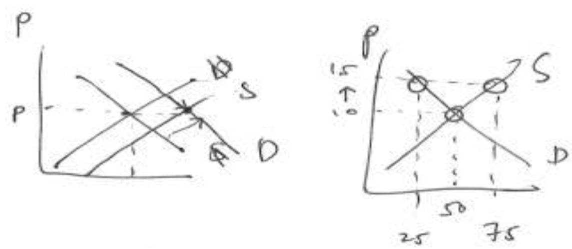

Figure 1: An example of *articulating relationships* helping students organize their thinking

Students' Nested Multivariational Reasoning

For this type of multivariation, as with the other types of multivariation, the students employed mental actions regarding *recognizing* relationships. In this case, the students attempted to *recognize a chain of influence* from one variable to the next. For example, with $z = sin^2(x)$, Student E imagined an intermediary quantity in the chain. She explained, "So, you put the *x* in, the *x* gets sine'd, and then that sine gets squared. So, it goes in like step one, it's turned into a sine, and then step two that sine is squared." Student E recognized that there was (a) the initial *x* value, (b) the sine of that *x* value, and (c) the square of that sine value. In the mass-velocity context, Student E similarly explained, "The velocity is never going to be more than the speed of light. So, this [points to v^2/c^2] is always going to be less than one, which means this [point to $1 - v^2/c^2$] will always be positive. But the more velocity increases, the closer this [points to v^2/c^2] is going to get to one, which means the smaller the denominator [e.g., $\sqrt{1 - v^2/c^2}$] is going to get, which means the larger the mass is going to be. So, the larger velocity gets, the larger mass gets." In this, Student E conceptualized explicitly the quantities (a) v, (b) v^2/c^2, (c) $\sqrt{1 - v^2/c^2}$, and (d) m. Several other students gave similar descriptions of these two contexts.

Once the chain of influence had been recognized, students again often used *decompose into isolated covariations*. To illustrate, as Student B thought about $z = sin^2(x)$ he described, "As x changes, we're going to end up with *y* having this oscillating pattern… between positive and negative one repeatedly. So, as you increase *x*, your *y* is going to be jumping between 1 and –1. If you decrease *x*, same thing… If *y* is positive, as we increase *y*, *z* will go up. If *y* is negative and we decrease *y*, *z* will also go up." Student B first examined *x* and *y* in isolation and then *y* and *z* in isolation. As before, these isolated covariations then needed to be *coordinated* into an overall image of the nested multivariation. Additionally, students also employed coordination of increases and decreases, including both *qualitative amounts* and *numeric amounts*.

Once students had completed mental actions of recognizing a chain of influence, decomposing into isolated covariations, and coordinating increase/decrease, these seemed to help students understand the direct relationship between the "initial input" variable (i.e. *x* and *v*) and the "final output" variable (i.e. *z* and *m*). They could take their new knowledge about the context and begin to work with direct covariation between the initial input and the final output. They did not necessarily need to work with the intermediary variables anymore. For example, after working through the nested reasoning, Student A summarized the velocity-mass context as follows, "As it's [v] changing, so if this gets bigger, then *m* would get bigger as well… So, if this [*v*] increases, *m* would increase and if this [*v*] decreases, *m* would decrease." Thus, one part of understanding nested multivariation structures might be to eventually conceptualize the direct two-variable covariation between the two most salient variables of interest.

As a last note, some students also attempted to circumvent the need for nested reasoning for $z = sin^2(x)$ by instead using visual reasoning on the graph of $\sin^2(x)$. They first took the graph of $\sin(x)$ and attempted to reason what the square of that graph looked like. Once they had a graph (whether correct or incorrect), the students then used that graph to vary x and report directly on its impact on the values of z.

Discussion

First, we observed that covariational reasoning was, in fact, an important part of these 10 students' reasoning. In all three multivariation types, students often *decomposed* the context to two-variable covariations at a time to organize their thinking. However, we note that this decomposition into covariations has connection to what Johnson (2015) termed simultaneous yet independent variation. In her work, she explained that students sometimes covaried two variables with *time* and then tried to coordinate the variables only through the intermediary of time. Johnson concluded that a full comprehension requires students to not need the intermediary of time, and to imagine the variables changing directly in relation to each other. Likewise in multivariation, it may be important for students to push past decomposition into covariations to imagining a single coherent image of the whole multivariational structure. Some of our students were able to compile the individual covariations to create a holistic image of the multivariation.

Next, our study extends covariational work by elaborating on *coordination of values* from the Thompson and Carlson (2017) framework. We suggest it be split into two types of coordination: *qualitative amounts of change* versus *numeric amounts of change*. Our students productively described "large" or "small" changes qualitatively to reason about a context. This mental action certainly goes beyond gross coordination. We even hypothesize that it may even be more complex than simply inserting numeric values into a formula and comparing resulting values (i.e. coordination of values), because it requires one to imagine relative sizes in changing values and coordinate them without the aid of specific numeric values. We perhaps even see chunky continuous coordination (Castillo-Garsow, 2012; Thompson & Carlson, 2017) as just adding intervals of a fixed size to *qualitative amounts of change*. Thus, we wonder if *qualitative change* may be between what is currently described as coordination of values and chunky continuous coordination. Of course, additional work would be required to examine if that is the case.

Third, another key idea from our study is that it requires cognitive work to *recognize dependence* and *independence* among the variables in multivariation. Students spent time imagining what variables could be held constant, which varied, which depended on which, and whether that dependence could be altered. In a recent paper, Kuster and Jones (2019) similarly noticed the importance of "recognize" in students using variational reasoning while discussing differential equations. They claimed that it may have been an oversight to drop "recognize" from the original covariational framework (Carlson et al., 2002) in the new framework (Thompson & Carlson, 2017). Our data concurs that it may be important to keep mental actions of "recognize" in variational reasoning frameworks, because of how important it is for more complicated variational structures. This suggests that in moving our students from covariation to multivariation, it may be useful to spend time engaging students in *recognizing* activities. It is possible we do not help students see, for example, the difference between independent and dependent contexts (see Bucy et al., 2007; Roundy et al., 2015).

Finally, we saw that there was much similarity in the types of reasoning across the different multivariation contexts. The good news is that it might not be necessary for students to learn about each type of multivariation separate from the others. By learning to reason about one type, they may simultaneously be developing reasoning abilities that transfers to other types. However, by being explicit about the different types, we as instructors might help students gain an appreciation for the nuances that exist between each type, enabling stronger reasoning.

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"DYSLEXIA IS NATURALLY COMMUTATIVE": INSIDER ACCOUNTS OF DYSLEXIA FROM RESEARCH MATHEMATICIANS

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Using neurodiversity as our theoretical framework, rather than a deficit or medical model, we analyze the narratives of five dyslexic research mathematicians to find common strengths and challenges for dyslexic thinkers at the highest level of mathematics. We report on 3 themes: 1) highly visual and intuitive ways of mathematical thinking, 2) pronounced issues with memorization of mathematical facts and procedures, and 3) resilience as a strength of dyslexia that matters in mathematics. We introduce the idea of Neurodiversity for Mathematics, a research agenda to better understand the strengths (as well as challenges) of neurodiverse individuals and to use that knowledge to design better mathematical learning experiences for all.

Keywords: Equity and Diversity; Advanced Mathematical Thinking; Geometry and Geometrical and Spatial Thinking; Representations and Visualizations

The neurodiversity movement, developed from activism of people with autism, dyslexia, and other cognitive differences, demands that such individuals be understood not as deficient, but as different: part of the natural and beneficial cognitive diversity of society (Robertson & Ne'eman, 2008). Cognitive neuroscience demonstrates evidence of both strengths and challenges for individuals with dyslexia, the largest group of students in the special education system. While some individuals with dyslexia have been highly successful in mathematics, students with LD/Dyslexic score on average lower than their neurotypical peers on mathematics achievement tests, with gaps widening over time (Wei, Lenz, & Blackorby, 2013). However, educational research has focused almost exclusively on identifying and remediating deficits of individuals with dyslexia, with a pronounced silence on how related strengths might matter for learning mathematics. Rejecting a deficit lens on the mathematical thinking of students with dyslexia/LD, we propose Neurodiversity for Mathematics, a research agenda to better understand the strengths (as well as challenges) of neurodiverse individuals and to use that knowledge to design better mathematical learning experiences for this large group of learners.

Using interviews and narrative analysis, this project investigated this issue from the perspective of *neurodiverse insiders* who have experienced learning mathematics with success at the highest levels. Using neurodiversity as our theoretical framework, rather than a deficit or medical model, we analyze the narratives of five dyslexic research mathematicians to find common strengths and challenges for dyslexic thinkers at the highest level of mathematics. We report on three themes: 1) highly visual and intuitive ways of mathematical thinking, 2) pronounced issues with memorization of mathematical facts and procedures, and 3) resilience as a strength of dyslexia that matters in mathematics.

We ground the proposed research in the academic field of Disability Studies (DS), which recognizes that although individuals have natural biological variations, it is the social effects of difference that disable rather than the impairments themselves (Linton 1998). From the DS perspective, the medical model and behaviorist tradition depict disability as deficits located within individuals resulting in identifying, pathologizing, and stigmatizing difference; thus requiring specialized knowledge (e.g., special educators) for individualized instruction and remedy. Academia continually reinscribes difference between children with and without disabilities in an unconscious effort to maintain the status quo, in which children with disabilities are conceptualized as fundamentally different from

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“normal children” (Linton 1998). In order to understand and improve the experiences of dyslexic students, we must ground our analysis in the perspectives of those with this difference. This study seeks to provide a new lens on the mathematical learning of neurodiverse individuals, grounded in the experiences of those with cognitive differences, rather than the perspectives of researchers who are often neurotypical. Our study is a collaboration between the first author, a cis white female former math teacher and special educator currently working in mathematics and disability studies at the university level, and the second author, a cis white male research mathematician with dyslexia.

Dyslexia is a hereditary neurobiological disability characterized by difficulties in reading, writing, and spelling, often unexpected in comparison to other academic skills (Lyons et al., 2003). While originally called dyslexia, these differences were reclassified “Learning Disabilities” when Specific Learning Disability became a category of special education services under US law. Learning Disabilities (LD) is a broader category that includes LD in the areas of reading (dyslexia), writing (dysgraphia), and mathematics (dyscalculia), as well as other variants of LD such as Auditory Processing. Individuals can experience LD in multiple areas. What tends to be consistent across LD is some form of processing and/or language difference that significant affects learning in school. A significant population of individuals with LD also have diagnoses of Attention Deficit Hyperactivity Disorder (ADHD). While LD is the term in US law, individuals often prefer the term “dyslexia.” Currently, laws in the US are shifting back towards dyslexia, specifically towards advocacy for multi-sensory, systematic reading instruction. Much of the research in LD and math is focused on students with dyscalculia, or significant difficulty learning mathematics. However, students with LD in general, which is most often most pronounced in reading, significantly underperform in mathematics (Wei et al., 2013).

Currently, there is little overlap between research in mathematics education and special education mathematics research (Lambert & Tan, 2020). These research traditions are largely separate because of pronounced theoretical and methodological differences over their history of development as fields (Woodward 2004). Based on a recent analysis of the literature on mathematics learning across the two fields (Lambert & Tan, 2020), we found that special education primarily understands mathematics learning through behavioral and information processing approaches, using quantitative methods that focus on large populations. In contrast, mathematics education is focused on constructivist and sociocultural approaches to understanding mathematics learning through a focus on individual thinking and classroom contexts. Research methodologies in mathematics education include both quantitative and qualitative methodologies. Special education research frames the achievement gap for students with learning disabilities as a problem of cognitive deficits in individuals and seeks interventions to remediate deficiencies. Recommended interventions are primarily explicit or direct instruction. In previous work, we have critiqued this focus on pedagogies as deficit-based and promoting narratives about students with learning disabilities being unable to learn through inquiry (Lambert 2018). These narratives, which we face continually in schools, are antithetical to the larger goal of increasing access to higher-level mathematics for students with learning disabilities.

There has long been speculation about the connection between dyslexia and visual-spatial talents, dating back to Orton in 1925 (Schneps, Rose, & Fischer, 2007). There is evidence that people with dyslexia have strengths in visual-spatial thinking, although not conclusively. Some of the differences in findings can be attributed to different ways to define and assess visual-spatial thinking (von Károlyi & Winner, 2004). One strength associated with dyslexia in several research studies is 3-D spatial thinking, connected to strengths in mechanics and complex visualization (Attree, Turner, & Cowell, 2009). Another strength is interconnected reasoning; many individuals with dyslexia tend to make unique associations between concepts, focused on the big picture (Everatt, Weeks, & Brooks, 2008). Individuals with dyslexia describe using this strength to analyze large data sets and recognize patterns. Dyslexic students scored higher than nondyslexics for original thinking (Akhavan Tafti, Hameedy, & Mohammadi Baghal, 2009) and creativity for tasks requiring novelty (Everatt, Steffert, & Smythe, 1999). There is evidence that these strengths are neurologically interconnected with the challenges of individuals with dyslexia (Eide & Eide, 2011). This research has been strongly supported by an increasing movement of adults with dyslexia to reject the medical, deficit model of dyslexia.

In studies on successful dyslexic adults (2007), Rosalie Fink has found that while the adults had several developmental pathways to becoming successful adult readers, a consistent thread was the importance of reading in areas that individuals were passionate about. Another pattern in Fink’s findings is that these successful adults continued to have significant early deficits in reading, difficulties with letter switching and decoding, yet were successful readers at a much higher level (both as self-reported and based on reading comprehension assessments). We know of only one study in the area of mathematics that is similar to our work; a collaboration between a mathematics education researcher and an individual with dyscalculia, investigating how the second author developed her own strategies to support learning in an undergraduate mathematics program (Lewis & Lynn, 2018). We believe that collaboration across difference is necessary to develop new understandings of the potential for dyslexic students in mathematics.

Research Questions

1. What have been the experiences and mathematical learning trajectory for individuals with dyslexia in higher mathematics?
2. How do individuals with dyslexia approach mathematics?

Methods

Participants

Participants were recruited using emails to professional organizations of research mathematicians by the second author. All participants were currently employed at universities in mathematics or related STEM departments. Four out of 5 participants reported being actively engaged in current mathematical research, with one participant focused on teaching. 3 out of the 5 participants were diagnosed with dyslexia and/or a reading learning disability during K-12 schooling. Another participant was given a diagnosis later in life related to the diagnosis of a child. Another participant had a diagnosis of a related disability (ADHD) but whose primary difficulty was reading and reading comprehension. Two out of the five participants reported a diagnosis of ADHD, one had significant speech and language delays as a child, and one participant self-identified as autistic. Two out of five participants identified as cis female, with 3 identifying as cis male, however we use the pronoun “they” throughout the document to avoid identification. Four out of 5 participants identified as white or Caucasian, with one Asian-American participant. We identify this lack of diversity as a significant limitation to our work and hope to expand the populations included in future studies. We report data on participants (P1-P5) in the aggregate to avoid identification, as not all participants were comfortable with disclosing their disabilities in the university setting. Thus, we purposefully report themes across the interviewees, rather than describing each individual as a case study. We also use the pronoun “they” to avoid identification of participants.

Data Collection

The first and second author together interviewed each participant for between 60 - 90 minutes. The interview was semi-structured with questions in the following categories: 1) description of current mathematical work, 2) school experiences in mathematics, 3) school experiences and diagnosis of dyslexia, 4) connections between dyslexia and mathematics. Interviews were video-recorded and then transcribed.

Data Analysis

We identified narratives within the interviews based on transcripts. We identified two kinds of narratives. The first was Life History Narratives, more typical retrospective narratives retelling a life event. We also identified a kind of narrative we called Narratives of Thinking, which we define as nontraditional narratives without a set sequence of events, with an experiential description of how it feels to think in a certain way, here related to mathematical thinking and problem solving (Lambert 2019). After identifying narratives, we coded thematically (Riessman 2007) with both apriori and emergent coding. All narratives were coded by the first author with discussion of coding categories with the second author.

There is a tension in our work between identifying commonalities between the participants and making diversity between them quite clear. When we report that 4 out of 5 interviewees noted their own strengths in visual thinking, we are not suggesting that 80% of people with dyslexia are visual thinkers. Our small set of studies is an opportunity to identify some common themes within a small, specific subgroup of the dyslexic population, dyslexic mathematicians. We would need different research to answer questions about how these preferences apply across people with dyslexia more broadly.

Findings

The research mathematicians were working in the following areas of mathematics with multiple participants in some categories: real analysis, three-dimensional geometry, topology, and algebraic topology. The life narratives of these research mathematicians describe a non-direct pathway to becoming a research mathematician. They describe barriers that could have limited their process, such as calculus focused on memorization, or classes such as organic chemistry focused on memorization. All participants noted that they moved forward in mathematics once they reached a place in which they were fascinated by the problems, most often, a visual-spatial set of problems to solve.

All participants described dyslexia as a set of strengths and challenges, although the specifics of those strengths and challenges varied between participants. P4 notes, “the dyslexia... I explain to people, it's sort of like you're strong in one thing, but it makes you weak in others.” Of course, our study attracted individuals who were interested in talking about their dyslexia, which possibly created a group that was more positive about dyslexia than a randomly constructed group. In this paper, we report on three themes: 1) highly visual and intuitive ways of mathematical thinking, 2) issues with memorization of mathematical facts and procedures, and 3) the development of resiliency as a strength.

Theme 1: Highly Visual and Intuitive Ways of Mathematical Thinking

Flexible, creative thinking. All participants noted that they had a history of unusual ways of solving mathematical problems. 4 out of 5 participants described visual thinking quite specifically. 4 out of 5 participants also noted that they were known for flexible, creative, “out of the box” solutions to complex problems. P3 shared that one of their collaborators once described them, saying,

> I talk in ghosts and mists. My brain seems to be really, really comfortable with just throwing out ideas. It just really is very flexible. It doesn't like boxes. It's just very, very flexible. And so, I get a sense that something is true, or something that I want, I need, is there. And then my brain really doesn't get bothered by the fact that some ideas don't work, it just will throw out lots and lots of ideas and sort of wander. And that drives co-authors nuts, because they'll say, "Oh, I see? That idea doesn't work." And it doesn't slow me down one bit. My brain just

> has like five other weird ideas, two of which you can throw out immediately, and the three others you have to spend time on. And it just sort of keeps working that way. (P3)

P1 noted their personal strengths in mathematics as:

> Coming out with the idea that pushes you beyond the routine. So thinking about things, especially a visual or spatial ideas. Questioning and poking at the routine to say how do we express this idea? So sort of coming up with ideas that are not in the routine, especially things that are related to images. (P1)

Visualization. Four out of 5 participants repeatedly brought up visualization as a personal strength in their own mathematical thinking. In multiple interviews, participants described a duality such as "verbal" vs. "visual", or "algebraic" vs. "geometric." Four out of 5 participants identified as visual and/or geometric mathematical thinkers, with P1 noting that they learn through "geometry first, thinking through space" and "I can do immensely technical work in images that others can do in language."

P4 describes a strong predilection for thinking visually, not just in mathematics but across topics; "Well my entire memory is sort of visual, it's like playing back little snippets of film." They became interested in "three-dimensional geometry and topology. Anything that I can draw or sculpt or anything that's like three dimensional and sort of visual-based." Once when struggling in a physics class, they saw a particular image in the text, of vector fields on a surface, and suddenly, the "pictures made it make sense." P4 prefers to not only think with visuals, but to write with them as well, noting a strong preference for storyboarding mathematical papers using a series of images. Another participant also indicated that they chose their mathematical topic based on their preference for visual thinking, specifically "picture drawing and the topology of it" and "I liked the fact that I could pin it down and think about it as something real" (P2). Another participant identified as both:

> A details kind of thinker and like a visual thinker. I can't get interested in the details unless I have the picture that I think I'm working out the details for. But, once I have a picture of what I think should be going on, then the details become interesting . . . It's not the other way around. (P5)

Symmetries and the commutative property. Three out of 5 participants at some point in their interviews suggested their dyslexia may have been connected to a way of seeing mirror images and/or symmetries in geometric shapes and algebraic equations, "Well I think I saw symmetries, I saw equations easily because of it. Because my brain would flip things around very easily. I understood equations quickly and easily because of my dyslexia" (P2). One participant wondered aloud if, "Somehow dyslexic thinking is naturally commutative?" (P1).

All participants noted difficulties with language in relationship to mathematics, each slightly differently. Some discussed primarily issues with communicating visual thinking through language, or through the more linear pathway of writing. One participant described how it was difficult for them to visualize symbols/mathematics when the only modality that is being used is talk.

> P5: This even happens when I'm with mathematician friends and they'll be vocalizing an argument. There's no white board and they'll say, you do this and then you ... Yeah, I'm not necessarily going to follow the point. But, I'll go back to my room later and I'll remember enough of the points that they were trying to make that I'll get it. And, I'm okay with that, I don't have to be as quick witted as some of my colleagues are in mathematics, and I don't mind that.
>
> A: So, it's really different for you if there's paper, or there's a white board?
>
> P5: Yeah, if I can visualize things I'm much better off. Well, is that true? ... I'm pretty good at visualizing, but what I'm not good is transcribing spoken language into notation . . . If someone were to read out loud the definition of continuity for all blah, blah, blah. I would say, "Yeah, that might be right. That might be wrong. I have no idea." So, it's this translation between spoken

> language, and it's kind of linear notation that mathematicians tend to use. I'm not good at that part. I'm good at visualizing geometric things, but not visualizing notation (P5)

This suggests that strengths in visualization may be connected to challenges around translating across different forms of language and/or modalities, particularly from visual thinking into verbal language, or vice versa.

Issues with Memorization of Mathematical Facts and Procedures

None of the participants noted significant difficulties with mathematics in elementary school, with one consistent exception across participants: difficulty with the times tables and/or memorizing mathematical procedures. Some brought this up spontaneously, and for others, they clearly did not connect memorization of facts to mathematics. In this exchange with P2, the first author is asking whether or not they had any difficulties with math in K-12.

> A: Was there any part of math, like in elementary school, middle school, or high school that was challenging for you?
>
> P2: No.
>
> A: So memorization of facts was not challenging for you?
>
> P2: Oh I never could memorize anything. I had to derive everything . . . Yeah, I've never been good at memorizing things, just like I couldn't memorize how to spell words, I couldn't memorize facts in math. So I paid attention in class, and I had good enough teachers that they derived everything. And I figured out how to derive everything I needed to know, and I just derived everything I needed to know. You take a trig class, for instance, okay ... I know the trig identity for sin of alpha plus beta. From that trig identity, I can derive all the other ones. And then if I needed any of them, I would just do that. But I never actually like memorized them. I still don't memorize them.

Similarly, P1 noted that their mother taught the multiplication tables through a smaller set of memorized facts, specifically the squares, and then encouraging P1 to build equations through the distributive property from known facts.

In addition to the multiplication tables, participants noted the difficulty of any kind of memorization “without structure.” P1 notes,

That is one of the reasons I'm slower. I have really good memory for connected facts. I can't remember phone numbers at all. Learning foreign languages was the one bit of school that I hated because you have this long list of words that had no connection to anything. So memorization without structure. So I memorized the structure. (P1)

Another participant noted that they had a history understanding “concepts” in mathematics and struggling with “the details.” When we asked what they meant by details, they told a story about being negatively judged for their lack of memorization of the multiplication tables in elementary school:

> I could've explained to you with a picture why nine times five was 45, and my friends could tell you that it was 45 but they couldn't tell you why. And it struck me as really upsetting that someone that, just memorizing that number, was valued more than me understanding why that was the right answer. And it's always been a problem. But it just seems to me that why something is true is much more important than knowing that it is true. (P3)

Several participants noted that their difficulties with memorization were connected to the expectation of speed connected to memorization. Not only was memorization without structure very challenging, participants were asked to do this task under time pressure, which made it feel impossible.

Theme 3: Developing Strengths Through Struggle

Four out of 5 participants specifically mentioned resilience as a strength of dyslexia. More specifically, participants noted that working through challenges made them more resilient and perseverant, which became a considerable strength for them in higher mathematics. When asked what a strength of dyslexia is, P2 said, “Resiliency, I guess. Just being able to kind of overcome things that are not necessarily the easiest for you.” P5 described their “coping mechanisms”:

> Basically being comfortable with the fact that I'm not going to be fast at a lot of things. And, being okay with not being fast, that's really pretty important because I think that this kind of reading comprehension and fluency stuff. The fluency tests really make you think that speed is the whole deal. And, it was really important for me to sort of realize that no, that's not what matters. (P5)

Success in mathematics, participants noted, comes with hard work. Because math gets hard for almost everyone, understanding what to do when that happens is a gift for a mathematician. As P1 notes,

> Sort of actually everyone is facing struggles. Calculus is hard for most people. And so what we can understand about how people get through it when they have greater struggles is really useful for the people who are having smaller versions of those same struggles. And that notion of motivation ... The point that people who want to do this material but it's really hard for them can do it then surely that's a principle for all education. (P1)

Discussion

This research contributes by challenging the deficit-based approach typically used in educational research with students with dyslexia, thus potentially opening new avenues of educational research that move beyond the medical model of disability which locates LD/dyslexia solely in individual students. While this paper focused on only 3 themes, these findings suggest ways in which we can make math classrooms more accessible for students with dyslexia. We describe these as initial tenets of *Neurodiversity for Mathematics*, and plan future research to both explore insider perspectives further and to test these ideas in the classroom.

1. Offer opportunities for visual thinkers to learn new concepts through visual thinking. Not only provide multiple modalities for learning mathematics, but explicitly connect different kinds of representations. For example, one participant explained how their own mathematics teaching relies on visuals, but also with explicit connections to algebraic representations for those who preferred to think that way.
2. Remove the focus on memorization and procedural learning for students with dyslexia. As P1, who attended school outside of the US, noted;
   The high school calculus in the U.S., I think I wouldn't be a mathematician if I'd taken that because it's so emphasized it's fast, technical things ... And I make a lot of errors when doing calculation, and when you have a test which is multiple choice which is designed to map you into all your errors, I would have got very poor scores (P1).

When mathematics focuses on speed and memorization “without structure,” the potential of those with dyslexia will not be realized in our schools.

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WHAT IS A FUNCTION?

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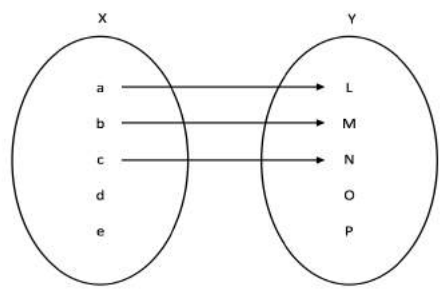
In the mathematical community, two notions of "function" are used: the set-theoretic definition as a univalent set of ordered pairs, and the Bourbaki triple. These definitions entail different interpretations and answers to mathematical questions that even a secondary student might be prompted to answer. However, mathematicians and mathematics educators are often not explicit about which definition they are using. This paper discusses these parallel usages and the related implications for the field of mathematics education.

Keywords: Curriculum, University Mathematics, High School Education

To frame the discussion of this paper, we invite the reader to answer the following questions that a secondary student might be asked about functions.

1. Does the following diagram represent a function?



2. Is the following set of ordered pairs a function? If so, what is its domain? $\{(-1,4),(0,7),(2,3),(3,3),(4,-2)\}$
3. Is $y = \sqrt{x}$ a function? If so, what is its domain?
4. Is $g(x) = \ln x$ the inverse function of $f(x) = e^x$?
5. Does $g(x) = \ln x$ have an inverse function? If so, what is it?
6. True or false: A function is invertible if and only if it is injective.

Two Definitions Of "Function"

Function is an important concept in mathematics, and students' understanding of function has been the subject of extensive research in mathematics education (Breidenbach et al., 1992; Leinhardt et al., 1990; Vinner & Dreyfus, 1989). Researchers have noted the mathematical definition of function has evolved: Initially, functions were characterized as explicit rules that assigned numbers to other numbers. Over time, the notion of function became more general— a function could take any type of object as its inputs or outputs (e.g., differentiation can be understood as a function that maps differential real-valued functions to other functions)— and *any* correspondence can be a function, regardless of whether the rule for the function can be explicitly stated or not. Finally, the modern treatment of function is usually provided in set theoretic terms (Kleiner, 1989). Research on students' understanding of function often shows that students hold varied conceptions of function that do not align with the formal theory, even when they can state the formal definition of function (Bardini et

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al., 2014; Leinhardt et al., 1990; Mirin, 2018; Sfard, 1992; Thompson, 1994; Vinner & Dreyfus, 1989).

It is unsurprising that the meaning of function has evolved. The definitions of many mathematical concepts have become more precise and more abstract over time, and several concepts are now defined in structural or set theoretic terms although they were not originally conceived of in this way (Sfard, 1992). Likewise, students often hold different understandings of the same mathematical concept, with their understandings being internally inconsistent or at variance with the formal theory. We might expect there to be an agreed upon modern definition of function - or, absent a uniform definition, that the different definitions that mathematicians use are logically equivalent. Surprisingly, this is not the case. In fact, the different definitions in use do not even yield the same answers on questions that a secondary student might be asked. In this paper, we describe two commonly used definitions of functions in high school and university mathematics. We show that these definitions actually lead to different answers to the six questions that we posed in the beginning of the paper.

One Definition Of Function: The Bourbaki Triple

One treatment of function is that of Bourbaki (1968), which defines a function as a triple (f, A, B), where A and B are sets and f is a univalent and total subset of $A \times B$. That is, for *all* x in A (*total*), there exists a *unique* y in B (*univalent*) such that (x, y) is a member of f. The set A is called the *domain* of the function, B is the *codomain,* and f is the *graph.* We refer to this as the *Bourbaki Triple* function definition, objects of this type as *Bourbaki Triple functions,* and people who use this definition as *Bourbaki Triple people.*

Another Definition Of Function: The Ordered Pairs

Forster (2003) notes, "some mathematical cultures… [say] a function is an ordered triple of domain, range, and a set of ordered pairs. This notation has the advantage of clarity, but it has not yet won the day" (pp. 10-11). Forster then refers to the alternative definition of a function as *any set of ordered pairs f* that satisfies the following criterion: if (x, y_1) and (x, y_2) are in f, then y_1 and y_2 are equal. In this case, the *domain* of f is the set of all numbers x such that there exists some y where (x,y) is a member of f. There is no unique *codomain;* **a** (rather than **the)** *codomain* is any superset of the range of f. We refer to this as the *Ordered Pairs* function definition, objects of this type as *Ordered Pairs functions,* and people who use this definition as *Ordered Pairs people*.

Comparing and Contrasting Definitions

It's worth emphasizing that a Bourbaki Triple function is a different sort of object than an Ordered Pairs function; a Bourbaki Triple function is an ordered triple, whereas an Ordered Pairs function is a set of ordered pairs. Hence, when a Bourbaki Triple person mentions a function, they are referring to a different type of object than that of an Ordered Pairs person. The way someone understands questions or statements about functions will be related to what type of object they understand a function to be – a triple, or a set of ordered pairs.

Dumas and McCarthy (2015), Bourbaki Triple people, assert the following:

> When you write $f: X \to Y$, you are explicitly naming the intended codomain, and this makes the codomain a crucial part of the definition of the function. You are indicating to the reader that your definition includes more than just the graph of the function. The definition of a function includes three pieces: the domain, the codomain, and the graph. (Dumas & McCarthy, 2015, p. 25)

Dumas and McCarthy are correct *if* one is using the convention of a function as a Bourbaki Triple. In this case, "$f: X \to Y$" names the function (f, X, Y) with domain X, codomain Y, and graph f. However, a look at works by authors who use the Ordered Pair notion of function suggests that adopting such notation does not necessitate endorsing the view that a function is an ordered triple (Devlin & Devlin, 1993; Goldrei, 1998; Halmos, 1960; Stoll, 1979). Using the Ordered Pair interpretation, these authors write "$f: X \to Y$" to mean that f is a function with domain X where $f(x)$ is a member of Y for all x in X (that is, Y is **a** codomain of f). Rather than X and Y being *part* of the function itself, they are *attributes* of the function.

We can see how the differences in the notion of function manifest themselves in interpreting a function definition. Consider the following sentence: "Let $f: N \to Z, f(n) = n + 1$". Under the Ordered Pairs definition, the function is the set f=$\{(n, n + 1): n \in N\}$, and this set (function) has the *property* that N is its domain and Z is **a** codomain. Under the Bourbaki Triple definition, the function actually at hand *is* the entire triple (f, N, Z), where f is still the set $\{(n, n + 1): n \in N\}$, the domain is N, and **the** codomain is Z. The important thing here is that in one interpretation, the function is just the set $\{(n, n + 1): n \in N\}$, while in the other interpretation, the function is the entire triple $(\{(n, n + 1): n \in Z\},\ N, Z)$. Now consider the notation "Let $g: N \to N, g(n) = n + 1$." Under the Ordered Pairs definition, the function g is the same as the function f. Under the Bourbaki Triple definition, the function at hand is the entire triple (g, N, N), which is a different triple than (f, N, Z) (as Z is a different set than N).

Notice that the domain of an Ordered Pairs function is not necessarily stipulated; it is derived as a consequence of the graph itself. If f is a function, x is in the domain of f exactly when there exists a y such that (x, y) is in f. For this reason, the only criterion a set of ordered pairs needs to satisfy is being univalent. It does not make sense to ask whether a relation f is "total" in the abstract; one would need to ask if f was total on a specified set. Similarly, it does not make sense to ask if f is "surjective"; one would need to specify a set (codomain) that f might be surjective upon. The notation "$f: X \to Y$" might be used to stipulate such sets, but this notation is not always used.

A sharp difference between the Bourbaki Triple definition and the Ordered Pairs definition relates to invertibility. With the Ordered Pairs definition, the inverse for a function f, denoted by f^{-1}, is the set $\{(y, x): (x, y)\ \in f\}$. This set f^{-1} is a function if and only if f is one-to-one, i.e., f is invertible as a function if and only if it is injective. With the Bourbaki Triple definition, if (f, A, B) is a function, we consider the inverse of the triple to be (f^{-1}, B, A) where f^{-1} is defined as above. In this case, for (f, A, B) to be invertible as a function, more than injectivity is required; the triple (f^{-1}, B, A) must also be a function. This means that f^{-1}must be total on B, requiring that f be surjective onto B. We will revisit this difference later.

How Are Functions Treated In The Literature?

In the mathematics literature, both definitions of function are common. The Bourbaki Triple definition appears in some introductory proof books (e.g., Dumas & McCarthy, 2015) and other domain-specific textbooks, such as Abbott's (2012) textbook on real analysis. On the other hand, the Ordered Pairs definition also appears in some introductory proof textbooks (e.g., Forster, 2003) and in set theory textbooks (e.g., Enderton, 1977; Halmos, 1960; Jech, 2003). Still, other textbooks offer both definitions (e.g. Eccles, 1997). Finally, we will illustrate how others (e.g. Stewart, 2003) are ambiguous.

In the mathematics education literature, we find the Bourbaki Triple definition to be more common. For instance, in their influential review of students' understanding of functions, Leinhardt et al. (1990) refer to the Bourbaki Triple as the modern definition of function that mathematics educators would like their students to master. Other studies have used this definition as their backdrop for what is normatively correct, but they use the *word* "Bourbaki" only, without referring to the internal structure of the triple itself (e.g. Breidenbach et al., 1992; Vinner & Dreyfus, 1989). These authors tend to focus on the arbitrariness of a function as a correspondence rather than as a rule or an equation. Nonetheless, there is variation, and other mathematics educators have adopted the set of ordered pairs definition as their background theory (e.g. Sajka, 2003). Also, it is noteworthy that mathematics educators often describe $g(x) = \ln x$ as the inverse function of $f(x) = e^x$ (e.g. Even, 1990; Mayes, 1994), which seems to suggest the Ordered Pairs definition (discussed below).

In algebra and precalculus texts, we have again found both definitions used. For instance, Hungerford and Shaw (2009) define a function explicitly consisting "of three parts— a set of inputs (called the domain); a rule by which each input determines exactly one output; a set of outputs (called the range)" (p. 155). However, there are also books that define functions as a set of ordered pairs; for instance, Marecek (2017) wrote:

> A relation is any set of ordered pairs (x, y). All the x-values in the ordered pairs together make up the domain. All the y-values in the ordered pairs together make up the range [...] A function is a relation that assigns to each element in its domain exactly one element in the range. (Marecek, 2017, pp. 314-317)

Consistent with the Ordered Pairs definition, Marecek (2017) notes that the domain and range are not specified but derived from the set of ordered pairs, and there is no notion of codomain.

What is especially interesting to us is that some definitions stated in textbooks, and in the education literature, define function in such a way that it is ambiguous as to whether they are using the Bourbaki Triple definition or the Ordered Pairs definition. For instance, consider the way that Stewart (2003) defines functions in his widely used calculus textbook: "A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E" (Stewart, 2003, p.12).

Stewart's (2003) definition contains an ambiguity. We can either (i) view D and E as part of the meaning of function and/or claim they need to be specified in advance, or (ii) view the statement as an existential statement where f is a function if there exist sets D and E that fit the definition. That is, the definition allows for similar interpretation as the notation "$f: D \rightarrow E$", under both types of function definition (Bourbaki Triple and Ordered Pairs). In the remainder of Stewart's (2003) text, he appears to treat a function as equivalent to its graph (p. 14) and claims all injective functions are invertible (p. 64). He always treats the range and image of a function equivalently and never concerns himself with codomains. He thus appears to be using the Ordered Pairs definition of function.

Not only is there ambiguity in what an author might mean by "function," but there is also ambiguity in what definition the "Bourbaki approach" to functions implies. Selden and Selden (1992) take a function to be a set of ordered pairs: "the formal *ordered pair definition* of function, first introduced in 1939, is often referred to as the *Bourbaki* approach" (p. 2). Based on the preceding discussion, one might think that a "Bourbaki function" is a Bourbaki Triple. However, the above quote illustrates that this interpretation is not so straightforward.

Our main point thus far is as follows: There are two different definitions of function that are not logically equivalent: (i) both definitions are used in advanced mathematics, secondary mathematics, and the writing of mathematicians; and (ii) even when an author defines functions in their text, it can sometimes still be ambiguous as to which definition they are using. In the next section, we discuss how the different definitions entail different interpretations of mathematical questions that a secondary student might encounter.

Revisiting Questions

We address each of the six previously posed questions using the most straightforward interpretation of the two definitions of function above:

1. Consider the diagram in the first question at the start of this paper. A Bourbaki Triple person might interpret the question to be asking if the triple $(\{(a, L), (b, M), (c, N)\}, X, Y)$ is a

function. The answer is "no"; there are members of X that are not assigned a value in Y (the structure is not total). We believe this is regarded as the normatively desired response. However, an Ordered Pairs person might interpret the question to be asking if the set $\{(a, L), (b, M), (c, N)\}$ is a function. Of course the answer is "yes", because it is a univalent set of ordered pairs (assuming *a, b,* and *c* are distinct).

2. This question was taken from Redden (2012), p. 249, Example 2. According to the Ordered Pairs definition, $\{(-1,4), (0,7), (2,3), (3,3), (4, -2)\}$ is a function because it is univalent. Its domain is $\{-1, 0, 2, 3, 4\}$. We believe this is regarded by most as the normatively desired response (as in Redden, 2012; Even, 1990). According to the literal Bourbaki definition, we would say this set is *not* a function, for the simple reason that functions are triples and not ordered pairs; that is, no domain and codomain are stipulated here. If we were less literal, this question would be undefined; we would need to know what domain and codomain were stipulated. It is possible to interpret the question to be asking if there *exists* a domain *D* and codomain *E* such that (f, D, E) is a function, where f is the given set of ordered pairs above. However, we would not expect this interpretation from a secondary student.
   A Bourbaki Triple person can adopt conventions so that their answer to this question agrees with that of an Ordered Pairs person. If a domain is not specified, the domain of a real-valued relation is stipulated to be the largest set of real numbers to which the relation can apply, and if the codomain is not specified, it is assumed to be *R* (in the context of secondary algebra, calculus, or real analysis).
3. According to the set of ordered pairs definition, $y = \sqrt{x}$ is a function. That is, we interpret the function at hand to be the set $\{(x, y) \in R{\times}R \colon y = \sqrt{x}\}$. As this set is univalent, it is a function. We believe this is regarded by most as the normatively desired response. However, interpreting the question using the Bourbaki Triple definition is less straightforward. What object are we asking is a function? If we are stipulating that the domain and codomain are both R, which is arguably a convention in the context of a calculus or real analysis course, then our question is "is $(\{(x, y) \in R{\times}R \colon y = \sqrt{x}\}, R, R)$ a function?" and the answer is a clear "no", since this triple is not total on R. On the other hand, we can interpret the domain to be $[0, \infty)$, the largest domain on which it can be defined. In this case, the answer is "yes", as the triple $(\{(x, y) \in R{\times}R \colon y = \sqrt{x}\}, [0, \infty), R)$ is a function.
4. Using the Ordered Pairs definition, $g(x) = \ln x$ is the inverse of $f(x) = e^x$. That is, the set $\{(x, y) \in R{\times}R \colon y = e^x\}$ has the inverse $\{(x, y) \in R{\times}R \colon y = \ln x\}$, which is a function. We believe this is regarded as the normatively desired response; see, for instance, Stewart (2003, p.67), who defines logarithmic functions as the inverses to exponential functions. However, a Bourbaki Triple person who assumed the convention that the codomain of a real-valued function is *R* unless otherwise stated would assume that the question is asking if $(\{(x, y) \in R{\times}R \colon y = e^x\}, R, R)$ has an inverse function, and such a structure is not invertible as a function because it is not surjective onto R.
5. For similar reasons as in the previous example, an Ordered Pairs person would straightforwardly believe that $g(x) = \ln x$ has an inverse function (it is injective) and its inverse is $f(x) = e^x$, which, as we noted above, we believe is the normatively desired response.
   The Bourbaki Triple person would have a less straightforward answer. If they accept that $g(x) = \ln x$ is a function, then they would interpret it as a function from R^+ to R to ensure that it were total. Now, since *g* is bijective between R^+ to R, *g* must have an inverse. In this case, they would be interpreting "$g(x) = ln\, x$" to name the triple $(\{(x, y) \in R{\times}R \colon y = ln\, x\}, R^+, R)$. However, the inverse is not $f(x) = e^x$ because they would interpret "$f(x) = e^x$" to name the triple $(\{(x, y) \in R{\times}R \colon y = e^x\}, R, R)$, and this function's

codomain (R) is not the domain of $(\{(x,y) \in R{\times}R: y = ln\, x\}, R^+, R)$. The inverse would be the function $h: R \to R^+$defined by $h(x) = e^x$, which is the triple $(\{(x,y) \in R{\times}R: y = e^x\}, R, R^+)$. There is nothing contradictory about this; the Bourbaki Triple definition implies that functions with the same graphs but different codomains are different functions. We only observe that we ordinarily would not expect a student to distinguish between the differing codomains to receive credit for identifying the inverse of $g(x) = ln\, x$.

6. As we noted in the previous section, the Ordered Pairs person agrees that a function is invertible whenever it is injective. The Bourbaki Triple person disagrees, claiming a function needs to be surjective as well. It appears there is no consensus in textbooks or amongst mathematics educators for whether injective functions are necessarily invertible. Some, like Abbott (2012, p. 155) and Stewart (2003, p. 64) assert that all injective functions are invertible. Others, like Mattuck (1999) and Friedberg et al. (1989), claim that injectivity and surjectivity are both necessary.

Two Approaches For Coping With Difference In Mathematics Education

Functions and their inverses are fundamental concepts. Naturally, mathematics educators would like students to develop productive and normatively correct understandings of functions and inverses. However, there are two different ways of defining functions that lead to divergent answers to basic questions from secondary mathematics. For instance, one would hope that there is a straightforward consensus answer as to whether $f(x) = e^x$ is invertible, at least in the context of secondary mathematics, but this is not the case. How, then, is an educator to teach students or evaluate the quality of a student's understanding of functions and their inverses?

There are at least two positions that an educator may adopt: *dogmatism* or *contextualism*. With dogmatism, we can insist that one of the two definitions is *the right definition,* argue that textbook writers and other researchers should use this definition, and regard those who do not act in accordance with this definition as being mathematically sloppy or incorrect. For instance, a Bourbaki Triple dogmatist might insist on using the Bourbaki Triple definition; the Bourbaki Triple dogmatist might call for textbooks to clarify Questions like 2 and 3 in the beginning of the paper to be mathematically accurate. This would involve rewrites such as "is there a set *D* (domain) and a set *E* (codomain) such that $(\{(-1,4),(0,7),(2,3),(3,3),(4,-2)\}, D, E)$ is a function?" If $f(x) = \sqrt{x}$ defines a graph of a function, what could its domain be?"). The Bourbaki Triple dogmatist acknowledges that some textbooks and even some mathematics educators use the ordered pairs definition of function, but that does not mean mathematics educators *should* be flexible with their definition of function. Indeed, it would be unwise policy to draw conclusions about the nature of mathematics based on errors that textbook writers and mathematics educators sometimes make. Similar dogmatism could be recommended by an advocate for the Ordered Pairs definition. We admit that a dogmatist approach has several advantages. For one, it would unify the differential treatment of functions in textbooks and mathematics education literature. Further, it would provide clear normative guidelines for how functions and inverses should be discussed and how students' understanding of these functions should be evaluated.

The alternative approach, *contextualism,* is to declare that there is no universal definition of function, but rather that the definition of function depends on context. Consider the following passage from Dumas and McCarthy's (2015) text in which they justify adopting the Bourbaki Triple definition:

> If you identified the function with its graph, then every function would have many possible codomains (take any superset of the original codomain). Set theorists think of functions this way, and if functions are considered as sets, extensionality requires that functions with the

> same graph are identical. However, this convention would make a discussion of surjections clumsy, so we shall not adopt it. (p. 25)

We highlight three ideas here. First, Dumas and McCarthy acknowledge that there was more than one definition of function that they could have used in their textbook. In particular, they do not say set theorists are wrong for defining a function as a univalent set of ordered pairs. Second, they view the decision on which definition to adopt as their choice. Third, they do not view their choice as arbitrary; they provide a reason justifying their choice; they adopted the Bourbaki triple definition because it made it easier to discuss and reason about surjective functions, which was one of their goals in the textbook.

Similarly, Joel David Hamkins, a mathematician and philosopher from Oxford University, justifies why set theorists like himself prefer to think of functions as ordered pairs for mathematically practical reasons. For instance, Hamkins explains why it is difficult to speak of sequences of ordinals in set theoretic terms using the Bourbaki Triple definition. Responding to a challenge that the concept of function is "imprecise", Hamkins responds:

> Many words lack meaning out of context, while becoming precise in a context. Why should you expect that there is a meaning for this word [function] outside of any context? [...] The function concept *has* been made absolutely precise. In fact, it has been made fully precise twice, in two different ways. Each group prefers to use their own precise definition, for sound reasons. (Hamkins, 2010)

The contextual position, for which we advocate, synthesizes the comments above. The contextualist acknowledges that for mathematicians, everything else being equal, it would be best for the same concept to be defined in the same way across all mathematical contexts and communities. This adds clarity and facilitates communication between different mathematical communities. However, mathematicians consider other factors to consider when choosing a concept's definition. In particular, mathematicians desire that their definitions should facilitate their communication, problem posing, and problem solving. Because the needs with respect to function vary by mathematical community, it is not surprising that different mathematical communities would define the function concept in different ways. The value of uniformity in these cases is not necessarily worth more than the value of utility.

In the case of *function* in secondary mathematics, functions are usually used for purposes of modeling and equation solving. In these cases, the totality of a function tends not to matter. The functions $f(x) = 1/(x^2 + 1)$ and $g(x) = 1/x^2$ are, for the most part, interpreted and acted upon in the same way, even though the former is total while the latter is not. It would be detrimental to the theory to exclude partial functions on R like $g(x) = 1/x^2$ or $h(x) = tan\,x$, and it would be cumbersome to constantly stipulate domains. For these reasons, we think it is prudent for textbooks to ignore the totality of functions in this context, except in the cases where the lack of totality matters. Similarly, inverse functions are generally used to assist in equation solving, graphing, and differentiation. In most cases, the use of inverses does not depend on whether the inverse is total on the codomain of the original function. It would be detrimental to the theory to eliminate inverse functions for functions that are not surjective, and it would be cumbersome to complicate the reasoning by introducing and changing codomains. Textbooks are justified in treating injective functions as always having inverses in this context. However, in other contexts in which non-total functions are not the object of consideration or would needlessly complicate the theory (e.g., group homomorphisms), it makes sense to adopt the Bourbaki Triple definition. Likewise, in contexts in which surjectivity or function composition play a central role, it might make sense to adopt the Bourbaki Triple definition.

We conclude by offering some recommendations to the mathematics education community for their investigations of functions and inverses:

- Be aware that there are two definitions of functions in mathematical practice and that these definitions entail different answers to questions that a secondary student might be asked.
- In research reports, state which conception you have in mind and justify your choice. How did your conception of function allow you to achieve your pedagogical or research goals? If your definition of function was not germane to the study (e.g., you were only focusing on univalence and issues of totality and inverses did not arise), explain that too.
- Avoid being a dogmatist when evaluating research papers. Just because a scholar used a different definition of function than you would prefer does not mean that they are mathematically incorrect. On the contrary, regardless of whether they adopted the Bourbaki Triple definition or the Ordered Pairs definition, they are in good company with many prestigious mathematicians.
- Avoid being a dogmatist when evaluating students. Regardless of whether a student asserted that "every injective function is invertible" or its negation, it would be a mistake to evaluate this comment as mathematically incorrect. It would be more appropriate to look at the reasonings and understandings that the student used to justify their assertion.
- In instruction, it is misleading to assert that mathematical definitions are always universal. There are multiple definitions for mathematical concepts that are not logically equivalent, even in secondary mathematics. Other examples besides functions include natural numbers (does this set include 0?) and trapezoids (is a parallelogram a trapezoid?). Rather than speak in absolutes, we suggest acknowledging that some mathematical concepts are defined in different ways, and to highlight the benefit of the definitional choice in the particular classroom context in question.

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UNDERSTANDING THE ROLES OF PROOF THROUGH EXPLORATION OF UNSOLVED CONJECTURES

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Mathematics majors, including future secondary teachers, should understand the work of pure mathematicians and the crucial role proof plays for the discipline of mathematics. Beyond the textbook proofs seen in most transition-to-proof courses, we conjectured that students might develop a deeper understanding of the discipline of mathematics and proof if they had the opportunity to do mathematical research—to try and prove an unsolved conjecture. As an added component of our transition-to-proof course, we designed an intervention so that students researched the Twin Primes conjecture or the Collatz conjecture. Students wrote reflections about their research and described how their perceptions of mathematics were influenced by the research. We analyzed the reflections and sought to understand how the students' views of mathematics and proof were enriched, if at all, through research on unsolved conjectures.

Keywords: Reasoning and Proof; University Mathematics; Problem Solving; Affect, Emotions, Beliefs, and Attitudes

Mathematics teachers should have a rich understanding of the discipline of mathematics. When a teacher possesses a healthy and informed conception of the discipline, she is well-positioned to pass on productive beliefs about mathematics to her students. Teachers and students benefit from understanding the important thought processes and practices that make mathematics the unique and enjoyable field that it is. But many mathematics majors and school teachers have naïve views of the discipline of mathematics (Pair, 2017; Thompson, 1992). Pair (2017) found that students in a transition-to-proof course were not familiar with conjectures in general and generally unaware of well-known conjectures such as the Twin Primes conjecture.

Students' Understanding of the Function of Proof

Research has shown that many students, even mathematics majors, struggle learning to prove (Bleiler, Thompson, & Krajčevski, 2014). This may be due in part to the fact that students do not experience the functionality of proof in the same way that mathematicians do (de Villiers, 1990). For instance, proof for a mathematician can a be a means to convince other mathematicians that a claim is true. Students may not experience proof in this way if they are only asked to prove textbook exercises which are assumed true from the get go.

De Villiers (1990) described five roles of proof for mathematicians that he conjectured would be productive for students to understand and experience: 1) Verification: Proof serves as a means of knowledge justification that enables mathematicians to obtain conviction that a claim is true; 2) Explanation: Proof provides insight into why a mathematical claim is true; 3) Systematization: Proofs serve to organize the deductive system of axioms, definitions, and theorems; 4) Discovery: Mathematicians make new discoveries through proof; and 5) Communication: A proof is a means by which mathematicians communicate mathematical knowledge.

We believe that a deeper understanding of the roles of proof corresponds to a deeper understanding of the discipline of mathematics. Some researchers have documented the types of course activities (e.g. critiquing classmates' proofs) that may engage students in these roles of proof (Bleiler-Baxter & Pair, 2017; Cilli-Turner, 2017). We conjectured that exploring unsolved conjectures may also provide students an opportunity to deepen their understanding of the discipline of mathematics and experience the functionality of proof as mathematicians do.

Transition-to-Proof Intervention Research Study

We conducted a research study in a transition-to-proof course at a university in Southern California. The course was required for students majoring in either pure mathematics or secondary mathematics education. There were thirty-three students enrolled in this class. Twenty-five of the students agreed to participate in the research project which was approved by our University's Institutional Review Board. Our University is a Hispanic-Serving Institution, and a diversity of races and genders are present in our sample.

As an added component of the course, students explored and invented their own methods to navigate one of two famous mathematical conjectures: either the Twin Primes conjecture or the Collatz Conjecture. Typically attributed to Euclid, the Twin Primes conjecture states that there are an infinite number of twin primes (Rezgui, 2017). Two prime numbers $x < y$ are twin primes provided $y = x + 2$. For instance, 3 and 5 is the smallest twin prime pair. The Collatz conjecture, named for the German mathematician Lothar Collatz, is also known as the $3n + 1$ problem (Bairrington & Okano, 2019). The conjecture concerns an iterative process on natural numbers. Given any natural number n, if n is odd multiply by 3 and add 1; or if n is even, then divide by 2; repeat the process on the resulting natural number; repeat again. The conjecture is that for any natural number, this iterative process will eventually reach 1. For instance, if we take n=3 the corresponding Collatz sequence is 3, 16, 8, 4, 2, 1 (if continued the sequence would cycle 4,2,1, 4,2,1…). These two conjectures have remained unsolved to the present day.

Students in the transition-to-proof course explored these conjectures for an entire semester, documenting their work and reflections in what was called their *mathematicians' notebooks*. Their work in the mathematicians' notebooks accounted for 5% of their course grades. In the first notebook assignment, students were tasked with exploring both the Twin Primes conjecture and the Collatz conjecture. For subsequent assignments, students chose which conjecture they would like to explore. Students were assigned to research teams with other members of the class based on their conjecture preferences.

About half of the students worked on the Collatz Conjecture[1] and half on the Twin Primes conjecture, with some students exploring both. Midway through the semester the students shared their findings with other members of the class. Students drew inspiration from each other and adopted their classmates' approaches in subsequent assignments. Twice during the semester, the instructor collected students' notebooks and provided feedback and direction to guide students in their explorations. One assignment also required the students to watch and reflect on videos of mathematicians addressing their own work on the conjectures.

See Figure 1 for an example of student exploration on the Collatz conjecture from a student's notebook. This student, a Hispanic male, was working backwards from 1, trying to show that all numbers will eventually cycle to 1. His first line shows powers of 2, which obviously will reach 1. For each power of 2, he considered if it was of the form 3n+1 by subtracting 1 from the number and dividing by 3. When he found such a number, he would find the value of n and use it to start a new number line—which started with n and increased by a factor of 2. He wrote "[the pink] represents how a new line in the form of $n \cdot 2^{x}$, where $x \geq 0$ and n is an odd integer, is made by subtracting by 1 and then dividing by 3 by another line in the same form. [The green line] represents the set of natural numbers in order starting from 1." This student recognized that if he could successfully show

[1] Slightly more students worked on the Collatz conjecture than the Twin Primes conjecture.

that every natural number appears in one of the number lines, then he could prove the Collatz conjecture.

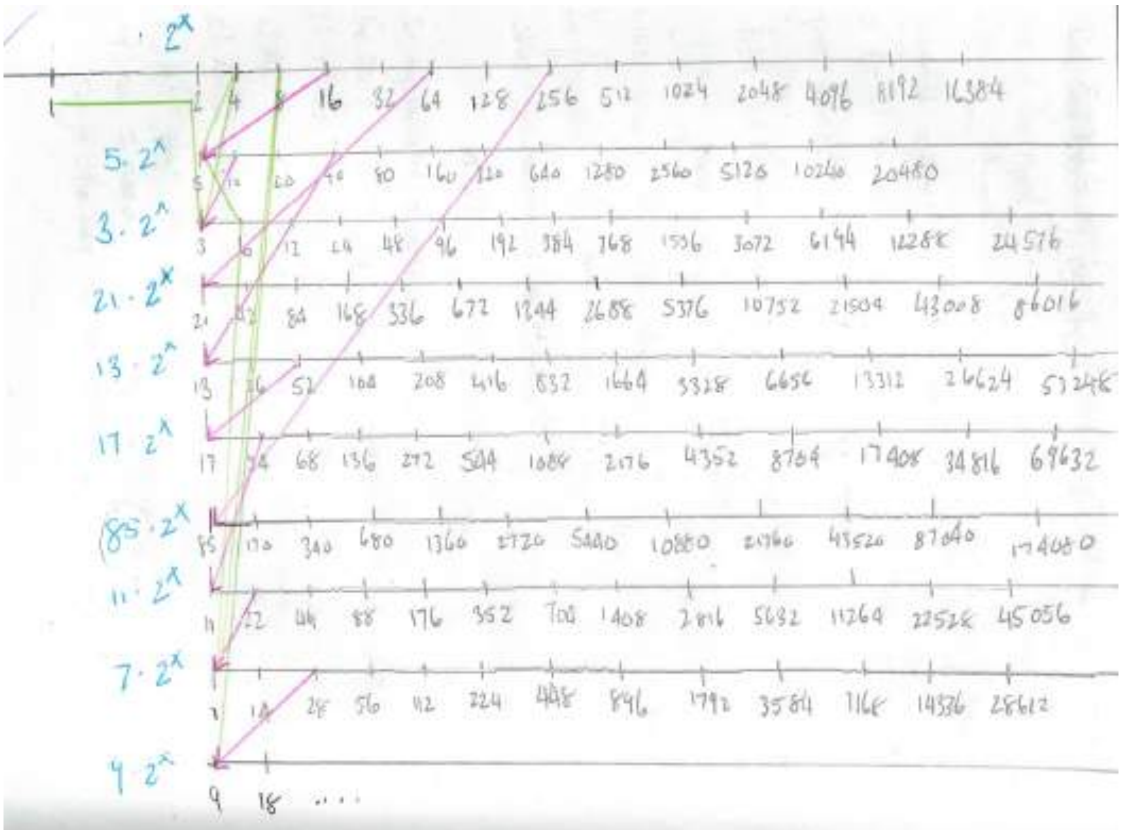

Figure 1: Example of Student Exploration of the Collatz Conjecture

We conjectured that through work on unsolved conjectures, mathematics majors would have novel opportunities to enrich their understanding of the nature of mathematics and the roles of proof. The first author (instructor of the course) designed the notebook assignments so that they would be a source of data useful for providing insight into what the students were learning about the nature of mathematics and proof as they explored the conjectures. Students had several reflective prompts they were required to address in their notebooks. For instance, at the beginning of the semester, students responded to the prompts *What is mathematics all about?* and *What is a mathematical proof, and how is it used by mathematicians?* The students responded to these prompts again at the end of the semester; they also responded to two additional prompts: (1) *How has your thinking regarding mathematics and mathematical proof developed and changed during this semester? Which changes were the result of engagement in the standard course activities (e.g. homework, tests, lectures) and which changes were the result of your experience conducting mathematical research (working on the unsolved conjectures)?* and (2) *What were the challenges and successes of your experience with mathematical research this semester?* The researchers analyzed student reflections to understand students' experiences working on the conjectures. In particular, we analyzed the notebook data to discern what students learned about the nature of mathematics and the roles of proof through their research.

Data Analysis

The data for the research study consisted of 25 student notebooks. The instructor (first author) and an undergraduate student from the course (second author) analyzed the data. First, each researcher individually read and examined a particular students' notebook. As we individually read through each student's notebook we took note of 1) Instances where students reflected about the nature of mathematics; 2) Possible evidence for changes in student's perceptions of the nature of mathematics (especially in regards to the five roles of proof); 3) Interesting mathematical ideas and approaches the student generated in working on the conjectures; and 4) Other interesting ideas expressed by the students. Each researcher then wrote a summary of his individual findings for the particular student's notebook.

After reading and analyzing a notebook, the researchers then shared their thoughts and findings with each other. We made notes on insights gained from the others' analysis and made note when a student demonstrated an understanding of any of the five roles of proof. We then repeated the process for the next student notebook.

Once we had analyzed all of the individual notebooks, the next step was for each researcher to create a holistic summary of the data that included broad themes in the students' responses as well as evidence (from the data) to back up our claims. The creation of these holistic summaries involved sorting student quotations into categories or themes (Ryan & Bernard, 2003), providing evidence that students' understandings of the nature of mathematics seemed to be enriched through the notebook project, and tallying how many students expressed certain ideas regarding the nature of mathematics in regards to the Roles of Proof framework. We also identified other recurring themes in the data related to students' experiences. We then shared our findings with each other, challenging each other to provide evidence for claims, which led to further refinement of the findings. We now present the results.

Results

We found that students alluded to four of de Villiers' (1990) five roles of proof while participating in this study: verification, explanation, systematization, and discovery.

Verification

Early in the semester, eight students' descriptions of the purpose of proof alluded to the notion of verification. For instance, a student wrote "A mathematical proof is a tool mathematicians use in order to determine if their statement is true or false." But by the end of the semester, seven additional students described the role of verification. These students wrote either about the role of proof in convincing others that a theorem is true, or the importance of proof in justifying and validating mathematical claims. For instance, one student wrote "Ideally, [a proof] should have no errors (holes) and must convince other mathematicians that it is correct and true for its purpose." Students were well aware of their inability to find a convincing proof for either the Twin Primes conjecture or the Collatz conjecture.

Explanation

Only two students alluded to the role of explanation in their initial descriptions about the purpose of proof, but by the end of the semester nine additional students alluded to this role of proof in their reflections. These students used language that emphasized proof's role in providing insight into why a mathematical claim is true. For instance, at the end of the semester a student wrote "At first, I feel like mathematics was more of just applying formulas and theorems to solve problems, but over the course I learned that it is more important to know why theorems work and how they work." Other students described how proof was needed to understand mathematics deeply. At the end of the semester a student wrote, "A mathematical proof is a way to understand how and why certain mathematical concepts exist. In math, there is always a reason for everything and math proofs help explain those reasons further. It is used by mathematicians to understand problems further and on a deeper level." Students recognized the surface level simplicity of the two conjectures (some students described how even an elementary student could understand them) but also recognized the complexity of what was required to understand *why* the conjectures were true.

Systematization

Four students alluded to the systematization role of proof at the beginning of the semester, and three additional students alluded to systematization at the end. These students described a building up of mathematics—results serve as the foundation for future results. A student wrote, "Proofs are facts so they have a crucial part to play in the development of other proofs where progression in the proof has more difficulty than normal." And another student, referencing a video she watched related to the twin primes, wrote, "Mathematicians use these proofs to help prove other conjectures. As Maynard's proof was influenced by Zhang's proof. Eventually, Maynard's proof will be used to help prove other conjectures." These students came to see that mathematicians work on conjectures; and proofs are used to build the body of mathematical knowledge.

Discovery

Four students alluded to the discovery role of proof, all at the end of the semester. These students seemed to better understand the mathematician's quest to discover new results. One student wrote, "Proofs are used by mathematicians to assist them in creating other proofs to eventually have a breakthrough that is groundbreaking in mathematics as well as the world." Another student wrote, "Mathematics is all about solving the world's greatest mysteries. Just from taking this class I have learned that mathematicians discover new problems and then spend their life trying to prove/understand it." See Figure 2 for a display of how many students alluded to a role of proof in their reflections.

Role of Proof	Alluded to at the Beginning of the Semester	Alluded to by the End of the Semester (but not at the Beginning)
Verification	8	7
Explanation	2	9
Systematization	4	3
Discovery	0	4
Communication	0	1

Figure 2: Number of Student Reflections on the Role of Proof

Other Results

We believe that most of the students had naïve views about pure mathematics and what it entails for mathematicians as they began the course. The notebook assignments provided students the opportunity to try their hand at proving conjectures that even famous mathematicians have not yet proven. This gave many of the students the opportunity to develop new ideas about what the discipline of mathematics and proof is all about, as many of them were dealing with a type of mathematics problem that they had never even conceived.

The notion of an unsolved conjecture provoked some of the students to start thinking more deeply about the topic, which in-turn led to creative and outside-the-box ideas. Eight of the students either showed a creative process in solving the conjectures, or stated they learned they could be more creative/look at conjectures from different angles and perspectives. Some forms of this creativity included working backwards on the Collatz conjecture, creating code to help find patterns and links within the Collatz conjecture, as well as finding a formula for the distance between twin prime pairs.

One student stated in their last reflection that "I learned we can look at an idea from many different angles." This student learned that some mathematical problems, such as conjectures, are not always a one-way road.

Another common trend amongst the students was that in the reflections, a number of the students took a step back and reflected from a broader point-of-view, and said they developed a deeper understanding of the nature of mathematics. Some students stated that they get the "bigger picture," or that they understand what happens "behind the scenes of mathematics." Of the 25 students, 10 of them had reflections in this vein. A few of those students were also ones that had a narrow idea of what mathematics was about in the beginning of the semester. These students originally had a concrete perception of mathematics being about calculations. For instance, a student wrote, "to me, mathematics is mainly about obtaining problem-solving skills through various mathematical problems," and, "mathematics is all about using formulas to solve problems." Working on an unsolved conjecture may have had an effect on their idea of mathematics and showed them that there is more to mathematics and it is not always about performing algebra with numbers. The same student, in their last reflection, wrote that "mathematics is all about solving the world's greatest mysteries."

The students' approaches and their perceptions of mathematics were not the only things they reflected about. Some of the students described a variety of emotions in their reflections. Eight of the 25 students expressed enjoyment in regards to the notebook. One student expressed how "math is getting more and more creative." Another student even stated that they were reassured and glad to be a mathematics major after completing the notebook, writing "I made the right choice in majoring in math since I love proofs so much."

Challenges

One of the most commonly noted student challenges was the struggle with finding where to begin proving the conjectures. Some of the students seemed overwhelmed at the prospect of exploring an unsolved conjecture that many mathematicians have tried and failed to solve. Some even expressed misconceptions, believing that they were being asked to prove something "impossible." Others were confused how they were supposed to go about proving something "unprovable." One student wrote, "I think the main challenge is just the fact that it's a conjecture. I could not find a way to write any proof because I did not understand or recognize the pattern behind it. I'm not sure if I'm doing the assignment correctly." Other students had a defeatist attitude, not believing that they would have anything positive to contribute: "Working on the conjecture was more irritating than exciting because I can't prove it. It took away all the satisfaction because it is a famously unproven conjecture and I couldn't solve it." Another student wrote, "Although it was intriguing, I gave up preemptively because I knew that I would not do anything that would help come to any conclusion." Although these students had trouble making progress, most were able to engage with the conjectures in some way. In subsequent semesters when implementing this project, the first author has incorporated more in-class discussion time for the conjectures. This has allowed the students more opportunities to get ideas from their peers for how to approach the conjectures.

Conclusions

Overall, we are encouraged that exploration of the unsolved conjectures was a productive experience for most students, and helped them to understand the role of proof and the discipline of mathematics in novel ways. We found that students had the opportunity to learn about what Hersh (1991) referred to as "the back of mathematics"—the messy informal work involved in proving conjectures. We believe that some students experienced and understood proof's role as verification, explanation, discovery, and systematization. Of the five roles of proof, explanation was most reflected on, with 9 students alluding to this role in their reflections (beyond the 4 that alluded to it at the beginning of the semester). We believe that working on unsolved conjectures provided students a special opportunity to understand this role of proof, as they were forced to come to the terms with the fact that even when they believe a conjecture to be true, it is not always easy to understand why it is true. We note that of the five roles of proof, communication was the only one we could not find addressed in the student reflections.[2] Other researchers have found that small-group work in inquiry-based classrooms brings to the forefront the communication role for students (Bleiler-Baxter & Pair, 2017; Cilli-Turner, 2017). Perhaps communication was not discussed by the students in our study as only one classroom day was devoted to student discussions of their work on the conjectures. We believe allotting more time for classroom discussion of conjectures may help students better understand the communication role of proof.

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[2] One student did address the communication role of proof; but she had previously participated in a summer research project exploring an unsolved conjecture, and thus had additional opportunities to learn about the importance of communication in mathematics.

PRECALCULUS, CALCULUS, OR HIGHER MATHEMATICS:

BRIEF RESEARCH REPORTS

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TOWARDS A DIDACTIC DISTINCTION BETWEEN CALCULUS AND ANALYSIS. THE CASE OF THE NOTION OF VARIABLE

HACIA UNA DISTINCIÓN DIDÁCTICA ENTRE EL CÁLCULO Y EL ANÁLISIS. EL CASO DE LA NOCIÓN DE VARIABLE

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This paper presents the results of a large documentary study, the purpose of which is to promote a research and teaching program for the didactic reconceptualization of Calculus, through detailed argumentation that makes visible the problems surrounding the profound influence of the organization of Mathematical Analysis on the teaching of Calculus, which makes it difficult for students to understand and use the central ideas of Calculus. The study was conducted using theoretical notions of the onto-semiotic approach (OSA) to mathematical knowledge and instruction. An inquiry on the objects of study of each mathematical discipline will be presented, situating the different problem-situations that each one addresses and the procedures, languages, properties, arguments, and concepts that differentiate them. Finally, these differences are illustrated in relation to the notion of variable.

Keywords: Calculus, Curriculum, Instructional Vision.

Introduction and problem

At the center of Calculus teaching lies a problem deeply close to the historical development of this discipline and to Mathematics itself. Authors such as Moreno-Armella (2014) have described that problem as a tension between intuition and formalism, related to the tendency to present the intuitive ideas of Calculus from the perspective of Mathematical Analysis (Ímaz & Moreno, 2010). Calculus textbooks, that materialize this teaching approach, usually organize mathematical content around the notions of function and limit, considered the basis for studying the derivative and the integral of a function. The same textbooks emphasize mathematical practices such as defining, arguing, and demonstrating, and not the foundational intuitive ideas of Calculus about variation and accumulation.

This work is part of the research of Jiménez, Grijalva, Milner, Dávila-Araiza and Romero, (in press), which continues the line drawn by the study of Ímaz and Moreno (2010) to make a clear distinction, with didactic purposes, between Calculus and Mathematical Analysis. The study of Jiménez et al. it is a documentary research. It presents, in a critical way, historical and epistemological facts and arguments found in books and research articles that help explain the current state of Calculus teaching, drawing a path for its necessary transformation. The current didactic route, implicit in the textbooks, constitutes a path full of difficulties for students, associated with the formalized notion of limit (Cory & Garofalo, 2011; Roh, 2008 & Nagle, 2013) and other mathematical notions, such as real numbers and functions (Artigue, 1998). These notions are presented in classrooms from a perspective that seeks to establish a basis for the study of Mathematical Analysis and not the understanding of the key aspects of Calculus.

As one of the results of this documentary research, this paper will present, without pretenses of exhaustiveness, a distinction of the contents of Calculus and Mathematical Analysis, which serves to show the strong influence of Analysis on the teaching of Calculus. Later, with respect to the notion of variable, a distinction will also be made from the perspective of Calculus and Analysis. These distinctions will be made using theoretical tools of the onto-semiotic approach (OSA) to mathematical knowledge and instruction (Godino, Batanero & Font, 2007), which allows a detailed analysis of mathematical content and the establishment of categories that facilitate their contrast.

Theoretical Elements

In OSA a pragmatic position is assumed to study the mathematical objects that intervene and emerge when solving problems, putting the focus on the notion of *mathematical practice* to refer to any practical or discursive performance that is carried out when solving problems or communicating the obtained results. Rather than hinting at isolated practices, OSA is concerned with the *systems of practices* that are carried out to deal with problem situations. With this idea, *mathematical objects* are characterized as emerging from systems of practice, allowing the following six types of *primary* mathematical objects to be distinguished, some of which are ostensive in nature and others non-ostensive: *Problem-situations* (more or less open problems, exercises, extra or intra mathematical applications, etc.); *language* (specific mathematical terms, algebraic expressions, number tables, graphs, diagrams, gestures, etc.); *procedures* (techniques, algorithms, etc., undertaken or executed by the subject faced with mathematical tasks); *properties and propositions* (attributes of the mentioned objects); *arguments* (which are used to validate and explain the propositions) and *concepts* (mathematical objects recognized as part of the mathematical structure, characterized by their essential properties, which allow them to be distinguished from others and are usually expressed through descriptions or definitions).

Contrast of the primary mathematical objects of Calculus and Analysis

Below is an analysis of four of the six types of primary mathematical objects, corresponding to Calculus and Mathematical Analysis, from which it is possible to show that the current approach to teaching Calculus does not fully consider the initial ideas of this discipline, but rather presents a version more in line with the purposes of Mathematical Analysis.

Table 1. Contrast of primary mathematical objects of Calculus and Analysis

Calculus	Analysis
Concepts: Variable magnitude, variation and covariation, instantaneous rate of change, infinitely small magnitude, differential, accumulation, evanescent quantity, infinite sum, derivative, integral, function.	*Concepts:* Real number, function (increasing, decreasing, bijective, continuous, differentiable, bounded, integrable), derivative of a function, definite integral of a function in an interval, limit of a function, succession, convergence.
Problem-situations (usually linked to physical or geometric contexts): To determine how much there is of a magnitude at all times, knowing the rate of change of such magnitude at all times, and inversely determine its rate of change at all times knowing how much there is of that magnitude in any given moment. To determine the quadrature of a figure formed by a curve and the slope of the tangent line to that curve at each point of it.	*Problem-situations*: To define the mentioned concepts and establish their properties. From their definitions, to determine when a relationship is a function, when a function is injective, bijective, increasing, decreasing, continuous, differentiable or integrable. To determine the limit of a function. To establish when the limit of a function exists, when a function is continuous, if a discontinuity is removable, when a function is differentiable or integrable.
Properties (they tend to be implicitly assumed): variable magnitudes vary continuously. Differential magnitudes are non-Archimedean in nature. Differential quantities depend on the phenomenon.	*Properties* (presented in isolation from their contexts of extra-mathematical use): Theorems of the mean value and the intermediate value. Properties of the limits of functions and differentiable functions. Conditions of

Language: Combination of the natural language with specific languages from the application areas. Algebraic, numeric and geometric representations. Algebraic treatment of differential magnitudes, such as $\boldsymbol{dx, dy, dz, dt}$, and of infinite sums represented as $\int \boldsymbol{x^2 dx}, \int \boldsymbol{senx\ dx}$, among others.	integrability of a function, fundamental theorem of calculus, etc. *Language*: Predominant use of analytical language of a formal nature and little use of numerical and graphic representations. The center of the language of analysis is that associated to functions and, linked to it, the criteria of convergence and continuity, representing the proximity between representative values by means of the absolute value of the difference of two magnitudes of interest.

The notion of variable from the Calculus perspective

The study of variable begins in high school, where two main meanings are promoted, one associated with the study and resolution of equations of two or more variables, and the other as *a generalized number* associated with the problems of numerical and figurative patterns. Subsequently, in the Calculus university courses, the concept of variable is not usually discussed (Biehler & Kempen, 2013). Only when defining function is it mentioned that *x* is an element in a set called *domain*, while *y* is an element in the set called *range*. Consequently, the variable is considered as “something” that takes all the values in a certain set of numbers or “something” that represents all the elements of a given set. This concept of variable corresponds to Set theory, a typical approach from Mathematical Analysis (see Apostol, 1979, p. 40), more importantly, a *decontextualized and static meaning*.

Favoring an static and decontextualized meaning of the variable generates difficulties for students in the modeling of physical phenomena with the tools of Calculus (López-Gay, Martínez Sáez & Martínez-Torregrosa, 2015), since for the students a variable is a letter that represents replaceable constants, that is, the variables *do not vary* in the students' thinking, as documented by Jacobs and Trigueros (cited in Thompson, Byerley & Hatfield, 2013). Thompson, Byerley and Hatfield (2013) affirm that to develop a variational meaning of the variable characteristic of Calculus, it is not enough to think of the variation as the simple substitution of one number for another; it is important to understand that this change in value is not arbitrary, but it occurs under a certain progression or sequence that is usually dependent on time. This sheds light on an important distinction: time is a central notion in Calculus; however, in Analysis it is not, as Bolzano expressed: “the concept of time, and even more that of motion, is as external to general mathematics as that of space” (quoted in Bottazzini, 1986, p. 98).

In Calculus, the variable is a dynamic notion that is intimately linked to time and physical contexts, since it emerges from the study of variable magnitudes and the relationships that can be established between them when trying to understand, and mathematize processes and variation phenomena that are present in the physical and social environment, in which a measurable changing property is manifested. In Calculus, the verb *to vary* refers to a change in progress (Thompson, Ashbrook & Milner, 2016), whether it is actually happening, or one can imagine that it is. In this sense, a variable quantity can be described as a *quantifiable property* of an object, process, or phenomenon of variation. In other words, “the variable magnitude that is measured or calculated takes progressively different values at different times, as the phenomenon or process in which it intervenes develops” (Jiménez et al., In press). It is important to highlight that the quantifiable quality not only takes each and every one of the numerical values in a set, but it does so sequentially, as time passes.

Consequently, the variable magnitudes in the Calculus, due to all the characteristics that they possess, particularly those described above, do not have the same meaning as the variable from the perspective of Mathematical Analysis, where the variables have no link to physical reality, nor with time or motion, as Bolzano explained it. In Mathematical analysis the variables are timeless, dimensionless, static (they do not involve motion). The way in which the analysis variable "takes" the values of a set needs to be completely arbitrary, it does not obey a temporal sequence.

Conclusions

The analysis of primary mathematical objects allowed us to identify that the systems of mathematical practices of Calculus and Mathematical Analysis are essentially different. Broadly speaking, the mathematical practices of Calculus are oriented to the study of the phenomena of variation by establishing relationships between variable magnitudes, while the mathematical practices of Analysis focus on the definition and study of functions, derivatives, integrals, and its properties, through the notions of real number and limit.

Regarding the notion of variable, it is important to highlight that Calculus students hardly have an approach to it from a variational perspective, which would allow them to develop the variational thinking necessary for understanding the fundamental ideas of this discipline that will become mathematical tools necessary for the prediction and control of change processes; the field of application of the Calculus.

The presented arguments aim to outline, for didactic purposes, some essential differences between Calculus and Analysis, taking as a particular case the notion of variable, to shed light on how Calculus teaching neglects aspects that are central to the understanding of its foundational ideas, by privileging a formal treatment of the mathematical content more attached to the perspective of Mathematical Analysis. These reflections remind us that the discussion on how to reorient the teaching of Calculus, as well as didactic research, towards understanding the central ideas of Calculus that would allow students to be efficient users of the mathematics of change is still pending.

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HACIA UNA DISTINCIÓN DIDÁCTICA ENTRE EL CÁLCULO Y EL ANÁLISIS. EL CASO DE LA NOCIÓN DE VARIABLE

TOWARDS A DIDACTIC DISTINCTION BETWEEN CALCULUS AND ANALYSIS. THE CASE OF THE NOTION OF VARIABLE

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Se presentan resultados de un estudio amplio, de carácter documental, cuya finalidad es el impulso a un programa de investigación y docencia para la reconceptualización didáctica del cálculo, a través de una argumentación detallada que visibiliza la problemática en torno a la influencia profunda de la organización del Análisis Matemático sobre la enseñanza del Cálculo, que dificulta a los estudiantes la comprensión y uso de las ideas centrales del Cálculo. El estudio se realizó usando nociones teóricas del Enfoque Ontosemiótico del conocimiento y la instrucción matemáticos (EOS). Se presentará una indagación sobre los objetos de estudio de cada disciplina matemática señalada, ubicando las diferentes situaciones problema que aborda cada una de ellas y los procedimientos, lenguajes, propiedades, argumentos y conceptos que las diferencian. Por último, se ilustran estas diferencias con relación a la noción de variable.

Palabras clave: Cálculo, currículo, visión de enseñanza.

Introducción

En el centro de la enseñanza del Cálculo yace una problemática profundamente ligada al desarrollo histórico de esta disciplina y de las matemáticas mismas, que autores como Moreno-Armella (2014) han relacionado con una tensión entre lo intuitivo y lo formal, y una tendencia a presentar el Cálculo desde la perspectiva del Análisis Matemático (Ímaz & Moreno, 2010). En los textos de Cálculo, que materializan el enfoque de su enseñanza, se suele organizar el contenido en torno a las nociones de función y límite, para llegar al estudio de la derivada y la integral, enfatizando prácticas matemáticas como fundamentar, definir y demostrar, y no en torno a las ideas intuitivas fundacionales del Cálculo sobre variación y acumulación.

Este trabajo se enmarca en la investigación de Jiménez, Grijalva, Milner, Dávila-Araiza y Romero, (en prensa), que da continuidad a la línea trazada por la investigación de Ímaz y Moreno (2010) para realizar una distinción clara, con fines didácticos, entre el Cálculo y el Análisis Matemático como disciplinas en la enseñanza. La investigación de Jiménez et al. es de tipo documental; recopila hechos y argumentos de corte histórico y epistemológico en libros y artículos de investigación que permiten explicar el estado actual de la enseñanza del Cálculo y esbozar una línea para su necesaria transformación; considerando que la ruta didáctica actual, implícita en los libros de texto, constituye para los estudiantes un camino plagado de dificultades asociadas a la noción formalizada de límite (Cory & Garofalo, 2011; Roh, 2008 y Nagle, 2013) y otras nociones matemáticas, como los números reales y la función (Artigue, 1998), que son presentados desde una perspectiva que busca a sentar bases para el estudio del Análisis Matemático y no la comprensión de las nociones centrales del Cálculo.

Como uno de los resultados de esta investigación documental, en este escrito se presentará, primeramente, de manera general (y sin pretensiones de exhaustividad) una distinción de contenidos matemáticos del Cálculo y del Análisis Matemático, que permiten mostrar la fuerte influencia del Análisis sobre la enseñanza del Cálculo. Posteriormente, con respecto a la noción de variable, también se realizará una distinción desde la perspectiva del Cálculo y del Análisis. Estas distinciones se realizarán mediante herramientas teóricas del Enfoque Ontosemiótico, EOS, (Godino, Batanero & Font, 2007), las cuales permiten analizar detalladamente contenidos matemáticos y establecer categorías que faciliten su contraste.

Elementos teóricos

En el EOS se asume una posición pragmática para estudiar los objetos matemáticos que intervienen y emergen al resolver problemas, poniendo el enfoque en la noción de *práctica matemática* para referirse a cualquier actuación práctica o discursiva que se realiza al resolver problemas o comunicar los resultados obtenidos. Pero, más que hacer alusión a prácticas aisladas, el EOS se preocupa por los sistemas de prácticas que se realizan para enfrentar las situaciones problema. Con esta idea, se caracteriza a los objetos matemáticos como los emergentes de los sistemas de prácticas, permitiendo distinguir los seis siguientes tipos de *objetos matemáticos primarios*, algunos de los cuales son de naturaleza ostensiva y otros son no ostensivos: *Situaciones problema* (problemas más o menos abiertos, ejercicios, aplicaciones extramatemáticas o intramatemáticas, etc.); *lenguaje* (términos específicos de matemáticas, expresiones algebraicas, tablas numéricas, gráficas, diagramas, gestos, etc.); *procedimientos* (técnicas, algoritmos, etc., emprendidos o ejecutados por el sujeto ante las tareas matemáticas); *propiedades* (atributos de los objetos mencionados); *argumentos* (que se usan para validar y explicar las proposiciones) y *conceptos* (objetos matemáticos reconocidos como parte de la estructura matemática, caracterizados por sus propiedades esenciales, las que permiten distinguirlos de otros y suelen expresarse por medio de descripciones o definiciones).

Contraste de objetos matemáticos primaros del Cálculo y del Análisis

A continuación, se presenta de manera general un análisis de cuatro de los seis tipos de objetos matemáticos primarios, sin pretender ser exhaustivos, correspondientes al Cálculo y al Análisis Matemático, a partir del cual es posible poner de manifiesto que el enfoque actual de la enseñanza del Cálculo no considera plenamente las ideas iniciales de esta disciplina, sino que presenta una versión del Cálculo mas acorde a los propósitos del Análisis Matemático.

Tabla 1. Contrastación de objetos matemáticos primarios del Cálculo y del Análisis

Cálculo	Análisis
Conceptos: Magnitud variable, (co)variación, razón instantánea de cambio, magnitud infinitamente pequeña, diferencial, acumulación, cantidad evanescente, suma infinita, derivada, integral, función.	*Conceptos:*Número real, función (creciente, decreciente, biyectiva, continua, diferenciable, acotada, integrable), derivada de una función, integral de una función en un intervalo, límite de una función, sucesión, convergencia, entre otras.
Situaciones problema (normalmente vinculadas a contextos físicos o geométricos): Determinar	*Situaciones problema*: Definir los conceptos mencionados y establecer propiedades de estos. A

cuánto hay de una magnitud en todo momento, conociendo la razón de cambio de esa magnitud en todo momento, e inversamente determinar su razón de cambio en todo momento conociendo en todo momento cuánto hay de esa magnitud. Determinar la cuadratura de una figura formada por una curva y, determinar la pendiente de la recta tangente a dicha curva en cada punto de ella.	partir de dichas definiciones, determinar cuándo una relación es función, cuándo una función es inyectiva, suprayectiva, biyectiva, creciente, decreciente, continua, diferenciable o integrable. Determinar el límite de una función. Establecer cuándo existe el límite de una función, cuándo es continua una función, si es removible una discontinuidad, cuándo una función es diferenciable o integrable.
Propiedades (tienden a ser asumidas de forma implícita): Por ejemplo, las magnitudes variables varían de manera continua, las magnitudes diferenciales tienen una naturaleza no Arquimediana, las cantidades diferenciales son propias de cada fenómeno.	*Propiedades* (se presentan aisladas de sus contextos de uso extramatemático): Teoremas del valor medio y del valor intermedio. Propiedades de los límites de funciones y de las funciones diferenciables. Condiciones de integrabilidad de una función, teorema fundamental del cálculo, etc..
Lenguaje: combinación de la lengua materna con lenguajes específicos de las áreas de aplicación. Expresiones numéricas geométricas y algebraicas. que representan magnitudes diferenciales como $\boldsymbol{dx}, \boldsymbol{dy}, \boldsymbol{dz}$, d$\boldsymbol{t}$ y de las sumas finitas, representadas por expresiones como $\int \boldsymbol{x^2\,dx}$, $\int \boldsymbol{senx\,dx}$ y otras.	*Lenguaje*: preponderantemente analítico, de carácter formal y poco empleo de las representaciones numéricas y gráficas. El centro del lenguaje del análisis es el de función y, ligado a ésta, los criterios de convergencia y de continuidad, representando la proximidad entre valores representativos mediante el valor absoluto de la diferencia de dos magnitudes de interés.

La noción de variable desde la perspectiva del Cálculo

El estudio de la variable inicia en la escuela secundaria, donde se promueven dos significados principales, uno asociado al estudio y resolución de ecuaciones de dos o más variables, y otro como número generalizado asociado a los problemas de patrones numéricos y figurales. Posteriormente, en los cursos universitarios de Cálculo no se suele discutir el concepto de variable (Biehler & Kempen, 2013). Únicamente, al definir la función, se menciona que *x* es un elemento en un conjunto llamado dominio, mientras que *y* es un elemento del conjunto llamado rango. Como consecuencia, la variable es considerada como "algo" que toma todos los valores en cierto conjunto de números o "algo" que representa a los elementos de un conjunto. Este concepto de variable corresponde a un significado conjuntista, propio del Análisis Matemático (ver Apostol, 1979, p. 40), un significado descontextualizado y estático.

Favorecer un significado estático y descontextualizado de la variable produce dificultades en los estudiantes para la modelación de fenómenos físicos con las herramientas del Cálculo (López-Gay, Martínez Sáez & Martínez-Torregrosa, 2015), pues para los estudiantes una variable es una letra que representa constantes reemplazables, es decir, las variables no varían en el pensamiento de los estudiantes, como lo documentaron Jacobs y Trigueros (citados en Thompson, Byerley & Hatfield, 2013). Thompson, Byerley y Hatfield (2013) afirman que para desarrollar un significado variacional de la variable propio del cálculo no es suficiente pensar en la variación como el cambio de un número por otro; es importante comprender que ese cambio de valor no es arbitrario, sino que ocurre bajo cierta progresión o secuencia normalmente temporal. Esto arroja luz sobre un aspecto importante: el tiempo es una noción central en el Cálculo; sin embargo, en el Análisis no lo es, como lo expresó Bolzano: "el concepto de tiempo, y más aún aquel de movimiento, es tan externo a las matemáticas generales como aquel de espacio" (citado en Bottazzini, 1986, p. 98).

En el Cálculo, la variable es una noción dinámica que está íntimamente ligada al tiempo y a los contextos físicos, pues ésta emerge del estudio de las magnitudes variables y de las relaciones que se pueden establecer entre ellas al tratar de comprender, y matematizar, procesos y fenómenos de variación que están presentes en el entorno físico y social, en los cuales se manifiesta una propiedad cambiante que puede ser medible. En el Cálculo, el verbo variar se refiere a un cambio en progreso (Thompson, Ashbrook & Milner, 2016), ya sea efectivamente está sucediendo o bien que se puede imaginar que lo hace. En este sentido, una magnitud variable se puede describir como una propiedad de un objeto, proceso o fenómeno de variación, la cual es cuantificable. Es decir, "la magnitud variable que se mide o calcula toma progresivamente distintos valores en distintos momentos, a medida que va desarrollándose el fenómeno o proceso en que ella interviene" (Jiménez et al., en prensa). Es importante resaltar que la cualidad cuantificable no solamente toma todos y cada uno de los valores numéricos en un conjunto, sino que lo hace de manera secuencial, a la par que transcurre el tiempo.

En consecuencia, las magnitudes variables en el Cálculo, por todas las características que poseen y que se describieron líneas arriba, no tienen el mismo significado que la variable desde la perspectiva propia del Análisis Matemático, donde las variables no tienen vínculo alguno con la realidad física ni con el tiempo o el movimiento, como lo expresó Bolzano. En el análisis Matemático las variables son atemporales, adimensionales, estáticas (no involucran el movimiento). La manera como la variable del análisis "toma" los valores de un conjunto es completamente arbitraria, no obedece necesariamente una secuencia temporal.

Conclusiones

El análisis de objetos matemáticos primarios permitió identificar que los sistemas de prácticas matemáticas del Cálculo y del Análisis Matemático son esencialmente distintos. A grandes rasgos, las prácticas matemáticas del Cálculo se orientan al estudio de los fenómenos de variación mediante el establecimiento de relaciones entre magnitudes variables, mientras que las prácticas matemáticas del Análisis se centran en la definición y estudio de la función, la derivada, la integral y sus propiedades, a través de las nociones de número real y límite.

Con respecto a la noción de variable, es importante resaltar que difícilmente los estudiantes de Cálculo tienen un acercamiento a ésta desde una perspectiva variacional, que les permitiría desarrollar un pensamiento variacional necesario para la comprensión de las ideas fundamentales de esta disciplina que son en herramientas matemáticas necesarias para la predicción y control de procesos de cambio; el campo de aplicación del Cálculo.

Los argumentos presentados brevemente pretenden esbozar, con propósitos didácticos, algunas diferencias esenciales entre el Cálculo y el Análisis, tomando como caso particular la noción de variable, para arrojar luz sobre cómo la enseñanza del Cálculo desatiende aspectos que son centrales para la comprensión de sus ideas fundacionales, al privilegiar un tratamiento formal del contenido matemático más apegado a la perspectiva del Análisis Matemático. Estas reflexiones nos recuerdan que sigue pendiente la discusión sobre cómo reorientar la enseñanza del Cálculo, así como la investigación didáctica, hacia la comprensión de las ideas centrales del Cálculo que permita a los estudiantes ser usuarios eficientes de las matemáticas del cambio.

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CONSTRUCTING RATES OF CHANGE THROUGH A UNITS COORDINATING LENS: THE STORY OF RICK

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In this research report, we describe the results of a paired-student constructivist teaching experiment with introductory calculus students focused on supporting their understanding of the derivative as rate of change. We focus on one student, Rick. We connect analyses of Rick's ways of assimilating and operating with numerical units with analyses of ways of conceptualizing rates. The results are conjectures about the relationships between levels of units students coordinate and their ways of quantifying rates.

Keywords: Cognition, Number Concepts and Operations, Rational Numbers, University Mathematics

Background

This report builds a connection between Johnson's (2015) work investigating the quantitative operations involved in constructing rates and research investigating the constraints and affordances of student's units coordinating activity as they develop understandings of calculus concepts (Boyce, Byerley, Darling, Grabhorn, & Tyburski, 2019; Boyce, Grabhorn, & Byerley, 2020). The goal of the current study is to identify connections between calculus students' units coordination and their understanding of rate of change.

Units Coordinating

Units coordination can be thought of as students' mental activity building and maintaining relationships of nested *levels* of units (Norton et al., 2015; Steffe, 1992). Some students bring a three-level units coordinating structure to bear when first encountering a task, what we call *assimilating* with three levels of units. Such students would be able to quickly reason through the Bar Task below (Figure 1) by recognizing that the orange bar is equivalent to nine ¼'s of a purple bar, thus 9/4 of a purple bar fits into the orange bar. Students that assimilate with two levels of units may construct an ephemeral third level of units in the midst of reasoning (what we call coordinating three levels of units *in activity*) by coordinating across two two-level units coordinating structures. Such activity requires perceptual reflecting on the outcomes of actions on physical or mental representations, often resulting in conflating or dropping units. For example, a student who assimilates with two levels of units may state 2 1/9 as an answer to the Bar Task by claiming that two full purple bars and one green bar (1/9 of an orange bar) fit into the orange bar.

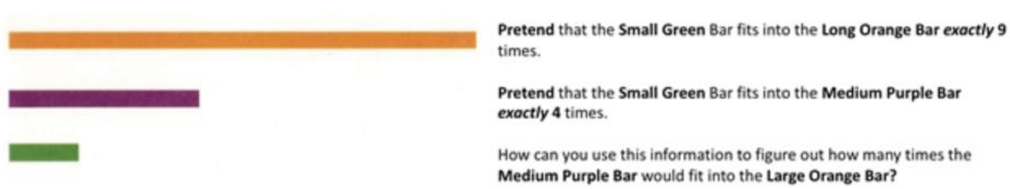

Figure 1: Units Coordinating Bar Task (Norton et al., 2015)

Rate of Change as a Ratio

Johnson (2015) investigated the affordances and constraints of secondary students' quantification of ratios in regard to their quantification of rate. The resulting Change in Covarying Quantities Framework distinguishes between quantitative operations involved in students' quantification of rate:

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

comparison refers to a quantification of rate as associations of changes of quantities (i.e., 12 miles per hour means associating a distance of 12 mile with an elapsed time of 1 hour), while *coordination* refers to quantification of rate as involving at least one continuously changing quantity (i.e. 12 miles per hour means as time increases, the distance traveled is 12 times as large). Johnson argued that "students' quantification of rate could help to explicate differences in students' conceptions of rate" (p. 86-87) and that the nature of how students might develop either operation of *comparison* and *coordination* is unanswered. We hypothesize that students' units coordinating activity may be helpful for understanding the nature of such operations.

Methods and Results

We conducted a paired-student teaching experiment (Steffe & Thompson, 2000) in summer 2019 at a large public university in the U.S. with the goal of producing models of introductory calculus students' developing understandings of rate of change. Our results and analysis will focus on one participant, Rick. Rick participated in five weekly one-hour teaching episodes concurrent with his enrollment in a differential calculus course. The first author served as teacher-researcher for each episode while the second author served as a witness. Each episode was video recorded and all written work was collected and scanned for analysis. Analysis methods included both ongoing (between session) and retrospective modeling of Rick's ways of coordinating units and Rick's ways of reasoning about rates of change (Steffe & Thompson, 2000). Rick was assessed as assimilating with two levels of units at the beginning of the teaching experiment. The following analysis details our attempt to support Rick in quantifying a rate via the *coordination* operation of Johnson's (2015) Change in Covarying Quantities Framework.

12 Meters In 3 Seconds

During the third teaching episode we focused on supporting Rick in quantifying a rate with the *coordination* operation. Rick's conception of a rate as the amount of change in a dependent quantity for a unit increase in an independent quantity was persistent. Rick was presented with the task displayed in Figure 2.

Compare and contrast the following statements.

A. I travel 12 meters in 3 seconds.
B. I travel at a constant rate of 12 meters per 3 seconds.
C. I travel at a constant rate of 12/3 meters per second.
D. It takes me ¼ seconds to travel 1 meter.

Figure 2: Comparing Rates Task

Rick first noted that statements B and C are similar because they were "a constant rate… over a certain interval of time", but that those statements are different from statements A and D because the latter pair did not reference a constant rate. When pushed to describe other similarities or differences, Rick claimed that "throughout the board… each second they would have traveled 4 meters". Even though Rick attended to the absence or inclusion of the phrase *constant rate*, he still compared statements by considering the amount of distance covered in 1 second (as if each statement referred to a constant rate of change between distance and time). Rick then stared at option A and claimed "actually, I don't know that". He then explained that each statement described traveling 12 meters in 3 seconds. The teacher-researcher then asked Rick to give an example where statement A is true, but does not describe traveling 4 meters in 1 second.

> Rick: Potentially within the first second maybe you're, uh… stopped the entire time. And then, so zero to one [seconds] you travel zero [meters]. Then one to two seconds you travel six [meters]. Then two to three [seconds] you travel another six [meters]. So, in one second it's not guaranteed to be four [meters] in that particular situation.

Rick was able to give several additional examples by considering individual changes in distance across three successive elapsed seconds, where the sum of changes in distance was 12 meters. Additionally, some of his examples included traveling four meters in one of the elapsed seconds but not all (e.g., 0 meters in the first second, 8 meters in the next second, 4 meters in the final second). According to Rick, "it could be any mixture of numbers leading to twelve".

We interpret Rick's response as indicating that, to him, a constant rate of 4 meters per second means that distance must change by four meters for *every* second he can consider throughout the trip. Additionally, Rick interprets any constant rate described with a non-unit change in independent quantity (in this case, time) by finding the associated change in dependent quantity per increase of one unit of the independent quantity that would maintain the originally stated ratio.

This explains why Rick singled out choice A as different than choices B and C but not D; For choice D, distance must change by four meters if we consider iterating ¼ second four times to let one second elapse. Even though Rick had an awareness that amounts of changes in distance can vary as time elapses for both statements A and D, Rick's image of that variation necessitated considering unit increases in time. Rick did not consider the variation of distance for individual ¼ second intervals of time, and thus statement D is consistent with statements B and C (in these three cases, for Rick, distance *must* increase by four meters for any increase of one second).

12 Meters In 0.8 Seconds

The previous task revealed that Rick could reason about rate of change by considering amounts of change in a dependent quantity constrained by an associated increase of the independent quantity by one unit. The following excerpt describes Rick attempt to reason about a rate with a non-unit change in the independent quantity. Specifically, we ask Rick to compare a statement similar to the previous task with a statement about instantaneous rate of change.

> Teacher-Researcher: Jim travels 12 meters in 0.8 seconds. Is it possible Jim traveled 12 meters per second at any point during his trip?
>
> Rick: Potentially… I don't want to say… Because I was thinking potentially Jim could go… travel 15 meters per second but stop at, you know, 0.8 seconds. And then Jim would be traveling… no… If you travel 15 [meters]… if Jim would stop at 0.8 [seconds] exactly after having traveled that, then 12 [meters] is eighty percent of 15 [meters].

Rick's activity is focused on relating a change in distance of 12 meters to a unit rate that describes traveling 12 meters in 0.8 seconds. One interpretation of his response is that Rick interprets "12 meters per second" as traveling 12 meters *within* one second, and thus it is possible to travel 12 meters per second during the trip.

Rick constructs (at least) three rates as he attempts to solve this task: 12 meters per 0.8 seconds (Figure 3a), 15 meters per second (Figure 3b), and 12 meters per second (Figure 3c). In each case, Rick can reason about a rate as a comparison of changes in distance and changes in time and appears to prefer reasoning about such rates over a unit interval (one second) of time. This may be due to Rick having not interiorized a conceptual structure for non-unit rates, thus necessitating activity to construct a unit rate with which to reason.

Rick can compare two speeds by constructing unit rates for speed and comparing the two changes in distance associated with a common unit increase in time. In doing so, Rick does not attend to time as a quantity that is necessary for his goal of comparing two speeds. This may explain his initial response that Jim could potentially travel 12 meters per second by traveling "15 meters per second but stop at, you know, point-eight seconds". Rick's suggestion links a unit rate with a change in distance of 12 meters.

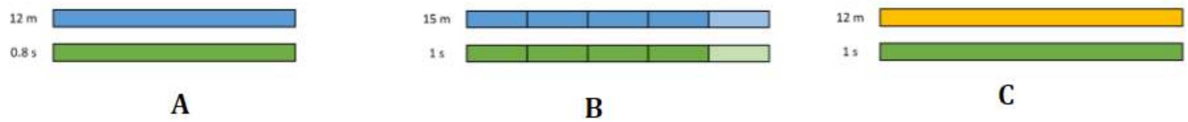

Figure 3: Author illustrations of rates that Rick constructs and compares

Ultimately Rick decided that it was not possible that Jim traveled 12 meters per second at any point during his trip because, "if we're going that specific interval [one second], that'd be fifteen meters per second. Not twelve".

Discussion

The goal of our research was to investigate how Rick, who we assessed as assimilating with two levels of units, reasoned about rate of change. We have focused analysis on two particular tasks to exemplify the powerful ways that Rick was able to reason about unit rates and able to coordinate three levels of units in activity involving known quantities. Still, throughout the teaching experiment, Rick did not exhibit behavior indicating that he constructed rates by engaging in the quantitative operation of *coordination*. Instead Rick was persistent in constructing rates through the *comparison* operation.

Rick is not an anomaly in that university students that assimilate with fewer than three levels of units exist in introductory calculus courses and appear to be at a higher risk of not finding success in such courses (Boyce, Grabhorn, & Byerley, 2020; Byerley 2019). Specific to our report, Johnson (2015) conjectured that sole reliance on the *comparison* operation could explain students' struggles with rates. This report builds a connection between Johnson's (2015) work investigating the quantitative operations involved in constructing rates and our previous research investigating the constraints and affordances of students' units coordinating activity as they develop understandings of calculus concepts. Further, Johnson left as an open question how students develop the *comparison* and *coordination* operations. We conjecture that engaging in the *coordination* operation (constructing a rate so that at least one of the quantities involved is continuously changing) requires assimilating with three levels of units.

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DETECTION OF DIFFICULTIES IN ENDOWING THE NOTIONS OF CHANGE AND VARIATION WITH MEANING THROUGH ITERATION

DETECCIÓN DE DIFICULTADES EN LA DOTACIÓN DE SIGNIFICADO DE LAS NOCIONES DE CAMBIO Y VARIACIÓN MEDIANTE ITERACIÓN

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Introduction

The concept of variable is of utmost importance since covariation, function, derivative and integral are built on it. However, as has been reported for decades (Rosnik, 1980), most textbooks tend to approach the concept of variable in an ambiguous and confusing way, due to the wide diversity of meanings that implies, causing difficulties in teaching (Díaz and Morales, 2005).

In this research is important to study in a particular way the difficulties that arise when working with a teaching model consisting of activities that involve the treatment of change and variation in situations of diverse contexts. Change is understood as the modification of states of the qualities involved in a phenomenon, and variation as the quantification of change; therefore, when carrying out these quantifications we are in the possibility of introducing the concept of variable.

The research central problem is related to reading / transformation of the textual spaces that involve the notions of change and variation, to give meaning to the arithmetic and algebraic variables.

We intend to characterize the way in which a group of future middle school math teachers give meaning to the notions of variation, function, variable and through experimentation with a Model of Teaching that incorporates work with dynamic phenomena and problem solving.

Based on these purposes, the following research questions were asked. What are the students' actions when solving problems related to variation and use of variables in different fields? How does the level of abstraction of the idea of change evolve, in order to give meaning to the notion of variable? And what are the main difficulties that arise when working iterative processes and solving problems related to variation and the concept of function?

Theoretical framework

The theoretical-methodological framework that guides this work is of the Local Theoretical Models (MTL) (Filloy, 1999), which allows to analyze the phenomena that occur during teaching / learning. Special attention is given to meanings of mathematical concepts thanks to its use more than its meaning in the abstract.

During the development of the teaching-learning processes of a mathematical content, four interrelated elements are presented: the person who teaches, the person who learns, the mathematical content to teach / learn and communication that is established between the participants in the process. Therefore, an MTL that contemplates the characterization of the phenomena and relationships between the aforementioned elements, is made up of the designed Teaching Model, the Model of cognitive processes under which the actions of the students will be characterized, the corresponding formal Model to the mathematical content and the Communication Model which analyzes the exchange of information between the participants (Filloy, Rojano, Puig, 2008).

In this research, the model of formal competence is supported in part by the ideas of Hans Freudenthal, exposed in his book Phenomenology Didactics of Mathematical Structures, specifically in the analysis he makes in the chapters "Algebraic Language" and "Functions" (Freudenthal, 1983). We also consider the formal study of the concept of iteration, which plays a prominent role in this work.

According to Freudenthal (1983), the concept of function is based on the concept of variable and the idea of dependency. The precision in which it is possible to describe the dependency between variables can be different, ranging from the use of notions of order (the more of this, the more of that), to relating one to the other more or less precisely, possibly numerically.

An idea that theoretically supports the notion of iteration, since it is considered to have the potential to contribute in the endowment of meaning to the concepts of variation and variable, highlighting its dynamic quality, from an approach with a low degree of abstraction (Choate, Devaney and Foster, 1999; Peitgen et al., 2004).

The formal model is an element that helps us to observe the students' productions, in addition to these references, we characterize the actions based on the categories: type of approach used by students when solving problematic situations, either qualitative or quantitative. We identify if they are productions of a dynamic or static type and we classify the uses of the variable, either arithmetic or algebraic.

Filloy (1999) states that there are theoretical bases that confirm that a first semantic approach to algebra is more convenient for the meaning endowment of concepts than a merely syntactic approach. In this research project, where the notions of change, variation, variable and function are worked on, it has been chosen to carry out a specific type approach based on a critical historical analysis of the concept of function, with the objectives to detect epistemological obstacles generated during its development (Cuevas and Díaz, 2014), and that students could present it during their learning process (Godino, 2003).

Methodology

The experimentation was carried out with 23 students (21-22 years old), who are currently studying the eighth semester of a Bachelor's Degree in Secondary Education with a specialty in Mathematics at Benemérita and Centennial Normal School of the State of Durango, located in Durango, Mexico.

The development of experimentation is based on the methodology defined in the Local Theoretical Models (Filloy, 1999), which establishes six stages: selection of the study population; design and application of a diagnosis; population classification, which was carried out based on the analysis of the level of syntactic, semantic competence and intuitive use of the population (Rojano, 1985); election of a representative sample of the population; case study through clinical interviews and elaboration of an observation report.

Experimentation results

Next, a representative problematic situation of the teaching model is presented, describing the students' productions during their resolution.

After the future teachers worked on a phenomenon of bank movements with simple interest, a similar problem was posed to them, but with a compound interest, since it is interesting to observe the way they treat phenomena where behaviors other than linear or quadratic are presented; in this case, exponential type. The situation raised was as follows.

> When in a loan the percentage is added to the capital and becomes part of the debt, the interest on this new capital is called compound interest.
>
> 1. Suppose we have a capital of 200 pesos at 20% compound interest for 5 months. Complete the following table to find out how much will be due at the end of each month

One of the main difficulties presented in this situation was detecting the type of variation involved. To achieve this, some students tried to help themselves with the graphical representation, first obtaining a tabular representation of the situation, and then graphing the values as ordered pairs. They proceeded by placing the corresponding points and then drew a line that passes through each of these. Thanks to this, linear variation began to be discarded as corresponding to the phenomenon worked.

Not being able to define the type of variation involved, some students carried out a different process, which consisted of calculating the differences between the consecutive terms and then calculating the differences of the differences, but when calculating the first and second differences they do not obtain constant values, which causes even more confusion. They even decide to make the calculation of the third differences without obtaining a result that makes it possible to make sense of the phenomenon. It can be seen that the method is rescued as a mechanized process.

There was another group of students who also used the method of differences and also did not allow them to determine the type of variation present in the phenomenon, but it was useful to discard the quadratic variation as a possibility, since they obtained the second difference, they did not have constant values.

The Numerical processing of the situation allowed two students to observe that, as the independent variable increases, the differences in the dependent variable become increasingly. It seems clear in their reasoning since they express in a qualitative way the characteristics that the graph would have if it were prolonged.

Conclusions

During the implementation of the teaching model, the students performed different productions in order to make sense of the change in dynamic phenomena, by observing and characterizing them, we can answer the questions posed initially.

Regarding the process of abstraction of notions of change, the following was identified. Generally, at first, a qualitative analysis is carried out in which the characteristics of the phenomenon that is modified is identified, which are expressed in the form of non-measurable qualities. By deepening in the analysis of the experiments, the need to measure these characteristics emerges. The qualities identified above are refined, to make way for the approach of magnitudes (variables) that can be measured.

After determining the magnitudes involved, those that change and those that remain constant are identified and, in turn, a class relationship approximation is carried out that maintains and the direction of said relationship (dependency).

By means of a quantitative analysis, consisting of operating arithmetically with the identified data, specific states of the variables are determined (use of the arithmetic variable) (regularly the initial state and one more).

By organizing the information obtained, it is possible to carry out a dynamic reading of it, which helps to determine the relationships between the variables.

By generalizing the arithmetic processes used, it is possible to produce an algebraic text that represent relationships (use of the algebraic variable), in other words: the statement of a function.

In some cases, the verification of the correspondence of the algebraic text with the set of states of the magnitudes were carried out, performing the substitution of data and observing if indeed the function represents the phenomenon.

Regarding the most frequent actions, there is a tendency to linearity: when the type of treatment that should be given to the phenomenon is not understood, a linear treatment was chosen. Students generate different strategies to define the kind of variation in phenomena, however, the most used resource is production is the production of Cartesian graphic texts. By observing the characteristics of these, they try to determine the type of function worked. When performing geometric text production with a low curvature, the variation tends to be defined as linear. However, when it is evident that the geometric text has a certain curvature, it tends to be defined as quadratic.

And finally, the most frequent difficulty arises when defining the type of behavior of the phenomena, resort to arithmetic text, generally in the form of tables of values. The frequent use of the method of differences was detected to define the variable dependent on the phenomena, Generally mechanically applied; the lack of meaning is evident when working with situations of exponential growth, where students state that they do not understand why they never obtain a constant value, despite making a large number of differences.

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En esta investigación experimental, observamos y caracterizamos el desempeño de un grupo de futuros docentes que cursan la licenciatura en educación secundaria con especialidad en matemáticas (20-21 años). Adoptamos la propuesta de los Modelos Teóricos Locales como marco teórico-metodológico. El objetivo es caracterizar los procesos de producción de significado de las nociones de variación y variable mediante la aplicación de un Modelo de Enseñanza que implica el trabajo con fenómenos de cambio.

Palabras clave: Modelo de enseñanza, cambio, variación exponencial, sucesiones, dificultades

Introducción

El concepto de variable es de suma importancia ya que sobre él se construyen ideas como la covariación, función, derivada e integral. Sin embargo, como se ha reportado desde hace décadas (Rosnik, 1980), la mayoría de los libros de texto suelen abordar el concepto de variable de forma ambigua y confusa, debido a la amplia diversidad de significados que implica, causando dificultades en su enseñanza (Díaz y Morales, 2005).

En esta investigación interesa estudiar, de forma particular, las dificultades que emergen al trabajar con un modelo de enseñanza constituido por actividades que implican el tratamiento del cambio y la variación en situaciones y contextos diversos. Entiéndase el cambio como la modificación de estados de las cualidades implicadas en un fenómeno, y la variación como la cuantificación del cambio; por lo tanto, al llevar a cabo dichas cuantificaciones estamos en la posibilidad de introducir el concepto de variable.

El problema central de la investigación está relacionado con la lectura/transformación de espacios textuales que involucran las nociones de cambio y variación, para dotar de significado a las variables aritméticas y algebraicas.

Nos proponemos caracterizar la forma en que un grupo de futuros docentes de matemáticas de nivel secundaria dan significado a las nociones de variación, variable y función, mediante la experimentación con un Modelo de Enseñanza que incorpora el trabajo con fenómenos dinámicos y la resolución de problemas.

Con base en dichos propósitos se plantearon las siguientes preguntas de investigación. ¿Cuáles son las actuaciones de los estudiantes al resolver problemas relacionados con la variación y el uso de variables en diversos ámbitos?, ¿de qué forma evoluciona el grado de abstracción de la idea de cambio, para lograr dar significado a la noción de variable? y ¿cuáles son las principales dificultades que se presentan al trabajar procesos iterativos y resolver problemas relacionados con la variación y el concepto de función?

Marco teórico

El marco teórico-metodológico que guía este trabajo es el de los Modelos Teóricos Locales (MTL) (Filloy, 1999), el cual permite analizar los fenómenos que se presentan durante los procesos de enseñanza/aprendizaje. Se otorga una especial atención a la dotación de significados de los conceptos matemáticos gracias a su uso, más que por su significado en abstracto.

Durante el desarrollo de los procesos de enseñanza aprendizaje de un contenido matemático se presentan cuatro elementos interrelacionados: el sujeto que enseña, el sujeto que aprende, el contenido matemático a enseñar/aprender y la comunicación que se establece entre los participantes en el proceso. Por lo tanto, un MTL que contempla la caracterización de los fenómenos y relaciones entre los elementos mencionados, se compone por el Modelo de enseñanza diseñado, el Modelo de procesos cognitivos bajo el cual se caracterizarán la actuaciones de los estudiantes, el Modelo formal correspondiente al contenido matemático y el Modelo de comunicación, con base en el cual se analiza el intercambio de información entre los participantes (Filloy, Rojano, Puig, 2008).

En esta investigación, el modelo de competencia formal está sustentado en parte en las ideas de Hans Freudenthal expuestas en su libro *Fenomenología Didáctica de las Estructuras Matemáticas*, de forma específica en el análisis que hace en los capítulos “Lenguaje Algebraico” y “Funciones” (Freudenthal, 1983). También se considera el estudio formal del concepto de iteración, que juega un rol destacado en este trabajo.

Según Freudenthal (1983), el concepto de función se constituye sobre el concepto de variable y la idea de dependencia. La precisión con que es posible describir la dependencia entre variables puede ser distinta, yendo desde el uso de nociones de orden (cuanto más esto, tanto más eso), hasta relacionar una con otra de forma más o menos precisa, posiblemente numérica.

Una idea más que sustenta teóricamente la investigación es la noción de iteración, ya que se considera que tiene potencial para contribuir en la dotación de significado a los conceptos de variación y variable, resaltando su cualidad dinámica, desde una aproximación con un grado de abstracción bajo (Choate, Devaney y Foster, 1999; Peitgen et al., 2004).

El modelo formal es un elemento que nos sirve para observar las producciones de los estudiantes, además de estos referentes, caracterizamos las actuaciones con base en las categorías: tipo de aproximación empleada por los estudiantes al resolver situaciones problemáticas, ya sea de tipo cualitativa o cuantitativa. Identificamos si son producciones de tipo dinámica o estática y clasificamos los usos de la variable, ya sea aritmética o algebraica.

Filloy (1999) afirma que existen bases teóricas que confirman que un primer acercamiento semántico al álgebra es más conveniente para la dotación de significado de los conceptos que un acercamiento meramente sintáctico. En este proyecto de investigación, en donde se trabajan las nociones de cambio, variación, variable y función, se ha optado por llevar a cabo un acercamiento de tipo concreto con base en un análisis histórico crítico del concepto de función, con el objetivo de detectar obstáculos epistemológicos generados durante su desarrollo (Cuevas y Díaz, 2014), y que podrían presentar los estudiantes durante su proceso de aprendizaje (Godino, 2003).

Metodología

La experimentación se realizó con 23 estudiantes (21-22 años), que actualmente cursan el octavo semestre de la Licenciatura en Educación Secundaria con especialidad en Matemáticas en la Benemérita y Centenaria Escuela Normal del Estado de Durango, ubicada en Durango, México.

El desarrollo de la experimentación se basa en la metodología definida en los Modelos Teóricos Locales (Filloy, 1999), la cual establece seis etapas: selección de la población de estudio; diseño y aplicación de un diagnóstico; clasificación de la población, la cual fue realizada con base en el análisis del nivel de competencia sintáctica, semántica y de usos intuitivos de la población (Rojano, 1985); elección de una muestra representativa de la población; estudio de casos mediante entrevistas clínicas y elaboración de un reporte de observaciones.

Resultados de la experimentación

A continuación, se presenta una situación problemática representativa del modelo de enseñanza, describiendo las producciones de los estudiantes durante su resolución.

Luego de que los futuros docentes trabajaron un fenómeno de movimientos bancarios con interés simple, se les planteó un problema más del mismo estilo, pero con un interés compuesto, puesto que resulta de interés observar la forma en que hacen el tratamiento de fenómenos donde se presentan comportamientos distintos al lineal o cuadrático; en este caso, de tipo exponencial. La situación planteada fue la siguiente.

> Cuando en un préstamo el porcentaje es añadido al capital y pasa a formar parte de la deuda, el interés de este nuevo capital se llama interés compuesto.
>
> 1. Supongamos que tenemos un capital de 200 pesos al 20% de interés compuesto durante 5 meses. Completa la siguiente tabla para averiguar cuánto se deberá al finalizar cada mes

Una de las principales dificultades presentadas en esta situación fue detectar el tipo de variación implicada. Para lograrlo algunos estudiantes trataron de auxiliarse de la representación gráfica, obteniendo primero una representación tabular de la situación, y en seguida graficando los valores como pares ordenados. Procedieron colocando los puntos correspondientes y luego trazaron una línea que pasa por cada uno de éstos. Gracias a esto se comenzaba a descartar la variación lineal como correspondiente al fenómeno trabajado.

Al no conseguir definir el tipo de variación implicada, algunos estudiantes realizaron un proceso distinto, que consistió en calcular las diferencias entre los términos consecutivos y luego calcular las diferencias de las diferencias, pero al calcular la primera y segunda diferencias no obtienen valores constantes, lo cual causa aún más confusión. Incluso deciden hacer el cálculo de las terceras diferencias sin obtener un resultado que permita dotar de sentido al fenómeno. Se puede apreciar que el método se rescata como un proceso mecanizado.

Hubo otro grupo de estudiantes que también usó el método de las diferencias y tampoco les permitió determinar el tipo de variación presente en el fenómeno, pero sí fue útil para descartar la variación cuadrática como posibilidad, ya que al obtener la segunda diferencias no obtuvieron valores constantes.

El procesamiento numérico de la situación le permitió observar a dos estudiantes que, a medida que la variable independiente aumenta, las diferencias en la variable dependiente se hacen cada vez mayor. Se observa claridad en su razonamiento ya que expresan de forma cualitativa las características que tendría el gráfico si este fuera prolongado.

Conclusiones

Durante la implementación del modelo de enseñanza los estudiantes realizaron distintas producciones con el objetivo de dar sentido al cambio en los fenómenos dinámicos, mediante la observación y la caracterización de éstas podemos responder a las preguntas planteadas en un inicio.

Respecto al proceso de abstracción de las nociones de cambio se identificó lo siguiente. Generalmente en un primer momento, se realiza un análisis cualitativo con el cual se identifican características del fenómeno que se modifican, las cuales son expresadas en forma de cualidades no medibles. Al profundizar en el análisis de los experimentos emerge la necesidad de medir dichas características. Las cualidades antes identificadas se refinan, para dar paso al planteamiento de magnitudes (variables) que pueden ser medidas.

Luego de determinar las magnitudes implicadas, se identifican las que cambian y las que se mantienen constantes y, a su vez, se realiza una aproximación a la clase de relación que mantienen y la dirección de dichas relaciones (dependencia).

Mediante un análisis cuantitativo, consistente en operar de forma aritmética con los datos identificados, se determinan estados específicos de las variables (uso de la variable aritmética) (regularmente el estado inicial y uno más). Al organizar la información obtenida, es posible llevar a cabo una lectura dinámica de ésta, que ayude a determinar las relaciones entre las variables.

Mediante la generalización de los procesos aritméticos empleados, es posible producir un texto algebraico que represente las relaciones (uso de la variable algebraica), dicho de otra forma: el planteamiento de una función. En algunos casos, se llevó a cabo la comprobación de la correspondencia del texto algebraico con el conjunto de estados de las magnitudes, realizando la sustitución de datos y observando si efectivamente la función representa el fenómeno.

Respecto a las actuaciones más frecuentes, se presenta una tendencia a la linealidad: cuando no se comprende el tipo de tratamiento que se debe dar al fenómeno, se optaba por un tratamiento lineal. Los estudiantes generan distintas estrategias para definir la clase de variación en los fenómenos, sin embargo, el recurso más utilizado es la producción de textos gráficos cartesianos. Mediante la observación de las características de éstos, tratan de determinar el tipo de función trabajada. Cuando se realiza la producción de un texto geométrico con poca curvatura, se tiende a definir la variación como lineal. Sin embargo, cuando es evidente que el texto geométrico presenta cierta curvatura, se tiende a definir como de tipo cuadrática.

Y por último, la dificultad más frecuente se presenta al definir el tipo de comportamiento de los fenómenos, recurren a textos aritméticos, generalmente en forma de tablas de valores. Se detectó la utilización frecuente del método de las diferencias para definir el comportamiento de la variable dependiente en los fenómenos. Aplicado de forma mecánica generalmente; la falta de sentido es evidente al trabajar con situaciones de crecimiento exponencial, donde los estudiantes manifiestan no comprender por qué nunca obtienen un valor constante, a pesar de realizar una gran cantidad de diferencias.

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FUNCTION IDENTITY AND THE FUNDAMENTAL THEOREM OF CALCULUS

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By analyzing the responses of 100 introductory calculus students to two questions, this study addresses how students understand the fundamental theorem of calculus as it relates to function identity. One question involves students' understandings of the fundamental theorem of calculus, and the other involves their concept definitions of function sameness. This analysis aims to better understand students' concept images of function sameness, both in the context of the fundamental theorem of calculus and in general.

Keywords: University Mathematics, Calculus

The fundamental theorem of calculus (hereafter "the FTC" or "the fundamental theorem") is an important aspect of mathematics that we would like calculus students to understand. The FTC provides a relatively fast way of calculating integrals, which are used in various quantitative situations. One way to view an integral is as a function, say $\int_a^x f(t)dt$, where x is a variable. This is, arguably, the manner in which Newton conceived it (Thompson & Silverman, 2008). In this light, the FTC is actually a statement of function identity: the function g defined by $g(x) = \int_a^x f'(t)dt$ is the same function as h defined by $h(x) = f(x) - f(a)$. This paper addresses student understanding of this concept; specifically, I investigate the following: how do students understand the fundamental theorem in relation to their conceptions of function sameness?

There are three broad topics that apply to this investigation: student understanding of *function*, *function identity*, and the FTC. Research suggests that secondary and university students often do not have a mathematically normative understanding of function (Bardini et al., 2014; Leinhardt et al., 1990; Mirin, 2017; Sfard, 1992; Thompson, 1994; Vinner & Dreyfus, 1989). Function and function identity are intertwined; how a student understands function identity is closely tied to how they understand function (Mirin, 2017). A student's concept of what a function is will be closely tied to how they understand when two functions are identical. For example, if a student thinks of a function as a process, then it would make sense for that student to think of functions as identical whenever they represent the same process. Relatedly, if a student thinks of a function as an equation, then they might therefore think of different but equivalent equations as necessarily representing different functions. Relatedly, some university students struggle with the notion of function identity, classifying functions represented differently as different functions (Mirin, 2017; Mirin, 2018; Mulhuish & Fagan, 2017).

There is little literature on how students understand the fundamental theorem. Thompson (1994) finds that students' issues grasping the FTC are grounded in underdeveloped understandings of rate of change and covariation. Orton (1983) reports the types of mistakes students make in doing problems with definite integrals. He focuses on how students understand definite integrals as limits. However, his study does not address integrals in the context of the fundamental theorem or as functions. Thompson and Silverman (2008) make the point that an integral *as a function* is conceptually different from a definite integral as a number. That is, conceptualizing $g(x) = \int_a^x f'(t)dt$ as a function is different than conceptualizing $\int_a^b f'(t)dt$ for a particular number *b,* in the same way that conceptualizing the squaring function is different from conceptualizing a particular number being squared. In this manuscript, I situate the FTC as a statement about function identity, and hence also as a statement about functions.

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I adopt the constructs described in Tall and Vinner (1981): A student's *concept image* is "the total cognitive structure that is associated with the concept, which includes all the mental attributes and associated properties and processes" (p.152). One component of a student's concept image is their *concept definition,* which is their stated definition of a concept. This study involves investigating student concept definitions for function sameness, while acknowledging that there is likely more to a student's concept image than their stated concept definition.

The overarching epistemology guiding this study is radical constructivism, as described in Thompson (2000). This epistemology takes the perspective that students construct their own mathematical realities. A guiding aspect of my research is to not assume that what is a representation of an abstract mathematical object to us is also viewed as an abstract mathematical object by a student (Thompson & Sfard, 1994). Similarly, what is the same to us (e.g. different representations of the same function) might not be the same to students. This mathematical reality of the students is not directly accessible to us as researchers – the best we can do is create models (explanations) that account for students' responses (Clement, 2000).

Task Design, Subjects, and Data Collection

This is part of a larger study, the first portion of which can be found in Mirin (2018). A quiz was administered by the instructor to 102 students during the last week of an introductory calculus course at Anonymous State University (ASU). The course followed the Stewart (2013) text, and students had, within the week prior, learned about the FTC and practiced textbook problems applying it. The tasks discussed here are in Figure 1 (below).

1. Let p be the function defined on all real numbers by

$$p(x) = \int_2^x 3t^2 dt$$

and let q be the function defined on all real numbers by

$$q(x) = x^3 - 8$$

(a) How are p and q related? (Select option i. or ii.).

i. p and q are the same function.

ii. p and q are not the same function.

(b) Provide an explanation for your answer for 1a

2. Suppose g is a function and h is a function. What does it mean for g and h to be the same function? Explain.

Figure 1: The FTC Question (1) and the Function Sameness Question (2)

The first part of the quiz involved questions regarding derivative at a point of a single function represented in two different ways. The results of that portion indicated that students might have a mathematically non-normative concept image of function sameness. Here, I address student responses to the tasks in the second part of the quiz, which are relabeled as "Question 1", the FTC question, and "Question 2", the function sameness question (Figure 1, above). Notice that Question 1 is an instance of the FTC. The normatively correct response to this question is that p and q are the same function. Question 2 asks the student to give their concept definition of *function sameness.* Note that there are at least two different normatively correct responses to this question; the first is that g and h are the same if and only if g and h share a graph (set of ordered pairs) and also share a codomain, and the second only requires that they share a graph (Mirin et et al., in review). However, codomain was not once mentioned by students.

Analysis and Results

Due to the multiple choice nature of the question, coding the results of the Fundamental Theorem question was straightforward. Two students did not answer the question, nor did they answer 1b or 2. For this reason, they are excluded from the remainder of this analysis, leaving us with a convenient sample size of 100. Of the remaining 100, 61 chose option (i) (that p and q are the same function), and 39 chose option (ii) (that p and q are not the same function). One thing to note is that the students were not asked to "evaluate" the integral, that is, put it in closed form (e.g. as a polynomial, in this case). This means that there is a possibility that some students might have evaluated the integral as something different and have assessed p and q as different for that reason. Of the 100 students, 46 attempted to evaluate the integral, and 29 did so correctly. Unsurprisingly, there is a strong correlation between those who evaluated the integral correctly and those who answered that p and q are the same function, with 27 out of 29 (93%) who evaluated the integral correctly also claiming that p and q are the same function, and 8 out of 17 (47%) who evaluated the integral incorrectly claiming that p and q are not the same function (χ^2=12.4883, $p<.05$).

The nature of students' incorrect integral evaluations was illuminating and not due to any sort of minor computational errors. In fact, only two students who incorrectly evaluated the integral did so in such a way that it was a function of x (e.g. writing $p(x)=x^3+12$). Instead, 14 out of 17 (82%) included a "+C" in their evaluation of the integral. Students' explanations in 1b have not yet been analyzed to their full potential, but a preliminary reading provides insight to student thinking. Their explanations seem to suggest that some students might have viewed the integral as representing a string of symbols. This is consistent with Musgrave and Thompson's (2014) and Sfard's (1992) findings suggesting that some students think of a function as a string of symbols. To many of the students who evaluated the integral as involving a C (e.g. x^3+C), it would make sense that these students would not think of x^3+C as being the same as x^3-8, as these are different strings of symbols. For example, one student explains "when you derive p(x) it becomes the generalized formula $3x^2+C$. This is not equal to q(x)." Similarly, the students who evaluated the integral correctly tended to find that the resulting string of symbols (x^3-8) was identical to that in the definition of q, and therefore q and p are identical: "once calculated, the integral in p(x) becomes the same expression as q(x)".

There's a sense in which 36 out of 46 gave consistent responses; they either (1) evaluated the integral correctly and wrote that p and q are the same function, or (2) evaluated the integral incorrectly and wrote that p and q are different functions. This is consistent with thinking of a function as a string of symbols; if a student evaluates the integral correctly, then they observe that the resulting string of symbols is the same as x^3-8, and if they evaluate it incorrectly then they observe that the resulting string of symbols is different from x^3-8 (discussed above). The remaining 10 students had mixed responses. Those students' explanations in 1b provide some insight into their understanding of function identity. For example, some students included a +C for the integral yet assessed p and q as the same on the grounds that they share a derivative. Relatedly, some students wrote that p and q are the same function while also stating that they had a different constant. For these students, sameness of derivative was sufficient for sameness of function, and this was reflected in their concept definitions (discussed below).

Coding Question 2 results involved partitioning student answers into "extensional" and "not extensional". "Extensional" includes the characterization of function identity as same graph, same ordered pairs, or same output for every input. Statements such as "g and h are the same when g(x)=h(x)" were not coded as "extensional"; this is because in the absence of a universal quantifier, students could view "g(x)=h(x)" to mean that g(x) and h(x) are identical as equations (strings of symbols) or that g(x) transforms to h(x) under certain rules (Mirin, 2017; Sfard, 1988). Additionally, students might not view g(x) as representing a number or value of a dependent variable and instead view it as a name of a function (Musgrave & Thompson, 2014; Thompson, 1994, 2013b). Thirty-five students' concept definitions were coded as "extensional". Coding the remaining concept definitions is an ongoing project, but it bears mentioning that, consistent with the previous paragraph, 11 students included sameness of derivative in their criteria for function sameness.

I had originally hypothesized that there would be a correlation between students who give extensional function sameness concept definitions and those who answer that p and q are the same function. This is because I expected students with other, non-normative understandings of function identity to claim that p and q are different. This was indeed the case with at least two students, who asserted that p and q differ because one represents an area under a curve, and the other does not. However, a chi square analysis revealed no such correlation. It seems that because p could be expressed in closed form, students' assessment of sameness of p and q was primarily about how they calculated the integral. This allowed for students to assess that p and q are the same on the grounds that they are expressed by the same equation, rather than requiring a robust understanding of function sameness. This resulted in the possibility that students who understand functions as strings of symbols answered that p and q are the same function.

That so many students evaluated the integral with a "+C" is especially revealing. This might suggest that, despite the function notation *p*(x) being used and the quiz explicitly telling them that *p* is a function, these students might not have viewed *p* as a function (perhaps, as one student above put it, "a formula"). This leaves open the possibility that, when these students were asked if *p* and *q* are the same function, they were not viewing *p* as a function at all. This is consistent with the results of the first part of the quiz, in which students appeared to not think of a particular piecewise function as a function (Mirin, 2018).

Conclusions and Future Directions

This preliminary report provides valuable data on students' concept images of function sameness. It is notable that only 61% of these calculus students identified a straightforward instance of the fundamental theorem of calculus as asserting function identity. However, it seems that for several of those students, their assessment was mostly about their calculation of an integral. Perhaps, for the reasons discussed above, we could investigate whether students understand an integral (such as in 1a) to even be a function. It might also be productive to provide a similar function sameness question as in 1a, but instead using the notation f '(x) rather than providing a specific derivative that the student can anti-differentiate procedurally. It might additionally be wise to see how students use and understand the notation "+C".

This study also gives insight into students' concept definitions of function sameness, with 35% providing an extensional (mathematically normative) answer. Interestingly, 11% included sameness of derivative in their criteria for function sameness. I hypothesize that the FTC question might have influenced students' function sameness concept definitions. Future research can address this hypothesis by providing the concept definition question in a different context.

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UNDERGRADUATE MATHEMATICS MAJORS' PROBLEM SOLVING AND ARGUMENTATION

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The transition to proof has been a heavily researched area in undergraduate mathematics research. As proof construction involved both formal-rhetorical and problem-solving aspects, this study investigated how undergraduate mathematics majors who recently completed a transition to proofs class engage with two different problems: one with formal rhetorical knowledge and one without. Overall, students exhibited behavior indicating that formal-rhetorical knowledge could continue to act as a barrier to approaching and solving problems. The study highlighted the importance of sense-making strategies and familiarity not only with mathematical content knowledge but also with the logical structure of mathematical arguments.

Keywords: Problem Solving, Reasoning and Proof, University Mathematics, Advanced Mathematical Thinking

Introduction

For much of schooling, mathematics education has often had computation and applications of theorems or formulas at the center. However, the academic discipline of mathematics is more concentrated on generalizations and abstractions; coming up with rigorous and correct proofs of previously unproved statements was the goal for most research mathematicians. This transition to formal proof usually took place during undergraduate studies, with many scholars having examined the cognitive difficulties in learning to do formal mathematical proofs (e.g., Brown, 2007; Moore, 1994; Thoma & Nardi, 2017). As a stark deviation from prior math experiences, this transition is abrupt and difficult (Moore, 1994). Despite the difficulty in learning proofs, research has recognized how proving and problem solving are closely related: It takes problem-solving in order to translate an informal argument into a valid formal proof (Mamona-Downs & Downs, 2009), and the proofs themselves can be divided into formal-rhetorical and problem centered parts (Selden & Selden, 2013). This translation is not an easy task, in that it requires both knowledge and instruction. Students often do not learn the necessary tools explicitly or rigorously until students are asked to construct formal proofs. In this study, we researched undergraduate mathematics students who recently completed their university's proof introductory course. Our research question was: How did these students engage with formal-rhetorical and problem-centered aspects of mathematical tasks?

Framing

The conceptual framework for this study drew from the following main areas: aspects of proof construction (Selden & Selden, 2013; Weber, 2005) and mathematical problem solving (Schoenfeld, 1985) to guide our data collection and analysis.

Aspects of Proof Construction

Though mathematical proofs have been studied for many years prior to their formulation, two considerations of proof construction procedures are particularly useful to this study. One delineates parts of a mathematical proof, whereas the other categorizes different types of proof constructions. Selden and Selden (2013) developed a framework of understanding proofs to be composed of two parts: the formal-rhetorical part and the problem-centered part. Between different statements and

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even different proofs of the same statement, there are different compositions when broken down into these constituent parts. Selden and Selden's idea of these two components of a proof will be the guiding main framework for this study.

Furthermore, Weber (2005) categorized three types of proof construction procedures demonstrated by undergraduate math students – procedural, syntactic, and semantic. Weber argued that learning opportunities are largely dependent on what type of construction students utilized. Together, the formal-rhetorical and problem-centered parts of proof and proof construction procedures provide a framework of understanding students' problem-solving pathways.

Mathematical Problem Solving

Schoenfeld (1985) developed a framework of understanding mathematical problem solving. He delineated four types of knowledge students draw from when engaging in mathematical problem solving: resources, heuristics, control, and beliefs. Others shortly categorized similar ideas in different ways. Considering related research together in a more cognition focused formulation, Schoenfeld (1992) recognized that "there appears to be general agreement on the importance of these five aspects of cognition: the knowledge base, problem solving strategies, monitoring and control, beliefs and affects, and practices" (p. 42). This framework has frequently been applied to college-level mathematics (e.g. Selden & Selden, 2003; Selden & Selden, 2013; Weber, 2005) and is useful for analyzing student problem-solving.

Methods

Ten undergraduate mathematics students, six females and four males, from a Hispanic-serving four-year research university in California participated in this study. All of the students completed a transition to higher mathematics course in the previous quarter. This course, as a prerequisite for many mathematics courses that follow, was described as an introduction to the elements of propositional logic, techniques of mathematical proof, and fundamental mathematical structures, including sets, functions, relations, and other topics.

We conducted and video-recorded task-based, think aloud interviews (Charters, 2003). The participants worked through two problems: a non-routine problem taken from the 2017 American Invitational Mathematics Examination II and a problem that asked students to construct an alternative proof of a well-known result given the argument's outline. The second was developed in consultation with a mathematics professor from the study's university, who taught a number of the participants in the quarter prior to data collection. For the rest of the paper, we will refer to these problems as P1 and P2 respectively. Students chose which problems to attempt first, but they worked through both tasks. The difference between the two tasks was that P1 was a "real-world" problem that needed no formal-rhetorical knowledge, and P2 required substantial formal-rhetorical knowledge (but concluded with the problem-centered part). After the tasks, we collected student work and finished with a quick debrief aimed at understanding the students' feelings about the problems (e.g., their ease and enjoyment) and clarifying the students' strategies. In relation to our research question: the student work informed and documented students' strategies and video data contributed both to our understanding of strategies as well as our understanding of students' feelings of working through the tasks.

To analyze the data, we did an initial round of focused coding (Maxwell, 2005), looking for students' feelings and strategies through the tasks while keeping track whether it related to formal-rhetorical or problem-centered aspects. After the initial coding, we developed another set of codes within strategies: sense-making, argumentation, and generality.

Findings

In general, students exhibited high levels of engagement with and interest in the tasks – every participant requested information about the solutions after the interview. Despite differences in students' feelings about the two tasks, we found similarities about what made the mathematics enjoyable for participants. Regarding strategies, we found that the presence and absence of formal-rhetorical aspects related to how students' sense-making and argumentation.

Students' Feelings: Familiarity, Fun, and Frustration

Familiarity played a role from the start in how students approached the problems. Excluding students whose decisions on which problem to attempt first was semi-arbitrary (i.e., left vs. right, top vs. bottom, etc.), the remaining students were split half and half. Notably, students chose P1 because of an expected enjoyment, whereas others began with P2 because it was more familiar. These reasons were consistent among P1-starters and P2-starters. Isaac, who began with P2, explained that he was, "more comfortable with this kind of thing … there were aspects of everything that [he] could relate to." Conversely, students who started with P1 did so because it seemed more puzzle-like and fun. Henry explained, he "[doesn't] know if [he] considers proof-writing fun," contrasting it with how P1 was "completely uncharted … never seen it before." While the task of constructing a proof felt familiar to students, only some were drawn to this comfort. Although there was certainly a problem-centered aspect to P2, and to proofs generally, the presence of formal-rhetorical aspects seemed to detract from it being seen as fun P1, even when they recognized the proof's puzzle-like aspect afterwards.

There was an interesting interplay between familiarity, fun, and frustration. A common theme among the students was that they enjoyed feeling productive without feeling as though they were doing tedious work. Esther considered P1, "not as much of a math problem." She expressed that sometimes she hard time motivating herself when she could think of how to code a computer program to compute it for her. Furthermore, students had a hard time engaging with the problem-centered part of P2 when they did not know how to approach it. That said, it should be highlighted that frustration was not inherently a bad feeling. Talia found P1 to be no only more frustrating but also more fun. Students found enjoyment when they had tools to approach the task and there was some familiarity, some frustration, and a feeling of progressing in the task without having it feeling immediately solvable by brute force.

Students' Strategies

Generality as a tool and obstacle. As a counting problem, P1 required thinking generally. All of the students recognized this: the students noticed that symmetry and choices were important aspects of the problem. For instance, Madison explained while solving, "If I can figure out how it works for one town, I can just apply that reasoning to other towns." Moreover, all of the students realized that the existence of a five-town loop would satisfy the conditions and attempted to count those configurations. However, though they all recognized it as a sufficient condition, not all of the students investigated whether the existence of a five-town loop was a necessary condition. In fact, Caleb was the only student to substantively explore this.

P2 had two major instances of generality in the desired proof. First, because the proof was a statement holding for every natural number, it sufficed to prove the result using a fixed but arbitrary natural number, maintaining its generality throughout the proof. Second, students had to represent two arbitrary elements of a set when proving that something was closed under multiplication. These uses of generality are routine in proof-writing. The students generally did not have an issue with the navigating the second instance, but there were instances where students reverted to considering all natural numbers simultaneously (as opposed to a fixed arbitrary natural number), which altered the argument's viability. Though students seemed aware of this generality's role, maintaining the correct level generality and applying it to sound argumentation proved to be an obstacle in both problems.

Sense-making to argumentation. Throughout the process, students employed different sense-making strategies. At the lowest level, every participant asked questions, reread the problems, and thought about the tasks to make sure that they understood the problems. However, the process of understanding and solving the problems were manifested differently between P1 and P2.

One of the most frequently used sense-making strategy was thinking of examples. Though this might have been due to the problems' context, every participant's primary approach to P1 entailed drawing examples. These examples then informed their understanding, helping students consider strategies as they constructed systematically. There were fewer instances of student example construction in P2: Isaiah and Gianna listed elements of the sets involved, and Megan drew a number line to explore parts of the problem. That said, the remaining students used symbol manipulation and, at most, tried thinking about the sets in their head. As another sense-making strategy in P1, two students considered a similar, yet simpler, problem (considering a system where there are fewer than the five towns). Additionally, Daniella calculated the total number of road configurations to provide an upper bound to the desired answer. Such a strategy not only provided a means towards an alternate solution path, namely subtracting the invalid configurations from the total to obtain the desired answer, but also provided a measure of reasonableness for future answers. Without this strategy, Henry and Esther got overwhelmed with the sheer number of possibilities and offered infinity as their initial proposed answer.

Largely due to the real-world context of P1, students used intuition to form their strategies. Half of the students clearly made and explored the realization of a working condition. As Talia said, "every town needs a way in and a way out." However, similar to student thinking of five-town loops, there was limited substantive work to show whether this was a sufficient and/or necessary condition. Isaac and Megan both recalled a puzzle that entailed drawing a certain figure without picking up their pencil. Because it seemed related enough, they took it as a potential solution strategy, taking time to realize that they had made an illogical jump to translate P1 into a nonequivalent problem. Conversely, many students explicitly unpacked the logical structure of the statement of P2 and how the argument outlined addressed the logical structure. Some went beyond doing this mentally or verbally, writing the statements translated into symbolic logic. Overall, students generally attended more to sense-making in P1 with sound argumentation taking a backseat, while the opposite was true for P2.

Discussion & Conclusions

The transition to proofs and higher mathematics in general is a complex process that deviates from what many students' previous mathematical experience. While they may not be strangers to mathematical problem solving and non-routine problems, students are asked to integrate formal-rhetorical knowledge when constructing proofs. The presence of this knowledge in problems may overpower the problem-solving that underlie mathematics' appeal. Making explicit connections between formal-rhetorical fluency to sound argumentation and problem-solving generally in instruction may aid understanding and value given to proofs.

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TEACHERS-IN-TRAINING'S REFLECTIONS ON THE TEACHING OF CALCULUS TO PEOPLE WITH DISTINCT CHARACTERISTICS

PROFESORES EN FORMACIÓN QUE REFLEXIONAN SOBRE LA ENSEÑANZA DEL CÁLCULO A PERSONAS CON CARACTERÍSTICAS DIFERENCIADAS

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We present the progress of a study that seeks to answer: What learning do teachers in training consolidate that reflect on the teaching of Calculus to people with distinct characteristics in higher education? We expect to answer this using the model of reflection and action of Parada (2011) as a theoretical-methodological guide. The study follows an action research methodology, planned in six phases; here the results of the first four are reported. The learning constructed by future teachers will be described in terms of the three components of reflective thinking established by the model. A meaning so far reached by future teachers is the need to approach the context of students to pose problems tailored to their distinct characteristics.

Keywords: Teacher Preparation in Training, Special Education, Calculus.

Colombian educational policies have promoted inclusion in the country's Higher Education Institutions (HEIs). One of them is the Universidad Industrial de Santander (UIS) context of the study reported here, in which there is an agreement for special admissions, which has been implemented since the first period of 2014. In addition, the UIS within its strategies to face the problems of desertion and permanence has created a policy of Academic Excellence (Agreement No. 018 of 2014) that is executed through accompaniment and monitoring programs for students. One of the programs, called SEA-ASAE, is led by the Mathematics School and is focused on serving and accompanying students who take math courses through peer tutoring and academic monitoring. The tutors who offer the accompaniment are mathematics teachers in training, who are studying a Bachelor's degree in mathematics from the institution.

The SEA-ASAE program has been developing since 2012 and has allowed the constitution of a Community of Practice (CoP), made up of teacher educators, professors of the mathematics subjects, practicing tutors and auxiliary tutors (graduates of the didactics of calculus course). Within the course, students who have practiced as Differential Calculus tutors with students from Science and Engineering programs have reported cases of students who: have Asperger, have a mild cognitive disability, are from indigenous communities, among others. The tutors show difficulty in attending to this population because they have not received the didactic training to do so; hence the need to promote reflection among mathematics teachers in training on teaching calculus to people with distinct characteristics.

The aforementioned, offers us two problems that can be articulated in favor of educational inclusion in HEIs: i) mathematics teachers in training require instruction around inclusion and, ii) people with distinct characteristics need support and attention from people that understand their condition and have mathematical knowledge. For this reason, in the research reported here, we focus our attention on the practitioners of the CoP, who begin their participation in it as practicing tutors who, in turn, develop the Calculus Didactics course. Course, in which a conceptual and methodological structure will be developed that allows reflection on the teaching of calculus and attention to distinct characteristics, an experience with which we hope to achieve the research objective: describe the

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learning built by training teachers reflecting on the teaching of calculus to people with distinct characteristics in higher education.

Background

According to Camargo (2018), since 1960 organizations such as UNESCO have pursued guiding principles with a view to promote inclusive education. As a result of these principles, policies have emerged in different countries, including Colombia.

The Colombian political constitution of 1991 and the General Law of Education of 1994 are the main policies that speak of inclusion in Colombia, establishing that education is a right for all and that Colombians should feel included, without differentiation of race, sex, religion, political beliefs, and other constituent elements of Colombian multiculturalism. These policies have been gradually implemented, in recent decades, at various levels of education, starting with basic and secondary education, and to a lesser extent in Higher Education Institutions (HEIs).

Because the inclusion policies for HEIs were not very precise, the Ministry of National Education (MEN), and the Higher Council for Higher Education (CESU, 2013), in the agreement for the upper 2034, establish a regulatory framework, in where it is stated that in order to achieve a quality education, the entrance, permanence and graduation of students with certain distinct characteristics must be guaranteed, which may be people: i) With Special Educational Needs (SEN), that is, they have some disability status or exceptional talents; ii) from indigenous communities or reservations; iii) from Afro-Colombian, *Palenquera* and *Raizal* populations; iv) from departments where HEIs do not exist; v) coming from municipalities with difficult access or with public order problems; vi) victims of the Colombian internal armed conflict; and viii) demobilized from the peace processes.

According to a study carried out by CESU (2013) in collaboration with the Development Research Center (CID) of the Universidad Nacional de Colombia (UNAL), it is established that students with distinct characteristics are the most likely to drop out of HEIs. Furthermore, it has been shown that these students have difficulties in understanding the mathematical objects of differential calculus. Moreover, according to MEN (2016), students with SEN present certain difficulties associated with their condition.

One of the strategies offered by the MEN to guarantee the permanence and therefore the graduation of the students with distinct characteristics is to have inclusive teachers in the HEI: to have a suitable training both in its disciplinary, didactic, as well as in inclusive education.

Silverman and Thompson 2008) mention that teachers must have a deep knowledge of mathematics; Ponte (2011) states that he must also have knowledge in the didactic; Llinares (2007), Flores (2009) establish that the teacher must be reflective; and Aké (2015) mentions the need to promote reflection among mathematics teachers about inclusive education.

Parada (2011) based on the reflection processes within the communities of practice, proposes the RyA reflection and action model, which aims to guide theoretically and methodologically the actions carried out by mathematics teachers before, during and after class, in order to increase the capacity of its members to reflect critically on their professional practices.

Theoretical and Conceptual Aspects

The research of which the first results are reported here, is based on the R-y-A model, whose main objective is to promote reflection processes in communities of practice (CoP) and whose main elements are: participation (which may be peripheral or full), reflection and action. The fundamental resource that is reflected on is action, understood as the performance of the mathematics teacher in her or his own professional practices.

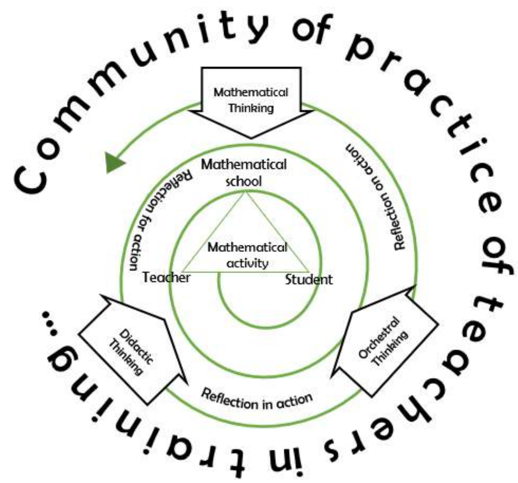

Figure 3. Adaptation to the R-y-A model of Parada (2011)

At the center of the Model is the **mathematical activity,** which is where professional development efforts are focused. Mathematical activity is found within the Saint-Onge pedagogical triangle (1997, cited by Parada, 2011),where there is a relationship between the student, the teacher and the mathematics school.

These relationships make the student's mathematical activity during class and the teacher's mathematical activity before, during and after class possible.

Following Dewey's ideas (1989), Parada (2011) understands **Reflection** as a process of resolving doubts, conflicts, and willingness to review one's performance; that is why it seeks to promote the **reflection processes** of mathematics teachers, which is a continuous process that favors professional development. It is broken down into three processes: **Reflection-for-action** (reflection made by the teacher before the class), **Reflection-in-action** (reflection made by the teacher during the class), **Reflection-on-action** (reflection made by the teacher after the class).

The three arrows around the spiral represent the three thoughts into which the **reflective thinking** of the mathematics teacher is broken down: **mathematical thinking** (the mathematical knowledge that the teacher uses to develop mathematical activity in the classroom), **didactic thinking** (arises when the teacher asks about the different ways of approaching the mathematical content to their students) and **orchestral thinking** (it is characterized around the conduct of the class and the way the teacher uses resources to promote mathematical activity in the classroom).

Research Methodology

The study reported here is of an action-research type, in the light of Kemmis and McTaggart (1988). It is carried out in six phases, which are briefly described below:

1) Phase 0 (preliminary study): study of national and international policies around inclusion in Higher Education, collection and analysis of data related to the number of students with distinct characteristics admitted to the UIS; 2) Phase 1: characterization of the community of practice (study context). Design of the intervention and participation plan with the CoP. Delimitation of the participants of the CoP on which the intervention will be carried out; 3) Phase 2: first approach to the CoP. Reflection and action processes with seven teachers in training who were developing the calculus teaching course in the first semester of 2019; 4) Phase 3: analysis of the results of the first approach; 5) Phase 4: second approach to the CoP, at the time of the writing this document. This phase is being developed with fifteen students who are ahead in the Calculus Didactics course; and 6) Phase 5: Data analysis and characterization of reflective thinking, which will answer the research

question taking into account the three dimensions of reflective thinking (variational thinking, didactic thinking and Orchestral thinking) of the mathematics teacher that the R-y-A model offers.

Some Results

The analysis of the institutional policies (Agreement No. 282 of November 7, 2017) and the data of the admissions of the University allowed us to know that between 2014 and 2019, the UIS has registered a total of 347 students who entered through special admissions, of which 220 (63%) students entered careers such as engineering, science and others, which have mathematics-related subjects, in particular differential calculus, in their study plan. Of these students: 81 come from municipalities with difficult access or problems of public order, 100 victims of the armed conflict, 15 from the Afro-Colombian, Palenquera and Raizal populations, 22 from indigenous communities and 2 from departments where there are no HEIs.

Regarding the first approach, through the negotiation of meanings that were made possible within the activities of the course, four projects were objectified, the purpose of which was to prepare a didactic design around an object of study of calculus aimed at a person with differentiated characteristics. It is from this first approach that the projects were made.

An interesting result was the one achieved by the teacher who carried out the design aimed at a student from the Misak indigenous community, with whom he worked on optimization based on problems contextualized to the needs and characteristics of his community. For this, the teacher in training initially made a documentary study of the community, then interviewed the student and later implemented its design. With the implementation of the design, it was found that from the contextualized problems the student managed to appropriate and get involved in the problem situations that were presented to him, thus favoring his learning regarding optimization.

First Reflections

Until now, some learning negotiated by teachers in terms of their reflective thinking have been evidenced: a) in mathematical thinking, they experienced confusion in their conceptual domains about the mathematical objects of differential calculus such as: function, variation, limit and derivative; which were resignified through reflection and discussion in CoP; b) in didactic thinking, they valued the need to make curricular adaptations, related to the use of language and the approach of contextualizingproblems adjusted to meet their student's needs and; c) in orchestral thinking, they managed to articulate different resources adjusted to the characteristics of the student and the mathematical objects of study.

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PROFESORES EN FORMACIÓN QUE REFLEXIONAN SOBRE LA ENSEÑANZA DEL CÁLCULO A PERSONAS CON CARACTERÍSTICAS DIFERENCIADAS

TEACHERS-IN-TRAINING'S REFLECTIONS ON THE TEACHING OF CALCULUS TO PEOPLE WITH DISTINCT CHARACTERISTICS

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Presentamos avances de una investigación que busca responder a: ¿Qué aprendizajes consolidan en sus prácticas profesores en formación que reflexionan sobre la enseñanza del cálculo a personas con características diferenciadas en la educación superior? La que se espera responder usando como guía teórico-metodológica el modelo de reflexión y acción de Parada (2011). El estudio sigue una metodología de investigación-acción, planificada en seis fases, aquí se reportan resultados de las primeras cuatro. Los aprendizajes construidos por los futuros profesores se describirán en términos de los tres componentes del pensamiento reflexivo establecidos por el modelo. Un significado hasta ahora alcanzado por los futuros profesores es la necesidad de acercamiento al contexto de los estudiantes para plantear problemas ajustados a sus características diferenciadas.

Palabras clave: Preparación de Maestros en Formación, Educación especial, Cálculo.

Las políticas educativas colombianas, han promovido la inclusión en las Instituciones de Educación Superior (IES) del país. Una de ellas es la Universidad Industrial de Santander (UIS) contexto del estudio que aquí se reporta, en la que se cuenta con un acuerdo para las admisiones especiales, que se viene implementando desde el primer periodo de 2014, Además, la UIS dentro de sus estrategias para afrontar las problemáticas de deserción y permanencia ha creado una política de Excelencia Académica (Acuerdo No. 018 de 2014) que se ejecuta por medio de programas de acompañamiento y seguimiento a estudiantes. Uno de los programas, denominado SEA-ASAE, es liderado por la Escuela de Matemáticas y está centrado en atender y acompañar a los estudiantes que cursan las asignaturas de matemáticas mediante tutorías entre pares y monitorias académicas. Los tutores que ofrecen el acompañamiento son profesores de matemáticas en formación, que cursan Licenciatura en Matemáticas de la institución.

El programa SEA-ASAE se viene desarrollando desde el año 2012 y ha permitido la constitución de una Comunidad de Práctica (CoP), conformada por formadores de profesores, profesores titulares de las asignaturas de matemáticas, tutores practicantes y tutores auxiliares (egresados del curso de didáctica del cálculo). Al interior del curso los estudiantes que han realizado la práctica como tutores de Cálculo Diferencial con estudiantes de programas de Ciencia e ingenierías han reportado casos de estudiantes que presentan: Asperger, discapacidad cognitiva leve, son provenientes de comunidades indígenas, entre otros. Los tutores manifiestan dificultad para atender a esta población pues no han recibido la formación didáctica para hacerlo, de allí la necesidad de promover la reflexión en los profesores de matemáticas en formación sobre la enseñanza del cálculo a personas con características diferenciadas.

Lo antes descrito, nos ofrece dos problemáticas que pueden articularse a favor de la inclusión educativa en las IES: i) los profesores de matemáticas en formación requieren instrucción alrededor de la inclusión y, ii) las personas con características diferenciadas necesitan apoyo y atención de personas que comprendan su condición y tengan conocimiento matemático. Por ello en la investigación que aquí se reporta centramos la mirada en los practicantes de la CoP, que inician su participación en ella como tutores practicantes que a su vez desarrollan el curso de Didáctica del Cálculo. Curso, en el que se desarrollará una estructura conceptual y metodológica que permita la reflexión sobre la enseñanza del cálculo y la atención a las características diferenciadas, experiencia con la que esperamos alcanzar al objetivo de investigación: describir los aprendizajes construidos por profesores en formación que reflexionan sobre la enseñanza del cálculo a personas con características diferenciadas en la educación superior.

Antecedentes

Según Camargo (2018) desde 1960, organizaciones como la UNESCO han procurado principios orientadores con miras a favorecer una educación inclusiva. Producto de estos principios han surgido políticas en diferentes países, entre ellos Colombia.

La constitución política de Colombia de 1991 y la Ley General de Educación de 1994 son las principales políticas en donde se habla de inclusión en Colombia, estableciendo que la educación es un derecho para todos y que los colombianos se deben sentirse incluidos desde la diversidad, sin diferenciación de raza, sexo, religión, creencias políticas, y demás elementos constitutivos del multiculturalismo colombiano. Estas políticas se han venido implementando de manera paulatina, durante las últimas décadas, en varios niveles de la educación, iniciando por la educación básica y media, y en menor medida en las Instituciones de Educación Superior (IES).

Debido a que las políticas de inclusión para las IES no eran muy precisas el Ministerio de Educación Nacional (MEN), y el Consejo Superior de Educación Superior (CESU, 2013), en el acuerdo por lo superior 2034, establecen un marco normativo, en donde se afirma que para lograr una educación de calidad se debe garantizar el ingreso, la permanencia y la graduación de los estudiantes con ciertas características diferenciadas, que pueden ser personas: i) Con Necesidades Educativas Especiales (NEE), es decir, que presentan alguna condición de discapacidad hasta las que presentan talentos excepcionales; ii) procedentes de comunidades o resguardos indígenas; iii) procedentes de población afrocolombiana, palenquera y raizal; iv) procedentes de departamentos donde no existen IES; v)procedentes de municipios de difícil acceso o con problemas de orden público; vi) víctimas del conflicto armado interno colombiano; y viii)desmovilizadas de los procesos de paz.

Según un estudio realizado por el CESU (2013) en colaboración con el Centro de Investigación de Desarrollo (CID) de la Universidad Nacional de Colombia (UNAL), se establece que los estudiantes con características diferenciadas son los más propensos a desertar de las IES. Además, se ha evidenciado, que estos estudiantes tienen dificultades para comprender los objetos matemáticos del cálculo diferencial. Además, según el MEN (2016) los estudiantes con NEE presentan ciertas dificultades asociadas a su condición.

Una de las estrategias ofrecidas por el MEN para garantizar la permanencia y por tanto la graduación de los estudiantes con características diferenciadas, es la de contar con docentes inclusivos en la IES: poseer una formación idónea tanto en su ámbito disciplinar, didáctico, así como en educación inclusiva.

Silverman y Thompson 2008) mencionan que los profesores deben tener un conocimiento profundo en las matemáticas; Ponte (2011) expone que además debe contar con conocimiento en lo didáctico; Llinares (2007), Flores (2009) establecen que el profesor debe ser reflexivo; y Aké (2015) menciona la necesidad de promover la reflexión en los profesores de matemáticas acerca de una educación inclusiva.

Parada (2011) basada en los procesos de reflexión al interior de las comunidades de práctica, plantea el modelo de reflexión y acción R-y-A, el cual pretende guiar teórica y metodológicamente las acciones llevadas a cabo por los profesores de matemáticas *antes, durante y después* de clase, con el fin de aumentar la capacidad de sus miembros para reflexionar críticamente sobre sus prácticas profesionales.

Aspectos Teóricos y Conceptuales

La investigación de la que aquí se reportan los primeros resultados, se apoya en el modelo R-y-A, que tiene como objetivo principal promover los procesos de reflexión en comunidades de práctica (CoP) y cuyos elementos principales son: la participación (que puede ser periférica o plena), la reflexión y la acción. El recurso fundamental sobre el cual se reflexiona es la acción, entendida como la actuación del profesor de matemáticas en sus propias prácticas profesionales.

Ilustración 1. Adaptación modelo R-y-A de Parada (2011)

En el centro del Modelo se encuentra la **actividad matemática**. Que es donde se centran los esfuerzos de desarrollo profesional. La actividad matemática se encuentra al interior del triángulo pedagógico de Saint-Onge (1997, citado por Parada, 2011). En donde existen relaciones entre el alumno, el profesor y la matemática escolar.

Estas relaciones posibilitan la actividad matemática del estudiante *durante* la clase y la actividad matemática del profesor *antes, durante y después* de la clase.

Siguiendo las ideas de Dewey (1989), Parada (2011) entiende la **Reflexión** como un proceso de resolución de dudas, de conflictos, y de disposición para revisar su actuación; es por ello que se busca promover los **procesos de reflexión** de los profesores de matemáticas, siendo este un proceso continuo que favorece el desarrollo profesional, y se descompone en tres procesos: **Reflexión-para-la acción** (reflexión que hace el profesor antes de la clase), **Reflexión-en-la acción** (reflexión que hace el profesor durante de la clase), **Reflexión-sobre-la acción** (reflexión que hace el profesor después de la clase).

Las tres flechas que se encuentran alrededor de la espiral representan los tres pensamientos en los que se descompone el **pensamiento reflexivo** del profesor de matemáticas: el **pensamiento matemático** (los conocimientos matemáticos que el profesor emplea para desarrollar la actividad matemática en el aula), el **pensamiento didáctico** (surge cuando el profesor se cuestiona sobre las diferentes maneras de acercar los contenidos matemáticos a sus estudiantes) y el **pensamiento orquestal** (se caracteriza alrededor de la conducción de la clase y la forma como el profesor usa los recursos para la favorecer la actividad matemática en el aula).

Metodología de la Investigación

El estudio que aquí se reporta, es de tipo investigación-acción, a la luz de Kemmis y McTaggart (1988). Se lleva a cabo en seis fases, que se describen brevemente a continuación:

1) Fase 0 (estudio preliminar): estudio de políticas nacionales e internacionales alrededor de la inclusión en la Educación Superior, recolección y análisis de datos relacionados con el número de estudiantes con características diferenciadas admitidos en la UIS; 2) Fase 1: caracterización de la comunidad de práctica (contexto de estudio). Diseño de el plan de intervención y participación con la CoP. Delimitación de los participantes de la CoP sobre los que se hará la intervención;3) Fase 2: primer acercamiento a la CoP. Procesos de reflexión y acción con siete profesores en formación que desarrollaban el curso Didáctica del Cálculo en el primer semestre del 2019; 4) Fase 3: análisis de resultados del primer acercamiento; 5) Fase 4: segundo acercamiento a la CoP, en el momento en que se escribe este documento, se está desarrollando esta fase con quince estudiantes que se encuentran adelantado el curso de Didáctica del Cálculo; y 6) Fase 5: Análisis de los datos y caracterización del pensamiento reflexivo. Se espera responder a la pregunta de investigación teniendo en cuenta las tres dimensiones del pensamiento reflexivo (pensamiento variacional, el pensamiento didáctico y el pensamiento Orquestal) del profesor de matemáticas que ofrece el modelo R-y-A.

Algunos Resultados

El análisis de las políticas instituciones (Acuerdo No. 282 del 7 de noviembre del 2017) y los datos de las admisiones de la Universidad permitieron saber que entre el 2014 y el 2019, la UIS ha registrado un total de 347 estudiantes que ingresaron por medio de las admisiones especiales, de los cuales 220 (63%) estudiantes ingresaron a carreras como ingenierías, ciencias y otras, las cuales tienen en su plan de estudio materias relacionadas con matemáticas, en particular cálculo diferencial. De estos estudiantes: 81 provienen de municipios de difícil acceso o problemas de orden público, 100 víctimas del conflicto armado, 15 provenientes de población afrocolombiana, palenquera y raizal, 22 pertenecientes a comunidades indígenas y 2 provenientes de departamentos donde no existen IES.

Respecto al primer acercamiento, a través de la negociación de significados que se posibilitaron al interior de las actividades del curso, se cosificaron cuatro proyectos los cuales tenían como finalidad elaborar un diseño didáctico alrededor de un objeto de estudio del cálculo dirigido a una persona con característica diferenciada. De ese primer acercamiento, que proyectos se hicieron.

Un resultado interesante, fue el logrado por el profesor que realizó el diseño dirigido a un estudiante de la comunidad indígena Misak, con quién trabajó la optimización a partir de problemas contextualizados a las necesidades y características de su comunidad. Para ello, el profesor en formación hizo inicialmente un estudio documental de la comunidad, luego entrevistó al estudiante y posteriormente implementó su diseño. Con la implementación del diseño se pudo encontrar que a partir de los problemas contextualizados el estudiante logró apropiarse e involucrarse en las situaciones problema que se le planteaban, favoreciendo así su aprendizaje respecto a la optimización.

Primeras Reflexiones

Hasta el momento se han podido evidenciar algunos aprendizajes negociados por los profesores en términos de su pensamiento reflexivo: a) en el pensamiento matemático, ellos experimentaron confusión en sus dominios conceptuales sobre los objetos matemáticos del cálculo diferencial como: función, variación, límite y derivada; los cuales resignificaron mediante la reflexión y discusión en CoP; b) en el pensamiento didáctico, valoraron la necesidad de hacer adaptaciones curriculares, relacionadas con el uso del lenguaje y el acercamiento al contexto para el planteamiento de problemas ajustados a sus necesidades y; c) en el pensamiento orquestal, ellos lograron articular diferentes recursos ajustados a las características del estudiante y a los objetos matemáticos de estudio.

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FROM RECURSION TO INDUCTION: STUDENTS' GENERALIZATION PRACTICES THROUGH THE LENS OF COMBINATORIAL GAMES

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In this study we describe the generalization and justification practices of students in a highly selective summer mathematics program as they explore a sequence of problems from combinatorial game theory. We find that while study participants readily generate examples and reason recursively when analyzing Nim-like two-player combinatorial games and are able to reach valid conclusions about winning strategies in these games, they do not readily formalize their justifications into proofs using mathematical induction. We describe some obstacles that we observe in the transition between recursive reasoning and proof by induction.

Keywords: Advanced Mathematical Thinking, Problem Solving, Reasoning and Proof

Research in undergraduate mathematics education has identified obstacles to students' acquisition and understanding of mathematical induction as a proof strategy (*e.g.*, Brown, 2008; Dubinsky, 1989; Harel & Brown, 2008; Movshovitz-Hadar, 1993) and described the understandings that undergraduate students, including preservice teachers, have of induction (*e.g.*, Stylianides, Stylianides, & Philippou, 2007). Harel and Brown (2008) point out that in many standard instructional treatments of proof by mathematical induction (PMI), problems that exemplify the utility of the proof strategy can be categorized into *recursion* and *non-recursion* problems based on whether they involve recursive representations of functions or processes. The authors note that insistence on the use of PMI to solve non-recursion problems, in the absence of genuine intellectual necessity (Harel, 2013), can reinforce students' authoritative proof schemes.

One genre of problems that receives little attention in introduction-to-proof courses in the U.S. deals with *combinatorial games*, deterministic two-player games with perfect information (nothing concealed from either player). In a two-player combinatorial game with a finite set of possible game states that must terminate after a finite number of moves with one player winning, each game position can be characterized as affording a winning strategy to the next player to move (a *winning position*) or not (a *losing position*). The process of classifying positions in a game as winning or losing often involves *recursive reasoning* (Lannin, Barker, & Townsend, 2006), reasoning about cases of a problem by referencing previously established cases. This recursive reasoning can then be generalized to describe the set of all winning or losing positions for the game, and this reasoning can be distilled into a proof by mathematical induction that formally verifies and explains this description. We hypothesize that the genre of problems about combinatorial games offers a potential setting in which PMI can fulfill the explanation function of proof (De Villiers, 1990; Hanna, 2000). While problems involving combinatorial games have some complexity not associated with typical "textbook" problems, such as requiring complex induction hypotheses (incorporating assumptions about both winning and losing positions), we view them as possessing the exploratory nature and recursive structure needed to create intellectual necessity for inductive proof.

In analyzing students' work on problems that invite recursive reasoning about examples and eventually call for generalization of this reasoning, we categorize students' generalization activities into *result pattern generalization*, in which a general insight is obtained by observing regularity in results of calculations, or as *process pattern generalization*, in which this insight is backed by an understanding of regularity in the processes by which these results occur (Harel, 2001). We view

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process pattern generalization as a potential way of transitioning from recursive reasoning about specific cases of a problem to the development of a general inductive argument.

Guided by this framework, we address the following question: In what ways do students engage in recursive reasoning and generalization as they work on problems involving two-player combinatorial games, and to what extent do they formalize their justifications using PMI?

Method of Study

We report results of a case study (Yin, 2017) of students' collaborative work on a sequence of problems involving combinatorial games, with each case consisting of the mathematical discussion, arguments, and written and visual representations of a group of four students. The study took place in an informal summer mathematics program for students ages 13-18.

We selected sixteen students in the program who reported low levels of prior familiarity with puzzles about Nim-like games (based on an initial questionnaire), and assigned them to four groups of four students each. Each group then participated in a video-recorded task-based interview lasting approximately an hour and a half. Each group worked collaboratively on a sequence of five tasks, each of which asked participants to analyze a Nim-like two-player combinatorial game. The first three tasks are shown in Figure 1. At the conclusion of each problem, a researcher asked the students to summarize and justify their conclusions orally, and asked questions as needed to clarify our understanding of students' reasoning.

Problem 1
Player A and Player B are playing a game where there is one pile of 10 marbles. Each player is allowed to take 1, 2, or 3 marbles from the pile on each turn. A player wins if they take the last marble. If Player A goes first and both players play as well as possible, which player will win, and how? **Justify your answer.**

Problem 2
Player A and Player B are playing a game where there is one pile of *N* marbles. Each player is allowed to take 1, 2, or 3 marbles from the pile on each turn, with Player A taking the first turn. A player wins if they take the last marble. For what values of *N* (the starting number of marbles in the pile) can Player A guarantee victory? How can Player A win in these cases? **Justify your answer.**

Problem 3
Player A and Player B are playing a game where there is one pile of *N* marbles. Each player is allowed to take any number of marbles that is a power of 2 (1, 2, 4, 8, ...); Player A has the first turn. A player wins if they take the last marble. For what values of *N* (the starting number of marbles in the pile) can Player A guarantee victory? How can Player A win in these cases? **Justify your answer.**

Figure 1: The first three problems in the task sequence.

Our analysis focuses on three groups' work on Problems 1 through 3 (Figure 1). The three groups in our analysis worked on these three problems for a total of 35 minutes, 29 minutes, and 45 minutes, respectively. Because our study focuses on students' generalization and justification practices, we transcribed the segment of each group's work on each of Problems 2 and 3 from the first time a group made a generalization or conjecture about the problem to the time when their work on the problem ended. We then analyzed each group's generalization processes and attempts to justify their conjectures, noting the degree to which each group formalized its reasoning using induction. We adopt the perspective that students' justification attempts offer some evidence of what they consider to be compelling arguments about winning strategies in combinatorial games, and in particular, of what aspects of PMI they are motivated and able to adapt to proving processes in this context.

Results and Analysis

In this section we provide a detailed description of one group's work on Problems 1 through 3 and briefly summarize the work of the two other groups in our analysis.

The Case of Group 1: Evidence of Process Pattern Generalization

Group 1, consisting of Bridget, William, Grace, and Ryan, used the names "John" and "Fluffy" in place of the letters A and B to refer to the two players. They used tables throughout the problem session to represent possible sequences moves in games, and gradually began to use these tables to represent branching cases that could occur within the same game.

After finding in Problem 1 that the first player can win by ensuring that the other player always receives a multiple of four marbles, the group moved on to Problem 2 and began categorizing possible starting positions as winning or losing for Player A. In the group's initial work on this problem, after identifying some examples of initial positions that are winning for Player A, Grace stated the conjecture, "So it's like -- basically, all the numbers except for the multiples of 4, because in that case, Fluffy would win." We interpret this as result pattern generalization, since Grace appeared to obtain this insight from a table of cases the group had considered, and because the group had not publicly justified all of its previous claims about winning positions for the first player.

The group chose to continue considering specific examples to gather evidence for this conjecture. Ultimately, the group returned to its conjecture and finalized it by writing it on the board: "if $N \equiv 1$, 2, or 3 (mod 4), then Player A will win (by generalization of P1). If $N \equiv 0$ (mod 4), then Player B will win." The group then debated whether to provide a written proof of this conjecture; while William remarked that "If we wanted to prove that, we could probably use induction or something," Grace indicated that this was not needed since they had already explained that their conjecture was true "by generalization of the first problem." We interpret this segment of discussion as indicating that the group saw an opportunity to formalize their argument using mathematical induction, but found such formalization to be unnecessary in this case, possibly because of the similarity between the reasoning used in Problem 2 and that used for the more specific Problem 1. We hypothesize that Problem 2 did not create intellectual necessity for PMI for this group; a proof by induction would not have done more to convince this group that their conjecture was universally valid.

On Problem 3, after testing the cases $N \leq 12$, the group correctly conjectured that Player A has a winning strategy if and only if N is not divisible by 3. The group then went on to write an argument justifying this conjecture:

> If 3 [does not divide] N, then Player A will win. If 3 [does not divide] N, then N ≡ 1 or 2 (mod 3), so Player A can leave Player B w/ a multiple of 3. If Player B has a multiple of 3, then B can only remove 1 mod 3 or 2 mod 3 marbles, leaving A w/ 1 mod 3 or 2 mod 3 marbles, so A can win.

Prior to the group's production of this argument, William had stated that each power of 2 is congruent to 1 or 2 modulo 3; this allowed the group to reason that if Player B receives a multiple of three marbles, then any move by Player B will reduce the number of marbles to a non-multiple of 3. In this argument we see evidence of process pattern generalization based on the group's work on specific examples: both the insight that a power of 2 cannot be a multiple of 3 (and that this is important in limiting what a player can do if given a multiple of 3), and the strategy of reducing the number of marbles to a multiple of 3. However, the argument as written does not explain why "A can win" after receiving a smaller number of marbles congruent to 1 or 2 modulo 3 (and one can envision this smaller number being outside of the range of specific examples that the group tested directly).

The Cases of Groups 2 and 3: Obstacles to Process Pattern Generalization

Like Group 1, Groups 2 and 3 correctly determined that in Problem 2, Player A has a winning strategy if and only if *N* is not a multiple of 4. Neither group used induction to ground claims that a first move for Player A would place Player B in a losing position; they instead referenced their prior work on Problem 1 and examples they explored in Problem 2. We hypothesize that the regularity of winning and losing positions in Problem 2 led both groups to the belief that a formal proof by induction was not essential for justification of their conjecture.

While both Groups 2 and 3, like Group 1, arrived at a correct answer to Problem 3, their attempts at justification differed significantly. In attempting to show in general that a multiple of 3 would be a losing position in this game, Group 2 did not take into account possible moves for the next player other than taking 1 or 2 marbles, even though they did account for other possible powers of 2 in the preliminary example work that led to their conjecture. Therefore, while Group 2 was able to render a partial explanation of how Player A could seize a winning strategy by taking 1 or 2 marbles if *N* is not a multiple of 3, their argument did not fully demonstrate that this first move would put Player B in a losing position. Group 3 attempted to prove its conjecture for Problem 3 using PMI, but in doing so, they attempted to establish values of *N* as winning or losing positions by proving the false claim that an integer *N* can be written as either a sum of an even number of powers of 2 or the sum of an odd number of powers of 2, but not both. They embarked upon this strategy despite the fact that expressing an integer as a sum of powers of 2 had not been a key part of their reasoning about specific examples that led to their conjecture.

Discussion and Implications

The results of our interviews are not necessarily indicative of students' ability to use PMI to generalize and formalize recursive reasoning. We did not require students to produce formal proofs during the interviews, so any attempts to use induction or other proof techniques reflected students' desire to confirm or formalize a result, or their sense that we wished for them to do so. In a future study we hope to ask groups of students to work on the same sequence of problems in one task-based interview, then ask them in another interview to write formal proofs of their results.

Nonetheless, we claim that observing students' work on combinatorial game problems provides useful insight about obstacles that may hinder students' efforts to make the cross-cultural translation between the empirical and recursive reasoning that often occurs naturally in exploratory mathematical activity, and the formal inductive justification accepted by the mathematics discipline. First, regularity in the structure of a problem may eliminate intellectual necessity for formal proof by induction. Second, our preliminary results suggest that students may have difficulty translating the recursive reasoning used in specific examples into a proof by induction; and in fact, they may use reasoning disanalogous to their prior reasoning when attempting a proof by induction. Finally, when students do successfully transfer their recursive reasoning into an induction argument, some work may be needed to impress upon students a sense of the importance of a base case, and the function of the structure of an induction proof in providing a foundation on which higher-order cases can rest on lower-order cases.

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STUDENTS' UNDERSTANDING OF LINEAR ALGEBRA CONCEPTS UNDERLYING A PROCEDURE IN A QUANTUM MECHANICS TASK

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As undergraduate physics students solve problems related to quantum mechanics, they often have to draw on their conceptual understanding of both linear algebra and physics concepts. One concept that is prevalent in both of these disciplines is change of basis. In this study, we focus our analysis on physics students' mathematical conceptual knowledge of principles that underlies their use of a procedure on a quantum mechanics task. As students perform this task, they demonstrated understanding of change of basis and properties of orthonormal bases. We share examples of students' reasons for performing a change of basis and demonstrate how students seem to draw on their conceptual understanding of properties of orthonormal bases as they implement the procedure. We thus exemplify how physics students apply their mathematical knowledge in interdisciplinary contexts.

Keywords: Interdisciplinary studies, STEM / STEAM, University mathematics

Undergraduate physics students are commonly required to take a course in linear algebra, seen as several concepts and procedures covered in a linear algebra course are also addressed in physics courses, particularly quantum mechanics. Physics students have to make connections between concepts covered in both linear algebra and physics contexts. This may be challenging for students, given that concepts included in both linear algebra and quantum mechanics courses are sometimes used differently in the two different contexts. One such example is that bases are typically orthonormal in quantum mechanics contexts (McIntyre, Manogue, & Tate, 2012), which is generally not the case in linear algebra. Furthermore, a change of basis in a quantum mechanics context may be performed differently than a change of basis in a linear algebra context, such as through formulaic substitution of commonly used bases rather than a change of basis matrix. Since instructors want students to have a cohesive understanding of basis and change of basis across these interdisciplinary contexts, it is useful to explore how students reason about change of basis in a quantum mechanics context.

As students solve problems related to quantum physics, they often have to draw on their conceptual understanding of both mathematics and physics. In this study, we focus our analysis on physics students' mathematical conceptual knowledge that underlies their use of a procedure on a quantum mechanics problem. We address the following research question: How do students use their mathematical conceptual knowledge as they perform a quantum mechanics task?

Theoretical Framing

Conceptual and procedural knowledge are constructs commonly used by researchers to characterize students' understanding of mathematical concepts. Rittle-Johnson, Schneider, and Star (2015) have theorized that bidirectional relations exist between students' conceptual knowledge and procedural knowledge. These researchers asserted that procedural knowledge can support students' development of conceptual knowledge and vice versa. Particularly, conceptual knowledge can support students' flexibility in choosing appropriate procedures (e.g., Baroody, Feil, & Johnson, 2007). Crooks and Alibali (2014) identified two main types of conceptual knowledge, general principle knowledge and knowledge of principles underlying procedures. *Knowledge of principles underlying procedures* involves understanding the "connections among the steps in a procedure and between individual

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steps and their conceptual underpinnings" (p. 367). In this study, we use Crooks and Alibali's (2014) construct of knowledge of principles underlying procedures to frame our analysis of students' conceptual knowledge of the mathematical concepts underlying their use of a procedure in quantum mechanics problem.

Brief Physics Background

Quantum mechanical systems and all knowable information about them are represented mathematically by normalized kets, symbolized in Dirac notation as $|\psi\rangle$. Kets mathematically behave like vectors, and a ket's complex conjugate transpose, called a bra, is symbolized as $\langle\psi|$. Spin is a measure of a particle's intrinsic angular momentum and is represented mathematically by an operator such as $\hat{S}_z$ (where the z indicates the particle's axis of rotation). In a spin-½ system, there are two possible results for the S_z measurement: $\pm\frac{\hbar}{2}$; they correspond to $|+\rangle$ and $|-\rangle$, which comprise a set of orthonormal basis vectors called the S_z basis. Any quantum state $|\psi\rangle$ is a linear combination of them: $|\psi\rangle = a|+\rangle + b|-\rangle$. In this study, we analyze the students' responses to the two interview questions presented in Figure 1. Interview question (a) asks students to determine the probability of obtaining $\frac{\hbar}{2}$ or $\frac{-\hbar}{2}$ in a measurement of the observable S_z on a system in state $|\psi\rangle$. This is calculated by $P_{\pm} = |\langle\pm|\psi\rangle|^2$, where $\langle\pm|\psi\rangle$ is an inner product between one of the basis kets and psi. Because $|\psi\rangle$ is written as a linear combination of the two vectors that comprise the z-basis, solving this problem requires no change of basis. The analogous information can be determined for other axes of rotation, such as y. To complete question (b), a change of basis is involved because the given state vector $|\psi\rangle$ is written in terms of the z-basis, but the prompt asks for the probability that the spin component is up along the y-axis. The two main approaches are to either change $|\psi\rangle$ to be written in terms of the y-basis (denoted $|\pm\rangle_y$), or change the y-basis vectors to be written in terms of the z-basis. In either change of basis approach, one would need to utilize the relations, $|\pm\rangle_y = \frac{1}{\sqrt{2}}|+\rangle \pm \frac{i}{\sqrt{2}}|-\rangle$.

Consider the quantum state vector $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$.

Calculate the probabilities that the spin component is up or down along the z-axis.

Calculate the probabilities that the spin component is up or down along the y-axis.

Figure 1: The interview questions analyzed in this paper.

Methods

Semi-structured interviews (Bernard, 1988) were conducted with 12 quantum physics students, of which eight were enrolled in a junior-level course at a large public research university (A) in the northwest United States, and four were enrolled in a senior-level course at a medium public research university (C) in the northeast United States. Students from these two universities were assigned pseudonyms of A# and C#. Both courses used a "spins first" approach, with McIntyre et al. (2012) as their course textbook. The interview questions were designed to elicit evidence of student understanding of linear algebra concepts used in their quantum mechanics course. We analyzed the students' responses to the two interview questions presented in Figure 1. A relevant follow-up question of particular interest was: "How do you see this problem relating to basis or change of basis?"

The data collected from these interviews include video recordings of the interviews and copies of student work. We transcribed the students' responses to these tasks and wrote descriptions of the procedures the students used and the mathematical concepts the students discussed as they explained their thought process and justified their choice of procedure. In writing these descriptions of the students' interview responses, we noticed the students commonly referred to mathematical concepts of orthonormal bases, inner product, and change of basis. We thus focused our qualitative analysis on students' conceptual understanding of these mathematical concepts underlying the probability procedure in this quantum physics context. We performed open coding (Miles, Huberman, & Saldaña, 2013) of the procedures the students used and the mathematical concepts they demonstrated an understanding of in their responses to the interview questions. We then wrote analytic memos (Maxwell, 2013) reflecting on how the students' understanding of linear algebra concepts seemed to support their use of the procedure.

Results

We present our results in two subsections. We first discuss the procedures the students employed to solve task. We then discuss the nature of the conceptual understanding the students exhibited as they performed this procedure and justified their procedure choice.

Students' Procedures for the Task Requiring a Change of Basis

In task (b), the students could either change $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ to be written in terms of the y basis or change $|+\rangle_y$ (and $|-\rangle_y$for spin down) to be written in terms of the z-basis. Three out of twelve students used the former approach, but only one did so correctly. This student, A8, added the change of basis equations (given on a reference sheet) $|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + i\frac{1}{\sqrt{2}}|-\rangle$ and $|-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle - i\frac{1}{\sqrt{2}}|-\rangle$ to find $|+\rangle_y + |-\rangle_y = \frac{2}{\sqrt{2}}|+\rangle$. He then multiplied both sides of the equations by $\frac{\sqrt{2}}{2}$ to find $\frac{1}{\sqrt{2}}|+\rangle_y + \frac{1}{\sqrt{2}}|-\rangle_y = |+\rangle$, which is a z-basis vector written as a linear combination of the y-basis vectors. He then subtracted the given change of basis equations to find $|+\rangle_y - |-\rangle_y = \frac{2}{\sqrt{2}}i|-\rangle$ and multiplied both sides by $\frac{\sqrt{2}}{2i}$ to find $\frac{1}{\sqrt{2}i}|+\rangle_y - \frac{1}{\sqrt{2}i}|-\rangle_y = |-\rangle$. He then substituted $|+\rangle = \frac{1}{\sqrt{2}}|+\rangle_y + \frac{1}{\sqrt{2}}|-\rangle_y$ and $|-\rangle = \frac{1}{\sqrt{2}i}|+\rangle_y - \frac{1}{\sqrt{2}i}|-\rangle_y$ into $|\psi\rangle = \frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle$ and simplified the equation to find $|\psi\rangle_y = \frac{5}{\sqrt{26}}|+\rangle_y + \frac{1}{\sqrt{26}}|-\rangle_y$. To calculate the probability that the spin component is up along the y-axis, he computed $\left|{}_y\langle+|\psi\rangle_y\right|^2 = \frac{25}{26}$ by squaring the coefficient of $|+\rangle_y$, taking for granted the fact that ${}_y\langle+|+\rangle_y = 1$ and ${}_y\langle+|-\rangle_y$= 0 because the basis vectors are orthonormal.

For the calculation in question (b), ten students changed $|+\rangle_y$ to be written in terms of the z- basis, and all of them used the appropriate change of basis formula $|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + i\frac{1}{\sqrt{2}}|-\rangle$. Seven of the ten students who performed the rest of the procedure found the conjugate transpose as ${}_y\langle+| = \frac{1}{\sqrt{2}}\langle+| - \frac{1}{\sqrt{2}}i\,\langle-|$ and substituted this into the probability formula $P_{+,y} = \left|{}_y\langle+|\psi\rangle\right|^2$ to find $P_{+,y} = \left|(\frac{1}{\sqrt{2}}\langle+| - \frac{1}{\sqrt{2}}i\,\langle-|)(\frac{3}{\sqrt{13}}|+\rangle + \frac{2i}{\sqrt{13}}|-\rangle\,)\,\right|^2$. They then distributed and used the orthonormality properties, $\langle+|+\rangle = 1$, $\langle-|-\rangle = 1$, and $\langle+|-\rangle = 0$ to simplify the equation to be $P_{+,y} = \left|\frac{5}{\sqrt{26}}\right|^2$, which gave a probability of $\frac{25}{26}$.

Students' Conceptual Knowledge of Linear Algebra Underlying These Procedures

As the students performed this procedure, they demonstrated conceptual knowledge of the linear algebra concepts of change of basis and properties of orthonormal bases. All of the students recognized a need to change the basis for this problem. They often explicitly acknowledged that they could not calculate this inner product without performing a change of basis. For instance, A13 claimed, "you can't just multiply out in, when they're [the vectors are] in different bases." As C6 described how this problem was related to change of basis, she claimed, "you can't do anything until you're in the same basis." The students each demonstrated conceptual understanding of why a change of basis was necessary in this task. Other students noted that performing the change of basis made the calculation simpler. For instance, in A8's change of basis procedure described above, changing the basis of $|\psi\rangle$ to express the ket as a linear combination of y-basis kets allowed him to use the simple procedure of squaring the norms of coefficients of y-basis kets. Students who used the second procedure discussed how changing basis makes the calculations simpler because they can use orthonormality property of the bases.

The students' motivation for selecting the procedure of changing the basis was the ability to take advantage of these properties of an orthonormal basis. A21 claimed it was necessary to change basis in order to make assumptions about $\langle +|-\rangle = 0$. C5 also suggested that a change of basis was necessary for the "inner products to be nice." The students demonstrated conceptual understanding of properties of orthonormal bases, which underlie their procedures for this task. Their procedures were particularly dependent on the fact that the y-basis and the z-basis are both orthonormal, which implies that $\langle +|+\rangle = 1$, $\langle -|-\rangle = 1$, and $\langle +|-\rangle = 0$.

Discussion

As the students used the change of basis procedure in this quantum mechanics problem, they demonstrated conceptual understanding of the orthogonality and normality properties of basis vectors and the associated inner product relations of $\langle +|+\rangle = 1, \langle -|-\rangle = 1,$ and $\langle +|-\rangle = 0$. The students seemed to draw on their understanding of the principles underlying this procedure as they performed the probability calculation and justified their choice of procedure. Their conceptual understanding of these mathematical properties seemed to support them in their choice of procedure and their implementation of it. Baroody et al. (2007) suggested that students' conceptual knowledge can supports students' flexibility in applying procedures. Our study further illustrates how students' conceptual understanding of linear algebra concepts can be useful in supporting their flexibility in performing procedures in quantum mechanics contexts.

In this quantum mechanics problem, the students' change of basis approaches involved employing several mathematical properties. However, their method for performing the change of basis though algebraic substitution is different from how they would perform a change of basis in a linear algebra course, which typically involves the use of a change of basis matrix. Physics students may experience challenges in using their understanding of linear algebra to solve problems in quantum mechanics contexts. Therefore, we suggest that future research can address how physics students transfer their understanding of change of basis across linear algebra and quantum mechanics contexts. Researchers can particularly focus on how physics students resolve differences they experience as they make connections across these disciplines.

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"SOLVING VERSUS RELATING": PRE-SERVICE TEACHERS' CONFLICTING IMAGES OF FORMULAS AND DYNAMIC CONTEXTS

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Researchers have identified both the affordances of engaging students in symbolization activities and students' difficulties in meaningfully representing contexts through algebraic expressions/formulas. In a semester-long teaching experiment, two pre-service teachers demonstrated their conflicting meanings for formulas with their images of a context when engaging in a task about a dynamic geometric object. The two students could construct both normative formulas by reasoning with a context and descriptions of covariational relationships between quantities within the context, but both still struggled to relate their formulas and quantitative relationships to one another. This result highlights the importance of attending to what students' formulas mean to them, which for the students in this study, could be either a way of "solving" or "relating" quantities.

Keywords: Precalculus, Algebra and Algebraic Thinking, Cognition, Teacher Education - Preservice

Several researchers have identified students' difficulties with symbolization within formulas, equations, etc. and others have illustrated students' ability to construct their own representational systems (e.g., Izsák, 2003). To support pre-service teachers in working with their future students, it is important to start with understanding what they know and similarly, understanding where perturbations (i.e., cognitive conflict) might occur. In this study, I explore two secondary mathematics pre-service teachers' (heretofore, students') meanings for formulas, particularly focusing on the relationship between students' images of context (i.e., the quantities they construct within contexts) and their associated formulas. To do so, I draw on two main bodies of research: symbolization activity and covariational reasoning—reasoning involving how two quantities change in tandem with each other (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). The research objective of this study builds on these researchers' findings through a teaching experiment with two students and is focused on learning about the mental operations involved in students' construction of formulas via reasoning with dynamic objects. Specifically, here, I focus on the students' meanings for an area formula for a parallelogram. The two students expressed conflicting meanings for a formula and its associated context. I describe these two students' meanings for their formulas and discuss the implications on students' symbolization activity based on their conflicting meanings and how they resolved them.

Background and Theoretical Perspective

In an effort to distinguish between terms used throughout the results section, I adopt Thompson and Carlson's (2017) definitions of constants and variables as students envisioning the following: a *constant* is an image involving a quantity as having a value that does not vary ever and a *variable* involves a quantity's value varying within a setting. I further define an *undetermined constant* as one in which the individual considering a quantity has not established a unit of measure, but anticipates needing to do so in order to produce a value (cf., *unknown* constant). The definition of variable relates to the notion of covariational reasoning proposed (i.e., the quantities co-varied are variables) and is also compatible with Küchemann's (1981) definition of a variable as a letter "seen as representing a range of unspecified values, and a systematic relationship is seen to exist between two such sets of values" (p. 104). Lastly, I emphasize that an individual using a letter (or any other marking) as a symbol for a constant, parameter, or variable requires the individual to re-present that

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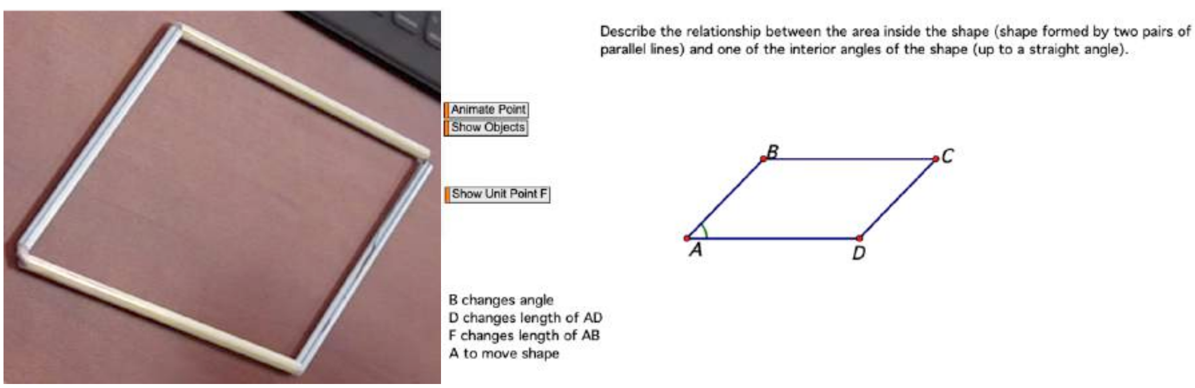
letter as a quantity or quantitative relationship within a situation. That is, a letter in itself is not a representation; the marking is the figurative material that results from an individual’s operations.

Methods and Task

This study was part of a semester-long teaching experiment (Steffe & Thompson, 2000) with two secondary mathematics pre-service teachers at a large public university in the southeastern U.S. The students were selected from a pre-calculus secondary content course based on their results of a modified version of the *MMTSM* assessment (Thompson, 2012) and a pre-interview showing that the students had differing ways of reasoning about quantitative relationships. This report focuses on a task that occurred in Lily’s hour-long teaching sessions 9-12 (of 12) and Dahlia’s teaching sessions 6-9 (of 10). As a result of open and axial coding of the video recordings and transcripts of the lessons and applying the definitions for constants and variables, I describe how Lily and Dahlia constructed and interpreted their conflicting images.

I use the *Moving Angles Task* to discuss students’ representational activity and construction of conflicting images. In this task, students were given the manipulative in Figure 1a and the prompt, “Describe the relationship between the area inside the shape (shape formed by two pairs of parallel lines) and one of the interior angles of the shape (up to a straight angle).” After initial discussions, both students received a sketch in a dynamic geometric environment (DGE) (Figure 1b) to support their exploration (which also included dynamic magnitude bars which are outside of the scope of this paper). The data was analyzed using generative and axial approaches (Corbin & Strauss, 2008) in order to construct models of the *mathematics of the students* (Steffe & Thompson, 2000). For a more detailed description of this task and insights into how this task has supported PSTs’ covariational reasoning with equations, see Stevens (2018).



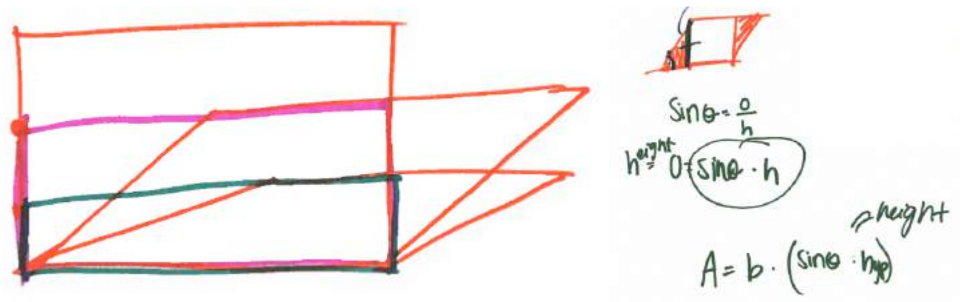
(a) **(b)**

Figure 1: The Moving Angles Task (a) manipulative and (b) sketch within a DGE

Results

Lily’s Conflicting Images

Lily originally focused on exploring the covariational relationships between angle measure (specifically for $\angle DAB$), height of the parallelogram, and the area of the parallelogram. After much deliberation, she concluded (as illustrated in Figure 2a) that equal changes in height corresponded with equal changes in area (i.e., “when the height was partitioned in decreasing equally, so was the area”), and in turning the angle clockwise from a right angle, the angle “decreases by decreasing amounts.”



(a) **(b)**

Figure 2: (a) Lily’s exploration of covariational relationships within a context (b) Lily’s reasoning with a static parallelogram

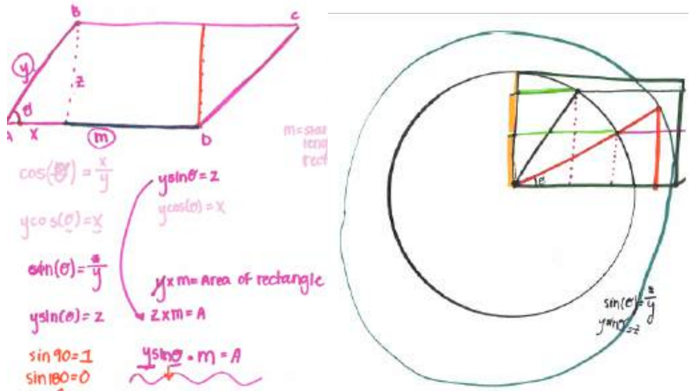
After her exploration, I asked Lily to “write an equation that represents the relationship that you’re talking about between the angle measure and the area of the parallelogram.” After several minutes, Lily drew a new parallelogram and stated that, “You can find this height [*of the parallelogram*] using sine and cosine” and produced the normative formula A=b(sin(θ)*hyp), where A=area of parallelogram, θ= m∠DAB in Figure 1b, and hyp= length of the hypotenuse in triangle in Figure 2b. In contrast to her drawing in Figure 2a, Lily said she did not see angle measure and height changing in her drawn parallelogram in Figure 2b. I interpret this description to indicate that Lily re-presented the symbols in her formula as undetermined constants of quantities she constructed from a static shape.

Lily indicated that her formula conflicted with her image of the relationships between quantities in the dynamic context, “Because I don’t know how to talk about it when I know this is true [*pointing to statement that from 0 to 90 degrees, angle measure increases so height increases so area increases*]. I don’t know how to relate it [*her statement*] to this part [*her parenthetical in her formula*].” She stated that the confusion stemmed from her understanding of her formula (Figure 2b) as “just solving, not relating,” where relating referred to seeing quantities (i.e., angle measure, height) as changing. In sum, to her, Lily, in constructing her formula, thought she was appropriately re-presenting a procedure for calculating area measures for static parallelograms but not the covariational relationships she constructed through her reasoning with the dynamic parallelogram.

Dahlia’s Conflicting Images

Like Lily, Dahlia identified a non-linear relationship between angle measure and area of the given shape, and she constructed a formula similar to Lily’s formula in Figure 2b using similar reasoning (Figure 3a). Unlike Lily, Dahlia also provided a unit circle meaning for sine and re-presented the segment corresponding to the hypotenuse of the right triangle in Figure 3a also as both a hypotenuse of a triangle and the radius of the circle (see Figure 3b). Moreover, she re-presented a relationship between *changing* quantities within her drawn parallelogram and formula; she described y as “not moving” and z and θ as “changing” in her figure and formula.

Nevertheless, Dahlia could not answer the question, “Why would we multiply a portion of the radius [*her description of* sin*(θ)*] by the radius [*her description of y*]?”. Thus, although Dahlia could construct a formula that re-presented varying quantities in a situation by reasoning with trigonometric relationships, her formula still conflicted with her image of the context. This conflict occurred because she thought that to calculate the area of a parallelogram, she would need to multiply the base length and height of the parallelogram together, but her formula indicated to her that the side length, h, of the parallelogram was also needed to obtain an area measurement. Thus, although Dahlia thought she appropriately re-presented her image of dynamic quantities in the context as a formula based on her reasoning with trigonometric ratios, she struggled to relate the symbols to her image of the context.



(a) **(b)**

Figure 3: (a) Dahlia’s formula for the area of a parallelogram with cosine related work dimmed for the reader (NOTE: m refers to AD, not the underlined segment in a different color) and (b) Dahlia’s diagram showing a unit circle approach for the sine relationship

Discussion and Implications

Researchers have often praised the symbolization activity of students developing their own symbols through representational activity, and this study is not an effort to discourage the use of contexts to support students’ meanings for formulas, equations, symbols, etc. Rather, this study indicates the importance of attending to the ways in which students’ symbolization activity re-presents their images of quantities and their relationships within given contexts. More specifically, Lily’s example indicates the importance of attending to students’ meanings for formulas (equations, functions, etc.) as ways to “solve for” or “figure out” values for constant quantities within static situations. This way of thinking about formulas was problematic for Lily even when she produced a normative formula because, for her, she was not re-presenting relationships between changing quantities with this formula. More generally, this view of formulas is problematic in students’ construction of variables because variables occur when a student re-presents values varying within a (dynamic) setting. Lastly, Dahlia’s example points to the importance of understanding students’ construction and role of units within their symbolization activity, particularly in regards to measurement contexts. By perturbing these meanings for formulas by attending to, for example, the role of units or the idea of a symbol as representative of a variable, students can accommodate their meanings for formulas to fit with their images of quantitative relationships in the context.

Acknowledgments

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PRECALCULUS, CALCULUS, OR HIGHER MATHEMATICS:

POSTER PRESENTATIONS

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FIRST FUNDAMENTAL THEOREM OF CALCULUS: HOW DO ENGINEERING STUDENTS INTERPRET AND APPLY IT?

PRIMER TEOREMA FUNDAMENTAL DEL CÁLCULO: ¿CÓMO LO INTERPRETAN Y APLICAN ESTUDIANTES DE INGENIERÍA?

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Palabras clave: Teorema fundamental del cálculo, Razonamiento, Método comparativo constante.

The first fundamental theorem of calculus relates differential and integral calculus, one of its important aspects according to Bressoud (2011) is that it shows the existence of two ways of calculating an integral: with the limit of a Riemann sum and by an antiderivative. Larsen, Marrongelle, Bressoud and Graham (2017) indicate that calculus is a barrier to the academic progress of many students and that there is a need for research that seeks to develop proposals for instruction to improve the understanding of its concepts. Therefore, with the idea of carrying out this type of research in the future, the present study seeks to identify the common interpretation of the first theorem of calculus and whether it is useful in solving contextual problems. Answering these questions will provide some elements to develop a proposal for instruction.

This study involved 18 students between the ages of 18 and 21 from engineering careers at a university located in Mexico City, who had completed a calculus course. The instrument was a set of three problems, in two, we propose contextual situations that can be solved by applying the first fundamental theorem of the calculus or by performing integration and derivation operations (one situation is about the ratio of change of the volume of water contained in a tank, with respect to time, where water falls to a variable ratio; the other is about the ratio of change of the volume of water contained in a cylindrical tank with respect to the height of water). In the last problem, the same type of situation is posed in abstract form: If $F(x) = \int_a^x f(t)dt$, obtain $F'(x)$, justify your answer.

The application of the instrument was done in a university classroom, participants were provided with the set of problems and paper sheets to write down their answers and were given 45 minutes to answer them. An analysis of students' responses to the problems was carried out using constant comparative method to form different categories, from which, one of the conclusions obtained is the following:

The common interpretation of the first fundamental theorem of calculus is that it establishes that integration and derivation are inverse operations, since in 71% of the answers of the shown problem (for which only 14 answers were given) used this argument. However, in 50% of these responses, mistakes are made by misusing notation when "cancelling" the inverse operations. This situation suggests that students, when applying the first fundamental theorem of calculus in the mentioned way, present a pseudo-conceptual behavior, (concept proposed by Vinner (1997)); since the way they proceed (making mistakes with the notation when "cancelling" the operations of integration and derivation when they appear together) seems to be linked to the algebraic representation of the situation ($\frac{d}{dx}\int_a^x f(t)dt$), not based on the concepts involved.

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PRIMER TEOREMA FUNDAMENTAL DEL CÁLCULO: ¿CÓMO LO INTERPRETAN Y APLICAN ESTUDIANTES DE INGENIERÍA?

FIRST FUNDAMENTAL THEOREM OF CALCULUS: HOW DO ENGINEERING STUDENTS INTERPRET AND APPLY IT?

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Palabras clave: Teorema fundamental del cálculo, Razonamiento, Método comparativo constante.

El primer teorema fundamental del cálculo relaciona el cálculo diferencial e integral, uno de sus aspectos importantes según Bressoud (2011) es que muestra la existencia de dos maneras de calcular una integral: con el límite de una suma de Riemann y con una antiderivada. Además, señala que se trata de un tema conceptualmente complejo. Larsen, Marrongelle, Bressoud y Graham (2017) indican que el cálculo es una barrera para el progreso académico de muchos estudiantes y que hace falta investigación que busque desarrollar propuestas de instrucción para mejorar la comprensión de sus conceptos, por lo que con la idea de realizar en un futuro una investigación de este tipo, se realiza el presente estudio buscando identificar cuál es la interpretación común que tienen los estudiantes sobre el teorema fundamental del cálculo y si ésta resulta útil al resolver problemas contextuales. Contestar estas preguntas aportará algunos elementos para desarrollar una propuesta de instrucción.

En este estudio participaron 18 estudiantes de entre 18 y 21 años de las carreras de ingeniería de una universidad ubicada en la ciudad de México, los cuales habían concluido un curso de cálculo. El instrumento fue una serie de tres problemas, en dos, se plantean situaciones contextuales que se pueden resolver aplicando el primer teorema fundamental del cálculo o realizando las operaciones de integración y derivación (una situación es sobre la razón de cambio del volumen de agua contenido en un depósito, respecto del tiempo, donde el agua cae a una razón variable; la otra es sobre la razón de cambio del volumen de agua contenido en un depósito cilíndrico respecto a la altura del agua). En el último problema, se plantea en abstracto el mismo tipo de situación: Si $F(x) = \int_a^x f(t)\,dt$, obtenga $F'(x)$, justifique su respuesta.

La aplicación del instrumento se realizó en un aula de la universidad, se les proporcionó a los participantes la serie de problemas y hojas de papel para escribir sus respuestas, y se les dio 45 minutos para contestarlo. Se realizó el análisis de las respuestas de los estudiantes a los problemas mediante el método comparativo constante para formar diferentes categorías, de donde, una de las conclusiones obtenidas es la siguiente:

La interpretación común de los estudiantes sobre el primer teorema fundamental del cálculo es que establece que la integración y derivación son operaciones inversas, pues en el 71% de las respuestas del problema mostrado (para el cual solo se tuvieron 14 respuestas) se utiliza dicho argumento. Sin embargo, en el 50% de estas respuestas se cometen errores al hacer uso inadecuado de la notación al momento de "cancelar" las operaciones inversas. Esto sugiere que los estudiantes al aplicar el primer teorema fundamental del cálculo de la forma mencionada, presentan un *comportamiento pseudo-*

conceptual, (concepto propuesto por Vinner (1997)); pues la forma en que proceden (cometiendo errores con la notación al "cancelar" las operaciones de integración y derivación cuando aparecen juntas) parece estar ligada a la representación algebraica de la situación ($\frac{d}{dx}\int_a^x f(t)\,dt$), no fundamentada en los conceptos involucrados.

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THE VARIATION, COVARIATION AND THE REFERENCE SYSTEM IN THE CONSTRUCTION OF CARTESIAN IDEAS

LA VARIACIÓN, LA COVARIACIÓN Y EL SISTEMA DE REFERENCIA EN LA CONSTRUCCIÓN DE IDEAS CARTESIANAS

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In the present investigation, the aim is to characterize and analyze the role of those elements necessary for the construction of Cartesian ideas in secondary school students (13-15 years old). After a bibliographic review about the construction and interpretation of Cartesian graphs, it was found that those elements that characterize a graphical functional thought are determined by: covariation, variation and unification of a reference system (Clement, 1989; Caballero, 2012; Radford, 2008). The notion of co-variation is defined as the relationship between the simultaneous variations of two quantities (Ferrari, 2005), while the variation in our research is characterized as the quantification of a change, that is, the modification of state, appearance, behavior, condition of a body, system or object (Caballero, 2012). Finally, the unification of a Reference System refers to unifying the phenomenological space with the Cartesian space, resulting in the starting point of a movement (Radford, 2009a). This refers to the point from which it is possible to define positions, organize actions and interpretations.

According to the above, in the present investigation the following question is posed: what is the role of variation, covariation and the unification of a reference system in the development of Cartesian ideas associated with graphs? So, the objectives that are set are: to propose a didactic alternative to start the study of the notion of function / graph different from the strategy that is commonly approached within the school mathematical discourse: the equation-table-graph triad.

Likewise, to develop meanings in the students regarding the notion of function and graph, articulating the notions of variation, co-variation and frame of reference, with the support of educational technology such as graphing calculators and motion sensors.

This ongoing research is supported by the Theory of Objectification, which conceives teaching and learning as a single process that involves both knowing and being, where the objective of mathematical education lies in a political, social, and historical effort. and cultural aimed at the creation of reflective, ethical and critical subjects in historically and culturally constituted mathematical practices, and that reflect on new possibilities of those practices (Radford, 2019). Knowledge is developed in human activity, which is called joint work since it is a social form of joint effort through which individuals produce their means of subsistence while producing themselves as human beings (Radford, 2009b).

This research has a qualitative cut, since it seeks to analyze the productions carried out by a group of secondary school students (13-15 years old) in four guided activities that involve the articulation of the three notions previously reported, with the support of motion sensors in order to link them and arrive at Cartesian notions associated with a graph.

Currently the research is in process and an experimental instrument consisting of an activity where the articulation of the three notions is proposed has been developed. In this instrument, it is proposed by asking students to freely describe the movement produced by a cyclist when going down a hill that will be previously drawn. Another type of task within the activity will be to ask students to draw two hills with the same height but different gradients and they will ask themselves in which of them the cyclist will be able to descend faster and why.

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For the analysis of the students' productions, a multimodal analysis will be carried out in which the cognitive, body and perceptual resources used by the students are considered, that is, dialogues, body movements and their written representations will be analyzed (Vergel, 2016). This is expected to have results that allow a reflection on the importance of the three elements described to strengthen the construction of Cartesian ideas associated with graphs in basic level students.

LA VARIACIÓN, LA COVARIACIÓN Y EL SISTEMA DE REFERENCIA EN LA CONSTRUCCIÓN DE IDEAS CARTESIANAS

THE VARIATION, COVARIATION AND THE REFERENCE SYSTEM IN THE CONSTRUCTION OF CARTESIAN IDEAS

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Palabras clave: Gráficas cartesianas, variación, covariación, Sistema de referencia

En la presente investigación se pretenden caracterizar y analizar el rol de aquellos elementos necesarios para la construcción de ideas cartesianas en estudiantes de nivel Secundaria (13-15 años). Posterior a una revisión bibliográfica alrededor de la construcción e interpretación de gráficas cartesianas, se encontró que aquellos elementos que caracterizan a un *pensamiento funcional gráfico* están determinados por: la covariación, la variación y la unificación de un sistema de referencia (Clement, 1989; Caballero, 2012; Radford, 2008). La noción de co-variación se define como la relación entre las variaciones simultáneas de dos cantidades (Ferrari, 2005), mientras que la variación en nuestra investigación se caracteriza como la cuantificación de un cambio, es decir, la modificación de estado, apariencia, comportamiento, condición de un cuerpo, sistema u objeto (Caballero, 2012). Por último, la unificación de un Sistema de Referencia se refiere a unificar el espacio fenomenológico con el espacio cartesiano, resultando en el punto inicial de un movimiento (Radford, 2009a). Esto se refiere al punto a partir del cual es posible definir posiciones, organizar acciones e interpretaciones.

De acuerdo a lo anterior, en la presente investigación se plantea la siguiente pregunta: *¿cuál es el rol de la variación, covariación y la unificación de un sistema de referencia en el desarrollo de ideas cartesianas asociadas a las gráficas?* De manera que los objetivos que se plantean son: proponer una alternativa didáctica para iniciar el estudio de la noción de función/gráfica diferente de la estrategia que comúnmente se aborda dentro del discurso matemático escolar: la triada ecuación-tabla-gráfica. Asimismo, desarrollar significados en los estudiantes con respecto a la noción de función y gráfica, articulando las nociones de variación, co-variación y marco de referencia, con el apoyo de tecnología educativa como calculadoras graficadoras y sensores de movimiento.

Esta investigación en proceso se apoya en la Teoría de la Objetivación, la cual concibe la enseñanza y el aprendizaje como un único proceso que implica tanto el *saber* como el *ser*, donde el objetivo de la educación matemática reside en un esfuerzo político, social, histórico y cultural dirigido a la creación de sujetos reflexivos, éticos y críticos en prácticas matemáticas constituidas histórica y culturalmente, y que reflexionan sobre nuevas posibilidades de esas prácticas (Radford, 2019). El saber se desarrolla en la actividad humana, que se denomina labor conjunta ya que es una forma social de esfuerzo conjunto a través de la cual los individuos producen sus medios de subsistencia mientras se producen a sí mismos como seres humanos (Radford, 2009b).

Esta investigación tiene un corte cualitativo, ya que se busca analizar las producciones que realice un grupo de estudiantes de nivel Secundaria (13-15 años) en cuatro actividades guiadas que involucren la articulación de las tres nociones reportadas anteriormente, con apoyo de sensores de movimiento para poder vincularlos y llegar a nociones cartesiandas asociadas a una gráfica.

Actualmente la investigación se encuentra en proceso y se ha desarrollado un instrument experimental consistente en una actividad donde se plantea la articulación de las tres nociones. En dicho instrument se plantea por solicita a los estudiantes que describan de manera libre el movimiento que produce un ciclista al bajar por una colina que se les dibujará previamente. Otro tipo de tareas dentro de la actividad será la de solicitarle a los estudiantes que dibujen dos colinas con misma altura pero gradientes diferentes y se preguntará en cuál de ellas el ciclista podrá bajar más rápido y por qué.

Para el análisis de las producciones de los estudiantes se llevará a cabo un *análisis multimodal* en el cual se consideran los recursos cognitivos, corporales y perceptuales que utilizan los estudiantes, es decir, se analizarán diálogos, movimientos corporales y sus representaciones escritas (Vergel, 2016). Se espera con ello, contar con resultados que permitan una reflexión sobre la importancia de los tres elementos descritos para robustecer la construcción de ideas cartesianas asociadas a las gráficas en estudiantes de nivel básico.

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MATHEMATICAL MODELING FOR STUDYING THE CONCEPT OF INTEGRAL THROUGH AN AUTHENTIC PROBLEM

MODELACIÓN MATEMÁTICA PARA EL ESTUDIO DEL CONCEPTO DE INTEGRAL A TRAVÉS DE UN PROBLEMA AUTÉNTICO

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Keywords: Calculus, Modeling, University Mathematics.

The findings of the research are shown from the perspective of mathematical modeling in the classroom, as a study process of phenomena or situations that could surface from everyday social and cultural contexts of the students, or from other sciences (Villa-Ochoa, 2010). To answer the question *how mathematical modeling of authentic problems contributes to the study on the concept of integral by university students,* we assume a mathematical model as a set of mathematical representations and relations for explaining, predicting and solving aspects of a phenomenon (Villa-Ochoa y otros, 2009a). An authentic problem is proposed (Kaiser y Schwarz, 2010) from the analysis of a simulation in GeoGebra of the download speed of a file, following the phases of ACODESA methodology. (Hitt, 2007).

The research was carried out with the participation of four students (aged 16-19) from a Colombian public university who passed the differential calculus course and had access to Aula Virtual GeoGebra. For the analysis of the data, the responses and written productions of the students in the Aula Virtual, and the recording of the responses to the structured interview (Goldin, 200) applied on the synchronic encounters were taken into account.

As an initial finding, the models built allowed to recognize the integral as the accumulated size of the file that has been downloaded until an instant of time t and was calculated as the area below the speed function graphic. Initially, the speed was simulated by a constant function, so the students resorted to the idea of inscribing a rectangle which base was the value of t and the height was the value of speed to find the size. Then, the speed was simulated by a lineal function with a slope of 0,1 for which the students divided the graphic into two regions and provided a model that represents the area below the curve. Afterwards, the speed was simulated by a sinusoidal function with a behavior similar to that generated by an internet connection, in which the speed increases or decreases, so the students considered the regions of the area they know and brought it neat the curve. Each student found a different model, by taking as a common significance that the area below the curve represented the size of the file until a particular time t.

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MODELACIÓN MATEMÁTICA PARA EL ESTUDIO DEL CONCEPTO DE INTEGRAL A TRAVÉS DE UN PROBLEMA AUTÉNTICO

MATHEMATICAL MODELING FOR STUDYING THE CONCEPT OF INTEGRAL THROUGH AN AUTHENTIC PROBLEM

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Palabras Claves: Cálculo, Modelación, Matemáticas de nivel universitario.

Se presenta resultados iniciales de una investigación desde una perspectiva de la modelación matemática en el aula de clase, como un proceso de estudio de fenómenos o situaciones que pueden surgir de contextos cotidianos, sociales y culturales de los estudiantes o de otras ciencias (Villa-Ochoa, 2010). Para dar respuesta a la pregunta ¿Cómo la modelación matemática de problemas auténticos contribuye al estudio del concepto de integral a estudiantes universitarios?, asumimos un modelo matemático como un conjunto de representaciones y relaciones matemáticas para explicar, predecir y solucionar aspectos de un fenómeno (Villa-Ochoa y otros, 2009a). Se propone un problema auténtico (Kaiser y Schwarz, 2010), a partir del análisis de una simulación en GeoGebra del fenómeno de la velocidad de descarga de un archivo, siguiendo las fases de la metodología ACODESA (Hitt, 2007).

La investigación fue realizada con cuatro estudiantes de una universidad pública colombiana (16-19 años) que aprobaron el curso de cálculo diferencial, y que contaban con acceso al Aula Virtual GeoGebra. Para el análisis de los datos se tuvo en cuenta las respuestas y las producciones escritas de los estudiantes en el Aula y las grabaciones de respuestas a la entrevista estructurada (Goldin, 2000) aplicada en los encuentros sincrónicos.

Como resultados iniciales se tiene que los modelos construidos, permitieron reconocer la integral como el tamaño acumulado del archivo que se ha descargado hasta un instante de tiempo t, y fue calculado como el área bajo la gráfica de la función velocidad. Inicialmente la velocidad fue simulada por una función constante, por lo que los estudiantes recurrieron a la idea de inscribir un rectángulo, donde la base fuera el valor de t y la altura el valor de la velocidad para hallar el tamaño. Luego la velocidad fue simulada por una función lineal con pendiente 0,1 para el cual, los estudiantes dividieron la gráfica en dos regiones y dieron un modelo que representa el área bajo la curva. Posteriormente la velocidad fue simulada por una función sinusoidal con un comportamiento similar al generado por conexión a internet, en los cuales la velocidad aumenta o disminuye, por lo cual los estudiantes tuvieron en cuenta las regiones de área que conocen y las aproximaron a la curva, encontrando cada estudiante un modelo diferente, pero tomando como significado en común el área bajo la curva estaba representando el tamaño de archivo hasta un determinado tiempo t.

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STATISTICS AND PROBABILITY

RESEARCH REPORTS

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COMPARISON STRATEGIES BY QUOTIENT AND THE URN MODEL IN THE CHOICE OF PROBABILITIES

ESTRATEGIAS DE COMPARACIÓN POR COCIENTE Y EL MODELO DE URNA EN LA ELECCIÓN DE PROBABILIDADES

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In this research, quotient strategies and their influence on decision-making in situations that involve the comparison of probabilities are analyzed. In order to achieve this, classical probability situations modeled with urns were designed. In each situation, two urns with simple extraction, involving or not proportional relationships, were proposed to third grade secondary school students. The analysis was carried out through the categorization of the results based on the relationships established between the components of the urns (favorable, unfavorable and possible cases). The study concluded that when comparing probabilities, students not only resort to the quotient (ratio): favorable case numbers among the number of possible cases to make their choice, but to others where they also include unfavorable cases.

Keywords: probability, problem solving

Background and the problem proposal

Proportional and probabilistic reasoning are closely related; both require quantitative and qualitative analysis, as well as inference and prediction of results. This statement follows from contrasting the definition of proportional reasoning by Lesh, Post and Behr (1988) with what Landín and Sánchez (2010) propose regarding the implications of probabilistic reasoning. In this way, both reasonings are not limited to numerical comparisons, and although some researchers have tried distinguishing these concepts from each other (see Hoemann and Ross, 1971), the work presented here does not focus on that distinction, but on incorporating them to extract and analyze those multiplicative strategies derived from comparison by quotient, considering as a means the resolution of situations modeled with urns.

The strategies used by students when solving probability situations modeled with urns, as Alatorre (1994) points out, have already been studied by researchers such as: Piaget and Inhelder (1951), Maury (1984, 1986), Lecoutre (1984), Fischbein et al (1970), and Thornton and Fuller (1981). However, although the authors refer to some indicators of why and how the selections were made, the uncertainty remains if other types of choices can be presented other than the ones they comment on or what other relationships the students establish, as well as what comparison mechanisms underlie in these strategies that can reveal their complexity, such as those derived from proportional reasoning that demand “the ability to recognize, to explain, to think about, to conjecture about, to graph, to transform, to compare, to make judgments about, to represent, or to symbolize relationships of two simple types ”(Lamon, 1999).

In the book for the secondary education teacher (Alarcón *et al.* 1994), as well as in works by Green (1988), Falk (1980), Aguas (2014), and Cañizares and Batanero (1997), situations with two urns have also been raised. Despite the fact that there are situations that can be modeled with them, the urns themselves are a rich model for working on probability or other topics in mathematics, as long as these are not seen as prerequisites for understanding it, but rather a context where they coexist and support each other. "use proportionality and a basic understanding of probability to make and test conjectures about the results of experiments and simulations" (NCTM, 2000, p. 248), leads to the development and strengthening of constantly interacting content. Thus, the "facility with proportionality develops through work in many areas of the curriculum, including ratio and proportion, percent, […] and probability" (NCTM, 2000, p. 212). Therefore, the present research consists of analyzing the quotients that arise when comparing probabilities. The strategies followed by secondary education students are classified in order to delve deeper into the processes they carry out to establish a result. It is proposed to identify what elements they consider when making their comparisons (favorable, unfavorable or possible cases), what types of choices they make based on the relationships established and what the particularities of these relationships are, which are fundamental for this study, because what is intended is to obtain indications of how the different quotients that they establish influence their decisions. The study is oriented by the following question and research purpose:

Research question: When solving probability situations, what are the quotients that secondary school students establish to make a choice?

Purpose: To identify, analyze and classify the strategies followed by third grade students of secondary education students when solving situations of classical probability contextualized with the urn model, focusing attention on those strategies that involve comparing by quotient to see how they influence the decision-making.

Method

The research is qualitative with an instrumental case (Stake, 1999) made up of a group of 35 third grade students from secondary education in Mexico City. Ten situations were designed, with and without proportionality, which involved the comparison of probabilities. The urn model was used to design the situations. From Lamon's (1993) classification, the problems posed correspond to those of *part-part-whole*, which Özgün-Koca (2009) includes in his study within those of numerical comparison, because most of the problems that were posed can be assumed as discrete sets where a *whole* corresponds to possible cases and subsets of that *whole* to favorable and unfavorable cases, respectively. For the implementation of the situations, three work sessions of 50 minutes were used. In each session, students solved three to four situations individually without instruction (prior or during implementation) of the resolution, which favored independent thinking and led to the identification and analysis of various strategies. In this paper it was interesting to illustrate those related to the comparison by quotient.

Theoretical Interpretative Framework

As a result of the analysis, an interpretative framework was constructed for the classification of the strategies derived from the quotient comparison that emerged from the implementation. Two types of comparisons were identified: a) additive — through a difference, and, b) multiplicative — through a quotient. The first implies an absolute thought, and the second a relative one (Lamon, 1999); the multiplicative one is interesting to illustrate in this paper. Table 1 shows sixteen expressions identified by comparing the elements of two urns by quotient, where F represents the number of favorable cases, D the number of unfavorable cases and $F + D$ corresponds to the number of possible cases. Subscripts 1 and 2, in F_1, F_2, D_1 and D_2 indicate the reference point, which states whether the favorable or unfavorable cases correspond to the first or second urn, respectively.

Table 1: Relationships that are established when comparing by quotient

Reference point F	Reference point D	Reciprocal of reference point F	Reciprocal of reference point D
Expression i	Expression ii	Expresion iii	Expression iv

$\frac{F_1}{F_2}$ with $\frac{D_1}{D_2}$	$\frac{D_1}{D_2}$ with $\frac{F_1}{F_2}$	$\frac{F_2}{F_1}$ with $\frac{D_2}{D_1}$	$\frac{D_2}{D_1}$ with $\frac{F_2}{F_1}$
Fraction: Part – Part		Fraction: Part – Part	
Ratio: When the antecedent is F and D and the consequent is F and D		Ratio: When the antecedent is F and D and the consequent is F and D	
Reference point Urn 1	Reference Point Urn 2	Reciprocal of reference point Urn 1	Reciprocal of reference point Urn 2
Expression v	Expression vi	Expresion vii	Expresion viii
$\frac{F_1}{D_1}$ with $\frac{F_2}{D_2}$	$\frac{F_2}{D_2}$ with $\frac{F_1}{D_1}$	$\frac{D_1}{F_1}$ with $\frac{D_2}{F_2}$	$\frac{D_2}{F_2}$ with $\frac{D_1}{F_1}$
Fraction: Part – Part		Fraction: Part – Part	
Ratio: When the antecedent is F and the consequent is D		Ratio: When the antecedent is D and the consequent is F	
Expression ix	Expression x	Expression xi	Expression xii
$\frac{F_1}{F_1+D_1}$ with $\frac{F_2}{F_2+D_2}$	$\frac{F_2}{F_2+D_2}$ with $\frac{F_1}{F_1+D_1}$	$\frac{F_1+D_1}{F_1}$ with $\frac{F_2+D_2}{F_2}$	$\frac{F_2+D_2}{F_2}$ with $\frac{F_1+D_1}{F_1}$
Fraction: Part-Whole. When the part is F		Fraction: Whole-Part. When the part is F	
Ratio: When the antecedent is F and the consequent is $F+D$		Ratio: When the antecedent is $F+D$ and the consequent is F	
Expression xiii	Expression xiv	Expression xv	Expression xvi
$\frac{D_1}{F_1+D_1}$ with $\frac{D_2}{F_2+D_2}$	$\frac{D_2}{F_2+D_2}$ with $\frac{D_1}{F_1+D_1}$	$\frac{F_1+D_1}{D_1}$ with $\frac{F_2+D_2}{D_2}$	$\frac{F_2+D_2}{D_2}$ with $\frac{F_1+D_1}{D_1}$
Fraction: Part-Whole. When the part is D		Fraction: Whole-Part. When the part is D	
Ratio: When the antecedent is D and the consequent is $F+D$		Ratio: When the antecedent is $F+D$ and the consequent is F	

In the comparison by quotient (see Table 1), considered as a fraction or ratio, they appear during the relationships established with the elements that make them up (*part-part*, *part-whole*, *whole-part* and antecedent-consequent), using strategies to compare them (see Table 2); where in the *part-part* relationship *a, b, c* and *d* are the parts; in the part-whole *a* and *c* are the parts and *b* and *d* are the whole; in whole-part *a* and *c* are the whole and *b* and *d* are the parts; in the antecedent-consequent relationship *a* and *c* are the antecedents and *b* and *d* are the consequents.

Table 2: Strategies to compare the relationships established in Table 1

1st Strategy: Fraction-ratio (part-part, part-whole and whole-part) with the use of:

Multiples: If multiplied $\left(\frac{a}{b}\right)\left(\frac{c}{c}\right)$; and $\left(\frac{c}{d}\right)\left(\frac{a}{a}\right)$, the expressions $\frac{ac}{bc}$ and $\frac{ac}{ad}$ are obtained, and as $bc = ad$ we can then establish the proportion $\frac{ac}{bc} = \frac{ac}{ad}$, which would be equivalent to: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable.

Cross products: Given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, when making the products ad and bc, equality is obtained as $ac = bd$, then the following proportion is established: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable. Cross products are a particular case of using multiples to compare ratios or fractions.

Submultiples: By simplifying $\frac{a}{b}$, $\frac{m}{n}$ is obtained, and by simplifying $\frac{c}{d}$, $\frac{m}{n}$ is also obtained; then the following proportion is established: $\frac{a}{b} = \frac{c}{d}$, which indicates that the sets represented by a and b, and c and d are equiprobable.

Adding and subtracting fractions or ratios: Given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, when subtracting $\frac{a}{b} - \frac{c}{d}$, the difference of $\frac{0}{bd} = 0$ is obtained; then the following proportion is established: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable.

Adding and subtracting fractions or ratios in their decimal form: It is part of the fraction-ratio relationship, which allows for representation in decimal form to compare and make choices. By simplifying $\frac{a}{b}$, m is obtained, and by simplifying $\frac{c}{d}$, m is also obtained, where m is a decimal number; then the following proportion is established: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable.

2nd Strategy: The relationship within. It is presented in an additive or multiplicative way.

Additive form: Given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, if $a \pm (a)\left(\frac{m}{n}\right) = b$, and $c \pm (c)\left(\frac{m}{n}\right) = d$, or if $b \pm (b)\left(\frac{m}{n}\right) = a$, and $d \pm (d)\left(\frac{m}{n}\right) = c$, then the following proportion is established: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable.

Multiplicative form: Given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, if $(a)\left(\frac{m}{n}\right) = b$ and $(c)\left(\frac{m}{n}\right) = d$, or if $(b)\left(\frac{n}{m}\right) = a$, and $(d)\left(\frac{n}{m}\right) = c$, then the following proportion can be established: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable.

3rd Strategy: Rule of three. A quaternary relationship is established where one of the four elements is unknown and the other three must be related to find its value. This value is compared with that obtained in another similar quaternary relationship.

To obtain percentages: Given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, if $\frac{a}{b}$ is multiplied by 100 in order to obtain the percentage of a with respect to the percentage of b, it is considered 100%. Also, when $\frac{c}{d}$ is multiplied by 100 to get the percentage of c with respect to the percentage of d, it is considered 100%. If $\frac{(a)(100)}{b} = \frac{(c)(100)}{d}$, then the following proportion could be established: $\frac{a}{b} = \frac{c}{d}$, which would indicate that the sets represented by a and b, and c and d are equiprobable.

To obtain a, b, c or d: given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, if $\frac{(cb)}{a} = d$ and $\frac{(ad)}{c} = b$, and d turns out to be a multiple or submultiple of b, then the following proportion can be established: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable.

4th Strategy: The relationship between. It is presented in an additive or multiplicative way.

Additive Form: Given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, if $a \pm (a)\left(\frac{m}{n}\right) = c$, and $b \pm (b)\left(\frac{m}{n}\right) = d$, or if $c \pm (c)\left(\frac{m}{n}\right) = a$, and $d \pm (d)\left(\frac{m}{n}\right) = b$, then the following proportion is established: $\frac{a}{b} = \frac{c}{d}$. This indicates that the sets represented by a and b, and c and d are equiprobable.

Multiplicative form: Given the ratios $\frac{a}{b}$ and $\frac{c}{d}$, if $(a)\left(\frac{m}{n}\right) = c$ and $(b)\left(\frac{m}{n}\right) = d$, or if $(c)\left(\frac{n}{m}\right) = a$ and $(d)\left(\frac{n}{m}\right) = b$, then the following proportion is established: $\frac{a}{b} = \frac{c}{d}$. This would indicate that the sets represented by a and b, and c and d are equiprobable.

If the previous relationships are not in proportion, then the sets represented by *a* and *b*, and *c* and *d* would not be equiprobable, therefore, it would be necessary to identify which of the two sets is more likely based on the cases (favorable, unfavorable or possible) that were considered and the relationships that were established to make the comparisons. In the aforementioned strategies, the relationships *within* and *between* are considered by Noelthing (1980) when working with mixing situations, and although this researcher only considers relationships *within* and *between* of a multiplicative nature, Alatorre (1994) extends them to order relations of an additive or subtractive nature, but she does not consider them as it is done in this study using quotient comparison, as she points out the differences without taking the basal amount into account, and from our perspective the *additive form* includes the basal amount when considering how many of how many. Quotient comparison is closely related to relative thinking, Lamon (1999) distinguishes this type of thinking as one where multiplicative structures are involved and differentiates it from absolute thinking where additive-type relationships are established.

Analysis of Results

For the purpose of this research, it is convenient to analyze the types of choices, the comparisons that are established, the strategies that are followed and the reference point that is considered in the comparison of probabilities, which is the meaning of the relationships that are given between the cases (favorable, unfavorable or possible). Table 3 illustrates the quantitative comparison by quotient and its implications in order to exemplify the monitoring of the strategies identified in the implementation.

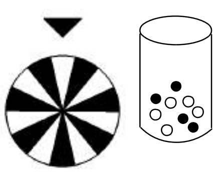
Table 3: Illustration of the type of choice, the strategy and the order relations identified in the quantitative comparison by quotient

Choice based on	Strategy (see Table 2)	Relationships (see Table 1)
Higher quotient; Lower quotient; or Equal quotient	1st Fraction-ratio (part-part)	Favorable-unfavorable v and vi Unfavorable-favorable vii and viii
	1st Fraction-ratio (part-whole)	Favorable-possible ix and x Possible-favorable xi and xii
		Unfavorable-possible xiii and xiv Possible-unfavorable xv and xvi

Consideration of favorable and unfavorable cases. The favorable and unfavorable cases of each set are compared by means of quotients (see expressions i, ii, iii, iv, v, vi, vii and viii in Table 1) and are chosen based on the highest, lowest or equal quotient. In the following, one of the strategies based on the choice of the highest quotient is exemplified.

Choice based on the highest quotient. *Strategy:* Fraction-ratio, part-part – with or without the use of submultiples and their representation in decimal form.

Order Relation: Favorable-Unfavorable.



Situation I. In a fair there are two games of chance: a roulette wheel and an Urn. You can spin the roulette wheel and bet to land on the white slot, or you can attempt to extract a white ball in urn mixed with black ones without looking. In which game is there a greater probability of winning? Explain how you determined your answer.

Student strategy. Example 1 (E1). Urn. There is a greater chance of winning at the urn. I was removing the median [half] to the colors of each game. In roulette they were the same colors and in the urn was a white ball rather than a black one and I saw that there was a better probability of winning.

$$\frac{7}{7}=\frac{3.5}{3.5}=1$$

$$\frac{5}{4}=\frac{2.5}{2}=1.25$$

$$\begin{array}{r} 1.25 \\ 2\overline{)2.5.} \\ 050 \\ 100 \\ 00 \end{array}$$

Figure 1: Student strategy. In example 1 (E1), the highest quotient is chosen. Quotient comparison (favorable-unfavorable) arriving at its decimal representation with the use of submultiples

In E1, a comparison of favorable and unfavorable cases is presented. To simplify the comparison of variables, submultiples are used and it ends with a decimal representation. In expressions v and vi (see Table 1), it is chosen based on a higher quotient, because the quotients in these expressions represent the number of favorable for each unfavorable. On the contrary, if expressions vii and viii are presented, the choices will be correct only if chosen based on the lowest quotient, which represents the number of unfavorable for each favorable. Hence, the significance of order relations: favorable-unfavorable or unfavorable-favorable.

Regarding the expressions i, ii, iii and iv of Table 1, correct choices would be provided if one considers the second quotient in i and iii; that is to say, this result is greater than the first quotient, whose elements are assigned the denominators of the comparative quotients. However, if it turns out to be less than the first, a correct choice is made if the set whose elements were assigned to the numerators is chosen. On the other hand, if expressions ii and iv are considered, the choice would also depend on the second quotient. If it turns out to be greater than the first quotient, the set that has remained in the numerators must be chosen. Yet, if it turns out to be less, a correct choice would be made if the set whose elements were in the denominators is chosen; this relates to the expressions i, ii, iii and iv, where the resulting quotients represent the number of favorable of a set for each favorable of the other, and in the same sense, the number of unfavorable of a set for each unfavorable of the other.

Consideration of unfavorable and possible cases. The possible and unfavorable cases of each set are compared by means of quotients (see expressions xiii, xiv, xv and xvi in Table 1) and the choice is made based on the highest, lowest or equal quotient. In the following, one of the strategies based on equal quotient is exemplified.

Choice based on equality of quotients. Strategy: Fraction-ratio, part-whole – with or without the use of submultiples and their representation in decimal form.

Order Relation: Unfavorable-Possible

Situation II. A teacher shows her students two black bags; the first bag contains 9 black and 6 white marbles, and the second bag contains 15 black and 10 white marbles. Which bag should they choose so that there is a greater probability that they will select a black marble on their first attempt? Explain the procedure you followed to make the choice.

Student strategy. Example 2 (E2). Any bag. Any bag has the same probability.

	Bag 1	Bag 1
White Marbles / Total Marbles	9 / 15	15 / 25
Probability	0.6	0.6

$$\begin{array}{r} .6 \\ 15\overline{)90} \\ 0 \end{array} \qquad \begin{array}{r} .6 \\ 25\overline{)150} \\ 0 \end{array}$$

Figure 2: Student strategy. In example 2 (E2), the equal quotient is chosen. Quotient comparison (unfavorable-possible) reaching its decimal representation without the use of submultiples

In E2, the comparison of unfavorable and possible cases in each set is presented. First, the cases of the first set were related and those of the second followed (see expression xiii of Table 1). It is important to note that in example 2 the decimal results obtained correspond to the probabilities of the unfavorable cases of each set. In expressions xv and xvi (see Table 1), it is chosen based on a higher quotient; this is because the resulting quotient determines the number of possible cases for each unfavorable one. Conversely, if expressions xiii and xiv are presented, the choices will be correct only if chosen based on the lowest quotient; this is because the quotient represents the number of unfavorable for each possible. In expressions ix and x, it is chosen based on a higher quotient; this is because they represent the number of favorable for each possible. On the other hand, if expressions xi and xii are presented, the choices will be correct only if chosen based on the smallest quotient; this is because the resulting quotient represents the number of possible for each favorable. It is noteworthy that expressions ix and x are the only ones that are used and considered in the 2011 Study Programs (SEP, 2011) to represent probabilities and make comparisons between them.

Final considerations

The results of this study concluded that with the probability of the event *A*, the students establish the quotient (ratio) $\frac{F}{P}$, determined by Laplace as the ratio of the number of favorable cases *F* to that of all the cases possible *P*. However, if they are presented with a probability comparison situation, they usually resort to other quotient comparisons where they also include unfavorable cases *D* such as: $\frac{D}{P}$ o $\frac{F}{D}$ or their reciprocals, and not necessarily the one established by Laplace. The research showed that these ratios led students to choices based on determining the urn with greater probability; hence, the importance of placing greater emphasis on quotient comparison strategies that result from comparing probabilities. The reference point is decisive for understanding the relationships that students establish and the interpretation they give to their quotients. For example, the interpretation of the quotient (ratio) $\frac{F}{D}$ is completely distinct from the one represented in the quotient (ratio) $\frac{D}{F}$, and in the same way, the interpretation is different $\frac{F_1}{D_1} > \frac{F_2}{D_2}$ to $\frac{F_2}{D_2} < \frac{F_1}{D_1}$. From a purely mathematical standpoint, these comparisons would have no greater difficulty, but from a pedagogical and didactic perspective of teaching and learning of probability, the change in the reference point when establishing the comparisons influences the treatment of the object of study and its interpretation. The incorporation of didactic orientations in the teaching materials for basic education curriculum is suggested, which includes an analysis of the strategies by quotient that the students can carry out when solving probability problems. A detailed analysis of the varied strategies presented by students would have a positive effect in the teaching and study of this branch of mathematics.

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ESTRATEGIAS DE COMPARACIÓN POR COCIENTE Y EL MODELO DE URNA EN LA ELECCIÓN DE PROBABILIDADES

COMPARISON STRATEGIES BY QUOTIENT AND THE URN MODEL IN THE CHOICE OF PROBABILITIES

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En esta investigación se analizan estrategias por cociente y su influencia en la toma de decisiones en situaciones que implican la comparación de probabilidades. Para ello, se diseñaron situaciones de probabilidad clásica modeladas con urnas. En cada situación se plantearon dos urnas con extracción simple, que implican o no relaciones de proporcionalidad, a estudiantes de tercer grado de educación secundaria. El análisis se realizó a través de la categorización de los resultados con base en las relaciones que se establecen entre los componentes de las urnas (casos favorables, desfavorables y posibles). El estudio arrojó que, al comparar probabilidades, los estudiantes no sólo recurren al cociente: números de casos favorables entre número de casos posibles para hacer su elección, sino a otros donde además incluyen a los casos desfavorables.

Palabras clave: probabilidad, resolución de problemas

Antecedentes y planteamiento del problema

Los razonamientos proporcional y probabilístico guardan estrecha relación; ambos requieren de un análisis cuantitativo y cualitativo, así como de la inferencia y la predicción de resultados. Esta afirmación se desprende de contrastar la definición de razonamiento proporcional de Lesh, Post y Behr (1988) con lo que Landín y Sánchez (2010) proponen respecto a las implicaciones del razonamiento probabilístico. De esta manera, ambos razonamientos no se limitan a comparaciones numéricas, y aunque algunos investigadores han intentado distinguir entre sí a estos conceptos (véase por ejemplo Hoemann y Ross, 1971); el trabajo que aquí se expone no se centra en distinguirlos sino en incorporarlos, para extraer y analizar aquellas estrategias multiplicativas derivadas de la comparación por cociente, considerando como medio la resolución de situaciones modeladas con urnas.

Las estrategias utilizadas por los estudiantes al resolver situaciones de probabilidad modeladas con urnas, como lo señala Alatorre (1994), ya han sido estudiadas por investigadores como: Piaget e Inhelder (1951), Maury (1984, 1986), Lecoutre (1984), Fischbein et al. (1970), y Thornton y Fuller

(1981). Sin embargo, aunque los autores refieren algunos indicadores del porqué y cómo se realizaron las elecciones, queda la incertidumbre si se pueden presentar otro tipo de elecciones distintas a las que ellos comentan o qué otras relaciones establecen los alumnos, así como qué mecanismos de comparación subyacen en estas estrategias y que pueden dejar ver la complejidad de las mismas, como las que se derivan de un razonamiento proporcional que demandan "la capacidad de reconocer, explicar, pensar, hacer conjeturas, graficar, transformar, comparar, emitir juicios, representar o simbolizar relaciones de dos tipos simples" (Lamon, 1999).

En el libro para el maestro de educación secundaria (Alarcón *et al.* 1994) así como en trabajos de Green (1988), Falk (1980), Aguas (2014), y Cañizares y Batanero (1997), también se han planteado situaciones con dos urnas. No obstante que existen situaciones que pueden ser modeladas con ellas, las urnas en sí mismas son un modelo rico para trabajar temas de probabilidad u otros de las matemáticas, siempre y cuando estos no sean vistos como prerequisitos para comprenderla, sino un contexto donde conviven y se apoyan mutuamente. "Utilizar la proporcionalidad y una comprensión básica de la Probabilidad para formular y comprobar conjeturas sobre los resultados de experimentos y simulaciones" (NCTM, 2000, p. 248), nos lleva al desarrollo y fortalecimiento de contenidos que interactúan de manera constante. Así, la "destreza con la proporcionalidad se desarrolla a través del trabajo con muchos temas del currículo: razón y proporción, porcentaje, […] y probabilidad" (NCTM, 2000, p. 212).

De esta manera, la presente investigación consiste en analizar los cocientes que surgen al comparar probabilidades. Para ello se clasifican las estrategias que siguen estudiantes de educación secundaria con la finalidad de profundizar en los procesos que llevan a cabo para establecer algún resultado. Es decir, se plantea identificar qué elementos consideran para hacer sus comparaciones (casos favorables, desfavorables o posibles), qué tipo de elecciones realizan con base en las relaciones que establecen y cuáles son las particularidades de estas relaciones, que para este estudio son fundamentales, porque lo que se pretende es obtener indicios de cómo los distintos cocientes que pueden establecer influyen en sus decisiones. El estudio está orientado por la siguiente pregunta y propósito de investigación.

Pregunta de investigación: Al resolver situaciones de probabilidad, ¿cuáles son los cocientes que estudiantes de educación secundaria establecen para hacer una elección?

Propósito: Identificar, analizar y clasificar las estrategias que siguen estudiantes de tercer grado de educación secundaria al resolver situaciones de probabilidad clásica contextualizadas con el modelo de urna, centrando la atención en aquellas estrategias que implican comparar por cociente para ver de qué manera influyen en la toma de decisiones.

Método

La investigación es cualitativa con un caso instrumental (Stake, 1999), conformado por un grupo de 35 alumnos de tercer grado de educación secundaria de la Ciudad de México. Se diseñaron 10 situaciones, con y sin proporcionalidad, que implicaron la comparación de probabilidades. Para el diseño de las situaciones se recurrió al modelo de urna. De la clasificación de Lamon (1993), los problemas planteados corresponden a los de Parte-parte-todo —que Özgün-Koca (2009) incluye en su estudio dentro de los de comparación numérica— porque la mayoría de los problemas que se plantearon pueden asumirse como conjuntos discretos donde un todo corresponde a los casos posibles y los subconjuntos de ese todo a los casos favorables y desfavorables, respectivamente. Para la implementación de las situaciones se utilizaron tres sesiones de trabajo, cada una de 50 minutos. En cada sesión los alumnos resolvieron individualmente de tres a cuatro situaciones sin instrucción (previa o durante la implementación) de la resolución, lo que favoreció un pensamiento independiente y conllevó a la identificación y análisis de diversas estrategias. En este documento interesó ilustrar las relacionadas con la comparación por cociente.

Marco teórico-interpretativo

Como resultado del análisis, se construyó un marco interpretativo para la clasificación de las estrategias derivadas de la comparación por cociente que surgieron de la implementación. Se identificaron dos tipos de comparaciones: a) aditiva —por medio de una diferencia— y, b) multiplicativa —por medio de un cociente—. La primera implica un pensamiento absoluto, y la segunda uno relativo (Lamon,1999); siendo la multiplicativa la que interesa ilustrar en este documento. En la Tabla 1 se muestran dieciséis expresiones identificadas al comparar por cociente los elementos de dos urnas, donde F representa el número de casos favorables, D número de casos desfavorables y F+D corresponde al número de casos posibles. Los subíndices 1 y 2, en F_1, F_2, D_1 y D_2 indican el punto de referencia, es decir, si los casos favorables o desfavorables corresponden a la primera o segunda urna, respectivamente.

Tabla 1: Relaciones que se pueden establecer cuando se compara por cociente

Punto de referencia F	Punto de referencia D	Reciproco del punto de referencia F	Reciproco del punto de referencia D
Expresión i	Expresión ii	Expresión iii	Expresión iv
$\frac{F_1}{F_2}$ con $\frac{D_1}{D_2}$	$\frac{D_1}{D_2}$ con $\frac{F_1}{F_2}$	$\frac{F_2}{F_1}$ con $\frac{D_2}{D_1}$	$\frac{D_2}{D_1}$ con $\frac{F_2}{F_1}$
Fracción: Parte – Parte		Fracción: Parte – Parte	
Razón: Cuando el antecedente es F y D y el consecuente es F y D.		Razón: Cuando el antecedente es F y D y el consecuente es F y D.	
Punto de referencia Urna 1	Punto de referencia Urna 2	Reciproco del punto de referencia Urna 1	Reciproco del punto de referencia Urna 2
Expresión v	Expresión vi	Expresión vii	Expresión viii
$\frac{F_1}{D_1}$ con $\frac{F_2}{D_2}$	$\frac{F_2}{D_2}$ con $\frac{F_1}{D_1}$	$\frac{D_1}{F_1}$ con $\frac{D_2}{F_2}$	$\frac{D_2}{F_2}$ con $\frac{D_1}{F_1}$
Fracción: Parte – Parte		Fracción: Parte – Parte	
Razón: Cuando el antecedente es F y el consecuente es D.		Razón: Cuando el antecedente es D y el consecuente es F.	
Expresión ix	Expresión x	Expresión xi	Expresión xii
$\frac{F_1}{F_1+D_1}$ con $\frac{F_2}{F_2+D_2}$	$\frac{F_2}{F_2+D_2}$ con $\frac{F_1}{F_1+D_1}$	$\frac{F_1+D_1}{F_1}$ con $\frac{F_2+D_2}{F_2}$	$\frac{F_2+D_2}{F_2}$ con $\frac{F_1+D_1}{F_1}$
Fracción: Parte-todo. Cuando la parte es F		Fracción: Todo-parte. Cuando la parte es F	
Razón: Cuando el antecedente es F y el consecuente es $F+D$.		Razón: Cuando el antecedente es $F+D$ y el consecuente es F.	
Expresión xiii	Expresión xiv	Expresión xv	Expresión xvi
$\frac{D_1}{F_1+D_1}$ con $\frac{D_2}{F_2+D_2}$	$\frac{D_2}{F_2+D_2}$ con $\frac{D_1}{F_1+D_1}$	$\frac{F_1+D_1}{D_1}$ con $\frac{F_2+D_2}{D_2}$	$\frac{F_2+D_2}{D_2}$ con $\frac{F_1+D_1}{D_1}$
Fracción: Parte-todo. Cuando la parte es D.		Fracción: Todo-parte. Cuando la parte es D.	
Razón: Cuando el antecedente es D y el consecuente es $F+D$.		Razón: Cuando el antecedente es $F+D$ y el consecuente es F.	

En la comparación por cociente (véase la Tabla 1), considerado como fracción o razón, se pueden presentar, durante las relaciones que se establecen con los elementos que las conforman (parte-parte, parte-todo, todo-parte y antecedente-consecuente), las siguientes estrategias para compararlas (véase la Tabla 2), donde en la relación parte-parte *a, b, c* y *d* son las partes; en la parte-todo *a* y *c* son las partes y b y d son los todos; en todo-parte a y c son los todos y b y d son las partes; y en la relación antecedente-consecuente a y c son los antecedentes y b y d son los consecuentes.

Tabla 2: Estrategias para comparar las relaciones que se establecen en la Tabla 1

1ª.- Estrategia: Fracción-razón (parte-parte, parte-todo y todo-parte) con el uso de: *Múltiplos:* Si se multiplica $\left(\frac{a}{b}\right)\left(\frac{c}{c}\right)$ y $\left(\frac{c}{d}\right)\left(\frac{a}{a}\right)$ se obtienen las expresiones $\frac{ac}{bc}$ y $\frac{ac}{ad}$, y si $bc = ad$ entonces se puede establecer la proporción $\frac{ac}{bc} = \frac{ac}{ad}$ que sería equivalente a: $\frac{a}{b} = \frac{c}{d}$. Esto significa que los conjuntos representados por a y b, y c y d son equiprobables. *Productos cruzados:* Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si al efectuar los productos ad y bc se obtiene la igualdad $ac = bd$, entonces se establece la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significa que los conjuntos representados por a y b, y c y d son equiprobables. Los productos cruzados son un caso particular del uso de múltiplos para comparar razones o fracciones. *Submúltiplos:* Si al simplificar $\frac{a}{b}$ se obtiene $\frac{m}{n}$, y al simplificar $\frac{c}{d}$ también se obtiene $\frac{m}{n}$, entonces se establece la proporción: $\frac{a}{b} = \frac{c}{d}$, lo que significa que los conjuntos representados por a y b, y c y d son equiprobables. *Suma y resta de fracciones o razones:* Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si al hacer la sustracción $\frac{a}{b} - \frac{c}{d}$ se obtiene la diferencia $\frac{0}{bd} = 0$, entonces se establece la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significa que los conjuntos representados por a y b, y c y d son equiprobables. *Suma y resta de fracciones o razones en su representación decimal:* Parte de la relación fracción-razón, y se llega a su representación en forma decimal para comparar y hacer la elección. Si al simplificar $\frac{a}{b}$ se obtiene m, y al simplificar $\frac{c}{d}$ también se obtiene m, donde m es un número decimal, entonces se establece la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significa que los conjuntos representados por a y b, y c y d son equiprobables.
2ª.- Estrategia: Relación dentro. Se presenta de forma aditiva o multiplicativa. *Forma aditiva*: Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si $a \pm (a)\left(\frac{m}{n}\right) = b$, y $c \pm (c)\left(\frac{m}{n}\right) = d$, o si $b \pm (b)\left(\frac{m}{n}\right) = a$ y $d \pm (d)\left(\frac{m}{n}\right) = c$, entonces se puede establecer la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significaría que los conjuntos representados por a y b, y c y d son equiprobables. *Forma multiplicativa*: Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si $(a)\left(\frac{m}{n}\right) = b$ y $(c)\left(\frac{m}{n}\right) = d$, o si $(b)\left(\frac{n}{m}\right) = a$ y $(d)\left(\frac{n}{m}\right) = c$, entonces se puede establecer la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significaría que los conjuntos representados por a y b, y c y d son equiprobables.
3ª.- Estrategia: Regla de tres. Se establece una relación cuaternaria donde uno de los cuatro elementos es desconocido y los otros tres deben ser relacionados para encontrar su valor. Este valor es comparado con el obtenido en otra relación cuaternaria similar. *Para la obtención de porcentajes:* Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si se multiplica $\frac{a}{b}$ por 100 para obtener el porcentaje de a respecto al porcentaje de b, considerado como el 100%. Y $\frac{c}{d}$ también se multiplica por 100 para obtener el porcentaje de c respecto al porcentaje de d, considerado como el 100%, si $\frac{(a)(100)}{b} = \frac{(c)(100)}{d}$, entonces se podría establecer la proporción: $\frac{a}{b} = \frac{c}{d}$, lo que significaría que los conjuntos representados por a y b, y c y d son equiprobables. *Para la obtención de a, b, c o d:* Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si $\frac{(cb)}{a} = d$ y $\frac{(ad)}{c} = b$, y d resulta ser múltiplo o submúltiplo de b, entonces se puede establecer la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significaría que los conjuntos representados por a y b, y c y d son equiprobables.

4ª.- Estrategia: Relación entre. Se presenta de forma aditiva o multiplicativa.
Forma aditiva: Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si $a \pm (a)\left(\frac{m}{n}\right) = c$, y $b \pm (b)\left(\frac{m}{n}\right) = d$, o si $c \pm (c)\left(\frac{m}{n}\right) = a$, y $d \pm (d)\left(\frac{m}{n}\right) = b$, entonces se puede establecer la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significaría que los conjuntos representados por a y b, y c y d son equiprobables.
Forma multiplicativa: Dadas las razones $\frac{a}{b}$ y $\frac{c}{d}$, si $(a)\left(\frac{m}{n}\right) = c$ y $(b)\left(\frac{m}{n}\right) = d$, o si $(c)\left(\frac{n}{m}\right) = a$ y $(d)\left(\frac{n}{m}\right) = b$, entonces se puede establecer la proporción: $\frac{a}{b} = \frac{c}{d}$. Esto significaría que los conjuntos representados por a y b, y c y d son equiprobables.

Si las relaciones anteriores no están en proporción, entonces los conjuntos representados por a y b, y c y d no serían equiprobables, por lo que habría que identificar cuál de los dos conjuntos tiene mayor probabilidad con base en los casos (favorables, desfavorables o posibles) que se consideraron y las relaciones que se establecieron para hacer las comparaciones. En las estrategias descritas anteriormente se encuentran las relaciones *dentro* y *entre* consideradas por Noelthing (1980) al trabajar situaciones de mezclas, y aunque este investigador sólo considera las relaciones *dentro* y *entre* de carácter multiplicativo, Alatorre (1994) las amplía a relaciones de orden y de carácter aditivo o sustractivo, pero no las considera como se hace en este estudio en la comparación por cociente, pues ella señala las diferencias sin tomar en cuenta la cantidad basal y desde nuestra perspectiva la *Forma aditiva* sí incluye esta cantidad basal al considerar cuántas de cuántas. La comparación por cociente está estrechamente relacionada con el pensamiento relativo, Lamon (1999) distingue a este tipo de pensamiento como aquel donde se involucran estructuras multiplicativas y lo diferencia del pensamiento absoluto donde se establecen relaciones de tipo aditivo.

Análisis de los resultados

Por la finalidad que esta investigación persigue consideramos conveniente analizar el tipo de elecciones, las comparaciones que se establecen, las estrategias que se siguen y el punto de referencia que se considera en la comparación de probabilidades, es decir, el sentido de las relaciones que se dan entre los casos (favorables, desfavorables o posibles). En la Tabla 3 se ilustra la comparación cuantitativa por cociente y sus implicaciones, para ejemplificar el seguimiento de las estrategias identificadas en la implementación.

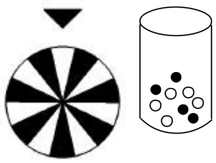
Tabla 3: Ilustración del tipo de elección, las estrategias y la relación de orden identificada en la comparación cuantitativa por cociente

Elección con base en el	Estrategia (véase Tabla 2)	Relaciones (véase Tabla 1)
Cociente mayor; Cociente menor; o Cociente igual.	1ª Fracción-razón (parte-parte).	Favorables-desfavorables v y vi Desfavorables-favorables vii y viii
	1ª Fracción-razón (parte-todo).	Favorables-posibles ix y x Posibles-favorables xi y xii
		Desfavorables-posibles xiii y xiv Posibles-desfavorables xv y xvi

Consideración de casos favorables y casos desfavorables. Se comparan por medio de cocientes los casos favorables y desfavorables de cada conjunto (véanse las expresiones i, ii, iii, iv, v, vi, vii y viii de la Tabla 1) y se elige con base en el mayor, menor o igual cociente. En lo que sigue se ejemplifica una de las estrategias con base en la elección del cociente mayor.

Elección con base en el cociente mayor. Estrategia: Fracción-razón parte-parte con o sin la utilización de submúltiplos y su representación en forma decimal.

Relación de orden: Favorables-Desfavorables.



Situ*ación II.* 1.- En una feria hay dos juegos azar: una ruleta y una urna, ¿en qué juego existe mayor probabilidad de ganar, en la ruleta si se apuesta a que al girarla se detenga en un sector blanco o en la urna si se apuesta a que en la primera extracción sin ver la bola será blanca? Explica como le hiciste para determinar tu respuesta.

Estrategia del estudiante. Ejemplo1 (E1). Urna 2. Hay más probabilidad de ganar en la urna. Pues fui sacando la mediana [mitad] a los colores de cada juego. En la ruleta eran los mismos colores y en la urna había una bola blanca más que negra y vi que había mejor probabilidad de ganar.

$$\frac{7}{7}=\frac{3.5}{3.5}=1 \qquad \frac{5}{4}=\frac{2.5}{2}=1.25$$

1.25
20)2.5.
050
100
00

Figura 1: Estrategia del estudiante. Ejemplo 1 (E1) se elige el cociente mayor. Comparación por cociente (favorables-desfavorables)
llegando a su representación decimal con el uso de submúltiplos

En E1, se presenta la comparación de casos favorables y desfavorables. Para simplificar la comparación de variables se utilizan submúltiplos y se termina con una representación decimal. En las expresiones v y vi (véase Tabla 1) se elige con base en cociente mayor, porque los cocientes en estas expresiones representan el número de favorables por cada desfavorable. Por lo contrario, si se presentan las expresiones vii y viii las elecciones serán correctas sólo si se elige con base en el cociente menor, que representan el número de desfavorables por cada favorable. De aquí la importancia de las relaciones de orden: favorables-desfavorables o desfavorables-favorables.

En cuanto a las expresiones i, ii, iii y iv de la Tabla 1 se tendrían elecciones correctas si se considera, en i y iii el segundo cociente, es decir, si este resulta ser mayor al primer cociente se debe elegir el conjunto cuyos elementos se asignaron a los denominadores de los cocientes comparados. Sin embargo, si resulta ser menor al primero, se realiza una elección correcta si se elige el conjunto cuyos elementos se asignaron a los numeradores. Por otra parte, si se consideran las expresiones ii y iv la elección también dependería del segundo cociente, si resulta ser mayor al primer cociente se debe elegir el conjunto que haya quedado en los numeradores. Pero si resulta ser menor, se haría una elección correcta si se elige el conjunto cuyos elementos quedaron en los denominadores, esto debido a que en las expresiones i, ii, iii y iv los cocientes resultantes representan el número de favorables de un conjunto por cada favorable del otro, y en este mismo sentido, el número de desfavorables de un conjunto por cada desfavorable del otro.

Consideración de casos desfavorables y casos posibles. Se comparan por medio de cocientes los casos posibles y desfavorables de cada conjunto (véanse las expresiones xiii, xiv, xv y xvi de la Tabla 1) y se realiza la elección con base en su mayor, menor o igual cociente. En lo que sigue se ejemplifica una de las estrategias con base en la igualdad de cociente.

Elección con base en la igualdad de cocientes. Estrategia: Fracción-razón parte-todo con o sin el uso de submúltiplos y su representación en forma decimal.

Relación de orden: Desfavorables-Posibles.

Situación IX. Una profesora muestra a sus estudiantes dos bolsas negras; en la primera deposita 9 canicas negras y 6 blancas, y en la segunda deposita 15 negras y 10 blancas ¿qué bolsa deben elegir para que tengan mayor probabilidad de que en la primera extracción sin ver la canica sea negra? Explica el procedimiento que seguiste para realizar la elección.	*Estrategia del estudiante.* Ejemplo (E2). Cualquier bolsa. En cualquiera de las bolsas se tiene la misma probabilidad.

	Bolsa 1	Bolsa 2
Canicas blancas / Total de canicas	$\frac{9}{15}$	$\frac{15}{25}$
Probabilidad	0.6	0.6

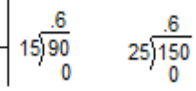


Figura 2: Estrategia del estudiante. Ejemplo 2 (E2) se elige el cociente igual. Comparación por cociente (desfavorables-posibles) llegando a su representación decimal

En E2, se presenta la comparación de casos desfavorables y posibles en cada conjunto. Primero se relacionaron los casos del primer conjunto y posteriormente los del segundo (véase la expresión xiii de la Tabla 1). Es importante señalar que en E2 se establece que los resultados decimales obtenidos corresponden a las probabilidades de los casos desfavorables de cada conjunto. En las expresiones xv y xvi (véase Tabla 1) se elige con base en cociente mayor; esto porque el cociente resultante determina el número de casos posibles por cada desfavorable. Por lo contrario, si se presentan las expresiones xiii y xiv, las elecciones serán correctas sólo si se elige con base en el cociente menor; esto debido a que el cociente representa el número de desfavorables por cada posible. En las expresiones ix y x se elige con base en cociente mayor; esto porque representan el número de favorables por cada posible. En cambio, si se presentan las expresiones xi y xii, las elecciones serán correctas sólo si se elige con base en el cociente menor; esto debido a que el cociente resultante representa el número de posibles por cada favorable. Es importante comentar que las expresiones ix y x son las únicas que se utilizan y consideran en los Programas de estudio de 2011 (SEP, 2011), para representar probabilidades y realizar comparaciones entre ellas.

Consideraciones finales

Con los resultados del estudio se concluye que, si se solicita la probabilidad del evento A, los estudiantes establecen el cociente (razón) $\frac{F}{P}$, determinado por Laplace como la razón entre el número de casos favorables F y el de todos los casos posibles P. Sin embargo, si se les plantea una situación de comparación de probabilidades, suelen recurrir a otros cocientes de comparación donde además incluyen a los casos desfavorables D como: $\frac{D}{P}$ o $\frac{F}{D}$ o sus recíprocos, y no necesariamente al establecido por Laplace. En la investigación se mostró que estas razones llevaron a los estudiantes a elecciones correctas con base en determinar la urna con mayor probabilidad. De aquí la importancia de poner mayor énfasis en las estrategias de comparación por cociente que resulten de comparar probabilidades. El punto de referencia es determinante para comprender las relaciones que los estudiantes establecen y la interpretación que le dan a sus cocientes. En este sentido, por ejemplo, la interpretación del cociente (razón) $\frac{F}{D}$ es totalmente distinta a la representada en el cociente (razón) $\frac{D}{F}$, y de igual manera es distinta la interpretación $\frac{F_1}{D_1} > \frac{F_2}{D_2}$ a $\frac{F_2}{D_2} < \frac{F_1}{D_1}$. Desde un punto de vista puramente matemático estas comparaciones no tendrían mayor dificultad, pero desde un sentido pedagógico y didáctico de la enseñanza y del aprendizaje de la probabilidad, el cambio en el punto de referencia al establecer las comparaciones influye en el tratamiento del objeto de estudio y su interpretación. Se sugiere la incorporación de orientaciones didácticas en los materiales curriculares de educación básica, donde se incluya un análisis de las estrategias por cociente que los estudiantes pueden llevar a cabo al resolver problemas de probabilidad. Se considera que un análisis detallado de las estrategias variadas que presentan los alumnos tendría un aspecto positivo en la enseñanza y estudio de esta rama de las matemáticas.

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HIGH-SCHOOL STUDENTS' PROBABILISTIC REASONING WHEN WORKING WITH RANDOM INTERVALS

RAZONAMIENTO PROBABILÍSTICO DE ESTUDIANTES DE BACHILLERATO CUANDO TRABAJAN CON INTERVALOS ALEATORIOS

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This work reports the results of a research aimed to know the probabilistic reasoning of high-school students when they deal with the notion of random intervals. An activity was carried out involving students between ages 16 and 17 who built random intervals through physical and computational simulations. The research question guiding this work was: Which reasoning do students exhibit when they estimate the probabilities of events related to the experience of creating random intervals from a frequentist approach? From the data analysis, partly based on the Grounded Theory, four categories were established. They suggest that the patterns observed in this work are likely present in situations demanding the frequentist approach to probability.

Keywords: Reasoning, Frequentist approach, probability, random intervals

Problem statement

The educational research on the probabilistic reasoning of students is a complex field since it involves an abundance of concepts, innovative instructional proposals, conceptions, misconceptions, and difficulties as well as a number of methodological approaches and conceptual frameworks (Jones et al., 2007; Chernoff & Sriraman, 2014). It has been recently stressed that educators and teachers must research and document the implementation of innovative approaches and materials in class to allow for a close integration of probability and statistics (Chernoff, Paparistodemou, Bakogianni, Petocs, 2016; Langrall, Makar, Nilson, Shaughnessy, 2017). It has been suggested to give probability teaching a modeling approach (Pfannkuch et al., 2016) starting from extra-mathematical situations or contexts of the natural or social reality. That is, modeling will allow students to acquire or create probabilistic concepts when solving problems emerged from real situations. For this supposition to be feasible, it is necessary to start with simple random situations that can be repeated under a set of well-defined conditions. Then, they will allow for the observation of patterns of outcomes. In our opinion, situations with coins, dice, urns, and roulettes can play a mediating role in the acquisition of probabilistic concepts and the fundamentals of modeling (Sharma, 2016). In this work, we assume the hypothesis that the situations based on random devices, with the aid of digital gadgets, can be mediators between abstract concepts and real situations. They can also be the support of fertile situations to contribute to the integrated learning of probability and statistics.

Traditionally, introductory high-school courses deal with probability and statistics separately. However, statistical analyses must include probabilistic reasoning since it allows to handle situations of uncertainty and variability intrinsic to the phenomena studied by statistics. Still, some approaches of probability teaching avoid developing reasoning on uncertainty and variability handling instead of promoting it and focus on more formal aspects (set theory), calculus (classic approach of probability), and technical elements (combinatorics). One way of including uncertainty and variability in probability classes is to organize situations and present problems that produce data following an unknown distribution. So, students have to analyze them to obtain conclusions, as in the estimation of the probability of events.

Saldanha (2016) and Prodromou (2016) argue that the study of samples of a distribution obtained through a digital simulation provides an adequate resource to link probability to statistics, especially from a frequentist approach. For that reason, we consider there should be research on how students reason when facing problems that link probability to statistics and that the frequentist approach of probability must be further studied.

Research question

Given that probability is studied in preuniversitary levels, it is important to explore the possibilities that technology offers to study its link to inference in high school. Particularly, this work explores the idea of introducing the notion of random intervals (RI). The aim is to set a background that helps reasoning on confidence intervals (CI), and so a research question was formulated:

Which reasonings do students show when they estimate the probability of events related to the experience of creating random intervals through a frequentist approach?

Background

The work presented here calculates probabilities from the frequentist approach of probability. So, we review some research that includes the frequentist approach of probability with technology and particularly focus on the work by Ireland and Watson (2009). The work explores the understanding of elementary-school students (ages 10–12) regarding the connection between theoretical probability and experimental probability (frequentist approach of probability) after students work with manipulatives (coins, dice) and the software ThinkerPlots. Ireland and Watson propose a framework to interpret the students' understanding as a continuum from concrete (experimental) to abstract (theoretical) in which manipulatives, the simulator, and the Law of Large Numbers are especially important. The findings lead them to conclude that it is necessary to explicitly teach the connection between theoretical and experimental probability; it is not enough for students to observe the behavior of the outcomes from simulations to achieve such connection.

Another study on the frequentist approach was that by Stohl and Tarr (2002). The authors report an instructional sequence with the aim of assessing how technological tools allow for and limit the development of the notion of inference from probabilistic situations. The participants were 23 students in sixth grade (ages 11–12) who worked in pairs and used the computational tool Probability Explorer to formulate and evaluate inferences during a 12-day teaching period. Among the conclusions, the authors state that the tasks designed in the context of games of chance and urns (one in the context of fishing in a lake) allowed the students to perceive relationships between empirical and theoretical probability as well as the role of the sample size in such relationships.

Conceptual framework

For the aims of this work, we have chosen concepts referring to two dimensions: content and cognition. From the first, content related to random intervals is presented and the same is done with probabilistic reasoning from the second.

Mathematical content relevant to the study

The content of this study refers to the estimation of probabilities through the frequentist approach of the probability of events related to the experience of generating random intervals. Then, some concepts on the frequentist approach of probability should be reviewed.

A repetitive phenomenon is that which can be repeated under a set of given conditions such that every repetition of the phenomenon is considered equivalent to its predecessors. Particularly, a random experience E is a repetitive phenomenon in which a characteristic is observed to change from one repetition to another and cannot be predicted; still, the set of all potential outcomes can be determined (sample space). Consider a random experience E and its sample space S while an event A is a subset of S. If experience E is repeated N times and n is the number of times event A occurs, then the quotient $n/_N$ is the relative frequency of A. The relative frequency of an event depends on N such that it varies as the values of N change. The most important characteristic of the relative frequencies of an event is that they converge in a given number as the number of repetitions grows indefinitely.

A random interval is defined as an interval in which at least one of the end points is a random variable, and so it gives rise to a family of intervals. Since we cannot assume that the students master the topic of distributions, the task design was carried out based on the experiment of drawing a ball from an urn containing 10 numbered balls from 0 to 9. If the random variable is the number printed on the ball, its distribution is the discrete uniform distribution taking the values $x = 0,1,2, \dots ,9$ and a probability $p = \frac{1}{10}$. Students are asked to observe the events of the type "E_C = the interval I_x contains number C, where C is any number between 0 and 9." Table 1 shows the probabilities of events E_C. These events are not mutually exclusive; hence the sum of their probabilities is higher than 1.

Table 1. Probabilities that the event E_c occurs

Event	E_0	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9
Probability	$\frac{9}{45}$	$\frac{17}{45}$	$\frac{23}{45}$	$\frac{27}{45}$	$\frac{29}{45}$	$\frac{29}{45}$	$\frac{27}{45}$	$\frac{23}{45}$	$\frac{17}{45}$	$\frac{9}{45}$

Probabilistic reasoning

Batanero et al. (1996) define probabilistic reasoning as a mode of reasoning that refers to judgements and decision making under uncertainty; therefore, it is relevant to real life. This reasoning includes the ability to: identify random events in nature, technology, and society; analyze the conditions of such events and derive suppositions to make an adequate modeling; construct mathematical models and explore different scenarios and outcomes; and apply mathematical methods and procedures of probability and statistics.

Methodology

Participants

The participants of this research were 16 students between ages 16 and 17 from a public school in Mexico City who had never taken a formal course on probability. One of the researchers was the class teacher and conducted the activity.

Activity

The activity was created using the principles of design to support the students' statistical reasoning proposed by Cobb and McClain (2004). Those principles must consider five aspects: 1) central statistical ideas (in this work, we focus on population, sample, random intervals, relative frequency, probability, and law of large numbers), 2) the instructional activity (the questions were formulated to observe whether the students created a conception of the central ideas), 3) the classroom activity structure (it must start by pointing out relevant aspects, variables to consider and how they will be mediated, the topic to cover, the activity development, and the students' discussion on the data obtained), 4) computer tools used by the students (in this work, an applet was made using Fathom), and 5) the classroom discourse (it refers to the language used, which should cover the possible judgements that students make on the central ideas).

For the development of the activity, the following situation was presented to the students:

Two balls are drawn from an urn containing 10 balls numbered from 0 to 9. Consider the interval formed by the integer values found between the minimum and maximum of the values drawn (considering the end points). Which is the probability that the value contain number 8?

*Containing number 8 means that 8 is between the minimum and maximum values or it is one of them.

Technology

Biehler (2013) states that, if it is to be considered in statistics teaching, a digital tool should allow students to perform several actions as: quickly dragging and dropping variables in a graph to visualize distributions and relationships between variables; visualizing in real time how data and parameters change dynamically, affecting measures and related representations; and linking multiple data representations to informally observe statistical trends. Fathom allows for these actions and more.

The applet provided to the students (Figure 2) works as follows:

In an urn called *collection 1*, 10 balls numbered from 0 to 9 are placed. In *Sample of Collection* 1 is a sample of size 2, symbolizing the two values drawn from the urn with which the interval *[min, max]* is created. A measure, *belongs to*, is defined and consists in using the function *if* (*Si*) when number 8 is in the interval or *No* in the opposite case. A collection of measures in different sizes is taken and the tool *Summary* shows the number of *If* and *No* in a certain number of repetitions of the event. The plot shows the behavior of the relative frequencies.

Results and data analysis

For the analysis of the students' responses, we sought words or ideas that were common and placed them in codes. This response grouping process is proposed in the Grounded Theory by Birks and Mills (2015).

For instance, when they created intervals in the applet, students were asked what they observed in the plot. Some of the responses were "the dots are too scattered," "the dots generated by the data are too separated." They were placed in the code *scattering* since the students described how the dots in the plot were placed using the words "scattered" and "separated." Another type of response was "The plot is more constant when we placed more intervals." In these responses, the student observed that the dots in the plot (relative frequencies) were close to a constant value when many intervals were generated. These responses were placed in the code *Tendency to regularity*. Finally, in responses as "In the plot, the dots seem closer when we place more intervals and they are separated when there are fewer," we observed that students managed to see there was a difference when the number of intervals increased. These responses were placed in the code *Variability*. This analysis process was used to group the responses to all the questions in the activity.

The activity was divided in two parts; in the first one, the students carried out a physical simulation. To do so, they were given a bag with 10 balls numbered from 0 to 9. Each student obtained 10 intervals by drawing two balls without replacement. They wrote down the interval formed by the two values obtained $[x_{min}, x_{max}]$ on a table (Figure 1.a) and determined whether number 8 belonged to the interval.

#	Intervalo	Sí	No
1	4 , 6		✓
2	3 , 9	✓	
3	0, 2		✓
4	6 , 7		✓
5	0 , 8	✓	
6	1 , 5		✓
7	3 , 6		✓
8	1 , 2		✓
9	6 , 8	✓	
10	4 , 8	✓	
	Total	4	6

a

$\frac{4}{10}, \frac{4}{10}, \frac{2}{10}, \frac{3}{10}, \frac{3}{10}, \frac{2}{10}, \frac{5}{10}, \frac{3}{10}, \frac{6}{10}, \frac{2}{10}, \frac{3}{10}, \frac{3}{10}$

b

Figure 1. a) Example of table filled in by students and b) group results.

Then, the students were asked to write down the relative frequencies of their classmates (Figure 1.b), and they were also asked about the probability that the interval contained the number 8. Some responded they had to obtain the mean or average (the best approximation), others expressed they had to pay attention to the one repeated the most (mode), and other students used the total relative frequency since they observed that 49 out of 160 intervals contained number 8. At the end of the physical simulation, the students were asked what would happen if they obtained 100 or 1 000 intervals to promote the use of the software. Three predictions were presented for this question: 1) proportionality (8/16)[1], which indicates that the number of favorable cases in 100 intervals must be proportional to 49/160; that is, approximately 30 in 100 intervals and 300 in 1 000; 2) approximation (3/16), where students propose a range of possible values around the proportional value of the frequency (values around 30 are suggested for 100 intervals while 300 are indicated for 1 000); and 3) the attention bias in favorable cases, in which only absolute frequencies, for instance, are said to "increase." The "approximation" responses are the most appropriate given that they take into account data on the relative frequency and predict a certain variability; attention bias is the response with the least quality.

In the second part of the activity, the students use an applet created in Fathom (Figure 2) by the authors and are explained how it works. Once they are familiarized with the software, the students are asked to create blocks of 5, 10, 20, … intervals, observe the corresponding plot, and write down their observations. Responses were classified in three codes, but it must be highlighted that the language used in their responses was mostly geometric and not probabilistic. They refer to the "dots" in the plot without providing any indication that they represent "relative frequencies:" 1) dispersion (5/16), they only say that the first dots in the plot are too separated; 2) a trend towards regularity (3/16), they only say that the "last" dots in the plot make a constant straight line; and 3) variability (4/16), they only deal with the separation between the dots at the beginning and their tendency to create a constant straight line at the end. The responses with the best quality are those classified in variability.

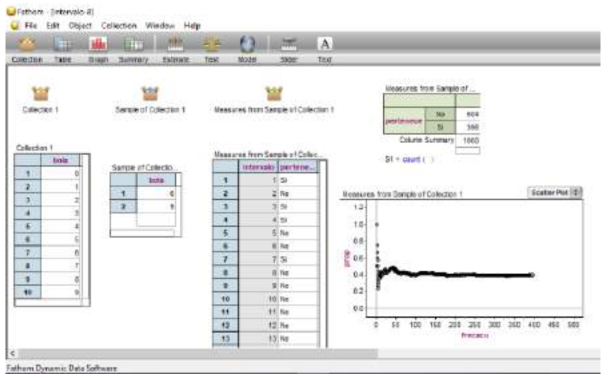

Figure 2. View of applet created by the students using Fathom.

Once the results of the applet were observed represented in the plot, the students were asked to tell the probability that an event contain the number 8. The responses to this question were classified in three codes: 1) relative frequency (8/16) when the students responded with the relative frequency that they obtained when they used the applet (considering 20, 100, and 1 000 tries); 2) approximation (6/16), when the students provided an interval or range within the relative frequency (as in "it is found around 0.306"). To know whether they were sensitive to the number of repetitions of the experiment (or number of intervals), the students were directly asked: What is the difference when there are a few and many intervals? The analysis of the responses led us to the codes above: 1) variability (7/16), 2) tendency to regularity (4/16), and 3) dispersion (1/16). Still, analyzing the responses according to the sense of their expressions, we proposed three additional codes: 1) geometric language (8/16) where they describe the behavior of the plot and not that of the relative frequencies, using terms as "dot," "straight line," or "constant" but not probabilistic terms; 2) variable probability (3/16), when they use the term probability in the same way as relative frequency, meaning that it changes as the number of experiments increases (for example "*The probability is more constant when the interval is greater*"); and 3) a priori probability (2/16), when, from their expression, we understand frequencies tend to a certain number ("*The more intervals there are, the better defined the constant is and also the probability we look for"* and "*The more dots there are, the closer we are to the probability*"). In these responses, students noticeably make a difference between relative frequencies and probability. Therefore, they are closer to the correct probability interpretation.

Finally, they are once again asked a prediction question: "If you had 1 000 intervals (without using the applet), how many of them would contain the number 8? Why?" The responses were classified in: 1) approximation (6/16), when students provide a range of 300 or say "*approximately around 300*;" 2) frequency (5/16), when they provide the frequencies obtained using the software; and 3) approximation to the value found using the software (4/16), when they say that they would obtain something similar to the result provided in the applet.

Discussion and conclusions

During the coding process, the synthesis of features present in several responses allow us to propose four categories that can provide a global notion of the progress and difficulties the students face in the process of conceptualizing not only the random intervals but also the probability in itself. They are:

Sensitivity to variability expressed in two ways. The first consists in accepting that, in prediction problems, it is impossible to accurately predict the number of favorable cases of an event in a series of repetitions of an experiment, but the relative frequency is known to be close to the probability. The second one consists in knowing that successive relative frequencies change greatly in a few repetitions of the experiment, while in the long run, they are stabilized around a constant. Our observations indicate that the activities using the software allow students to perceive variability. This is especially revealed when the simulation process is accompanied by the representation of the trajectory of the relative frequencies and also when, in prediction problems, several students do not provide exact values but intervals or ranges in which results can be found. This achievement is important since it is the basis to subsequently understand the law of large numbers.

No conservation of probability. It consists in believing that the probability of an event changes according to the implementations of the random experience. This phenomenon is similar to that of no conservation of the quantities described by Piaget (Gisnburg & Opper, 1988, p. 149). In the present case, the students do not use the term "relative frequency", preferring instead to use the term "probability." By doing so, they accept the notion that probability changes according to the number of experiments done. Although this is apparently a matter of terminology, it reflects the fact that the students are confused and do not conceive probability as a constant number related to an event.

Descriptive probability. It consists in believing that probability only offers information on the random experiences that have already occurred without reporting the future implementations of such experience. It is related to the previous category in that probability only describes a past state, and, when making new experiments, the state will change; therefore, probability will also change. Under this belief, there can be the misconception that probability does not allow for predictions. It is believed that the constant achieved when many experiments are repeated only occurs when a series of experiments are conducted. However, there is no assurance that the constant will be the same when carrying out other repetitions of the same random experience. This conception can be even more present in real random experiences different from games of chance; for example, social, medical, or weather problems.

Absence of a probabilistic language. It consists in describing a procedure or probabilistic result using non-probabilistic terms associated to a representation. In the students' responses to the question What is observed? Several students use geometric terms (dots, closeness/distance, constant, straight line) in the trajectories of relative frequencies without making any reference to relative frequencies and probability. The tendency to not use probabilistic terms to say what trajectories means raises the question of whether students understand the probabilistic meaning of trajectories of relative frequencies; that is, whether they interpret such representations as relative frequencies trending towards probability.

The four categories that emerged from the data of the exploration of the students' reasoning are more general to the particular situation of random intervals studied. They can also make sense in probability situations where there are problems to be solved through digital probabilistic simulations and a frequentist approach of probability. Indeed, in any situation where a computational simulation is used, it is suitable to consider and remember variability. In any preuniversitary teaching design, it should be considered that students can be at a stage where they do not accept the continuity of probability. They might also believe that probability only describes outcomes that have already occurred without any future consequences. Furthermore, care must be taken so that students interpret computational representations of probabilistic objects, as trajectories of relative frequencies, in probabilistic terms.

We also conclude that the use of technology was important because students managed to observe that relative frequencies in the plot generated by the applet converged in one value. Although they used geometric language, we think that better responses could be obtained by stressing what the constant and each point represent.

Finally, as a result of the analysis, we observed several ways in which the study can be improved to continue with another research cycle. Particularly, we have seen that the formulation of some questions should be improved, and more questions must be added to obtain further information on some aspects. For instance, it would be suitable to include an additional question to find to what extent do students who make geometric descriptions of the trajectories of relative frequencies understand the probabilistic background of the situation.

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COVARIATIONAL REASONING PATTERNS OF HIGH SCHOOL STUDENTS IN PROBLEMS OF CORRELATION AND LINEAR REGRESSION

PATRONES DE RAZONAMIENTO COVARIACIONAL DE ESTUDIANTES DE BACHILLERATO EN PROBLEMAS DE CORRELACION Y REGRESION LINEAL

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The topics of correlation and linear regression constitute a complex and subtle system of statistical and mathematical ideas whose teaching-learning raises numerous practical and theoretical problems. In this research paper, the patterns of reasoning that students exhibit, under the approach of informal inferences when they face problems of correlation and regression line are identified. To achieve this, activities were implemented in two stages: in the first stage, two problems (one of estimation and other of best-fit line) were applied to be solved using pencil and paper; the second stage incorporates the use of the Fathom software.

Keywords: Correlation, Linear Regression, Technology, Informal Inferences, Reasoning.

Background and Issues

Covariational statistical reasoning consists of the processes that allow subjects to perceive, describe and justify the relationships between statistical variables. These processes occur in two ways, on the one hand they occur in the subject's mind, and on the other, in the spoken or written discourse that occurs when relationships between variables are described or justified (Moritz, 2004). When the solid arguments or generalizations that students make are highlighted, based on the information they have, instead of only the representations they carry out, we speak of inferences; in this study we will refer to informal statistical inferences (without a formal instruction or explicitly formal procedure). Several authors have reported relatively recently, studies on informal inferential reasoning using the informal statistical inferences made by students as the premise (Ben-Zvi, 2006; Pfannkuch, 2005). For example, in the aforementioned Ben-Zvi work, the informal statistical reasoning of 5th grade students is analyzed and developed within a technological environment, reporting that the use of technological tools showed argumentative advantages in the way students presented ideas.

In this sense, we focus on covariational reasoning starting with informal inferences that students make, taking as reference some relevant clues about this type of reasoning. For example, Zieffler and Garfield (2009, p. 11) summarize some findings: often, students, 1) are significantly influenced by their personal beliefs regarding their covariational judgments; 2) they frequently assume that there is a correlation between two events that is non existent (illusory correlation) ; 3) they are liable to imply causal relationships when dealing with covariation tests; and 4) have difficulty reasoning about covariation when the relationship between variables is negative. Regarding the determination of the line of best fit, some studies, such as those by Casey (2014) and Casey and Wasserman (2015), show that: 1) it is necessary to induce students to convert the data collected in tables into a graphic representation (Dispersion diagram); 2) Students have difficulty observing the global trend of a data set when reading a scatter plot, because they focus their attention on isolated points and perceive the data as a series of individual cases, rather than considering them holistically (with invisible characteristics for isolated points); 3) Many students, when drawing a line of best fit, focus their attention on characteristic points such as the first or last, the highest or the lowest, or a subset of these.

Theoretical framework

This work uses the theory of Rubin, Hammerman and Konold (2006) on informal inferences reasoning seen as statistical reasoning that considers the dimensions of: *Aggregate*, *Signals and Noise*, and *Various Forms of Variability*. An aggregate vision involves the added characteristics of individual cases, that when seen together, enable properties to emerge that are different from those of the individual cases.. Signals and noise refers, on the one hand, to constant elements in statistics such as the mean or the best fit line (signals) and on the other to variable elements that serve to introduce variability around any signal (noise). The idea of considering the various forms of variability refers to the fact that when making judgments from a set of data, the variability that underlies the situation must be considered. Various studiesin the area of statistical education show that students at all levels have difficulties when reasoning about these ideas (Ben-Zvi and Garfield, 2004; Bakker *et al*., 2004).

Based on these ideas, we propose as a central objective of this research: To determine and characterize the reasoning patterns of high school students in the face of correlation problems and the line of best fit, as informal inferences.

Method

First stage of the study

Participants. The questionnaire was applied to a group of 30 high school students (16 to 18 years old) who were taking the subject of Statistics and Probability at the College of Sciences and Humanities, Plantel Vallejo, located in Mexico City. Two instructor-researchers, co-authors of this article participated in the application of the questionnaire; they provided worksheets containing the problems and guided the dynamics of the sessions, in particular clarifying doubts about the instructions of the activities without delving into the content.

Instruments and Execution. The questionnaire was applied to students who had not received formal and expository instruction on these topics. Two problems were chosen to report on, which are shown in Figure 1. The first priblem is an Estimation problem taken from Moore (1988) and which was modified to the present version. The second is a best fit line problem for which a scatter diagram of the variables Height and Weight is presented, in a situation in which they are directly correlated and with a linear trend. Given the diagnostic nature of the questionnaire, a didactic sequence was not elaborated as such, instead the students were given a series of problems that were solved individually during two sessions of 90 min each, only allowing the use of pencil and paper.

Estimation Problem

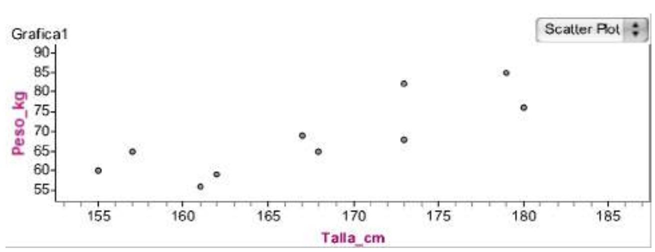
The Morales family is about to install solar panels at their home to cut down on heating costs. To better understand the savings that installing these panels can mean, the Morales have been recording their gas consumption for the last year and a half. The table shows the data with the average gas consumption and the average ambient temperature for each month:

Average Temperature (° C)	5.2	-9.8	-5.4	0.2	4.1	11.3
Gas Consumption (m^3)	17.6	30.5	24.9	21	14.8	11.2
Average Temperature (° C)	16.3	18.5	18.5	18	15.2	11.8
Gas Consumption (m^3)	4.8	3.4	3.4	3.4	5.9	8.7
Average Temperature (° C)	1.8	0.7	-10.4	1.8	17.1	10.7
Gas Consumption (m^3)	17.9	20.2	30.8	19.3	6.3	10.7

If the average temperature recorded in a month is 8 °C, what is the gas consumption expected by the Morales family in that month? Explain your answer:

Line of Best Fit Problem

The graph below shows the height and weight data for 10 high school students. Draw the line that you think best fits the data.



Explain the criteria you used to draw the line:

Figure1. Estimation and Best Fit Line Problem

Analysis Methodology. The analysis of the responses was carried out through a coding process (types of reasoning or inferences) of the responses, gradually refining through comparative analysis. Each researcher carried out an open coding of the data, generating codes that represent patterns of reasoning according to the similarities in the procedures and arguments declared by the students, which were then compared with each other. Comparison of these coding proposals resulted in a more consistent and abstract set of codes than the descriptions initially developed; this preliminary scheme facilitated inducing some central ideas of the students' covariational reasoning.

Second stage of the investigation

Participants. The activities were applied to a group of 40 high school students (20 couples) (16 to 18 years old) from the Colegio de Bachilleres Plantel 2, which is also located in Mexico City. Given the uncontrollable nature of the classroom, certain students were absent from some class sessions. Similarly, the two researchers involved in the first stage of the study participated in the experiment. In addition to guiding the class sessions, their participation included supporting the students in manipulating the Fathom software. As in the previous stage, the professor-researchers were prevented from delving into the thematic content.

Instruments and Execution. Again, for this report we have chosen two problems that we want to focus on: the first, which deals with the estimation of a response value, which was the same as in the first stage, only in this case the use of the Fathom software was incorporated as a tool for the student to be able to build a scatter diagram on and obtain the least squares line, to make the requested estimate. The second (figure 2) also deals with proposing the line that best fits the data, but the context of the situation was modified. Unlike the first stage, in this stage the students collected statistical data by measuring some of their own physiological attributes (namely, for the same student, the measurement of their height against that of their arm). In addition, they used the software to draw and manipulate a line that they considered best fit the data.

Line of Best Fit Problem

According to studies of the anatomy of the human body, there is a certain relationship between the height of people and the measurement of some parts of the body.

With the data of the measurement of the arms (from elbow to shoulder) and the height of your colleagues, which were collected in the first work session, and which are shown in the table, answer what is asked of you.

X (arm, cm)	30	33	35	36	32	38
Y (height, cm)	153	164	175	177	160	175
X (arm, cm)	34	31	35	29	28	38
Y (height, cm)	167	154	180	162	155	180

Figure 2. Problem of best fit line.

Analysis Methodology. The coding process of the data obtained in this stage was extended to the search for solid and consistent relations between the codes identified in both phases of the study; this process was carried out by investigating possible coincidences and patterns that we considered emerged with sufficient sense and coherence. Once the responses from the two stages of the study were coded, conceptual connections were identified between the proposed codes, to finally define the reasoning patterns as informal inferences made by the students.

Results

Analysis of the first stage of the study

During this stage, a system of codes representing the types of reasoning or inferences showing the student responses was developed, these codes are defined in terms of the arguments exhibited, that is to say the identification correlation between variables, use of all available data, or perception of the uncertainty that underlies the data. Some evidence of student responses is presented.

For the problem of estimation, a type of reasoning was defined as *Arithmetic Interpolation* (4/28 responses) representing the responses where the student takes a range of values in which the data of the given temperature (8 ° C) is included and with its corresponding gas consumptions, the student obtains the average to make the estimate. In this case the student uses data from the two variables, which we consider a slight statistical approach towards obtaining the arithmetic average. The response of the student E10 is presented as evidence in figure 3, where it can be seen that he chooses two temperature values (5.2 ° C and 10.7 ° C) among which is the data of 8 ° C - of which the estimate is requested - and with their corresponding gas consumption values (17.6 m^3 and 10.7m^3) calculate their average to give their estimated answer.

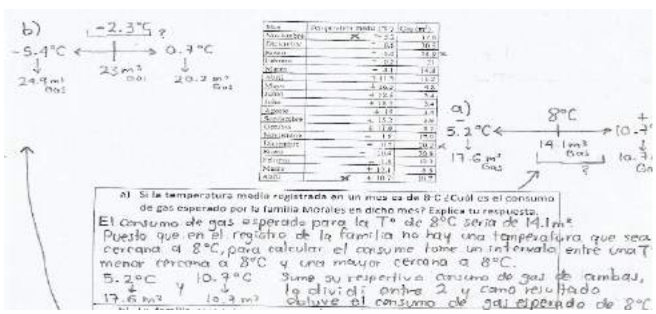


Figure 3. Student response E10. Arithmetic Code Interpolation

The following code we call *Proportional Arithmetic* (8/28) includes the answers where students look for a proportionality factor; choosing a pair of data (X-Temperature, Y-Consumption) and with the given temperature of 8 ° they form a rule of three, assuming that there is a proportional relationship between the variables. The student's answer E9 is shown as an example, in which he chooses the pair of values that correspond to the first month of the table (5.2 ° C, 17.6m^3) and with the value of 8 ° C he forms a rule of three.

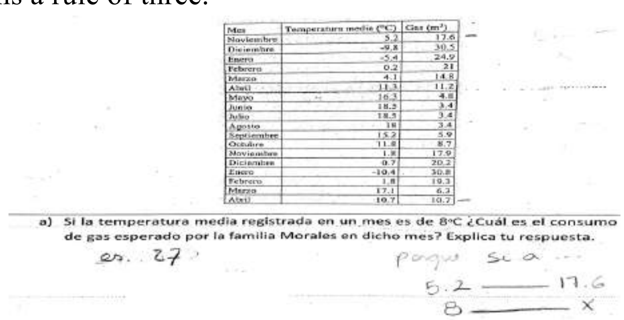


Figure 4. Student answer E9. Proportional Arithmetic Code

The *Arithmetic Reasoning Following A Pattern* (2/28) includes the answers where the students take as a reference the given temperature value (8 °) and try, from the data in the table, to “complete” this value by means of some arithmetic procedure, and once they get it they use that data in order to obtain their answer.

In the *Arithmetic Type Without Defined Pattern* (6/28) the students used some basic operations (addition, subtraction, multiplication and division; in one case the square root is used) but without being able to deduce a well-defined procedure.

In the *Perception Of The Trend* code (6/28) the student does not carry out any operation and only focuses his attention on the data in the table and his answers are based on a visual analysis of the trend of the data, in particular in the meaning of their behavior.

Finally, the code *Without Argument* (2/28) was defined, which represents those responses where the student only provides the result, without arguing or making their procedure explicit.

For the problem of drawing the line of best fit, two characteristic types of reasoning were found, the *Partition* code (6/21 responses) where the responses were classified in which the student draws the line or refers to the fact that their position must be such that passes through the middle of the cloud of points, following its direction, that is to say, traced diagonally, leaving the same number of points on one side and the other of the line, as shown in Figure 5.

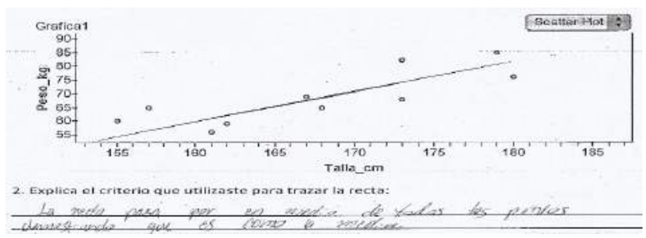


Figure 5. Student response E14. Partition Code

The *Belonging* code (15/21) includes the responses that show two types of behavior, on the one hand, those where the student draws or refers to the fact that the line must pass through as many points as possible or through all of them, and there are also the answers where the student draws the line insuring that it passes through specific points of the cloud, as in the answer shown in the following Figure, where it is argued that it must pass through two points (the lowest and the highest).

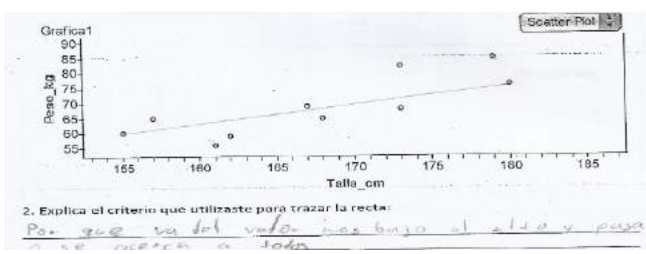


Figure 6. Student response E28. Membership Code

Analysis of the second stage of investigation

For the estimation problem, most of the identified reasoning was presented in the same way as in the diagnostic questionnaire, however, at this stage it stands out that the Arithmetic Interpolation code is absent in the students' responses and was replaced by the Use code of Software (3/15 pairs), in which the student uses the software to modify the position of a point in the cloud up to the given temperature value (8 ° C) and, following the trend of the data set, provides the estimated value of gas consumption. An evidence of this code is shown below:

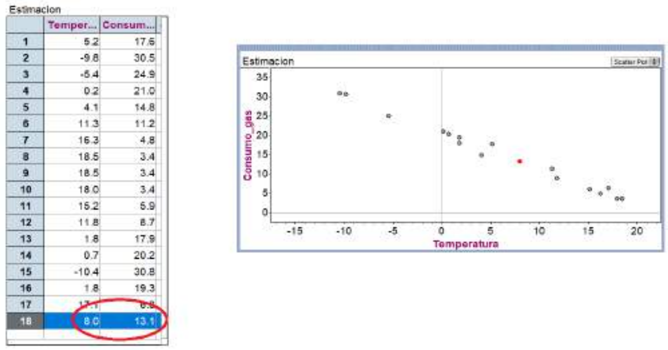


Figure 7. P4 partner response. Software Use Code

In the problem of the line of best fit, some students took the measurement of the arm (from elbow to shoulder) and with its corresponding height they formed a new bivariate database; It is from this set that the respective scatter diagram with which the students worked with was constructed.

The *Partition* and *Belonging* codes emerged with a frequency similar to that observed in the previous stage, with the exception of a new argument that we called *Closeness* (7/19 couples); The code includes the answers in which the students position the line with the help of the software in such a way that it is as close as possible to most of the points. The response of the pair of students P6 is included as evidence.

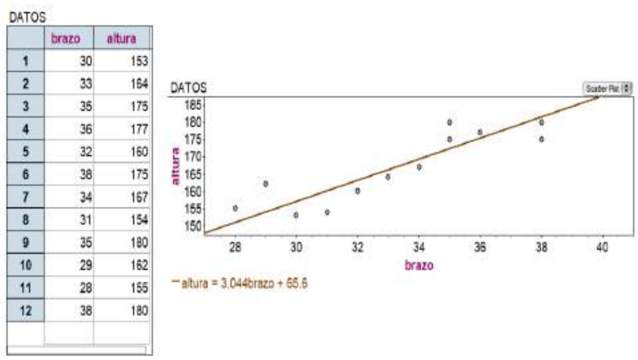


Figure 8. Pair answer P6. Closeness Code

A summary table of the analysis and initial coding is shown for the two stages of the study:

Table 1. Coding Process

Code	First stage	Second stage	Relative frequency	
Arithmetic Interpolation	✓		0.143	0.000
Use of Software		✓	0.000	0.200
Arithmetic Proportionality	✓	✓	0.286	0.267
Arithmetic Following a Pattern	✓	✓	0.071	0.267
Arithmetic Without Following a Pattern	✓	✓	0.214	0.267
Perception of the Trend	✓		0.214	0.000
No Argument	✓		0.071	0.000
Partition	✓	✓	0.286	0.211
Closeness		✓	0.000	0.368
Belonging	✓	✓	0.714	0.421

The next phase of the analysis was to compare the codes defined in the two stages of the study and to identify common covariational reasoning traits and some arguments raised in the second stage; the process of conceptualization of reasoning (inference) of student responses are described below.

Based on the inferences made by the students, conceptual connections were identified between the reasoning they exhibit, hence the codes *Perception of Tendency* and *Use of Software* (estimation problem) were reclassified in the reasoning pattern *Notion of Aggregate,* since in both students use the complete set of values that they are given to argue their responses; regardless of whether or not their results are normatively correct. On the other hand, the codes *Interpolation*, *Proportionality*, *Arithmetic following a pattern, Arithmetic without following a Patern* (estimation problem), *Partition* and *Belonging* (problem straight adjustment) will be reclassified as a *search for a signal*. The interconnection between these codes lies in the fact that the students infer that there must be a kind of clue to solve the problem, and they carry out a search in the data that was provided. It seems that the student suspects that there is a hidden pattern or structure in a subgroup of data (he does not use the totality of the data or contemplate the set of points) that will lead him to constant structures, absent of uncertainty or variation, that are familiar to him. Finally, the *Closeness* code (best fit line problem) was renamed as *Sense of Variability*, since features are perceived, albeit in a spurious way, that students infer a relationship between the fit model and the points of the cloud, in the scatter diagram. The *No Argument* code that was presented in the estimation problem, by not providing evidence of some type of inference made by the students, was not considered at this stage of the analysis.

Table 2. Process Conceptualization

Code	Informal Reasoning Pattern / Inferences
Perception of the Trend Use of Software	Notion of Aggregate
Arithmetic Interpolation Arithmetic Proportionality Arithmetic Following a Pattern Arithmetic Without Following a Pattern Partition Belonging	Search for a Signal
Closeness	Sense of Variability

Table 2 shows a summary of the conceptualization and definition of the identified reasoning patterns.

Conclusions

From our theoretical perspective, an informal inference is a type of reasoning that includes considerations in several dimensions (Rubin, Hammerman and Konold, 2006) and in specific study conditions. In our case a first dimension is the *Notion Added*, where a holistic perception of the problem situation over the contemplation of individual cases is privileged. In this sense, the initially defined codes, *Perception of the trend* and *Use of software*, present as a pattern that the students consider all the data they have available.On the one hand, they estimate the value of the requested response variable (gas consumption) by making a purely visual analysis based on the trend of the data set, that is, they argue their response based on the global behavior of the set of values. On the other hand, when they use Fathom, when constructing the scatter diagram and observing the trend of the point cloud again, they identify that there must be a certain relationship between the points, so they choose one and modify its position in such a way that the value matches the independent variable (8 ° C temperature) with that of the corresponding response variable (gas consumption), following the general "shape" of the cloud. Although it is true that in none of the above cases the response of the students is normatively adequate, it does allow us to appreciate that under certain circumstances they are able to perceive that in bivariate relationships it is necessary to consider the characteristics and behavior of the data as a whole and, based on this, infer the value of some particular point of interest.

Another dimension to consider is reflected with the pattern that we define as *Search for a signal* , where the answers show that given the difficulty represented by the uncertainty or the intrinsic variation in this type of problem, the students try to solve the situation in a familiar terrain for them or with which they feel comfortable, possibly for this reason they mostly use arithmetic procedures (rule of three, proportionality factor or additions and subtraction) to make the estimation; where they also only use part of the data. It is also the case of the best fit line problem, in which in the absence of the notion of uncertainty or of this aggregate view, they partially use the available data, referring to the linear function model as an alternative to fit a line to a distribution of points that presents a linear trend, considering only some of these points in their choice, ignoring the influence of all of these and their joint variation, and above all, defining two types of data: those that do or do not belong to the linear model that they choose to plot.

As a third dimension we propose, *Sense of Variability*, represented by the code that was initially defined as *Closeness*. In this pattern of reasoning, the answers that refer, albeit briefly, to the perception that there is some variation that underlies these types of problems and that that must be considered when proposing a line of adjustment for a set of points., just as the students did when plotting and arguing that the line should be positioned as close to most of the points as possible.

We trust that the identification of these reasoning patterns as the way in which students make informal inferences in the face of statistical association problem situations adds to the body of knowledge in the study of bivariate data, without neglecting the importance of exploring obstacles of learning, such as the apparent disconnection that the student has between the predictive or inferential nature inherent in the linear regression model and its identification as the line that best fits the set of points, as well as the difficulty to conceive the data set as an aggregate , that is, as a system in which they are linked to each other and have the property of being deviations from the same model.

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PATRONES DE RAZONAMIENTO COVARIACIONAL DE ESTUDIANTES DE BACHILLERATO EN PROBLEMAS DE CORRELACION Y REGRESION LINEAL

COVARIATIONAL REASONING PATTERNS OF HIGH SCHOOL STUDENTS IN PROBLEMS CORRELATION AND LINEAR REGRESSION

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Los temas de correlación y regresión lineal constituyen un sistema complejo y sutil de ideas estadísticas y matemáticas cuya enseñanza-aprendizaje plantea numerosos problemas prácticos y teóricos. En esta investigación se identifican patrones de razonamiento que exhiben los estudiantes, bajo el enfoque de inferencias informales cuando enfrentan problemas de correlación y regresión lineal; para esto se implementaron actividades en dos etapas: en la primera se aplicaron dos problemas (uno de estimación y otro de recta de mejor ajuste) para ser resueltos a lápiz y papel; en la segunda se incorpora el uso del software Fathom.

Palabras clave: Correlación, Regresión Lineal, Tecnología, Inferencias Informales, Razonamiento.

Antecedentes y Problemática

El razonamiento estadístico covariacional consiste en los procesos que le permiten a los sujetos percibir, describir y justificar las relaciones entre variables estadísticas; estos procesos se presentan en dos sentidos, por un lado ocurren en la mente del sujeto, y por otro, en el discurso hablado o escrito cuando se describen o justifican relaciones entre variables (Moritz, 2004). Cuando se destacan las argumentaciones sólidas o generalizaciones que los estudiantes realizan, a partir de la información con que cuentan, en lugar de solo las representaciones que llevan a cabo se habla de inferencias: en este estudio nos referiremos a las inferencias estadísticas informales (sin una instrucción formal o procedimiento explícitamente formales). Varios autores han reportado en fechas relativamente recientes, estudios sobre el razonamiento inferencial informal tomando como premisa las inferencias estadísticas informales que realizan los estudiantes (Ben-Zvi, 2006; Pfannkuch, 2005). Por ejemplo en el trabajo de Ben-Zvi mencionado, se analiza y desarrolla el razonamiento estadístico informal de estudiantes de 5° dentro de un ambiente con tecnología, reportando que el uso de herramientas tecnológicas mostró ventajas argumentativas en la presentación de ideas por parte de los estudiantes.

En este sentido, nos enfocamos en el razonamiento covariacional partiendo de inferencias informales que los alumnos hacen, tomando como referencia algunas pistas relevantes sobre este tipo de razonamiento; por ejemplo, Zieffler y Garfield (2009, p. 11) resumen algunos hallazgos: a menudo, los estudiantes, 1) se encuentran influenciados significativamente por sus creencias personales con respecto a sus juicios covariacionales; 2) suponen frecuentemente que existe correlación entre dos eventos que no lo están (correlación ilusoria); 3) son susceptibles a implicar relaciones causales cuando tratan con pruebas de covariación; y 4) tienen dificultad para razonar acerca de la covariación cuando la relación entre las variables es negativa. Sobre la determinación de la recta de mejor ajuste, algunos estudios, como los de Casey (2014) y Casey y Wasserman (2015), manifiestan que: 1) es necesario inducir a los estudiantes convertir los datos recopilados en tablas en una representación gráfica (diagrama de dispersión); 2) los estudiantes tienen dificultad para observar la tendencia global de un conjunto de datos cuando leen un diagrama de dispersión, debido a que enfocan su atención en puntos aislados y perciben los datos como una serie de casos individuales, en lugar de considerarlos de manera holística (con características invisibles para puntos aislados); 3) muchos estudiantes, al trazar una recta de ajuste, enfocan su atención en puntos característicos como el primero o el último, el más alto o más bajo, o en un subconjunto de estos.

Marco Teórico

Este trabajo toma ideas de la teoría de Rubin, Hammerman y Konold (2006) sobre inferencias informales vistas como razonamientos estadísticos que implican considerar las dimensiones de: *Agregado*, *Señales y Ruido*, y *Diversas Formas de Variabilidad*. Una visión de agregado implica que las características de los casos individuales, al ser vistas en conjunto, permiten que emerjan propiedades que son diferentes de las que tienen los casos individuales por sí mismos. Señales y ruido, se refiere por un lado a elementos constantes en estadística como la media o la recta de ajuste (señales) y por otro a elementos variables que sirven para introducir variabilidad alrededor de cualquier señal (ruido). La idea de considerar las diversas formas de variabilidad, se refiere a que al elaborar juicios a partir de un conjunto de datos se debe considerar la variabilidad que subyace en la situación. Diversas investigaciones en el área de la educación estadística manifiestan que estudiantes de todos los niveles presentan dificultades al razonar sobre estas ideas (Ben-Zvi y Garfield, 2004; Bakker *et al*., 2004).

Con base en estas ideas planteamos como objetivo central de la investigación: Determinar y caracterizar los patrones de razonamiento de estudiantes de bachillerato ante problemas de correlación y de recta de mejor ajuste, como inferencias informales.

Método

Primera etapa de la investigación

Participantes. El cuestionario se aplicó a un grupo de 30 alumnos de bachillerato (16 a 18 años de edad) que se encontraban cursando la asignatura de Estadística y Probabilidad y pertenecientes al Colegio de Ciencias y Humanidades, Plantel Vallejo, localizado en la Ciudad de México. Participaron en la aplicación del cuestionario dos profesores-investigadores coautores del presente artículo; proporcionaron hojas de trabajo que contenían los problemas y guiaron la dinámica de las sesiones, en particular aclarando dudas sobre la instrucción de las actividades sin profundizar en los contenidos.

Instrumentos y Ejecución. El cuestionario se aplicó a estudiantes que no habían recibido instrucción formal y expositiva sobre dichos temas. Se eligieron dos problemas para reportar que se muestran en la figura1, el primero es un problema de Estimación tomado de Moore (1988) y que se modificó a la presente versión. En el segundo problema, de recta de ajuste, se presenta un diagrama de dispersión de las variables Talla y Peso, en una situación en que están correlacionadas directamente y con una tendencia lineal. Dado el carácter diagnóstico del cuestionario, no se elaboró como tal una secuencia didáctica, sino una serie de problemas que fueron resueltos individualmente por los alumnos durante dos sesiones de 90 min cada una, permitiéndose únicamente el uso de lápiz y papel.

Problema de Estimación

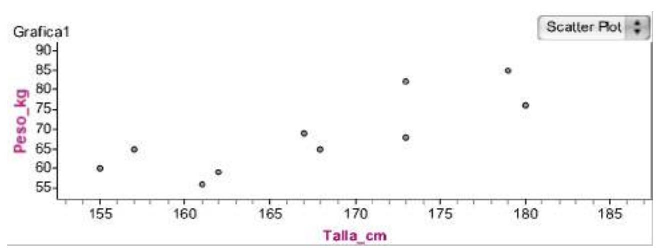
La familia Morales está a punto de instalar paneles solares en su casa para reducir el gasto en la calefacción. Para conocer mejor el ahorro que puede significar la instalación de dichos paneles, los Morales han ido registrando su consumo de gas durante el último año y medio. En la tabla se muestran los datos con el promedio del consumo de gas y de la temperatura media ambiental de cada mes:

Temperatura	5.2	-9.8	-5.4	0.2	4.1	11.3	16.3	18.5	18.5
Consumo de Gas (m^3)	17.6	30.5	24.9	21	14.8	11.2	4.8	3.4	3.4
Temperatura	18	15.2	11.8	1.8	0.7	-	1.8	17.1	10.7
Consumo de Gas (m^3)	3.4	5.9	8.7	17.9	20.2	30.8	19.3	6.3	10.7

Si la temperatura media registrada en un mes es de 8°C ¿Cuál es el consumo de gas esperado por la familia Morales en dicho mes? Explica tu respuesta:

Problema de Recta de Mejor Ajuste

La siguiente gráfica muestra los datos de la talla y el peso de 10 estudiantes de bachillerato. Traza la recta que piensas se ajusta mejor a los datos.



Explica el criterio que utilizaste para trazar la recta:

Figura 1. Problema de Estimación y de Recta de mejor ajuste

Metodología de Análisis. El análisis de las respuestas fue llevado a cabo mediante un proceso de codificación (tipos de razonamiento o inferencias) de las respuestas, refinándose gradualmente a través de un análisis comparativo. Cada investigador realizó una codificación abierta de los datos generando códigos que representan patrones de razonamiento según las semejanzas en los procedimientos y argumentos declarados por los estudiantes, que luego fueron comparados entre sí. La comparación de estas propuestas de codificación dio lugar a un conjunto de códigos más consistente y abstracto que las descripciones inicialmente elaboradas; este esquema preliminar facilitó inducir algunas ideas centrales en el razonamiento covariacional de los estudiantes.

Segunda etapa de la investigación

Participantes. Las actividades se aplicaron a un grupo de 40 estudiantes (20 parejas) de bachillerato (16 a 18 años) del Colegio de Bachilleres Plantel 2, que también se encuentra ubicado en la Ciudad de México; dada la naturaleza poco controlable del aula escolar, ciertos estudiantes se ausentaron de algunas sesiones de clase. De igual manera, en el experimento participaron los dos investigadores involucrados en la primera etapa del estudio; además de guiar las sesiones de clase, su participación incluyó el apoyar a los alumnos en la manipulación del software Fathom. Al igual que en la etapa previa, se evitó que los profesores-investigadores profundizaran en los contenidos temáticos.

Instrumentos y Ejecución. Nuevamente, para este reporte hemos elegido dos problemas en los que deseamos enfocarnos: el primero, que trata sobre la estimación de un valor respuesta, fue el mismo que en la primera etapa, solo que en este caso se incorporó el uso del software Fathom como una herramienta para que el estudiante estuviera en posibilidad de construir un diagrama de dispersión y obtener la recta de mínimos cuadrados, para realizar la estimación pedida. El segundo (figura2) también trata sobre proponer la recta que mejor ajusta a los datos pero el contexto de la situación fue modificado. A diferencia de la primera etapa, en esta los estudiantes recolectaron los datos estadísticos a través de la medición de algunos atributos fisiológicos propios (a saber, para un mismo estudiante, la medida de su talla contra la de su brazo); además, utilizaron el software para trazar y manipular una recta que consideraran se ajustara mejor a los datos.

Problema de Recta de Mejor Ajuste

Según estudios de la anatomía del cuerpo humano, existe cierta relación entre la talla (altura) de las personas y la medida de algunas partes del cuerpo.

Con los datos de la medida de los brazos (del codo al hombro) y de la altura de tus compañeros, que se recolectaron en la primera sesión de trabajo, y que se muestran en la tabla, contesta lo que se te pide.

X (brazo,cm)	30	33	35	36	32	38	34	31	35	29	28	38
Y (altura,cm)	153	164	175	177	160	175	167	154	180	162	155	180

Figura 2. Problema de Recta de Mejor Ajuste

Metodología de Análisis. El proceso de codificación de los datos obtenidos en esta etapa se extendió a la búsqueda de relaciones sólidas y consistentes entre los códigos identificados en ambas etapas del estudio; dicho proceso se realizó indagando posibles coincidencias y patrones que consideramos emergieron con suficiente sentido y coherencia. Una vez que se codificaron las respuestas de las dos etapas del estudio, se identificaron conexiones conceptuales entre los códigos propuestos, para finalmente definir los patrones de razonamiento como inferencias informales que realizan los alumnos.

Resultados

Análisis de la primera etapa de investigación

Durante esta etapa se desarrolló un sistema de Códigos que representan los tipos de razonamientos o inferencias que muestran las respuestas de los estudiantes, estos códigos se encuentran definidos en función de las argumentaciones que exhiben, es decir, la identificación de la correlación entre las variables, la utilización de todos los datos disponibles o la percepción de la incertidumbre que subyace en los datos. Se presentan algunas evidencias de las respuestas de los estudiantes.

Para el problema de estimación, un tipo de razonamiento se definió como *Aritmético Interpolación* (4/28 respuestas) que representa las respuestas donde el estudiante toma un intervalo de valores en el que se encuentra incluido el dato de la temperatura dado (8°C) y con sus correspondientes consumos de gas obtiene el promedio para realizar la estimación. En este caso el alumno utiliza datos de las dos variables, lo que consideramos un ligero acercamiento estadístico al obtener el promedio aritmético.

Se presenta como evidencia la respuesta del estudiante E10 en la figura3, donde se aprecia que elige dos valores de temperatura (5.2°C y 10.7°C) entre los que se encuentra el dato de 8°C –del que se pide la estimación- y con sus correspondientes valores de consumo de gas (17.6 m^3 y 10.7m^3) calcula su promedio para dar su respuesta estimada.

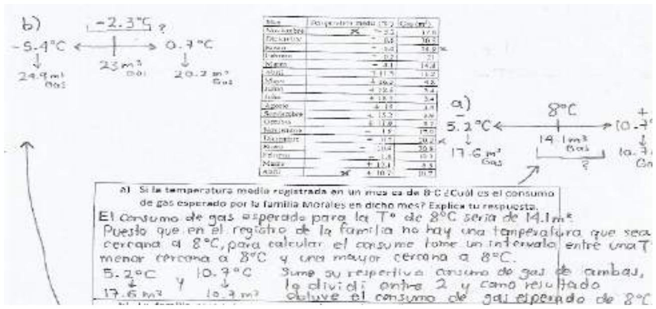


Figura 3. Respuesta estudiante E10. Código Aritmético Interpolación

El siguiente código lo llamamos *Aritmético Proporcional* (8/28) incluye las respuestas donde los estudiantes buscan un factor de proporcionalidad; eligiendo una pareja de datos (X-Temperatura, Y-Consumo) y con la temperatura dada de 8° forman una regla de tres, asumiendo que entre las variables existe una relación proporcional. Se muestra como ejemplo la respuesta del estudiante E9, en la que elige la pareja de valores que corresponden al primer mes de la tabla (5.2°C, 17.6m^3) y con el valor de 8°C forma una regla de tres.

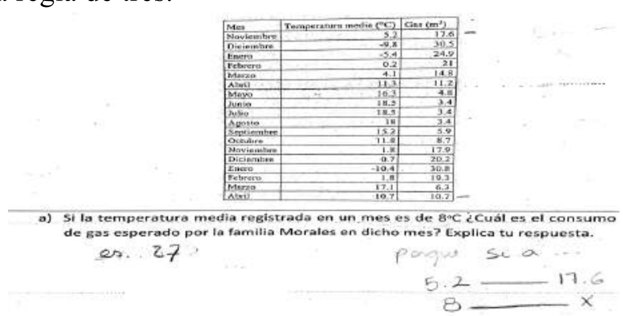


Figura 4. Respuesta estudiante E9. Código Aritmético Proporcional

El razonamiento *Aritmético Siguiendo Un Patrón* (2/28) incluye las respuestas donde los estudiantes toman como referencia el valor de la temperatura dado (8°) y tratan, a partir de los datos de la tabla, de "completar" este valor mediante algún procedimiento aritmético, y una vez que lo consiguen utilizan esos datos para obtener su respuesta.

En el tipo *Aritmético Sin Patrón Definido* (6/28) los estudiantes utilizaron algunas operaciones básicas (suma, resta, multiplicación y división; en un caso se utiliza la raíz cuadrada) pero sin que se pueda deducir un procedimiento bien definido.

En el código *Percepción De La Tendencia* (6/28) el estudiante no realiza operación alguna y sólo enfoca su atención en los datos de la tabla y sus respuestas se basan en un análisis visual de la tendencia de los datos, en particular en el sentido de su comportamiento.

Finalmente, se definió el código *Sin Argumento* (2/28) que representa aquellas respuestas donde el estudiante solo aporta el resultado, sin argumentar o hacer explícito su procedimiento.

Para el problema de trazar la recta de mejor ajuste se encontraron dos tipos de razonamiento característicos, el código *Partición* (6/21 respuestas) donde se clasificaron las respuestas en las que el estudiante traza la recta o hace referencia a que su posición debe ser tal que pase por en medio de la nube de puntos, siguiendo la dirección de ésta, es decir trazada de forma diagonal, dejando de un lado y del otro de la recta, el mismo número de puntos, como lo muestra la Figura 4.

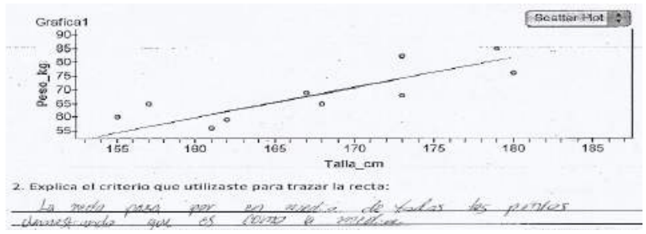


Figura 5. Respuesta estudiante E14. Código Partición

En el código *Pertenencia* (15/21) se incluyen las respuestas que muestran dos tipos de comportamiento, por un lado aquellas donde el estudiante traza o refiere que la recta debe pasar por el mayor número posible de puntos o por la totalidad de estos, y también están las respuestas donde el estudiante traza la recta cuidando que pase a través puntos específicos de la nube, como en la respuesta mostrada en la siguiente Figura, donde se argumenta que debe pasar por dos puntos (el más bajo y más alto).

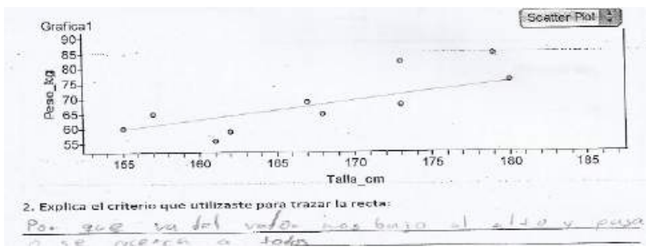


Figura 6. Respuesta estudiante E28. Código Pertenencia

Análisis de la segunda etapa de investigación

Para el problema de estimación, la mayoría de los razonamientos identificados se presentaron de igual forma que en el cuestionario diagnóstico, sin embargo en esta etapa destaca que el código *Aritmético Interpolación* se encuentra ausente en las respuestas de los estudiantes y fue sustituido por el código *Uso de Software* (3/15 parejas), en el cual el estudiante utiliza el software para modificar la posición de un punto de la nube hasta el valor de temperatura dada (8° C) y, siguiendo la tendencia del conjunto de datos, proporciona el valor estimado del consumo de gas. Una evidencia de este código se muestra a continuación:

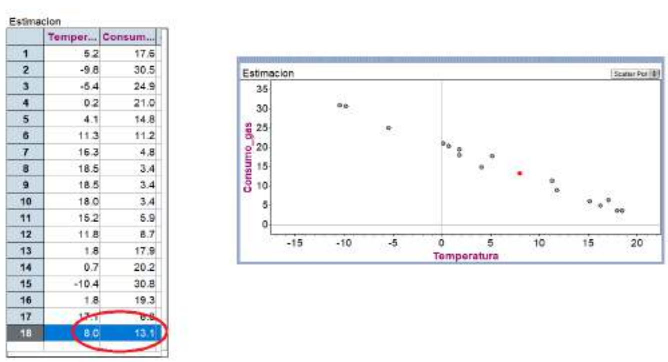


Figura 7. Respuesta pareja P4. Código Uso de software

En el problema de recta de mejor ajuste algunos alumnos se tomaron la medida del brazo (del codo al hombro) y con su correspondiente altura conformaron una nueva base de datos bivariados; a partir de este conjunto es que se construyó el respectivo diagrama de dispersión con el que trabajaron los estudiantes.

Los códigos *Partición* y *Pertenencia* emergieron con una frecuencia similar a la observada en la etapa previa, con la excepción de un nuevo argumento que denominamos como *Cercanía* (7/19 parejas); el código engloba a las respuestas en las que los estudiantes posicionan la recta con ayuda del software de tal manera que se encuentre lo más cerca posible de la mayoría de los puntos. Se incluye como evidencia la respuesta de la pareja de estudiantes P6.

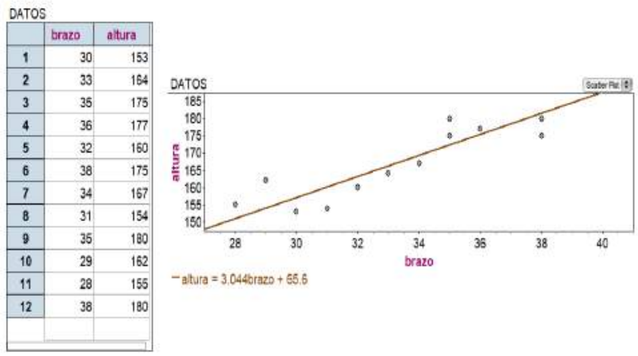


Figura 8. Respuesta pareja P6. Código Cercanía

Se muestra una tabla resumen del análisis y codificación inicial, para las dos etapas del estudio:

Tabla1. Proceso de Codificación

Código	Primera Etapa	Segunda Etapa	Frecuencia Relativa	
Aritmético Interpolación	✓		0.143	0.000
Uso de Software		✓	0.000	0.200
Aritmético Proporcionalidad	✓	✓	0.286	0.267
Aritmético Siguiendo un Patrón	✓	✓	0.071	0.267
Aritmético Sin Seguir un Patrón	✓	✓	0.214	0.267
Percepción de la Tendencia	✓		0.214	0.000
Sin Argumento	✓		0.071	0.000
Partición	✓	✓	0.286	0.211
Cercanía		✓	0.000	0.368
Pertenencia	✓	✓	0.714	0.421

La siguiente fase del análisis consistió en comparar los códigos definidos en las dos etapas del estudio e identificar rasgos de razonamiento covariacional comunes, así como algunos razonamientos que surgieron en la segunda etapa; el proceso de conceptualización de los razonamientos (inferencias) de las respuestas de los estudiantes a continuación se describe.

En función de las inferencias que realizan los estudiantes se identificaron conexiones conceptuales entre los razonamientos que exhiben, de ahí que los códigos *Percepción de la Tendencia* y *Uso de* Software (problema de estimación) se reclasificaron en el patrón de razonamiento Noción de *Agregado* pues en ambos casos los estudiantes hacen referencia a la utilización del conjunto completo de valores de los que disponen, para argumentar sus respuestas; independientemente de que sus resultados sean o no correctas normativamente. Por otro lado los códigos *Interpolación*, *Proporcionalidad*, *Aritmético Siguiendo un Patrón, Aritmético Sin Seguir un Patrón* (problema de estimación), *Partición* y *Pertenencia* (problema de recta de ajuste) se reclasificaron como *Búsqueda de una Señal*. La interconexión entre estos códigos radica en que los estudiantes infieren que debe existir una especie de clave o pista para resolver el problema, y realizan una búsqueda en los datos que se le proporcionaron. Parece que el alumno sospecha que existe un patrón o estructura ocultos en un subgrupo de datos (no utiliza la totalidad de datos ni contempla el conjunto de puntos) que lo llevarán a estructuras constantes, ausentes de incertidumbre o variación, que le son familiares. Finalmente se renombró al código *Cercanía* (problema de recta de ajuste) como *Sentido de Variabilidad,* ya que se perciben rasgos, aunque de manera espuria, de que los estudiantes infieren una relación presente entre el modelo de ajuste y los puntos de la nube, en el diagrama de dispersión. El código *Sin Argumento* que se presentó en el problema de estimación, al no aportar evidencias de algún tipo de inferencia realizada por los alumnos, no se consideró en esta etapa del análisis.

Tabla 2. Proceso de Conceptualización

Código	Patrón de Razonamiento /Inferencias Informales
Percepción de la Tendencia Uso de Software	Noción de Agregado
Aritmético Interpolación Aritmético Proporcionalidad Aritmético Siguiendo un Patrón Aritmético Sin Seguir un Patrón Partición Pertenencia	Búsqueda de una Señal
Cercanía	Sentido de Variabilidad

La tabla 2 muestra un resumen de la conceptualización y definición de los patrones de razonamiento identificados.

Conclusiones

Desde la perspectiva teórica que abordamos, una inferencia informal es un tipo de razonamiento que incluye consideraciones en varias dimensiones (Rubin, Hammerman y Konold, 2006) y en condiciones específicas de estudio. En nuestro caso una primera dimensión es la *Noción de Agregado*, donde se privilegia una percepción holística de las situación-problema por sobre la contemplación de los casos individuales. En este sentido los códigos inicialmente definidos, Percepción de la tendencia y Uso de software, presentan como patrón el que los estudiantes consideran la totalidad de los datos con que disponen; por un lado hacen la estimación del valor de la variable respuesta pedido (consumo de gas) haciendo un análisis puramente visual a partir de la tendencia del conjunto de datos, es decir que argumentan su respuesta con base en el comportamiento global del conjunto de valores. Por otro lado cuando utilizan Fathom, al construir el diagrama de dispersión y observar nuevamente la tendencia de la nube de puntos, identifican que debe existir cierta relación entre los puntos, por lo que eligen uno y modifican su posición de tal manera que coincida el valor dado de la variable independiente (8°C de temperatura) con el de la variable respuesta (consumo de gas) correspondiente, siguiendo la “forma” general de la nube. Si bien es cierto que en ninguno de los casos anteriores la respuesta de los estudiantes es la normativamente adecuada, si permite apreciar que bajo ciertas circunstancias son capaces de percibir que en las relaciones bivariadas es necesario considerar las características y comportamiento de los datos como un todo y, con base en esto inferir el valor de algún punto particular de interés.

Otra dimensión a considerar se ve reflejada con el patrón que definimos como *Búsqueda de una señal*, donde las respuestas arrojan que ante la dificultad que les representa la incertidumbre o la variación intrínseca en este tipo de problemas, los alumnos tratan de resolver la situación en un terreno familiar para ellos o con el que se sienten cómodos, posiblemente por esto utilizan en su mayoría procedimientos aritméticos (regla de tres, factor de proporcionalidad o sumas y restas) para hacer la estimación; donde además solo utilizan una parte de los datos. También es el caso del problema de la recta de ajuste, en el que ante la ausencia de la noción de incertidumbre o de esa visión de agregado, utilizan parcialmente los datos disponibles, haciendo referencia al modelo de la función lineal como alternativa para ajustar una recta a una distribución de puntos que presenta una tendencia lineal, considerando sólo algunos de estos puntos en su elección, ignorando la influencia de la totalidad de estos y su variación conjunta, y sobre todo, definiendo dos tipos de datos: los que pertenecen o no al modelo lineal que ellos eligen trazar.

Como tercera dimensión proponemos, *Sentido de Variabilidad*, representada por el código que se definió inicialmente como *Cercanía*. En este patrón de razonamiento se incluyeron las respuestas que hacen referencia, aunque de manera somera, a la percepción de que existe cierta variación que subyace en este tipo de problemas y que debe considerarse al momento de proponer una recta de ajuste para un conjunto de puntos, tal y como lo hicieron los estudiantes al trazar y argumentar que la recta debe posicionarse lo más cerca posible de la mayoría de los puntos.

Confiamos en que la identificación de estos patrones de razonamiento como la manera en que los estudiantes llevan a cabo inferencias informales ante situaciones-problema de asociación estadística abona al cuerpo del conocimiento en el estudio de datos bivariados, sin dejar de lado que es importante explorar obstáculos de aprendizaje como son la aparente desvinculación que tiene el estudiante entre la naturaleza predictiva o inferencial inherente al modelo de regresión lineal y su identificación como aquella recta que mejor se ajusta al conjunto de puntos, así como la dificultad para concebir al conjunto de datos como un agregado, es decir, como un sistema en el que están ligados unos a otros y tienen la propiedad de ser desviaciones de un mismo modelo.

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TYPES OF ORCHESTRATIONS IN A CASE STUDY OF A STATISTICS CONSTRUCTIONIST TEACHING PRACTICE

TIPOS DE ORQUESTACIÓN EN UN ESTUDIO DE CASO DE UNA ENSEÑANZA CONSTRUCCIONISTA DE LA ESTADÍSTICA

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This paper presents a case study that is part of a research studying how university statistics teachers integrate digital tools in their practice. The case study is of a teacher who claims to follow constructionist ideas for her teaching of statistics, using the R programming language. The teacher developed a series of activities with R, aiming to promote the understanding of statistical concepts. We observed 11 of her classes, in a three-week period. These classes were analyzed using as framework the notion of Instrumental Orchestration and Papert's construccionist principles. We identified four ways in which this teacher orchestrates her class and the constructionist elements present in them.

Keywords: University Mathematics, Data Analysis and Statistics, Technology, Programming and Coding

Introduction

For the study, at university level, of statistics and its advanced models, the use of digital technologies (DT) is necessary, due to the high computing and graphing requirements of the discipline. Additionally, mathematics education researchers point out that digital technologies can be used to rethink the teaching and learning of statistics concepts, by supporting students' statistical reasoning (Biehler et al., 2013) and facilitating access to the fundamental ideas of the discipline (Burrill, 2014).

In contrast, the results of a 2017 survey (Ruiz & Sacristán, 2019) of Mexican university statistics teachers showed that, although they use different digital resources in their teaching practice, most tend to limit their use to computation, graphing and visualization in practical sessions and/or in tasks, without changing their teaching, nor achieving what they themselves believe can be achieved using DT.

However, in the survey data, the responses of a particular teacher caught our attention: this teacher uses the R[1] programming language to engage her students in expressive exploratory activities to confront their intuitions regarding statistical concepts and practice. This teacher pointed out that she changed her use of technology in her classes after she became familiar with the constructionism paradigm (Papert, 1991), which we describe below. Thus, unlike most of the other participants in the survey (Ruiz & Sacristán, 2019), this teacher does not restrict the use of DT to perform calculations and draw graphs to solve tasks; rather, her aim is to promote conceptual learning in her students, through R-based programming. That is why we decided to conduct a case study of her practice, in order to identify how she integrates R, designs and implements activities using this language, in order to promote a conceptual learning of statistical ideas, and identify which of constructionist ideas are present in her practice.

For the case study, we used as a theoretical framework, the notion of Instrumental Orchestration (Guin & Trouche, 2002), derived from Rabardel's (1995) Instrumental Approach. In this regard,

[1] https://www.r-project.org/

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Drijvers et al. (2009) note that the different forms of orchestration of teachers are related to their views of mathematics learning and teaching. Through the case study presented in this paper, we aim to identify the relationship between the studied teacher's view and her forms of orchestration, in particular, the research question were: What types of orchestrations can we observe in this teacher's practice? And what are the construccionist elements in those orchestrations?

Conceptual and theoretical framework

The construccionism proposal

The constructionist paradigm proposed by Papert (1991), follows the premise from constructivism that learning involves the construction of knowledge structures. Constructionism adds to that premise the idea that learning is facilitated when an individual consciously engages in the creation of a public entity – that is, of an object or product that can be shared with others (Papert, 1991). Thus, constructionism emphasizes the active role of the individual in the construction of their knowledge.

Sacristán et al. (2020) identified a long list of the principles and ideas that are part of constructionism. From their perspective, these are organized around four themes: (1) Epistemology, and conceptions of mathematical knowledge and of mathematics; (2) Conception of learning and of the role of the student; (3) Pedagogy and design; and (4) Computer programming and microworlds.

From a constructionist perspective, the use of DT artifacts in the classroom requires a rethinking of the teaching practices: emphasis is placed on inquiry and on the learner, instead of on a specific curriculum or on facts to be learned. For this reason, the use of the computer is conceived as an object-to-think-with (Martinez & Stager, 2013): Papert (1980b) believes that computers can provide environments to develop and work, especially through programming with powerful ideas and/or intellectual skills, that respond to one's own interests and needs. He also considers that the challenges that arise when programming can be learning opportunities.

Statistics education researchers (e.g., Chance et al., 2007) agree that the use of DT artifacts in teaching must be accompanied by changes in teaching style; they further point out that the teacher plays an important role in leveraging DTs to achieve such rethinking.

Instrumental Orquestration

The notion of Instrumental Orchestration was proposed by Guin and Trouche (2002) to take into account how a teacher directs and conducts his classes. It refers both to the different ways in which a teacher organizes and uses available artifacts, whether technological or not, and the teacher's performance during class seeking an effective use of those artifacts.

Drijvers et al., (2009) define instrumental orchestration as "the teacher's intentional and systematic organisation and use of the various artefacts available in a […] learning environment in a given mathematical task situation" (p. 482). It is composed of three elements: (i) the *didactical configuration* consisting of the arrangement of the artifacts, whether technological or not, and the learning setting; (ii) the *exploitation mode* which refers to the ways the teacher decides to carry out a given task, in order to exploit the proposed configuration; and (iii) the *didactical performance* , which involves the teacher's decisions and interventions taken while teaching in the chosen didactic configuration and exploitation mode (Drijvers et al., 2009).

The first form of instrumental orchestration mentioned in the literature (by Guin & Trouche, 2002) is the so-called *sherpa-student* orchestration, in which students carry out a task with an artifact (a calculator), while the work of one of them is projected and the teacher guides the actions of that student. In this orchestration, the didactical configuration is an arrangement that allows students to observe the projection while working on their own calculator, following the actions of the sherpa student. This mode of exploitation is determined by the teacher when guiding the sherpa student in his/her actions.

Trouche (2004) also distinguishes the *sherpa-at-work* orchestration, which has the didactical configuration described above and where the sherpa student is used to guide the work of the class in the exploitation mode. Other types of orchestrations identified by Drijvers et al. (2009) are: *technical-demo*, *explain-the-screen*, *link-screen-board*, *discuss-the-screen*, *spot-and-show*. In the exploitation modes of the first three orchestrations, the teacher has a central role; while the other orchestrations promote student intervention. On her part, Tabach (2011, 2013) distinguishes other types of orchestration: *monitor-and-guide*, *not-use-tech* and *discuss-tech-without-it*. In the *monitor-and-guide* orchestration, the teacher uses a learning management system to guide students to perform tasks similar to those of the *demo-technical* and *explain-the-screen* orchestrations. In the *not-use-tech* orchestration, technology is available but the teacher decides not to use it. The last form of orchestration proposed by Tabach (2013) is the *discuss-tech-without-it*, which corresponds to settings where it is not possible for students to have the technology element. Drijvers (2012) distinguishes one more type of orchestration, which he calls *work-and-walk-by*. In that orchestration, students work with a computer, either individually or in pairs, while the teacher walks between them to monitor their work and guide them when needed.

For Drijvers et al. (2009), teachers' different types of orchestration relate to their views on mathematical learning and teaching. For example, for one of his observed teachers it was important to achieve certain mathematical learning objectives, stressing the relationship between what happens in the technological and paper-and-pencil environments; thus, the type of orchestration that he used most was the *link-screen-board* one.

Drijvers et al. (2009) also point out that the *demo-technical* orchestration was the most prevalent among the observed teachers, because they "felt the need to familiarize students with basic techniques, in order to prevent technical obstacles from hindering the mathematical activities" (p. 6).

Based on these proposals, we analyzed the orchestration of the case study teacher. Next, we present the methodology of the study.

Methodology

As noted in the introduction, in 2017 we conducted a survey, using an online questionnaire, to get an insight into how university teachers use DT in their teaching of statistics. From the responses of 31 teachers, and interviews of three of them, the teacher, Mayra (pseudonym), was selected for a qualitative case study. As also noted, this teacher was chosen because she claimed to have a constructionist view of learning and teaching, which is why she uses technological tools in order to promote in her classes student-centered activities that will help them develop their conceptual learning (an appropriation) of statistical concepts.

We carried out an initial semi-structured interview with Mayra; observed 11 of her classes; and also had 6 conversations with her, in which she discussed details of her classes, the decisions she made and how her practice compares to other courses. All interviews (both formal and informal), as well as her classes, were recorded (audio for the interviews and video for the classes). We also analyzed the activities and assessments that she designed and implemented with her students.

The analysis of the data from the observations of her classes, focused on identifying the instrumental orchestration elements of her practice. Data from the interview and of her comments were used to complement the analysis of the classes and determine her vision of the teaching and learning of statistics and of the discipline itself.

The observation of the classes was divided into episodes when changes, in either the didactic configuration or in the mode of exploitation of Mayra's orchestration, were identified. We attempted to distinguish Mayra's types of orchestrations, using as reference those reported in the literature, and taking into account what Tabach (2013) pointed out, in terms of identifying whether an orchestration is a variant of another one, or a new orchestration.

In addition to that, guided by Papert's ideas (1980a, 1980b, 1991) and the synthesis of constructionist principles carried out by Sacristán et al. (2020), we attempted to identify the constructionist aspects in Mayra's classes.

Case study

Mayra's background, her teaching practice and her current activities design

Mayra is a biologist but has been teaching probability and statistical courses for 17 years, both at the undergraduate and graduate level. The topics covered in her courses have a strong practical orientation, as they address students of biology and environmental sciences.

DT resources always have been part of Mayra's teaching, because statistics needs them (for data processing, calculations and graphing). Initially, she used programs such as Excel, Statistica, and MiniTab, mainly to perform the calculations required in the formulas associated with a concept or statistical procedure. When using those resources, her classes used to have a traditional format, and only when necessary she showed students how to use those DT resources to perform calculations and get some results. The type of orchestration of those classes corresponded to the demo-technical one, where DTs were used to get a result that helped respond to a specific statistical problem.

When Mayra was introduced to the R programming language in 2007, she felt that her possibilities to statistically analyze the data of her professional practice were extended: "the window that opened to me when I got to know R, was precisely the possibility of adapting a set of analytical procedures to the data, and not the other way around".

A short time later, Mayra was introduced to the constructionist paradigm and was struck by the fact that learners did not need to be experts in programming to become creators of (increasingly complex) projects, be able to work independently and with, implicit or explicit, mathematical concepts. It was then that her view of statistics education changed and she decided to use R to develop a new way of teaching.

Thus, adopting the idea that students could construct their own knowledge, without the need of an excessively guided instruction but, rather, through a more construccionist pedagogy, Mayra designed a series of R-based activities so that students could explore data, using numerical, graphical and tabular records. In the course of several school cycles, Mayra refined the design of the activities: In her first proposals, the activities aimed to introduce the R programming language, so that later the students could use it for calculations and graphs; but in that case the tool was just integrated, without changing her teaching practice. The activities were then modified in order to explore statistical concepts (e.g. mean, standard deviation), or sets of concepts (included in the curricula) through different types of representations. For that, the activities present, through several parts, a particular problem involving a specific statistical concept. It should be noted that each activity leads the student to continuously explore and link the different types of representations, so that they can infer the result of an action, justify their decisions, explore the commands and propose different solutions. Mayra says: "The aim of the exercises became one that I had not been able to see[:] find in each of these thematic units, the essential concepts that they had to learn." She pointed out that her aim became to promote a more conceptual learning in her students through the activities that they had to solve, discuss and share. She explained that she is now aims to use DT resources, not to make easier what has always been done, but to rethink her teaching and change the usual roles in class of both teachers and students, as well as of technology.

Her activities with R are designed to be solved in class, in teams of two or three students. Through these activities, she aims to promote in students exploratory work with questioning and inference; in other words, students do a similar job to that of a statistical user. In this way, her activities coincide with what Papert promoted (1980a, 1980b): i.e., to use programming as an activity for students *to do* *math,* instead of learning about math. Thus, as Kynigos (2015) explains, whereas traditional school practices may impose an artificial picture of mathematics, by presenting the products of mathematical activity, the constructionist paradigm focuses on the mathematical activity and the expression of meanings through the use of and tinkering of representations in the form of digital artifacts.

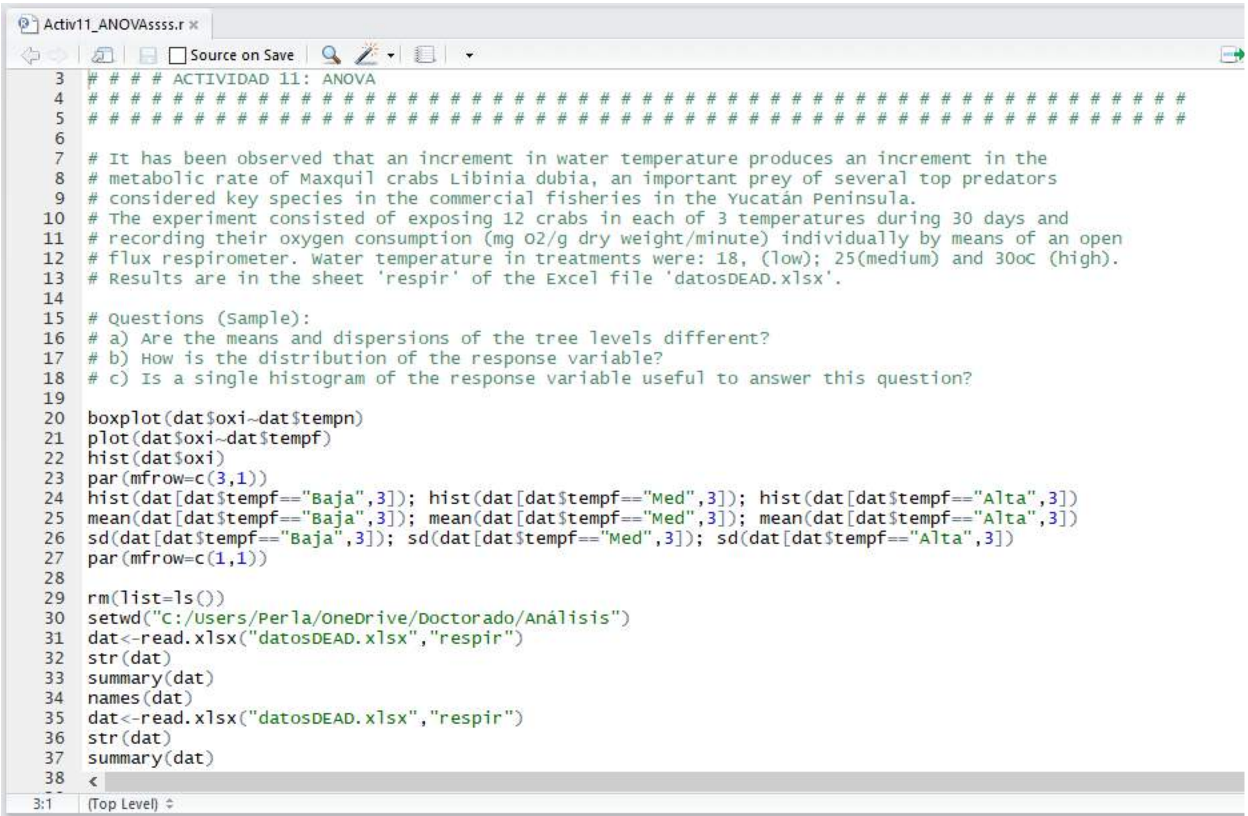

Figure 1: Script of an R-based activity on Analysis of Variance (ANOVA)

For example, one of Mayra's activities (see Figure 1) focuses on the analysis of variance (ANOVA) by presenting a real situation of the study of crabs under three temperature treatments. Some of the aims of the activity include exploring the components of the variance through the different representations and relating them to various elements in the ANOVA table (sum of squares, degrees of freedom, etc.); and familiarizing students with the F-distribution and the parameters that define it. The teacher explained that the purpose of this activity is for students to have to build representations themselves using R; that is, for

> students to have to use an ANOVA table... to look there for certain values that will help them give an answer [to the situation raised]. Do the same graphically, forcing them to relate what they are seeing on a graph to the numerical value; make them reflect on the magnitude of that difference, or on that variation, or on that number in terms of the problem, in general, [or] in terms of the units being used.

Although Mayra's approach focuses on students solving activities, we also observed some of her regular lessons (without technology), and others where group discussions of the activities (with technology) were carried out. In her regular lessons she presents a statistical concept or method and the symbology to be used. She also leads discussions on the relationships of that concept with others previously studied, and on relevant theoretical aspects. Mayra then assigns students an R-based activity to be carried out in teams of two or three students. When students complete the activity in R, or when Mayra needs to help them with their progress, she leads a group review of the activity.

Types of orchestration in Mayra's observed classes

Throughout our observations, of her regular lessons, the sessions where students carried out the R-based activities, and the group reviews of those activities, we identified four types of orchestrations in Mayra's practice, as presented below.

Discuss-tech-without-it orchestration. In her regular lessons, Mayra does not use technology; however, there are times when either she, or the students, refer to the R commands to talk about a concept or statistical process. For example, in an episode where the teacher asked for a reference value needed to interpret the F-value present in an ANOVA, under a null hypothesis, a student's answer was in terms of the R (qf) command used for calculating quantiles in hypothesis tests for the F-distribution. The teacher used this idea to discuss how R presents the results of an ANOVA. In this episode we observed the *discuss-tech-without-it* orchestration, where she explained how statistical processes in the R environment are presented and how R commands are used for expressing statistical ideas. In this orchestration, the didactical configuration corresponds to a traditional class arrangement, in which students follow the explanations of the teacher on the blackboard. In addition to this material artifact (the blackboard), the didactic configuration includes symbolic artifacts, such as graphic representations, symbols, and terms that denote certain statistical concepts and the contexts in which the statistical problems are presented (the problems are usually taken from real situations and data). The exploitation mode includes Mayra's explanations, using the blackboard for annotations and to illustrate concepts. Part of the exploitation mode relates to how Mayra's explanations are characterized, where she aims to discuss the relevance of a newly presented statistical concept and/or the approach of a particular method.

Although her regular lessons refer to the statistical content of the activities to be carried out by her students, these are not theoretical sessions with activities as a practical component; on the contrary: the discussions and contexts addressed in these regular lessons serve as a scaffolding for students, so that they can develop the activities in R. In this sense, the contexts of the proposed problems play an important role, and by referring to them, Mayra links her different types of class-formats. The orchestration of Mayra's regular lessons do not show constructionist aspects, since they are more instructionist (Papert, 1980b) ways of teaching, although she does promote student participation, so that students are not passive and they may benefit from what is discussed, which is fundamental for carrying out the R-based activities.

Work-and-walk-by orchestration. In classes dedicated to solving the activities, we observed the work-and-walk-by orchestration. Some of the artifacts involved in this didactical configuration are the resources shared in computer folders by the teacher, which include the designed activities as R-scripts and databases to be used with those activities. The symbolic artifacts include the (previously studied or newly presented in the lessons) statistical concepts. Other artifacts include the programming language, in particular the commands related to those statistical concepts. Following Papert's (1980a, 1980b) ideas, these artifacts become objects-to-think-with and for exploring the mathematical and statistical ideas.

The didactical configuration of this orchestration requires students to have access to the computer resources, and for the classroom setup to allow them to gather as teams on one or more computers. In classes that we observed, almost every student used their own computer. In the exploitation mode of the work-and-walk-by orchestration, the teacher promotes student teams to investigate, on their own, the R-based activity, looking for their own solutions, discussing them within each team and also with other teams; Mayra interacts with one team at time, seeking to: (i) emphasize important aspects and the relationship between the procedures/results of the activity and the theoretical concepts involved; and (ii) address students' particular doubts and difficulties.

How the teacher's aims for the programming activities are fulfilled, depends on how students are able to outline a solution proposal to the problem and refine it through different attempts, until reaching an appropriate response in terms of the given context. According to Mayra, the activities' approach generates a need for students to ask themselves "what do I want to do? What do I want to do it for? What is the information I'm going to get with that particular step, through that method?" Mayra states that when students fail an attempt, they need to find out the reason for such a result and that reflection exercise leads to a more profound learning than if they didn't make mistakes, where the result simply confirms their reasoning. These different roles of the teacher and students –the latter having an active role in their learning process– that involve collaboration and communication, are central constructionist aspects (as described in Sacristán et al., 2020) in this orchestration, which also includes other constructionist aspects such as: problem solving, exploration, the construction of new objects and/or ideas, the need to overcome obstacles and debug, etc..

One of the advantages of Mayra's proposed activities is that they are conceived and designed taking into account a problem from real biological/environmental practices (the students' study area), so that the context and the results of that problem become references for students. This helps articulate the explanations given in the regular lessons with what is done in the R environment. Sometimes, discussions that emerge during the activities are taken up in subsequent regular lessons to make connections between topics.

In Mayra's didactical performance, there are times when she decides to lead a plenary discussion of some proposed solution, of a result or of a doubt, either to facilitate the understanding of the concepts or to help the pace of the different teams in solving the problems.

Sherpa-at-work and link-screen-blackboard orchestrations. When Mayra leads classes to review the activities that were carried out, we observe a variation of the sherpa-at-work orchestration, in which a team of students participates to present their work. The didactical configuration consists of an arrangement where a team projects their work so that it can be shared with the other students. In the exploitation mode, the teacher selects a team of students to present and discuss how they followed the R-based activity guidelines, and their solutions to the problem –thus fulfilling the constructionist aspect of sharing and discussing "public entities" (Papert,1991).

The didactical performance in this orchestration includes the validation by the teacher of the different solution proposals, confronting the misconceptions and difficulties that arose in the activity, and the assessment of students' understandings of the concepts under study.

The review classes also involve the link-screen-blackboard orchestration. For example, during the review of an activity on the Additive Model, the first part focused on exploring the corresponding dataset and indicating its difference with datasets that were used in the T-test. In her explanations, she emphasizes how data needs to be arranged in order to perform the statistical method. She uses a whiteboard to show how data needs to be arranged, noting how subscripts can serve to distinguish between the numbering of a data observation and the treatment for each observation, and how this is reflected in the R t.test command inputs.

Conclusions

In this study we observed how Mayra, through her constructionist teaching approach, implemented activities, as promoted by Papert (1980a), to get her students involved in *doing* statistics through programming and by generating R-based representations, rather than learning *about* statistical concepts. Other constructionist aspects in Mayra's orchestrations were observed in her classroom dynamics and in the participants' roles (except in the regular lessons). In Mayra's three class formats, we identified four types of orchestrations. In the orchestration types presented in the research literature (Drijvers et al. 2009, Guin & Trouche, 2002; Trouche 2004, Tabach, 2011, 2013) there are few examples where teamwork is encouraged. Mayra's orchestrations involve much more varied aims and considerations than those illustrated in Drijvers et al. (2010). Also, while the demo-technical orchestration is the most frequently observed in the study by Drijvers et al. (2010), Mayra’s classes do not present this type of orchestration.

Pratt et al. (2011) state that immersion in the use of statistics software is important for teachers to gear student learning of statistical ideas, and helps them appreciate the role that DTs play in promoting understanding of statistical concepts and methods.

Mayra, using her own experience, generated activities that reflect a greater use of technology in the classroom. In this document we show a novel way to use the R programming language in the teaching of statistics, in which students have a much more active role in the construction of their knowledge.

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TIPOS DE ORQUESTACIÓN EN UN ESTUDIO DE CASO DE UNA ENSEÑANZA CONSTRUCCIONISTA DE LA ESTADÍSTICA

TYPES OF ORCHESTRATIONS IN A CASE STUDY OF A STATISTICS CONSTRUCTIONIST TEACHING PRACTICE

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En este artículo se presenta un estudio de caso que forma parte de una investigación con profesores de estadística de nivel universitario para estudiar su incorporación de herramientas tecnológicas digitales en su enseñanza. El estudio de caso corresponde a una profesora que dice adoptar ideas del construccionismo para su enseñanza de la estadística con el lenguaje de programación R. La profesora desarrolló una serie de actividades con R, para promover la comprensión de conceptos estadísticos. Observamos 11 de sus clases, en un periodo de 3 semanas. Sus clases se analizaron considerando la noción de Orquestación Instrumental y los principios construccionistas de Papert. Identificamos cuatro formas en la que esta profesora orquesta su clase y los elementos construccionistas presentes en éstas.

Palabras clave: Matemáticas de Nivel Universitario, Análisis de Datos y Estadística, Tecnología, Programación y Codificación Computacional

Introducción

La estadística que se estudia en el nivel superior, y sus modelos avanzados precisan del uso de tecnologías digitales, debido en gran parte a las elevadas necesidades de cómputo y graficación de la disciplina. Además, investigadores en educación matemática señalan que su uso puede aprovecharse para replantear la enseñanza y el aprendizaje de conceptos estadísticos, promoviendo el desarrollo del pensamiento propio de la disciplina (Biehler et al., 2013) y permitiendo que los alumnos accedan a las ideas estadísticas fundamentales (Burrill, 2014).

En contraste con lo anterior, los resultados de una encuesta que realizamos en 2017, con profesores de estadística de nivel universitario (Ruiz & Sacristán, 2019) mostraron que, si bien los profesores utilizan distintos recursos tecnológicos digitales (TD), en su práctica docente, la mayoría tiende a limitar su uso al cómputo, graficación y visualización en las sesiones prácticas y/o en tareas, sin cambiar su forma de enseñanza ni llevando a cabo lo que ellos consideran que puede lograrse al usar TD.

Sin embargo, de entre los resultados, nos llamó la atención una profesora que utiliza el lenguaje de programación R[2] para involucrar a sus alumnos en actividades expresivas de exploración para confrontar sus intuiciones sobre los conceptos y la práctica estadística. Esta profesora señaló que hubo un cambio en la forma en que usa la tecnología en sus clases a partir de que conoció el planteamiento del construccionismo (Papert, 1991), enfoque que describimos más abajo. Así pues, a diferencia de la mayoría de los encuestados (Ruiz & Sacristán, 2019), esta profesora no se restringe a utilizar las TD para realizar cálculos y obtener gráficas para resolver tareas, sino que su principal objetivo es promover un aprendizaje más conceptual por parte de sus alumnos a través de la programación con R. Por ello, la forma de enseñanza de esta profesora llamó nuestra atención, y decidimos hacer un estudio de caso de ella para identificar de qué forma integra R, diseña y conduce las actividades con este lenguaje, para fomentar el aprendizaje conceptual de ideas estadísticas y cuáles de las ideas construccionistas son parte de su práctica.

Para el estudio de caso, hemos utilizado como marco teórico, la noción de Orquestación Instrumental (Guin & Trouche, 2002), la cual se deriva de la Aproximación Instrumental de Rabardel (1995). Al respecto, Drijvers et al. (2009) señalan que las diferentes formas de orquestación de los profesores se relacionan con su visión sobre el aprendizaje y la enseñanza de las matemáticas. En la parte del estudio de caso, presentada en este documento, pretendemos identificar, en particular, la relación entre la visión de la profesora y sus formas de orquestación. Por ello, las preguntas de investigación que guían lo aquí descrito, es: ¿Qué tipo de orquestaciones se observan en la enseñanza de esta profesora?; y como sub pregunta ¿cuáles son los elementos construccionistas en esas orquestaciones?

Marco conceptual

La propuesta del Construccionismo

El paradigma construccionista propuesto por Papert (1991), parte de la premisa constructivista de que el aprendizaje implica una construcción de estructuras de conocimiento. La propuesta construccionista añade a esto, la idea de que el aprendizaje se promueve cuando un individuo se involucra conscientemente en la creación de una entidad pública –es decir, un objeto o producto que pueda ser compartido con otras personas (Papert, 1991). Así, esta propuesta enfatiza el rol activo del individuo en la construcción de su propio conocimiento.

Sacristán et al. (2020) identificaron una larga lista de los principios e ideas que forman parte de la propuesta construccionista. Desde su perspectiva, éstos se organizan alrededor de cuatro temas: (1) Epistemología y concepciones sobre el conocimiento matemático y sobre la matemática; (2) Concepciones del aprendizaje y el rol del estudiante; (3) La pedagogía y el diseño; y (4) La programación computacional y los micromundos.

Desde la perspectiva construccionista, el uso de artefactos TD en el aula requiere un replanteamiento de la forma de enseñanza, de manera que se hace hincapié en la indagación y en el aprendiz, no en un currículo específico o en hechos a ser aprendidos. Por esta razón, se promueve el uso de la computadora como algo con qué pensar (Martinez & Stager, 2013): Papert (1981) considera que la computadora puede proveer ambientes que permitan desarrollar y trabajar con ideas poderosas y/o habilidades intelectuales, que respondan a los intereses y necesidades de cada individuo, sobre todo a través de la programación. También considera que las eventualidades que surjan al programar, pueden ser oportunidades de aprendizaje.

Investigadores en educación estadística (e.g., Chance et al., 2007) coinciden que el uso de artefactos TD en la enseñanza debe ir acompañado de cambios en las formas de enseñanza; señalan, además,

[2] https://www.r-project.org/

que el profesor juega un rol importante en el aprovechamiento de las TD para alcanzar tal replanteamiento.

La Orquestación Instrumental

La noción de Orquestación Instrumental fue propuesta por Guin y Trouche (2002) para dar cuenta de cómo un profesor dirige y lleva a cabo sus clases. Se refiere tanto a las distintas formas en las que un profesor organiza y usa los artefactos disponibles, ya sean tecnológicos o no, así como al desempeño del profesor durante la clase para sacar provecho de dicha configuración.

La orquestación instrumental se define como la organización y uso sistemático e intencional de los artefactos a disposición del profesor en el aula, durante la realización de una tarea determinada en un ambiente de aprendizaje (Drijvers et al., 2009). Se compone de tres elementos: (i) la *configuración didáctica* que consiste en la disposición de los artefactos, ya sean o no tecnológicos, y del ambiente de aprendizaje; (ii) los *modos de explotación* que se refieren a las formas indicadas por el profesor para llevar a cabo la tarea dada, en busca de sacar provecho de la configuración propuesta; y (iii) el *desempeño didáctico* del profesor, incluye sus decisiones e intervenciones desarrolladas en el ambiente de aprendizaje (Drijvers et al., 2009).

La primera forma de orquestación instrumental mencionada en la literatura fue la llamada orquestación *alumno-sherpa*, en el cuál los alumnos desarrollan cierta tarea ayudados de un artefacto (una calculadora), mientras que el trabajo de uno de ellos se proyecta y el profesor, regula la intervención de dicho estudiante (Guin & Trouche, 2002). En esta orquestación la configuración didáctica consiste en un arreglo que permite a los alumnos observar la proyección y manejar su propia calculadora, siguiendo las acciones del alumno sherpa. El modo de explotación lo determina el profesor, indicando al alumno sherpa que acciones realizar.

Trouche (2004) también distingue la orquestación sherpa-en-el-trabajo, la cual tiene una configuración didáctica como la descrita antes y en cuyo modo de explotación se guía el trabajo de la clase a través del alumno sherpa. Otros tipos de orquestaciones identificadas por Drijvers et al. (2009) son: *demostración-técnica*, *explica-la-pantalla*, *enlaza-pantalla-pizarrón*, *discute-la-pantalla*, *indica-y-muestra*. Los modos de explotación de las primeras tres orquestaciones implican que el profesor tome un papel central, mientras que las siguientes orquestaciones promueven la intervención de los alumnos. Por su lado, Tabach (2011, 2013) distingue otros tipos de orquestación: *monitorea-y-guía*, *no-usa-tecnología* y *discute-la-tecnología-sin-ella*. En la de monitorea-y-guía, el profesor utiliza un sistema de administración del aprendizaje para guiar a los estudiantes o llevar a cabo tareas similares a las de las orquestaciones demostración-técnica y explica-la-pantalla. En la orquestación no-usa-tecnología, la tecnología está disponible pero el profesor decide no hacer uso de ella. La última forma de orquestación propuesta por esta autora, *discute-la-tecnología-sin-ella,* corresponde a escenarios donde no es posible que los alumnos dispongan del elemento tecnológico. Drijvers (2012) distingue un tipo de orquestación más, al que denomina como circula-mientras-trabajan (*work-and-walk-by*). En esa orquestación, los estudiantes trabajan con una computadora, ya sea individualmente o en pares, mientras el profesor circula entre ellos para monitorear su trabajo y guiarlos cuando es necesario.

Para Drijvers et al. (2009), las diferentes formas de orquestación de los profesores se relacionan con su visión sobre el aprendizaje y la enseñanza de las matemáticas. Por ejemplo, para uno de los profesores que observaron era importante alcanzar ciertos objetivos de aprendizaje matemático, vinculando lo que sucede en los ambientes tecnológicos y de papel-y-lápiz; así, utilizaba más frecuentemente la forma de orquestación enlaza-pantalla-pizarrón.

Drijvers et al. (2009) también señalan que la orquestación demostración-técnica era la más frecuente entre los profesores que observaron, debido a que éstos “sienten la necesidad de familiarizar a los alumnos con técnicas básicas, para prevenir obstáculos técnicos que inhiban las actividades matemáticas" (p. 6).

Es con base en estas propuestas, que analizamos la orquestación de la profesora estudio de caso. A continuación presentamos la metodología del estudio.

Metodología

Como se señaló en la sección de introducción a este artículo, en 2017 realizamos una encuesta, mediante un cuestionario en línea, para tener un panorama sobre cómo profesores de nivel superior utilizan las TD en su enseñanza de la estadística. A partir de las respuestas de 31 profesores, y de las entrevistas a tres de ellos, se seleccionó a la profesora, que llamamos Mayra, para realizar un estudio de caso de tipo cualitativo. Como también se señaló, esta profesora fue elegida ya que decía tener una visión construccionista del aprendizaje y la enseñanza, razón por la cual utiliza las herramientas tecnológicas de manera que en sus clases se da un papel primordial a la actividad de los alumnos; además, busca promover un aprendizaje más conceptual en sus alumnos (una apropiación de los conceptos estadísticos).

Se realizó una entrevista inicial semi-estructurada a Mayra, se observaron 11 de sus clases, y se tuvieron 6 conversaciones fuera de clase en las que Mayra comentaba detalles de la clase, de las decisiones que tomaba y algunos contrastes de su práctica con otros cursos. Tanto las entrevistas (formales e informales) como las clases, se grabaron (en audio las entrevistas, en video las clases). También se recolectaron las actividades y evaluaciones que Mayra había llevado a cabo con sus alumnos.

El análisis de los datos de las observaciones de sus clases se centró en la identificación de los elementos de su práctica que componen el modelo de orquestación instrumental. Los datos de la entrevista y los audios de sus comentarios se utilizaron para complementar el análisis de las clases y para determinar la visión de la profesora respecto a la enseñanza y aprendizaje de la estadística y de la disciplina misma.

La observación de las clases se dividió en episodios, según se identificara un cambio, ya sea en la configuración didáctica, o en el modo de explotación de la orquestación de Mayra. Tratamos de distinguir sus tipos de orquestaciones, a partir de las señaladas en la literatura, considerando lo señalado por Tabach (2013) para identificar si se trata de una variante de cierta orquestación o de una nueva orquestación.

Adicionalmente, se llevó a cabo una identificación de los aspectos construccionistas de las clases de Mayra, guiado por las ideas de Papert (1980, 1981, 1991), y la síntesis realizada por Sacristán et al. (2020) en donde se distinguen los principios construccionistas.

Estudio de caso

Antecedentes de Mayra, sus formas de enseñanza y su diseño actual de las actividades

La profesora Mayra es bióloga pero cuenta con 17 años de experiencia docente impartiendo cursos de probabilidad y estadística, tanto a nivel licenciatura como posgrado. Los temas abordados en sus cursos tienen una fuerte orientación práctica, ya que se dirigen a alumnos de carreras biológico-ambientales.

Los recursos TD siempre formaron parte de la enseñanza de Mayra, debido a que en estadística no se puede prescindir de ellos (para procesamiento de datos, cálculos y graficación). Al principio, usaba programas como Excel, Statistica y MiniTab, principalmente para realizar los cálculos prescritos en las fórmulas asociadas al concepto o procedimiento estadístico. Cuando utilizaba esos recursos, las clases mantenían un formato tradicional y cuando era necesario, mostraba cómo utilizarlos para realizar cálculos y obtener algún resultado. El tipo de orquestación de esas clases, correspondía a la *demostración-técnica*, donde las TD eran el medio para obtener un resultado a partir del cual responder a una situación estadística concreta.

La percepción de Mayra, al conocer el lenguaje de programación R en el año 2007, fue que se le ampliaron las posibilidades de analizar estadísticamente los datos de su práctica profesional: “la ventana que se me abrió cuando conocí R, fue justamente la posibilidad de adecuar un conjunto de procedimientos analíticos a los datos, y no al contrario”.

Poco tiempo después, Mayra conoció el planteamiento construccionista y le llamó la atención que los aprendices no necesitaban ser expertos en lenguaje de programación para convertirse en creadores de proyectos (cada vez más complejos), poder trabajar de forma independiente, y abordar, implícita o explícitamente, conceptos matemáticos. Fue entonces que su visión de la educación estadística cambió y decidió usar R para desarrollar una nueva forma de enseñanza.

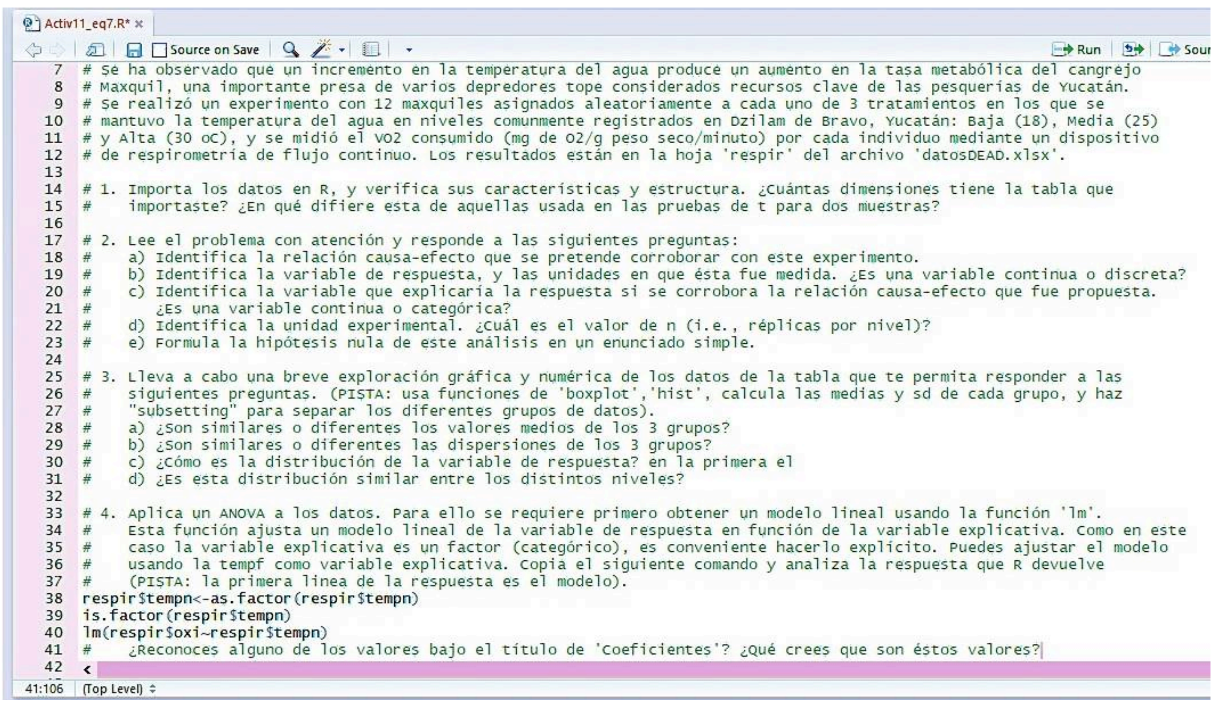
Así, con la idea de que los alumnos pudieran construir su propio conocimiento, sin necesidad de una instrucción demasiado guiada, Mayra diseñó una serie de actividades buscando una pedagogía más construccionista, basadas en el uso del lenguaje de programación R para la exploración de datos, por parte de los alumnos, usando los registros numérico, gráfico y tabular. Durante varios ciclos escolares, Mayra depuró y refinó el diseño de las actividades: En sus primeras propuestas, las actividades buscaban introducir el lenguaje de programación R, para que después los alumnos pudieran utilizarlo para obtener cálculos y gráficas; siendo ésta, una incorporación de la herramienta sin cambiar la forma de enseñanza. Luego las actividades se modificaron para explorar conceptos estadísticos (e.g., media, desviación estándar), o conjuntos de conceptos (incluidos en los planes de estudio) a través de distintos registros de representación. Para ello, las actividades plantean una problemática particular, cuya resolución se lleva a cabo por etapas donde se pone en acción un concepto determinado. Cabe mencionar que cada actividad lleva al alumno a explorar y vincular continuamente los distintos registros de representación, a tratar de inferir el resultado de una acción, a justificar sus decisiones, a explorar los comandos y a plantear distintas propuestas de solución. Al respecto Mayra comenta: “el objetivo de las practicas se convirtió en un objetivo que yo no había sido capaz de ver[:] encontrar en cada una de esas unidades temáticas, cuáles eran los conceptos esenciales que tenían que aprender.” Señaló como objetivo promover en sus alumnos un aprendizaje más conceptual a través de las actividades que resuelven, discuten y comparten. Explicó que busca ahora un uso de los recursos TD, no como auxiliares para hacer más fácil lo que siempre se ha hecho, sino para replantear su enseñanza y cambiar los roles usuales en clase tanto del profesor y los alumnos, como de la tecnología.

Sus actividades con R están diseñadas para realizarse en clase, en equipos de dos o tres alumnos. A través de dichas actividades, buscaba promover en los alumnos un trabajo de exploración, de prueba y error, de cuestionamiento e inferencia; en otras palabras, que realizaran un trabajo parecido al de un usuario de la estadística. De esta manera, coincide con las propuestas de Papert (1980, 1981) de utilizar la programación como una actividad para que los alumnos *hagan matemáticas* en lugar de aprender acerca de las matemáticas. Así, mientras que en la escuela tradicional se impone una imagen artificial de las matemáticas, con objetos acabados de la actividad matemática, la propuesta construccionista se enfoca en esa actividad matemática y en la expresión de significados a través del uso y re-creación de representaciones en forma de artefactos digitales (Kynigos, 2015).

Por ejemplo, una de las actividades (ver Figura 1) trata sobre el análisis de la varianza (ANOVA): presenta una situación real en torno a cangrejos sometidos a tres tratamientos de temperatura. Algunos de los objetivos de la actividad incluyen explorar los componentes de variación en los distintos registros y relacionarlos con diversos elementos en la tabla de ANOVA (suma de cuadrados, grados de libertad, etc.), además de familiarizarse con la distribución de F y los parámetros que la definen. La profesora explicó que el propósito de esta actividad es que los alumnos tengan ellos mismos que construir las representaciones y programar; es decir, que

> los alumnos [tengan] que ir a una tabla de ANOVA… a buscar ahí, ciertos valores que… [les permitan] dar una respuesta [a la situación planteada]… Hacer lo mismo gráficamente, obligarlos a relacionar lo que estaban viendo en una gráfica con ese valor numérico, hacerlos reflexionar sobre la magnitud de esa diferencia, o de esa variación, o de ese número en términos del problema en general, en términos de las unidades que estaban siendo utilizadas.

Si bien el planteamiento de Mayra se dirige hacia la resolución de las actividades por parte de los alumnos, también se observaron clases de tipo magistral (sin tecnología) y otras donde se hace una revisión grupal de las actividades (con tecnología). En las clases de tipo magistral se presenta un concepto o método estadístico y la simbología a utilizar. Se discuten las relaciones de este concepto con otros previamente estudiados y los aspectos teóricos pertinentes. Después, la profesora asigna a los alumnos una actividad en R, para realizarse en equipos de dos o tres alumnos. Al finalizar los alumnos la actividad en R, o en momentos donde la profesora requiere de uniformizar su progreso, se realizan las revisiones grupales de la actividad considerada.



```
# Se ha observado que un incremento en la temperatura del agua produce un aumento en la tasa metabólica del cangrejo
# Maxquil, una importante presa de varios depredores tope considerados recursos clave de las pesquerías de Yucatán.
# Se realizó un experimento con 12 maxquiles asignados aleatoriamente a cada uno de 3 tratamientos en los que se
# mantuvo la temperatura del agua en niveles comunmente registrados en Dzilam de Bravo, Yucatán: Baja (18), Media (25)
# y Alta (30 oC), y se midió el VO2 consumido (mg de O2/g peso seco/minuto) por cada individuo mediante un dispositivo
# de respirometría de flujo continuo. Los resultados están en la hoja 'respir' del archivo 'datosDEAD.xlsx'.

# 1. Importa los datos en R, y verifica sus características y estructura. ¿Cuántas dimensiones tiene la tabla que
#    importaste? ¿En qué difiere esta de aquellas usada en las pruebas de t para dos muestras?

# 2. Lee el problema con atención y responde a las siguientes preguntas:
#    a) Identifica la relación causa-efecto que se pretende corroborar con este experimento.
#    b) Identifica la variable de respuesta, y las unidades en que ésta fue medida. ¿Es una variable continua o discreta?
#    c) Identifica la variable que explicaría la respuesta si se corrobora la relación causa-efecto que fue propuesta.
#       ¿Es una variable continua o categórica?
#    d) Identifica la unidad experimental. ¿Cuál es el valor de n (i.e., réplicas por nivel)?
#    e) Formula la hipótesis nula de este análisis en un enunciado simple.

# 3. Lleva a cabo una breve exploración gráfica y numérica de los datos de la tabla que te permita responder a las
#    siguientes preguntas. (PISTA: usa funciones de 'boxplot','hist', calcula las medias y sd de cada grupo, y haz
#    "subsetting" para separar los diferentes grupos de datos).
#    a) ¿Son similares o diferentes los valores medios de los 3 grupos?
#    b) ¿Son similares o diferentes las dispersiones de los 3 grupos?
#    c) ¿Cómo es la distribución de la variable de respuesta? en la primera el
#    d) ¿Es esta distribución similar entre los distintos niveles?

# 4. Aplica un ANOVA a los datos. Para ello se requiere primero obtener un modelo lineal usando la función 'lm'.
#    Esta función ajusta un modelo lineal de la variable de respuesta en función de la variable explicativa. Como en este
#    caso la variable explicativa es un factor (categórico), es conveniente hacerlo explícito. Puedes ajustar el modelo
#    usando la tempf como variable explicativa. Copia el siguiente comando y analiza la respuesta que R devuelve
#    (PISTA: la primera linea de la respuesta es el modelo).
respir$tempn<-as.factor(respir$tempn)
is.factor(respir$tempn)
lm(respir$oxi~respir$tempn)
#    ¿Reconoces alguno de los valores bajo el título de 'Coeficientes'? ¿Qué crees que son éstos valores?
```

Figura 1: Guión de una actividad en R sobre Análisis de Varianza (ANOVA)

Tipos de orquestación en las clases observadas de Mayra

A lo largo de nuestras observaciones, tanto de las clases de tipo magistral, de las de resolución de actividades y de las de revisión, se distinguieron cuatro tipos de orquestación de Mayra, como se presenta a continuación.

Orquestación discute-tecnología-sin-ella. En las clases de tipo magistral de Mayra, ella no utiliza tecnología; sin embargo, hay ocasiones en que, ya sea la profesora o los alumnos, hacen referencia a los comandos en R relacionados con un concepto o proceso estadístico involucrado en la explicación. Por ejemplo, en un episodio donde la profesora preguntaba por un valor de referencia para interpretar el valor F obtenido en un ANOVA, bajo cierta hipótesis nula, un alumno dio su respuesta en términos del comando de R (qf) asociado al cálculo de cuantiles para la distribución F en pruebas de hipótesis. La profesora aprovechó esta idea para discutir la forma en que R despliega los resultados de un

ANOVA. En este episodio se observó la orquestación *discute-tecnología-sin-ella*, donde se explica el manejo de los procesos estadísticos en el ambiente de R y se utilizan sus comandos para expresar ideas estadísticas. En esta orquestación, la configuración didáctica corresponde al arreglo tradicional de una clase en la que los alumnos atienden la explicación del profesor, auxiliado del pizarrón. Además de este artefacto material (el pizarrón), en la configuración didáctica, se incluyen artefactos simbólicos, como las representaciones gráficas, los símbolos y los términos que denotan ciertos conceptos estadísticos y los contextos en los que plantea las problemáticas estadísticas (generalmente tomadas de situaciones y datos reales). El modo de explotación incluye las explicaciones de la profesora, usando el pizarrón para anotar e ilustrar. Parte del modo de explotación se relaciona con las características de las explicaciones de la profesora, las cuales buscan discutir la pertinencia de un nuevo concepto estadístico y/o del planteamiento de un determinado procedimiento.

Aunque las clases de tipo magistral hacen referencia al contenido estadístico de las actividades a desarrollar por los alumnos, no se trata de sesiones teóricas cuya componente práctica se encuentra en las actividades; al contrario: las discusiones y contextos abordados en esas clases de tipo magistral, sirven como andamiaje para los alumnos, para que puedan desarrollar las actividades en R. En este sentido, los contextos de las problemáticas planteadas juegan un papel importante, ya que la profesora conecta, a través de referencias a éstos, los distintos tipos de clases. En estas clases de tipo magistral de Mayra, no es posible señalar aún aspectos construccionistas en su orquestación, al ser más un modo más instruccionista (Papert, 1981) de enseñanza, aunque Mayra busca la participación de los alumnos, para que no estén pasivos y sus clases le son valiosas, ya que lo discutido es fundamento de las actividades con R.

Orquestación circula-mientras-trabajan. En las clases dedicadas a la resolución de actividades, se observó la orquestación circula-mientras-trabajan *(work-and-walk-by)*. Parte de los artefactos que componen esta configuración didáctica son los recursos electrónicos que la profesora comparte en carpetas con los alumnos, los cuales incluyen las actividades diseñadas en R y las bases de datos utilizadas. Entre los artefactos simbólicos se encuentran los conceptos estadísticos mismos (previos y los recién presentados en las clases de tipo magistral), además del lenguaje de programación, en particular los comandos que engloban dichos conceptos estadísticos. Estos artefactos constituyen entonces objetos con los cuales pensar y un medio para la exploración de ideas matemáticas, en particular, las estadísticas, siguiendo las ideas propuestas por Papert (1980, 1981). La configuración didáctica de esta orquestación requiere que los alumnos tengan acceso al recurso computacional y que el mobiliario del aula permita que se reúnan en equipo para utilizar una o varias computadoras. En las clases observadas, casi todos los alumnos utilizaban su propia computadora. Como modo de explotación, los alumnos realizan la actividad planteada en R, discutiendo entre ellos, en ocasiones también con otros equipos. Esta forma de orquestación, se centra en las actividades de los equipos de alumnos, permitiéndoles desarrollar sus propios intentos de solución. Mayra interactúa con un equipo a la vez, buscando: (i) enfatizar aspectos importantes y la relación entre los métodos/resultados de la actividad y los conceptos teóricos abordados; y (ii) abordar dudas y dificultades particulares de los alumnos.

El logro del objetivo de la profesora en las actividades de programación, se basa en la necesidad de los alumnos, de esbozar una propuesta de solución al problema e irlo refinando a través de distintos intentos, hasta alcanzar una respuesta que sea adecuada en términos del contexto dado. Siguiendo a Mayra, el planteamiento de las actividades genera en los alumnos la necesidad de preguntarse "¿qué es lo que quiero hacer? ¿Para qué quiero hacerlo? ¿Cuál es la información que voy a obtener con ese particular paso, en el procedimiento?" Mayra afirma que cuando los alumnos hacen un intento fallido, se enfrentan con la necesidad de encontrar el porqué de tal resultado y que este ejercicio de reflexión conlleva un tipo de aprendizaje distinto al obtenido en los intentos donde no se cometen errores, en los cuales el resultado ayuda a confirmar su razonamiento. El cambio en los roles del profesor y los alumnos, el rol activo de éstos últimos en su proceso de aprendizaje, la colaboración y la comunicación, son aspectos construccionistas ligados a esta orquestación (Sacristán et al., 2020). También se observan otros aspectos construccionistas relacionados con la resolución de problemas, la exploración, la construcción de nuevos objetos y/o ideas y la superación de obstáculos o depuración.

Una de las virtudes de las actividades planteadas por Mayra es que se desarrollan considerando una problemática asociada a la práctica biológica/ambiental (el área de formación de los alumnos), de manera que el contexto y los resultados obtenidos se convierten en referentes para los alumnos. Ello da pie a la articulación de las explicaciones dadas en la clase magistral y lo realizado en el ambiente R. En ocasiones, las discusiones surgidas durante las actividades son retomadas en clases magistrales subsecuentes para relacionar los temas discutidos.

En su desempeño didáctico, hay momentos en que la profesora decide hacer un discusión plenaria de alguna propuesta de solución, resultado o duda, ya sea porque tiene un interés particular para facilitar el trabajo o la comprensión de los conceptos o porque busca que el trabajo de los distintos equipos sea un poco más homogéneo.

Orquestaciones sherpa-en-el-trabajo y vincula-pantalla-pizarrón. Finalmente, Mayra conduce clases en las que se revisan las actividades realizadas. En ellas, se observa una variación de la orquestación sherpa-en-el-trabajo *(sherpa-at-work)*, en la cual participa un equipo de alumnos para presentar su trabajo. La configuración didáctica consiste de un arreglo que permita proyectar el trabajo de un equipo y que los demás alumnos lo observen. Como modo de explotación los alumnos de un equipo utilizan R para discutir sus respuestas a la actividad y seguir los planteamientos de la profesora – cumpliendo el aspecto de compartir y discutir "entidades públicas" que señala Papert (1991). El desempeño didáctico relativo a esta orquestación incluye la validación por parte de la profesora de las distintas propuestas de solución, la confrontación de las ideas erróneas y dificultades que surgieron en la actividad y la evaluación de la comprensión, por parte de los alumnos, de los conceptos abordados.

En las clases de revisión también se presenta la orquestación vincula-pantalla-pizarrón. Por ejemplo, durante la revisión de una actividad sobre el Modelo Aditivo, la primera parte se centraba en explorar el conjunto de datos e indicar la diferencia de éste con los conjuntos de datos usados en la prueba de T. Durante esta explicación, el énfasis estuvo en el arreglo de los datos para realizar el procedimiento estadístico. La profesora utilizó el pizarrón para representar los datos arreglados, señalando cómo los subíndices sirven para distinguir el número de dato y el tratamiento al que corresponde, y cómo esto se refleja en las entradas del comando t.test.

Conclusiones

En nuestro estudio se observó cómo Mayra, a través de una propuesta construccionista, puso en práctica actividades para que sus alumnos se involucraran en hacer estadística a través de la programación y generación de representaciones en R, en lugar de aprender acerca de sus conceptos, como propuso Papert (1980). Otros aspectos construccionistas en la orquestación de Mayra se observaron en la dinámica del aula, y en los roles de los participantes (excepto en las clases magistrales). En sus tres tipos de clases, se distinguieron cuatro tipos de orquestación de Mayra. En pocos de los tipos de orquestación presentados en trabajos previos (Drijvers et al. 2009, Guin & Trouche, 2002; Trouche 2004, Tabach, 2011, 2013), se fomenta el trabajo en equipos. Las orquestaciones de Mayra implican una planeación con intenciones y consideraciones mucho más variadas que las referidas en Drijvers et al. (2010). Por ejemplo, mientras que la orquestación-demo-técnica es la más frecuentemente observada en el estudio de Drijvers et al. (2010), en las clases de Mayra no se presenta este tipo de orquestación.

Pratt et al. (2011) señala que la inmersión en el uso de tecnología ayuda a los profesores a replantear el papel que las TD juegan en promover la comprensión de conceptos y métodos estadísticos. Mayra, por su experiencia propia, generó actividades que reflejan un mayor aprovechamiento de la tecnología en el aula. En este documento mostramos una manera novedosa de utilizar el lenguaje de programación R en la enseñanza de la estadística, en la cual los alumnos tienen un rol mucho más activo en la construcción de su conocimiento.

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HIGH SCHOOL STUDENTS' MISCONCEPTIONS ABOUT SIGNIFICANCE TESTING WITH A REPEATED SAMPLING APPROACH

DIFICULTADES DE ESTUDIANTES DE BACHILLERATO SOBRE PRUEBAS DE SIGNIFICACIÓN A TRAVÉS DE UN ENFOQUE DE MUESTREO REPETIDO

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In this paper, we address the following questions: What misconceptions do high school students exhibit in their first encounter with significance test problems through a repeated sampling approach? Which theory or framework could explain the presence and features of such patterns? With brief prior instruction on the use of Fathom software to generate empirical sampling distributions, 18 pairs of high school students participated in a series of lessons involving four significance test problems addressed by a repeated sampling approach. Based on the analysis of students' responses to the first problem, we identified four misconceptions about significance testing. A framework to explain the misconceptions is conjectured.

Keywords: Misconceptions, significance testing, empirical sampling distribution (ESD), verificationism, skepticism.

Sampling and inference are fundamental statistical ideas (Burril & Biehler, 2011). Statistical inference involves using a sample to make a claim about the population from which it has been drawn, quantifying the uncertainty in the process, then making decisions based on the information it provides. It arises from the necessity to evaluate experimental outcomes in any scientific domain. To fulfill this purpose, statistical inference relies on mathematical knowledge and thinking. This implies that the notions of sampling and inference are important components of statistical literacy and should be considered as candidates for compulsory topics in mathematics education curricula (Watson, 2006). However, the topic of statistical inference is usually not introduced until the university level, where it is compressed in an introductory course on statistics and probability. This is a concern, since teaching the wide scope of concepts that constitute statistical inference in just one semester often leads to poor understanding by students, who may only retain a set of terms and procedures.

A potential solution to this problem is to introduce the notion of statistical inference at different levels of education, with the degree of formality adapted to each level. Recent calls from the statistics education community to promote and advance the understanding of statistical inference (Pratt & Ainley, 2008) have spurred interest in exploring ways to introduce concepts related to statistical inference at the primary and high school levels. This research trend is consistent with the idea that Heitele, invoking Bruner, formulated 45 years ago: "…that any subject can be taught effectively in some intellectually honest form to any child at any stage of development" and that fundamental ideas "differ on the various cognitive levels, not in a structural way, but only by their linguistic form and their levels of elaboration" (1975, p. 187). Additionally, the rapid development of technological tools

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in education have allowed for new ways of representing both mathematical and statistical ideas that make these ideas more accessible at pre-university levels (Biehler, Ben-Zvi, Baker, Makar, 2013).

The notion of significance testing is a critical element in the understanding of statistical inference, and is an application of an important principle of scientific thinking: "one does not have evidence for a claim if nothing has been done to rule out ways the claim may be false" (Mayo, 2018, p. 5). However, the idea of validating the outcomes of scientific research by searching for ways to demonstrate the falseness of the claims is contrary to intuition: According to Fischbein (1987), people's natural inclination is to look for confirmatory evidence. Given the importance of and difficulties in understanding the concept of significance testing, therefore, we contend that this concept should be introduced at the high school level, when students are capable of reasoning in a formal way. By developing an understanding of some pertinent components of significance testing earlier in their mathematical education, we propose that students would be in a better position to tackle problems involving statistical inference at the university and professional levels. Accordingly, we formulated the following hypotheses: 1) It is possible to design and implement a series of lessons for high school students that involve reasoning with and about significance testing. 2) The use of technological resources is key to the implementation of the former. 3) The understanding gained about students' reasoning in response to such problems may point to a better teaching approach for significance testing. Using these hypotheses, we formulated our research question:

What misconceptions do high school students exhibit in their first encounter with significance test problems through a repeated sampling approach using computational simulation? What intuitions may explain these misconceptions?

We are motivated to investigate students' reasoning about significance testing by the fact that on the one hand, significance tests are extensively used in a wide range of scientific research domains to evaluate the results of experimental outcomes (Haig 2016; Winch & Campbell, 1970), while on the other hand, many students and researchers have a tendency to misunderstand the objective of significance tests and to misuse the associated concepts and results (Cohen, 1994; Goodman, 2008; Morrison & Henkel, 1997). This tension has given rise to two contrasting reactions to the use of significance tests in experimental research: a) The development and refinement of concepts embedded in the test, as well as the ways they are used in experimental procedures (Mayo, 2018); b) a strong opposition to the use of significance tests and calling for their retirement from experimental research practices (Amrhein, Greenland, & McShane, 2019). This situation points to the need for a better understanding of the ways in which students reason about significance testing.

Background

One of the considerable challenges in the teaching of statistics at the college level is enabling students to rationally interpret the system of ideas that constitute significance testing (Castro-Sotos et al., 2009; Vallecillos, 1996). Batanero (2000) identified three main concepts involved in significance testing that students often misunderstand: 1) the nature of the test, 2) the nature of the *p*-value, and 3) the significance level. In their own study, Castro-Sotos et al. (2009) described two frequent misconceptions about significance testing: 1) it is viewed as a mathematical proof that establishes the truth of one of the two hypotheses, and 2) it is viewed as a probabilistic proof by contradiction: that is, if the null hypothesis is rejected, the *p*-value is the probability of making the wrong decision. Castro-Sotos et al. (2009) also detailed several misconceptions about the concepts of the *p*-value and significance level, including: a) the *p*-value is the inverse of the normative definition (the *p*-value represents the probability of the hypothesis being true, given the outcome of the sample); b) the *p*-value is the probability of a simple event (the *p*-value represents the probability of obtaining the observed outcome); and c) the *p*-value is the probability of the null or alternative hypothesis being true.

Lane-Getaz (2017) explored how early education may decrease the incorrect use (and possible abuse) of significance testing. The researcher examined learning outcomes related to statistical inference for social science students in an introductory course on statistics that incorporated randomization and simulation tasks. The study demonstrated progress made by the students during the course and highlighted several difficulties and misconceptions that could be tackled through instruction.

Other than the previous study, however, the following have not received much attention in the research on students' understandings and misunderstandings about significance testing: 1) students' reasoning about significance testing in the context of a classroom intervention, and 2) the use of digital resources to generate empirical sampling distributions (ESDs) as a support for learning about significance testing. Both are addressed in our current experiment.

Conceptual framework

Reasoning is any process whose objective is to determine the validity or plausibility of a proposition or result by means of certain premises (data or propositions). Mathematical reasoning, in a broad sense, can be elaborated at different levels of formality, given that premises can range from intuitions to firmly instituted axioms, and derivation processes can range from persuasive argumentations (examples, induction, analogies) to well-established mathematical and logical procedures.

Reasoning is accompanied by *sense-making*, which involves linking a new proposition or unexpected result to previous knowledge or beliefs held by the learner (Shaughnessy et al., 2009). Sense-making is supported by intuition which, according to Fischbein, "…expresses the fundamental need of human beings to avoid uncertainty" (1987, p. 28). Doubtful information and uncertain propositions prevent actions and reasoning. Intuitions are necessary to act and reason because they are considered to be true by the subject. While intuitions derived from perceptions of reality are generally correct, "mental representations, hypothetical ideas and solutions may be biased, distorted, incomplete, vague or totally wrong. To believe, however, at least temporarily, in such mental productions, a certain excess of confidence is required" (p. 28). In summary, in the reasoning process, people need to start from certainties, which is why they tend to trust their intuitions (beliefs or conceptions) more than what would be justified under objective evaluation. According to Fischbein, the excess of confidence in intuitions is a mechanism that allows reasoning to be carried out, but at the expense of the risk of spurious conclusions. Fischbein goes on to explain that people's cognitive mechanism for acting and reasoning in certain environments consists of producing a coherent structure (Gestalt) while preserving "facts and segments which fit together and to discard those which may disturb the unity" (p. 35). In particular, drawing from the psychology literature, Fischbein explains that people manifest a bias to confirmation and are reluctant to seek non-confirmatory evidence; when it does appear, they tend to ignore it. They also tend to not consider other plausible scenarios nor that the very same evidence can also account for alternative hypothesis. We can add that hypothetical deductive reasoning is a formal expression of the need to start from (assumed) certainties in a scientific and controlled manner. However, such reasoning requires maturity and practice, given that it does not usually appear in a spontaneous way in specific situations, even when subjects have reached the formal operational stage of development proposed by Piaget (Schmid-Kitsikis, 1983).

The logic of significance testing requires overcoming the cognitive tendencies described by Fischbein. Indeed, a null hypothesis is a hypothesis that one tries to reject, rather than confirm, as is often believed: for Fisher (…), the null hypothesis is "the hypothesis that the phenomenon to be demonstrated is in fact absent." An experimental outcome is deemed significant when the null hypothesis is rejected, and not significant when the null hypothesis is not rejected. A statistic is a function that assigns a number to each sample of a given size. In the simplest case (the one used in this study), this may be the proportion of an attribute in the sample. Significance testing is made possible by the ability to model the sampling distribution of the statistic under the assumption that the null hypothesis is true. It should be noted that in order to understand the role of the sampling distribution, one must apply hypothetical deductive reasoning. Once a sample has been drawn from the population and a statistic is calculated, a probability of the given event occurring or a more extreme value is computed, under the assumption that the null hypothesis is true. This probability is known as *p*-value. If the *p*-value is low (generally <5%), the null hypothesis is rejected, and the result is deemed significant.

A repeated sampling technique using software and random simulations allows students to construct an approximation to the sampling distribution of the statistic, which is usually referred to as an empirical sampling distribution (ESD). To simplify presentation, we define the following terminology as related to ESDs: assume that N samples of size k are simulated with H_0:P=θ, where θ is the probability of success. In this instance, success means selecting an element from the population that presents the feature R, and θ represents the proportion of elements of the population with the feature R. We define the statistic X as the "number of success cases in the sample" and use the notation ESD(N,k,P=θ) to denote an ESD of that statistic.

Method

Participants and data. The following results arise from data corresponding to responses provided by a group of 36 high school students to the first of four activities in a series of lessons on significance testing. Students were arranged in 18 pairs throughout the intervention, each with access to a computer equipped with Fathom. At the time of the intervention, the students were enrolled in their second year of high school (16-17 years old) and had not previously taken a course on statistics or probability. Activities were selected from statistics textbooks and then modified to align with the participants' school level and the repeated sampling approach. While solving the problems, each pair wrote a report that detailed their analysis and solution to the task; these reports constitute our main data source.

Instruments. The first problem of the series of lessons appears below; Table 1 summarizes the statistical data and the solution to the task.

> Coca-Cola's advertising campaign claims that the majority of people (more than 50%) prefer Coca-Cola to Pepsi. To corroborate this, an experiment was conducted, where 60 randomly-selected participants tasted both beverages in a blind test. Thirty-five of the participants preferred the Coca-Cola. Based on these results, would you accept the hypothesis that more than 50% of people prefer Coca-Cola to Pepsi?

Table 1. Data and solution to the first problem

Null Hypothesis (H_0)	N	Rejection freq.	*p–value*	Conclusion
$P = 0.50$	60	$X \geq 37$	$p = 0.126$	H_0 is not rejected

The intervention began with an introductory session aimed to enable students to use Fathom to generate ESDs. This was followed by a series of four lessons, each of which featured a problem involving significance testing. From the second lesson on, the lessons unfolded as follows: First, the instructor facilitated a discussion about students' solutions to the previous problem, inviting students to explain the motives and rationales behind their procedures; second, students were tasked with a new problem that gave them the opportunity to apply newly-gained skills and understandings about significance testing. Students worked on the problems in self-selected pairs and described their solutions on a worksheet. During this time, the instructor's role mainly involved posing questions to stimulate inquiry and helping students overcome technical difficulties (not to provide normatively correct answers).

We rely on principles of the grounded theory methodology (Birks & Mills, 2011; Glaser & Strauss, 1967/2008) for coding students' answers to the problems. This is a general research methodology in the social sciences (Holton, 2008) that involves constructing theories or frameworks through the gathering and analysis of data (as opposed to analyzing the data using an existing theoretical framework). As previously stated, the data analyzed in this study were students' responses to significance test problems. These responses were digitally transcribed, then analyzed in order to generate a set of codes and categories. Several misconceptions were identified using this procedure.

Results

Students' responses were analyzed and coded in three general categories: reasoning, hypothesis, and conclusion. In this report, we will only examine the four misconceptions identified in the first category (reasoning) exhibited in responses to the first problem. A more thorough analysis of the data can be found in García (2017); however, we consider the following evidence sufficient in exemplifying the identified misconceptions.

It should be noted that the descriptions of the emergent patterns use the authors' language and not necessarily the terminology students actually used when describing these ideas. Each identified misconception is a result of the constant comparative method, which allows for abstraction of the subtle differences in students' answers.

Misconception 1: *Majority in the ESD*. Students generate an ESD (500, 60, P=0.50, or similar) and identify the number of samples in which the statistic X takes a value greater than 30. If this number is greater than 250 (N/2), the null hypothesis is rejected, and if it is lower than 250 (N/2), then it is not. It should be noted that this procedure ignores information given by the sample and the significance level. This misconception is consistent with the belief that the ESD represents the population, or an approximation of the population. This misconception was the most frequent in the data, identified in 13 out of 18 responses, and can be exemplified by R6's answer:

> …we considered surveys in which we had at least 31 favorable cases for Coca-Cola, which was a total of 233 surveys… while surveys with 30 favorable cases, that is, 50% or less, represented 267 surveys… a quantity greater than 50%.

The pair concluded that it is not true that there is a greater preference to Coca-Cola over Pepsi and included Figure 1a in their response.

Figure 1: a) Majority in the ESD (pair R6); b) Mode (pair R13)

Misconception 2: *Mode*. Students generate an ESD (500, 60, P=0.50) and identify its mode. If this value is greater than 30, the null hypothesis is rejected, and if it is equal to or lower than 30, it is not. As in the previous case, this reasoning ignores both the observed outcome and the significance level, and is also compatible with the belief that the ESD provides some information about the probability of the null hypothesis being true: In other words, it is assumed that $p = frequency(H)$, where H is an independent variable. This misconception emerged in three responses. An example is provided by R13:

> …our most frequent value in the surveys was 28 people (out of 60) that liked Coca-Cola more, so we can see there are less than 50% that liked Coca-Cola the most.

The pair concluded that Coca-Cola's advertised claim is false and supported their argument with Figure 1b).

Misconception 3: ***Majority in the sample***. This is the only misconception in which students completely ignore the ESD and rely exclusively on the observed outcome (35/60 that preferred Coca-Cola). According to this line of reasoning, if the proportion of favorable to unfavorable outcomes is greater than 0.5, the null hypothesis is rejected, otherwise it is not. R2's response exhibits this misconception:

> …we can say that a majority is 51% or more, and the experiment's result showed that 35 out of 60 people preferred Coca-Cola, which is 59% of the total [(59% x 60) = 35.4]. Thus, we can conclude that they are not wrong in their conclusion.

The pair concluded that Coca-Cola's advertised claim is true, but unlike other students who participated in previous explorations of the task (Garcia & Sanchez, 2014), these students were unable to apply the ESD results generated with the software to support their argument.

Misconception 4. ***Extreme hypothesis***. In this case, $P > 0.5$ is taken to be the alternative hypothesis. Students arbitrarily establish a margin of error, M (e.g., 5% or 10%). Then, they pick a value for P such that $0.5 - M < P < 0.5$ and generate an ESD (500, 60, $P \neq 0.50$) to observe the values within the range of this distribution. If the value of the statistic at hand is in this range (which is the case for this problem), then the hypothesis $P > 0.5$ gets rejected. R10's response demonstrates this line of reasoning:

> …in general, we must take a greater number of surveys of the population so we can conclude that indeed more than 50% like or prefer Coca-Cola, because in these surveys there must be a range of around 10 values more and 10 values less around the expected value […] in the simulated survey, despite the percentage being lower than 50% (because P = 44%) and is expected to have a value of 26, we obtain results that go from 16 (the lowest value of X with a non-zero frequency) to 38 (the highest value of X with a non-zero frequency), from which we can see a greater value (for X) than in the original problem, which is why the 35 do not assure that most people like Coca-Cola.

As previously stated, these responses were provided to the first of the four problems in the series of lessons. It should be noted that the process of addressing students' ideas and solutions may have allowed them to gradually incorporate some previously absent but significant elements involved in significance testing, such as using all of the appropriate facts in the problem and determining if the observed outcome could be labeled as an outlier or not. Nevertheless, students were ultimately unable to provide solutions to this problem that were consistent with the logic of significance testing.

Conclusions and discussion

We propose an explanation based on Fischbein's (1987) theory about intuition for the first three observed misconceptions. According to Fischbein, intuition is related to people's tendency to avoid uncertainty and to look for confirmatory evidence. This idea, coupled with the traditional focus on proof in mathematics teaching, led us to propose a category in the context of our experiment called *naïve verificationism*, which consists of the belief that the objective of a significance test is to demonstrate the veracity of the null hypothesis. *Critical verificationism*, on the other hand, consists of the belief that the objective of the test is to compute the probability of the null hypothesis being true (or false). However, as students continue to participate in discussions about the problems and analyze solution strategies, they develop a kind of skepticism, which involves recognizing that it isn't possible to conclusively verify any hypothesis based on the evidence provided by a sample.

According to Fischbein (1987), people try to produce coherent structural schemata in which they can integrate their intuitions (certainties), beliefs (theories), and observations (evidence). Such structural schemata underpin the reasoning process—justifying the solution of a problem or explaining a phenomenon. In this sense, the misconception of mistaking an ESD with the population (or its approximation; Garfield & Ben-Zvi, 2008) is compatible with naïve verificationism, given that if the sampling distribution represents the population, then analyzing the ESD alone would be enough to verify the validity of the hypothesis. We interpret Misconception 1 (Majority in the ESD) as a manifestation of students' efforts to align the ESD, a new concept for our participants, to their expectations of verifying the null hypothesis. In a similar way, another way to create a structural schema is to believe that the ESD is a distribution in which each value for the statistic is taken as a possible hypothesis, with the frequency of each hypothesis (statistic) allowing for the computation of the probability of the truth (or falsehood) for each one; such reasoning demonstrates critical verificationism. The mode of such a conceived distribution would be the most likely hypothesis (statistic), and this is consistent with Misconception 2 (Comparing the mode of the ESD with the null hypothesis).

In Misconceptions 1 and 2, students do not assign a specific role to the information provided by the sample, given that their conception of the ESD makes this task an unnecessary one. These results show that the intended purpose and meaning of an ESD is not clear to students, despite their participation in activities in which they produced and interacted with different ESD's using physical and computational simulations.

In Misconception 3 (Majority in the sample), students do not ignore sample results as in previous cases, but create a simpler structural schema: they assume that because the information provided by the sample is the only information available when making a decision, it alone contains the key to the solution of the problem. Therefore, it is assumed that the proportion of favorable to unfavorable outcomes in the sample closely mirrors the corresponding proportion of the population. In this misconception, the information provided by the ESD is completely ignored, and therefore demonstrates naive verificationism. Shaughnessy (1992, p. 478) referred to two false conceptions that are related to this misconception: "people inadequately believe that there's no variability in the real world" and that "people often have an unjustified overconfidence in small samples." Nickerson (2000, p. 254) and, in a similar way, Castro-Sotos et al. (2009) report that one of the most common beliefs about significance tests among researchers and students is that "by rejecting the null hypothesis, a theory that predicts the falseness of the null hypothesis is established." Such a claim is compatible with naïve verificationism, because it responds to a desire of establishing the falseness of an hypothesis. This particular misconception emerged only in the first problem and was abandoned as students gained experience in generating and analyzing ESDs.

Furthermore, in Misconception 4 (Extreme hypotheses), students incorporate the idea of a margin of error by defining an interval on one side of the null hypothesis, with the opposite side representing the alternative hypothesis. An extreme hypothesis is selected in this interval. If the distribution generated by the extreme hypothesis captures the value of the statistic, there is no reason to reject the null hypothesis. That is, if the value of the statistic (in this case, 35) can plausibly occur in a more extreme ESD ($P < 0.5$), then the event is also plausibly occurring for the null hypothesis ($P = 0.5$) as well, and the null hypothesis should therefore not be rejected. In using such reasoning, students exhibit intuitions that are partially aligned with the logic of significance testing: for example, the idea of "taking more extreme values" is used when defining a p-value, and considering "what would happen if the hypothesis was…" demonstrates hypothetical-deductive reasoning that is used in the establishment of the hypothesis. However, such intuitions are not appropriately applied. An important feature of this line of reasoning is that it relies on an arbitrary, but well-intended criterion:

the null hypothesis is rejected when the statistic is not contained in a certain range of the simulated ESD.

In summary, the misconceptions that students exhibited in their solutions to the first problem correspond to the intuitive idea that the objective of a significance test is to verify the null hypothesis, or to estimate the probability that it is true. It is likely that a tendency to verificationism also explains some of the misunderstandings and misuses of ESDs evidenced even by experimented learners. This is why it is critical in the teaching of statistics to enable students to develop alternative reasoning schemes—that is, schemes based in reasonable skepticism.

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STATISTICS AND PROBABILITY:

BRIEF RESEARCH REPORTS

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FUNDAMENTAL STATISTICAL IDEAS IN PRIMARY, SECONDARY AND HIGH SCHOOL MEXICAN CURRICULUM: REFLECTIONS FROM THE INTERNATIONAL PERSPECTIVE

IDEAS FUNDAMENTALES DE ESTADÍSTICA EN PRIMARIA, SECUNDARIA Y BACHILLERATO EN EL CURRICULO MEXICANO: REFLEXIONES DESDE LA PERSPECTIVA INTERNACIONAL

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In this article we analyze the fundamental ideas of statistics in the Mexican curriculum of basic and high school education, with the purpose of establishing relationships with some curricula and recommendations of organizations that promote statistical education. The results show that statistics are present from kindergarten to high school, as established by international recommendations; however, statistical inference is absent from the secondary school curriculum, including high school. The methodology of teaching in basic education gives importance to real contexts and to the posing of statistical questions to respond with the data, but in high school there is a greater emphasis on statistical procedures. Technology for data analysis and simulation is practically absent in the curriculum of all levels.

Keywords: Probability, data analysis and statistics, curriculum, technology

Study objectives

In the last three decades, statistics has had the greatest growth in the mathematics curriculum of basic education and high school in many countries, due to its importance as a methodological tool, but also due to the relevance of literacy and statistical thinking in modern society. In this way, organizations and researchers that promote statistics education have issued recommendations on the fundamental statistical ideas that should be in the curricula, as well as the teaching perspectives more according with the needs of literacy and statistical thinking of today's society.

In this context, it is pertinent to analyze the status and significance of statistics in pre-university educational levels in Mexico. It is interesting to analyze contents and didactic orientations, with the purpose of characterizing their teaching and contrasting with the recommendations and trends of the international curriculum.

Background

Holmes (2003) points out that in England since 1961, contents of data and chance were introduced in high school curriculum, in the late 1960s were extended to primary and secondary school. In the 1980s and 1990s the Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) and Principles and Standards for School Mathematics (NCTM 2000) were published in the United States, curricular documents in which data Analysis and Probability appears as an integral part of the Mathematics curriculum. Guidelines for Assessment and Instruction in Statistics Education (GAISE) (Franklin et al., 2005) was published in 2005 as a framework for statistics education from kindergarten to high school. Meanwhile, in 1992 in New Zealand, Statistics and Probability were included for the first time as a teaching area at all educational levels; in 2007 three main statistical areas were included in the eight levels of pre-university education: statistical research, statistical literacy and probability.

In Mexico, the Statistics and Probability appear for the first time in secondary education curriculum in 1975, and in the case of some high school programs since the end of the 1970s. The high school

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curriculum consists of three components formative (basic, propaedeutic and professional). Statistics is part of the propaedeutic component, that is, it is not part of the basic table of subjects, but most educational institutions integrate at least one course of Statistics and Probability into their curriculum, either optional or compulsory.

Theoretical perspective

Statistical fundamental statistical ideas

In the search to define a set of fundamental statistics and probability ideas that must be learned by students before finishing high school, Burrill & Biehler (2011) propose the following themes:

1. Data.
2. Variability.
3. Distribution.
4. Representations.
5. Association and modeling between two variables.
6. Probability models.
7. Sampling and inference.

Guidelines for Evaluation and Instruction in Statistical Education

The Guidelines for Assessment and Instruction in Statistics Education (GAISE Report) (Franklin, et al., 2005) proposes a central teaching model based on solving statistical problems as a four-component research process: posing questions to answer with the data, collect the data, analyze the data and interpret results to answer the questions. The GAISE Report visualizes probability as a tool for statistics; in this sense, it is considered that the relationship between classical and frequency approach should be explored to compare theoretical probabilities and observed frequencies. Use probability to understand randomization in statistical work in the case of sampling and design of experiments.

Methodology

The methodology we have used is documentary in nature (Bardin, 2006). The sources of information were the most current official curricular documents. The contents, expected learning and didactic guidelines were analyzed to contrast them with the same entities proposed by Burrill & Biehler (2011) and Franklin et al., (2005). In each curriculum document, the presence of the fundamental ideas was reviewed in detail; Likewise, statements that inform about the contents, expected learning and didactic guidelines were analyzed in the text.

Results and Discussion

The study of statistics appears from the kindergarten, while the study of probability begins in the fifth grade of primary school. In basic education, both areas are included as themes in the mathematics subject, while at the high school are part of a subject by itself, generally called Probability and Statistics, or simply Statistics. In the lower grades of primary education, the importance to distinguish particular questions (referring to an element) from statistical questions (referring to a group) is pointed out. These teaching guidelines are in accordance with the recommendations of the GAISE report for the primary level. In grades 5 and 6, the notions of qualitative and quantitative variables are introduced, and the calculation of descriptive measures of center and dispersion of the data (mode, mean and range) appears for the first time. The guidelines suggest collecting data from the classroom and school context, and asking questions that require the use of such descriptive measures. The study of probability begins in the fifth year of primary school, beginning by the distinction of random and non-random experience, the idea of sample space and the determination of its elements through the use of tree diagrams. The expected learning consists of identifying chance games and carrying out experiments to record the results in tables of relative and absolute frequencies (frequency approach to probability).

Meanwhile in high school, the expected learnings in statistics emphasize again the collection and recording of data, and representations for the organization and interpretation of data; in this way, in addition to the bar and circular diagrams, histograms, frequency polygons and line graphs (time series) are considered. In the calculation of descriptive measures of the data, the mean, median, mode, range and mean deviation are considered. The instructional guidelines emphasize collecting data from school contexts such as the classroom or issues of interest to students that appear in the media. Teachers are directed to construct the graphs manually so that students understand how they are constructed. Instead, the use of spreadsheet to construct frequency histograms and polygons is suggested in an isolated and superficial way. Regarding probability, students are expected to carry out random experiments and record results as an approximation to frequency probability. It is further proposed that they determine the theoretical probability of a random experiment and of two mutually exclusive events using the rules of addition and multiplying probabilities.

Chance games and experiments are proposed in the didactic orientations where both are contrasted. In the second grade, the use of simulation of random phenomena by means of some software is proposed only as a recommendation, which is in full agreement with the recommendations of various authors and the international curriculum (Chaput, Girard and Henry, 2011; Burrill & Biehler, 2011).

In high school there is not unique curriculum; however, from 2008 there is a proposal for an official curriculum that we will take as a reference for the analysis. The course of Statistics and Probability I offers a review and deepening of the contents of the secondary school; the variance and standard deviation, quartiles, deciles and percentiles are added. An introduction to two-variable data analysis (bivariate data), which includes scatterplots, linear correlation, and simple linear regression. This topic is highly recommended in the GAISE Project and is one of the fundamental statistical ideas for high school (Burrill and Biehler, 2011) for its importance in developing multivariate thinking and prediction theory. In some countries such as Costa Rica and Chile this subject does not yet appear in the high school curriculum. In the course of Statistics and Probability II, the topics of sets, counting techniques and mutually exclusive events appear before calculating the theoretical and frequency probability; Bernoulli, Binomial, Normal and Poisson probability distributions and Bayes' theorem are included. Statistical inference is not part of the official curriculum, it is only considered in the case of CCH (a particular high school system offered by the National University) optionally in the last semester.

Regarding the use of technology in the statistics and probability teaching, there are superficial references in the all levels of Mexican curriculum. Its use is suggested for the simulation of random phenomena as a means of estimating theoretical probabilities, but there are no references for its use in the analysis and exploration of data, a situation that is widely recommended in the international recommendations and curricula of New Zealand, Spain, United States and other countries.

Conclusions

The study of the data is present from kindergarten to the high school, while the study of chance begins in the fifth grade of primary education, which is very similar to the international curriculum, thus claiming that the statistics and probability they have an important status in the education of citizens. However, there are some differences in some levels, particularly in the high school, where the absence of statistical inference in the official curriculum, -widely recommended in the international curriculum-, is observed.

The data collection, organization and representation of data, and descriptive measures of the data, constitute the backbone of the statistical content throughout all basic education and high school. Data collection techniques (surveys, observation, interviews, information consultation) are common at all levels, but increase in complexity in high school with a general description of random sampling methods and not random.

In the same way, graphical representations evolve from simple pictograms in kindergarten to bar and circle diagrams in primary school, to histograms, frequency polygons, and line graphs to visualize quantitative data. However, stem and leaf plots, box plots, and dot plots are absent in the curriculum, which are common in curricula from other countries. Descriptive measures begin with calculating frequencies in kindergarten, calculating mode in primary, and calculating mean, mean, mode, and mean deviation in high school, to expand standard deviation, variance, and correlation in high school.

For its part, probability in primary school begins with the study of notions of chance through simple random experiments and calculation of the frequencies of the results. In secondary education, the classical approach is introduced and its contrast with the frequency approach is promoted, this is an aspect highly recommended in the international curriculum. However, in high school that link between the two approaches is not encouraged and more emphasis is placed on the classical approach using combinatorial techniques.

Regarding teaching methodology, we observe a uniform trend in basic education, suggesting the use of real and meaningful contexts for students and the posing of statistical questions that must be answered with the data, which represents, in our opinion, an innovative trend, according to the perspective of development of statistical thinking. This trend is interrupted in high school, which focuses more attention on later stages of the statistical research cycle, such as data analysis.

The use of computer technology in data analysis and the simulation of random phenomena only has superficial references at all educational levels, which constitutes the greatest difference with the international curriculum. This is undoubtedly a pending subject, as is the absence of statistical inference in high school, which must be improved in the Mexican curriculum.

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FUNDAMENTAL STATISTICAL IDEAS IN PRIMARY, SECONDARY AND HIGH SCHOOL MEXICAN CURRICULUM: REFLECTIONS FROM INTERNATIONAL PERSPECTIVE

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En este artículo analizamos las ideas fundamentales de estadística en el currículo mexicano de educación básica y bachillerato, con el propósito de establecer relaciones con algunos currículos y recomendaciones de organismos que promueven la educación estadística. Los resultados muestran que la estadística está presente desde preescolar hasta bachillerato, tal como lo establecen recomendaciones internacionales; sin embargo, la inferencia estadistica está ausente del currículo de secundaria, incluso del bachillerato. La metodología de enseñanza en educación básica otorga importancia a contextos reales y al planteamiento de preguntas estadísticas para responder con los datos, pero en bachillerato se hace mayor énfasis en el cálculo estadístico. La tecnología para análisis de datos y simulación está prácticamente ausente en el currículo de todos los niveles.

Palabras clave: Probabilidad, análisis de datos y estadística, currículum, tecnología

Objetivos del estudio

En las últimas tres décadas, la estadística ha tenido el mayor crecimiento en el currículo de matemáticas de la educación básica y bachillerato de diversos países, debido a su importancia como herramienta metodológica, pero también por la relevancia que tiene la cultura y el pensamiento estadístico en la sociedad moderna. De tal forma, organismos e investigadores que promueven la educación estadística han emitido recomendaciones sobre los fundamentos y las ideas centrales de la estadística que deben estar en los currículos de cada nivel educativo, así como las perspectivas de enseñanza más acordes a las necesidades de alfabetización y pensamiento estadístico de la sociedad actual.

En este contexto, es pertinente analizar el estatus y significado que tiene la estadística en los niveles educativos pre-universitarios en México. Interesa analizar contenidos y orientaciones didácticas, con el propósito de caracterizar su enseñanza y hacer un contraste con las recomendaciones y la tendencia del currículo internacional.

Antecedentes

Holmes (2003) señala que, en Inglaterra, desde 1961 se introdujeron contenidos relacionados con datos y azar en cursos de bachillerato, a finales de la década de 1960, estos contenidos fueron extendidos a la escuela primaria y secundaria inglesa. En la década de 1980 y 1990 se publican en los Estados Unidos los Curriculum and Evaluation Standards for School Mathematics (NCTM, 1989) y Principles and Standards for School Mathematics (NCTM 2000), documentos curriculares en los que el eje de Análisis de Datos y Probabilidad aparece como parte integral del currículum de matemáticas. En 2005 se publica Guidelines for Assessment and Instruction in Statistics *Education* (GAISE Report) (Franklin et al., 2005), como un marco para la educación estadística desde preescolar hasta el bachillerato. Mientras tanto, en 1992 en Nueva Zelanda se incluye por primera vez la estadística y probabilidad como área de enseñanza en todos los niveles educativos. En 2007 se incluyen tres grandes líneas de contenido estadístico en los ocho niveles de la educación preuniversitaria: investigación estadística, cultura estadística y probabilidad.

En México, aparecen contenidos sobre estadística y probabilidad por primera vez en los programas de estudio de educación secundaria en 1975, y en el caso de algunos programas del nivel medio superior desde finales de la década de 1970. El currículo de bachillerato consta de tres componentes formativas (básica, propedéutica y profesional). La estadística es parte de la componente propedéutica, es decir, no forma parte del cuadro básico de materias, pero con base en su autonomía curricular, la mayoría de las instituciones educativas integran al menos un curso de estadística y probabilidad en su currículo, ya sea de manera optativa u obligatoria.

Perspectiva teórica

Ideas estadísticas fundamentales de estadística

En la búsqueda por definir un conjunto de ideas fundamentales de estadística y probabilidad que deben ser aprendidas por los estudiantes antes de concluir el bachillerato, Burrill & Biehler (2011) proponen las siguientes:

1. Datos.
2. Variabilidad.
3. Distribución.
4. Representaciones.
5. Asociación y modelación entre dos variables.
6. Modelos de probabilidad.
7. Muestreo e inferencia.

Lineamientos para la Evaluación e Instrucción en Educación Estadística

El reporte Guidelines for Assessment and Instruction in Statistics Education (GAISE) (Franklin, et al., 2005) propone un modelo central de enseñanza basado en la resolución de problemas estadísticos como un proceso investigativo de cuatro componentes: formular preguntas para responder con los datos, recolectar los datos, analizar los datos e interpretar resultados para responder las preguntas planteadas. En cuanto al enfoque de enseñanza y el contenido de probabilidad en el currículo, el reporte GAISE visualiza a la probabilidad como una herramienta para la estadística. En este sentido, se considera que se debe explorar la relación entre enfoque clásico y frecuencial para comparar probabilidades teóricas y frecuencias observadas. Usar la probabilidad para comprender la aleatorización en el trabajo estadístico en el caso del muestreo y diseño de experimentos.

Metodología

La metodología que hemos utilizado es de carácter documental (Bardin, 2006). Las fuentes de información fueron los documentos curriculares oficiales más actuales. Se analizaron los contenidos, aprendizajes esperados y orientaciones didácticas para contrastarlos con los mismos entes que proponen Burrill & Biehler (2011) y Franklin et al., (2005). En cada documento curricular se revisó con detalle la presencia de las ideas fundamentales; asimismo se analizaron en el texto, enunciados y declaraciones que informan sobre los contenidos, aprendizajes esperados y orientaciones didácticas.

Resultados y discusión

El estudio de la estadística aparece desde el nivel preescolar, mientras que el estudio de la probabilidad inicia en quinto grado de primaria. En la educación básica, ambos contenidos se incluyen como temas en la materia de matemáticas, mientras tanto en el nivel bachillerato los contenidos son parte de una materia por sí sola, denominada por lo general como Probabilidad y Estadística, o simplemente Estadística. En los grados inferiores de la educación primaria se señala la importancia de que los alumnos aprendan distinguir preguntas particulares (referidas a un elemento) de preguntas estadísticas (referidas a un grupo o colectivo). Estas orientaciones didácticas están concordancia con las recomendaciones del reporte GAISE para el nivel primaria. En los grados superiores se amplía el uso de representaciones de los datos a diagramas de barras y circulares. En los grados 5 y 6 se introducen las nociones de variables cualitativas y cuantitativas, y aparece por primera vez el cálculo de medidas descriptivas de centro y dispersión de los datos (moda, media y rango). En las orientaciones se sugiere la recolección de datos del contexto del salón de clases y la escuela, y el planteamiento de preguntas que requieren del uso de tales medidas descriptivas. El estudio de la probabilidad inicia en quinto año de primaria, se empieza introduciendo la distinción experiencia aleatoria y no aleatoria, la idea de espacio muestral y la determinación de sus elementos mediante el uso de diagramas de árbol. Los aprendizajes esperados consisten en identificar juegos en los que interviene el azar y realizar experimentos para registrar los resultados en tablas de frecuencias relativas y absolutas (enfoque frecuencial de la probabilidad).

Mientras tanto en secundaria, los aprendizajes esperados en estadística enfatizan de nuevo en la recolección y registros de datos, y un repertorio más amplio de representaciones para la organización e interpretación de los datos; de tal forma, además de los diagramas de barras y circulares, se contemplan los histogramas, polígonos de frecuencias y gráficas de línea (series de tiempo). En el cálculo de medidas descriptivas de los datos se contempla la media, mediana, moda, rango y la desviación media. Las orientaciones didácticas hacen hincapié en recopilar datos de contextos escolares como el salón de clases o asuntos de interés para los estudiantes que aparecen en los medios. Se orienta a los docentes a construir las gráficas manualmente para que los estudiantes comprendan cómo se construyen. En cambio, el uso de hoja de cálculo para construir histogramas y polígonos de frecuencia se sugiere de manera aislada y superficial.

En cuanto a probabilidad, se espera que los estudiantes realicen experimentos aleatorios y registren resultados como un acercamiento a la probabilidad frecuencial, se propone además que determinen la probabilidad teórica de un experimento aleatorio y de dos eventos mutuamente excluyentes utilizando las reglas de la suma y de la multiplicación de probabilidades. Se proponen juegos de azar en las orientaciones didácticas y experimentos donde se contrasten ambas. En segundo grado se propone el uso de simulación de fenómenos aleatorios mediante algún software solo como una recomendación, lo cual está en plena concordancia con las recomendaciones de diversos autores y el currículo internacional (Chaput, Girard y Henry, 2011; Burrill & Biehler, 2011).

En cuanto al nivel bachillerato no existen programas únicos a nivel nacional, sin embargo, a partir de 2008 hay una propuesta de currículo oficial por parte de la SEP que tomaremos como referencia para el análisis. El curso de Estadística y Probabilidad I ofrece un repaso y profundización de los contenidos de la escuela secundaria, se agregan la varianza y desviación estándar, cuartiles, deciles y percentiles. La novedad en este nivel es la introducción al análisis de datos de dos variables (datos bivariados), en el que se incluyen los diagramas de dispersión, correlación lineal y regresión lineal simple. Este tema es muy recomendado en el proyecto GAISE y es una de las ideas fundamentales de estadística para el bachillerato (Burrill y Biehler, 2011) por su importancia para desarrollar el pensamiento multivariado de los estudiantes e introducirlos a la teoría de la predicción. En algunos países como Costa Rica y Chile aún no aparece esta temática en el currículo.

En el curso de Estadística y Probabilidad II, aparecen los temas de conjuntos, técnicas de conteo y eventos mutuamente excluyentes previo al cálculo de la probabilidad teórica y frecuencial, distribuciones de probabilidad Bernoulli, Binomial. Normal y Poisson, probabilidad condicional y teorema de Bayes. La inferencia estadística no forma parte del currículo oficial definido por la SEP, solo es contemplada en el sistema de bachillerato CCH (sistema de bachillerato ofrecido por la Universidad Nacional), de manera opcional en el último semestre.

En cuanto al uso de tecnología en la enseñanza de la estadística y probabilidad en todos los niveles, apenas hay referencias superficiales en el currículo mexicano. Se sugiere su uso para la simulación de fenómenos aleatorios como un medio para estimar probabilidades teóricas, pero no hay referencias de su uso en el análisis y exploración de los datos, situación que es ampliamente recomendada en las recomendaciones internacionales y currículos de Nueva Zelanda, España y otros países.

Conclusiones

El estudio de los datos tiene presencia desde el nivel preescolar hasta el bachillerato, mientras que el estudio del azar inicia en el quinto grado de la educación primaria, lo cual guarda mucha similitud con el currículo internacional. Sin embargo, existen algunas diferencias en cuanto a lo contenidos en algunos niveles, particularmente en el bachillerato, donde se observa la ausencia de la inferencia estadística en el currículo oficial, ampliamente recomendada en el currículo internacional.

Los contenidos temáticos recolección de datos, organización y representación de datos, y medidas descriptivas de los datos, constituyen la comuna vertebral del contenido estadístico en educación básica y bachillerato. Las técnicas de recolección de los datos (encuestas, observación, entrevista, consulta de información) son comunes en todos los niveles, pero aumentan de complejidad en el bachillerato con una descripción general de los métodos de muestreo aleatorio y no aleatorio.

De la misma manera, las representaciones gráficas evolucionan desde los pictogramas sencillos en preescolar a diagrama de barras y circulares en la primaria, a histogramas, polígonos de frecuencia y gráficas de línea para visualizar datos cuantitativos. Sin embargo, se encuentran ausentes en el currículo gráficas de tallo y hoja, gráficas de caja y gráficas de puntos, que son comunes en currículos de otros países. Las medidas descriptivas inician con el cálculo de frecuencias en preescolar, cálculo de la moda en primaria, y el cálculo de la media, media, moda y desviación media en secundaria, para ampliar a la desviación estándar, varianza y la correlación en el bachillerato.

Por su parte, la probabilidad en la escuela primaria inicia con el estudio de nociones de azar a través de experimentos aleatorios sencillos y cálculo de frecuencias de los resultados. En la educación secundaria se introduce el enfoque clásico y se promueve su contraste con el enfoque frecuencial, un aspecto muy recomendado en el currículo internacional. Sin embargo, en el bachillerato esa liga entre ambos enfoques no es fomentada y se hace mayor énfasis en el enfoque clásico con uso de técnicas combinatorias.

En cuanto a la metodología de enseñanza observamos una tendencia uniforme en la educación básica, sugiriendo el uso de contextos reales y significativos para los estudiantes y el planteamiento de preguntas estadísticas que se deben responder con los datos, lo cual representa a nuestro juicio una tendencia innovadora y acorde con la perspectiva de desarrollo de pensamiento estadístico en los estudiantes. Esta tendencia es interrumpida en el bachillerato, que centra mas su atención en etapas posteriores del ciclo de investigación estadística, como es el análisis de los datos.

El uso de la tecnología computacional en el análisis de datos y la simulación de fenómenos aleatorios solo tiene referencias superficiales en todos los niveles educativos, lo que constituye la mayor diferencia con el currículo internacional. Sin duda esta es una asignatura pendiente, al igual que la ausencia de la inferencia estadística en el bachillerato, que se debe mejorar en el currículo mexicano.

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REASONING ABOUT ASSOCIATION FOR CATEGORICAL DATA USING CONTINGENCY TABLES AND MOSAIC PLOTS

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With the implementation of Common Core in most states, the pre-k-12 mathematics curriculum now contains a significant amount of probability and statistics, mainly situated in middle and secondary grades. Statistical association is a challenging concept, and secondary students are expected to use contingency tables to begin to reason about association of categorical variables. This requires proportional reasoning, which is a focus of middle grades mathematics and necessary for more advanced study but remains a struggle for even most adults. Researchers call for use of multiple representations to develop conceptual understanding, and I consider a traditional contingency table in addition to a mosaic plot to see how students reason through a series of tasks.

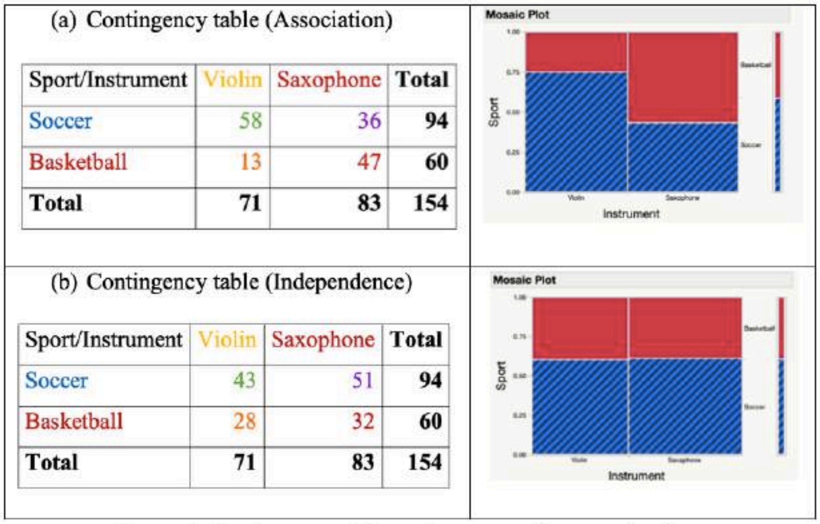
Keywords: Cognition, Data Analysis and Statistics, Representations and Visualization, Geometry and Geometrical and Spatial Thinking

Statistical skills develop over time and in order for all high school graduates to have statistical literacy, instruction should begin early and expand through middle and high school (Bargagliotti et al., 2020). Association can exist between variables that are quantitative, like a person's height in centimeters, as well as variables that are categorical, like a person's eye color. Association of two categorical variables is included in the eighth and ninth grade curriculum in most states. Statistical (in)dependence can be determined numerically or visually and it certainly requires proportional reasoning, which researchers have identified as a "major connecting idea" when reasoning with probability and statistics where it is important to help students make explicit connections between data and proportions (Watson & Shaughnessy, 2004).

When considering bivariate data that are quantitative, there are well developed and standard graphical methods to aid students in determining (in)dependence, namely the use of a scatterplot (Friendly, 1999). The widely accepted Cartesian plane serves as a structure to visualize the data and determine if a linear or other type of association might be present. When the data are categorical, a two-way contingency table is used, but a standard graphical display does not exist for considering association of categorical variables (Friendly, 1999). Bar graphs, either segmented or side by side are often used to display frequencies in contingency tables; however, a mosaic plot is a default display used in some statistical software. Researchers in Australia (Pfannkuch & Budgett, 2017) recently noted some promising results of students working with an interactive mosaic plot, which assisted students in appropriately applying proportional reasoning with problems dealing with probability, especially when considering independence. A mosaic plot is based on a unit square that is vertically divided in proportion to marginal frequencies of one variable and further divided into rectangular regions that are proportional in area to each of the joint frequencies (see Figure 1).

Mosaic plots are often non-numerical and can be used to understand what the displayed data implies quantitatively and determine independence. Visualization can aid engagement with meanings and concepts that are not readily available through symbolic representation and when information is displayed visually, we are able to "see" the story, picture a cause-effect aspect of a relationship, and vividly remember it (Arcavi, 2003). Visuals can "group together clusters of information that can be apprehended at once" (Arcavi, 2003, p. 218), and "visualization at the service of problem solving, may also play a central role to inspire a whole solution, beyond the merely procedural." (p. 224).

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020



Sport/Instrument	Violin	Saxophone	Total
Soccer	58	36	94
Basketball	13	47	60
Total	71	83	154

Sport/Instrument	Violin	Saxophone	Total
Soccer	43	51	94
Basketball	28	32	60
Total	71	83	154

Figure 1. Contingency tables and corresponding mosaic plots.

Although elementary students are not likely to reason proportionally, previous studies indicate that younger students can reason correctly about association when only doubling and halving are required. The present study investigates how pre-k-12 students reason about association of categorical variables using contingency tables with and without mosaic plots.

Framework

Seminal work aimed to understand how students reason with complete contingency tables (Batanero, Estepa, Godino, & Green, 1996) provides the basis of my framework. Proportional reasoning requires the comparison of ratios and the use of all four cells in a multiplicative manner. I developed a framework which includes eight conceptions of reasoning with contingency tables (see Table 1), which are based on the five levels (L1-L5) identified by Perez-Echevarria (1990, as cited in Batanero et al., 1996).

Table 1 Graph Type and Variable Values for Different Items

Code	Description and features
N0	No cells in the table are used to decide about independence or association.
N1	No interior cells and one or more marginal cells are used to decide about independence or association.
L1	Localist-1: One interior cell in the table is used.
L2	Localist-2: Two cells in the table are used.
L3	Localist-3: Three cells in the table are used.
A1	Localist-4: All four cells in the table are used in an additive way.
P1	Proportional-1: All four cells in the table are used with proportional reasoning that compares risk (part to whole ratios). One conditional relative frequency is compared to another, focusing on the interior cells.
P2	Proportional-2: All four cells in the table are used with proportional reasoning that compares risk (part to whole ratios), and compares one conditional relative frequency to a marginal relative frequency.

P3	Proportional-3: All four cells in the table are used with proportional reasoning that compares odds (part to part ratios) and compares the odds for one category to another category for the same variable through subtraction or a ratio.

When considering the problems where the mosaic plots were provided, I used the same base codes and appended an additional code to indicate how the mosaic plot seemed to function. I considered whether the mosaic plot was a hindrance (M-), seemed to have no impact on the solution (M), or was helpful (M+).

Research Design

Since my interest is of the "how" and "why" nature, a qualitative, multiple-case study design is appropriate (Patton, 2005). I conducted think-aloud interviews (Charters, 2003) with seven participants that ranged in age from seven to 17 because I wanted to get a sense of ways that students across upper elementary, middle school and high school would respond to the same tasks. The reasoning about contingency tables of students in this age range has been underrepresented in past studies.

Each interview was semi-structured and used a protocol I develop based on the literature. The tasks all used the same context, but varied in the completeness of the contingency tables, numerical values of the frequencies, and whether there was an association among the variables. The words "association" and "independent" were used in the questions along with an additional explanation of their meaning. For the first part of the interview, I provided them with a series of problems with contingency tables and asked questions to ascertain their understanding of what the values in the tables represent. Then I introduced a mosaic plot by having them create one with a simple example. After verifying they could reason with it in conjunction with a contingency table, I presented two of the initial problems along with an accompanying mosaic plot. This study focuses on these two tasks. I concluded by asking them questions about the mosaic plot in general.

I recorded each interview with two video cameras, capturing both a close-up view of the student work as well as a broader view of the student to include gestures and facial expressions. Each of the seven interviews was transcribed and both an augmented transcript, noting participant actions, and a lesson graph, including notes of interesting moments, were created. The framework was used to code the data.

Results and Significance

The mosaic plot was never a distraction and was most often helpful. The two tasks considered in this study proved difficult for most students, and only two of the older students, Scott and Klaus, were able to provide comprehensive explanations for the tasks (Scott: Task 1 P2/P2M, Task 2 A1/P2M+; Klaus: Task 1 N)/P2M+, Task 2 A4/P2M+). These correct explanations occurred each time a mosaic plot accompanied the problem, but only once when it did not (Scott, Task 1). When the mosaic plot was provided, it improved all students' reasoning with the exception of this one problem Scott was able to solve without it.

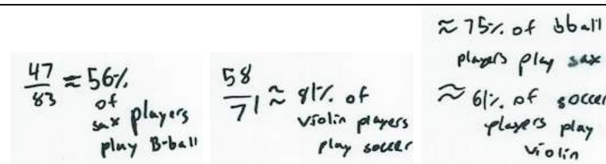
The younger students always showed improved reasoning when the mosaic plots were provided, but because of their limited proportional reasoning, they were not able to provide a completely correct solution and justify their reasoning. As the literature suggests, they were able to use numbers and benchmark fractions to reason about the data, but they conflated percentages with frequencies. Additionally, they did not have an understanding of the structure of a contingency table, often indicating the marginal frequencies for the rows and columns were a different group of people.

Some of the improvements in performance on the problem when the mosaic plot was included could be due to other factors like seeing the problem for a second time and working with different contingency tables between reasoning without and with the mosaic plot. However Scott and Klaus both verified they were looking at the mosaic plot and used it to solve the problem and overall, explanation and justification was more limited when there was no mosaic plot provided. Clear and complete explanations more frequently occurred when the mosaic plot was included. For example, when Scott was using the mosaic plot in the second task, he said, "just look at the mosaic, it's pretty clear" and provided a succinct explanation.

Interestingly, the mosaic plot seemed to help Klaus as he reasoned through the second task. With the contingency table alone, his additive understanding was apparent and he remained with his initial reasoning relying on the larger numbers. (see Figure 2).

Sport/Instrument	Violin	Saxophone	Total
Soccer	58	36	94
Basketball	13	47	60
Total	71	83	154

"The majority of soccer players play violin. And the majority of basketball players play saxophone."



"So, I don't think those percents mean anything in this situation."

Figure 2. (a) Klaus's work considering greatest numbers in comparison with smaller numbers in a contingency table and (b) computation of conditional probabilities.

Although he computed percentages that he could compare to reason proportionally, he did not recognize their usefulness. However, when he later reasoned with the mosaic plot accompanying the same problem, he clearly used the mosaic plot to compare the two categories and their proportions.

Conclusion

Overall, the mosaic plots appeared to be accessible, appealing, and useful to students. Scott claimed it allowed him to visualize the total as a whole and the percentages better than the table. In addition, all participants agreed mosaic plots were helpful and that having to draw it themselves helped them to understand it. Cici, the youngest participant mentioned it helped her to "memorize it a little more in your head."

Mosaic plots may be a useful representation for students when reasoning about (in)dependence of categorical variables. Future work might consider different aspects of contingency tables, how students understand and work with the constituent components, and how they reason across different representations.

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STUDENTS' "MULTI-SAMPLE DISTRIBUTION" MISCONCEPTION ABOUT SAMPLING DISTRIBUTIONS

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The sampling distribution (SD) is a foundational concept in statistics, and simulations of repeated sampling can be helpful to understanding them. However, it is possible for simulations to be misleading and it is important for research to identify possible pitfalls in order to use simulations most effectively. In this study, we report on a key misconception students had about SDs that we call the "multi-sample distribution." In this misconception, students came to believe that a SD was composed of multiple samples, instead of all possible samples, and that the SD must be constructed by literally taking multiple samples, instead of existing theoretically. We also discuss possible origins of this misconception in connection with simulations, as well as how some students appeared to resolve this misconception.

Keywords: Statistics, Sampling Distribution, Process and Object, Multi-sample Distribution

It is important to help pre-service mathematics teachers develop their own conceptual understanding of statistics content (Conference Board of the Mathematical Sciences, 2001), because their conceptual understanding impacts learning opportunities available for their students (Ball, Lubienski, & Mewborn, 2001). In statistics, one concept of critical importance is the sampling distribution (SD). It forms the conceptual basis of much of elementary statistics, including confidence intervals, hypothesis testing, and correlation testing (Lipson, 2003). Thus, if we want students to develop strong conceptual understandings of elementary statistics, it is essential to help our pre-service math teachers develop strong understandings of SDs, as well.

Much research aimed at conceptual understanding of SDs focuses on exploration activities in which students repeatedly sample from a population (Aguinis & Branstetter, 2007; Chance, delMas, & Garfield, 2004; delMas, Garfield, & Chance, 1999; Glencross, 1988; Mills, 2002; Peck, Gould, Miller, & Zbiek, 2013; Watkins, Bargagliotti, & Franklin, 2014). These simulations are meant to show the emerging properties of the SD that: (a) the shape of the distribution is approximately normal, (b) $\mu_{\bar{x}} = \mu$, and (c) $\sigma_{\bar{x}} = \sigma/\sqrt{n}$. However, some have realized that these same simulations might inadvertently be misleading (Watkins et al., 2014). It is beneficial for teacher educators to know what misunderstandings their students might develop from such simulations in order to use them most effectively. Our study seeks to build on this research by describing a previously undocumented misconception seen in pre-service teachers, which gets at the heart of what a SD even is. We also examine how this misconception might be resolved.

Background on the Sampling Distribution

Brief Recap of Sampling Distributions

Many types of statistical studies are based on using a sample to estimate or test certain population parameters, such as the population's mean (μ) or standard deviation (σ). Sampling distributions (SD) are what underlie the statistical methods used to do this. The basic idea of a SD is that, given a fixed sample size n, if all possible samples of size n are taken from the population, then the statistic of interest from all those samples creates a distribution in and of itself (Triola, 2010). For example, a SD for means is constructed by taking the sample means ($\bar{x}$) of all samples of the same size n from the population and putting them together to create a new distribution (see Figure 1). Note that there is a different SD for every sample size n that is chosen, because n is the same for all samples within a given SD. The Central Limit Theorem (CLT) then guarantees that a SD for means will always have the same mean as the population, $\mu_{\bar{x}} = \mu$, and a standard deviation given by $\sigma_{\bar{x}} = \sigma/\sqrt{n}$. If n is sufficiently large, often cited as $n > 30$ (e.g., Triola, 2010), the CLT states that the SD will be an approximately normal distribution. Similar properties hold for SDs for proportions. It is important to note that SDs are *theoretical* in nature, in that they do not need to be empirically constructed to be used in statistical analysis. The CLT guarantees those properties that are needed for statistical analysis.

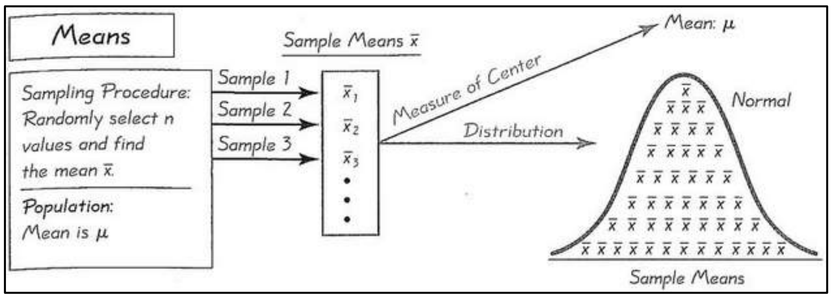

Figure 1: Creation of a Sampling Distribution, taken from Triola (2010, p. 281)

Brief Literature Review on Sampling Distributions

A common tool for teaching SDs and the CLT are simulations, in which physical enactment or computer software is used to create the results of many samples and to partially construct a SD. Using simulations to help students learn about SDs has been recommended as far back as the 1970s (e.g., Committee on the Undergraduate Program in Mathematics, 1972). Some have demonstrated that these simulations can give insight into otherwise theoretically intractable ideas (e.g., Mills, 2002; Simon, 1994). Yet other research has shown that simulations are insufficient by themselves. delMas et al. (1999) found that when students were allowed to experiment with simulations, their understanding did improve by a little bit, but not by as much as expected. They realized that the simulations alone did not force students to notice relevant features, and that activities needed to carefully scaffold student noticing (see also Chance et al., 2004). Lipson (2003) explained that there is a jump between an empirically constructed approximation to a SD using simulations and the actual theoretical SD. However, Lipson's focus was more on the influence that disconnect had on students' understanding of inference. In this paper, we examine how that disconnect directly impacts students' understanding of SDs themselves. Further, because simulated distributions are not perfect representations of the theoretical SD, Watkins et al. (2014) saw that students were sometimes misled by simulations. They observed students who incorrectly believed that the SD's mean got "closer" to the population mean as n increases. In fact, the CLT guarantees that $\mu_{\bar{x}} = \mu$ exactly, regardless of sample size. The misconception we discuss is related to what Watkins et al. observed, and might even be a root cause of it.

Taken together, this literature shows that simulations can be a useful tool in statistics education, so long as they are used carefully. We must be fully aware of potential pitfalls simulations might contain. We should continue to unpack possible issues in understanding SDs with simulations, in order to most effectively use simulations. This study adds a key, previously undocumented misconception, related to what a sampling distribution fundamentally is, that we observed in pre-service teachers who experienced this type of simulation activity.

Theoretical Perspective: Processes versus Objects

We view the concept of SDs as having a close connection to the theoretical notion of *processes* versus *objects* (Sfard, 1991, 1992). This lens gives valuable insight into possible SD misconceptions, and can help produce paths toward their resolution. In short, a process means an activity that can be conceptualized as being carried out, like imagining counting up to 1 million. It does not necessarily need to be enacted to be a process, but imagined. An object, then, is the encapsulation of such a process into a single cognitively entity. The "size" 1 million is a conceptual object, which can emerge out of imagining the process of counting up to 1 million.

SDs inherently deal with the *process* of repeated sampling. That is, one can conceptualize taking a sample and recording its sample mean (or proportion), and then taking another sample and recording its sample mean (or proportion), and so on. However, the full comprehension of SDs is to realize that this process can be encapsulated into a final result: the distribution of *all* sample means (or proportions). The SD is the *object* that results from the *process* of repeated sampling. We view simulations, through this lens, as essentially a representation of the *process* aspect of a SD. It permits the process to be quickly viewed over a large number of samples (as seen in Figure 2). However, in these simulations the *object* aspect of the completed SD is typically not reached. This limitation comes because such simulations usually do not depict when every sample has been represented exactly one time in the simulation, or at least represented in exactly equal proportion to every other possible sample. This matches Lipson's (2003) assertion that there is a jump from the empirically-simulated approximate SD to the actual theoretical SD. In this study, we examine how this issue led pre-service teachers to make incorrect conclusions about the fundamental nature of what a SD is.

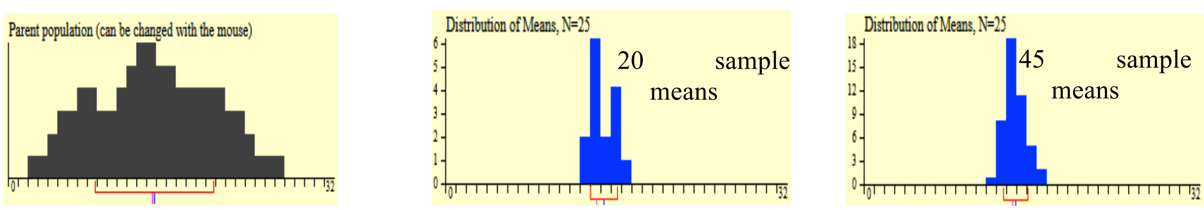

Figure 2: Example of a simulation (http://onlinestatbook.com/stat_sim/sampling_dist/)

Methods

This report emerged from a broader study we were engaged in regarding pre-service mathematics teachers resolving their misconceptions about confidence intervals. Students for the study were recruited from an undergraduate "Teaching Statistics and Probability" course for mathematics education majors, focused on conceptual understanding and on task exploration. The pre-service teachers had all completed a pre-requisite undergraduate statistics course, or AP statistics. In the education class we recruited from, the students used a simulation of repeated sampling, as discussed in the literature review, to develop the ideas of SDs.

The major purpose of the larger study was to understand how students might resolve misconceptions they held and to document the route they took in doing so. To recruit students for interviews, the students were given a quiz in their class regarding misconceptions on SDs and confidence intervals. Because the misconception we report on in this paper had not previously been documented, our specific misconception was not tested for, but emerged during the interviews while the students discussed other aspects of their understanding. From the quiz results, five students with varying levels of misconceptions about confidence intervals were selected to participate in two, hour-long interviews. We give the students the pseudonyms Danielle, Ethan, Corinne, Tiana, and Anna. During the interviews, the students were asked to explain how confidence intervals are constructed, to design their own hypothetical study that would use confidence intervals, and to discuss various aspects of SDs and confidence intervals. While conducting the interviews, the interviewer (Author 1) noticed a trend in terms of how all five students seemed to be talking about SDs. Thus, the interviewer began to follow up on this trend as well, and to make sure each student was asked about it. As the purpose of the larger study was to help students resolve misconceptions, the interviewer also attempted, impromptu, to document instances of students resolving this misconception during the interviews.

To analyze this trend, we went through all of the parts of the interviews where students made statements or gave explanations regarding SDs. In examining all instances of normatively incorrect statement or explanation, we realized they typically dealt with one main misconception about SDs. That is, most incorrect statements or explanations about SDs seemed rooted in the same misunderstanding. Once we singled out this misconception, we went back to the interviews to try to identify where the misconception came from in terms of prior knowledge or in-class activity. In doing so, we saw an important likely connection to the in-class simulation. Finally, as the interviewer had attempted, in the moment, to understand and document these confusions, we tracked the students' evolving conceptions of SDs over the interviews and looked for what was discussed in conjunction with changes in their understanding. This aided us in identifying what might help resolve the underlying misconception.

Limitations

There are some limitations to our analysis of this misconception. First, we had only five students in the sample, which is few. However, this report focuses only on documenting and discussing the misconception, rather than on establishing how common it is. Yet, since all five students in this study did share this same misconception, we posit that it is likely to be more widely shared. Second, this study was not originally designed to uncover this misconception, but it rather emerged from the data. Future work can be done to examine this misconception more systematically and to identify how common it might be among typical pre-service teachers.

Results

The Multi-sample Distribution Misconception

All five students suggested that to do statistics, one essentially uses a "sampling distribution" that contains *some* of the samples of size *n* from the population, as opposed to a completed sampling distribution of *all* possible samples. For example, consider Corinne's explanation.

> Corinne: If you took a bunch of samples and you found their means, you would get a sampling distribution.
> Interviewer: How many samples?
> Corinne: More than 30.

Corinne implied that if one has a certain amount of samples (i.e. "30"), then the distribution of those sample means *is* a SD. In another example, Ethan was describing how his hypothetical statistical study could be done. He seemed to imply that to obtain a SD, one would literally collect several samples in practice and compile them into a SD. The interviewer was initially unsure what he meant, and asked about how he imagined a SD.

> Interviewer: What's a sampling distribution?
> Ethan: The distribution of the means of your samples.
> Interviewer: How many?
> Ethan: Are you talking about in my thing [i.e. hypothetical study], or just in general? … [The SD is] the means of how many samples you take.
> Interviewer: But what if you only take one [sample]?
> Ethan: If you only take one sample then, [pause] I'm lost.

…
Interviewer: [The SD] is the distribution of the means of all your samples. Okay. All the samples that you take, or all the samples that you could take?
Ethan: That you take.

Here, Ethan explained that a SD is created by *empirically* collecting multiple samples during a statistical study. That is, he did not conceptualize a *theoretical* SD with the properties guaranteed by the CLT. In fact, at one point Ethan suggested that "all possible samples" really meant all the samples it was possible for someone to practically take. Ethan was not alone in believing one must collect multiple samples to empirically create a SD to do statistics. Most of these students described that to carry out their hypothetical study, they would need to take many samples to create a SD, as seen in the following statements.

Tiana: I want to take 30 samples of size 100.
Corinne: I would take 100 samples of size 30.
Ethan: I would get at least 200 samples just to be realistic.
Danielle: You take a large number of samples, like say 1,000, to get a sampling distribution of $\bar{x}$.

It is clear that these students were all thinking of a SD as a collection, not of *all* possible samples, but of several literally collected samples. We call the conceptualization of such a distribution of many, but not all, samples the *multi-sample distribution*, denoted *M-SD*. We consider it a misconception when the M-SD is seen as being *the* SD. We claim that the M-SD misconception is closely connected to perceiving only the *process* part of the SD. That is, the process of repeated sampling is understood, but it does not have a theoretically completed end of all possible samples that is the *object* SD. In this way, M-SD is not "wrong," but incomplete in a critical way. The students even seemed to understand that this process could continue, with more samples, to create a "better" M-SD, but they typically did not understand that the process has an end-result *object* that is the theoretical SD.

In conjunction with the M-SD misconception, the students in our study exhibited some misconceptions previously reported on in the research literature. For example, many claimed that $\mu_{\bar{x}} \approx \mu$ rather than $\mu_{\bar{x}} = \mu$ (cf. Watkins et al., 2014), as in the following excerpt from Corinne.

Corinne: [$\mu_{\bar{x}}$] is the mean of the means you sampled… In the real world, we never get to work with the distribution where μ and $\mu_{\bar{x}}$ are equal. We just get closer and closer [with more samples].

In fact, we believe previously reported misconceptions like this regarding $\mu_{\bar{x}}$ may really be a symptom of an underlying M-SD misconception. Note that the mean of the M-SD is technically $\bar{x}_{\bar{x}}$, as opposed to $\mu_{\bar{x}}$ because it is only a *sample* of sample means rather than the *population* of all sample means (where $\bar{x}$ refers to a sample and μ to a population). In this perspective, it is true that $\bar{x}_{\bar{x}} \approx \mu$ as the students claim. It is only in the SD of *all* samples where $\mu_{\bar{x}} = \mu$ exactly.

Possible Origins of the M-SD Misconception

During the interviews, the students described aspects of their thinking that matched with the simulation activity used in their class to discuss SDs. In the classroom activity, each student randomly selected samples from a population and computed the sample mean of them. The activity culminated in the students plotting their sample means together to create a visual representation of what is, essentially, a M-SD. This activity may have fostered M-SD thinking in the students. For example, in her interview, Tiana recounted this simulation activity and explained that she understood the image she saw – of the multiple sample means plotted together – as being *the* SD. Regardless of whether the instructor may have mentioned that the image was not the SD, the strong visual that represented the culmination of the simulation activity seemed powerful enough that she interpreted it as though she were seeing a SD.

By contrast, Corinne did recognize in her interview that there is such a thing as a distribution of *all* sample means. However, she discarded it as anything practically useful in doing statistics, explaining instead, as seen in her excerpt above, that "in the real world, we never get to work" with the actual SD. She explained that you could only use the actual SD "in something like manufacturing where you have data on every item or when you have a small population. But in that case it would be pointless because you could just do a census and know the population parameters." She explained that, practically speaking, in order to do statistics one would need to create the type of distribution seen in their class, that was made up of a collection of multiple sample means rather than all sample means.

Another root of the M-SD misconception may lie in classroom discussion of the CLT. One property of SDs given by the CLT is that if the common sample size for all samples is sufficiently large, often given as $n > 30$, then the SD is approximately normal. However, students may have confused this with believing that they need at least 30 samples for the (M)-SD to be approximately normal. The simulation activity may have inadvertently led them to focus on the wrong thing for "$n > 30$." In the simulation, the students saw that with each new sample mean added, the distribution began to resemble a normal distribution more. For example, when Ethan was explaining his hypothetical statistical study, he settled on wanting to collect 30 samples.

> Interviewer: Why is 30 a magical number to you?
> Ethan: The central limit theorem wants 30 [samples] for the sampling distribution to be normally distributed.

Notice that Ethan is justified in asserting that "30" is connected to the normality property given by the CLT. But, he did not appear to connect $n > 30$ as representing the *sample size* of each of those samples, as opposed to the *number of samples* needed to create a reasonably normal (M)-SD. This result is supported in the students excerpts from the previous section about wanting 30, 100 or 1,000 samples to make a sampling distribution.

Possible Resolutions of the M-SD Misconception

We defined the resolution of this misconception as recognizing that (a) the SD is a theoretical distribution from *all* possible samples and (b) that it does not need to be empirically constructed to be used. From our process-object perspective, the M-SD misconception essentially lacks the object component. Thus, resolution of this misconception is based on extending their process-oriented conception to include an object. Danielle, Ethan, and Corinne each gave some evidence of resolving this misconception. First, consider Danielle. One important part of her resolution of this misconception involved clearly distinguishing between sample size and number of samples.

> Interviewer: How many samples do we need to take before we can use the sampling distributions and assume that they are normal?
> Danielle: I think generally they say it's supposed to be like 35 or 30.
> Interviewer: Samples?
> Danielle: Yeah. That's the size of the sample [pause]. So wait, your question is?
> Interviewer: How many samples?
> Danielle: Oh, how many samples do we need to take. So, usually when we are using these types of things like our equations [refers to a formula sheet] we just take one sample!

Here, Danielle seemed to have realized the mismatch in thinking that multiple samples are needed to literally create an SD versus the fact that the statistical formulas use only one sample. Then, by thinking of just this single sample, she began to create for herself the ideas of a SD.

> Danielle: Any kind of sample you take is going to fall… somewhere. It is possible to get one that is farther away from the population mean… [Draws Figure 3]. If you were to take a sample, just one

sample, then it will fall somewhere along here in this range that is close to the population mean [gestures toward the middle a population distribution in Figure 3].

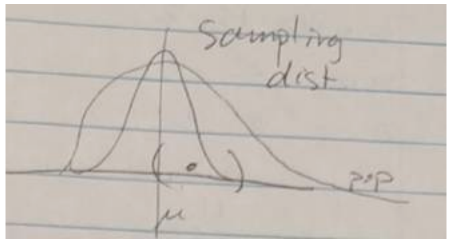

Figure 3: Danielle building on the simulation to now imagine all possible sample means

By reasoning with only a single sample, Danielle began to think more theoretically about where that single sample mean could be. In fact, this theoretical thinking seemed to help her imagine all possible samples, without having to literally collect all of them.

Danielle: So, if you could possibly take every single sample of that certain size and you were to be able to plot that, the sampling distribution would be normal and so we, since that concept is true, then we can just pull one sample point and it will be a point from somewhere on the sampling distribution.

We can see that Danielle had now conceptualized a SD as having all possible samples, and that it was theoretically, not empirically constructed. The single sample used in statistics was a member of this theoretical distribution. In fact, thinking of a single $\bar{x}$ more abstractly appeared helpful for some students in transitioning from the empirically-grounded M-SD to the theoretically-based SD. Corinne also used single $\bar{x}$'s to help make this transition.

Interviewer: What happens if someone only picks one sample? Let's make this the smallest possible $\bar{x}$ and this the largest possible $\bar{x}$. [Here the interviewer writes a number line and marks two points along it.]

Corinne: Without even knowing anything about this, most of them are going to be in the middle. So chances are that this one single [$\bar{x}$], it's here somewhere [gestures to the middle of the number line.]

Here, Corinne made an assertion about where a given $\bar{x}$ might be, without trying to create multiple samples. The interviewer tapped into this by then asking Corinne to imagine where all possible $\bar{x}$'s might fall along this number line. Corinne began to piece together where they might be, including that many $\bar{x}$'s would fall toward the middle. She eventually drew a SD similar to Figure 3. The interviewer asked about some of the specific properties of this new distribution.

Interviewer: Is μ the same as $\mu_{\bar{x}}$? [i.e. assuming all possible samples]
Corinne: I think at this point they are the same.
Interviewer: Why?
Corinne: Because at this point, if we have taken every possible sample, and take their means, and we are finding the mean of all those means, that is mathematically the same as finding the mean of all of those at once, which is finding μ.
…
Interviewer: So you are saying we can use just one sample [to do statistics]?
Corinne: But you're basing it off of information about all possible samples.

By leaving the empirical enactment from the simulation, Danielle and Corinne could begin to reason theoretically about where one $\bar{x}$ might lie, and then to where multiple $\bar{x}$'s might lie, to then where all $\bar{x}$'s might lie. This seemed to help them extend the *process* seen in the simulation to an imagining of a completed SD *object* with all possible $\bar{x}$'s being represented.

Discussion

We agree with the body of research that claims simulations are important for developing students' understanding of SDs (e.g., Mills, 2002; Simon, 1994). We also believe our study helps us better understand why simulation activities might be misleading in some ways, as noted by Watkins et al. (2014). Our process-object perspective suggests that simulations can only account for the process part of the conception of SDs, and cannot adequately portray the object part. We believe this is the reason for the possible disconnect Lipson (2003) described between the empirical simulation and the theoretical SD. If a simulation can only achieve the process component, the M-SD becomes a possible misconception students might develop. To be clear, we see the process component of a conception of SDs as essential, and simulations as a valuable way to develop the process component. That is, if one tried to simply create the *object* SD without first developing the process behind it, one might be left with a pseudostructural conception instead (Sfard, 1992; Sfard & Linchevski, 1994). In other words, the students might conceive of an SD object, but without understanding the underlying process that leads to it. Thus, we promote simulations as a useful way to develop the process, but claim that instruction must, at some point, move past the empirical simulation into a theoretical SD. Of course, simply "telling" students that there is a completed theoretical SD after observing a simulation might be insufficient to bridge the gap between process and object. Rather, it seemed important for some of our students who resolved the misconception to reason more theoretically about the distribution of a single hypothetical $\bar{x}$. This led to where multiple $\bar{x}$'s, and eventually all $\bar{x}$'s, might be distributed.

It was also important for our students to explicitly confront the difference between the sample size and the number of samples, which we believe is related to the misconception that $\mu_{\bar{x}}$ gets closer to μ as sample size increases (Watkins et al., 2014). As sample size increases, it is true that a M-SD will have an $\bar{x}_{\bar{x}}$ closer to μ. However, since a SD deals with all samples, *not* an increasing number of samples, it will always have $\mu_{\bar{x}} = \mu$.

Finally, we wish to emphasize that the M-SD misconception is not "wrong," but simply incomplete. In the spirit of perceiving misconceptions as useful building blocks, rather than faulty thinking that must be removed and replaced (see Smith, diSessa, & Roschelle, 1993/94), we find that resolving this misconception deals with adding on to what is already there, rather than taking away. Viewing the M-SD misconception in this light makes the path toward its resolution clearer, in that we can take the students' understanding as is and help them extend it to a completed *process-object* conception of SDs.

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CONDITIONAL PROBABILITY IN EARLY CHILDHOOD: A CASE STUDY

PROBABILIDAD CONDICIONAL EN LA INFANCIA INICIAL: UN ESTUDIO DE CASO

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This study investigated the intuitive knowledge of conditional probability in one four-year-old child. Six clinical interviews were video-recorded for analyzing transcripts and expressions from the child. Findings from this research suggest that the child has a pre-operational intuition about change in sample space in non-replacement situations. He also seems to have an intuitive understanding of independent events. Furthermore, although his judgments are mainly subjective, he does use quantitative justifications, although, inconsistently.

Keywords: Cognition, Early Childhood Education, Probability.

According to Nikiforidou, Pange and Chadjipadelis (2013) and Antonopoulos and Zacharos (2013), the study of probabilistic thinking in preschoolers is very limited. Jones, Langrall, Thorton and Mogill (1997) list sample space, probability of an event, probability comparisons and conditional probability as four key constructs in probability thinking; however, while the first three constructs have been investigated by several researchers, they point out that there are few studies related to conditional probability in young children. As one of the key constructs in probability thinking, it would be worthwhile to identify how this construct first begins to develop in young children. In this way, we can better understand how to advance children's early probabilistic reasoning. Thus, we examined the following research question: How does a preschooler think about conditional probability? In this study we investigated the intuitive knowledge of one four-year old child in situations involving conditional probability and independence. The study focused on these concepts from an informal perspective.

Literature Review/Framework

Xu and Tenenbaum (2005) demonstrated that preschoolers might possess some intuitive notions of conditional probability despite having not reached the formal operational stage identified by Piaget and Inhelder (1975). Jones and Tarr (1997) validated a framework for assessing probabilistic thinking through a characterization of four levels in conditional probability and independence. Following are the first two levels briefly described:

- Level 1. Subjective: Students' judgments are based on their own construction of reality, looking for non-existent patterns and imposing their own system of regularity. Students believe past outcomes will always affect the future outcomes, and they "deny the existence of independence" (Jones & Tarr, 1997, p. 51). The level of subjectivity does not allow any meaningful attention to independence and conditional probability.
- Level 2. Transitional: Students in this level occasionally use quantitative information for making conditional probability judgments.

Some researchers (Nikiforidou & Pange, 2010, Nikiforidou, et al. 2013) highlight the importance of considering intuition when investigating children's probabilistic thinking. In this way, Fischbein (1975) argued that children have intuitive probabilistic knowledge. These intuitive notions are implicit forms of cognition beyond cognitive structures and conceptual systems. Fischbein (1975) categorized intuitions into pre-operational, operational and post-operational. Specifically, pre-operational intuitions can be influenced by subjective or perceptual considerations, and operational intuitions are basic intuitions that are expressed through formal rules of logic.

Methods

In this study we used a case study analysis to investigate notions of conditional probability in one four-year-old child, Mack. Six clinical interviews of 5 minutes each were conducted and video-recorded. Tasks designed as games were used in each interview. In these games, Mack was asked to predict the outcomes in different scenarios. Correct predictions were reinforced with tokens. We analyzed each of the activities focusing on Mack´s predictions and his reasons for providing his answers. After transcribing each interview, in the video records we identified Mack´s expressions, actions, and fluency in his answers. From these records, we selected the video-segments that met one of the following criteria: showed any justification given by Mack, revealed specific indicators of a particular kind of intuition, or showed specific features associated with his notions of conditional probability. For the chosen segments, we included descriptions of Mack´s actions associated with his verbal expressions in the transcriptions. Later, we organized the completed transcriptions into two non-disjunctive groups: those that enabled us to infer Mack's intuitions, and those that reveal different kinds of judgments.

Results

We describe the results from the interviews according to Mack's intuitions and judgements. We begin with a discussion of the BALLS situation. In this situation, we mixed two green balls and two red balls in a bag. We asked Mack to predict the color of the ball that he thinks will have the greatest chance of being chosen from the bag. He answered "red," giving the reason that it is his favorite color. He then chose a red ball without looking inside. After that, he recognized that there were two green balls and one red ball remaining in the bag. Here, we suggest a pre-operational intuition associated with the chance of selecting a green ball, because after reviewing the bag, he quickly predicted the selection of a green ball for a follow-up selection. He gave as his reason for this prediction, that he likes all the colors (subjective consideration).

His pre-operational intuition of chance is further confirmed in the next trial in which there was one ball of each color left. In this case, he predicted that the red ball would be chosen, arguing that red is his favorite color. Similarly, in a scenario involving two red and two green balls in the bag, he selected a red ball which was then replaced; when he was asked which ball had the greatest chance of being selected, he answered red again arguing that it was his favorite color. Then, the red ball was not replaced, and he predicted that a green ball would be more likely to be chosen, again arguing that he likes all the colors. In this case, the number and color of the balls were not confirmed as in the previous situation. In a subsequent scenario, we added one green ball to the two red balls and two green balls. In this case, Mack predicted the green ball again arguing that he liked all the colors. In summary, when the chance of choosing the different colored balls was equal, he predicted a red ball arguing that red is his favorite color; but, when the chance of choosing a green ball was greater, he selected that color, arguing that he likes all of the colors. These situations reveal that Mack has an intuitive recognition that the sample space changes. For example, after selecting some of the balls from the bag, he changed his decisions according to the remaining number of colored balls without checking which balls were left in the bag. Furthermore, these decisions reflect an intuitive notion of conditional probability as the probability of the subsequent choice of ball appears to be conditioned on the knowledge of the previous choice.

We now turn to a discussion of Mack's judgements in situations involving independent events. In the COINS tasks, Mack was asked to predict the resulting color (yellow or red) after flipping a coin.

He was asked to first toss the coin without making a prediction. The outcome was yellow. On the next toss, he predicted the coin to be red. In the third toss, he looked more confident in his guessing than previously. Then, the following discussion took place:

Interviewer (I): Now, which color do you think that you will get if you throw the coin?
Mack (M): Mmm, red
I: Why do you think is red?
M: Because it's my favorite color.
I: Ok, it's your favorite color, very nice, let's see
M: [tossed the coin and got yellow]
I: Yellow! Now, let's do it again. What color do you think you will get?
M: Red
I: Red again? Ok, why red?
M: Dark

Mack's choices in the coin tossing situation seems to reveal an intuitive recognition that previous outcomes do not affect subsequent outcomes. In a follow-up scenario, we used two coins that were flipped one immediately after the other. Mack was asked to predict the color he would get tossing the second coin once he had tossed the first coin; he answered yellow, and the reason that he said was "shine!" In different scenarios, the main justification of his predictions continued being "my favorite color!" when his predictions were red. However, he seemed to change his justifications when he had a clearer intuition about the possible chance of the outcomes, as in the BALLS situation. He also included subjective arguments like "because is dark," "I am sweety," and "because it's my mom's favorite color."

In the FIGURES situation (Figure 1), Mack was asked to predict from which piece of cardboard the figure was chosen (situation inspired by Lucas, Bridgers, Griffiths, and Gopnik, 2014). Covering the pieces of cardboard, we selected a diamond and asked Mack from which piece of cardboard the image was taken. He linked the selection of the color (red cardboard) with the red flowers in the interviewer's blouse. Repeating the experiment twice, he changed the kind of judgment in the second repetition:

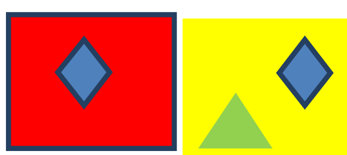

Figure 1. Pieces of colored cardboard

I: I take the blue diamond.
M: The yellow one.
I: Do you think it's from the yellow one?
M: (nodded)
I: Why?
M: Because there is more.
I: There is more what? There is more…
M: Shapes.

Although the quantity of shapes did not affect the chance, Mack´s attention moved away from subjective arguments to the quantity of shapes. In a similar situation in which there were two triangles on the yellow piece of cardboard and one on the red piece, Mack's first guess was that the triangle came from the yellow piece of cardboard, because "there is two of them." Although this was not technically a prediction because Mack was looking when we selected the triangle, he seemed to be appealing to the number of triangles as a justification for his choice. In, yet, another situation where two triangles were on the yellow piece of cardboard and three on the red piece, his judgements associated with the number of triangles became clearer. We took one triangle without letting him see where it came from:

I: I took this triangle from one of these two cardboards, what cardboard do you think this triangle came from?
M: Red.
I: Why do you think is red.
M: Because there is more.
I: There is more in red?
M: More triangles.

However, in a following task, the triangle was not replaced, and he expressed the same argument in selecting the red piece of cardboard when they were the same number of triangles in each one.

Conclusions

This study identified the intuitive knowledge of a four-year-old child in situations involving conditional probability and independence. Although the child does not have any formal knowledge about sample space, he has a pre-operational intuition about changes in the sample space in non-replacement situations. In other words, Mack quickly adapted his choices and justifications depending on the different options he had once the sample space changed. Additionally, as seen in the COINS task, Mack did not seem to consider past outcomes when making future predictions. These findings seem counter to Jones and Tarr's (1997) framework related to the lack of attention to independence and conditional probability for children who are in the subjective level.

Mack's judgments were mainly subjective as Jones and Tarr (1997) characterize for the subjective level and Fischbein (1975) characterizes for pre-operational intuitions. However, over the course of the interviews, Mack changed his judgments from subjective to quantitative without instructional interventions. Although his quantitative reasoning was inconsistent and sometimes irrelevant, notions of quantity did influence his decisions. The trajectory of Mack's judgments provides an intriguing situation for further research. In particular, when he felt more confident about the possible answer, he appealed to the quantity of objects to justify his predictions. It seems that Mack has an intuition about the likelihood of an object being chosen that is associated with the quantity of objects involved. This suggests that intuitions about chance in conditional situations associated with quantitative reasoning may be stronger than Jones and Tarr (1997) argue for the subjective level. Nevertheless, although Mack mentioned number of objects in his justifications, he did not associate such numbers in his predictions.

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THE IMPACT OF SELF-COLLECTED DATA ON STUDENTS' STATISTICAL ANALYSIS

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Context plays a crucial role when students engage in analysis and interpretation of data, and is broadly recognized as one of the main elements that separate the studies of mathematics and statistics (Cobb & Moore, 1997). The ways that students engage with the environment surrounding data within a statistical task are of interest to researchers because of this. In this analysis of a teaching experiment with third grade students, interactions with self-collected wingspan data led to insights about graphs as students attended to individual values, but students varied in their willingness to use the data to describe/infer more broadly.

Keywords: Data analysis and Statistics, Measurement, Elementary School Education

When working with data, the context surrounding it plays an essential role in developing students' understanding. Context is one feature that can separate the study of statistics from the study of mathematics (Cobb & Moore, 1997). Watson (2009) posited that one of the goals of statistical literacy is for "students to be able to tell a story from a context with a distribution" (p. 33). With the potential for students' statistical investigations to involve the collection and use of survey or measurement data (Lovett & Lee, 2016; Makar, 2014), it is of particular importance to carefully unpack the impact of self-collected data (which may include the students themselves) on their statistical analysis. In this work, we investigated the ways that students in a third grade classroom engaged with context as they completed an activity involving wingspan data. The guiding research question included: how would the use of self-collected measurement data impact students' descriptions and their ability to infer based on their data collection and display?

Conceptual Framework

According to Watson (2007), a learner's ability to engage productively with context is one of the hallmarks of statistical literacy. When "meaningless or nonexistent" context is provided for data, students will often apply incorrect or inappropriate procedures, while when the context is "interesting, and relevant to students' worlds" they are more likely to use statistical ideas (Doerr & English, 2003, p. 111). However, when students engage with the context too personally, they may also make assumptions that are not supported by statistical evidence (Watson, 2007). Students may focus so closely on the story behind data that they prefer making conclusions based on their informal knowledge of the data's context over patterns in the data itself (Ben-Zvi et al., 2012; Biehler et al., 2018; Pfannkuch, 2011).

Within statistics, there are two broadly defined categories; descriptive and inferential. Descriptive statistics focuses on the "organization, summarization, and presentation" of data (Paparistodemou & Meletiou-Mavrotheris, 2008, p. 83). Inferential statistics looks at how patterns for a particular group could be viewed more broadly. Recently, research has focused on the importance of exposure to tasks which build inferential reasoning for students as young as the elementary grades (Makar, 2014; Pfannkuch, 2011; Watson, 2001). Graphical displays are an important part of both descriptive and inferential statistics. They can help students see the distribution of data more clearly, with distribution being defined as the overall picture of data based on expectation (center), variation, and shape (Watson, 2009). Watson (2007) concludes based on her longitudinal work with students in both the elementary and secondary grades that learners can benefit from opportunities to create their own displays and compare/contrast them with displays made by others to investigate their "success in telling the story of data" (p. 61). Despite this much remains unknown about the intricacies and impact of such an approach. This work looks specifically at the context of self-collected measurement data and how the nature of the data may have impacted elementary students' display creation and descriptions/inferences.

Methods

This research aimed to investigate the ways that a group of third grade students from a Midwestern State engaged in a task related to class wingspans. Using characteristics of a teaching experiment methodology (Steffe & Thompson, 2000), a focus group of five students were recruited to participate in the study. The wingspan task unfolded in three stages. First, the focus group were asked to measure each other's wingspans, and to consider reasons for observed differences. Second, the focus group collected measurements from the rest of their class and used these measurements to create displays individually or in pairs. Students had previously been introduced to bar graphs and pictographs involving categorical data but had not used them to represent quantitative data. Finally, participants presented their displays to the focus group. The purpose of this discussion was to consider the ways that different displays conveyed similar information, examine the overall distribution of class wingspans, and hypothesize about the distribution of wingspans for other classes or grade level groups. Video recordings of discussions, transcriptions of the activity, and student work served as data sources for analysis.

Data analysis consisted of three phases. First, video-data were segmented to isolate instances during which students seemingly contemplated the self-collected or personal nature of the data. These were broadly defined to consist of occasions where they referenced specific data values or general characteristics of specific groups (i.e. their class, other classes, other grades). The second phase of analysis focused on identifying ways that the context shaped students' interactions with the data towards description or inference. The third phase concentrated on analyzing the type of descriptions/inferences identified during the second phase of analysis.

Findings

The context of the physical measurement of wingspans impacted the ways that students engaged with the data in a descriptive sense. It led to a focus on individual students' measurements which both helped and hindered students' perceptions as they attempted to describe the data. Including self-collected data based on their own classroom seemingly led to an appreciation for how individual values fit within the data, but at times this focus on individuals appeared to inhibit students from looking at their displays more holistically.

Personalization of data

Locating self-based measurements amongst the data set was critical to enhancing students' understanding of the characteristics of displays produced. This approach to data reading seemed to have motivated them to both create and understand their graphs (Lovett & Lee, 2016). Finding one's own measurements in the display anchored their interpretations and descriptions of the overall data as depicted in this vignette (all names are pseudonyms):

> Julie: Everybody's was in the 50s but mine – but that kind of makes sense because I'm shorter than everybody.
> Researcher: Oh, so you think that maybe it's connected to your height?
> Katelyn: Yeah, because people say that if you're really tall then you have a big wingspan
> Julie: Yeah like Katelyn and Bob are the tallest people and they have the longest – um
> Bob: Wingspan

In both this and other instances, students referenced the data values of both themselves and others in the class. One wingspan, which was lower than all the others, is given an interpretation when Julie recognizes that the value represents her and makes a potential connection with height. This is pursued further by others who recognized that the higher wingspans also represent those in the group who are taller. In this way students were able to give some justification for the variation in the data and make a potential connection with height. Their prior experiences may have impacted this interpretation, as evidenced by Katelyn's statement about what "people say" about height and wingspan.

Interpreting based on individual measurements or personal experiences led to varying approaches to inference. When asked what the graph for another, unspecified third grade class might look like, students first believed that the graphs might be similar, as when Bob and Julie referenced their displays (Figure 1) and said, "It would look a lot like this/ours", or when Jeff said, "Most of them would be in the 50s". This provides evidence that some students viewed their data as representative of third graders in general. Others believed the graphs would look different, but for varying reasons. Abbie based her analysis on another specific class, saying, "If I were to measure Ms. Johnson's class, she has some really tall kids," leading her to conclude that they would have more values on the right side of the graph. Katelyn was unwilling to describe what a graph for another class would look like because they have "different people – one class could have all the tall people". In this case, envisioning specific classes led to inferences regarding a single case (Abbie) or an unwillingness to make any conclusion (Katelyn), characteristics that have been seen previously in young students with regards to informal inference (Ben-Zvi et al., 2012; Makar, 2014). Abbie's and Katelyn's observations also provide further evidence that students inferred a more general relationship between height and wingspan; for them, referencing height appears to be sufficient to justify potential variation in wingspan.

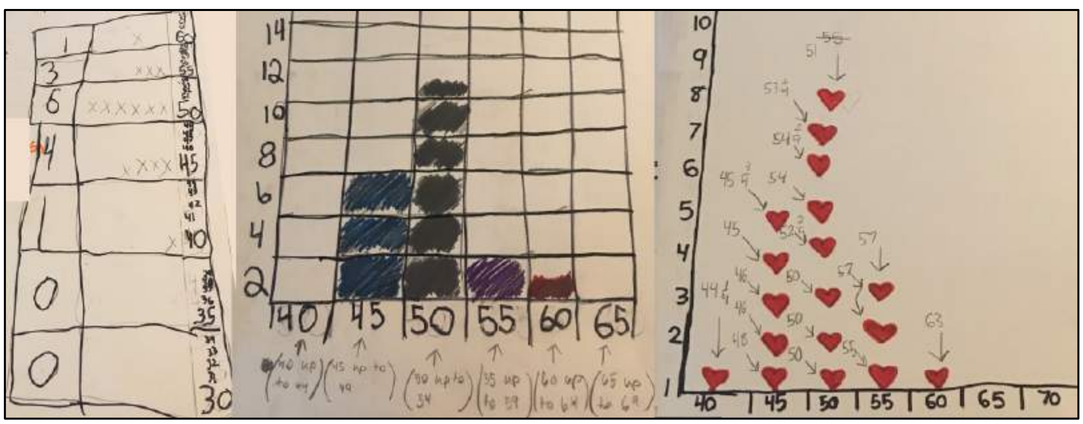

Figure 1: Jeff/Bob's display (left), Julie's display (middle), Katelyn's display (right)

Differences in Displays

The self-collected data were helpful for engaging students in interpreting their wingspans at an individual level, but the data collection and display creation process also led to some issues which kept students from seeking structural characteristics of the data. Students were intent on critiquing the format of others' displays (things like labels and titles), and characterized them as either correct or incorrect. This tendency can plausibly relate to the procedural and standalone nature of instruction on graph creation that students often experience (Friel et al., 2001). This led to a focus on minor discrepancies in students' data lists. Some values were left out accidentally by one or more of the participants, while other values differed slightly based on measurement. While this led to a discussion of measurement variability, students were also intent on tracking and noting these differences. Questions like, "Did you get John?" or "Which number is Maggie?" were common. When asked to compare their graphs, two students said, "They're all different". The following captures the difficulty students experienced in viewing and interpreting their displays holistically:

> Researcher: Look at these side by side and look at the shape. What do you notice?
> Abbie: Oh, it's the same
> Jeff: Oh they're the same
> Bob: It's the same thing
> Jeff: It's the same thing except these are pictures and those are like bars (pointing)
> Abbie: Wait wait wait!
> Researcher: What do you mean it's the same thing?
> Abbie: It's not the same thing – it's not the same thing
> Katelyn: It's not the same thing because Julie is missing a 40

Note that while students' graphs had similar structures (Figure 1), Julie's graph does not include a value in the 40-44 inch category, whilst the other two graphs do. Thus, an exchange that initially seemed to be moving students towards a discussion of the more general characteristics of their displays transitioned to a focus on an individual value. Discrepancies related to the self-collected nature of the data seemed to inhibit their ability to summarize the wingspan characteristics in this instance. Despite this, students did note some of the general characteristics of the data based on their displays. A consensus was reached that the "average" for the class was "50-55 inches" because it was "the highest" value amongst the data set. Thus, students' conception of "average" in this case appeared more closely aligned with ideas related to the formal ideas of "median" and "mode", a phenomenon previously reported by researchers who had considered students of the same age group (Makar, 2014; Mokros & Russell, 1995). Katelyn also noted during the discussion how the data "all go up and down", a reference to the unimodal shape of the distribution. Students determined that potential values could be anywhere between 40 and 65 inches, and believed this to apply to third graders in general. While some students envisioned that the graph for another class might look different in terms of "average", students also believed that the wingspans of another class would fall within this range, demonstrating awareness of reasonable variation between samples.

Discussion/Limitations

This research investigated how the use of self-collected measurement data would impact students' statistical analysis. The results suggest that positioning themselves within the data at times helps students distinguish relationships between values. However, the personal nature of the data, along with slight differences in representation due to measurement/sampling variability, at times kept students' attention on the differences between their graphs and the individuals represented by each element as opposed to aggregate characteristics of the distribution. However, the exploratory/preliminary nature of this study limits the scope of its conclusions. A future study comparing students engaged in tasks utilizing self-collected versus teacher-provided (not self-collected) data would be essential to gain further evidence of whether the characteristics observed in this study are truly related to the use of self-collected data.

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STATISTICS AND PROBABILITY:

POSTER PRESENTATIONS

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DECISION-MAKING PROBLEM FOR INTERPRETING ALGEBRAIC INEQUALITIES

PROBLEMAS DE TOMA DE DECISIÓN PARA INTERPRETAR LAS INECUACIONES ALGEBRAICAS

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Keywords: Inequalities, Constraints, Solutions, Number Line

The research about inequalities has reported that students treat them as equations; this parallelism also has an effect on the syntax that has a repercussion on the semantics of the literals used in algebra. The student often makes algebraic transformations without regard to the constraints of inequality. There are also conflicts with the type of solution, finally, variability can represent a problem from the treatment with the intervals, since it is related to the idea of multiple solutions.

We consider that the study of numerical inequalities and the variability of intervals are basic resources for the adequate use of linear inequalities. In the first, inequalities reflect the relationship that gives meaning to the use of signs of inequality by comparing quantities, while in the second they help to establish a relationship of dependency associated with the set of solutions.

We propose to develop this research as part of a reflective practice within a collective work that allows the student to "meet" with knowledge in a process of semiotic mediation and ethical collaboration, Radford (2020).

This ongoing research aims to develop the transit of the interpretation of the unequal sign from the numerical field, through the treatment of intervals and variability through the use of tables, to support the solution of linear inequalities associated with problem constraints for decision making, carrying out a treatment of variables as a generalized number.

To justify these relationships we will first use the model of the real line and the symmetry of the position of values, in particular, to make sense of the product by negative numbers, and then focus on the algebraic treatment of the inequalities and then make sense of the solutions over the previously used model. We would be developing this research with populations of high school students in Honduras and Mexico.

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PROBLEMAS DE TOMA DE DECISIÓN PARA INTERPRETAR LAS INECUACIONES ALGEBRAICAS

DECISION-MAKING PROBLEM FOR INTERPRETING ALGEBRAIC INEQUALITIES

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Palabras clave: Inecuaciones, restricciones, soluciones, recta numérica

Las investigaciones desarrolladas sobre las desigualdades han reportado que los estudiantes las tratan como las ecuaciones, este paralelismo también tiene un efecto en la sintaxis que repercute en la semántica de las literales utilizadas en el álgebra. El estudiante con frecuencia hace transformaciones sin tener en cuenta las restricciones de la desigualdad dominando la regla del despeje. También hay conflictos con el tipo de solución, por último, la variabilidad puede representar un problema a partir de un tratamiento con los intervalos, ya que está relacionada con la idea de múltiples soluciones.

Consideramos que el estudio de las desigualdades numéricas y la variabilidad de los intervalos son recursos básicos para el uso adecuado de las inecuaciones lineales. En el primero las desigualdades reflejan la relación que da sentido al uso de los signos de desigualdad comparando cantidades, en el segundo este contribuye a establecer una relación de dependencia asociada al conjunto solución.

Planteamos desarrollar esta investigación como parte de una práctica reflexiva al interior de una labor conjunta que permita que el estudiante se "encuentre" con el saber en un proceso de mediación semiótica y colaboración ética, Radford (2020)

Esta investigación en curso pretende desarrollar el tránsito de la interpretación del signo desigual desde el ámbito numérico, pasando por el tratamiento de intervalos y la variabilidad a través del uso de tablas, para sustentar la solución de las inecuaciones lineales asociadas a restricciones de problemas para la toma de decisiones, llevando a cabo un tratamiento de las variables como un número generalizado.

Para justificar estas relaciones usaremos inicialmente el modelo de la recta real y la simetría de la posición de valores, en particular, para dar sentido al producto por números negativos, para luego centrarse en el tratamiento algebraico de las inecuaciones y después darles sentido a las soluciones sobre el modelo previamente usado. Esta investigación la estaríamos desarrollando con poblaciones de estudiantes de educación media en Honduras y México.

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MODEL ELICITING ACTIVITY FOR LEARNING STATISTICAL TECHNIQUES FOR PROCESS CONTROL

ACTIVIDAD PROVOCADORA DE MODELOS PARA EL APRENDIZAJE DE TÉCNICAS ESTADÍSTICAS DE CONTROL DE PROCESOS

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Keywords: Data Analysis and Statistics, Modeling, University Mathematics

The research reported here sought to answer the question: How do students' ways of thinking, about statistical concepts (data, distribution, variability, and sample) and process quality control techniques, changes while solving a Model Eliciting Activity [MEA] proposal? (Lesh, Hoover, Kelly, Hole & Post, 2000). Learning process quality control techniques is important for industrial engineering students because "these techniques are useful for identifying where, how, when and how often problems occur (statistical regularity)" (Gutiérrez-Pulido, 2005, p. 146).

We implemented a MEA proposal designed to identify the model's students generate in their problem-solving approach; which allows them to test the quality of their solutions, maintain productive thinking, in a short period of time and with a minimum of interventions (Lesh et al., 2000). In the activity, a discrepancy was raised between a pet food factory and a government institution due to anomalies in the weight of the product bags, hence the student is asked to write a letter to the editor of the magazine where the problem was reported. The letter should describe a method for verifying the packaging of the product. It was sought that with the models created the students would reinforce the concepts related to process control techniques, mainly descriptive techniques and control charts for variables.

Eighteen students participated in the activity, working first individually, and then in groups of three; at the end of the session, the participants share their work in a plenary session to the rest of their classmates. We qualitatively analyze the models and modeling process by reviewing voice and video recording from each session and categorized the level of understanding of the statistical concepts as described by Garfield & Ben-Zvi (2008).

A preliminary analysis of the models showed a change in the way concepts are thought of as data and dispersion. Students shift from perceiving data as individuals to data as a group. It was also noted that they began with a partial recognition of variation, and grew to think about the application of variation (Watson, Kelly, Callingham & Shaughnessy, 2003). Furthermore, students' ideas about concepts related to control techniques for variables were noted and need further refinement.

Acknowledgments

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ACTIVIDAD PROVOCADORA DE MODELOS PARA EL APRENDIZAJE DE TÉCNICAS ESTADÍSTICAS DE CONTROL DE PROCESOS

MODEL ELICITING ACTIVITY FOR LEARNING STATISTICAL TECHNIQUES FOR PROCESS CONTROL

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Palabras clave: Análisis de Datos y Estadística, Modelación, Matemáticas de Nivel Universitario

La investigación que aquí se reporta buscó responder a la pregunta ¿cómo cambia la forma de pensar acerca de los conceptos estadísticos (dato, distribución, variabilidad y muestra) y de las técnicas de control de calidad de procesos (carta de control de variables) de un grupo de estudiantes universitarios al resolver una propuesta de Actividad Provocadora de Modelos [APM]? (Lesh, Hoover, Kelly, Hole y Post, 2000). Aprender las técnicas de control de calidad de procesos es importante para los estudiantes de ingeniería industrial debido a que "son útiles para identificar dónde, cómo, cuándo y con qué frecuencia se presentan los problemas (regularidad estadística)" (Gutiérrez-Pulido, 2005, p. 146).

Se diseñó e implementó una propuesta de APM para identificar los modelos que los estudiantes generan en su proceso de solución; proceso que les permite probar la calidad de sus soluciones, mantener un pensamiento productivo, en un periodo corto de tiempo y con un mínimo de intervenciones (Lesh et al., 2000). En la actividad se planteó una discrepancia entre una fábrica de alimento para mascotas y una institución gubernamental por anomalías en el producto, por lo que se solicita al estudiante escribir una carta al editor de la revista donde se reportó la situación, en la carta debe describir un método para verificar el envasado del producto. Se buscó que los modelos creados los estudiantes reforzaran los conceptos relacionados con las técnicas de control de procesos, principalmente las cartas de control de variables.

La actividad se implementó con 18 estudiantes que trabajaron primero de manera individual, posteriormente en grupos de tres integrantes, los cuales finalmente presentaron su trabajo en una plenaria al resto de sus compañeros. Para analizar cualitativamente los modelos y el proceso de modelación se revisaron grabaciones de audio y video de la implementación, y se categorizaron niveles de comprensión de los conceptos estadísticos con base en Garfield y Ben-Zvi (2008).

En un análisis preliminar de los modelos se observó un cambio en las formas de pensar de los conceptos como dato y dispersión, que permitió pasar de pensar en cada dato de manera individual a concebir el dato como un grupo. Asimismo, se notó que comenzaron con un reconocimiento parcial de la variación y llegaron a pensar en la aplicación de la variación (Watson, Kelly, Callingham y Shaughnessy, 2003). Además, se advirtieron ideas de los estudiantes acerca de conceptos relacionados con las técnicas de control de variables que necesitan refinarse posteriormente a la implementación de la APM.

Agradecimientos

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THE ROLE OF SIMULATION IN SOLVING PROBABILITY TASKS

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Keywords: Probability, Computational Thinking, Programming and coding, Problem Solving.

One of the key components of probabilistic thinking is attention to outcomes within a sample space (Chernoff & Zazkis, 2011). Researchers are also interested in the notion of computational thinking, including the use of simulation and programming (Weintrop et al., 2016). According to Lockwood and De Chenne (2020), coding can "facilitate mathematical learning" in combinatorics, an area with close connections to the concept of sample space. In addition, simulations can lead to solutions which are very close numerically to those found using theory. This study investigates how learners interpret these two solution types within a probability task.

Methods

Using a clinical interview methodology (Clement, 2000), a preservice secondary mathematics teacher engaged with a probability task. Greg (a pseudonym), was asked to determine the likelihood of obtaining 3 blue marbles in a situation where 10 marbles are drawn without replacement from a bag containing 50 green, 20 yellow, and 30 blue marbles (theoretical solution using hypergeometric distribution approximately 0.2812). The interview was video recorded and transcribed, and analysis focused on Greg's two solution strategies.

Preliminary Findings

Greg's first strategy was to think about different "possibilities", referencing the task's sample space. He listed different "configurations" that would be possible, saying, "I could go through and do this systematically by hand." Greg stated that it would take a long time to list all of these out and seemed perplexed about how he would actually get to some sort of numerical solution, but that it might involve permutations or combinations. He then stated that his next step "would probably be to run some sort of simulation," going on to describe how he would "code" the situation using random numbers, conditionals, and loops, and run 10,000 trials in Excel to get an idea of what the likelihood might be. According to Greg, "What that would do is it would give me some sort of framework to judge my work." His results from the simulation would allow him to check whether his theoretical approach was going "in the right direction". The researcher then asked whether he would consider giving the results of his program as his answer to the task. He said, "Yes and no." While he would give the answer as an approximation, he believed that to answer the question "fully" he would need to go back to his theoretical method which involved enumerating all the possibilities. It appeared that he viewed the simulation approach as a way to verify a potential theoretical solution, not as a true solution in its own right.

Discussion

This research suggests that learners may view experimental or simulated solutions to probabilistic tasks differently than those they arrive at using theoretical methods involving concepts like sample space. In this case, it appeared that a theoretical solution was valued more highly by Greg than one which utilized programming, and that his simulation was primarily useful as an approximation or verification tool. This brings up questions about what it means to "solve" a problem involving likelihood. However, these conclusions are extremely preliminary, based on one individual's responses. More research needs to be done to see if others respond similarly to solutions found using theory versus those obtained using simulation.

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STUDENT LEARNING AND RELATED FACTORS

RESEARCH REPORTS

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MATHEMATICS ANXIETY AS A MEDIATOR FOR GENDER DIFFERENCES IN 2012 PISA MATHEMATICS SCORES

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Although gender differences in mathematics are smaller than they have been in the past, prominent voices still attribute these differences to a variety of fixed individual factors, such as genetic characteristics of men and women. We hold the alternative view that these differences can be ultimately attributed to malleable factors. From this vantage, societies could influence gender differences in mathematics by changing students' experiences in school. In this study, we built on prior work suggesting that mathematics anxiety causes lower mathematics scores. In particular, we found that mathematics anxiety entirely explains the gender differences evident in mathematics scores from the 2012 US Programme for International Student Assessment (PISA). Furthermore, we found that gender moderates the mediating role of mathematics anxiety: math anxiety is more detrimental for male than for female students. Because math anxiety is a malleable individual characteristic, we conclude that gender differences reveal more about gendered societal experiences than they do about innate characteristics of men and women.

Keywords: Gender and Sexuality; Equity and Diversity; Assessment and Evaluation; Affect, Emotion, Beliefs, and Attitudes

Headlines such as "Men in science think they are more intelligent than female counterparts, study reveals" (Richards, 2018) and "Study: 6-year-old girls say they are less 'brilliant' than boys. Why?" (Botkin-Kowacki, 2017) and "Why don't young girls think they are smart enough?" (Cimpian & Leslie, 2017) are not uncommon in the media today. Many researchers have challenged claims that "boys are better at math than girls" (e.g., Ding et al., 2006; Hyde et al., 2008). However, small average score differences between boys and girls on large-scale mathematics tests remain, particularly at the top of the score distribution, leading some to assume that girls are not as talented as boys in mathematics.

In this paper, we use the 2012 Programme for International Student Assessment (PISA) dataset to examine this small but persistent gender gap on mathematics exams. Although gaps analyses have been criticized for deficit thinking and not supporting individual students' mathematical identities (e.g., Gutiérrez, 2008), they often inform researchers and policymakers about necessary adjustments to educational systems. Lubienski (2008) notes, "It is dangerous for the mathematics education community to refrain from gaps analyses and allow others to speak in our place" (p. 352). Two recent cases illustrate this point. Mark Perry (2016), an economics professor at the University of Michigan Flint Campus used boys' and girls' average score differences on the mathematics portion of the SAT to assert that "closing the STEM gender degree and job gaps may be a futile attempt in socially engineering an unnatural and unachievable outcome" (para. 12). Likewise, Eric Rasmusen (2019), a professor in the highly-ranked Kelley School of Business at Indiana University, publicly and repeatedly agreed with an article entitled "Are Women Destroying Academia? Probably." In the article, differences in IQ scores were used to say that "geniuses" were most often men, and women's empathetic and emotional nature is the "enemy" of genius and, therefore, academia (Welton, 2019; see also Brice-Saddler & Paul, 2019). We find these views problematic because they frame female achievement in terms of fixed, innate characteristics.

Over the years, much scholarly work has been dedicated to understanding boys' and girls' mathematics score differences, and has produced clear evidence for a wide variety of contributing

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factors. Nearly all of these factors—high-stress and timed environments, decreased curricular alignment, negative depictions in media—are related to high mathematics anxiety, which itself is a malleable factor. If educators can lessen mathematics anxiety in early grades, then more girls might increase their confidence in mathematics and subsequently pursue STEM careers. This can have a global effect, as Smith and colleagues (2015) note, because more women in STEM increases creativity and innovation. As we discuss below, researchers have also documented that mathematics anxiety affects a larger percentage of girls than boys and operates to reduce mathematics performance. Looking across these findings suggests the question that guided our research: *To what extent does mathematics anxiety explain the small but persistent gender gap in mathematics exam performance?*

Perspectives

Mathematics Gender Gaps Analyses

Researchers have found that girls outperform boys on many school measures, from grade point average to number of undergraduate, masters, and doctoral degrees in their postsecondary years (e.g., Carnevale et al., 2018). Although several researchers have found no significant gender differences on mathematics tests like state-mandated end of course assessments, National Assessment of Educational Progress (NAEP), and Pearson's Stanford Achievement Test (e.g., Ding, et al., 2006; Hyde et al., 2008), some suggest that there still might be a slight gap in scores at the higher end of the score distribution, specifically, on more challenging items that may not have been explicitly taught in school (e.g., Downey & Vogt Yuan, 2005, Lubienski & Ganley, 2017). For instance, a 2016 study by Stewart et al. found no overall gender difference for math calculation, geometric concepts, basic math concepts, and addition. The researchers did find a significant difference in solving "real-life complex math problems," which had multiple steps and required the test-taker to respond orally (p. 53). In addition, researchers have found significant score differences on the AP Calculus exam, the mathematics SAT, and the quantitative portion of the Graduate Record Exam (Niederle & Vesterlund, 2010). Cimpian and colleagues (2016) found that score differences on the Early Childhood Longitudinal Study were significant as early as first grade at the higher end of the distribution and widened throughout elementary school. Other researchers, even as early as Aiken's 1972 publication, have found little to no score differences for elementary-aged children—especially on curriculum-aligned tests, but find that a gap in scores expands in the high school years (e.g., Aiken, 1972; Casey et al., 2001, Hyde et al., 2008).

We first discuss factors that have been shown to contribute to boys' and girls' score differences on mathematics tests. A number of researchers note that timed tests can be particularly damaging to math-anxious students (e.g., Walen & Williams, 2002; Whyte & Anthony, 2012). Neiderle and Vesterlund (2010) suggest that, on average, boys are more competitive and confident than girls and use this competitive nature in high-stress test-taking environments. As mentioned, some scholars suggest that girls perform better on school-taught material (Downey & Vogt Yuan, 2005; Lubienski & Ganley, 2017) as well as tasks that involve computations and memorized procedures (Ganley & Vasilyeva, 2014). In other words, if the test is not aligned to curriculum to which they have been exposed, boys tend to score better. Others suggest differences in the ways girls and boys solve problems, with studies showing that, as early as 1st grade, girls tend to use concrete strategies like modeling and counting while boys use more creative problem-solving strategies (Fennema et al., 1998) Lubienski and Ganley (2017) state, "Girls' teacher-pleasing behavior is likely a consequence of gender socialization and […] is likely linked to later differences in mathematical problem-solving approaches, with girls following teacher-given rules more often than boys" (p. 74). In contrast, many researchers suggest that boys tend to have better spatial skills, which could improve scores on problems with measurement, space, and shape (e.g., Halpern et al., 2007; Lubienski & Ganley, 2017). Cimpian and colleagues (2016) found that teachers consistently rated girls' mathematical proficiency lower than boys with similar achievement and behaviors. Some researchers suggest that stereotype threats and negative depictions of girls and math in the media might deter girls from pursuing STEM courses and careers, with the "math is for boys" stereotype influencing students as early as 2nd grade (e.g., Cvencek et al., 2011; Gunderson et al., 2012). Halpern and colleagues (2007) believe sociocultural forces such as parent beliefs and expectations, teacher encouragement, and peer influences contribute to score differences. Van Langden and colleagues (2006) suggest that the more gender equality in a country, the smaller the score differences. They also posit that girls do better in math when they are in "integrated classrooms" instead of "differentiated systems," i.e., separate classes in which students are placed according to ability. In sum, competitive environments, timed tests, decreased curricular alignment, teacher-pleasing behaviors, less gender equity in a country, a "math is for boys" viewpoint—almost all factors mentioned—could be related to mathematics anxiety.

Mathematics Anxiety

Mathematics anxiety has been defined as "a feeling of tension and anxiety that interferes with the manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary and academic situations" (Richardson & Suinn, 1972, p. 551). It differs from general anxiety and test anxiety; math anxiety can be exhibited in people who excel in testing situations in other subjects, and it can occur even during anticipation of interacting with numbers.

Dowker (2019) describes two dimensions of mathematics anxiety: cognitive and affective. The cognitive dimension closely relates to test anxiety and involves performance anxiety, worry, and "fear of failure." The affective dimension is often labeled as "emotionality" and refers to fear, nervousness, tension, and their related physiological reactions. Most importantly, the affective dimension occurs in the presence of numerical situations, with or without a test. Mathematics anxiety is not just correlated to performance deficits, but many researchers have suggested that mathematic anxiety has a causal relationship with performance deficits (e.g., Hembree, 1990; Ma, 1999). Ma (1999) describes two theoretical models that typically guide mathematics anxiety research: an interference model and a deficits model. In the interference model, mathematics anxiety disturbs the recall of prior knowledge and experiences, often causing lower scores; some scholars refer to this as the "debilitating anxiety model" (Carey et al., 2016). In the deficits model, researchers believe repeated poor performances produce increased levels of mathematics anxiety. In this model, lower scores are often attributed to poor test-taking skills and study habits. Several researchers have posited a bidirectional relationship between these two models, creating a vicious cycle of high mathematics anxiety and low mathematics scores (e.g., Carey et al., 2016; Dowker, 2019).

Working memory and the brain. Math anxiety has been shown to deprive people of their working memory, leading to lower scores, supporting Ma's interference model (e.g., Beilock & Willingham, 2014; Dowker, 2019). One might think that students with larger amounts of working memory could memorize every fact and formula and thus have lower levels of mathematics anxiety, but a counterintuitive result has been found. Children with high levels of working memory have a more pronounced negative relationship between math anxiety and math achievement (e.g., Beilock & Willingham, 2014; Dowker, 2019; Lubienski & Ganley, 2017; Lyons & Beilock, 2012; Maloney & Beilock, 2012). Dowker (2019) posits that those with higher levels of mathematics anxiety have more preoccupying thoughts and mental rumination, depleting crucial working memory resources. Consistent with Dowker's findings, Maloney and Beilock (2012) say, "Math anxious individuals tend to worry about the situation and its consequences. These worries compromise cognitive resources, such as working memory, a short-term system involved in the regulation and control of information relevant to the task at hand" (p. 404). In other words, students might be quite talented at memorizing formulas and procedures for use in normal classroom activities or homework, but the effects of mathematics anxiety might break down these rote procedures during stressful situations. Although other students might be able to solve the problem in a creative or novel way, students who have relied on rote memorization could further deplete their cognitive resources.

Neurologically, children with math anxiety show more activity in their right amygdala when performing mathematical tasks, and this area of the brain is known for processing negative emotions and fear (e.g., Beilock & Willingham, 2014; Chang & Beilock, 2016). When the right amygdala was active, there was a decrease in activity in the areas of the brain responsible for mathematical reasoning (Chang & Beilock, 2016). Psychologists at the University of Chicago refer to this as "working memory disruption," hypothesizing that students with higher working memory resources often use algorithms and problem-solving strategies with multiple steps, which are more susceptible to disruption during situations that induce anxiety (Ramirez et al., 2013). Further, Lyons and Beilock (2012) showed that the anticipation of doing math triggered the dorso-posterior insula in the brain, which is the area of the brain associated with bodily threat but also visceral pain; therefore, mathematics anxiety can be neurologically associated with physical pain.

Teacher, peer, and parent influences. Several researchers have noted how others might influence a student's level of mathematics anxiety. As early as 1959, Banks suggests that repeated failure, parents' and peers' unhealthy attitudes towards mathematics, and teacher insecurities can all contribute to students' negative associations with mathematics. Beilock et al. (2010) agree that mathematically anxious teachers at the elementary level impact students' mathematics achievement, especially when girls mimic the anxieties of their female teachers. Unfortunately, this can be a cyclical problem as the majority of elementary teachers are female, and a disproportionate number of preservice teachers have mathematics anxiety or worry that they will be unable to teach mathematics effectively. Brown and colleagues (2011) found this number to be 60% of preservice teachers, while Jackson and Leffingwell (1999) found that only 7% of preservice teachers in their study had only positive mathematics experiences in their K-16 schooling. Whyte and Anthony (2012) suggest that parents who suffer from math anxiety can transfer this to their children, and parents who give math a low status or, contrastingly, apply extra pressure to their children may contribute to mathematics anxiety as well.

Interventions to Decrease Math Anxiety

Some researchers have suggested that there is an optimal amount of mathematics anxiety that will positively affect performance. The graph is a curvilinear relationship, an inverted U-shaped curve. Cognitive reactions to mathematics anxiety might include "blanking out" or self-doubt, while an affective reaction might include the fear of looking stupid. Physical reactions for both dimensions might include perspiring, an increased heart rate, or nausea (Frieberg, 2005). In a 2010 study, researchers informed one group of participants that anxiety could help to improve performance but said nothing to the control group (Jamieson et al., 2010). When students took a practice Graduate Record Examination (GRE), the control group had more salivary alpha amylase (sAA)—an indicator of stress—than the test group. In addition, the group that was told anxiety improves performance had higher scores both on the practice test and on the actual GRE, taken 1-3 months later.

Jamieson et al. had a simple approach for lessening the impact of mathematics anxiety on high-stakes standardized tests like the GRE, but others have posited interventions to help with mathematics anxiety in everyday situations. In recent years, Ganley et al. (2019), have proposed the use of the Math Anxiety Scale for Teachers (MAST) to reduce the harmful effects of teachers' mathematics anxiety on students' learning. They note that teachers with high levels of mathematics anxiety avoid the subject, spending less time on mathematics lessons, especially in whole class discussions. Whyte and Anthony (2012) suggest promoting a positive classroom culture with effective teaching practices, such as having students share creative approaches to problem solving. They also suggest utilizing math fiction books, journal writing, and math autobiographies in the classroom. Researchers who noted the negative impact of timed tests (e.g., Walen & Williams, 2002; Whyte & Anthony, 2012) suggested providing ample time for assessments. Recently, Schaeffer et al. (2018) used an app to help mathematically anxious parents creatively interact with their children while discussing mathematics. After using the app for even a short period of time, parents reported feeling less anxious about math, and these results were maintained even two years after the study. This was an important finding, as parents' negative associations with math are often reflected in their children. Maloney and Beilock (2012), also note the importance of journal writing, saying that writing about emotions for 10-15 minutes before a test can boost the scores of students with mathematics anxiety. Lastly, students develop a negative relationship with mathematics when they are told there is only one right way to do a problem. When less emphasis is put on memorization and more weight is placed on creative problem-solving, students are less dependent on their working memory when working on mathematics problems.

Methods

Data for the present study come from the public release version of the 2012 US PISA data set. These data were selected because they are nationally representative, include trustworthy instruments measuring the focal constructs of our investigation (i.e., mathematics anxiety and of student achievement) as well as important covariates. More recent PISA data has not included the variables of interest. Participants were 4978 students aged 15 to 16 years old. The sample was 50.7% male, 52% White, 13.1% Black, 24.0% Hispanic, 4.6% Asian, 4.3% identified as multiracial, and 2.1% identified as other (n = 67, missing).

We used the PISA mathematics score as the dependent variable in our analysis. Because not all students answer the same questions when taking the PISA assessment, five plausible values are reported for each student instead of a single score. The differences between the plausible values for a specific student capture the uncertainty about the student's estimated math score. We completed the analyses described below with each of the five math scores, then used accepted methods to pool results (Rubin, 1987).

The first independent variable was a math anxiety instrument based on 5 rating items (e.g., "I often worry that it will be difficult for me in mathematics classes," "I get very tense when I have to do mathematics homework," "I get very nervous when doing mathematics problems," "I feel helpless when solving a mathematics problem," and "I worry that I will get poor grades in mathematics"). The composite score for the five-item math anxiety questionnaire was available within the PISA dataset. According to the technical report (OECD, 2014), the variable was transformed to an international metric with a mean of 0 and a standard deviation of 1 for all OECD countries that participated in the PISA 2012 (OECD, 2014). The internal validity of this multi-item construct was evaluated with Cronbach alpha = 0.88 and deemed to have good internal consistency. The sample mean was -0.104 and ranged between -2.370 and 2.550. By design, not all students answered each section of the survey. Thus, there were 1720 students with missing scores on the math anxiety measure. We used full information maximum likelihood estimator which is robust to missing data and this technique allowed us to include all students in the analysis. Self-reported gender was the second independent variable, and a summary of this variable is provided above. Finally, we used the PISA standardized index of economic, social and cultural status (ESCS) which summarizes occupational status, parent education, family wealth, home educational resources, and an index of possessions relative to each country context. The sample mean was 0.188 and ranged between -3.80 and 3.12 (n = 63, missing).

A mediation effect is said to occur when a mediating variable helps explain or account for the relationship between an independent and dependent variable. For example, in this study we wanted to know if the relationship between gender and mathematics score is mediated by math anxiety. A moderation effect occurs when a moderating variable influences the effect of the dependent variable on the independent variable. For example, gender is said to moderate the relationship between math anxiety and math scores if the relationship between these variables differed by gender. Moderated mediation describes a theoretical model that includes a variable that moderates the mediating effect of another variable on the outcome. In this study, we wanted to know whether the extent to which math anxiety accounts for the relationship between gender and math scores differs by gender—a moderated mediation model.

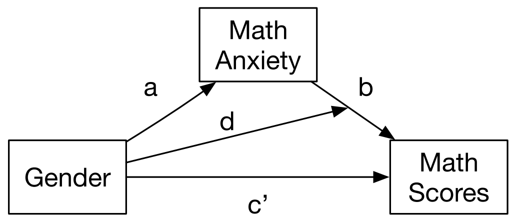

Figure 1: Conceptual model for moderated mediation.

As shown in Figure 1, the total effect of gender on math scores is the sum of the direct effect, c', and the indirect (or mediated) effect, a x b. The coefficient d describes how gender moderates the mediating effect of math anxiety on the relationship between gender and math scores. Following Shevlin et al. (2015) we operationalized moderated mediation by testing the path diagram in Figure 2 which includes an interaction term between gender and math anxiety. In this figure, the moderated mediation is investigated by examining coefficient d, which quantifies the gender difference in the math anxiety, math score relationship. We used lavaan, an R package, to estimate two structural equation models for (1) mediation and (2) moderated mediation for each of the five plausible values for the outcome variable. We used another R package (mice) to combine the results across the five models we estimated. We compared these models with a regression model that used gender and other covariates to predict math scores.

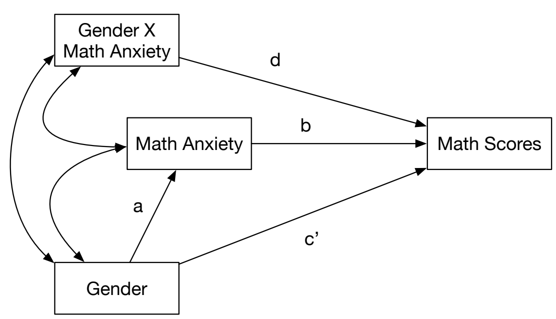

Figure 2: Path diagram used for analyzing moderated mediation.

Results

In the baseline regression model (see Table 1), we found a small (0.093), statistically significant (p < .001) relationship between gender and math scores in our sample after controlling for race and ESCS. The goal of our subsequent analysis was to understand how much of this relationship could be explained by mathematics anxiety. The second model shows that once anxiety is included as a mediating variable, the relationship between gender and math scores is no longer statistically significant (0.020, p = 0.42). Thus, in this sample, anxiety completely mediates the relationship between gender and math scores. The total effect is still the same (0.093, p < .001), but most of this effect is indirect via math anxiety (0.074, $p < .001$). The size of this effect can be quantified as the percent mediation: the indirect effect explains 80% of the total effect.

In the final model, the relationship between gender and math scores is not statistically significant (0.009, $p = 0.71$). In this model we can compare the indirect effects by gender. For males, the indirect effect is 0.085 ($p < .001$) and for females the indirect effect is 0.064 ($p < .001$). Moreover, the difference is statistically significant (.021, $p < .01$). We concluded that gender does influence the extent to which math anxiety explains the relationship between gender and math scores, and in particular, we found a stronger indirect relationship between these variables for male students.

Table 1: Regression and path coefficients for the baseline regression model and for the mediation and moderated mediation models.

	Regression Model	Mediation Model	Moderated Mediation
predicting Math Scores	β (SE)	β (SE)	β (SE)
Male	0.093 (0.025) ***	0.02 (0.024)	0.009 (0.024)
Anxiety		-0.333 (0.013) ***	-0.288 (0.018) ***
Male by Anxiety			-0.096 (0.027) ***
ESCS	0.334 (0.014) ***	0.283 (0.014) ***	0.283 (0.014) ***
Black	-0.833 (0.039) ***	-0.817 (0.038) ***	-0.81 (0.038) ***
Hispanic	-0.273 (0.034) ***	-0.277 (0.032) ***	-0.276 (0.032) ***
Asian	0.47 (0.061) ***	0.427 (0.058) ***	0.425 (0.058) ***
Multicultural	-0.083 (0.063)	-0.12 (0.06) *	-0.131 (0.06) *
Other race	-0.647 (0.09) ***	-0.604 (0.086) ***	-0.596 (0.086) ***
predicting Anxiety			
Male		-0.222 (0.036) ***	-0.222 (0.036) ***
ESCS		-0.151 (0.02) ***	-0.151 (0.02) ***
Black		0.048 (0.057)	0.049 (0.056)
Hispanic		-0.011 (0.048)	-0.011 (0.048)
Asian		-0.13 (0.087)	-0.129 (0.086)
Multicultural		-0.113 (0.091)	-0.113 (0.091)
Other race		0.13 (0.131)	0.13 (0.131)
Direct & Indirect Effects			
Male Indirect			0.085 (0.014) ***
Female Indirect			0.064 (0.011) ***
Difference Male/Female Indirect			0.021 (0.007) **
Indirect Effect		0.074 (0.012) ***	0.064 (0.011) ***
Total Effect		0.093 (0.025) ***	0.073 (0.025) **

Limitations

This study used cross-sectional data, therefore only associations—not causal relationships—can be investigated. Furthermore, mediation relationships depend on theory—not statistics. If the mediation model we presumed for this study is not accurate, then the results of the model are not meaningful (Maxwell & Cole, 2007). Finally, our conceptual model has anxiety causally preceding mathematics scores. Although we have argued in our review of the literature that this assumption is reasonable, not all scholars agree. Moreover, the data we used in this study cannot be used to settle this question because they are cross-sectional and non-experimental.

Discussion and Implications

The goal of this study was to better understand the influence of mathematics anxiety on the small but persistent gender gap in mathematics achievement. We applied moderated mediation to rigorously investigate these relationships, attending to missing data, and controlling for potential confounding variables at the same time. This rigorous method enabled us to accurately evaluate the associations between multiple variables in a large sample data set and draw robust conclusions.

Based on a large sample of 15-year-old students in the US, the results show that math anxiety entirely mediated the association between gender and math scores. In particular, the higher math anxiety of female students entirely explained female students' lower math scores. We also examined a moderated mediation model, in which we allowed the relationship between math anxiety and math scores to vary by gender. The results show that gender does moderate the mediation relationship between math anxiety and math scores. In particular, the anxiety of male students had a stronger negative association with mathematics scores than did the math anxiety of female students. Further, the moderated mediation finding suggests that although there are fewer male than female math-anxious students, the male students who have math anxiety may benefit more than similar female students from interventions because their experience of anxiety is more debilitating. Our results suggest that known interventions that decrease mathematics anxiety might be helpful in narrowing or eliminating the gender gap in mathematics achievement. For example, de-emphasizing rote memorization and encouraging creative problem-solving can lessen mathematics anxiety in early grades. As girls' mathematics anxiety has been linked to their math-anxious teachers in elementary grades, further studies using tools like the Mathematics Anxiety Scale for Teachers (Ganley et al., 2019) would be beneficial. In addition, it will be important to investigate math anxiety's relationship to other factors related to boys' and girls' score differences, such as high-stress environments, decreased curricular alignment, and negative depictions in media.

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ELEMENTARY STUDENTS AND THEIR SELF-IDENTIFIED EMOTIONS AS THEY ENGAGED IN MATHEMATICAL PROBLEM SOLVING

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In this study, we investigated two students', ages ten and eleven, emotions while they engaged in mathematical problem solving. During three task-based interviews, the students explored parts of the unsolved problem the Graceful Tree Conjecture. While they were engaged in the interviews, they self-identified the emotions of frustration and joy they were feeling using the Wong-Baker Scale. The students displayed the interplay of the emotions of frustration and joy or which we consider to be productive struggle. A descripted case of Georgia is included to describe her emotions while problem solving.

Keywords: Affect, Emotion, Problem Solving

Past researchers have documented that students experience both positive and negative emotions while engaged in mathematics (Hannula, 2015; O'Dell, 2017) and Else-Quest, Hyde, and Hejmadi (2008) and Williams (2002) have found an association between students having positive emotions during problem-solving and the development of mathematical understanding. However, much of the research completed on emotions have been documented through surveys and not while students are engaged in mathematical problem solving (Hannula, 2015).

O'Dell (2017) documented that when students are experiencing the emotion of frustration followed by the emotion of joy while the student is engaged in mathematical problem solving, they are experiencing productive struggle. Struggle is when "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent" (Heibert & Grouws, 2007, p. 387). It has been acknowledged allowing students the opportunity to struggle is beneficial (Hiebert & Grouws, 2007; Kapur, 2010). Kapur (2010) said when students are allowed to struggle they are able to significantly outperform students of a similar ability who have not been granted the opportunity to struggle. Kapur further found when a student has engaged in productive struggle they are able to better transfer that knowledge to mathematical concepts to which they have not yet been exposed.

Warshauer (2015) and Zeybek (2016) stated researchers know productive struggle is beneficial but there is limited research on what productive struggle looks like. O'Dell (2017) has documented that productive struggle is the interplay of the emotions of frustration and joy, but we want to examine first how students express the emotions of joy and frustration, second how they self-identify their emotions of frustration and joy, and lastly if they self-identify as having more frustration during problem solving, do they then experience more joy.

With these ideas, the following questions guided our research study:

1. How do students display the emotions of joy and frustration while they are engaged in problem solving?
2. How do students self-identify the emotions of frustration and joy they experience while problem solving?

Theoretical Framework

To examine how students display and identify emotions, we draw on the concept of positioning theory (Van Langenhove & Harré, 1999). Positioning theory is "the study of local moral orders as ever-shifting patterns of mutual and contestable rights and obligations of speaking and acting" (van

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Langenhove & Harré, 1999, p. 1). Thus, positioning theory allowed us to examine how students position themselves through their engagement with the mathematics and the other participants. In mathematics education positioning theory has been used to examine social interactions (e.g., Turner, Dominquez, Maldonado, & Empson, 2013; Wood, 2013; Yamakawa, Forman, & Ansell, 2009) and researchers have identified that students' emotions are linked to their positioning (e.g. Daher, 2015; Evans, Morgan, & Tsataroni, 2006). Further, Wood (2013) used positioning theory to examine micro-level moment-to-moment interactions that allow a researcher to document the exact moment an identity was enacted. A person positions themselves through conversations, actions, and dispositions. Through the events and moment-to-moment interactions storylines are created. These storylines allow a researcher to document the exact moment a student displayed a disposition or in our case, an emotion.

Methods

The participants of the study were two students using the pseudonyms Luna and Georgia. Luna was 10 and in Grade 4 and Georgia was 11 and in Grade 5. The study took place on a university campus close to their elementary school in the Midwestern United States. The two students participated in three semi-structured, task-based interviews (Goldin, 2000). The interviews took place over three weeks and each interview lasted approximately 60 minutes. For the three interviews, the students explored an unsolved graph theory problem, the Graceful Tree Conjecture. This is a problem that has been used in previous studies that have documented students display productive struggle while engaged with the Graceful Tree Conjecture.

Graceful Tree Conjecture

During the three task-based interviews, the students explored the Graceful Tree Conjecture. The Graceful Tree Conjecture is an unsolved problem in graph theory that is accessible to young children. While the parts of the problem the students explored have been previously solved, the entire problem remains unsolved. The students explored different classes of tree graphs which are connected graphs without a cycle. This means all tree graphs contain one less edge than node (vertex). It is believed that all tree graphs can be assigned a graceful labeling. This means when the nodes are distinctly labeled 1 through *n* or the number of nodes in the graph. The edges are labeled with the absolute value of the difference between the two connecting nodes. If the edges are labeled distinctly 1 through *n*-1, the graph is labeled gracefully (see Figure 1 for an example of a graceful and non-graceful labeling). The Graceful Tree Conjecture states that all tree graphs can be labeled gracefully.

Graceful Labeling **Non-Graceful Labeling**

Figure 1: Example of Graceful Labeling and Non-Graceful Labeling

Overview of Task-Based Interviews

During the three task-based interviews, Luna and Georgia explored different classes of tree graphs in increasing sophistication (see Figure 2). We not only challenged the students to find a graceful labeling for the first four distinct graphs but to find a pattern or justification that would show they could label any graph in the given class gracefully.

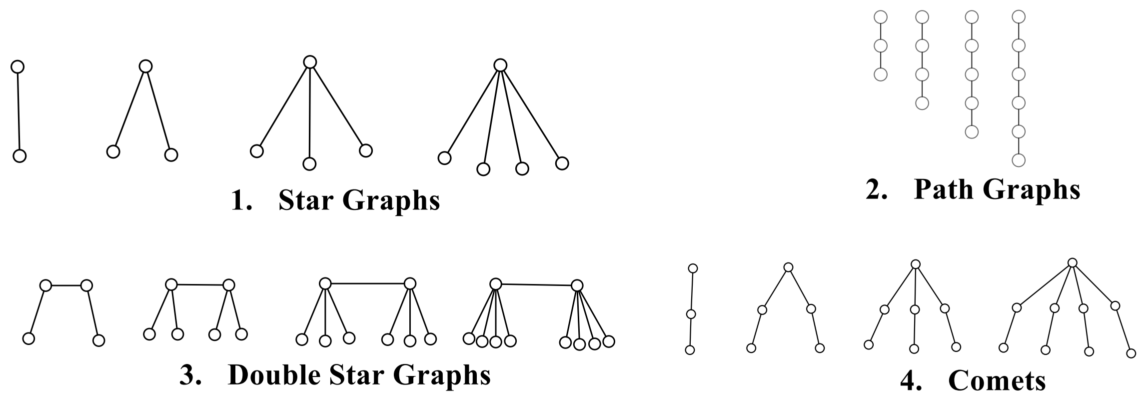

Figure 2: Graphs Students Explored in Increasing Sophistication

While the students explored the different classes of tree graphs, they were given a page on which to record labels for the first four distinct graphs in the class, a question that asked them to draw the next graph in the class, and statement to record their pattern. They were also given enlarged copies of each graph with numbered square and circle chips (see Figure 3 for an example of the students using the enlarged copy of each graph and numbered chips). The enlarged copies were given to the students so they could find solutions to the graphs without having to erase.

Figure 3: Example of enlarged copy of the graph and chips and image of how students self-identified emotions

During the first interview, the students were introduced to the meaning of the word conjecture, tree graphs, edges and nodes, what a graceful labeling was, and the Graceful Tree Conjecture. Next, they explored Star Graphs and Path Graphs. During the second interview, the students were reminded of the task, reviewed their solutions for Path Graphs, and explored Double Star Graphs. During the final interview, the students reviewed the problem, the Double Star Graphs, and explored Comets.

During each of the three interviews, the students self-identified their emotions or feelings using the Wong-Baker Faces (see Figure 3). Under the happy face, the words extreme joy was listed (we refer to this as level 6) and under the most upset face the words extreme frustration was listed (we refer to this as level 1). Students were instructed to move a chip to whatever they were feeling. An alarm sounded every three minutes as a reminder to mark their feelings.

Analysis

The task-based interviews were video recorded and the student work was collected. Next, the three interviews were transcribed using the program Transana (Woods & Fassnacht, 2016). This included the non-verbal actions, such as hand or arm motions, facial expressions, and the Wong-Baker level the students self-identified as. We then used a framework created by Else-Quest et al. (2008) and adapted by O'Dell (2017) to examine the emotions of frustration and joy the students displayed. Because of O'Dell's (2017) prior findings that the most common emotions students displayed were frustration and joy, we only examined the transcripts and video for those emotions (see Table 1 for modified framework).

Table 1: Analytic Framework

Emotion	Definition	Example
Frustration/Distress	Disappointment, discontent, displeasure	I am stuck. Man, this is confusing.
Joy/Pleasure	Delight, amusement, pride	I got it!

We examined the transcripts using the analytic framework and documented anytime an emotion of joy or frustration was displayed using Transana (Woods & Fassnacht, 2016). If the study displayed several statements or emotions of joy in a row, each individual statement or motion was documented as its own occurrence. We also document every time a student changed their self-identified emotion on the Wong-Baker Faces. Both authors both did this and discussed any discrepancies until we both agreed. Next, we created reports through Transana to account for each emotion and self-identified level displayed by the student.

Results

During analysis, we examined the emotions of frustration and joy the students displayed while working (see Table 2) and what level on the Wong-Baker Scale they marked. First, we will share the overall instances of frustration and joy, next we will give a description of Georgia during the third session to show how her emotion was displayed with her self-identified emotions, and lastly, we will share how the students self-identified on the Wong-Baker Scale.

Frustration and Joy

After analysis of the data, we used Transana to run reports of the instances of frustration and joy. We found 51instances of frustration and 92 instances of joy (see Table 2 for instances).

Table 2: Frustration and Joy Displayed by Interview

	Interview 1		Interview 2		Interview 3		Total	
	F	J	F	J	F	J	F	J
Georgia	6	12	5	16	22	19	33	47
Luna	2	10	13	18	3	17	18	45

Joy was typically displayed by both students when they made progress on the Graceful Tree Conjecture. Luna displayed joy, most commonly by verbal statements such as "I got it! So, I just have to figure out this pattern" when she found a graceful labeling. Another example of joy for Luna was, "that would be three. Yay!" with a big smile and saying "Okay, I have got to make sure I did this right." She also displayed joy when she was sharing her pattern verbally to one of the researchers and Georgia.

Georgia also displayed joy when she found a graceful labeling. Her joy was displayed more often by physical movement. One example is when she found a solution to a graph, she circled her arms around her head smiling and saying “my brain powers.” Another instance of joy she motioned her hands over her page and said: “Ta-da!” She often clapped her hands with excitement, danced in her seat, or made a fist of joy.

Frustration was typically displayed while attempting to find a graceful labeling for a specific tree graph, searching for a pattern in a class of tree graphs, and when we pushed the students to create a generalization for each specific class of tree graphs. Luna gave verbal statements in a similar way that she displayed joy. She often things such as, “That wouldn’t work” or “Wait, that wouldn’t work because the five is supposed to be there.”

Georgia again displayed her frustration more visually and kinesthetically. She would make faces of frustration, show frustration through her eye movements, leaning her body back in the chair, and toss chips down on the table when she got stuck.

Descriptive Case Study of Georgia

Georgia was a Grade 5 student. Her story from the third interview was chosen to be documented because she displayed the largest amount of frustration and joy. She worked on the Comet Graphs (see Figure 3). She began the session by identifying herself as a level five on the Wong-Baker Scale. Showing her joy of starting the interview session. The researcher then introduced the class of tree graphs called Comets. Georgia displayed joy by stating, “Those look really cool.” She began to attempt to label the first distinct graph in the class (see Figure 4 for her labelings) and displayed several signs of frustration. After making an angry face at the camera, she said (time is shown as minutes and seconds into the interview):

(6:15) I am trying to remember my line graph pattern.

She then continued to move chips around. Several seconds later she made a motion with her hands and gave a big smile showing joy. She asked:

(6:40) Is that graceful?

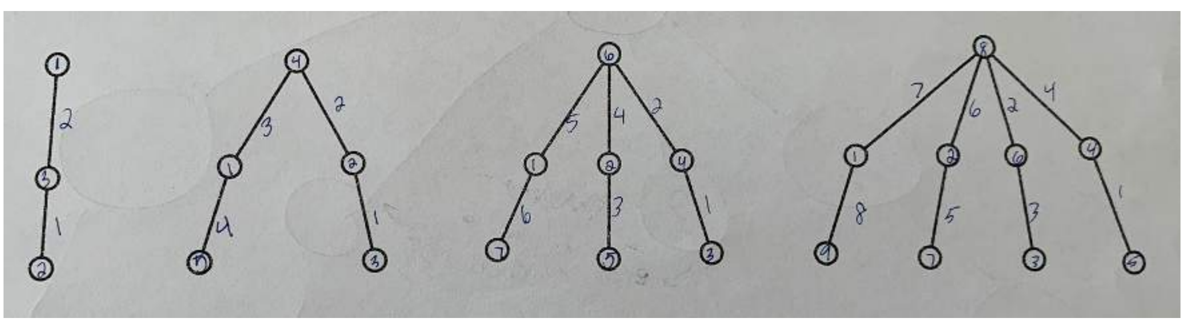

Figure 4: Georgia’s solutions to the first four distinct comet graphs

After deciding that it is and recording her solution, Georgia began to work on a graceful labeling for the next graph. The timer sounded and she moved down to level four on the Wong-Baker Scale. Georgia stated:

(7:27) I am feeling a little confused.

She continued to work and at nine minutes into the interview, she shook her head displaying clear signs of frustration. Georgia stated with a frustrated undertone:

(9:05) Wait, how would you label it gracefully? Like, how would you?

Georgia continued to work on the labeling and thirty seconds later she gave a sly smile and said:

(10:15) Got it!
(10:21) That is graceful right?

The researcher told her she was correct and the three-minute timer sounded. Georgia moved her tile to level 5 or second highest level on the Wong-Baker Scale and stated:

(10:31) I feel good!

Georgia then began working on the third distinct graph in the class. The next timer went off and Georgia moved down to level 4. She continued to work. At 15:23, Georgia tossed a chip with an angry look on her face showing a sign of frustration. Next, she said:

(15:23) I am stuck.

She continued to work making two more facial movements of frustration and several seconds later said:

(16:25) Ugg

The timer went off again, Georgia moved down to level 3. She said:

(17:12) This is confusing.
(17:31) So I was just doing it line by line trying to figure it out.

Georgia continued to move chips around looking for a pattern and making visual signs of frustration but no verbal signs. At twenty minutes in, the researcher told both girls they stuck at the same place. Georgia put a big smile on her face showing a sign of joy. She continued to work on finding a label but did not show any signs of frustration. The timer went off at 24 minutes and Georgia moved her level up to four.

At 26 minutes in, Luna found a graceful labeling for the third graph. Georgia examined Luna's paper, followed suit quickly, and stated she found a solution that was the same as Luna's solution. Both girls began working on the next graph, repeatedly saying, "I think I can" and laughing. Several minutes later, Georgia stated with frustration:

(37:16) Urg, this is where I got confused last time.

When the timer went off next, Georgia moved her level down to 3. She continued to work on her graph. She stated:

(38:41) Oh, wait, no, no, no. Four minus what equals five. Four. Five. Six
(39:03) So I want this to be four so two minus what equals four?
(39:16) Ten. Six.
(39:40) And I want my three to be here. Brain is working.

At 39:51, Georgia began clapping and smiling and then stated:

(39:52) This is my only hiccup. That is what my teacher says. That is my only hiccup.

She thought she had an almost graceful labeling but had one edge that did not have the proper number for a graceful label (she was missing a five). She continued:

(40:18) I know but, wait, wait, wait.

Georgia made an angry face and made a joke about not liking fives. The timer went off and Georgia moved her tile up to four and continued to move chips around. Two minutes later, Georgia exclaimed:

(42:19) Eight! Ah. One is always here and then it goes. Wooo (hands on her head). That is the second biggest number because that is the third biggest number.

She continued to work. Next, she said excitedly:

(43:19) Oh, I am so excited!

She found a solution to the fourth graph and after prompting from Luna, Georgia moved her tile to level six but said:

(43:45) It is kind of confusing because I have mixed emotions.

And then she moved her tile back down to level four. She examined her solutions looking for a pattern and stated:

(44:23) So this is one and then and then it counts by twos. This is the second biggest number. That is the biggest. That is the third biggest number. This is the fifth biggest number.

Georgia continued to examine her solutions and found a pattern to generalize that any Comet graph can be labeled gracefully (see Figure 5 for her generalization). She shared that the bottom row of nodes were odd numbers and the biggest, the middle row of nodes was one and the even numbers, and the center node was the second biggest number.

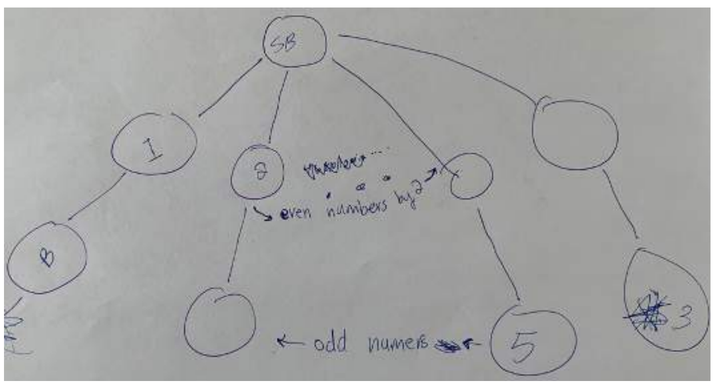

Figure 5: Georgia's generalization to Comet Graphs

Self-Identified Emotions

While the two students were engaged in the task-based interviews they self-identified their emotions of frustration and joy using the Wong-Baker Scale. The face of the most frustration we assigned a numeric value of one and the face with the most joy has a value of 6. Both subjects expressed familiarity with the scale from visiting the doctor's office. A timer went off every three minutes to remind them to mark how they were feeling and they were encouraged to move the tile to a new face if their emotion changed. We used Transana to run reports for every time the students self-identified using the tile and the Wong-Baker Scale (see Table 3)

Table 3: Frequency of tile movement on the Wong-Baker Scale

Level	Luna				Georgia			
	Day 1	Day 2	Day 3	Total	Day 1	Day 2	Day 3	Total
1	0	0	0	0	1	0	0	1
2	0	0	0	0	0	0	0	0
3	1	3	2	6	1	1	3	5
4	2	4	5	11	4	3	5	12
5	3	5	3	11	6	3	3	12
6	1	0	1	2	3	0	1	4

When either Luna or Georgia changed their tile to a higher level of frustration (lower number) on the Wong-Baker Scale they typically made some type of comment to discuss why they were moving down. The comment almost always had to do with them being “confused” or as Luna would say, “cornfused.” Neither student ever identified being more frustrated than a level 3 besides Georgia on the first day. This happened at the end of the session and with the tile movement Georgia stated, “I feel sad because we have to leave.”

Midway through the first interview, Luna decided that they should treat level six (extreme joy) as if they are experiencing similar feeling to riding a roller coaster and only move to that level if they felt that way. After that statement, there were only two more instances of extreme joy. Luna’s extreme joy came at the beginning of the third interview and said, “I feel good.”

Luna and Georgia tended to move their tile up when they were finding success on the graphs but that was not always the case. Several times both students would shift their tile higher when they reached a similar place on finding graceful labelings and were encouraged by the researcher to work together. Luna also tended to move her tile up at times while engaged in struggle and showing frustration but would not give a reason. For example, while trying to find a labeling for the third distinct Comet Graph she stated, while moving her tile up from three to four, “I am feeling better because” but did not give a reason. Both students never stated they reached extreme frustration besides, Georgia when she was sad they had to leave or even the level two. They also seldom reached level six of extreme joy.

Discussion and Conclusions

The results of this study are similar to Else-Quest et al. (2008) and O’Dell (2017) that students displayed both frustration and joy while engaged in mathematical problem solving. When O’Dell (2017) completed a similar study, she found students displayed more frustration than joy; however, we found the students to display more joy than frustration. Both studies still contained the oscillation between frustration and joy while engaged in problem solving. While Georgia and Luna were struggling through finding a graceful labeling they displayed several instances of frustration and when they found a successful labeling they displayed instances of joy. Other times they found joy in working together on the problem and through making jokes, such as “I think I can” while working.

When self-identifying their emotions, the two students repeatedly moved their tile to more frustrated when they stated they were “confused” and moved their tile toward joy when they found solutions. At other times we were not sure why they moved their tile toward joy while they were clearly still frustrated during the problem solving. Interestingly, they never—with the exception of Georgia being sad about the interview was over—moved to high frustration, level one or two. They only documented level three eleven times even though we documented 51 instances of frustration. The ratio of joy to frustration (levels 1-3 frustration and 4-6 joy) was 52 to 13. This demonstrated that even though the mathematical task was rigorous and an unsolved mathematics problem the students reported significantly more joy than frustration by a four to one ratio.

Overall, both of the students in this study were able to preserve through the struggle and frustration and were able to find joy and pride in their work on an unsolved graph theory problem. Unsolved problems are not typically included in elementary school but we found these problems to give students the opportunity to experience mathematics more similar to how mathematicians experience mathematics as a quest to describe patterns and relationships.

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STUDENT LEARNING AND RELATED FACTORS:

BRIEF RESEARCH REPORTS

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A CULTURE OF CHANGE: STUDENTS STORIES IN UNDERGRADUATE REFORM MATH

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A student's perceptions of, feeling towards, and beliefs about mathematics have long been known to be associated with their learning and performance in mathematics. But how might we observe and document changes in these attitudes and dispositions? How might a student's affects respond when the nature of mathematics changes? To answer these questions, we adopt a life stories theoretical perspective and read these stories through an affect lens. In so doing, we observe patterns of change in students' affects that correspond to milestones in their concurrent mathematics course. We present a vignette that illustrates one student's redemptive story of her transformation towards productive affects. Taken together, we suggest that curricula can be enacted in ways that afford such redemptive changes.

Keywords: Affect, Emotion, Beliefs, and Attitudes.

Introduction

In 2017, in the California State University (CSU) system, over 25% of entering, full-time freshman (16,628 students) were placed into a remedial mathematics sequence. Their stories within remedial mathematics courses were bleak, with students describing experiences of increased anxiety and negative dispositions towards mathematics (Maciejewski, Tortora, & Bragelman, under review). A change to this placement system was needed.

Later that year, the CSU Chancellor's office issued Executive Order 1110, which abolished remediation across its 23 campuses, leaving the institutions to individually determine reform of their general education mathematics curriculum for the 2018 academic year. San José State University embarked on a curricular redesign, Math 1, that emphasized four principles: an inclusive environment where students were no longer tracked by ability; content emphasizing contemporary mathematics rather than the traditional algebra sequence; a student-centered classroom experience; and a curriculum that also targeted non-cognitive components to support students' disposition towards mathematics (Maciejewski et al., accepted). In short, the reform targets multiple processes of the instructional dynamic (Cohen, Raudenbush, & Ball, 2003). In this work, we orient on the interactions between students and the reform curriculum by exploring students' change in affect towards mathematics as evidenced in their emergent life stories during Math 1.

Life Stories

We approach learners' experiences through a life narrative or life story methodology (McAdams, 1985, 2008; McAdams, 2018; McAdams & McLean, 2013; McLean et al., 2018. Indeed, "stories are the best vehicles known to human beings for conveying how (and why) a human agent, endowed with consciousness and motivated by intention, enacts desires and strives for goals over time" (McAdams, 2008, p. 244). They capture both consistent and inconsistent patterns over time (McAdams, 1985), such as a person repeatedly identifying as 'bad at math' across interviews or as a person describing what sequence of events led them to changing their major. Stories evolve over time, implying individuals' meanings attributed to important events may also change (Singer & Salovey, 2010). Last, stories are contextual (McAdams, 2013), suggesting stories are both created and discontinued within established cultural norms and traditions.

Affect and Stories

The affective domain in mathematics education encompasses varying concepts and theories and manifold approaches to its study (Goldin et al., 2016), and there is a clear alignment between life stories as a methodology and affect as a theoretical lens. Di Martino and Zan (2010) provide an entry point to examining students' affects towards mathematics through stories, with the Three-dimensional Model of Affect (TMA) having been developed from thousands of students' life stories with mathematics. We approach affect in this work through the TMA of Di Martino and Zan (2010); in particular, we restrict our attention to the categories of emotion and competence. Rather than approaching students' affect towards mathematics through more traditional life stories approaches (cf. McAdams and McLean, 2013), we explore students' change in affect by capturing multiple narratives across a semester of a reform introductory university mathematics course, in situ. Unlike Di Martino and Zan (2010), we explore change in affect across multiple narratives from each participant.

Method

In this multiple case study, we draw on empirical data collected during the first semester implementation of Math 1 in Fall 2018 at San José State University. The case study focus was twofold: an examination of students' mathematics experiences and affect in a reform general education mathematics course and an examination of a specific institutional implementation of a new curriculum (Yin, 2009). Within a life story methodology, qualitative methods were employed for the inquiry, most notably classroom observations and semi-structured interviews. Data was collected across two course sections of Math 1, and a subset of this data, ten students' stories about their mathematics learning experiences, is the primary unit of analysis here.

Setting and Participants

Under Executive Order 1110, incoming freshmen at San José State University are no longer required to take a basic skills assessment, which would normally have determined the starting point in their course sequence. Rather, multiple academic indicators are used to place students into the mathematics sequence appropriate for their degree focus. Math 1 served as the first of several mathematics courses for mathematics-intensive degree programs. However, students in non-mathematics intensive programs had the option to self-enroll in Math 1 as it satisfied their general education mathematics requirement. Of the 288 students enrolled in Math 1 in Fall 2018, 10 consented to the interviews.

The first author conducted three to four interviews across the semester, depending on students' availability, with a total of 53 interviews. Interview questions focused on personal background and school experiences, with an emphasis on school mathematics experiences both before and during Math 1. The first set of interviews was conducted during Unit 1, before the first assessment. The second set was conducted during Unit 2. The third set was conducted during Unit 3, and the fourth set, if availability allowed, was conducted in the last week of the semester, before finals. This work focuses on the participants who interviewed at all four time points, amounting to five participants in total.

Coding

Interview data was transcribed using an online transcription service, and the results were transferred into Microsoft Excel for coding. The interview data was separated into time points by participant, and each time point was broken into narrative utterances between interviewer and interviewee. Di Martino and Zan (2010) inform our coding for affect. Drawing on their descriptors for emotion and competence, we assigned codes of emotion (positive or negative) and competence (high or low) to narrative utterances, when applicable. Each time point was reduced to a single descriptor of affect dependent on the frequency of emotion and competence codes: more negative affect, less negative affect, less positive affect, and more positive affect. McAdams and McLean (2013) inform our coding for life story constructs.

Results

Table 1 displays participants' change in affect across a semester of Math 1. Participants' affect across the semester changes in relation to the reform curriculum and their perceived performance in the course. While changes in affect do not necessarily align with changes in perceived performance, emergent life stories seem to coincide with changes in participants' affect and perceived performance. Whereas we identify three types of changes in Table 1, we discuss one in this proposal due to space constraints.

	Entering Affect	Test 1	Affect After Test 1	Test 2	Affect After Test 2	Test 3	Affect After Test 3
Brian	more negative	30	more negative	50	less negative (**redemption**)	80	less positive
Alejandra	more negative	50	negative (**redemption**)	80	less positive	100	more positive
Becky	more positive	86	more positive	84	less positive (**contamination**)	78	less positive
Cici	more negative	32	more negative	34	more negative	56	less negative (**redemption**)
Frank	more negative	100	less positive (**agency**)	75	less positive (**contamination**)	91	less negative

Table 1. Affect and perceived performance across a semester of Math 1

Redemption - Alejandra's Story

Alejandra is the first in her family to attend college. While not required to take Math 1, she enrolled in it due to her more negative affect towards math, in her words, 'to start anew'. Her narrative suggests her negative perception of math developed in high school. Moving into Math 1, she struggles initially, as the course's focus on explanation and reasoning was difficult for her to understand. She explains that she felt lost with her first group. In her second interview, Alejandra describes how she felt during the first test:

> Alejandra: I did feel more confident, after the end. Right, I knew I failed this. After the end I felt like, I don't even know if I could say for certain I did it, wrong or right. Cause I was like, this answer ... you can answer the question, you can answer really simply even with equations or make it as complicated as you want to. At the end of the day, I guess you could say it was a flip of the coin.

In the same interview, she later describes her work with a new group, newly assigned after the first assessment, particularly one member, who has a noticeable impact on her understanding of the concepts.

> Alejandra: I think just surrounding myself with people who understand the material has made me understand that material more. The previous test, I think what happened was my group, we didn't talk about anything, so I was very confused, as were they.

Alejandra suggests the negative experiences of the first test and her first group pushed her to find what was missing for her success in the course. These narrative excerpts alone would not indicate a redemptive story; also necessary was her shift in performance and in her affect in the subsequent interviews. For example, in her fourth interview, she describes the end of her redemptive story:

Alejandra: I think when we were doing the disease population problem, and I was telling people where to move and stuff. And they'd be like, "I don't understand." And I would explain to them and in my head I was like, "You know this, like you're explaining to them. They're not explaining to you."

Alejandra experiences a change in perceived competence, part of her overall shift in affect across the semester. For her, Math 1 was the context for her redemptive story with mathematics.

Discussion

We highlight the theoretical underpinnings of our work as opening up potential new perspectives and approaches to understanding the development and evolution of students' affects towards mathematics. To assess change in affect, we could have used an established instrument with a pre/post design (e.g. the Mathematics Attitudes and Perceptions Survey (Code, et al., 2016)). However, any such instrument is inherently limited in scope: though they might capture the "first order" affects that bear on performance, say, they cannot capture the full lived experience of the students in mathematics. What's more, there are no objectively positive affects in mathematics, but rather productive affects; persistence during "effortful struggle", for example. A life story methodology captures these.

Participants' stories unfolded during a mathematics course that was intended to be a new mathematical experience, different from the traditional curriculum from high school. Our analyses demonstrate transformations of negative affects into productive ways of working with mathematics. This presents an opportunity: curricula can be designed and enacted to afford such productive transformations. The culture of a course can be a culture of change.

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DISCOVERING SQUARE ROOTS: PRODUCTIVE STRUGGLE IN MIDDLE SCHOOL MATHEMATICS

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Using videos and transcripts from a lesson on square roots and follow-up conversations with students and teachers, we analyze factors that can facilitate episodes of productive struggle, including student and teacher dispositions, task features, and classroom conditions. We highlight choices and task features that honored student autonomy and maintained students' engagement in, and success with, the lesson. We also discuss students' equitable access to the mathematics of the lesson.

Keywords: Problem Solving, instructional activities and practices, and communication.

Background and Theoretical Framework

The phenomenon of productive struggle is a relatively new research focus in the study of classroom mathematical activity. Warshauer (2015) characterized productive struggle in terms of students' "effort to make sense of mathematics, to figure something out that is not immediately apparent" (Hiebert & Grouws, 2007, p. 287). Her findings suggest that teacher interactions with students in moments of productive struggle can either maintain or subvert the cognitive demands of tasks, thus strengthening or hindering student learning and understanding.

Sengupta-Irving and Agarwal (2017) discuss the implications of collective effort and decision making in persistence through challenging episodes, as well as indicators of these occurrences that allow educators to facilitate productive peer interactions. Granberg (2016) investigates the role of student observation and analysis of errors and mistakes in turning struggle into a productive event. She relates her definition of unproductive struggle to Schoenfeld's (1985) description of the behavior of novices, in which ideas are not revisited and knowledge is not reconstructed. Zeybek claims that tasks with high-level cognitive demands, specifically those which lend themselves to multiple approaches and allow for more than one correct answer, are essential to students' development of deep understanding (Zeybek, 2016).

In keeping with Harel's Necessity Principle (Harel, 2013), we hypothesize that students' acquisition of mathematical ideas is both more likely and more robust when animated by an intellectual need to understand a situation or solve a problem. Therefore, we conceptualize productive struggle as follows: we say that students are engaged in *productive struggle* when they autonomously attempt to use resources - including their own knowledge, knowledge of their peers and teachers, and physical resources such as technological tools - to overcome an intellectual obstacle, and when this process leads to the discovery or consolidation of a mathematical idea, technique, or problem-solving strategy.

Guided by this framework, we aim to address the question: *What student and teacher dispositions, task features, and classroom conditions are conducive to productive struggle?*

Study Method and Participants

This study was conducted at a two-week summer mathematics program for upper-elementary and middle school students in 2018. Our studied focused on one course which covered concepts of area, perimeter, and the Pythagorean theorem. The course was taught by an inservice seventh grade algebra teacher (Rita) and included seventeen students entering grades 6 and 7, two preservice

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teachers (PSTs), and a doctoral student with experience teaching in special education programs. Sixteen students, Rita, and her three assistants (the two PSTs and the graduate student) participated in our study. Eight students were male and eight were female; all but one student belonged to ethnic minority groups.

The classroom was videotaped at all times, with a second camera used to capture episodes of small-group work. During breaks, we often conducted brief audiotaped interviews with Rita to capture her perspective on activity that had just occurred. After each day's class, Rita and her assistants completed written reflections and participated in a videotaped small-group debrief of the morning's events.

Throughout the program, we observed and recorded instances in which the entire class's activity met our criteria for productive struggle: that students are engaged in a sustained effort to overcome an intellectual obstacle; that students autonomously use resources to overcome the obstacle; and that the effort results in the discovery or consolidation of a mathematical idea, technique, or strategy. In this report, we present our analysis of one such instance, informed by observations from both students and teachers, in order to illuminate characteristics of a lesson and learning environment that can stimulate productive struggle.

Data Analysis

Our analysis of classroom, debrief, and interview transcripts focuses on an episode that occurred during the seventh day of the program. Following an activity on the Pythagorean theorem, Rita presented students with a planned sequence of problems on squares, square roots, and areas and perimeters of rectangles. Each time Rita presented a problem, students were asked to work out the problem on individual whiteboards at their seats, concealing their work until Rita prompted the entire class to reveal their answers. Students' work on the problems revealed some lingering confusion about the distinction between squaring a number and taking the square root of a number and the distinction between the area and the perimeter of a square; Rita and her assistants addressed these confusions as the class progressed through the tasks.

For the last problem in the sequence, Rita drew a picture of a square, labeled the area as "$A = 40$ cm^2", and asked students for the length of a side ("$l = ?$"). Videos of students' work revealed that most students approached this problem by observing that the length must be between 6 and 7 centimeters, then attempting to obtain increasingly precise approximations for the side length by iteratively selecting decimal numbers between 6 and 7, squaring them by hand, and using the result to decide whether to guess a higher or lower value for the side length.

Rita led a brief discussion of side lengths that students had tried so far. This discussion organically reverted to small group and individual work, with students attempting to refine approximations. During this process, a student discovered a bag of four-function calculators (with square root keys) behind the teacher's desk, and Rita encouraged students to use them to continue their work on the problem. The episode closed with a discussion of how the square root function on the calculator could quickly provide an approximation for the side length of the square, and why no terminating decimal value would yield an area of exactly 40.

We selected four students for one-on-one interviews at the end of the lesson based on the key roles they played in the development of the episode. During interviews, students had access to rulers, four-function calculators with square root keys, grid paper, and sheets of blank copy paper, as well as their personal electronic devices, when they were available.

In reviewing classroom and interview data, we conducted a thematic analysis (Braun & Clarke, 2006), coding themes in classroom discourse and in students' and teachers' descriptions of the episode that corresponded to elements of our framework for productive struggle or suggested factors that contributed to the episode's development. We organized our analysis according to the extended instructional triangle of Boerst and Ball (2007), which suggests we may gain insight by analyzing the work of students, work of teachers, mathematics in which students and teachers are engaged, and learning environment. We analyzed video and transcripts of the classroom episode, and then triangulated our observations with insights from our interview with Rita and the post-class debrief. We summarize the themes we identified in Table 1.

Table 1: Factors Supporting Productive Struggle in the Square of Area 40 Task

Student Factors	**Teacher Factors**
Curiosity about the value of an unknown quantity* Willingness to persist through lengthy calculations* Disposition to use tools strategically*+ Approximation sense: understanding when a guess is too high or too low; deciding when to move closer to one endpoint of an interval	Commitment to growth mindset (shared by Rita and assistants)* Commitment to probing rather than directing student thinking Disposition to enlist students in addressing each other's thinking Disposition to allow students to use technology to make work more efficient* Strategies for sustaining student motivation and engagement
Task Factors	**Learning Environment Factors**
Problem was "not too easy, not too hard"+ Problem came at the end of a sequence of tasks that gradually increased in difficulty* Problem addressed students' conceptions of perimeter and area* Problem rewarded sustained effort with successively more accurate approximations	Calculators were available in room, but not immediately visible* Classroom norms permitted students to move about the room and share ideas and approximations with each other
* Indicates an observation supported by teachers' comments during the interview or debrief + Indicates an observation supported by students' comments during post-task interviews	

Based on our observations as well as those of Rita and the assistants, the children's work on the Square of Area 40 task satisfied our criteria for productive struggle. Their work was productive in that it created learning opportunities that were realized in the collaboration among students and teachers: the co-construction of a process for interpolating a solution x to the equation $x^2 = a$, where a is a given real number, the consolidation of this process into the notion of the square root, and a discussion of how a calculator rounds the results of calculations and can appear to give a whole number output, even when the true value is not a whole number.

Teachers' discussions after the lesson revealed several student dispositions that contributed to the lesson's success in stimulating productive struggle. Foremost among these was genuine curiosity and excitement about the problem situation, as the doctoral student pointed out during the post-class debrief: "[the] question created so much engagement … intense discussion, action, like everybody's running around, digging in to get that exact number." Another factor was the students' willingness to persist through laborious pencil-and-paper calculations as they tested their guesses. During the episode, several students tested guesses that had three significant digits of precision. In the post-class debrief, the doctoral student recalled that "they were just … very excited, trying to be precise." Also important, in our own observation, was the students' strategic knowledge for approximation. For example, at least one student observed that since 40 was closer to 36 than to 49, her first guess for the side length should be closer to 6 than to 7. Students' approximation sense helped them obtain more accurate guesses for the square's side length, allowing them to obtain areas closer to 40 and bolstering their motivation to continue.

Both Rita and one of the PSTs spoke about the importance of helping students interpret struggle and temporary adversity through a growth mindset, so that students were not discouraged by temporary setbacks. Several choices by Rita and her colleagues bolstered students' sense of mathematical agency, and their disposition to encourage students to help one another through obstacles. This included heavy use of probing questions and tendencies to frequently have students describe their work in detail. We hypothesize that such facilitation moves by teachers can help students reinforce the conceptual content of a lesson for themselves and others, and allows them to enlist their peers as resources in pursuing a shared strategy.

Characteristics of the classroom environment also played a role in the episode's development. In their initial work on the task, students performed calculations by hand. Gradually, some students began to use the calculator functions of their phones to explore more efficiently; some students noticed this and questioned whether the use of calculators was contrary to class norms. The initial lack of calculators encouraged students to perform calculations by hand, which helped them become accustomed to the parameters and goals of the problem. However, the availability of calculators to only students with personal devices threatened to limit some students' access to the full depth of the task. The class set of calculators helped to resolve this, though some students still had access to more precise approximations.

Technology played an additional role in the resolution of the task: one student discovered that when she squared one of her estimates on her phone, the device reported that the square was 40. However, switching the phone to landscape orientation revealed more digits, showing that the value was equal to 40 to only thirteen decimal places. This helped the student to see that the device was automatically rounding values, even when it appeared to report a whole-number answer. Rita, noticing this insight, asked the student to share it with the entire class, again helping one student's access to the mathematics in the task become a resource for others.

Discussion

Granberg (2016) posits that when students are prematurely offered procedures for solving mathematics problems, the intellectual challenge of problem solving is lost, and students' inquiry into the nature of such problems is cut short. The Square of Area 40 task led to an episode of productive struggle in which students refined their conceptions of squares and square roots in an exploratory way without being guided to use a particular procedure or strategy. We hypothesize that the autonomy afforded to students during this activity strengthened their sense of agency as well as their understanding of the nature and purpose of the task. This sense of autonomy was a result of choices made by both students and teachers during the activity.

We hypothesize that the conditions we have described - student and teacher dispositions, task features, and classroom conditions - can be tailored to create a learning environment which facilitates productive struggle. These factors can further be considered in bringing productive struggle from a thing we strive to achieve into something that can be operationalized in creation of tasks to be facilitated in the research environment. However, some care is necessary to ensure that the benefits of productive struggle accrue to all students and that the knowledge and resources enjoyed by some students are marshalled for the benefit of the entire class. In orchestrating the lesson, Rita and her colleagues explicitly reinforced productive processing of adversity, encouraging students when they made mistakes or when their calculations did not turn out as anticipated. They also noticed some of the insights available to students with personal handheld devices and worked to draw other students into discussions about these insights. By attending to issues of equity, the teachers enhanced the lesson's effectiveness and deepened students' opportunities for learning.

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EVERYDAY EXPERIENCES AND SCHOOL KNOWLEDGE OF MATHEMATICS. AN ENACTIVE APPROACH

EXPERIENCIAS COTIDIANAS Y CONOCIMIENTOS ESCOLARES DE MATEMÁTICAS. UNA APROXIMACIÓN ENACTIVISTA

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This text details the advances of a study based on the enactivist perspective of knowledge embodiment (Maturana and Varela, 1984; Varela, 2000). The purpose is to document the current knowledge about number, space and measuring, of children in a multigrade primary school located on a coffee plantation, where they also live and coexist with daily coffee-related activities. The recorded repertoire of these children's enacted learnings will be the foundation for teachers to propose situations that can help retrieve knowledge obtained beyond classrooms, through a corporal and sensitive experience. As Varela argues, this approach challenges the common classroom ethos and calls for a transformation of the research community's modus and values.

Keywords: Elementary Education, Embodiment and Gestures Studies, Rural Education, Concept of Number and Calculations.

Evidence derived from own experience and formal research has shown for years that children build knowledge (Resnick, 1989; Resnick & Greeno, 1990), which in most cases is not related to formal school learnings. This can be observed particularly among the contents in the field of mathematics prescribed by formal education and the skills that are acquired outside of it (Nunes, T., 1993; Padilla, 2015). In cases where out-of-school knowledge is documented, a distinction is made between this kind of knowledge and the one school code considers relevant to teach or learn within its boundaries. The idea that "real math" is learned only by interacting and appropriating the "true mathematical objects" presented by study programs, seems to underlie.

Our consideration, grounded on the ideas of the theory of enactivism, is that direct experience is a significant source of knowledge; that in the exchanges we experience via our actions, with and through our body, we produce experiences that may enhance or limit our conceptual understanding, while this last one also hinges on the different cognitive domains that we reach progressively.

Thus, girls, boys, youths, adults… we all build explanations and procedures which, from the formal perspective of the school institution, may be considered insufficient or not compatible with the criteria of "appropriate behavior", even if they constitute initial, provisional and effective explanations. When juxtaposed to time constraints, fragmented content, grading, planning, prescriptions, study programs, textbooks, and other formal devices that require attention at each school year, the students' knowledge, intuitions, incomplete procedures, affections, emotions, and interactions are interpreted as obstacles unrelated to school activity -more so with respect to mathematical content- and thus they are often avoided. Taking this into account, we assert that learning through concrete action is an approach of greater complexity, validity, and tradition.

The studies carried out by Piaget and his collaborators display how cognitive structures arise from recurring action patterns with a perceptual basis. Lozano, when referring to the enactivist approach, states: "The activities we carry out with different objects when interacting with the world result in cognitive structures such as mathematical concepts and categories" (2014:170). Knowing through concrete action is applicable to different fields, particularly in the field of mathematics as it is held in this research.

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Purposes of the study

Identify and document -from an enactivist point of view- the understanding of numbers, space, and measure created by girls and boys from their own corporeal and interactive experience with their environment.

Work with the teachers of these children to build frameworks that integrate knowledge built from the corporeal, sensitive and interactive experience, linking it with the mathematical knowledge prescribed by the school while also making this last one more flexible.

Theoretical perspective

Even though the field of mathematical education is assisted by numerous theoretical perspectives that aid in analyzing what occurs inside and outside the classroom with respect to teaching processes and mathematical content learning, for our work we chose a theoretical reference based on the enactivist perspective (Maturana and Varela, 1984; Varela, 2000), which sustains that knowledge is built from the actions that we carry out in our daily experience and by the particular constitution of each individual, in a specific social and cultural context. This perspective of constructivist orientation conceives cognition in a broad sense, and therefore recognizes and addresses other dimensions -such as emotion, affection, and social interaction- in and for learning.

Lozano (2014:179) synthesizes substantial aspects of this approach: "Through enactivist ideas, individual learning is reconciled with social interaction, mind with body, reason with emotion, and knowledge with mathematical knowledge. Its concepts, as I understand them, allow for a broader perspective on learning and teaching mathematics, while taking its complexity into account ".

Enactivism, is separated from mind-body dualism and offers a theoretical perspective that coincides with what our experience and research work has shown us: knowing entails interacting with the world, starting from the individual story of each person and their context. Therefore, learning can only be conceived from the multiplicity of factors coexisting within it. Although more complex, this approach encompasses substantial dimensions that are omitted by other perspectives. Its understanding of perception, the fact that it does not recognize representations as a product of cognition, the relevance given to intuition, and its consideration of common sense (Varela, 2000:275), are significant disruptions that expand the perspectives regarding cognition understanding and analysis, and set it apart from the compartmentalized, formal, rigid and "disembodied" knowledge. We ponder that this approach will facilitate the creation of alternatives for students to regain interest in school mathematics content and in the knowledge about this subject produced by themselves.

Research method

Our work is a qualitative research, which process is thought of as a participant training space. The researcher joins as an apprentice and collaborator in both school and community activities, as well as with the teacher, the parents, and students. Consequently, there is no room for "observers", as the enactive approach holds that, as cognitive agents, we are active participants who enact in the world.

Context

The farm where the study is carried out, like others in the area, is located in a mountainous region of the state of Chiapas; the natural environment is exuberant, located in a rural area only connected to urban settlements through public transportation options that run a limited number of times a day. No public transportation gets directly to the farm. Internet connection is unstable to nil. Housing for workers and their families are equipped with basic drinking water services, electricity, a sanitary network, cement flooring. Health and education services are present. The elementary school is multigrade as a result of the limited number of students, while preschool is monograde.

Data Sources

Both the in and out-of-school activities deployed by the children of the farm constitute the sources. Recreational scenarios are designed to promote varied interactions where enacted knowledge can be deployed. Multilayered and semi-structured interviews are also carried out.

Analysis

We work through triangulation, with the purpose of complementing and enriching information collected through different activities and people. The criteria for analysis are in process of elaboration since they are put together from the previously mentioned scenarios.

Results

Up until now (February 2020) two visits have been made to the farm cumulating a total of three weeks living and sharing with the community -with the children, in particular- from Monday to Sunday. The following visits are designed to enhance participation and gradually foster an environment of confidence. Through these stays, the development of a wider perspective on the dynamics, roles, activities, needs, and complex knowledge required for the cultivation and harvest (*tapisca*) of coffee, as well as for orientation, recognizing harmful and edible plants -to name just a few domains- has been possible. Records of interviews and informal talks are being reviewed.

Discussion

We think it is fundamental that, since cognition is intimately related to action and actions occur in specific settings, learning has to be investigated in accordance to the situation in which it occurs. Knowledge in a specific location is associated with appropriate behavior or effective action in that location (Maturana, 1987:66). We believe that this perspective is promising for discovering new knowledge and that, through its formation, the active experience of the students can be brought out.

The conditions of multigrade groups are commonly perceived as deficient as they are associated with isolated communities with limited resources and even with indigenous populations. However, our approach highlights the opportunities that a multigrade classroom provides for the creation of knowledge in an enactive and collaborative environment. Associating age with school grade and defining the contents applicable to that entire group of individuals through it is not that reasonable. Even more so considering all the evidence that supports the case for making the students' approach to any knowledge more flexible. Multigrade, and its wide possibilities as a learning context, is a promissory area with great potential which, we believe, can be successfully articulated to the enactive approach.

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Se reportan los primeros avances de un estudio sustentado en la perspectiva enactivista, de corporización del conocimiento (Maturana y Varela, 1984; Varela, 2000). El propósito es documentar los conocimientos sobre número, espacio y medida, que tienen niñas y niños de una escuela primaria multigrado enclavada en una finca cafetalera, donde residen y conviven con la actividad laboral. El repertorio documentado de conocimientos enactados de esos niños, sustentará el acompañamiento para que sus maestros propongan situaciones que recuperen los conocimientos construidos más allá del salón de clase, mediante la experiencia corpórea y sensitiva. Este enfoque constituye, como sostiene Varela, un desafío a la cultura del salón de clases, y convoca a transformar el estilo y valores de la comunidad de investigadores.

Palabras clave: Educación primaria, Estudios de embodiment y gestos, Educación Rural, Concepto de Número y Operaciones.

Evidencias derivadas de la propia experiencia, así como otras basadas en procesos de investigación formal, han mostrado desde hace años que las y los niños construyen conocimientos (Resnick, 1989; Resnick & Greeno, 1990), que, en la mayoría de los casos, no tienen cabida en la escuela. En particular, esta situación se observa entre los contenidos de matemáticas prescritos en la educación formal y aquellos conocimientos que se aprenden fuera de ella (Nunes, T., 1993; Padilla, 2015). En los casos en que se identifica el conocimiento surgido más allá de la escuela, se establece una distinción entre ese conocimiento y lo que la cultura escolar considera relevante de enseñar o aprender en su territorio. Al parecer, subyace la idea de que las “verdaderas matemáticas” se aprenden al interactuar y apropiarse de los “verdaderos objetos matemáticos”, incluidos en los programas de estudio.

Nuestra consideración, sustentada en los planteamientos de la teoría de la enactividad, es que la experiencia directa es una importante fuente de conocimientos; que en el intercambio que tenemos con el mundo, a partir de nuestras acciones con y a través de nuestro cuerpo, producimos experiencias que posibilitan o limitan nuestra comprensión conceptual que, a su vez, depende de los diferentes dominios cognitivos que progresivamente alcanzamos.

Así, niñas, niños, jóvenes, adultos, todos, construimos explicaciones y procedimientos, que, desde la perspectiva formal de la institución escolar, pueden considerarse insuficientes, o no cumplen los criterios de un “comportamiento adecuado”, aunque para quien los genera constituyen explicaciones iniciales provisorias y efectivas. Frente a prescripciones de tiempo, contenidos fragmentados, graduados, planeaciones, procedimientos, programas de estudio, libros de texto y otros dispositivos formales que requieren cubrirse durante cada ciclo escolar, los conocimientos de los estudiantes, sus intuiciones, procedimientos incompletos, afectos, emociones e interacciones, particularmente respecto de contenidos matemáticos, se interpretan como irruptores, poco pertinentes a la actividad escolar, por lo que es preferible obviarlos. Ante esa situación, sostenemos que conocer a través de la acción concreta, es un planteamiento cada vez de mayor complejidad, vigencia y larga tradición.

Los estudios realizados por Piaget y sus colaboradores, mostraron en su momento cómo las estructuras cognitivas surgen de pautas recurrentes de acciones, con una base perceptual. Lozano, al referirse a la postura enactivista, plantea: “Las actividades que llevamos a cabo sobre los objetos al interactuar con el mundo dan lugar a estructuras cognitivas tales como los conceptos y categorías matemáticas” (2014:170). Conocer a través de la acción concreta, aplica a diferentes campos, particularmente en el campo de las matemáticas como se sostiene en este estudio.

Propósitos del estudio

Identificar y documentar desde un enfoque enactivo, los conocimientos sobre número, espacio y medida, que construyen niñas y niños a partir de su experiencia corporal e interaccional con su entorno.

Generar con los profesores de esos niños, andamiajes que incorporen los conocimientos que ellos han construidos a partir de la experiencia corpórea, sensitiva e interaccional, vinculando y flexibilizando el conocimiento matemático que se prescribe en la escuela.

Perspectiva teórica

Si bien en el campo de la educación matemática existen numerosas perspectivas teóricas para analizar lo que sucede dentro y fuera del aula con respecto a procesos de enseñanza y aprendizaje de contenidos matemáticos, elegimos como referente teórico la perspectiva enactivista (Maturana y Varela, 1984; Varela, 2000), que sostiene que el conocimiento surge de las acciones que realizamos en nuestra experiencia cotidiana y por la constitución particular de cada individuo, en un contexto social y cultural específico. Esta perspectiva, de referentes constructivistas, concibe la cognición en un sentido amplio, por lo que reconoce y aborda otras dimensiones -como la emoción, el afecto y la interacción social- en el y para el aprendizaje.

Lozano (2014:179) sintetiza aspectos sustanciales del enfoque: “A través de las ideas enactivistas, el aprendizaje individual se reconcilia con la interacción social, el cuerpo con la mente, la razón con la emoción y el conocer con el conocimiento matemático. Sus conceptos, desde mi punto de vista, permiten tener una perspectiva amplia acerca del aprendizaje y la enseñanza de las matemáticas, tomando en cuenta su complejidad.”

El enactivismo, se deslinda del dualismo mente-cuerpo y ofrece una perspectiva teórica que coincide con lo que la experiencia y el trabajo de investigación nos ha expuesto: conocer implica interactuar con el mundo, desde la historia de cada persona y su contexto, por lo cual el aprendizaje no puede concebirse sino a partir de la multiplicidad de factores que en él concurren. Aunque resulta más compleja, esta aproximación incorpora dimensiones sustanciales, que otras posturas omiten. Su concepción de percepción, el no reconocer a las representaciones como producto de la cognición, el dar relevancia a la intuición y ponderar del sentido común (Varela, 2000:275), son rupturas importantes que amplían perspectivas para la comprensión y análisis de la cognición, así como para diferenciarse de la concepción de conocimiento compartamentalizado, formal, rígido y “desencarnado”. Consideramos que esta aproximación contribuirá a generar alternativas para que los alumnos recuperen el interés por los contenidos escolares de matemáticas y por los conocimientos que sobre el tema, ellos mismos construyen.

Método de investigación

Se trata de una investigación de corte cualitativo, cuyo proceso se prevé como espacio de formación para los participantes. Quien investiga se incorpora como aprendiz y colaborador en las acciones de la escuela y la comunidad, con el maestro, padres y los alumnos. Siguiendo el enfoque de esta investigación, no hay cabida a "observadores", ya que la perspectiva enactiva sostiene que los agentes cognitivos somos activos participantes que enactamos en el mundo.

Contexto

La finca donde se lleva a cabo el estudio como otras de la zona, se encuentra en el área de montaña del estado de Chiapas; el entorno natural es exhuberante. En un área rural, comunicada con asentamientos urbanos a través de varias opciones de transporte público terrestres, en ciertos horarios del día, sin embargo, ningún transporte colectivo llega directamente a la finca. La conexión a internet, es de inestable a nula. En esa finca las viviendas para los trabajadores y sus familias, cuentan con servicios básicos de agua potable, electricidad, red sanitaria, piso de cemento, servicio de salud, y educativo. Por el número de alumnos, la escuela primaria es multigrado y la de educación preescolar, unitaria.

Fuentes de datos

Las diferentes actividades escolares o desplegadas en otros escenarios, por los niños de la finca, constituyen las fuentes. Se diseñan situaciones lúdicas para promover interacciones variadas donde sea posible desplegar conocimientos enactados. También se realizan enttrevistas a profundida y semiestructuuradas.

Análisis

Se trabaja mediante triangulación, con el propósito de complementar y enriquecer información recabada a través de diferentes actividades y personas. Los criterios para un análisis más fino, están en proceso de elaboración, ya que se construyen a partir de las situaciones que se generan.

Resultados

A la fecha (febrero 2020) se han realizado dos estancias en la Finca, un total de 3 semanas conviviendo con las personas de lunes a domingo, con las niñas y niños. Las siguientes estancias, presentando, ante la familia, participar e ir construir paulatinamente un clima de confianza. De ambas estancias se ha podido tener un panorama más amplio de las dinámicas, roles, actividades, exigencias y conocimientos complejos, que demanda el trabajo de cultivo y cosecha (tapisca) de café. Así como de los múltiples referentes que ponen en juego para orientarse, reconocer plantas dañinas y comestibles, por citar solo algunos dominios. Se cuenta con registros de entrevistas y pláticas informales que esttán en proceso de revisión.

Discusión

Consideramos fundamental que, dado que la cognición está íntimamente relacionada con la acción, y las acciones ocurren en lugares determinados, entonces el aprendizaje tiene que ser investigado en relación a la situación en la que ocurre. El conocimiento, en un lugar determinado, está asociado con la conducta adecuada o la acción efectiva en ese lugar (Maturana, 1987:66). Consideramos que esta persspectiva es promisoria para develar conocimientos y que mediante su construcción propiciar que se despliegue la experiencia activa de los estudiantes.

La condición de trabajo de los grupos multigrado en muchos espacios se valora como deficitaria porque se asocia a comunidades aisladas, con recursos limitados e incluso con población indígena. Sin embargo, nuestra postura reivindica las condiciones que ofrece un aula multigrado para la construcción de conocimientos de forma enactiva y colaborativa. Asociar una edad a un grado escolar y suponer por ello que eso determina los contenidos pertienentes para toda esa franja de individuos, es poco sostenible. Aún más si se cuenta con tanta evidencia que argumenta a favor de flexibilizar el acercamiento de los estudiantes a cualquier conocimiento. Multigrado, y sus amplias posibilidades como contexto de aprendizaje, es una línea promisoria y con gran potencial, que perfilamos, puede articularse con éxito al enfoque enactivo.

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MOVING TOWARDS MEANING MAKING IN MULTIPLICATION: A PRELIMINARY REPORT OF AN INTERVENTION IN NUMBER SENSE

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Rejecting pedagogies of deficit built around deficit mythologies about the mathematical potential of students with learning disabilities (Lambert 2018), this study will document an intervention designed to engage students in mathematical problem-solving and discussion, building student computational skills as well as number sense and participation in the mathematical practices. In this paper, we provide close analysis of the development of a fourth-grade student who demonstrated growth in participation and conceptual understanding of multiplication across the intervention.

Keywords: Equity and Diversity; Special Education; Number Concepts and Operations; Instructional Activities and Practices.

Research on the mathematical learning of students with disabilities has focused on medical deficits within children and prescribed behavioral mathematics teaching to remediate these deficits (Lambert & Tan, 2020). Locating the problem not within individual students, but within limited access to opportunities to make mathematical meaning, we designed an intervention for students with disabilities ages 8 through 11 significantly underperforming in multiplication and division and including students with disabilities. This intervention is designed to engage students in mathematical problem-solving and discussion, building both student number sense and participation in the mathematical practices. Our full study explores the growth of student strategic thinking, accuracy for multiplication and division, and participation in mathematical discourse. In this brief report, we focus on one student whose measurable math score did not increase during and after the intervention. While she displayed no measurable growth in her math score, she displayed growth in participation and strategy development. Our research question for this brief report focused on one student: How did one student participate in a mathematics intervention designed to promote student meaning-making and discussion? What shifts in participation are in evidence? What shifts in conceptual and procedural understanding of multiplication?

Schools in the US are being asked to provide intervention within Multi-Tiered System of Support (MTSS) in mathematics. However, interventions are often not aligned with classroom instruction based on Common Core State Standards, creating difficulties for students who must make sense of different approaches to mathematics. Our intervention using number strings is designed to align with a focus on meaning making in the curriculum. The intervention consisted of 8 sessions of number strings (Lambert, Imm & Williams, 2107) designed and facilitated by undergraduate tutors after 6 hours of professional development led by the first author. Each tutor was observed 2-4 times by a member of the research team and offered feedback. In addition, all tutors participated in a session in which they analyzed the participation of the students in their small group. Future analysis will focus on the teaching moves of the novice tutors.

A number string is a short (15–20-minute) daily instructional routine in which a teacher presents a carefully designed sequence of problems one at a time for children to solve mentally (Lambert, Imm & Williams, 2017). Instead of interventions that focus on direct instruction, number strings provide opportunities for students to engage in mathematical discourse, both in describing their own strategies and connecting with the mathematical strategies of others. Research on number strings has found that students participating in number string routines are able to adopt new strategies (O'Loughlin 2007) and make connections between conceptual understanding and procedures. Studies

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on number strings have not previously focused on students with disabilities or students who are significantly underperforming in mathematics.

One issue in the assessment of students with disabilities is the difficulty of capturing growth that may be unusual in its learning trajectory or slower than peers. Van Geert and van Dijk (2012) describes the importance of moving beyond group-level data in understanding the variability for students that may exist at the level of strategies and engagement, recommending collecting data on inter-individual variability to better understand strategic change. We document conceptual growth through analysis of student participation and discourse, including attention to non-verbal communication. We used aspects of the coding scheme by Ing at al. (2015). A Complete Share was an answer that was accurate and explained in enough detail that we could confidently code the strategy. A Partial Share was either inaccurate or did not include enough detail that researchers could determine the exact strategy path of the student. We added the last two categories to track students who had nonverbal engagement in the problem. Nonverbal captured moments in which we could see evidence of nonverbal engagement, yet students did not verbally share in discussion (such as students counting on fingers). No Engagement was coded if the student did not demonstrate verbal or nonverbal engagement.

Methods

The study was situated in grades 3 – 5 at an elementary school in California. Demographics are as follows: 76.3% are Socioeconomically Disadvantaged, 14.4% are Students with Disabilities, 58.9% are English Learners, and 9.9% of students are Homeless. The majority of students at the school are Hispanic (88.3%) with the second largest demographic category being White students (7.5%). The full study included 12 student participants in 3rd grade, 6 students in 4th grade, and 18 students in 5th grade. 12 students had current IEPs, with an addition 4 students in the referral process. Each group met for 8 sessions, twice a week for 4 weeks. We collected two primary kinds of data: a) researcher-created multiplication and division paper and pencil assessments (Multiplication + Division CCSS CBM Math Assessment) with all students taking the assessment three times (pre, during and post intervention), and b) video records of the tutors teaching the number strings to document student participation and strategy development. In order to ascertain growth in student accuracy, we scored the MD-CBM before, during and after the intervention. After analysis of the first MD-CBM assessment, the first researcher met with the classroom teachers to decide the students who would be placed into the Tier 2 intervention. We assessed student use of strategies and participation in mathematical problem-solving and discussion through analysis of transcripts. Two authors each coded the small group we present in this paper, resolving any discrepancies. We will determine intercoder reliability for the final paper.

Findings

This paper is a case study that focuses on one student (Inez) within one small group of 6 students in a fourth-grade class taught by undergraduate tutor Yola (all names are pseudonyms). Comprised of students with and without disabilities, the students in this small group had the lowest scores on multiplication in their class. Inez is Latina and classified as an English Learner, as well as a student of significant concern for her classroom teacher. The teacher noted that Inez rarely shared in math class and seemed to have significant issues with number sense. Inez appeared eager to participate in this small group, even when she did not share. She seemed particularly to enjoy talking to Yola.

In the first two sessions, Inez did not volunteer to answer questions. She shared twice when called on. Unlike her peers, she did not use her fingers to keep track as she skip counted. Instead, we could see her subvocalizing her counting and losing track. In discussions, we wondered if Inez needed support to help her keep track of her count. Starting in the second session, Yola passed out card stock arrays for the students. Inez began using these arrays to keep track of her counting. However, Inez seemed to need additional experience with the arrays. Initially, Inez counted each square as 2. For much of the first 6 sessions, Inez demonstrated her ability to count by 2s and 5s, but not other numbers. Inez appeared to prefer counting by 2s and 5s so much that she used this strategy to solve problems that neither 2 or 5 were factors. Asked to solve 9 x 5, she got the answer of 10 by skip counting by 2s, because "I thought it would be easier to count by twos." In discussions with Yola, we decided at the end of Session 2 that Inez needed to sit closer to Yola, who supported her in using her fingers or the array as a tool. Yola also spoke to Inez during turn and talks, which seemed to support Inez sharing in the small group. This shift seemed to mark a pronounced difference in engagement (Table 1) from a lack of engagement in the first two problems to a more sustained engagement in mathematical discussion in the subsequent sessions.

Table 1: Shifts in Inez's Participation in Mathematical Discussion

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Complete Share	0	0	1	1	0	1	1
Partial Share	1	1	4	4	6	5	3
Nonverbal	0	3	0	2	0	0	1
No Engagement	5	1	1	0	0	0	0

Through close analysis of Inez's strategies across the 7 sessions (one session was not video recorded), we saw evidence that Inez developed her understanding of arrays. While in the beginning she did not count squares by ones successfully (by counting boxes as 2s or 5s), she was able to do so by Session 5. She also developed an understanding of the connection between skip counting and multiplication by groups. Twice, Yola represented Inez's skip counting numerically and connected that to the representation of the array. The first time Yola did so, Inez stopped, stared at the array and the skip counting represented next to it, and said, "What the heck?" The next session, Inez again counted an array by 2s, and then miscounted, getting an answer of 62 for 6 x 5. Yola listened to Inez's strategy, and then remodeled it on the array keeping track of the numbers. Inez again appeared to be provoked into disequilibrium by the tutor's representation of her strategy, saying, "I went really really far." Connecting visual and numerical representations of her own strategy appeared to make Inez's own thinking visible to her, thus allowing her to understand her own thinking as reflected by the tutor's representations.

While Yola seemed to make supportive moves to increase Inez's participation, as well as to model her thinking to make it visible, Yola described having significant difficulty understanding and representing Inez's strategies. While there were instances in which Yola pressed for explanation, there were more instances in which Inez shared an incorrect answer and Yola did not ask her to elaborate. In further analysis, we will determine which teacher moves within the number strings routine were most challenging for novices to enact. We suspect that pressing a student for further explanation when that student has a pattern of strategies that do not make sense to the teacher might be a particularly challenging teaching move to enact.

Discussion

Our intervention aims to increase the mathematics achievement of students with disabilities and students whose performance is significantly below grade level, but not with instructional practices that focus on memorization or procedural learning. Instead, we investigated the use of a number string to develop multiplication and division computation simultaneously with number sense. In this paper, we demonstrate how one student significantly below grade level in mathematics grew in her use of mathematical strategies and her engagement through participation in a number string routine.

Further analysis will include latent class modeling to determine growth patterns for particular subgroups of students. We also will document overall growth, using group averages as well as close qualitative analysis of strategy growth. Finally, we will analyze the teaching moves of the novice teachers to determine the effectiveness of the professional development we provided for tutors. We also plan to analyze how Inez's emergent bilingual status could better have been leveraged in her learning. Most importantly, we seek to better understand how to provide mathematics intervention for students who need more support engaging in meaningful mathematics.

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GENDER DIFFERENCES IN ATTITUDES TOWARDS MATHEMATICS AND STEM MAJOR CHOICE: THE IMPORTANCE OF MATHEMATICS IDENTITY

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The building of a diverse and highly capable population of young people for careers in the science, engineering, technology, and mathematics (STEM) fields remains a critical issue in the United States. The researchers employed data from a nationally representative sample of high schoolers to better understand the relationships between students' learning experiences, attitudes towards mathematics, and STEM major choice. The focus of this paper is on how these relationships differ for males and females. The findings suggest that the underrepresentation of female students in STEM majors can be partially explained by a tendency for females to have less positive attitudes towards mathematics as compared to their male counterparts. Mathematics identity may be the most important attitude explaining this difference.

Keywords: STEM, Attitudes, Equity and Diversity, Gender

This research is part of a larger project aimed at better understanding factors that motivate U.S. students' decision to major in STEM fields and ultimately guiding efforts for broadening participation in STEM. The focus of this paper is on gender differences in attitudes towards mathematics and how these differences can help to explain why female students are underrepresented in STEM majors. Expectancy-value theory and data from the High School Longitudinal Study of 2009 were employed.

Theoretical Framework

The expectancy-value model of motivated behavioral choice (Eccles, 2009) holds that students' achievement-related choices (such as college major choices) are directly determined by the expectancy for success and the subjective value they attach to the tasks involved in those choices. Expectancy for success is similar to the notion of *self-efficacy* (Wigfield & Eccles, 1992). Subjective task values include the following three aspects: the relation of the task to one's self-image (*identity value*); the anticipated enjoyment from engaging in the task (*interest value*); and the perceived usefulness of the task for fulfilling personal goals (*utility value*). *Cost* is a fourth aspect of value that will not be examined in this study. Expectancies and values themselves are determined by various factors, including personal background characteristics and past learning experiences (Eccles et al., 1983). This model guides the research questions for the current study: (a) How are U.S. high school students' prior educational experiences, mathematics expectancy-value attitudes, and STEM major choice related? (b) How do these relationships differ for males and females?

Methodology

To answer these questions the present study employed data from the High School Longitudinal Study of 2009 (HSLS:09; Ingles et al., 2011). HSLS:09 is the most recent in a series of surveys administered by the National Center for Education Statistics (NCES) that follow nationally representative samples of young people as they transition from high school to postsecondary years. The first wave of HSLS:09's data collection began in the fall of 2009 with over 23,000 ninth-graders from 944 public and private schools throughout the United States. Sampling involved a complex, two-stage design in which eligible schools were first randomly selected and then students within those schools were randomly selected (Ingles et al., 2011). The students were followed up in the spring of 2011, when most were in the eleventh grade (Ingles et al., 2014) and a third time in 2013, after most had completed high school (Ingles et al., 2015).

Variables

The present study utilized variables from the base-year, first follow-up, and 2013 update of HSLS:09's public-use data file. Ninth-grade variables included mathematics achievement, STEM extracurriculars, and mathematics teacher support. Eleventh-grade variables included mathematics expectancy-value attitudes. The twelfth grade-variable was STEM major choice.

STEM major choice. STEM major choice was measured with a dichotomous variable indicating whether the student was planning to major in a STEM field at a postsecondary institution (reported in 2013 college update). Majors considered STEM included computer, physical, and natural sciences; engineering; mathematics and statistics; and military and science technologies/technicians.

Mathematics self-efficacy. Four items were used to measure students' confidence in their mathematics ability, including the degree to which they were confident that they can do an excellent job on tests and assignments; understand the most difficult material presented in the textbook; and master skills in their first follow-up mathematics course. All mathematics attitudes were standardized (mean = 0, SD = 1).

Mathematics identity. Two items were used to measure students' mathematics identity value, including the degree to which they agreed that they were a math person and other people saw them as a math person.

Mathematics interest. Five items were used to measure students' mathematics interest value, including the degree to which they agreed that their first follow-up mathematics course was enjoyable, a waste of time (reverse coded), and boring (reverse coded); that they were taking the classes because they enjoy math; and whether math was their favorite school subject.

Mathematics utility. Three items were used to measure student's mathematics utility value, including the degree to which they agreed that what they were learning in their mathematics course would be useful for everyday life, college, and a future career.

Mathematics achievement. Prior achievement in mathematics was measured using the ninth-grade algebraic reasoning assessment score. The assessment was developed by NCES using an item-response theory (IRT) design and standardized (mean = 0, SD = 1).

STEM extracurriculars. Extracurricular participating in STEM-related activities were measured by the number of mathematics or science related activities (clubs, summer camps, competitions, study groups) the student reported participated in from the base-year to the first follow-up.

Mathematics teacher support. Nine items were used to measure students' perceived support from their ninth-grade mathematics teacher. Some example items include the degree to which the student agreed that their teacher values and listens to students' ideas; thinks every student can be successful; and makes math interesting. This scale was also standardized.

Covariates. Covariates included student's gender (1 = female, 0 = male), race/ethnicity (1 = underrepresented minority [URM], 0 = white or Asian), and socioeconomic status.

Missing Data

Missing data were handled using multiple imputation. In total, five datasets were imputed for the analysis using Blimp 1.1 (Keller & Enders, 2018). The five datasets were analyzed and then the estimates and standard errors were averaged into a single set of results (Rubin, 1987). Multiple imputation has been shown to be robust against departures from normality and to provide unbiased results even for high rates of missing data (Enders, Keller, & Levy, 2018).

Analytic Plan

To analyze the relationship between the variables, the researchers used structural equation modeling (SEM; Byrne, 2011). The analysis was conducted in Mplus 8.2 with robust maximum likelihood (MLR) estimator and logit link. (Muthén & Muthén, 1998-2017). Figure 1 displays the path model. To asses model fit, the chi-square statistic (χ^2), comparative fit index (CFI), Tucker–Lewis index (TLI), and the root mean square error of approximation (RMSEA) were used. SEM literature typically considers CFI and TLI values greater than .95 and .90 to indicate excellent and acceptable fits, respectively. For RMSEA, values less than .05 and .08 are considered excellent and acceptable fits, respectively (Byrne, 2011). Design effect adjusted weights were applied to account for the nested structure of the survey (Hahs-Vaughn, 2005). As per Ingles et al. (2015), the raw weight adjusted was W3W1W2STU.

Results

The sample was comprised of 50.4 percent male and 49.6 percent female. The proportion of students who pursued STEM was 14.7 percent. Males chose STEM at a 19.4 percent rate compared to 9.9 percent for females. Whites and Asians chose STEM at a 17.5 percent rate compared to 11.2 percent for URMs. The (unweighted) sample size was N = 15,860 consisting of students who participated in the base-year, first follow-up, and 2013 update.

The SEM analysis began by testing the measurement model. The CFA indicated excellent fit of the measurement model: $\chi^2(217)$ = 2,898.685, CFI = .958, TLI = .951, RMSEA = .023. The CFA estimated that all four of the mathematics attitudes were significantly pairwise correlated ($p < .001$). However, examining variance inflation factors (VIFs) did not provide evidence of significant multicollinearity (VIF < 5). Next, the SEM model from personal background and educational experiences to mathematics attitudes was tested. This model also had adequate fit: $\chi^2(312)$ = 3,820.444, CFI = .951, TLI = .942, RMSEA = .022. Lastly, the full SEM model was tested. The above fit statistics are not available for models with categorical outcomes (Muthén & Muthén, 1998-2017). The pseudo R^2 for the logistic regression on STEM was .267.

Table 1 contains the estimates for the direct effects of personal background and prior educational experiences on mathematics attitudes. Table 2 contains the estimates for the direct, indirect, and total effects for the full path model, including the odds ratios (OR) for the direct effects on STEM major choice.

Table 1: Direct Effects for Paths from Educational Experiences to Mathematics Attitudes

	On Math Attitudes			
Predictor and Covariate	Self-Efficacy	Identity	Interest	Utility
Educational Experiences				
Math achievement	.23***	.39***	.24***	.11***
STEM extracurriculars	.14***	.14***	.15***	.10***
Math teacher support	.09***	.05**	.08***	.10***
Personal Background				
Female	-.25***	-.21***	-.11***	-.09***
URM	-.09	.07***	-.02	-.11***
SES	.33***	-.03**	.30***	-.11***

Table 2: Direct, Indirect, and Total Effects for Paths from Educational Experiences to Mathematics Attitudes and STEM Major Choice

	On STEM Major Choice		
Predictor and Covariate	Direct [Odds Ratio]	Indirect	Total

Educational Experiences			
Math achievement	.58*** [1.79]	.16***	.74***
STEM extracurriculars	.16*** [1.17]	.08***	.24***
Math teacher support	-.03 [0.97]	.04***	.01
Mathematics Attitudes			
Math self-efficacy	.11 [1.12]	—	—
Math identity	.31*** [1.36]	—	—
Math interest	-.01 [1.00]	—	—
Math utility	.16** [1.18]	—	—
Personal Background			
Female	-.77*** [0.46]	-.11***	-.88***
URM	-.09 [0.92]	.07***	-.02
SES	.33*** [1.39]	-.03**	.30***

Focusing on the first research question, Table 1 shows that the analysis found that higher mathematics achievement in ninth, participation in a greater number of STEM extracurriculars, and greater teacher support early in high school predicted more positive attitudes towards mathematics later in high school ($p < .001$ for all paths expect math teacher support on math identity which had $p < .01$). In turn, higher mathematics identity and utility predicted greater odds of majoring in STEM (OR = 1.36, $p < .001$ and OR = 1.18, $p < .01$ respectively).

For the second research question, Table 2 shows that after accounting for race/ethnicity, SES, prior educational experiences, and attitudes towards mathematics, females' odds of choosing a STEM major where half that of males (OR = 0.49, $p < .001$). From Table 1, the effect of female was negative on each mathematics attitude ($p < .001$ for all four), with the largest magnitudes on self-efficacy and identity. Thus, after accounting for personal background and prior educational experiences, female students tended to have less positive attitudes towards mathematics.

Discussion

This study provides evidence representative at the national scale for the expectancy-value model: higher mathematics achievement, greater participation in STEM-related extracurriculars, and more supportive mathematics teachers early in high school predict more positive attitudes towards mathematics later in high school, which in turn predict greater odds of majoring in STEM. The findings suggest that the underrepresentation of females in STEM in the U.S. can be partially explained by less positive attitudes towards mathematics with a sense of identity as a math person having the largest gender disparity. Given that mathematics identity was also the attitude most predictive of STEM major choice, this study supports the growing focus on identity in mathematics education research (see Graven & Heyd-Metzuyanim, 2019). Future work for this project is planned to better understand why female students' attitudes towards mathematics tended to be lower than that of males. Existing literature suggests that negative stereotypes about STEM professionals are partially responsible, including unattractive appearances and socially awkward personalities, which are typically at odds with female gender identity and cultural expectations (Eccles & Wang, 2016; Starr, 2018).

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INVESTIGATING SELF-EFFICACY, TEST ANXIETY, AND PERFORMANCE IN COLLEGE ALGEBRA

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College Algebra students who struggle to pass the course could face delayed graduation or fail to obtain their degree. A key part to academic performance is self-regulated learning which includes self-efficacy and test anxiety as parts of motivation, as well as learning strategies. This study aims to investigate the change in motivation and learning strategies over the course of a semester and the relationship of this change to performance in College Algebra, as measured by final course grade. Test anxiety and self-efficacy were measured at the beginning and end of the semester using the Motivated Strategies for Learning Questionnaire. During the semester, self-efficacy decreased, and test anxiety increased. Moreover, the increase in test anxiety predicted performance. An important finding from the study was that the students who experience more stress on exams are the ones whose grades suffer the most.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Post-Secondary Education

Introduction

A large number of students take College Algebra each year in the United States. Generally, College Algebra fulfills a graduation requirement for non-math intensive majors and has historically been structured to prepare students for calculus (Gordon, 2008). While required for many students, only about 50% of students earn an A, B, or C in the course (Ganter & Haver, 2011). This could result in a graduation delay or failure to obtain a degree. With so many students unable to pass College Algebra, research is needed to investigate what occurs over the course of a semester to impact student performance.

A key aspect of academic performance is self-regulated learning (Berger & Karabenick, 2011; Pintrich & De Groot, 1990; VanderStoep, Pintrich, & Fagerlin, 1996). Students who are self-regulated learners utilize cognitive and metacognitive strategies to be successful in their learning. However, the knowledge of strategies is not enough; a student must also have the motivation to use them (Pintrich & De Groot, 1990; VanderStoep et al., 1996). Therefore, both motivation and learning strategies could be crucial to improving achievement in College Algebra.

This study aims to investigate the self-regulated learning of College Algebra students and answer the following research questions:

- What are the changes in College Algebra students' motivation and learning strategies over the course of a semester? And are the changes the same for all students?
- What is the relationship between the changes in motivation and learning strategies and performance in College Algebra?

While motivation and learning strategies are both important components of self-regulated learning, this paper focuses on two motivation components: self-efficacy and test anxiety. This controlled focus is due to test anxiety being a significant finding for both research questions, and self-efficacy is prevalent in the literature.

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Literature Review

Over the last few decades, self-efficacy and test anxiety have been studied extensively. Self-efficacy is a key component in social cognitive theory and refers to "beliefs in one's capabilities to organize and execute the courses of action required to produce given attainments" (Bandura, 1997). Test anxiety is a multi-dimensional construct defined by Zeidner (1998) as "the set of phenomenological, physiological, and behavioral responses that accompany concern about possible negative consequences or failure on an exam or similar evaluative situation" (p. 17).

At the college level, many studies have examined how self-efficacy or test anxiety change over time but with mixed results. For two online courses, no statistically significant changes in self-efficacy (Hodges & Kim, 2010) and test anxiety (Chapman, 2013) were detected during the semester. Others found decreases in self-efficacy (DiBenedetto & Bembenutty, 2013) and test anxiety (Fournier, Couret, Ramsay, & Caulkins, 2017) in science courses. Some reported increases in self-efficacy in biology (Ainscough et al., 2016) and test anxiety for medical students (Kim & Jang, 2015) over the course of a semester.

Self-efficacy and test anxiety have also been studied in relation to academic performance at the college-level. Several studies found self-efficacy (Hodges & Kim, 2010; Roick & Ringeisen, 2018) and test anxiety (Gibbens, 2019; Hieb, Lyle, Ralston, & Chariker, 2015) to be statistically significant predictors of performance. However, these studies measured self-efficacy and test anxiety at a single time point, and change over time was not used as a predictor of performance.

Only two studies were found to look at change over time as a predictor of performance. Fournier et al. (2017) found a decrease in test anxiety was not a statistically significant predictor of performance, and DiBenedetto and Bembenutty (2013) found that the decrease in self-efficacy was negatively correlated with final course grade. In light of these mixed results and other studies, this paper sought to examine the change in self-efficacy and test anxiety for College Algebra students over the course of a semester, and if the observed changes impact the final course grade.

Methodology

This study was conducted at a public university in the northeast region of the United States. During the spring semester of 2017, six sections of College Algebra were included in the study with 166 out of 227 students (73%) consenting to participate. To measure the changes in students' motivation and learning strategies, the Motivated Strategies for Learning Questionnaire (MSLQ) was used; an instrument considered to be both valid and reliable for this population of undergraduate students (Duncan & McKeachie, 2005). The MSLQ consists of two sections, motivation and learning strategies, with 15 scales total. These scales can be used together or individually for a total of 81 survey items, each with a Likert-scale from 1 to 7.

The students were asked to complete the MSLQ during the third week of the semester (T_1) and again on the last day of classes (T_2). Students' final course grade was also collected. Changes in all 15 MSLQ scales were considered by final course grades using a MANOVA with the paired differences ($T_1 - T_2$) as a response variable and final letter grade as the factor. In order to investigate how these changes in motivation and learning strategies relate to the students' final course grade, multiple linear regression was performed using changes in MSLQ scales over time and final numerical course grade. Statistical analysis was conducted using the R software package. Familywise false coverage probabilities and error rates were controlled using an adjustment based on the multivariate *t*-distribution that is implemented in the emmeans (Lenth, 2019) and glht (Hothorn, Bretz, & Westfall, 2008) R packages.

Results

The MANOVA results indicate that final letter grade in College Algebra is statistically significant (Pillai Test Statistic = .888 and p-value < .0001). Post-hoc comparisons using a 95% familywise confidence level indicate that students who earn a final course grade of "D" or "F" are the only ones to experience statistically significant changes in the MSLQ scales. Students who earn a "D" in College Algebra on-average experience a decline in self-efficacy between 0.59 and 1.79 points. Students who earn an "F" in College Algebra experience a decrease in self-efficacy between 0.78 and 2.07 points on-average, and an increase in test anxiety by 0.67 and 2.25 points on-average. When controlling for familywise error rate, multiple linear regression analysis showed that test anxiety is the only statistically significant predictor of final course grade at the 5% significance level. For students that experienced a one-point increase in test anxiety during the semester, it is expected their mean final course grade in College Algebra will decrease by between 0.21 and 6.81 percentage points.

Discussion

The present paper examined the changes in motivation and learning strategies for College Algebra students over the course of a semester, and if the observed changes impacted the final course grade. During the semester, self-efficacy decreased while test anxiety increased; findings that are consistent existing research (DiBenedetto & Bembenutty, 2013; Kim & Jang, 2015). However, not all students experienced these changes. Students who earned a "D" or an "F" in College Algebra felt less capable of being successful in the course as the semester went on, similar to findings by VanderStoep et al. (1996). Additionally, the students who earned an "F" experienced increased worry and had a preoccupation with performance over the course of the semester, consistent with Fournier et al. (2017). This is an interesting result as the students who need the most support were the ones to experience negative changes in their motivation during a semester of College Algebra.

For this study, there was a relationship between the changes in motivation and learning strategies and performance for College Algebra students. As a semester goes on, the students who experience more and more stress on exams are the ones whose grades suffer the most. This is in contrast to work by Fournier et al. (2017) and DiBenedetto and Bembenutty (2013). Paired with the other results, students who end up earning an "F" in course experience increases levels of test anxiety along with decreased self-efficacy; this increase in test anxiety has a direct, negative effect on their course grade.

While there is ample research on both self-efficacy and test anxiety, the findings are not always consistent and vary from subject to subject. This highlights a need for continuing research on both motivation components. To improve the passing rates of College Algebra, additional research is needed to investigate how to improve self-efficacy and decrease test anxiety, especially for the students likely to earn a non-passing grade. With the key finding that test anxiety increases over the semester and significantly predicts performance, future studies could attempt interventions throughout the semester to curb anxiety.

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MONOPOLIZING TEACHER ATTENTION: A CASE OF MULTILINGUAL LEARNERS' COMPETENCE EMBODIED IN SOCIAL FORMATIONS

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The rapid growth of multilingual mathematics learners in the US creates an urgent need for researchers and teachers to pay close attention to the complex social negotiation of resources in their classroom because these students are some of the most vulnerable to continual dehumanizing practices. Researchers and teachers need to understand how multilingual learns access resources but also how they manage the social risk of incompetence ascribed to students utilizing these resources. The current work presents an interaction as a case of the negotiation of resources, access, and competence during of a breakdown in the expectations of whom is allowed to make meaning of mathematics. Through the analysis, we see both how multilingual students are othered and excluded from mathematical discussion and how these students can reconstruct themselves as competent.

Keywords: Classroom Discourse, Marginalized Communities, Embodiment and Gesture, Learning Theory

Introduction

The rapid growth of multilingual mathematics learners in the US creates an urgent need for researchers and teachers to pay close attention to the complex social negotiation of resources in the classroom (Barwell, Chapsam, Nkambule, & Phakeng, 2016). Multilingual students are some of the most vulnerable to continual dehumanizing practices (Gutiérrez, 2018). Furthermore, their competence is constantly in question around both their mathematical conceptual knowledge and their language communications (Moschkovich, 2002), yet competence is a co-constructed phenomenon within classrooms (Gresalfi, Martin, Hand, & Greeno, 2009) meaning all students' power and identity play a major role in the formation of competence and incompetence of multilingual learners. In order to design and facilitate educational spaces that support multilingual learners, teachers and researchers need to understand how students manage both their access to resources but also the social risk of incompetence if/when resources are needed (c.f. Gibbons, 2003). As multiple students balance these two factors within the classroom system, they socially negotiate the resources present to assert power and reify their mathematical identity. The current work unpacks this negotiation by presenting a case where meaning-making resources are managed, and language incompetence is used to bar access. Within the case, I seek to answer the research questions: What role does language competence play in student-student negotiations of resources? and How are these negotiations embodied in students' social formations? My analysis breaks down how a dyad of 7th-grade students competes for resources, including the attention of a researcher-teacher, and negotiates each other's and their own competence around an educational digital narrative environment designed for single player. I discuss the implications of this type of negotiation in undermining educators' efforts to support all learners to make meaning of mathematics.

Literature and Conceptual Framework

Across a diverse array of approaches to research in mathematics learning, researchers increasingly recognize the themes of identity and power as urgent, especially where equity is foregrounded (e.g. Aguirre, Mayfield-Ingram, & Martin, 2013; Boaler & Greeno, 2000). These constructs are especially important as global immigration and language diversity in classrooms grow (Barwell et al., 2016). Mathematics plays a politically significant role in our society both as a gatekeeper to educational

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success and in our culture overall (Gutiérrez, 2013), so the imprints left on students' mathematical identities, self-efficacy, and confidence from negotiations of power in math class greatly impact their lives. Multilingual learners must balance their need to draw on language resources with others' perceptions of their incompetence. This tension permeates their experiences of mathematics and school in general through discourse, both at the interactional level and through larger Discourses (Gee, 2004). Greater understanding of how narratives of incompetence are perpetuated and used to further alienate these students is a vital component of efforts to support teacher noticing of the power dynamics at work in their classroom and then to disrupt microagressions within those dynamics.

To unpack these dynamics, I present a conceptual framework of discourse understood as layers of the semiotic field that can be 'read' by participants in the class which are laminated together in interaction (c.f. Goodwin, 2017). I parse these layers into three forms of discourse to better understand student-student negotiations of resources and how they make use of different aspects of the semiotic field in those negotiations. The first form is the active communication, both verbal and nonverbal, among people within the classroom. In many ways, the active communication between the teacher and the students and amongst the students is the most important form of discourse because it encompasses all the in-the-moment teacher moves to scaffold student thinking (Cazden, 1988). Yet, another form of discourse exists, the historically situated narratives around the discipline, the classroom/school, the students, and the teacher, conceptualized as Discourses (Gee, 2004). These narratives permeate the classroom, framing interactions and relational identities as they are invoked, enacted, and inscribed. These two forms of discourse make up most discourse analysis approaches, but I argue for considering the institutional infrastructure in place around the classroom as an additional form of discourse. Key aspects of infrastructure are its seeming invisibility and also its deep relation to ongoing practiced (Star & Ruhleder, 1996). Together, these aspects of infrastructure generate a form of discourse which communicates what is normal and what isn't, what is allowable and what isn't, what is supported and what isn't. Each form of discourse plays a distinct role in the classroom, and understanding their relation allows us to understand the complexities of students' social negotiation and formation and the impacts of these on student learning and identity.

To understand each form of discourse, I build on different constructs within discourse analysis. I used two constructs to analyze in-the-moment, dynamic communication among individuals: footing (Goffman, 1981) and formations (Kendon, 1990). Footing provides a space to deeply examine roles of speaker and listener. Specifically, I employ Goffman's distinction between ratified and unratified participants. Ratified participants are those with the access and opportunity to fully engage in the social dynamics and enact any of the three roles of the speaker (animator, author, and principal). Unratified participants are persons present but expected not to engage directly with ratified participants. I use this distinction with Kendon's conceptualization of formations to analyze the embodied practices at work and the relational configurations of bodies, resources, and gaze. Formations are flexible patterns of physical arrangements of bodies during interaction that can be categorized, and F-formations, or formations where participants are facing each other, are a common one. By combining the principles of footing and formations, I present representations of bodies, people and objects, to understand and convey the direct communication and relational dynamics of social negotiation moment to moment.

Gee's Discourses (2004) provides a framework to understand the larger social, political, and historical contexts of the classroom within which interactions occur. Discourses are constantly at work in and through direct discourse where they are used and operated on to exercise power and manipulate positionality. By unpacking how students apply, perpetuate, or challenge different Discourses meaningful to the situation, my analysis in sensitive to Discourses role in social resource negotiation, specifically the Discourse of incompetence of (some) multilingual learners.

Finally to understand the role of infrastructure and its impacts on classroom interactions and social negotiations, I draw on conceptualizations of infrastructure as temporal and enacted through local practice (Star & Ruhleder, 1996). Star and Ruhleder (1996) characterize infrastructure with eight properties, two of which are especially significant in educational contexts. First, infrastructure becomes most visible upon breakdown, when the system does not function as it normally does. This property is especially important upon disruptions of the (classroom) system when new elements like digital games are introduced (c.f. Barab, Gresalfi, & Ingram-Goble, 2010). Another key feature is how infrastructure embodies standards which perpetuates what is 'expected' and what is 'normal.' In many ways, multilingual learners continuously grapple with both of these features of the infrastructure of 'normal school' in their daily lived experience. Unlike Discourses and moment-to-moment communication, infrastructure is an expression of the embedded norms within and assumptions of the classroom system. Important to note, both technological formations or socio-cultural formations play the role of infrastructure, and considering this social infrastructure is especially important in math classrooms (c.f. Yackel & Cobb, 1996) and multilingual learners (c.f. Langer-Osuna, Moschkovich, Norén, Powell, & Vazquez, 2016).

Data and Methods

The data for the current paper came from a multi-year design-based research study of how educational story games support students' mathematical engagement (XXX, 2017; XXX, 2016). The classroom of focus was within an ethnically diverse school serving a primarily low-income community (92% free and reduced lunch) and many multilingual learners (30% of the school population) located in a medium-sized city in the Southeastern United States. The classroom teacher, Ms. Lynn (pseudonym), was a seventh-grade mathematics teacher having seven years of teaching experience at the time of the study and in her second year of participation with the research team. The current work focuses on a class of thirty-two students, Ms. Lynn, and two to four researcher-facilitators in the room (including the author). The role of the researcher-facilitators was to support the teacher by assisting with both technical concerns and students' conceptual questions. Ms. Lynn implemented the game in a four-day unit on rates, ratios, and proportional thinking. I focus on a dyad of two students, X and LM. X was a female Latinx multilingual learner with Spanish as a first language and seemed socially active with other Spanish speakers but rarely in whole class discussions. LM was a female black student active in the class and seemed to have a positive relationship with Ms. Lynn and other students, including A, another multilingual learner.

While data collected for the larger project encompassed much for each of the four days, I focus on data of a focal dyad working on a single computer. I use video data collected from three sources. First, a standalone camera captured the table at which students were working. Second, a camera embedded in the computer provides a view from the computer's perspective to give both an additional angle and to show who is framed in front of the computer. Finally, a screen capture software records students' digital actions on the game. Audio is provided from both the computer microphone and a table microphone, but because of the proximity of another group, not all speech is captured, especially simultaneous utterances. Coordinating these different sources allows for a bird's eye view representation of the dyad's dynamic.

To analyze the interactions of this dyad, I first watched their complete progress through the four days selecting the focal case of social negotiation. I chose this interaction because it captured a breakdown in interaction when X attempts to participate and this creates an activation of a social infrastructure of other multilingual learners to "help" X. After bounding the focal interaction, I transcribed intelligible talk and noted the occurrence of any unintelligible talk and (when possible) the speaker. Next, I transcribed each participant's body language and then coordinated these multimodal transcripts in a single transcript. By coordinating this transcript with the video, I generated a series of bird's eye views of the relevant space using representations similar to those employed by Kendon (1990). These temporally discrete snapshots were created at significant changes in the dyad's formations during the interaction. Lastly, I analyzed these representations with both the video and transcript to contextualize the formation changes, document salient Discourses, and understand the enacted infrastructure.

Analysis

Overview

Evident from the first day and throughout the implementation, LM and X appear to be working together as a dyad against their wishes (at least in part). As directed by the teacher on the first day of gameplay, students choose partners to work on the game together with the condition that at least one person in the dyad reads English, the exclusive language within the game. LM and X seem to have been joined largely based on this latter requirement. Two other individuals are also the main actors within the interaction of focus, A and F. A is a female Latinx multilingual learner positioned as bilingual and a translator for multilingual learners, including X, within the class. A and three other multilingual learners sit at a table group, referred to here as G1, directly at the back of LM and X. F entered the classroom for the first time only a days prior and is a female Thai research assistant working in the class as a researcher-facilitator.

The scene starts with LM communicating frustration with a specific part of the game where students are pushed to solve unit rate calculations using a double numberline tool before moving on. LM attempts to engage both the teacher, Ms. Lynn, and G1 to little effect. F approaches and offers to help LM and X. LM expresses her confusion with this part of the game, and F provides scaffolds via clarifying and probing questions. Throughout this first part of the exchange, X seems to follow the interaction and, in a lull, makes a bid to participate in F and LM's meaning-making around the problem and the tool. Upon X's attempt (and the ending of segment 1), LM draws on A in G1 to aid in the interaction by translating. The start of the final segment is the inclusion of A. LM continues to engage directly with F to solve the problem and move forward in the game while X and A converse inaudibly in Spanish. As A finishes translating and returns to G1, LM figures out the answer to the problem and inputs it into the computer. F asks the question "Does 8 [the answer] make sense to you?" and as LM responds with "Yes," X points to the computer bidding for participation once again but this time to close the interaction.

During this time, LM is acting as both a pivot point for the formation and the arbiter for whom can join in it while X remains slouched toward computer gazing downward. F, a somewhat new resource to the environment, offers to aid the pair, and she and LM solidify the formation into an F-formation to include the three individuals (LM, F, and X) and the computer. F positions herself as a ratified listener and lets LM take up the role of ratified speaker. At the same time F is joining the dyad, X engages in sideplay (Goffman, 1981) with members of G1 from between F and LM.

As LM talks with F (the third picture of Figure 1), F constricts the F-formation around the computer, and this constriction seems to draw X into the interaction. She begins to follow the interaction between F and LM, and as LM pauses to think of an answer to F's main question, X makes a bid to participate by offering an alternative answer and looking to LM as a meaning making partner. X's attempt to connect with LM as partner is in stark contrast to the behavior of LM, who has yet to even look at X, let alone make eye contact. This difference is further expressed in the subsequent segment when LM reacts to this bid as a disruption and a violation.

This moment holds a lot of tension for the group, which can be seen in the contortion of their formation, and for the individuals, which can be seen in the LM's twisted body. Furthermore, highlighting these contortions helps us to see the breakdown of 'normal' within the scene which reveals the social infrastructure to 'support' multilingual learners. LM enacts this social responsibility for A to leave her formation and mathematics learning in G1 and 'deal' with X's constructed incompetence. Unpacking this further, X's constructed incompetence is twofold, combining both language incompetence and social incompetence, because she did not activate the infrastructure of G1 and A in the first place (unlike during the sideplay of segment 1). Following this fraught moment, LM turns her body back towards the remnants of the previous formation and makes a slight, but distinct, motion with her posture and elbow between X and F. In this microaggression, LM simultaneously recovers the formation with F capturing her attention, absolves F of responsibility to engage with X, and further positions X as a non-member of the formation and an unratified speaker.

An analytic finding problematizes Kendon's F-formations. Kendon defines F-formations as semi-static spaces of interaction where each participant has equal access to the resources within the formation. Presumably, unratified participants could not be included in F-formations because they would not have equal access. Yet in this instance, we see just that, an F-formation where an unratified participant attempts to contribute and is re-positioned out of the formation. In the final picture of Figure 1, the closeness of F, X, and LM seems to show resoundingly that they form a single formation, at least as how F and X enacts it. Yet in segment 2, LM's surprise at X's bid implies LM's enactment of the formation rendered X as virtually invisible, merely meant to engage in sideplay and animate others authorings. Such a finding pushes on Kendon's F-formation definition and brings up the question of whose formation is being described.

..	= short pause, 2-4 secs
…	= long pause, more than 4 secs
()	= inaudible
(words?)	= sounds like "words"
((actions))	= Speaker is performing "actions"
[ ]	= Simultaneous speakers, always comes in sets of at least, but not limited to, 2
-	= Latched talk (see Dressler and Kreuz's "=")
?	= Rising intonation
.	= falling intonation

Thick Dotted line	= formation
Solid line	= F-formation
Thin Dotted line	= gaze
-People-	
Complete white center	= unratified participate not included in formation (i.e. bystander)
Gradient with white center	= unratified participate in formation
Gradient with black center	= ratified participant in formation

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MOTHER-CENTERED UNDERSTANDING OF MATHEMATICAL INTERACTIONS WITH CHILDREN: PURSUING POSITIVE INTENT

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How past research considers the influence of parents on their children's mathematical understanding does not often focus on the experience and intent of the parents themselves. In a cross-case analysis, I address how two mothers' experience shapes their mathematical positioning and the resulting interactions with their children in the subject. The attention on parents in mathematics takes on particular gendered roles, which is not often considered in research. There are more complex factors that shape how parents see and interact with mathematics. This study begins to show what alternative possibilities exist for engagement by U.S. parents in the content that still positively support children's early learning.

Keywords: Informal Education; Gender and Sexuality; Affect, Emotion, Beliefs, And Attitude

Objectives

Mathematics is taught and learned across a wide array of environments and situations, certainly not limited to the school classroom. Parents are often involved in early childhood mathematics learning through authentic engagement in everyday activities. However, across studies on parental influence on children's learning is the assumption of mother as a proxy for parent (Posey-Maddox, 2017). In research and media, mothers are frequently assumed to be doing the work of parenting. Maloney and colleagues (2015) claim that "parents are their children's first and most sustained teachers" (p. 1480). This statement coupled with the assumption of mothers doing the work of parenting implies that *mothers* are these first teachers. What is missing across literature on parents involved in mathematics learning is a focus of perspective on the parents and exploration of the parental (or specifically maternal) intentions.

The connections between parenting and mothering create additional tensions within the often-masculinized environment of mathematics. Mathematics is often framed as being for men and unwelcoming to women (e.g., Connell, 2010; Hottinger, 2016). Given the parallels between women and mothers (Arendell, 2000), potentially negative assumptions are made about mothers engaging with their children in mathematics. I explore the following questions: How does past experience in mathematics impact mothers' interactions with their child's early learning? What does this say about a mother's intent to support mathematical learning? Exploring the factors that direct mothers' action in mathematics provides a richer context for researchers in understanding parents' action in mathematical learning.

Background Literature

To understand how experience impacts mothers' interactions in mathematics requires an exploration of the larger context of motherhood expectations, parents' experience in mathematics, and the types of alternative activities parents can use to engage their children in early mathematical learning. The expectations within motherhood show what societal factors function to direct a mothers' actions in mathematical engagement. For example, adults in the United States believe that the more time women spend with their children, specifically giving their time to their children's development, the better mothers they are (Dillaway & Pare, 2008; Gorman & Fritzsche, 2002). These beliefs frame a societal expectation that mothers should be caring for and teaching their children, which mothers may identify as an obligation in their role as mothers.

Previous research that has studied parents' experience in mathematics has focused on the activities parents might do at home (e.g. Elliott & Bachman, 2018; LeFevre, Skwarchuk, Smith-Chant, Fast, Jamawar, & Bisanz, 2009) or the transfer of parental anxiety, affecting performance (e.g. Maloney et al., 2015). Research indicates a focus on authentic problem-solving (Pattison, Rubin, & Wright, 2016) and hands-on activities (Elliott & Bachman, 2018) in everyday tasks are common strategies for effectively engaging children in mathematical learning. While mathematical activity at home can be enriching for children, many parents do not recognize these problem-solving tasks as forms of mathematics (Goldman, 2005). Rich mathematics learning can happen at home, but much research on parents in mathematics is centered around how their anxiety impacts children's performance, often focused on mothers (e.g. Else-Quest, Hyde, & Hejmadi, 2008; Soni & Kumari, 2017). What is missing from this body of work is analysis from the perspective of mothers, to understand what has shaped their (gendered) experience and how they use it to create the best experiences for their children.

Theoretical Framework

I use positioning theory to frame mothers' experience in mathematics and how they represent themselves today in actions with their children. This theory uses experience and broader context to position people or be positioned as mathematical or not. Positioning theory thus becomes a way to understand the influences across cultural norms, history, experiences, and interactions with others (Harré & Van Langenhove, 1999). How mothers are positioned and position others will speak to a larger context of expectations for mothers in mathematics and what they do to still support their children in learning. Positioning oneself as mathematically able or not is frequently paralleled with concepts of power and agency (e.g. Kotsopoulos, 2014; Langer-Osuna, 2017). Using these themes of agency in positioning in mathematics education can support the agency mothers may have in positioning themselves to teach their children.

Methods

I use life history to capture the experience of my participants as a means to express the complexity of their experiences, informing their actions. Life history, an analysis of life stories, provides context to the situation and factors that make up the decisions of someone's life. "Life stories express our sense of self: who we are and how we got that way" (Linde, 1993, p. 3). This methodology supports a more intentional exploration of mothers and the reasons behind their mathematical interactions, which is frequently missing in mathematics education literature. Additionally, life stories can be characterized "as relational, as both personal and social, and as grounded in places" (Steeves, Clandinin, & Caine; 2013, p. 225). Life history offers a way to examine the connections to social and personal expectations more explicitly.

Two white, middle-class mothers, Ella and Corinne (names have been changed), who are roughly the same age, have young children, and spouses working in similar jobs, are the participants for this study. Their current life situations are similar, but their past mathematical experience and current interactions with their children differ in interesting ways. A comparison of their positioning in relation to their experiences demonstrates how mothers in different situations use available resources to provide the best mathematical opportunities they can for their children. Both participants were interviewed, with a focus on stories about their past mathematics experience and the current mathematical interactions they have with their children. Particular attention is given to the context of their stories, relating to feelings, relationships, and reflection. Interviews were audio-recorded and transcribed. A final meeting with each participant was made to go over their stories from the transcription, in order to confirm the stories best represented their experience. Interviews were then coded for themes around their positioning as mathematical thinkers, gendered roles as women and mothers, and the resultant activity they had with their children in mathematics.

Results

The difference in past experience and positioning of each mother frames how they see themselves today and ultimately engage with mathematics learning for their young children. While the specific activity differs, both mothers demonstrate attempts for positive and authentic engagement in activity. Ella came from a larger Midwest town, with an interest in reading and sports. The stories Ella shared continually related to her lack of confidence in her mathematical ability and in activity with mathematics today. In many stories related to her schooling, Ella described how teachers would put her on the spot to solve problems and she could not keep up with other students, never seeing herself as the smart one in class. Gendered roles from her past show how mathematics was understood and supported by her father, but not by her mother:

> My mom's like me, she was, my mom was probably fine with elementary school but when I got past that she was like "you have to ask your dad." And my dad, he's an engineer, so he's got a great math brain.

Ella frequently connected to her mother and her lack of fluency in mathematics problem solving. Her experience with her parents in gendered roles of mathematics support extended to how she expected her and her husband to divide work in supporting their children's learning, saying "I feel like when we got married it just like, we fall into certain roles." She claimed it was more likely that her husband would help their children with math and she could assist them in other learning.

The interactions Ella currently has with her daughter in mathematics are focused primarily on counting activities. She explained how her daughter learned to count higher than expected by stating, "we count a lot, at home. Like, she learned to count to 13 because every time we'd go up and down the stairs we would count the stairs." Ella's experience working directly with mathematics and her daughter is based on activities that have an authentic connection to their everyday lives. In moments where Ella wants more mathematical exposure for her daughter, she relies on resources in other areas, such as math-focused games from the library, and specific initiatives directed to her husband about helping in mathematics.

Corinne grew up in a small Midwest town, with a love of mathematics and learning, encouraged by her parents and supported by teachers. The stories Corinne shared related to her interest in math and dedication to engage her children in math at every available opportunity. Similar to Ella's experience, Corinne's parents had an influence on her mathematical positioning, but it was her mother who associated with mathematics and her father who was hesitant: "My mom was an accountant. So we already had like the math background and she pushed school, like school was a pretty high priority in my house. Like math just came very easy to me." The gendered roles of her parents' connection to mathematics, coupled with the positive positioning of Corinne as mathematical, supported Corinne's current activity with mathematics and her children. She is engaged often with her children in mathematical tasks and sees that as an important feature for parents who know math, connecting to a shared future role in teaching for her and her husband, where they both have mathematical backgrounds.

The activities Corinne uses to engage her children in mathematics have similar authentic connections to everyday life. However, Corinne feels comfortable asking her children questions in the moment and engaging in mathematics across any activity where a mathematical opportunity may present itself. She shared the following story about her youngest daughter:

> I was cutting up a banana for her and counting as I did it and I would pause and she would sometimes fill in the next number and I was like oh, okay you know way more than I thought you did, you can just count things.

Corinne's comfort in asking her children mathematics questions allows for opportunities to push for deeper connections in authentic settings, extending from counting to arithmetic and geometric reasoning.

Discussion and Implications

The experience and action of both Ella and Corinne demonstrate the impact of family and teacher positioning on a person's association with mathematics. Ella's association with her mother, who did not connect with math, and repeated interactions with teachers and peers that she was not learning math quickly enough have shaped how Ella characterizes herself today as uncomfortable with the subject. Corrine's past connections with her mother, who did connect with math, and the repeated affirmations by teachers of her success have shaped Corinne's continued positive associations with the subject. Literature on mathematical development in children often stresses the influence parents and teachers can have (e.g. Maloney et al., 2015; Pea & Martin, 2010). While this influence is apparent in how Ella and Corinne position themselves today, it also informs the influence of their mathematical activity with children. In this sense, mathematics interaction by parents can have a generational impact.

How Ella and Corinne interact with mathematics and their children today has marked differences, and reflects on their prior experience in the subject. Ella relies on outside or common resources, such as directives for her spouse or everyday tasks, to support engagement in mathematics for her daughter. She uses these elements as a way to counteract her positioning of uncertainty and discomfort with the subject. Corinne relies on authentic moments in everyday tasks to ask more math-specific questions of her children, divided equally with her mathematically-capable spouse. Her actions reflect similar exposure to the subject she received from her mother and repeated positioning by parents and teachers as mathematically capable. While there are variations in the kinds of activities Ella and Corinne choose, both reflect positive forms of mathematical engagement.

Ella and Corinne's interactions with children are based on authentic activity. As Pattison and colleagues (2016) suggest, this attention to authentic, at-home mathematics learning in diverse settings effectively supports mathematical reasoning. In both cases, the mothers are seeking out positive forms of mathematical engagement with their available resources and reflective of the positioning that they have. These interactions may have elements of difference, but they both rely on the use of everyday activity and are done with the best of intentions for their children. The cases of Ella and Corinne demonstrate alternative understanding of mothers' engagement with mathematics and recognition of context informing their interactions.

While there are limitations in the current study due to a lack of diversity in the participants, this study acts as a gateway to further exploration of parents (not just mothers) and their mathematical interactions. Literature that addresses Black mothers' perspectives in engagement to promote academic success for their older children (e.g. Jackson & Remillard, 2005; McGee & Beale Spencer, 2015) offer further validation to the possibilities of mothers' positive interactions in mathematics. Future work can strengthen connections between school and early childhood mathematics while alleviating the assumptions about mothers negatively impacting their children. Additional attention is needed to understand the complexity of experience and the factors that shape parents' positioning and resultant interaction with mathematics.

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DIDACTIC PURPOSES AND FUNCTIONS OF SECURITY AND DOUBT IN MATHEMATICAL CONTENTS

FINES Y FUNCIONES DIDÁCTICAS DE LA SEGURIDAD Y LA DUDA EN CONTENIDOS MATEMÁTICOS

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The security and doubt about mathematical facts (F) that a high school student experiences are examined. Rigo-Lemini (2009; 2013) has called those states 'epistemic states of conviction' (ESC) around F. It is argued that: i) the ESCs fulfill a didactic purpose, related to the satisfaction of certain epistemic needs and that ii) those ESC function as an alarm system that informs the person about the status of those needs and as a system that prepares the person to act accordingly.

Key words: Affect, Emotion, Beliefs; Research Methodologies; Cognition.

Background, Problem and Research Questions

The study focuses on phenomena related to security and doubt in mathematical contents F (eg, results of mathematical tasks or operations, solving tasks' strategies). Rigo-Lemini (2013) calls these states "epistemic states of conviction around F's", denoted as "ESC". Researchers have provided evidence that security and doubts about some F are present in school mathematics (Fischbein, 1987; Segal, 2000). Likewise, experts have suggested that, in certain cases, these ESCs adequately guide mathematical work, promoting advances in learning. However, in other cases, ESCs come to have unfavorable effects on the student's decisions and actions: for example, if a student doubts a mathematical concept or rule, it is very likely that he will not use it in a problem-solving activity (Fischbein, 1987; Foster, 2016). It would be highly desirable to understand the conditions under which the phenomena described above occur. However, experts acknowledge that the phenomena of conviction around F's have been little studied in the field of mathematics education (Fischbein, 1987; Foster, 2016; Inglis, Mejia-Ramos and Simpson, 2007; Segal 2000). So, to make didactic interventions focused on the ESCs, and well-founded, it is necessary to expand the current knowledge on this topic. In order to increase these understandings about ESCs in the field of mathematics education, this manuscript answers the following two research questions: What are the didactic purposes of the ESCs that a student experienced during the resolution of a mathematical task? And what functions did those ESCs perform?

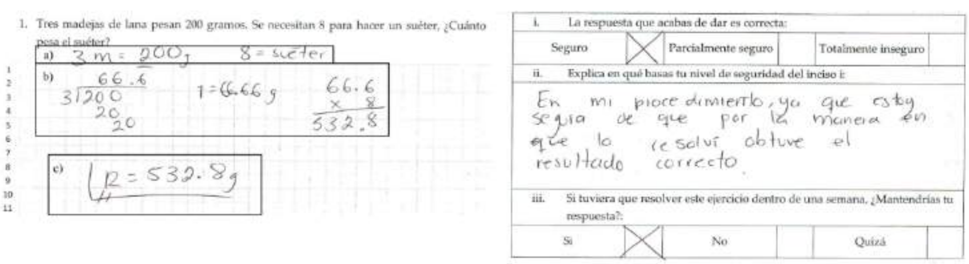
Methods and Methodology

The research adheres to the guidelines of the Grounded Theory (GT), in the version by Corbin & Strauss (2015). GT is a form of qualitative research that offers tools to build theoretical categories, based on empirical data, in order to develop explanations about the phenomena under study. For this reason, the GT does not start from a theoretical framework; it is about developing one. To elaborate these theoretical explanations, Corbin & Strauss (2015) suggest going to what they call *context analysis* (CA). In CA it is assumed that when people act or have some internal experience they are seeking to respond to events that are significant to them, in order to cover some unmet need or to maintain circumstances to preserve satisfied needs; these events are called *conditions*. From the conditions and from the actions or experiences that they promote, some result is usually derived, called *consequences*. In GT, explanations are constructed by applying the categories developed in the research and following the principles of CA. One of the objectives of the present investigation is to transcend the description to construct theoretical explanations; it was for this reason that GT was chosen. In addition to CA and other analytical techniques, microanalysis was applied, a tool that seeks to explore in depth the meaning of some pieces of data and that is usually used in early exploration stages.

What is reported in this manuscript is focused on a case study, about Hannia. She, like the five classmates who participated in the research (between 14 and 15 years old), were in third grade. The choice of subjects was made by the math teacher, who was asked to choose students with academic excellence. Empirical data was retrieved through a questionnaire, applied individually, that included 6 problems of missing value, 5 of them about proportional reasoning. At the end of each item students were asked to report the ESC they experienced (on a scale that included secure, partially secure, and totally unsecure) with respect to the answer given, and to explain in what reasons they based their level of secure. After the questionnaire, an unstructured interview was applied individually. Hannia was chosen because her questionnaire resolutions and her interview pronouncements and testimony provided valuable information on the nodal concepts examined in this investigation. In the following the case of Hannia is exposed; the data from the case is taken from the resolution that she offered to one of the tasks proposed in the questionnaire (the task of the skeins), as well as from her interventions in the interview.

Description of Security and Doubt Experiences around F's. Case Report

The researchers proposed Hannia the following task: Three skeins of wool weigh 200 grams. It takes 8 to make a sweater. How much does the sweater weigh? Three elements can be distinguished in Hannia's production: strategy approach (a), the operations carried out (b) and the resolution reached (c).


1. Tres madejas de lana pesan 200 gramos. Se necesitan 8 para hacer un suéter, ¿Cuánto pesa el suéter?

a) 3 m = 200 g   8 = suéter

b) 66.6 3|200 20 20   1 = 66.66 g   66.6 × 8 = 532.8

c) R = 532.8 g

i. La respuesta que acabas de dar es correcta:

Seguro	X	Parcialmente seguro		Totalmente inseguro	

ii. Explica en qué basas tu nivel de seguridad del inciso i:

En mi procedimiento, ya que estoy segura de que por la manera en que lo resolví obtuve el resultado correcto.

iii. Si tuviera que resolver este ejercicio dentro de una semana, ¿Mantendrías tu respuesta?:

Sí	X	No		Quizá	

Figure 1. Hannia's resolution

Although the strategy approach is not orthodox and despite omitting the reference to the quantities involved in each measurement space, Hannia's resolution is correct. Its resolution consists of two steps. In the first one, the unit value is calculated using the proportionality factor. In the second, she applied the external factor to the unit value (Cf. Vergnaud, 1985). The analysis of the episode, which is set out below, has been divided into three segments.

First Segment: Security in Strategy

In the questionnaire, Hannia explicitly stated her security in the strategy (See Figure 1). In the interview, when the researchers asked her "(. . .) What is your security about?" (311), she confirmed that security and expressed what it was linked to: "(. . .) I know that the procedure [strategy] that I carried out to solve it, I know it was correct ..., so that is what my security is based on" (312). By reporting that her security in the procedure is based on the fact that she "knows that it is correct", she reveals what she needs to experience that security: she needs to know that she did the right thing. She did not base her security only on supposing that her procedure was correct, or on believing, intuiting, or imagining it (Cfr. Villoro, 2002); to be sure, she demands to know that she acted properly.

But how did Hannia know that her procedure was correct? What reasons did she rely on? Although the guarantees on which she based her knowledge does not make them explicit, in the interview she hinted at some clues about what it means for her to trust. When the researchers asked her: "Hey, for you, what is it to be sure?" (317), she responds: "(. . .) Trusting what you did or what you said or what you do… that you do not regret what you have already done, you have to be focused" (318), "(. . .) be ... like ... zero nerves, relaxed, as sure haha with what you did, that you feel satisfied "(320); to this, the researchers ask: "Satisfied with what? (321) and she responds: "(. . .) satisfied with what you achieve or what you already did" (322). She immediately takes up the topic: "(. . .) Of what I have already done, [that I did] what they asked me to do, [and] what I did was good and therefore I am relaxed" (324), and then she assures: "(. . .) [when] (. . .) I don't follow a procedure that was taught to me, I feel more unsure of the result I obtained "(340). From this, it is plausible to suppose that 'doing what they asked' (324) or 'applying a procedure they taught' (340) are the reasons on which Hannia possibly based her 'knowing' that her strategy was correct.

Second Segment: Insecurity in Operations (carried out on a first attempt)

Following the chosen strategy, Hannia carried out the corresponding operations (See Figure 1). In the interview, she also outsourced the ESCs that she experienced during the execution of those operations and clarified what she based those ESCs on. Hannia comments that her security is based on the procedure performed, "(. . .) I know that the procedure I carried out to solve it, I know that the calculations I made were correct or well done, so my security is based on that" (312) and when asked her about how she knows that her calculations are correct, she expresses: "(. . .) I rectified them several times when I had already obtained the final result and I was able to make my conclusions and that's it" (314). By the type of response, it was considered important to ask if this is something that she usually does, to which she replied: "yes, when I don't feel very sure, or things like that (. . .) sometimes I do rectify my operations" (316).

As in the case of strategy, in the case of operations Hannia clarifies that she needed to know that they were correct in order to experience security, imposing on herself the epistemic need to know the correctness of the calculations to experience confidence. But, unlike the first, in this segment the student did specify the reasons on which her knowledge rested: in the verification of operations. In the interview, Hannia reveals that if she does not verify the calculations, she cannot know that they are correct and therefore she experiences insecurity around them. So, regarding her resolution of the task of the skeins, it is possible to suppose that in the first attempt, that is, in this second segment, Hannia felt insecure of her operations, because as she clarified, she verified them to be sure of the final result (314).

Third segment: Security in operations (performed on a second attempt) and security in the final result

Both in its production and in the interview, Hannia left no evidence of the rectification of operations. We assume that work was done mentally. Upon achieving the goal of 'rectifying the operations multiple times', she knew they were correct (met her epistemic need to know) and felt a security experience reporting in 312: "I know (. . .) That my calculations are correct, in that is based on my security". Supported by that security, she was already able to make her conclusions (314), that is, she was able to propose with confidence her final result, 'without regret' (318), concluding the exercise.

Theoretical Empirical Findings: Explanations on the Didactic Purposes and Functions of Security and Doubt about H's

From the above, it can be said that the overlapping of conditions that generated security (eg, about the strategy, or the operations) in Hannia included having achieved some *objectives* (eg, verifying operations, applying what they taught her), which it allowed her to satisfy her *epistemic need* to know that (her work was correct). As a consequence, security led her to *accept the results* that are the object of her security and to *continue with the mathematical work* in accordance with this acceptance trend. A consideration, in a sense symmetrical to the previous one, can be made regarding the insecurity that Hannia felt during her productions. The overlapping of **conditions** that generated insecurity (e. g, in operations) includes (among other factors) not having reached certain *objectives* (e. g, to verify), which prevented her from meeting her *epistemic need* to know that (her work it was well done). As a **consequence**, insecurity led Hannia *to distance* herself from the object of her doubt and led her *to carry out certain mathematical works* (in order to achieve her goals and satisfy her epistemic need to know). The categories corresponding to the conditions and those corresponding to the consequences that have been introduced in the framework of the investigation whose partial results are presented here have been highlighted with italics.

In accordance with this approach, the ESCs function as signals that inform the student whether or not an epistemic need is met. The study also reveals the effects of ESCs: when Hannia experiences security (epistemic need fulfilled), she generates a certain commitment to F and specifically directs her mathematical work towards F accordingly. When she builds an experience of doubt, in the face of unmet epistemic needs, she generates a distance or reluctance towards F and specifically guides her mathematical works in accordance with this trend.

We are then able to answer the first question, about the aims of ESCs in Hannia, or about the kinds of situations that her ESCs experiences allowed Hannia to face and resolve. A remarkable purpose of the ESCs in Hannia is that they allowed her to face and solve situations where epistemic needs are involved: when they are fulfilled, to preserve them and when they are not, to settle them.

And it is also possible to answer the second: What didactic functions did her ESCs play in solving these events? Her ESCs allowed Hannia to identify if her epistemic needs were met or not, functioning as an alarm system that keeps her informed about the status of those epistemic needs. Additionally, her ESCs also functioned as a system, coordinated with the previous one, that prepares her for action, that is, that drives her to carry out the corresponding mathematical works.

Final thoughts

Research based on GT methodology is not intended to establish generalizations; however, GT allows generating understandings that can be generalized, under certain restrictions, to similar cases that adjust to conditions analogous to those studied under GT (Corbin & Strauss, 2015; Reichertz, 2007). So, it is feasible to suppose that the explanations given here can be applied to cases analogous to those of Hannia. These explanations are novel and pertinent for research in mathematics education and for its teaching and may represent the basis for the gradual theoretical construction of a compendium of understandings on the phenomenon, empirically based, that allow, in the medium term, to make didactic interventions focused on the ESCs, well oriented and ethically responsible.

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FINES Y FUNCIONES DIDÁCTICAS DE LA SEGURIDAD Y LA DUDA EN CONTENIDOS MATEMÁTICOS

DIDACTIC PURPOSES AND FUNCTIONS OF SECURITY AND DOUBT IN MATHEMATICAL CONTENTS

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Se examina la seguridad y la duda en torno a hechos de las matemáticas (H) que experimenta una estudiante de secundaria. A esos estados Rigo-Lemini (2009; 2013) los ha denominado 'estados epistémicos de convencimiento' (EEC) en torno a H. Se argumenta que: i) los EEC cumplen una finalidad didáctica, relacionadas con la satisfacción de ciertas necesidades epistémicas y que ii) esos EEC funcionan como un sistema de alarma que informa a la persona sobre el estatus de esas necesidades y como un sistema que prepara a la persona para actuar en consecuencia.

Palabras clave: Afecto, Emoción, Creencias; Metodologías de Investigación; Cognición.

Antecedentes, Problema y Preguntas de Investigación

El estudio se centra en fenómenos relacionados con la seguridad y la duda en hechos de las matemáticas H (e. g., resultados de tareas matemáticas o de operaciones; estrategias de resolución de tareas). A esos estados Rigo-Lemini (2013) los denomina *estados epistémicos de convencimiento* en torno a H's', denotados como EEC. Investigadores han aportado evidencias de que la seguridad y las dudas en torno a algún H están presentes en la matemática escolar (Fischbein, 1987; Segal, 2000). Asimismo, los expertos han sugerido que, en ciertos casos, esos EEC orientan adecuadamente el trabajo matemático, promoviendo avances en los aprendizajes. Sin embargo, en otros casos los EEC llegan a tener efectos desfavorables en las decisiones y acciones del alumno: por ejemplo, si un estudiante duda de un concepto o regla matemática, es muy probable que no la utilice en una actividad de resolución de problemas (Fischbein, 1987; Foster, 2016). Sería del todo deseable comprender las condiciones en las que se dan los fenómenos antes descritos. No obstante, los expertos reconocen que los fenómenos de convencimiento en torno a H's han sido muy poco estudiados en el ámbito de la educación matemática (Fischbein, 1987; Foster, 2016; Inglis, Mejia-

Ramos y Simpson, 2007; Segal 2000). De modo que, para hacer intervenciones didácticas centradas en los EEC, y bien fundamentadas, resulta necesario ampliar el actual conocimiento sobre ese tema. Con el fin de incrementar esas comprensiones en torno a los EEC en el ámbito de la educación matemática, en este manuscrito se responde a las siguientes dos preguntas de investigación: ¿Cuáles son los fines didácticos de los EEC que experimentó una estudiante durante la resolución de una tarea matemática? Y ¿Qué funciones desempeñaron esos EEC?

Métodos y Metodología

La investigación se apega a los lineamientos de la Teoría Fundamentada (TF), en la versión de Corbin & Strauss (2015). La TF es una forma de investigación cualitativa que ofrece herramientas para construir categorías teóricas, fundadas en datos empíricos, con el fin de construir explicaciones sobre los fenómenos bajo estudio. Por eso, en la TF no se parte de un marco teórico; se trata de desarrollar uno. Para elaborar esas explicaciones teóricas, Corbin & Strauss (2015) sugieren acudir a lo que ellos llaman *análisis de contexto* (AC). En el AC se supone que cuando las personas actúan o tienen alguna experiencia interna están buscando dar respuesta, a sucesos o eventos que son significativos para ellas, con el fin de cubrir alguna necesidad no satisfecha o de mantener circunstancias para preservar necesidades satisfechas; a esos eventos se les denominan *condiciones*. De las condiciones y de las acciones o experiencias internas que propician, se suele desprender algún resultado, denominado *consecuencias*. En la TF, las explicaciones se construyen aplicando las categorías desarrolladas en la investigación y siguiendo los principios del AC. Uno de los objetivos de la presente investigación es trascender la descripción para construir explicaciones teóricas; fue por ello que se eligió la TF. Además del AC y de otras técnicas analíticas, se aplicó el microanálisis, herramienta en la que se busca explorar en profundidad, el significado de algunas piezas de datos y que se suele utilizar en estadios de exploración temprana.

Lo que se reporta en este manuscrito está centrado en un estudio de caso, el de Hannia. Ella, al igual que los cinco compañeros que participaron en la investigación (de entre 14 y 15 años), cursaban tercero de secundaria. La elección de los sujetos la realizó la maestra de matemáticas, a quien se le solicitó que fueran estudiantes con excelencia académica. Se recuperaron datos empíricos a través de un cuestionario, aplicado de manera individual, que incluía 6 problemas de valor faltante; 5 de ellos eran de proporcionalidad. Al final de cada reactivo se les pidió a los alumnos que reportaran el EEC que experimentaron (en una escala que incluía seguro, parcialmente seguro y totalmente inseguro) con respecto a la respuesta dada, y que explicaran en qué basaban su nivel de seguridad. Después del cuestionario se aplicó, de manera individual, una entrevista no estructurada. Se eligió a Hannia porque sus resoluciones al cuestionario y sus declaraciones en la entrevista brindaron información valiosa sobre los conceptos centrales que se examinan en esta investigación. En lo que sigue se expone el caso de Hannia; los datos del caso se toman de la resolución que ella ofreció a una de las tareas propuestas en el cuestionario (la tarea de las madejas), así como de sus intervenciones en la entrevista.

Descripción de Experiencias de Seguridad y Duda en torno a H's. Relato de un Caso

Las investigadoras le propusieron a Hannia la siguiente tarea: Tres madejas de lana pesan 200 gramos. Se necesitan 8 para hacer un suéter ¿Cuánto pesa el suéter? En la producción de Hannia se pueden distinguir tres elementos: el planteamiento de su estrategia (a), las operaciones realizadas (b) y la resolución a la que llegó (c).

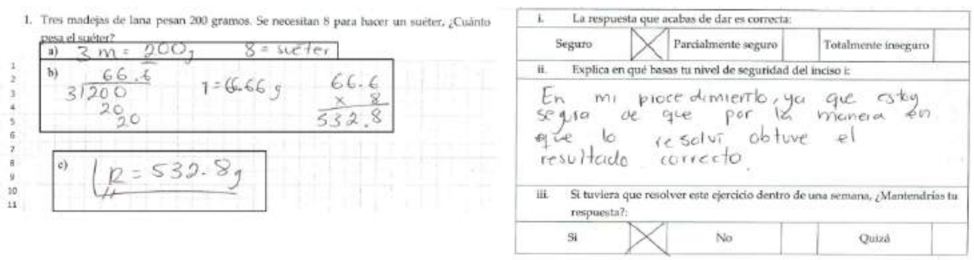

Figura 1. Producción Hannia

Aunque el planteamiento de la estrategia no es ortodoxo y a pesar de que omite la referencia a las cantidades involucradas en cada espacio de medida, la resolución de Hannia es correcta. Su resolución consta de dos pasos. En el primero calcula, mediante el factor de proporcionalidad, el valor unitario. En el segundo aplica el factor externo al valor unitario (Cf. Vergnaud, 1985). El análisis del episodio, que se expone en lo que sigue, se ha dividido en tres segmentos.

Primer Segmento: Seguridad en la Estrategia

En el cuestionario Hannia manifestó de manera explícita su seguridad en la estrategia (V. Figura 1). En la entrevista, cuando las investigadoras le preguntaron "(. . .) ¿A qué se debe tu seguridad?" (311), ella confirmó esa seguridad y expresó a qué estaba ligada: "(…) sé que el procedimiento [estrategia] que realicé para solucionarlo, sé qué pues fue correcto…, entonces en eso se basa mi seguridad" (312). Al reportar que su seguridad en el procedimiento está basada en que ella 'sabe que es correcto´, deja ver lo que necesita para experimentar esa seguridad: ella necesita saber que procedió de manera acertada. Ella no basó su seguridad sólo en suponer que su procedimiento fue correcto, o en creerlo, intuirlo o imaginarlo (Cfr. Villoro, 2002); para estar segura, ella se exige saber que actuó adecuadamente.

Pero ¿Cómo supo Hannia que su procedimiento era correcto? ¿En qué razones se apoyó? Aunque las garantías en las que sustentó su saber no las explicita, en la entrevista insinuó algunas pistas sobre lo que para ella significa confiar. Al preguntarle: "Oye y para ti ¿Qué es estar segura?" (317), ella responde: "(. . .) confiada de lo que tú hiciste o lo que dijiste o lo que haces… que no te arrepientas de lo que ya hiciste, tienes que estar como centrada" (318), "(. . .) estar… como que… cero nervios, relajada, como segura jaja con lo que tú realizaste, que te sientas satisfecha" (320); a esto se le pregunta: "¿Satisfecha de qué? (321) y ella responde: "(. . .) como de lo que logras o lo que ya hiciste" (322). En seguida retoma: "(. . .) de lo que ya hice, [que hice] lo que me pidieron, [y] lo que hice estuvo bien y por ello estoy relajada" (324), y después asegura: "(. . .) [cuando] no (. . .) sigo un procedimiento que me enseñaron sí me siento más insegura del resultado que obtuve" (340). De esto, es plausible suponer que el 'hacer lo que le pidieron' (324) o el 'aplicar un procedimiento que le enseñaron' (340) son las razones en las que posiblemente Hannia basó su 'saber' que su estrategia era correcta.

Segundo Segmento: Inseguridad en las Operaciones (realizadas en un primer intento)

Siguiendo la estrategia elegida, Hannia realizó las operaciones correspondientes (V. Figura 1). En la entrevista ella también externalizó los EEC que experimentó durante la ejecución de esas operaciones, y aclaró en qué basaba esos EEC. Hannia comenta que su seguridad la basa en el procedimiento realizado, "(. . .) sé que el procedimiento que realicé para solucionarlo, sé que pues fue correcto, o estuvo bien los cálculos que yo hice, entonces en eso se basa mi seguridad" (312) y al preguntarle sobre cómo sabe que sus cálculos están bien, ella expresa: "(. . .) los rectifiqué varias veces cuando ya había obtenido el resultado final y ya fue que pude hacer mis conclusiones y ya" (314). Por la respuesta, se consideró importante preguntar si eso es algo que ella suele hacer, a lo que ella respondió: "sí, cuando no me siento muy segura, o cosas así (. . .) a veces sí rectifico mis operaciones" (316).

Al igual que en el caso de la estrategia, en el de las operaciones Hannia aclara que necesitaba saber que eran correctas para poder experimentar seguridad, imponiéndose la necesidad epistémica de saber de la corrección de los cálculos para experimentar confianza. Pero, a diferencia del primero, en este segmento la alumna sí especificó las razones en las que descansó su saber: en la verificación de operaciones. En la entrevista, Hannia revela que, si ella no verifica los cálculos, no puede saber que son correctos y por tanto ella experimenta inseguridad en torno a ellos. Así que, respecto a su resolución de la tarea de las madejas, es posible suponer que en el primer intento, es decir, en este segundo segmento, Hannia se sintió insegura de sus operaciones, pues como ella aclaró, las verificó para estar segura del resultado final (314).

Tercer segmento: Seguridad en las operaciones (realizadas en un segundo intento) y seguridad en el resultado final

Tanto en su producción como en la entrevista, Hannia no dejó evidencia de la rectificación de operaciones. Suponemos que ese trabajo fue hecho de manera mental. Al lograr el objetivo de 'rectificar varias veces' las operaciones, supo que eran correctas (satisfizo su necesidad epistémica de saber) y sintió una experiencia de seguridad que reporta en 312: "sé (. . .) que mis cálculos son correctos, en eso se basa mi seguridad". Pertrechada en esa seguridad, ya pudo hacer sus conclusiones y ya (314), es decir, pudo proponer con toda confianza su resultado final, 'sin arrepentirse' (318), dando por terminado el ejercicio.

Hallazgos Teórico Empíricos: Explicaciones sobre los Fines y las Funciones Didácticas de la Seguridad y la Duda en torno a H's

De lo antes expuesto se puede decir que el imbricado de **condiciones** que generaron seguridad (e.g., sobre la estrategia, o las operaciones) en Hannia incluyó el haber alcanzado algún *objetivo* (e.g., verificar operaciones, aplicar lo que le enseñaron), lo cual le permitió satisfacer su *necesidad epistémica* de saber que (su trabajo era correcto). Como **consecuencia**, la seguridad la llevó a *aceptar los resultados* objeto de su seguridad y a *continuar con el trabajo* matemático en concordancia con esa tendencia de aceptación. Una consideración, en cierto sentido simétrica a la anterior, se puede hacer con respecto a la inseguridad que Hannia sintió durante sus producciones. El imbricado de **condiciones** que en ella generaron inseguridad (e.g., en las operaciones) incluye (entre otros factores) el no haber alcanzado ciertos *objetivos* (e.g., el de verificar), lo que le impidió cubrir su *necesidad epistémica* de saber que (su trabajo estaba bien hecho). Como **consecuencia**, la inseguridad llevó a Hannia a *tomar distancia* del objeto de su duda y la llevó a *realizar ciertos trabajos matemáticos* (con el fin de alcanzar sus objetivos y satisfacer su necesidad epistémica de saber). Se han resaltado con letras itálicas las categorías correspondientes a las condiciones y las correspondientes a las consecuencias que se han introducido en el marco de la investigación cuyos resultados parciales aquí se exponen.

De acuerdo con este planteamiento, los EEC funcionan como señales que le informan a la estudiante si una necesidad epistémica está o no solventada. El estudio también deja ver los efectos de los EEC: cuando Hannia experimenta seguridad (necesidad epistémica cumplida) genera un cierto compromiso con H y orienta específicamente sus trabajos matemáticos hacia H conforme a ello. Cuando construye una experiencia de duda, ante necesidades epistémicas insatisfechas, genera una distancia o reticencia hacia H y orienta específicamente sus trabajos matemáticos de acuerdo con esa tendencia.

Se está entonces en condiciones de responder a la primera pregunta, sobre los fines de los EEC en Hannia, o sobre el tipo de situaciones que sus experiencias de EEC le permitieron a Hannia afrontar y resolver. Un propósito destacable de los EEC en Hannia, es que le permitieron enfrentar y solventar situaciones donde están involucradas necesidades epistémicas: cuando están cumplidas, para preservarlas y cuando no lo están, para saldarlas.

Y también es posible responder a la segunda: ¿Qué funciones didácticas desempeñaron sus EEC en la resolución de esas incidencias? Sus EEC le permitieron identificar a la estudiante si sus necesidades epistémicas estaban satisfechas o no, funcionando como un sistema de alarma que la mantiene informada sobre el estatus de esas necesidades epistémicas. Adicionalmente, sus EEC funcionaron también como un sistema, coordinado con el previo, que la prepara para la acción, esto es, que la impulsa a la ejecución de los trabajos matemáticos correspondientes.

Consideraciones finales

Las investigaciones basadas en la metodología de la TF, no tienen la finalidad de establecer generalizaciones; no obstante, la TF permite generar comprensiones que se pueden generalizar, bajo determinadas restricciones - a casos semejantes que se ajusten a condiciones análogas a los estudiados bajo la TF (Corbin & Strauss, 2015; Reichertz, 2007). De modo que es viable suponer que las explicaciones aquí dadas se pueden aplicar a casos análogos a los de Hannia. Esas explicaciones resultan novedosas y pertinentes para la investigación en educación matemática y para su enseñanza y pueden representar la base para la paulatina construcción teórica de un compendio de comprensiones sobre el fenómeno, fundamentadas empíricamente, que permitan, a mediano plazo, hacer intervenciones didácticas centradas en los EEC, bien orientadas y éticamente responsables.

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COMPARING TRANSFER AND NON-TRANSFER COLLEGE STUDENTS' MATHEMATICS PROFESSIONAL VISION

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This research focuses on the importance of attending to transfer students in mathematics departments at research universities. This study explores the development of professional vision among mathematics transfer students, through examining community, "student" skills, and students' future career aspirations. A bundle of three transitional mathematics course for transfer students offered concurrently at a four-year research university provided the setting, and we compared transfer students enrolled in bundle, non-bundle transfer students, and non-transfer students. Overall, students identified differences in their mathematical communities, their development as mathematics students, and their resources for career pathways.

Keywords: University mathematics; Affect, emotion, beliefs, and attitudes

The recent increase of transfer student enrollment mandated by many universities drives a need to understand how we can better prepare faculty to support this growing population of students. There is currently little research about how to support mathematics majors as they make the transition from two-year institutions to four-year institutions of higher education, even though there is general scholarship about transfer students (e.g., Melguizo et al., 2011). Understanding mathematics transfer students is important for three key reasons. First, the mathematics transfer population includes large numbers of underrepresented minorities, low-income, and first- generation college students (>50%), who start their college education at a community college—students who are often left out of the STEM pipeline (Carnevale et al., 2011). Second, we have seen a high level of attrition of transfer students from the mathematics major at our university, with just over half completing the major. It is important to understand whether retaining students in the major with the proposed intervention, a bundle of course to support transfer students helps. Third, transferring and adjusting to a four-year institution can be a complex process for students, requiring adjustments on many different levels (Laanan, 2001). In addition to a drop in GPA, transfer students often report feelings of alienation and isolation at their new institution (Laanan, 2001), with women and minorities majoring in STEM more likely to face this particular struggle (Espinosa, 2011). For these reasons, we studied how transfer mathematics students compared to non-transfer mathematics students enrolled in the aforementioned bundle of courses designed to support transfer students.

Framing

We use Lave and Wenger's (1991) *legitimate peripheral participation* and Goodwin's (1994) *professional vision* as a conceptual framework to consider mathematics transfer students' development along a novice to expert continuum of professional vision. In particular, our research question was: How do community college transfer students differ from non-transfer students enrolled in a bundle of support courses for transfer students in a mathematics department in terms of: (a) feeling that they are part of a community; (b) learning to be a college mathematics student; and (c) preparing for a future mathematics career?

Legitimate Peripheral Participation

We drew on Lave and Wenger's (1991) ideas of legitimate peripheral participation, as related to apprenticeships, and active participation in a community of knowledge. In particular, we were interested in how individuals engaged with one another to feel like they were part of a community.

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Students learned content not simply by receiving factual knowledge, but within and engaged as part of a community of learners, developing facility with content and community (Lave & Wenger, 1991). As students learned to be college students and prepared for their future careers, they engaged in peripheral participation, which was full participation in the community of practice (Lave & Wenger, 1991). This allowed four-year mathematics students to engage in novice-expert relationships, being apprenticed into new skills and knowledge of expertise, such as becoming four-year university college students or mathematics professionals.

Professional Vision

Such novice-expert interactions as those previously described were organized for the development of professional vision (Goodwin, 1994). We drew on Goodwin's professional vision to understand the development of the professional vision needed for being a four-year student and a future mathematics professional, which he defined using three characteristics: 1) being perspectival; 2) being situated; and 3) being learned, because it was situated.

Research Methodology

This research took place in the context of a research university mathematics department and involved students engaged with at least one "bundle" course. Three mathematics courses made up the bundle: (1) special topics in mathematics (a course to develop problem solving and the expectations for upper level mathematics courses)—Course A, (2) group studies in mathematics (an academic and career advising course)—Course B, and (3) transition to higher mathematics—Course C. This third course served as a comparison course across all three groups of participants in the study, as it was a prerequisite for many mathematics courses that followed. Course C was an introduction to the elements of propositional logic, techniques of mathematical proof, and fundamental mathematical structures, including sets, functions, relations, and other topics. This was the second year of this study. In the first year of our study (Roberts et al., 2019), the mathematics department selected students into the bundle and placed all students into the same section of Course C. During the second year, reported here, students voluntarily selected into the bundle, and the mathematics department placed students into two separate sections of Course C.

Three groups of students participated in this study: transfer students who voluntarily chose to take the "bundle" courses (concurrently); transfer students who took only Course C; and non-transfer students who took only Course C. This allowed for comparative groups to understand how groups of students engaged with the content in the bundle courses.

We conducted and audio-recorded semi-structured focus groups (Yin, 2016; 40-60 minutes) and individual interviews (25-35 minutes) to understand how participants developed professional vision around being a mathematics major and mathematics professional. The interview covered the following: background information, three-course bundle, being a mathematics student, ways of thinking and doing mathematics, and preparation for a career in mathematics.

We used focused coding (Maxwell, 2005), applying three themes: community, being a student, and future as a mathematician, to code the interviews and focus groups. We then looked within and across each set of coded data and research question, looking for consistencies and inconsistencies. We also drew on data from our Year 1 data from the 2018-2019 school year, which was coded similarly and used the same conceptual framework but only examined bundle transfer student (See Roberts et al., 2019, for more information), to make longitudinal comparisons, to triangulate, and to look for further consistencies and inconsistencies.

Findings

We found that there were differences between bundle students, transfer students, and non-transfer students across the students we interviewed. We describe key differences in ways transfer and non-transfer students viewed community, office hours, and their future career opportunities.

Role of Community

There was a difference in the way that transfer students discussed the role of community compared the non-transfer students, especially those students who were enrolled in the bundle courses. This resonated with our Year 1 findings, where we found that students were very connected with other students in the bundle course. In our Year 2 findings, students in the bundle noted the role of the community. For instance, Megan shared, "[So,] the community I feel, like, it kind of empowered me a bit" (p. 6), when she was asked about the benefits of taking the bundle courses. Similarly, Talia, also from the bundle, explained, "[T]o reiterate, the community…I feel like I can always text someone, even if they're not in the same class. I can always message and be like, 'Does anyone, you know, wanna help?' Or whatever" (p. 21). These students felt like there was a strong sense of community in the bundle, where they could go to other students for help and draw on the collective wisdom of other students—there was collective professional vision situated in their bundle (Goodwin, 1994). However, Caleb, a bundle student, also mentioned that he had community through another outlet, a club sport, and did not need to draw on the community from other transfer students and the bundle. Therefore, this was not consistent with all bundle students. Additionally, not all transfer students mentioned that they felt that they were part of this community. For example, a non-bundle transfer student, Madison, explained, "I agree that there was community, but I also felt very excluded from it, so I didn't feel like I was part of that community. I did see that there was a community" (p. 17). For this student, there was an issue with being excluded from the community.

The non-transfer students also mentioned community, but this community did not appear to emanate from the courses, like it did with the bundle courses. Instead this community was from friends made externally from the courses or friends made within the courses but not with the help of the courses. In the bundle courses, the community appeared to be built from within the course—with the instructors helped to develop that community, as described below. Non-transfer student, Esther, described a group of friends she with whom had taken earlier mathematics classes and with whom she had studied previously. However, those students were not in Course C, so there was not a feeling of community, because she did not have interactions with any of those same students any longer. Bundle courses, in contrast, forced students to work together, which likely allowed most students to develop community.

Being a Student

Approaching faculty for support was the key component of being a student we explored in this study. Transfer students were the only students who mentioned visiting professors' office hours. Bundle students were the only ones who discussed their visits in detail. Others noted that these were resources for help in their mathematics course, but none of the other students we interviewed, besides the bundle students, discussed, in detail, attending office hours. Non-transfer students even went so far as to mention not going to office hours or not even interacting with faculty. This is notable, because office hours can be a useful tool for students to learn how to ask for help, especially for transfer students, who are more likely to change majors and leave universities. As we noted in our Year 1 paper, Course B required students to attend office hours. Talia, from the bundle, shared, "I think the first time I went to office hours last quarter, I was so terrified, but it kind of made me realize…how approachable people are" (p. 7). Even though going to visit professors was difficult at first, students found this valuable. As Talia explained, "And, so, now, I have no problem going to office hours if I need to" (p. 7). This was consistent with our Year 1 findings, where those students who were "forced" to go to office hours in the bundle were thankful for developing familiarity with the process and with getting to know faculty and TAs.

Future Career Opportunities

The Course B instructor, who was also a mathematics department advisor, was highlighted as key in sending out information to all student participants about future mathematics career opportunities. There was a key difference between transfer students and non-transfer students in their considerations about their future careers and what they had accomplished toward those careers by the time we had met for our interviews. Second year non-transfer students had already applied for internships, were participating in research on campus, had attended mathematics conferences off-campus, had secured funding, and had other such opportunities. Additionally, they seemed to feel like they had plenty of time to figure out their future plans. In contrast, transfer students were just beginning to learn about these opportunities, but they also felt the pressure to figure out their future more quickly as third-year students. Bundle students had, as part of their Course B coursework, weekly panels on graduate school, jobs in industry, and other opportunities, whereas, non-bundle transfer students were dependent upon weekly department emails, clubs, and reaching out for individual emails. Even so, bundle students asked for more, such as how to write in LaTeX, what K-12 careers might entail, and how to write cover letters for research internships (which those non-transfer students had already secured). We noted in our Year 1 findings that bundle students were similarly reaching out for more internship opportunities. The fact that non-transfer students had already applied for these placements supports this finding that more needs to be done to work with transfer students to level the playing field for future opportunities. Finally, because not all students had access to the bundle, we see that transfer students are missing out on Course B and this future career advising.

Discussion and Conclusions

There were three key differences between transfer students and non-transfer students in the bundle and non-bundle courses. First, the bundle course provided a community for transfer students to develop a mathematics community. Second, those transfer students in the bundle were more likely to attend office hours, which meant they were more likely to reach out for support in their coursework. Third, transfer students were more likely to be behind non-transfer students in their future mathematics plans; however, bundle students were more likely to have had exposure to more future possibilities. This bundle of courses offered transfer students with connections that helped them adjust to life as a mathematics student, as well as professional vision for their future mathematics careers.

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SPONTANEOUS MATHEMATICAL MOMENTS BETWEEN CAREGIVER AND CHILD DURING AN ENGINEERING DESIGN PROJECT

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Research studies support caregiver's involvement in their child's mathematical journey as foundational to their cognitive development and academic success as mathematical learners (e.g., Sheldon & Epstein, 2005). The purpose of this intrinsic case study was to understand how a caregiver initiated and/or continually engaged their child in spontaneous mathematical moments during the engineering design process. Through the analysis of approximately 13.5 hours of video data, we noted several ways in which Tonya guided, supported, and challenged Cindy through a shared endeavor of designing a remote-controlled delivery robot – questioning that promoted reflection and advanced Cindy's mathematical understanding, affording Cindy opportunities for decision making, and providing Cindy with the mathematical language to describe her approaches within the engineering design process.

Keywords: Caregiver as Educator, Informal Education, Mathematical Moments

Objective

Previous research suggests that interest and engagement in science, technology, engineering, and mathematics (STEM) can be triggered at a young age, and caregivers are considered to be one of the most significant influences in this development (e.g., Maltese & Tai, 2011). Additionally, the benefits of out-of-school learning experiences for youth is well documented and include positive dispositions toward STEM, greater likelihood of pursuing a STEM-degree and career, and development of interest and confidence in STEM (e.g., Bell et al., 2009; Denson et al., 2015). Engaging in teaching and learning of mathematics within home environments and other out-of-school contexts are framed as shared family experiences and tend to include budgeting, home improvement projects, games, proportions of ingredients when using recipes, and verbal exchanges during mealtime (e.g., Esmonde et al., 2012; Pea & Martin, 2010). As such, caregivers, regardless of their own experiences, are able to act as mathematics educators in engaging their child(ren) in mathematical moments (Sheldon & Epstein, 2005). In this study, these mathematical moments are defined as a spontaneous experience to engage with and/or explore mathematical ideas and concepts (Cunningham, 2015), and situated within a project aimed at developing, implementing, and refining a program for integrating engineering design practices with an emphasis on emerging technologies (i.e., making, DIY electronics) into home environments of families. Research has shown that participating in engineering design principles support students' application of mathematical concepts (e.g., Berland et al., 2014). Yet, we know very little of how such mathematical moments in the engineering design process arise in out-of-school learning contexts between caregiver and child. We address this gap in the literature by addressing the following research question – How does a caregiver initiate and/or continually engage their child in spontaneous mathematical moments during the engineering design process? We contend that caregivers and other family members should be recognized for their ability to enhance school mathematics within out-of-school learning contexts.

Perspective

In this study, we utilized a socio-cultural perspective, which views learning as active participation and engagement in cultural and social activities (Rogoff et al., 1993). More specifically, we employed Rogoff and colleagues' (1993) guided participation in which participation is guided, supported, and challenged from another in a shared endeavor; in this study, this other referred to the caregiver and the shared endeavor is the development of a robot (see below). As such, Rogoff (2008) defined participation as an interpersonal process in which individuals are actively observing and/or communicating with their words and hands. It builds upon the notion of zone or proximal development as it involves "not only the face-to-face interaction, which has been the subject of much research, but also the side-by-side, joint participation that is frequent in everyday life" (Rogoff, 2008, p. 60). Similar to Vedder-Wiess (2017), we contend that the caregiver's role within in the process of guided participation is through modeling and engaging in spontaneous mathematical moments with their child. Collectively, the caregiver and child are employing their knowledge and understanding of mathematics.

Methods

The larger research project was conducted between January to May. We met with caregiver-child dyads once a month for approximately 3 hours in length. This particular study is an intrinsic case study of a caregiver-child dyad (Tanya and Cindy) engaging in mathematical moments during an engineering design project developed and designed by the dyad (Stake, 1995). As stated by Cindy, "My project is a remote-controlled delivery robot to help people who can't get out of bed or are sick…I was thinking about someone in a nursing home." At the time of the study, Cindy was a third-grade student who aspired to be an artist.

Data Collection

The main source of data was video recordings of each monthly session and home video recordings of Tonya and Cindy working alongside a member of the team. Cameras were stationed as to capture the interactions between Tonya and Cindy, as well as interactions with facilitators and engineers who volunteered their time to assist the dyad. This amounted to approximately 11 hours of video data from the monthly sessions and approximately 2.5 hours of video data from the home visits.

Data Analysis

The analysis was conducted in two phases. During the first phase, both authors watched all the videos, individually looking for mathematical moments. We each noted the time range and provided a brief overview of the interaction in terms of engagement with mathematical ideas and/or concepts. Our goal was not to establish inter-rater reliability, but to capture identifiable mathematical moments, or ethnographic chunks, for further analysis (Jordan & Henderson, 1995). We met five times to discuss our observations as we acknowledged these identifiable moments to be influenced by our cultural understandings of and experiences with mathematics as a mathematics teacher educator and STEM education researcher, and science education doctoral student respectively. The final meeting focused on identifying specific moments that addressed the research question, which were transcribed verbatim and included non-verbal acts of communication. During the second phase, we individually read through the transcripts and noted the ways Tonya initiated and continually engaged Cindy in spontaneous mathematical moments. When we met to discuss, we were similar in our understanding of these spontaneous moments such as the manner in which Tonya posed questions to both initiate and advance Cindy's engagement as a mathematician. We also developed a shared language (i.e., agency).

Findings

We present two specific instances in which Tonya initiated and/or continually guided Cindy in spontaneous mathematical moments during the engineering design process. Both examples occurred during the last workshop when Tonya and Cindy are brainstorming how to construct the tray with the materials on hand. The first transcript begins as they are discussing the appropriate height for the tray once mounted on top of the rumba, which served as the base of the robot.

T: Okay. And so you were talking about the height of your stands and what you, you had said that- oh, well maybe you'll do it a certain way.
C: Yeah, in the middle of the three beds.
T: Okay. So what would that measurement be here? How would you figure out that measurement?
C: That would, wait…it would be all the beds to get all of it? No, it'd be the biggest height and then split that in half. So 32 in half is…
T: Are you trying to find the average?
C: Yeah.
T: So if you are going to take an average, you would take the three numbers. You would add them together and then you would divide them by three, if you're trying to get the average. Is that what you want? Or are you trying to do it one particular height to get to the person that…it's kind of your choice here.
C: No. I want it to be the average. So then it could get to anything. And it would either be a little too tall or a little too short. They [people in bed] would have to reach down a little bit or reach up, or like sit up.
T: Okay. So you think we should do the measurement or do you want to figure out the actual height?
C: I want to figure out the average.

The transcript highlights several things. One, Tonya provided Cindy with an opportunity to decide whether the average of the height of the three beds or the height of one bed was preferred (e.g., "It's kind of your choice here."). While Tonya more than likely knew the most appropriate approach within this context, she allowed Cindy to make her own decision (i.e., agency; Norén, 2015). Further, Cindy revealed her reasoning of why the average was appropriate in that the person in bed would have to reach down or up to gain access to food on the tray. Two, Tonya provided Cindy with the definition and language to describe the approach, which Cindy adopted as part of her language throughout the transcript (e.g., last line). Three, this example illustrates how Tonya was "with" Cindy in these moments as she gathered evidence of Cindy's thinking and made in-the-moment and intentional decisions regarding the project and Cindy's process and progress. This was often done through questioning.

In the next transcript, Tonya encouraged Cindy to find an alternative to converting inches to centimeters, which would be needed for her code.

C: *(Speaking into a tablet.)* Centimeters to inches.
T: *(Reaches across the table to grab a tape measure.)* Instead of using that, there's a way that you can figure it out using this. What do you think it is?
C: *(Grabs tape measure and pulls the tape from the housing. Smiles.)*
T: Yeah, you don't always need that. You can figure it out without just trying to get the quick answer.
C: Eight and a half. *(Let's go of the end of the tape and it retracts.)* I mean, no. *(Pulls the tape out again and seems to examine.)*
T: Yeah, that doesn't… Does that make sense to you? [Asking - How can 27 inches equal 8 centimeters?]
C: It said eight. *(Continues looking at the tape.)* Oh no, I get it. I get it. Sixty…sixty…sixty-eight and a half.
T: *(Takes the tape measure.)* These are decimals, so it actually would be 68 and six-tenths. When you're doing measurements, sometimes that tenth of a centimeter is going to make a big difference.

This mathematical moment was sparked through Tonya's question that pushed Cindy to think of another conversion strategy, namely, reading the tape measure. We also observed Tonya questioning the reasonableness of Cindy's first response of eight, indicating that 27 inches was the same as 8 centimeters. This question, as noted in the previous example, was intentional; it served a purpose as Cindy was encouraged to reflect upon her response (NCTM, 2014). Lastly, Tonya explained to Cindy the importance of accuracy and precision appropriate to this particular context (i.e., Mathematical Practice 6; CCSO, 2010).

Significance

The two examples presented here illustrated how one caregiver initiated and engaged their child in mathematical ideas and concepts that spontaneously arose within and throughout a self-identified engineering design problem. Tonya guided, supported, and challenged Cindy through a shared endeavor, designing a remote-controlled delivery robot (Rogoff, 1993). These spontaneous mathematical moments afforded authentic sense making between caregiver and child, which may be harder to attain in structured learning environments and other out-of-school contexts such as STEM-focused afterschool programs and summer camps (e.g., Vedder-Weis, 2017). For example, Cindy gained a different perspective and strategy of how to convert centimeters to inches; a strategy that was authentic and spontaneous to the design of the tray in this instance. Such mathematical moments were often initiated through questions for Cindy to explore within the design of the robot. These questions were not always answered orally, but addressed through physically engaging in mathematical ideas and concepts. Tonya further provided Cindy with a sense of agency in that Cindy was allowed to make mathematical decisions regarding the project. As such, we contend that this case highlighted how children can engage in mathematics in out-of-school learning contexts through the support and encouragement of caregivers. As a field, we should continue to think about ways to engage caregivers as mathematical partners, both within mathematics and STEM fields more broadly (e.g., engineering projects). As mathematical partners, researchers and educators should consider what is required for caregivers to actively and productively engage their children in spontaneous mathematical moments. Archer and colleagues (2015) made a similar argument in respects to science capital or the "level of scientific literacy and access to plentiful, high quality science-related cultural and social resources" (p. 15).

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MANIFESTATIONS OF BIAS WITHIN PRESERVICE TEACHERS PROFESSIONAL NOTICING OF CHILDREN'S MATHEMATICAL THINKING

MANIFESTACIONES DE PARCIALIDAD ENTRE LOS MAESTROS PROSPECTIVOS DE LA MIRADA PROFESIONAL DEL PENSAMIENTO MATEMÁTICO DE LOS NIÑOS

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The purpose of this investigation was to examine connections between noticing and bias with a particular focus on how perceptions of race and gender may influence such noticing. Our primary research question was: How and to what extent does bias emerge within preservice teachers' professional noticing of children's mathematical thinking of differing races and genders? Results from this study indicate that bias most frequently occurs within the interpreting component of professional noticing, and that manipulating visual stimuli, such as adding a photo to a professional noticing vignette, influences manifestations of bias across race and gender.

Keywords: Teacher Education-Preservice, Equity and Diversity.

Equity concerns in mathematics have been of interest to researchers for a considerable period of time (Breshlich, 1941, DiME, 2007; NCTM, 2014). More recently, though, there has been renewal of interest regarding the support and flourishing of students from diverse backgrounds within mathematical contexts (Aguirre et al., 2017; Gutierrez & Dixon-Roman, 2011), and such differences in experiences are especially evident in STEM disciplines where students from non-dominant groups have received implicit and explicit messaging regarding their inclusion (or lack thereof) within such disciplinary settings (Goffney, Gutierrez, & Boston, 2018; Museus, Palmer, Davis, & Maramba, 2011). In some instances, inequities arise due to reduced expectations or lack of cultural considerations (Savage, Hindle, Meyer, Hynds, Penetito, & Sleeter, 2011; Zavala, 2014). We aimed to investigate *how and to what extent does bias emerge within pre-service teachers' professional noticing of children's mathematical thinking with respect to differing perceived races and genders?*

Conceptual Framework

Professional Noticing

Professional noticing (of children's mathematical thinking) (PN) has been of interest to mathematics educators for some time (Schack, Fisher, & Wilhelm, 2017; Sherin, Jacobs, & Philipp, 2011). The professional noticing framework incorporates three interrelated components, attending, interpreting and deciding (Jacobs, Lamb, & Phillip, 2010). Although the relationship and temporal ordering among these components is still being explored (Castro-Superfine, Fisher, Bragelman, & Amador, 2017), there appears to be some stability within the literature regarding the general nature of these components (Thomas, 2017). Specifically, *Attending* refers to the attention given to observable aspects of a mathematical context, *Interpreting* refers to the internal mediating and sense-making of that which is attended and *Deciding* refers to an intended response which is informed by some interpretation of a mathematical context.

Equity and Professional Noticing

There is emerging interest in connecting and studying aspects of equity in conjunction with PN (Jong, 2017). PN, as a practice situated within classroom contexts, is inherently braided with equity constructs and frameworks, and study thereof is appropriate and necessary. Jackson, Taylor, and

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Buchheister (2018) synthesized professional noticing with four dimensions of equity (i.e., access, achievement, identity, power) put forth by Gutierrez (2009; 2013) resulting in a framework for intersections between noticing and equity. Louie (2018), investigating relationships between equity concerns and noticing, makes a case that strict cognitive orientations of professional noticing (Schack et al., 2013) may overlook significant cultural dimensions of a mathematical moment. For example, PN may be focused upon equitable concerns such as power distribution and student positioning (Louie, 2018). Such practices are consistent with asset-oriented perspectives where students' backgrounds and their contributions are valued. Further, in the area of asset/deficit perspectives, Harper (2010) presents an anti-deficit framework of research on students of color in STEM by shifting questions to focus on assets. While subtle, the reframing of questions to be anti-deficit can empower how students of color are studied. These findings suggest that biases influence, to some extent, the manner in which teachers enact PN in the mathematics classroom. Thus, investigations of the nature and thresholds of such biases are necessary.

Methods

Measures

To examine emergence of bias (i.e., asset/deficit perspectives), an electronic survey was constructed. The primary element of this survey was an adaptation of a video-based professional noticing measure used by Schack et al. (2013) in their study of preservice teachers' professional noticing capabilities. Rather than using a video clip of a teacher asking a student to solve a story problem as the anchor for professional noticing enactment, we substituted a transcription of the video. Similar to Schack et al., preservice teachers (PSTs) were asked to respond to three prompts, except in this study, the picture was also visible – each aligned with a particular component skill of professional noticing: 1) "Please describe in detail what [Student Name] did in response to the problem" (*attending*), 2) "Please explain what you learned about [Student Name]'s understanding of mathematics" (*interpreting*), and 3) "Pretend that you are [Student Name]'s teacher. What problems or questions might you pose next? Provide a rationale for your answer" (*deciding*). Further, PSTs were prompted to provide some basic demographic data (i.e., gender, ethnicity, age, home state) as well as their familiarity with professional noticing. The transcript case names were *Margaret* (perceived white female), *William* (perceived white male), *Shaquan* (perceived African American male), and *Miguel* (perceived Latino male) (see Figure 1). We limited ourselves to these four cases as we wanted to maximize opportunities to examine differences across gender (i.e., male/female – William/Margaret) and race (i.e., African American/Latino/white – Shaquan/Miguel/William). While more cases would have allowed for additional comparisons (e.g., Latino Female/Latino Male), they would also have necessitated a much larger data set to ensure that each case had an adequate number of survey respondents. Note, a prior study was scored using the same asset/deficit scale, *without a picture* of a child that matches the perceived ethnicity and gender of the name, showed that bias tends to manifest significantly in only the interpreting stage of professional noticing (Thomas et al., 2019).

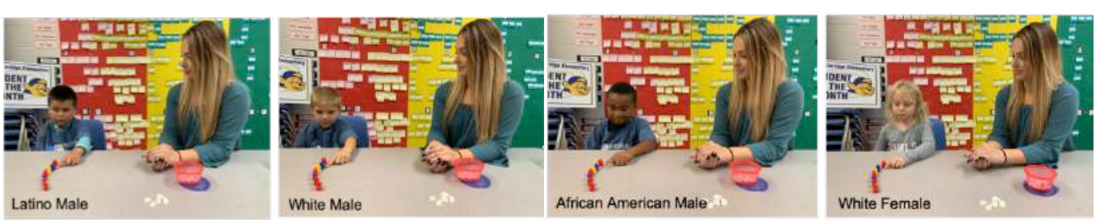

Figure 1: Case visuals

Participants

The electronic survey was fielded in teacher education programs across 18 states. Target respondents were PSTs who were in various stages of their respective teacher education programs. 315 PSTs initiated the survey, and incomplete surveys were discarded resulting in 170 completed responses with distribution across cases as follows: Margaret (n=44), Miguel (n=39), Shaquan (n=45), William (n=42). We used an electronic apportioning tool to ensure an approximately equivalent distribution of cases across respondents. Among the 170 participants, as one might expect of a preservice teacher sample, the largest gender and racial demographic was 18-24-year-old white females (59% of respondents)

Analysis

The asset/deficit perspective of the participant responses to the three questions were evaluated using a flow-process tool (AMSE, 1947) to determine the presence or absence of asset-oriented or deficit-oriented language describing the child's mathematical strategy. Each response was ascribed one of four different codes – asset, deficit, both [asset and deficit], and neutral. A previous study measuring equity in professional noticing, scored using the same asset/deficit scale, without a picture of a child (Thomas et al., 2019). Note, neutral responses contained no asset/deficit-oriented descriptions of the child's thinking/activity. Two raters used the flow process tool to calibrate with sample data, from a previous data set, until an 80% interrater reliability was achieved. The data from the current study were then blinded and combined into one list and scored independently by the two raters. Per previous studies of PN, rating differenes were resolved via discussion (Jacobs et al., 2010; Krupa, Huey, Lessieg, Casey, & Monson, 2017).

Findings

Adding the feature of a picture of a child produced similar results. The percentage of responses across perspectives with and without picture can be found in Table 1.

Table 1. Percentage of Responses Across Perspectives

		% With Picture	% Without Picture
Attending	Asset	19	6
	Deficit	4	5
	Neutral	75	87
	Both	2	2
Interpreting	Asset	38	31
	Deficit	13	27
	Neutral	24	12
	Both	25	30
Deciding	Asset	2	1
	Deficit	2	3
	Neutral	96	95
	Both	0	1

Delving more deeply into these data, we conducted chi-square tests to determine whether there are any relationships between case, survey type (picture, no picture), and bias categorizations (asset, deficit, neutral, both) (see Tables 3 and 4). Specifically, for each of the four cases, chi-square tests of independence were performed to test whether the different survey types were associated with a different distribution of attending, interpreting, or deciding bias categories. Furthermore, for each of the two survey types, chi-square tests of independence were performed for each noticing facet (attending, interpreting, deciding) to test whether each case was associated with bias categorization.

Table 2. Chi-square independence for each survey type between case and noticing component

Survey Type	Case vs. Attending	Case vs. Interpreting	Case vs. Deciding
No Picture	13.508 (9)	8.061 (9)	7.628 (9)
Picture	7.890 (9)	20.575 (9)*	5.719 (6)

Note. * $p < .05$, ** $p < .01$. Results are reported as χ^2 (df).

Table 3. Chi-square independence for each case between survey type and noticing component

Case	Survey Type vs. Attending	Survey Type vs. Interpreting	Survey Type vs. Deciding
Margaret	7.168 (2)*	9.025 (3)*	3.606 (3)
Miguel	0.935 (3)	4.826 (3)	0.781 (2)
Shaquan	2.430 (3)	1.204 (3)	1.506 (3)
William	7.294 (3)	14.629 (3)**	N/A

Note. * $p < .05$, ** $p < .01$. Results are reported as χ^2 (df).

A test of independence could not be calculated for the William case in the deciding component because all responses for this condition were categorized as "neutral". From these results, for picture survey, we see a significant association between case (Margaret, William, Shaquan, Miguel) and bias categorization within the professional noticing component of interpreting. Further, the Margaret case showed significant association between survey type and the attending bias categorization, and both the Margaret and William cases showed significant association between survey type and the interpreting bias categorization.

Discussion

The mere changing of names (e.g., Shaquan, William, Margaret, Miguel) does not appear to provoke the directly negative biases observed in other studies (Bertrand & Mullainathan, 2004; Hanson, Hawley, Martin, & Liu, 2016). Looking toward the relationship between bias and action, Rudman (2004) describes the relationship between individuals' implicit and explicit biases as connected but somewhat distant. As one's explicit biases exist consciously downstream of one's implicit biases, they are inherently more malleable by the individual. With respect to teacher noticing, implicit biases may (and likely do) influence teacher decision-making (i.e., deciding) and the situating of such biases appears to be within the interpreting component.

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MIDDLE SCHOOL STUDENTS' CONTEXTUALIZED GEOMETRIC SPATIAL UNDERSTANDINGS

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This study examined the spatial-scientific understandings of students from Kentucky (6th graders) and Nevada (8th graders). Quantitative data consisted of students completing a content survey as well as two spatial assessments at the conclusion of Earth-space instruction. Qualitative data involved student interviews concerning 2D Earth/Moon/Sun modeling. Findings showed Kentucky and Nevada students shared similar misconceptions regarding geometric motions, configurations, and spatial awareness to explain the physical phenomenon of lunar phases. Post data revealed significant differences in favor of Kentucky on lunar phases understanding related to the spatial domain of spatial projection (ability to visualize the Moon from multiple Earthly locations). Significant differences were also found in favor of Kentucky on the Geometric Spatial Assessment. No significant differences were found between students on mental rotation ability.

Keywords: Geometrical Spatial Thinking, STEM

Objectives

This research with middle level students from Kentucky (N=238) and Nevada (N=138) explored how well students from two geographically different locations understood lunar-related spatial-scientific content. The Next Generation Science Standards (NGSS Lead States, 2013) and Common Core State Standards-Math (National Governors Association, 2010) iterate the importance of student understandings related to spatial-scientific learning (i.e. scale, patterns, and geometric modeling). Previous research (Plummer et al., 2014, Black 2005) has linked increased spatial ability with an increased understanding of lunar phases. This study examined students' geometric spatial ability and how students developed and contextually applied this ability to their understandings concerning the phenomenon of lunar phases. The research question was: What geometric spatial factors might hinder or facilitate moon phase understanding? Factors could include students' understanding of scale of the Earth/Moon/Sun system, students' geographic perspective as they observe the moon, students' ability to recognize patterns, and students' aptitude to visualize in both 2D and 3D spaces.

Perspective: Spatial Reasoning and Scientific Performance

Students with high spatial reasoning tend to perform better on science assessments than students with low spatial ability; this has been found true on science assessments concerning chemistry, geoscience, physics, astronomy, calculus, and anatomy (Cole, Cohen, Wilhelm, & Lindell, 2018; Wilhelm, Toland, & Cole, 2017; Sorby, Casey, Veurink, & Dulaney, 2013). Wilhelm, Cole, Cohen, and Lindell (2018) argued that when spatial reasoning ability is advanced via an intervention or spatial experiences within a particular discipline, this spatial development should lead to improved understanding in other scientific disciplines. For example, in the Sorby et al. (2013) study, freshmen engineering students were separated into two groups (an intervention group and a comparison group) based on results of a mental rotation (MR) test. Students who scored low on the MR test were assigned to a spatial intervention course and those who scored above a passing cutoff grade were

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assigned to a comparison group. Sorby et al.'s findings showed the treatment group's scores increased after the intervention as shown on a post MR test, and even more interesting, treatment students displayed ***transfer*** effects as displayed in increased calculus performance.

Other studies have shown correlations between spatial reasoning and science performance as well as gender differences on spatial reasoning assessments. Guillot et al. (2006) researched the relationship between visuo-spatial representation, MR, and functional anatomy examination results. Guillot et al. (2006) measured visuo-spatial skills using the Group Embedded Figures Test (GEFT; Demick, 2014) which contains 18 complex figures. The test taker must identify a simple form by tracing the simple form within the complex form. MR was measured using the PSVT-Rot (Bodner & Guay, 1997). Guillot et al. found that males scored better than women in GEFT and the MR test; however, this "gender effect was limited to the interaction with MRT ability in the anatomy learning process. The correlations found between visual spatial and MR abilities and anatomy examination results underscore the advantage of students with high spatial abilities" (p. 504).

Spatial Thinking in an Astronomical Context

People interact with many aspects of astronomy on a daily basis, often without noticing them. They develop their own ways of knowing and explaining astronomical phenomena from their conscious and unconscious daily glances at the Moon and sky. In reality, these ideas are more complex than most people realize. In order to understand many aspects of astronomy, developed spatial thinking ability is required. The necessary spatial thinking skills vary by astronomy topic, but studies show that spatial reasoning ability contribute to understanding of astronomy (Wilhelm, et al., 2018). Spatial reasoning ability, as stated earlier, has been linked to performance in both mathematics and science (Black, 2005; Lord & Rupert, 1995; Wilhelm, 2009; Wilhelm, Jackson, Sullivan, & R. Wilhelm, 2013). In terms of lunar phases, spatial thinking ability in the domain of mental rotation is particularly important (Wilhelm et al., 2018). Historically, males have shown an advantage in spatial thinking, particularly in the area of mental rotation. "Countering this view is substantial evidence that environmental influences, in the form of experience in spatial activities from an early age and explicit training can eliminate sex differences on spatial tasks" (Linn & Petersen, 1985; Casey et al., 1999). Thus, it is important that spatially rich curricular experiences be examined to better understand how we can foster the development of factors that encourage students' geometric spatial understanding of scientific phenomena such as lunar phases.

Methods

Study Design

In order to determine what geometric spatial factors hinder or facilitate middle level students' lunar phases understanding, we utilized a mixed methods design. Students were purposefully selected from two different geographic locations so that we might be able to determine if sky viewing in a mountainous terrain would affect students' ability to accurately note Moon motion, Moon rise/set times, and visualization of relative positions of the Earth, Moon, and Sun as compared to Kentucky students in comparatively flat terrain. Quantitative data included the Lunar Phases Concept Inventory (LPCI; Lindell & Olsen, 2002), the Geometric Spatial Assessment (GSA; Wilhelm et al., 2007), and the Purdue Spatial Visualization Test-Rotations (PSVT-R; Bodner & Guay, 1997). The LPCI is a 20 question multiple choice test that assessed eight science domains as well as four spatial domains. The PSVT-Rot was a 20-item multiple choice survey that assessed the level of mental rotation reasoning. The GSA was a 16-item multiple choice test that assessed the same spatial domains addressed by the LPCI, but outside of a lunar context. The qualitative data included semi-structured interviews, where four students were chosen by each teacher for the interviews. Teachers were asked to select the girl and boy with the highest and lowest spatial ability in their classes.

Participants

Participants were from two states, Kentucky and Nevada. Subjects were drawn from one public school in Kentucky with three 6th-grade teachers and their students (N=238). The teaching experience of the Kentucky teachers ranged from 6 to 16 years. The Kentucky teachers' Earth-space curricular unit is outlined in Table 1. Three 8th-grade teachers in a public school in Nevada participated along with their students (N=138). Their teaching experience ranged from 3 to 14 years. Nevada teachers taught their Earth-space curricular unit as shown in Table 1. Grade years were chosen based on the grade lunar phases content was required to be taught in each state (6th grade for Kentucky and 8th grade for Nevada). Both Kentucky and Nevada teachers implemented their units in approximately 5 weeks and both curricula asked students to keep a Moon journal. The main difference in the two curricula was Nevada's emphasis on eclipses which was embedded within lessons on phases and scaling. Kentucky lessons incorporated ***Stellarium*** (software) to examine views and motions from both Northern and Southern hemispheres.

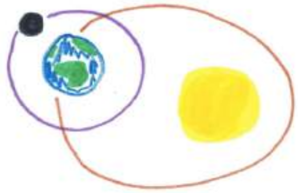
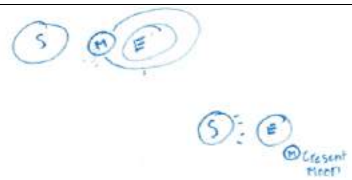
	Kentucky Curricular Unit	Nevada Curricular Unit
Lesson 1	***Can I see the Moon every night and why does it appear to change shape? -*** *Moon Journals* - Students keep daily Moon observation journals for 5 weeks.	***Ancient Civilization and the Moon***
Lesson 2	***How do I measure the distance between objects in the sky?***	***What's Up with the Moon? –*** Students record Moon observations for a month
Stellarium	Students observe the apparent motion of the Moon over the course of a day and compare this motion for locations in the Northern and Southern hemispheres.	
Lesson 3	***How can I say where I am on the Earth?***	***Earth's Moon Vocabulary***
Lesson 4	***How can I locate things in the sky?***	***Determining Hours of Daylight***
Lesson 5	***Why do we have Seasons?***	***Origin of the Moon***
Lesson 6	***What can we learn by examining the Moon's surface?***	***The Sun-Earth-Moon system and Eclipses***
Lesson 7	***What affects a crater's size?***	***Eclipses***
Lesson 8	***The scaling Earth/Moon/Mars NASA Activity***	***Scaling the Sun-Earth-Moon system and Solar and Lunar Eclipses***
Lesson 9	***Moon Finale*** - **Lunar Phases Modeling**	

Table 1: Kentucky and Nevada Curricular Units

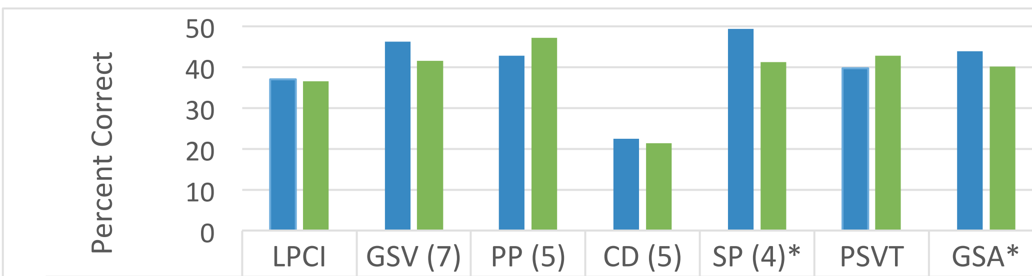
Results

Although both Kentucky (KY) and Nevada (NV) teachers asked their students to keep Moon journals for at least 4 weeks, a large portion of students in both locations failed to do so. As noted in Table 1, Nevada teachers placed a heavier emphasis on eclipses and taught this concept within lessons on phases and scaling. Qualitative interviews with high and low spatial ability NV and KY students showed similar ideas regarding geometric positioning of the Earth, Moon, and Sun for various lunar phases as well as how the Moon orbits the Earth. Table 2 illustrates representative samples of NV and KY high and low spatial ability students' geometric orientations and motions of the Earth/Moon/Sun system. Table 2 shows a High Nevada student modeling correctly the Moon's orbit around the Earth and the Earth's orbit around the Sun, and a High Kentucky student illustrating correctly the Earth/Moon/Sun geometry for New Moon and Waxing Crescent phase (although, neither representation is to scale). A Low Nevada student shows an incorrect understanding of the geometric configuration of a Waxing Crescent phase by demonstrating either an Earth blocking notion or an Earth's shadow misconception, and a Low Kentucky student revealed similar ideas to the Low Nevada student.

Table 2: Students' Geometric Spatial Orientations for Various Lunar Phases

High Nevada	High Kentucky
	
Low Nevada	Low Kentucky
	

Quantitative KY and NV survey results are shown in Figure 1 for the LPCI and the LPCI spatial domains: Geometric Spatial Visualization (GSV), Periodic Patterns (PP), Cardinal Directions (CD), and Spatial Projection (SP). Other results shown in Figure 1 are the PSVT-Rot test and the Geometric Spatial Assessment (GSA). Kentucky 6th grade students scored significantly higher on the SP domain items of the LPCI test and significantly higher on the GSA test than the 8th grade Nevada students. Test results showed KY and NV students had similar percentages of students holding classic misconceptions regarding cause of lunar phases explanations (object blocking (~10%), Sun's shadow (~25%), and Earth's shadow (~42%)).



	LPCI	GSV (7)	PP (5)	CD (5)	SP (4)*	PSVT	GSA*
KY (N=238)	36.97	46.22	42.77	22.44	49.37	39.68	43.88
NV (N=138)	36.45	41.51	47.1	21.3	41.12	42.71	40.13

Figure 1: NV and KY students post scores on the LPCI by domain, PSVT, and GSA (*p < 0.05)

Discussion and Conclusion

Regardless of geographic region, students held similar misconceptions concerning the causes of the lunar phases (i.e. object blocking, Sun's shadow, and Earth's Shadow); however, KY students scored significantly higher on the SP domain that concerns visualizing how the Moon appears from various Earthly perspectives on same day. KY students also scored significantly higher on the Geometric Spatial Assessment. The GSA is not in a lunar context and assesses all four spatial domains (PP, GSV, CD, and SP). Possible explanations for the differences could be due to the heavy emphasis on eclipses in the NV curriculum which could have confused students since they were also trying to comprehend/visualize cause of lunar phases.

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STUDENT LEARNING AND RELATED FACTORS:

POSTER PRESENTATIONS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

PRESCHOOL-AGED CHILDREN'S EMERGENT DISPOSITIONS WITH RESPECT TO MATHEMATICS

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Beginning to understand children's mathematical dispositions during the preschool years may be beneficial as early childhood experiences can predict student's achievement in subsequent educational settings (Fleer & Raben, 2005). Findings suggest that mathematical dispositions can be related to academic success in mathematics (Beyers, 2011a, 2012; Kusmaryono, Suyitno, Dwijanto, & Dwidayati, 2019; O'Dell, 2017). However, there are relatively few studies examining mathematical dispositions among preschool-aged children. However, it is critical to explore the nature of these dispositions and corresponding behaviors at this stage of development, because these behaviors may be directly related to mathematical gains in subsequent educational settings (Hofer, Farran, & Cummings, 2013). The current study aims to address the question of whether preschool-aged children demonstrate observable mathematical dispositions in a pre-kindergarten school environment.

The authors draw on a conceptual framework which includes three primary areas of mathematical dispositions: cognitive, affective, and conative dispositional functions (Beyers, 2001a). Within those three areas of dispositional functions reside 10 total dispositional functions[1], such as attitudes, beliefs, argumentation (Beyers, 2011b). The authors developed an observation rubric based on construct definitions and examples of dispositional functions in previous work. For example, within the affective dispositional function attitude, a child who has a positive attitude about mathematics may gesture gleefully or offer relevant verbal excitement when engaged in or about to engage in mathematical activity, when he or she had just previously not been excited. Conversely, a student's behavior might shift from excitement to a more apathetic state when a mathematical task or discussion is introduced.

The second author was a teaching assistant in an early childhood center. Math activities were done informally as part of games and more formally as part of instruction. The teaching assistant took copious field notes annotating her observations throughout the day. Any activity directly or indirectly involving mathematical content was highlighted in her field notes. Both authors then reviewed and coded a portion of the field notes together, and then the remainder independently. Evidence could have been of a verbal or non-verbal nature. Evidence was coded to reflect whether evidence of a dispositional function was present, and which dispositional function was observed. The authors achieved interrater reliability over .80.

The data show that some dispositions with respect to mathematics are present among preschool-aged children. There is evidence of dispositional functions from each of the three primary areas of dispositional functioning: cognitive, affective, and conative, but not all 10 dispositional functions within those three areas. Data include examples of the cognitive function argumentation, the affective function attitude, and the conative function persistence. Evidence for other dispositional functions, such as making connections, or beliefs about the nature of mathematics were not observed. It is possible that other functions were not observed because opportunities to engage or demonstrate those dispositional functions did not present themselves or those dispositional functions are not yet emergent. Further examination is warranted.

[1] For a complete discussion of the 10 dispositional functions (e.g., while engaging in mathematical activity, demonstrating a tendency to: make connections, make mathematical arguments, hold certain attitude or beliefs, etc…) please refer to Beyers, 2011b.

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A PRELIMINARY MODEL OF INFLUENCES ON ADOLESCENTS' PERCEPTIONS OF USEFULNESS IN SCHOOL MATHEMATICS

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Keywords: Affect, Emotions, Beliefs, & Attitude; Middle School Education

Students commonly question teachers about the *usefulness* of mathematics, and the Common Core State Standards in Mathematics encourages teachers to help students "see mathematics as…useful" (National Governors Association Center for Best Practices, 2010). In fact, perceiving a subject as useful can have numerous positive benefits for students including enhanced interest in a subject and improved course performance (Hulleman et al., 2010; Hulleman & Harackiewicz, 2009). However, adolescents use a range of criteria to evaluate whether something is useful (Dobie, 2019), and little is known about when or why these various criteria are employed. This research explores the question, *What factors influence the decisions adolescents make about what is or is not useful in the context of school mathematics?*

Expectancy-value theory highlights that one's goals and self-schemata influence perceptions of usefulness (Eccles & Wigfield, 2002), yet a black box remains regarding the mechanisms that mediate this relationship. The current research begins to build theory around influences on adolescents' perceptions of usefulness by drawing on data from interviews with 11-14-year-old students in two large cities in the United States. In particular, adolescents responded to card-sorting tasks depicting images of students engaging with varied mathematics content in a range of ways and described whether or not the mathematics seemed useful and why. Those responses were used to identify criteria adolescents used to make judgments about usefulness, and additional questions probed into the factors that influenced the criteria students applied.

Figure 1 illustrates a preliminary model unpacking the relationship between one's goals and self-schemata, and perceptions of usefulness. Emergent influences include whether usefulness was considered at the level of the subject (mathematics), specific topic (e.g., linear equations), or particular task (e.g., worksheet). Additionally, some adolescents attended to the form of engagement (e.g., individual vs. collaborative). Others considered the novelty of what they were learning, the usefulness of a specific practice (e.g., justifying thinking) or strategy (e.g., making graphs to represent data), or how engaging with the mathematics made them feel. These features attended to in turn influenced the specific criteria used to make judgments about usefulness, such as whether the mathematics applies to everyday life or enhances one's understanding of mathematics. Individual cases of students will be shared along with quotes to illustrate each factor in action. Future work will explore how different features influence the criteria applied, as well as the outcomes afforded by different pathways to perceiving mathematics as useful.

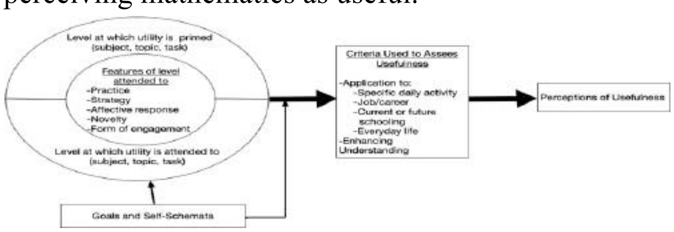

Figure 1: Influences on Adolescents' Perceptions of Usefulness

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EXPLORING THE RELATIONSHIP BETWEEN MATH ANXIETY, WORKING MEMORY, AND TEACHER PRACTICES

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Keywords: Affect, Emotion, Beliefs, Attitudes, Instructional activities and practices

Betz (1978) proposed that 68% of students in mathematics classes experience high levels of math anxiety. This is most unfortunate as it is a well-established fact that math anxiety is negatively correlated with mathematics performance (Ashcraft & Kirk, 2001; Ashcraft & Moore, 2009; Foley et al., 2017). This does not necessarily imply that math anxiety is an indicator of lower potential to succeed in mathematics. Arnsten (2009) and Diamond et al. (2007) have shown that moderate levels of anxiety can help focus attention and enhance working memory which is known to be a major factor in math competence. It has also been shown that the negative correlation between math anxiety and math performance is stronger for those with high working memory capacity (Foley et al., 2017). Though there has been much research on working memory and situational factors associated with math anxiety, there is not much research which synthesizes the data on working memory with classroom experiences relating to math anxiety. Furthermore, few studies on math anxiety include participants with a broad range of math anxiety levels.

In this study, we sample students in a year-long calculus course. We dig deeper into how students experience math anxiety and how they interpret past classroom experiences. The study utilizes tests for both math anxiety and general anxiety. Interviews are conducted in order to examine past classroom experiences and how these experiences helped to shape the students' belief of math anxiety. We use the interpretation framework developed by Ramirez et al. (2018) to explore the impact of classroom experiences on the development of math anxiety. Under this framework, we hope to discover ways in which the instructor can construct rigorous and engaging classroom activities which would ultimately fashion a favorable impression upon the student.

We also use the disruption account framework proposed by Ashcraft & Kirk (2001) to interpret the role in which working memory affects math anxiety and math performance. The interviews include various working memory tests along with written mathematical procedures. We hope to synthesize the information we gain from these activities with the data we collected for math anxiety and experiences in order to gain deeper insight into how we understand math anxiety.

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PROBLEM POSING IN PARTITIVE AND QUOTITIVE DIVISION

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Keywords: Modeling, Problem Solving, Number Concepts and Operations

Problem posing (Brown & Walter, 2005) can improve problem-solving (e.g., English, 1998), expand mathematical creativity (e.g, Voica & Singer, 2012), and provide instructors with valuable insight into student understanding (e.g, Silver, 1994). Problem posing can also give students ownership in learning and provide students with authentic modelling processes (Hanson & Hana, 2015). We aim to investigate problem posing with partitive and quotitive models of division. In quotitive division, the divisor is the number in each group. In partitive, the divisor is the number of groups (Neuman, 1999). Asking students to make the distinction between the two models of division has been shown to help students better understand place value (Bicknell, Young-Loveridge & Simpson, 2017), division with decimals (Okazaki & Koyama, 2005), division with remainder (Lamberg & Wiest, 2012), and the division algorithm (Silver, 1987).

Methods

In a 100-level math course at a private liberal arts university, which incorporated various problem posing tasks, students (n=38) were given the task "For 14÷2=7, write (a) one word problem that demonstrates a partitive model and (b) one that demonstrates a quotitive model." Students were then asked to create children's books that contained both models of division, to be donated to the Girls & Boys Club as a local outreach project. We seek to address: how do students respond when asked to pose problems demonstrating the two models of division?

Results

In the written task, 14 (36.8%) correctly demonstrated both models of division. We observed gaps in understanding and nuanced misconceptions, which inform teacher education. For example, students commonly provided partitive when asked for a quotitive model, provided problems not solved by 14÷2=7, and struggled with the concept that word problems from both partitive and quotitive models can correspond to the same equation.

Table 1: Summary of Student Response Themes

	Incorrect Model	Used Own Numbers	Rearranged Given Equation	Phrased as Statement
Part (a) [Partitive]	4 (10.5%)	5 (13.6%)	1 (2.6%)	15 (39.5%)
Part (b) [Quotitive]	14 (36.8%)	6 (15.8%)	10 (26.3%)	16 (42.5%)

Furthermore, we observed that all (14 of 14) students who drew an image with their response on part (a) provided a correct model. The analysis of the children's book task, which revealed what contexts the students chose, yielded themes of real-world relevance, fairness, and novelty. Future research includes exploring the impact of problem posing on problem solving in division and student perceptions of a course that utilizes problem posing tasks.

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MODAL CONTINUITY IN DEAF STUDENTS' SIGNED MATHEMATICAL DISCOURSE

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Keywords: Deaf, Design experiments, embodiment and gesture, mathematical discourse

Despite numerous studies pointing to the delay in mathematical achievement of deaf and hard-of-hearing (DHH) students as compared to their hearing peers (e.g., Traxler, 2000; Pagliaro & Kritzer, 2013), only little is known about the peculiarities of mathematical learning processes of DHH learners. At the same time, studies both in psycholinguistics and mathematics education portray sign language (SL) as playing a key role in their individual and social processes of concept formation (e.g., Kurz & Pagliaro, 2020; Krause, 2019).

From both embodied and semiotic perspectives, the iconicity of SL signs may be relevant for mathematical thinking and learning: emerging from action, some signs carry and sustain enactive and/or depictive features of source sensorimotor forms, thus spontaneously schematizing individuals' situated enacted experience. As this, it becomes part of the modal hybrid of gestures and signs in signed discourse and thereby shapes the development of socially negotiated mathematical meaning. At the same time, the gestural expression develops from idiosyncratic gestures towards locally conventionalized signs that refer to the situated mathematical meaning. What is first action then influences a gestural representation of the action in a new mathematical context and eventually a gestural representation for a developing mathematical idea. Signed discourse thus facilitates modal continuity in the gestural modality from individual manual action to expression in social interaction, vice versa feeding into the individual's situated understanding. How modal continuity affects mathematical thinking and learning, we submit, is important for the theory and practice of mathematics education, both for DHH and for hearing learners. More broadly, theorizing modal continuity could illuminate the relationships between embodiment, representations, and language in processes of teaching and learning mathematics.

The poster reports on an ongoing design-based research project in which we develop a mathematical learning opportunity that considers sign language as a resource for learning mathematics. More concretely, we adapt a well-established embodied design—the Mathematical Imagery Trainer for Proportions (Abrahamson & Trninc, 2015)—with an eye on emerging manual movement patterns that foster a common ground for mathematical discourse in a way that is conducive to linguistic accuracy of (American) SL. In a two-step design, students first each develop sensorimotor schemes through solving a dynamic interaction problem, then explore in pairs a related mathematical problem, negotiating mathematical meaning in signed discourse. The poster will expand on the design's rationale and elaborate on the theoretical construct of modal continuity in light of data collected with Deaf and hearing students.

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ADOLESCENTS' MATHEMATICS IDENTITIES IN A MATH CIRCLE

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Keywords: Informal education; Affect, emotions, beliefs, and attitudes

Identity can be conceptualized as a function of participation in the various activities of communities of practice (CoPs), which are defined by mutual engagement, a joint enterprise, and a shared repertoire amongst its members (Wenger, 1998). An individual's mathematics identity development, then, is inextricably linked to their participation in the CoPs associated with the contexts in which they learn mathematics (Boaler & Greeno, 2000).

One such context is a Math Circle, a type of out-of-school mathematics program in which adolescents, guided by "mathematically sophisticated leaders, […] work on interesting problems or topics in mathematics […] through problem-solving and interactive exploration" (What is a math circle?, n.d.). The participants' participation in the Math Circle CoP is mediated through their participation in various activities of the CoP, where activity is defined as "a socially recognized and institutionally or culturally supported endeavor that usually involves sequencing or combining actions in certain specified ways" (Gee, 2014, p. 95). Therefore, identifying the Math Circle CoP activities is key to understanding (1) how these adolescents participate in the CoP and (2) how their participation affects the development of their mathematics identities.

I interviewed three City Math Circle (CMC) participants who had completed at least three years in CMC programs by Spring 2020. The interview addressed their current and past participation in CMC programs, what occurs in a typical CMC session, and their relationships with mathematics in different environments. I analyzed the transcripts using Gee's (1991; 2014) narrative structure, identified CMC CoP activities and the adolescents' participation therein using social practice analysis (van Leeuwen, 2008), and analyzed their participation in these activities using Wenger's (1998) modes of belonging.

The adolescents' descriptions of the CMC CoP activities were consistent with each other, and most activities involved both mathematical and social interactional components (i.e. program participants gave each other feedback [social interactional] on their problem solutions [mathematical] as part of the "sharing solutions" activity). Participation in such activities tended to be inversely related to how the adolescents participated in the mathematical and social interactional components of activities in other mathematics learning environments. For example, one adolescent who felt that "school mathematics" did not allow for collaboration with peers described the social interactional components of the CMC CoP activities in more depth than the mathematical components, where another who did not feel challenged by "school mathematics" described the mathematical components of the CMC CoP activities in more depth than the social interactional components.

These findings suggest that the development of these adolescents' mathematics identities due to their participation in City Math Circles programs is complementary to the development of their mathematics identities due to participation in other mathematics learning contexts. That is, through their voluntary participation in City Math Circles, an out-of-school mathematics program, these adolescents are developing their (mathematics) identities with agency they are not typically allowed in more institutionalized mathematics learning environments such as school.

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TUTORING LAB ATTENDANCE AND TIME SPENT ON HOMEWORK: IMPACT ON STUDENT PERFORMANCE IN COLLEGE ALGEBRA

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The aim of this research is to investigate how College Algebra students utilize a peer-staffed tutoring lab during a spring semester. Records of students' ID cards swipes as they enter the lab were analyzed to explore trends in attendance. In addition, the relationship between final course grades and the time spent studying math content using ***A****ssessment and* ***LE****arning* ***K****nowledge* ***S****paces (ALEKS), a web-based, adaptive learning system, was explored. The results found 79% of the students who used the tutoring lab made a D or better, a passing grade, versus 48% of the students who did not visit the lab. Also, for every 100-minute increase in time spent in ALEKS, a student's final course grade increased on average by 1%.*

Keywords: Post-secondary, Instructional activities and practices

While a large number of students take College Algebra each year, only 50% earn a C or higher (Ganter & Haver, 2011). Research shows that peer-tutoring has a positive impact on student performance in College Algebra (Xu, Hartman, Uribe, & Mencke, 2001), and time spent on homework out-of-class has a substantial effect on grades (Keith, Diamond-Hallam, & Fine, 2004). Based on this existing literature, this study aims to investigate 1) student attendance in a peer-staffed tutoring lab, 2) lab attendance's impact on final course grade, and 3) the relationship between time spent studying in ALEKS and final course grade.

During a spring semester, six College Algebra classes were included in the study. The students were expected to spend three hours each week completing homework assignments in ALEKS (with one-hour goals staggered throughout the week). The students could complete the time requirement from anywhere but were encouraged to attend a peer-staffed tutoring lab to work on homework. Lab attendance was incentivized by earning extra credit on the exams if a certain amount of time was met before each exam. Overall, students were encouraged to increase the time spent on studying math content in the lab and stagger that time throughout the week.

Lab attendance was collected through swipe-card access using students' university ID cards when students entered the lab. Total time spent working in ALEKS and final course grades were also collected for each student. The statistical software R was used for linear regression to model the relationship between final course grade and time spent working in ALEKS.

Among the 166 research participants, 57% attended the tutoring lab at some point during the semester. Half of the students who attended the lab came only once or twice during the semester with very few students coming on a weekly basis. During the week of exams, there were small peaks in lab attendance. Five to eight more students came during exam weeks, and there were 10 to 16 more swipes. Of the students who came to the lab, 79% made a D or better, a passing grade for the course. For the students who did not attend the lab, only 48% made a D or better. There was a positive relationship (r = 0.639) between final course grade and time spent studying in ALEKS. The r-squared value was 0.4083, so about 41% of the variance can be explained by the model. For every 100-minute increase in time spent in ALEKS, a student's grade will increase by about 1%. The preliminary results of this study point to the peer-tutoring lab having a positive impact on students' College Algebra course grades.

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STUDENT INQUIRY IN INTERESTING LESSONS

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Keywords: Classroom Discourse, Communication, Curriculum

What impact, if any, do interesting lessons have on the types of questions students ask? To explore this question, we used lesson observations of six teachers from three high schools in the Northeast who were part of a larger study. Lessons come from a range of courses, spanning Algebra through Calculus. After each lesson, students reported interest on a Likert scale via lesson experience surveys (Riling et al., 2019). The average interest measures were then used to identify each teachers' highest and lowest rated interest lessons. The two lessons per teacher allows us to compare across the same set of students per teacher.

We compiled 145 student questions and identified whether questions were asked within a group work setting or part of a whole class discussion. Two coders coded 10% of data to improve the rubric for type of students' questions (what, why, how, and if) and perceived intent (factual, procedural, reasoning, and exploratory). Factual questions asked for definitions or explicit answers. Procedural questions were raised when students looked for algorithms or a solving process. Reasoning questions asked why procedures worked, or if facts were true. Exploratory questions expanded beyond the topic of focus, such as asking about changing the parameters to make sense of a problem. The remaining 90% of data were coded independently to determine interrater reliability (see Landis & Koch, 1977). A Cohen's Kappa statistic ($K=0.87$, $p<0.001$) indicates excellent reliability. Both coders then reconciled codes before continuing with data analysis.

Initial results showed differences between high- and low-interest lessons. Although students raised fewer mathematical questions in high-interest lessons (59) when compared with low-interest lessons (86), high-interest lessons contained more "exploratory" questions (10 versus 6). A chi-square test of independence shows a significant difference, χ^2 (3, N = 145) = 12.99, p = .005 for types of students' questions asked in high- and low-interest lessons. The high-interest lessons had more student questions arise during whole class discussions, whereas low-interest lessons had more student questions during group work. By partitioning each lesson into acts at points where the mathematical content shifted, we were able to examine through how many acts questions remained open. The average number of acts the students' questions remained unanswered for high-interest lessons (2.66) was higher than that of low-interest lessons (1.68). Paired samples t-tests suggest that this difference is significant $t(5)=2.58$, $p = 0.049$.

Therefore, student interest in the lesson did appear to impact the type of questions students ask. One possible reason for the differences in student questions is the nature of the lessons students found interesting, which may allow for student freedom to wonder and chase their mathematical ideas. There may be more overall student questions in low-interest lessons because of confusion, but more research is needed to unpack the reasoning behind student questions.

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PROBLEM SOLVING DISPOSITIONS IN RURAL COMMUNITIES

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Even as the mathematics education field has highlighted the importance of community-based resources, most of the attention has focused on local knowledges and practices. There has been comparably less attention paid to community-based dispositions. In ethnographic fieldwork in rural communities, we identified 5 community-based problem solving dispositions.

Keywords: Culturally Relevant Pedagogy, Rural Education

The mathematics education field has shown much interest in community-based, "everyday" practices (e.g., Civil, 2016). Research has documented the rich and varied ways that people understand and participate in the world (Lave, 1988). The field has also been interested in understanding disciplinary dispositions and advancing the notion that developing particular dispositions ought to be one goal of schooling (Gresalfi & Cobb, 2006). We bring these lines of inquiry together by sharing 5 community-based problem solving dispositions we found via ethnographic fieldwork in rural communities in the Rocky Mountain West.

Findings

We documented 5 CPSDs which were durable across data sources and communities: (1) *Self-reliance*: Community members were inclined to "do it themselves" or learn how to do it themselves. (2) *Resourcefulness*: Even as community members were disposed towards self-reliance, they were also inclined to seek out and use material and social resources to help them solve problems. (3) *Care/Helpfulness*: We found that community members were inclined to care for one another and accept care from others when faced with problems. (4) *Try something and iterate*: Community members were inclined to dive right in and "try" something. The notion of trying captures the willingness to attempt a particular approach, without full confidence that it will work. (5) *Practical wisdom*: Community member were inclined to use knowledge developed through experience, or which circulated and was taken for granted in the community.

Implications

We agree with scholars who argue that schools should not seek to replace community-based resources but rather should seek to *build with* and *strengthen* them. Our work has implications for this effort. For example, many classrooms are organized to purposefully restrict access to resources, especially during consequential activities such as testing. This seems to work against a community-based disposition of resourcefulness. Our work prompts us to wonder how schools might be reorganized to build with and strengthen this and other community-based dispositions.

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THE EFFECTS OF WORKED EXAMPLE FORMATS ON STUDENT LEARNING OF ALGEBRA

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Keywords: Algebra, Cognition, Problem Solving, Worked Examples

Worked examples consist of step-by-step derivations of problems, and are often provided to students as a way to learn the procedures for solving problems. An extensive body of literature has shown that worked examples can help students learn mathematics (e.g., Atkinson, Derry, Renkl, & Worham, 2000; Booth, Lange, Koedinger, & Newton, 2013; Sweller, 2006) but how the presentation of worked examples influence student learning remains unclear. For instance, new educational technology tools that demonstrate the dynamic process of solving algebraic equations (e.g., dragging 4 across the equal sign to initiate the inverse operations and divide both sides by 4 in $4x = 6$) may provide the opportunity to explore the potential benefits of dynamic worked examples.

To study how variations in worked example presentations influence learning, we varied the format of worked example presentations of algebraic equations in an online platform. Specifically, we compare the impact of viewing worked examples in animated dynamic presentations, traditional static presentations, and sequential line-by-line presentations. Within these variations, we also compare concise worked examples (e.g., Rittle-Johnson & Star, 2007), which display only the important steps of solving algebra problems, and extended versions, which display all steps in a derivation.

Our research questions are as follows: 1) *Do concise or extended worked examples lead to larger gains from pretest to posttest?* 2) *Do students show differential gains from viewing dynamic, static, or sequential worked examples?* 3) *If dynamic worked examples lead to higher learning gains, which form of dynamic worked examples are best for learning?* Currently twelve middle school teachers with a total of 146 students have completed this study and data collection is ongoing. Students complete a pretest on algebraic equation solving, six worked example-practice problem pairs, and a mirrored posttest in a 45-minute session. Our estimated sample of over 400 affords at least 80% power to detect small to medium effects of worked example length ($f = 0.13$) and presentation ($f = 0.15$).

Data analysis for this study will be conducted once all participants have completed the study in Fall 2020. To explore our research questions, a 2 (pretest vs. posttest) × 2 (concise vs. extended) ANOVA will be conducted to test the effects of work example length, and a 2 (pretest vs. posttest) × 3 (static, sequential vs. dynamic) ANOVA, and a 2 (pretest vs. posttest) × 2 (dynamic concise vs. dynamic extended) ANOVA will be conducted to test the effect of each worked example format on learning gains.

We expect that the results from this study will help the field better understand the best way to present worked examples. Results and implications will be presented at the PME-NA conference.

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TEACHER EDUCATION (PRE-SERVICE)

RESEARCH REPORTS

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USING VIDEO ANALYSIS TO IMPROVE PRESERVICE ELEMENTARY TEACHERS' NOTICING SKILLS

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Prospective elementary mathematics teachers (PTs) were asked to analyze 28 videos of cognitive interviews. The purpose of this study was to determine if experiences analyzing videos would lead to improvements in PTs' professional noticing skills. Using a coding schema that reflected three levels of understanding (periphery, transitional, and accomplished), a frequency table was constructed that allowed PTs' use and understanding of a noticing framework to be analyzed. Findings indicate that experiences analyzing videos leads to improvements in PTs' professional noticing skills.

Keywords: Teacher Education - Preservice, Instructional activities and practices, Mathematical Knowledge for Teaching

Professional noticing, the skill of making complex, in-the-moment decisions regarding children's mathematical thinking has been introduced as a means to improve overall mathematics thinking and instruction (Jacobs, Lamb & Philipp, 2010). The research reported in this paper analyzed prospective elementary mathematics teachers' (PTs) use of the noticing framework, while promoting the exchange and enrichment of mathematics education research across learning environments. The research team consisted of three mathematics teacher educators (MTEs) who work at three different institutions, one in the south, one in the southeast, and one in the western part of the United States. Each MTE's unique experiences and perspectives enhanced the research study.

Theoretical Framework

Despite the fact that many teacher preparation programs require PTs to spend a significant amount of time observing classroom teaching and learning, researchers cannot make specific claims about what they learn as a result of these observations (Brophy 2004). Conducting observations may not benefit PTs, because they may not know what key features to focus on while conducting their observations. MTEs, using video recordings of teaching may provide an opportunity for PTs to develop their noticing skills (Berliner et al. 1988).

According to Barnhart and van Es (2015), without structured support, PTs' analyses of student knowledge typically focus on aspects of the classroom related to management rather than on students' mastery of the content. It is critical that mathematics teacher educators guide PTs in making instructional decisions that align with student understanding (Darling- Hammond & Bransford, 2005; Davis, Petish & Smithey, 2006; Zeichner & Liston, 1996). A body of research has found that PTs can learn to attend to, interpret, and make decisions on the basis of student thinking, skills related to analyzing teaching (Jacobs, Lamb, & Philipp, 2010; Mitchell & Marin, 2014; Santagata, 2011).

MTEs may use video to provide PTs with the knowledge and skills they will need to be effective mathematics teachers. Using videos as a teaching tool saves time, money, and provides PTs the opportunity to learn new skills and to craft their practice without placing real students at risk during the learning process. Star & Strickland (2008) found that viewing videos led to significant increases in PTs' observation skills, particularly in teachers' ability to notice features of the classroom environment, mathematical content of a lesson, and teacher and student communication during a lesson. This study aimed to explore the following question: In the context of a four-week online mathematics course for pre-service elementary teachers, does the experience of analyzing videos of

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students' discussing their mathematical thinking lead to improvements in PTs' professional noticing skills?

The Instructional Activity

In an attempt to answer the research questions, a MTE at a regional public university located in southeastern United States used videos as an instructional strategy to study PTs' noticing skills. Fifteen female PTs in their second to fourth year of their studies to become elementary school teachers were a part of the study. Eleven of the PTs were White, three were Black, and one was Hispanic. Thirteen of the PTs were 20-25 years old, one was 42 years old, and one was 53 years old. PTs previously took between one and two math pedagogical content courses. PTs previous classroom experience includes 30 hours of focused observations in the areas of diversity, classroom management, and teaching strategies in a Survey of Education with Field Experiences course. They also have completed 10 hours of focused observations in diverse classroom settings related to classroom management along with small group teaching assignments in a Classroom Management course.

PTs were enrolled in an online course, Math Through Problem Solving. The study took place during Summer Session II. Therefore, the content of the course, which is typically taught over the course of a 15-week semester, was compressed into four weeks. The course focused on the following units: Number Theory, Fractions, Decimals, and Integers. One week was spent on each unit. In each unit, the first assignment was for the PTs to read the sections in the unit. The information from the book was also summarized in power points on a Supplemental Resources Page. PTs were required to take a Readiness Assurance Test (RAT) focused on the reading. Subsequently, PTs completed the unit homework assignments. Then, PTs took the unit test, there was a practice test to help them prepare for the test. Throughout the week PTs were asked to work on their cognitive interview video analysis assignment worth 10% of PTs' final grade. This sequence of assignments was repeated by PTs in each of the four units. Before the end of the course PTs were required to "pass", 80% or better, a Rational Numbers proficiency test that they have up to three attempts to pass. At the end of Unit 4, the PTs took a final exam.

The MTE developed the Cognitive Interview Video Analysis assignment to capture PTs' professional noticing skills. Over the course of the class the PTs were asked to watch and analyze 28 videos of cognitive interviews. The videos were focused on Number Theory (five videos), Fractions (eleven videos), Decimals (six videos), and Integers (five videos). The research team decided to focus their study on PTs' analysis of three videos, all focused on the concept of ordering, but completed across the span of the course. More specifically, the first video was focused on ordering fractions, the second video on ordering decimals, and the third video on ordering integers.

Before engaging in professional noticing, the PTs read the article A New Lens on Teaching: Learning to Notice (Sherin & van Es, 2003). In this article, the authors provide examples of how in-service teachers reflect on their teaching through noticing. Reading this article helps the PTs realize that noticing will help them make in-the-moment decisions and that there are a variety of ways to use noticing in their future classrooms.

The prompts that the interviewer asked the children in these three videos can be found in Figure 1. For each video, the assignment directions were:

> Post one (1) initial post where you answer each question below. Grading is based on effort (thoughtful and thoroughly explained answers) not accuracy. You are encouraged to read your peers' posts (you must make your initial post before being able to read others' posts) and post replies based on your reactions. Embedded in the book you will find the following video (there is a movie icon in the reading). Watch the video and then respond to each of the following prompts.

Attending:

- What did the student do?
- What strategies did the student use?
- Interpreting:
- What does this mean about the student's understandings or misconceptions of the mathematics?
- Deciding:
- Based on what you attended to and interpreted, what are the best steps to take next with this student?
- What questions would you ask this student?

The MTE structured the assignment as a discussion for three main reasons. For one, since the course was an online class, there are no opportunities to discuss the videos face to face. Secondly, the instructor wanted to be able to provide PTs with feedback about the expectations of the assignment as well as feedback and guidance for meeting those expectations. Finally, the MTE believed that PTs could benefit from seeing each others' responses and the MTE's feedback on their responses.

$\frac{1}{6}$ $\frac{1}{3}$ $\frac{1}{7}$ $\frac{2}{7}$ 1 $\frac{4}{3}$ $\frac{3}{10}$ $\frac{1}{2}$ $\frac{3}{6}$ $\frac{1}{2}$ 1.973 19.73 0.567 0.3 0.87 0.276 0.7 0.70 0.087 0.87 Fraction and Decimal Video Prompt: "For each pair, circle the larger of the two numbers. If they are equal, write =."
|6| |-5| -7 2 -1 11 0 -13 Integer Prompt: Order the numbers from least to greatest

Figure 1: Image of the Tasks Presented to Students

Methodology

This paper reports on the integrated findings of an exploratory sequential mixed methods research design (Figure 2). In sequential exploratory design, qualitative data is first collected and analyzed, and themes are used to drive the development of a quantitative instrument to further explore the research problem (Creswell & Plano Clark, 2011). As a result of this design, three stages of analyses were conducted: after the initial qualitative phase, after the secondary quantitative phase, and at the integration phase that connects the two strands of data and extends the initial qualitative exploratory findings (Creswell & Plano Clark, 2011). In this paper the authors share the results of final integration phase of the research.

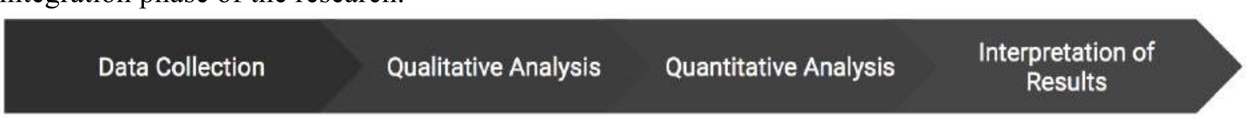

Figure 2: Exploratory Sequential Design (Creswell & Plano Clark, 2011)

The first goal of the coding process was to establish a standard exemplar response to the assignment for each video. To accomplish this goal, we, the team of three MTEs, each completed the Cognitive Interview Analysis assignment individually. Next, we met to discuss any discrepancies in the exemplar responses. We resolved any differences that existed and merged responses to create a standard exemplar response for each video. This task provided us with a standard exemplar to reference during coding. It also provided us with a thorough understanding of the content of the videos.

Next, we decided to use the coding scheme developed by Author (2019). The coding scheme is described in Figure 3. To analyze PSTs' responses, we used open coding (Corbin & Strauss, 2014) to determine the noticing level for each attending, interpreting, and deciding prompt in the Cognitive Interview Analysis assignment.

Noticing Level	Description
Periphery	Made general impressions (e.g. "Student understands the questions.")
Transitional	Highlighted noteworthy events, general impressions, but included why they believed something occurred (e.g. "The student used logic to reason through the problem."
Accomplished	Used evidence to elaborate on student understanding, made connections between the work and the next steps.

Figure 3: Coding Scheme for Professional Noticing

Prior to coding, all PTs names were removed and the responses were randomized in the spreadsheet to avoid any potential coding biases. The research team calibrated coding by discussing our inferences and interpretations of one PT's responses to each of the five items. Subsequently, each MTE independently double-coded all PTs' responses for two of the three videos, in a blinded format, to ensure the data from each video was analyzed by two MTEs. The percent agreement for the two raters across all items was 75%, suggesting substantial inter-rater agreement. Having computed a satisfactory percent agreement, we reconciled our coding through discussion of the data and the professional noticing framework coding scheme (Figure 3).

Findings

Through the lens of the noticing framework (Attending, Interpreting, and Deciding), the results of the study show patterns of growth related to the noticing levels of periphery, transitional, and accomplished. Once PTs' responses were coded using the scheme (Figure 3), the results of the noticing levels were analyzed for each of the three noticing assignments in the given semester (Comparing Fractions, Comparing Decimal Numbers, and Ordering Integers). The bar graph in Figure 4 shows the overall frequency of each response coded as periphery, transitional, and accomplished within each noticing assignment.

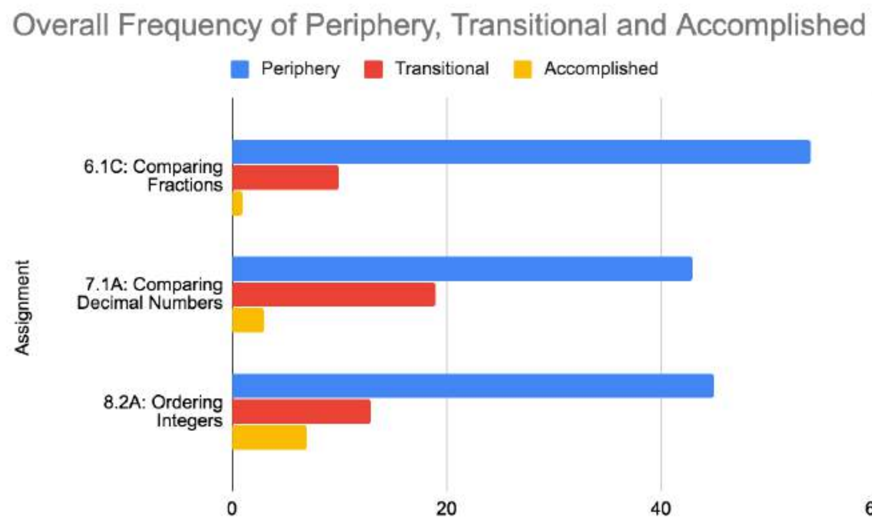

Figure 4: Overall Frequency of Periphery, Transitional, and Accomplished on each Assignment

Table 1 provides overall percentages of responses coded as periphery, transitional, and accomplished within each noticing assignment.

Table 1: Overall Percentages of Periphery, Transitional, and Accomplished

Assignment	Periphery	Transitional	Accomplished
6.1C: Comparing Fractions	0.83	0.15	0.015
7.1A: Comparing Decimal	0.66	0.29	0.046
8.2A: Ordering Integers	0.69	0.2	0.107

The percentage of responses coded as "periphery" decreased from 83% to 69%, whereas "transitional" increased from 15% to 20% "accomplished" increased from 1.5% to 10.7%, indicating that as the semester progressed, PTs' responses moved toward a transitional and accomplished level of interpreting student thinking. As seen in Figure 4, 54 of the PTs' responses were at a "periphery" level on the first assignment (Comparing Fractions) and only 1 PT response was at the "accomplished level on the first assignment. However, 45 PT responses were at the "periphery" level on the last assignment (Ordering Integers) and 7 PT responses were at the "accomplished" level on the last assignment.

These results indicate that the PTs have little experience with examining student mathematical thinking as seen on videos at the onset of this course. But, with practice, PTs' abilities to professionally notice improved as the course progressed. Based on experience and coursework, this seems to be a natural consequence of interacting with the ideas related to the framework. As shown in Figure 5, there is an overall increase of transitional responses in two of the three categories when considering Activity 7.1A. The attending percentage remained constant between Activity 6.1C and 7.1A. There is a slight decrease in transitional responses related to activity 8.2A but it is still down trending related to the introductory activity.

For example, one PT's response to the deciding piece of the framework changed over time. In the first assignment, the student offered the following suggestion, "And for both students I would ask the same questions the instructor did, and I would ask for more examples and review questions". Then, on a later video, the PT's level of sophistication changed and their response became, "What strategies are you using?", "How did you come up with that strategy?", "How did you figure out the answer?". As you can see, the later response is focused on conceptual understanding and is student-centered.

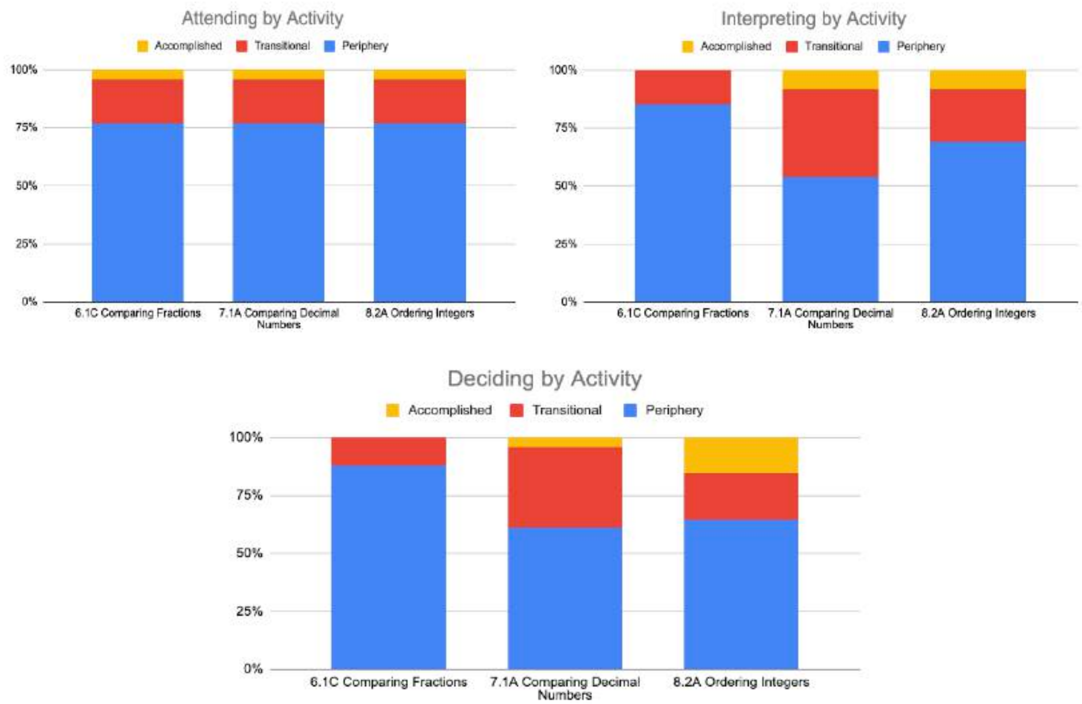

Figure 5. Percent Periphery, Transitional, Accomplished by Video

Three of the fifteen PTs did not complete all three assignments (chapters 6, 7, and 8). Therefore, their data were removed from the participant level analyses, resulting in n = 12. At the participant level, we (the MTEs) counted how many of the questions (out of the five questions) each PT answered at each noticing level for the three (comparing fractions, decimals, and integers) assignments. Next, we calculated the frequency and percent changes in the number of questions each PT answered at each noticing level from the first to the second, second to third, and first to third assignments. Finally, we found the average frequency and percent changes for all PTs.

We found a decrease in PTs' performance at the peripheral level during the course. All twelve PTs answered 49 questions at the peripheral level on the chapter 6 assignment. On the chapter 7 assignment all twelve PTs answered 43 questions at the peripheral level. All twelve PTs answered 40 questions at the transitional level on the chapter 8 assignment. The average number of questions that PTs answered at the peripheral level decreased by a frequency of 0.5, 0.25, and 0.75 from the first to the second assignment, second to the third, and the first to the third assignments, respectively. The average number of questions that PTs answered at the peripheral level decreased by 13.19, 0.69, and 19.44 percent from the first to the second assignment, second to the third, and the first to the third assignments, respectively

We found an increase in PTs' performance at the transitional level from beginning to middle and beginning to end of the course, but a slight decrease at the transitional level from the middle to the end of the course. Nine of the twelve PTs answered ten questions at the transitional level on the chapter 6 assignment. On the second assignment eight PTs answered fifteen questions at the transitional level. Seven PTs answered fourteen questions at the transitional level on the third assignment. The average number of questions that PTs answered at the transitional level changed by a frequency of 0.41, -0.17, and 0.25 from the first to the second assignment, second to the third, and the first to the third assignments, respectively. The average number of questions that PTs answered at the transitional level changed by 22.73, -2.27, and 4.55 percent from the first to the second assignment, second to the third, and the first to the third assignments, respectively.

We found an increase in PTs' performance at the accomplished level during the course. Only one PT answered one question at the accomplished level on the first assignment. On the second assignment two PTs each answered one question at the accomplished level. Five PTs answered six questions at the accomplished level on the third assignment. The average number of questions that PTs answered at the accomplished level increased by a frequency of 0.08, 0.33, and 0.42 from the first to the second assignment, second to the third, and the first to the third assignments, respectively. In most cases, the percent change could not be calculated because zero PTs initially answered at the accomplished level (causing a dividing by zero error in the percent change calculations).

Conclusion and Implications

This study, framed by research on noticing, the Coding Scheme for Professional Noticing (adapted from Van Es, 2011) was used to assess the development of PTs' use and understanding of noticing. Since PTs' noticing skills as they participated in the instructional activity was the focus of the study, the results are promising that the use of video to support PTs' understanding of student thinking may be a viable strategy for supporting growth. Preliminary findings indicate that through a deliberate scaffolding of course activities and projects, MTEs can help PTs develop their noticing skills.

Although many PTs' professional noticing skills improved, some PTs' skills did not show an overall increase, and few PTs reached the accomplished level. More research is needed to determine how to scaffold all PTs' skill development to the accomplished level. Analysis of the instructional activity indicates that prior to this course PTs have had little experience describing students' work, interpreting students' understandings, and then deciding how to proceed. PTs' initial interpretations seemed to rely on their own content understanding related to the task and limited the PTs in their ability to apply appropriate strategies to promote conceptual understanding for students. These results indicate the need for MTEs to spend more time reflecting on and discussing implications for teaching. Engagement in this work allowed us to see the PTs' reasoning so that we, mathematics educators, can improve our practice and our PTs' professional noticing skills.

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COUNTERSTORIES OF PRESERVICE ELEMENTARY TEACHERS: STRATEGIES FOR SUCCESSFUL COMPLETION OF THEIR MATH CONTENT SEQUENCE

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Master narratives exist in many forms within mathematics education. Preservice elementary teachers often are seen as having high levels of math anxiety while students in developmental mathematics are seen as being deficit in their mathematical understanding. This study uses counterstories to understand the experiences of two women of color, who are enrolled in math content courses for preservice elementary teachers. Students share strategies that they learned from one math content course in order to succeed in their math course sequence.

Keywords: Instructional Activities and Practices; Affect, Emotion, Beliefs, and Attitudes; Equity and Diversity

Master narratives about preservice elementary teachers' (hereon labeled as PSTs) relationship with mathematics often paints a story that PSTs have high levels of math anxiety both during their teacher preparation program and also beyond as in-service teachers (Gresham, 2007). In particular, of all college majors, female elementary PSTs show the highest levels of mathematics anxiety (Beilock et al., 2010). Math anxiety can often be debilitating, leading to feelings of helplessness, tension, or panic, and can affect PSTs' experiences and success in their math content courses. "Mathematics anxiety affects learning and causes individuals to perform at lower levels than their capabilities" (Brown, Westenskow, & Moyer-Packenham, 2012, p. 366). It has been recommended that mathematics teacher educators incorporate strategies in the PSTs' math content classes that can help students alleviate their anxieties and to help PSTs succeed in their future courses and ultimately in their careers (Vinson, 2001). In fact, Johnson & vanderSandt (2011) showed that taking a math content course in the freshman year of college can contribute to reducing math anxiety in PSTs.

Another master narrative that can apply to PSTs is the deficit perspective that often comes with being required to complete developmental math courses. Developmental mathematics courses "target underprepared students with the purpose of improving their abilities to handle college-level material and succeed in college" (Bettinger, Boatman, & Long, 2013, p. 94). Oftentimes these courses focus on what students *do not* know and attempt to remediate topics in arithmetic and algebra. According to Sitomer (2014), a deficit view of students in such courses is problematic because it does not distinguish between different ways of knowing mathematics, seeks to blame instead of finding solutions, and does not acknowledge that the mathematical experiences of students might differ in significant ways from the mathematical experience of those who "define" the content of mathematics courses. Valencia (2015) argues that many remedial programs fail because they are quick fixes based on a deficit view, placing the burden of acquiring the needed knowledge on marginalized students and their families without much support for accomplishing learning. Oftentimes the coursework does not align with future courses that specific majors, like liberal studies majors, will need to take. Gutiérrez (2008) contends that in order to move away from the negative view of student achievement gap, more research should be done to investigate effective teaching and learning environments for marginalized students.

These master narratives are important to dismantle. Larnell (2011) defines *identity infiltration* where students who often continue to hear the master narrative can begin to replace their experience with that of the master narrative. This can be troublesome for PSTs. If a PST believes that math anxiety is a fixed feature of being an elementary teacher or believes that they are deficit in their

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mathematical knowledge, this can be transferred onto their future students (Vinson, 2001). Therefore, we need to understand strategies that help students to succeed. The research questions that this paper addresses are: 1) What are the counterstories of two PSTs who enrolled in a support course? 2) What strategies do PSTs utilize to succeed in their math content course sequence? 3) How does a mathematics support course support students to forge new mathematical identities?

Theoretical Framework

This study uses a Critical Race methodology (Solórzano & Yosso, 2002) to understand the ways that two students experienced their math content courses as part of their teacher preparation. We chose this methodology because it focuses on the stories and experiences of marginalized students, viewing these experiences as sources of strength. Solórzano and Villalpando (1988) define those who are marginalized as having "less access to opportunities and resources [and] experience different barriers, obstacles or other forms of individual and societal oppression than those at the center" (p. 212).

PSTs who are also placed into developmental mathematics face more obstacles than those students who are placed in a traditional, introductory mathematics course. First, PSTs are expected to deeply understand, engage with, and convey deeper conceptual understandings of mathematics than a student taking, say, a pre-calculus course. Elementary teachers must balance the challenges of not only understanding mathematical content, but they must have a level of specialized content knowledge (mathematical knowledge that may not be familiar to mathematicians), paired with pedagogical content knowledge (knowledge of how the content interacts with students and within teaching) (Ball, Thames & Phelps, 2008). Second, more underrepresented minority students are placed in developmental mathematics courses (Bahr, 2010) which is problematic because such remediation often places barriers to access to higher levels of education (e.g., receipt of financial aid, extended time to degree), which effectively places many underrepresented students on the periphery of educational opportunities and advancement (Valencia, 2015).

In this study we specifically focus on counternarratives of the second and third authors. *Counterstories* "serve as a method of telling the stories of those people whose experiences are not often told" (Solórzano & Yosso, 2002, p. 32) and can be seen as a tool to challenge the dominant stories of marginalization. "Storytelling has been used to provide a venue for the marginalized to voice their knowledge and lived experiences" (Rodriguez, 2010, p. 493). Solórzano and Yosso (2002) argue that stories can build a sense of community among marginalized populations, giving voices to those who may be overshadowed by the dominant stories within education. Because schooling privileges some students and not others, understanding a student's narrative can aid in understanding the way that those students who are under-supported within our educational system experience mathematics and also find ways to support their learning and success. As scholars, we need to understand how students who may be marginalized in our educational system "make sense of their own experiences and how they feel empowered to act to learn mathematics" (Zavala, 2014, p. 62).

Methods

This study took place at California State Polytechnic University, Pomona, a large public university on the West Coast of the United States. The second (Hazar) and third (Samantha) authors enrolled as freshmen in Math 1900 (described in the next section) at the university Fall 2018. Both students enrolled as Liberal Studies majors, preparing themselves to complete a Bachelor's degree supporting elementary school teaching. Hazar is in the fourth and final course of the mathematics content sequence while Samantha is enrolled the third course. Students who are deemed as needing extra support enroll in Math 1900 and then move onto the three-course math sequence. Students that do not need extra support enroll directly into the three-course math sequence which covers numbers and operations, algebra, statistics, and geometry. Hazar identifies as a first-generation, Lebanese-American woman. She is the first in her family to attend public schooling in the United States. She is 20 years old and plans to finish her multiple subject credential with the goal to ultimately get a masters degree. Samantha is a 19-year-old woman who identifies as a Hispanic woman with Native American heritage, focusing on early childhood education. She would like to one day open a pre-school.

The Quantitative Reasoning Course

Based on a statewide change in the California State University (CSU) system in Fall 2018, developmental mathematics coursework could no longer be offered, and instead general education courses needed to be modified to provide support for students who would have been placed in developmental mathematics courses. Because research has shown developmental mathematics courses hinder student graduation, this change was implemented to help students complete general education mathematics in their first year, shortening their time to degree completion and to lower dropout rates. When a student enrolls at a CSU, they are suggested to take specific courses that provide the right level of support. Through an algorithm that utilizes multiple measures, such as high school GPA, cumulative high school math GPA, SAT score, and whether or not students took math their senior year, students are placed into their math courses.

Math 1900 was a newly designed course, developed by author one, to fulfill the CSU mandate. As a credit-bearing course, the quantitative reasoning course was specifically designed to support students who are enrolled in the liberal studies sequence aiming to become elementary teachers. Previously, these students would have been asked to enroll in two developmental mathematics courses focusing on algebraic skills before starting the liberal studies course sequence. The course was structured around three factors designed to: 1) help students alleviate their mathematics anxiety by focusing on course reflections, strategies, and growth mindset, 2) provide students with opportunities to have authority in their learning and self-regulation of their ideas, and 3) engage students in the eight standards for mathematical practice.

Analysis

This study focuses on the storytelling of these two students during their fourth semester in the liberal studies program. Both students responded to reflection questions asking them to relate their experiences in the math content courses they have taken. In particular, they were asked to reflect on their relationship with math coming in as a freshmen, their perspectives on teaching mathematics, how they attempt new problems or content areas, strategies they have developed along the way to support their success, and their sense of belonging in the liberal studies program. Through constant comparative methods, we pulled the following themes from the narratives: minoritized experiences in mathematics, change in relationship with math, relying on peers for deeper learning, and looking to oneself for validation.

This study, like all, has its limitations. First, the second and third authors, the participants of this study, were students in the first author's course. This may lead to some bias in the responses of the students. However, these students discuss their experiences after taking several math courses in the liberal studies math sequence, reflecting back on their overall experience. Second, the course has only run for two years, and therefore, much is to be learned, yet about its long-term implications for other students. However, it is worthwhile to gain an understanding of how it is addressing PSTs' needs in the moment.

Findings

Through their reflections, Hazar and Samantha related their experiences in the content sequence for liberal studies. In particular, the two students described how being in Math 1900, specifically, helped them to feel confident in their math abilities and to persevere in the subsequent course sequence. Both students indicated that their peers who did not start in Math 1900 struggled in their coursework when placed in the next class. From their narratives, both students discussed how they felt minoritized in previous math experiences, strategies that they have used to be successful, which include discussing their past and current relationship with mathematics and teaching mathematics, relying on peers for deeper learning opportunities, and relying on themselves to validate their work.

Minoritized Experiences in Math

Both women described being minoritized racially or because of their gender in their high school educational experiences, while Hazar also felt minoritized because of her learning disabilities. Both Hazar and Samantha went to schools that had a high proportion of Asian students. Samantha indicated that she was a Hispanic female attending school with a predominantly Korean population of students. She felt that because she was Hispanic, her peers automatically thought that she would not perform well in mathematics. Hazar attended nine different elementary schools, and felt that this caused gaps in her education. When she attended high school, she had a similar experience to Samantha:

> As a middle eastern woman I had less in common with my peers. Our school had a reputation of being academically advanced…Students felt the need to be competitive, and outperform one another instead of lift each other. I felt a need to prove my abilities, not only to my peers, but also my teachers. I did not feel as comfortable raising my hand, and asking questions. I had the fear of being thought of as stupid, or unintelligent. I wouldn't have the confidence to raise my hand or ask the teacher to slow down. On the first day of classes, I felt the need to stay after class and explain I had ADHD, and admitting that I may need extra guidance to perform at the same levels of my peers.

Hazar felt the pressure to perform at her high school, and felt that her lack of consistent learning in elementary school paired with her learning disability contributed to how people thought of her. Samantha described how in high school, she was told that because she was female, she was not good at mathematics. "I had always been told that my brain simply did not function in a mathematical standpoint…I was told that because I am a woman, I would not be as good at math as a male. I noticed that many times, the teachers would pair us with a male partner."

In contrast, both students described how they did not feel this way in their Math 1900 course. For example, Hazar reflected on how she felt when she walked into the class for the first time.

> When I walked into Math 1900, I was taken back to how female dominant the class was. I was heard, and felt that people cared about what I had to say…I was surrounded by a group of diverse and powerful women (and men) who had similar high school experiences to me. I thrived in Math 1900. It was the first time I felt my mental disability was not a crutch. I was able to embrace my differences, and got to be up and moving, and talking. I was no longer subjected to sitting down and lectured at. I am the first generation within my family to be born and raised in the United States. I have had to navigate the education system, which is very different than education in Lebanon. When coming to Cal Poly, specifically Math 1900, I was surrounded by a group that shared the same experiences I have had.

Samantha also indicated that her experience in college was different than in high school, "we were embraced for our differences and told that we were capable regardless of race or ethnicity."

Changes in Relationship with Math

Hazar and Samantha had relationships with math prior to starting college. Samantha particularly had a high level of anxiety when it came to her relationship with math.

> I did not believe that I would be able to pass a college level math class nor understand it. I was most likely one of the most pessimistic human beings ever when it revolved around math. Throughout my pre-college years, math was truly one of my biggest struggles…although I was excited to finally learn math in which I would be passionate about, I could feel my heart beating out of my chest as I entered into my first ever college class.

Samantha was very insecure about her ability to perform in the set of courses she was expected to pursue as a liberal studies major. She realized that her biggest fear was that she felt that if she could not comprehend the basics of mathematics, how would she be able to teach children? Before college, Hazar felt that mathematics gave her "a lot of anxiety and stress". While she enjoyed math in her early years, she felt that she started to fall behind in high school because she lacked the basic understanding of concepts. She feels that mathematics is important and is something that is used regularly in everyday life, however, she felt a lot of pressure to perform prior to starting college.

Both students described ways that their relationship with math changed after starting the liberal studies program. Samantha described learning about growth mindset in Math 1900 and decided that it might be worth it to change her perspective of mathematics and her ability to learn.

> I began to learn and grow faster than I could have ever imagined. Astonishingly, I finally understood math to the point where I could educate my peers during study review. Never have I walked out of a class with such confidence…Now, I am able to use critical thinking to attempt to comprehend [math] rather than [searching] the internet immediately…[the class] offered me multiple strategies of comprehending different forms of math.

Her relationship with mathematics started to change and her confidence grew. Hazar also indicated seeing similar changes.

> I relearned foundations of math that helped me to understand it better. I was no longer blindly doing problems and hoping I was doing it right. I understand what I am doing, and why I am doing math the way I am. And the moment I don't understand a topic, I've gained the confidence to raise my hand and ask for clarification. I had many misconceptions about math, I thought it was independent, and all memorization…I genuinely want to know and learn more. It has increased my appreciation for teaching…As cliché as it sounds, math is really fun and I want my students to get the appreciation of it earlier than I did.

Both women found a way to relate to mathematics again. Samantha found that changing her mindset provided her the opportunity to engage in mathematics more than she did in high school. She recognized that now she was learning mathematics in order to teach, and therefore found joy when she was able to explain to others. Hazar recognized that with support to understand basic foundations of mathematics, she was able to understand and gain more confidence. She reflected on a moment when she was selected to present her explanation for how to add two fractions:

Peer Interactions

Samantha and Hazar distinguished that interacting with peers was an essential strategy that they learned from Math 1900 that helped them succeed and persevere in the subsequent math sequence. As noted above, Hazar had always been in classes where she sat in rows, faced forward, and was lectured at. Both students indicated that because group work was a daily routine in Math 1900 and that they were expected to change group members regularly, they were able to create a bond with their peers. Samantha and Hazar found it extremely rewarding to be able to help their peers. Hazar stated

> More than anything, I learned how to think critically, which is very important as a Liberal Studies major. I even took a different approach to how I help my peers. I wanted to create an encouraging and positive setting, but I couldn't do that by saying "no that's wrong". Just changing my verbiage ended up being a learning curve, and I have gone a long way. I now see dozens of students and tutor them at my school's Learning Resource Center…It gave me confidence I didn't know I had.

At the end of her first year, Hazar had gained so much confidence through working with others, that she decided to apply for a position as a tutor for the first two math courses in the sequence. She has now been tutoring for almost a year, and finds the experience very rewarding.

Both Hazar and Samantha started group study sessions outside of class. They found that by being able to practice teaching concepts in front of one another, they were able to clarify their language and understanding of the material. For example, Samantha indicated that by becoming close with her classmates through groupwork in class, she was able to develop closer bonds with her peers and feel comfortable meeting outside of class to study and develop their mathematical understanding. "We were able to…develop the proper communication skills and be able to know the process to overall work through problems with one another…[which I used] not only my math courses, but my other courses in general."

In particular, Hazar indicated that because of her widespread interactions with peers in Math 1900, she felt she finally found her calling.

> I knew I was meant to be a teacher because I have never bonded so instantly with any group in my life. We are like-minded and have similar goals. I've met some of the most encouraging and uplifting people I will ever meet. I made a mistake on a worksheet and I said "I'm stupid" and the whole table responded "No you're not" so instantly. Teaching is exactly what I want to do, and I've never been so excited to start my future.

While both students indicated that the type of group work they experienced in Math 1900 was not the same in the subsequent math courses, they decided to carry on the tradition. For both students, their interactions with peers was a strategy that they took into their future math classes. For example, Hazar mentioned how she volunteers and shares her work with her peers, even though her instructor had not expected her to, because she understood the value of learning from peers. Samantha continues to meet with peers outside of class, even if they are not in the same course section, "with the community [from Math 1900], we still have study sessions, coffee dates, a hand to hold onto, and overall encouragement when we are feeling down."

Validating their Own Work

Hazar and Samantha both indicated the value of struggle in their Math 1900 course. Hazar described scenarios when in Math 1900, the instructor never explicitly answered her questions and instead would ask her to think through her reasoning.

> I found it important to have a professor that allows you to struggle and make mistakes. [My professor] was patient, she didn't give us all the answers right away. She gave a lot of time to communicate and troubleshoot within our groups. If I asked for help she would ask me to go through the work I had out loud, and upon doing so I would naturally answer my own question.

Hazar indicated that this strategy of asking "why?", "what?", or "how?" helped her immensely in her subsequent math courses. She does so for questions like: "Why did I convert fractions to decimal in this word problem? Why are we using base 5? In ⅚ why, is 5 part of the 6? What does the 3 represent in 532, and what is its relation to the other numbers?

Samantha described a similar takeaway from that course. She recalled that for the first time she was genuinely asked to explain her reasoning behind how she did specific problems.

> [My professor] instigated the challenge of questions such as "*why* do we do this that way?" or "*how* did you get that outcome?" This ultimately helped me to find the deeper meaning to why or how I got an answer which is an important aspect to becoming an educator.

Samantha recalled a moment when she was presenting at the board and when she was asked by her professor how she arrived at her answer, she "[stared] at the numbers scribbled all over the board for what seemed to be hours, I finally…said 'I do not know.' This taught me that children will ask why and how, and an answer is a necessity." Samantha understood that in future courses and her transition to becoming a teacher, she needed to rely on herself to validate her own work.

Discussion

In this study, we discussed the counterstories of two women enrolled in their math sequence for future elementary teachers. Both women described how their previous high school experiences positioned them as inferior to their peers in their mathematics courses. Their experience in Math 1900 was transformational to these experiences; they were able to create and utilize strategies in their subsequent math courses that promoted their success while also creating more positive mathematical identities. The master narratives that PSTs struggle with high anxiety and also that developmental math students are deficient in math understanding are being challenged through their experiences. An essential take-away from the findings is that the two students felt a sense of community in their Math 1900 class (e.g., did not feel minoritized, felt like they were part of a team). On the first day of class, all students in the course contributed towards a classroom set of norms, which were followed all semester. Part of the list was that students were expected to hold authority of their learning, to give respect to others, and to provide positive feedback to others. This could also contribute to why both women experienced a change in their relationship with mathematics and the joy they found working with others. Students were expected to be creative and to work on problems together, requiring them to only ask questions once the entire group needed help. Positive experiences for PSTs can contribute to positive attitudes towards mathematics (Kalder & Lesik, 2011). Part of the design of the course was to be transparent in the rationale for why the instructor did what she did, revealing underlying reasons to classroom decision making of the content and the mathematical practices.

Hazar and Samantha may have found purpose and utility in the mathematics they were learning because it was the first time they saw how it could be connected to their field. Anderson (2007) argues that an important feature of mathematics identity is to be able to *imagine* how mathematics fits into a student's broader life. Students in developmental mathematics courses often find that the math they learn is not connected to their career goals or futures and therefore do not feel motivated to learn the material (Cawley, 2018). Hazar and Samantha found a transformative change to enjoying mathematics and wanted to better understand it once they saw its purpose: to teach future children. Because the class was built around the standards for mathematical practice, students were constantly asked to make sense of problems and persevere in solving them while also constructing viable arguments for the conclusions they reached in problems and to critique the reasoning of others in their group. Both women found the strategy of asking themselves *how* and *why* they chose certain ways to work on problems to be a positive learning strategy that they carried forward. They also found that asking for help from others supported their learning. These strategies appeared to increase their confidence to engage with and present their mathematical work.

It is important to provide the supports early on for students in order to help them succeed. Different from developmental math, this course was grounded in math content that was purposeful for future teachers, intentional in the design, provided an early intervention for students to build confidence, connected to mathematical practices, and created positive learning experiences for students to have continued success.

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PRODUCTIVE SEEDS IN PRESERVICE TEACHERS' REASONING ABOUT FRACTION COMPARISONS

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Reasoning about fraction magnitude is an important topic in elementary mathematics because it lays the foundations for meaningful reasoning about fraction operations. Much of the research literature has reported deficits in preservice elementary teachers' (PSTs) knowledge of fractions and has given little attention to the productive resources that PSTs bring to teacher education. We surveyed 26 PSTs using a set of 9 fraction-comparison tasks. We report the frequency of complete strategy-arguments and the perspectives (ways of reasoning) used for each item. We further examine incomplete strategy-arguments, noting substantial evidence for productive seeds of reasoning. Using data from interviews with 10 of these PSTs, we identify evidence suggesting these seeds are, in fact, productive in that they provide foundations for further development. We argue that this type of research is needed in order to further mathematics teacher education.

Keywords: Number Concepts and Operations, Rational Numbers, Teacher Education-Preservice

The mathematics education research community is concerned with the mathematics content knowledge of preservice elementary teachers (PSTs; Thanheiser et al., 2014). The research literature has tended to characterize PSTs' mathematics content knowledge as poor (Graeber, Tirosh, & Glover, 1989; Green, Piel, & Flowers, 2008; Putt, 1995; Tsao, 2005; Thipkong & Davis, 1991; Widjaja, Stacey, & Steinle, 2011; Yang, 2007). A synthesis of the literature reveals that there is insufficient research that seeks to make sense of PSTs' conceptions, or that views their conceptions as resources for further learning (Thanheiser et al., 2014; Whitacre, 2013). The literature on PSTs' knowledge of fractions is a prime example of such characterizations (Olanoff et al., 2014). Researchers have found PSTs to be inflexible in their reasoning about fractions, relying heavily on standard procedures, while at the same time having difficulty justifying such procedures and difficulty relating fraction operations to contexts (Olanoff et al., 2014).

Preservice elementary teachers may not typically reason flexibly about fractions when they come to teacher education (Ball, 1990; Yang et al., 2009), but how far are they from doing so? We operate from the assumption that PSTs possess fundamental mathematical resources with which to reason productively about fraction magnitude, but they may not have had sufficient opportunities to exercise such reasoning. In that vein, we analyzed 26 PSTs' responses to fraction-comparison tasks in order to identify the variety of ways of reasoning that they might bring to such tasks. In keeping with our perspective, we went beyond tabulating correct responses and coding for strategies. We also examined how PSTs reasoned through comparisons that they ultimately answered incorrectly or incompletely. Even in these cases, we find evidence of PSTs reasoning about fractions in productive ways. In interviews with 10 of the PSTs, in which we provided low-level support and encouragement, we investigated which strategy-arguments for comparing fractions were readily learnable depending on their current knowledge.

This research highlights the valuable prior knowledge and the potential for growth in PSTs' knowledge of fractions. We believe our findings offer a fresh perspective that contrasts with the vast majority of literature on this topic by highlighting the strengths of PSTs that can be leveraged into effective reasoning strategies for fraction-comparison tasks.

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Background

Knowledge Framework

We have found Smith's (1995) framework useful for categorizing PSTs' ways of reasoning about fraction magnitude (Whitacre & Nickerson, 2016). It consists of four *perspectives*, which are categories of comparison strategies: Transform, Parts, Reference Point, and Components. Below, we briefly describe each perspective.

The *Transform* perspective involves use of procedures such as converting to a common denominator or converting to a decimal. These strategies involve transforming one or both fractions in some way in order to facilitate the comparison (e.g., comparing 6/7 and 7/8 by converting to a common denominator and then recognizing that 49/56 is greater than 48/56).

The *Parts* perspective involves interpreting fractions in terms of parts of a whole. This approach works especially well in certain cases, such as when comparing fractions that have the same numerator or same denominator. For example, 3/4 is greater than 3/5 because 1/4 of a whole is larger than 1/5 of the same-sized whole. Thus, three larger parts are greater than three smaller parts.

The *Reference Point* perspective involves reasoning about the magnitudes of fractions on the basis of their distance from reference points, or *benchmarks* (Parker & Leinhardt, 1995). In particular, Reference Point strategies relates to the number line. For example, to compare 7/8 and 6/7, a student may notice that 7/8 is 1/8 away from 1, whereas 6/7 is 1/7 away from 1. Since a distance of 1/8 is less than a distance of 1/7, 7/8 is closer to 1, and therefore larger.

The *Components* perspective involves noticing additive or multiplicative relationships in the numerators and denominators of the given fractions. For example, in order to compare 13/60 and 3/16, a student may notice that $13 \times 5 = 65 > 60$, whereas $3 \times 5 = 15 < 16$. Thus, 13/60 is greater because the numerator is larger relative to the denominator.

Our coding scheme for fraction-comparison strategy-arguments represents a revised version of that of Smith (1995). Length limits prevent us from providing operational definitions for each strategy-argument here. See Whitacre and Nickerson (2016) for a similar coding scheme.

Previous Research

We note three points that concern us about the state of the literature regarding PSTs' mathematical knowledge: (1) The body of literature tends to emphasize deficiencies, rather than to regard PSTs' prior knowledge as a productive resource (Thanheiser et al., 2014; Whitacre, 2013). This emphasis runs the risk of promoting low expectations of PSTs' abilities to learn. (2) There are few articles that provide specific, qualitative descriptions of PSTs' mathematical thinking that could provide useful information from which to design instruction. The work of Thanheiser (2009) is a notable exception. (3) There is a tendency to overgeneralize about the mathematical thinking of PSTs, rather than to recognize the variety in their thinking.

We view PSTs as sense-makers who are ready and able to improve their mathematical knowledge. Unfortunately, there is scant literature that helps the field to understand *how* PSTs' knowledge of fractions can be improved (Olanoff et al., 2014). Thanheiser et al. (2014) assert that the field of mathematics teacher education needs studies that document successful approaches to improving PSTs' content knowledge and that illuminate the processes by which such changes can occur. We agree. In particular, the field needs studies that find value in PSTs' prior knowledge and that demonstrate how PSTs can and do use that knowledge as they learn, because "The key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know" (Conference Board of the Mathematical Sciences [CBMS], 2001, p. 17). In the literature on K-12 students' mathematical thinking and learning, much attention has been given to students' mathematical conceptions and to the productive ways in which they make use of their prior knowledge as they learn new mathematics (e.g., Carpenter et al., 2015; Clements & Sarama; 2014;

Fuson et al., 1997). Unfortunately, such a perspective has rarely been applied in the literature concerning PSTs' mathematical thinking and learning (Thanheiser et al., 2014; Whitacre, 2013). In this study, focusing on the challenging topic of fraction magnitude, we examined how PSTs made use of their prior knowledge, including to develop new strategies, when comparing fractions.

Theoretical Framework

This study is informed by the notion of the *zone of potential construction (ZPC)* (Norton & D'Ambrosio, 2008; Steffe, 1991). The ZPC refers to the range "determined by the modifications of a concept a student might make in, or as a result of, interactive communication in a mathematical environment" (Steffe, 1991, p. 193). In the case of our work, to say that a strategy for comparing fractions is in a learner's ZPC is to say that the learner can hypothetically extend or reorganize her current schemes or mental operations to compare fractions in this new way.

Informed by the above literature, together with our previous experience, we expected PSTs to approach the fraction-comparison tasks primarily by drawing upon Parts and Transform reasoning. In particular, we expected them to be able to apply Parts reasoning to compare fractions in cases of a common denominator or common numerator. We did not expect many PSTs to compare complements initially, but we conjectured that doing so might be within their ZPCs. We expected many PSTs to default to Transform procedures, such as converting to a common denominator, in cases in which there was not a common numerator or common denominator in the given fractions.

Method

The research questions that we address are the following: (1) How do elementary PSTs reason about a set of fraction-comparison tasks? (2) What productive seeds of reasoning are evident in their responses? (3) Given the opportunity to explore a set of fraction-comparison tasks in an interview setting, which strategy-arguments are PSTs able to construct, and how do these relate to their current ways of reasoning?

This study took place at a large, public university in the Southeastern United States. The participants were a cohort of 26 PSTs in an elementary mathematics methods course. They were senior-level Elementary Education majors enrolled in a credential program.

Collection of Survey Data

Participants were given a fraction-comparison survey early in the semester (prior to instruction related to fractions). The cover page had nine pairs of fractions and asked the PSTs to mentally decide which fraction in the pair was greater or whether the two fractions were equal. The subsequent pages revisited each of these nine comparisons, asking participants for a "Description of Method" and a "Justification" for each comparison. We chose this format in order to encourage the participants to exercise their number sense, although they were free to approach the tasks in any way that they chose.

The same survey was administered at the end of the fraction unit. In both cases, time to complete the survey was limited to 25 minutes. We note that some participants left items toward the end of the survey blank. It is possible that more attempts would have appeared on later items if there had not been a time limit, or if participants had been given significantly more time.

Analysis of Survey Data

Note: We do not assume that the participants performed all of their work mentally and then reported that work in writing. In fact, some participants explicitly noted that they changed some answers. Whenever a participant gave one answer on the cover page but gave a different answer when describing or justifying their method, we regarded the latter response as the final answer.

Whitacre and Nickerson (2016) used a modified version of Smith's (1995) framework to code fraction-comparison strategies. In this study, we further developed that coding scheme to capture the wider variety of strategies that we observed. Two authors separately coded every response for perspective and strategy. The data were coded in batches (e.g., data from 8 participants) and inter-rater reliability was checked after each batch. By coding in batches, we were able to make revisions and additions to the coding scheme along the way and to code each subsequent batch with an updated scheme. This approach also enabled us to identify any interrater reliability issues early and to clarify our interpretations. Overall, the authors initially agreed on the perspective for 91% of the participants' responses and agreed on the specific strategy for 89% of the responses. Consensus was reached through further discussion of the disagreements until the authors were satisfied with the final coding decision.

In addition to coding for a perspective and specific strategy-argument, we also focused on comparisons that fell short of being complete strategy-arguments, yet demonstrated what we judged to be productive seeds of reasoning. Thus, all comparisons were coded into one of five categories: (a) complete strategy-argument and correct solution, (b) complete strategy-argument with minor errors, (c) incomplete strategy-argument with productive seeds, (d) incomplete strategy-argument with no apparent productive seeds, and (e) no strategy-argument evident. Incomplete strategy-arguments were given credit for productive seeds if the characteristics of the work, together with the perspective, were consistent with a complete strategy-argument (i.e., the participants' reasoning was headed down a productive path but stopped short of the complete argument). We believe that this approach to the study of PSTs' mathematical thinking provides a more comprehensive picture than that which has typically been reported in the literature.

Collection of Interview Data

Prior to the fraction unit in class, PSTs were invited to participate in one-on-one interviews. In contrast to the typical interview style in which the interviewer refrains from providing any form of support, these interviews were designed to allow for minimal support. The purpose of the interview design was to investigate which strategy-arguments were in participants' ZPCs. Thus, we specified in the interview protocol allowable types and levels of intervention. The intervention strategies that interviewers used included emotional encouragement, requests to solve a comparison task in a different way, requests or encouragement to continue down a path of reasoning, and offering counter arguments or pointing out evidence that was relevant to determining whether a solution was correct or incorrect. Ten of the 26 PSTs participated in these video-recorded interviews. The first and second author each interviewed five of the participants. In these interviews, PSTs were given nine fraction-comparison tasks that each mapped closely to the nine comparisons on the survey, but with different components (see Table 1).

Analysis of Interview Data

To analyze the interview videos, we targeted comparisons in which participants activated a productive seed in their work. We first identified comparisons from the videos that demonstrated use of productive seeds, we then wrote short narratives on what transpired in each case, and finally chose representative cases that highlight successful progressions in reasoning from productive seeds with low-level support. In our analysis, we applied three criteria as evidence that a strategy-argument was within a participant's ZPC: (1) the PST had not previously used that strategy-argument, as determined from pre-assessment and interview data; (2) the PST produced the strategy-argument during the interview with no more than minimal intervention from the interviewer; (3) the PST later used the same strategy-argument independently.

Table 1: Fraction-comparison Tasks from the Survey and Interview

Item	Survey Comparisons	Interview Comparisons
1	2/8 vs. 3/8	4/6 vs. 5/6
2	3/4 vs. 3/5	5/8 vs. 5/9

3	6/7 vs. 7/8	7/8 vs. 8/9
4	14/13 vs. 13/12	11/10 vs. 12/11
5	8/24 vs. 13/39	6/24 vs. 13/52
6	13/60 vs. 3/16	17/42 vs. 8/23
7	7/28 vs. 13/50	4/20 vs. 11/56
8	2/7 vs. 12/43	5/12 vs. 30/71
9	35/832 vs. 37/834	25/287 vs. 28/290

Results

Survey Results

We summarize the survey results in terms of three themes: (a) PSTs know what they were expected to learn in school, (b) when encouraged to do so, PSTs explore different perspectives and strategies, and (c) PSTs exhibit productive seeds for reasoning about fraction comparisons.

First, unsurprisingly, we find that the participants tended to use Parts and Transform strategies. As expected, those two perspectives are most familiar to PSTs, and they were the perspectives most commonly used. Figure 1 tabulates the number of instances of strategy-arguments for each perspective, broken down by item. Parts was a common perspective across items involving smaller, easier fraction comparisons. Transform was also a common strategy perspective across most of the items, with many PSTs frequently converting both fractions to a common denominator explicitly or using cross multiplication. Components strategies occasionally appeared on the later items, but were predominant on the last item with many PSTs noting the common difference of two in that comparison. Reference Point strategies were rare. Strategy arguments coded as "Other" were not developed enough to code, and comparisons coded as "None" had no work shown (either a simple answer or completely blank).

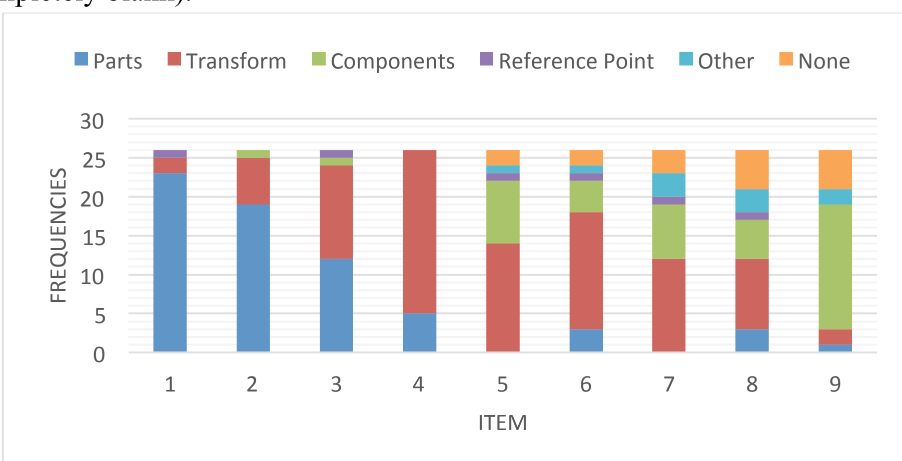

Figure 1: Perspectives used by comparison item

Second, in contrast to descriptions in the literature, the participating PSTs did exhibit flexibility in their reasoning about fraction magnitude. Recall that the instructions for the survey asked participants to first make comparisons mentally and that the numbers chosen for the comparison items lent themselves to different strategies. Nonetheless, the participants could have defaulted to converting to a common denominator for every task. They did not. Instead, PSTs used an average of 5.5 distinct strategies across the 9 tasks, and they averaged 3.73 distinct strategies that were accompanied by complete arguments. Of the 26 participants, 20 used five or more distinct strategies. All of the participants used at least three distinct strategies.

Third, we see substantial evidence of productive seeds of reasoning. The comparisons items ranged from easy to difficult for the participants, especially given that they were instructed to make their judgments mentally and that the survey was administered under time constraints. Each participant was given a correctness score for each item: correct answers scored 1 point, and incorrect answered scored 0 points. The mean total score was 5.96 (of a maximum of 9 points) with a standard deviation of 1.66. The average number of correct answers accompanied by complete arguments was 4.38 with a standard deviation of 2.17. Thus, the participants answered most items correctly, but there was substantial room for improvement in correctness and especially in producing complete arguments.

Candidate responses to be coded for productive seeds were those that did not constitute complete arguments. Of the 87 incomplete arguments, 39 (approximately 45%) included evidence of productive seeds. (There were another 6 responses with no written work provided, and there were 17 items left blank, which may have been due to time constraints.) Thus, even in cases of incomplete arguments, the participants often reasoned about fraction comparisons in productive ways. This result indicated the potential for the interview participants to construct new strategy-arguments with minimal intervention during the interviews.

Interview Results

Due to length constraints, we focus here on the interview participants' responses to the third comparison item, 7/8 vs. 8/9. This item was intended to invite PSTs to consider the complements of the given fractions (i.e., 1/8 vs. 1/9) and to construct a strategy-argument based on comparing complements (e.g., 1/9 is smaller than 1/8, so 8/9 is larger than 7/8 because it is closer to whole). Indeed, 8 of the 10 interview participants were able to construct a complete argument that involved comparing the complements and reasoning from a Parts perspective. Most participants did not compare complements initially. Instead, they began with a more familiar strategy such as converting to a common denominator. They then compared complements in response to an interviewer's request, such as to try to find a "different way" of making the comparison. Alternatively, the interviewer followed up on something that the participant had mentioned (e.g., the possibility of thinking in terms of "parts" or "pies"). Given such requests and encouragement, 80% of the participants constructed a complete Comparing Complements strategy-argument, supporting the correct conclusion that 8/9 was greater than 7/8. By contrast, only 1 of the 10 participants had compared complements for the corresponding item (6/7 vs. 7/8) on the pre-survey. On the post-survey—without assistance and free to choose any strategy they wished—7 of the 10 interview participants used comparing complements for 6/7 vs. 7/8.

Thus, we see evidence that the strategy of comparing complements was in the ZPCs of the majority of the interview participants. This was especially the case for those who took the size of the parts into account in comparisons involving a common numerator (5/8 vs. 5/9 in the interview). Those who explained that 5/8 was greater than 5/9 because eighths are larger than ninths (using Parts: Denominator Principle) appeared to be ready to reason in terms of complements for 7/8 vs. 8/9, even if doing so was novel and somewhat challenging. For example, Jane used the Denominator Principle to correctly compare 5/8 and 5/9. When she was posed 7/8 and 8/9, she noted that "the numerators are each one less than the denominator" and that eighths were larger than ninths. However, she was not immediately sure what conclusion to draw. She made rectangular area drawings of 7/8 and 8/9. Her drawings were sloppy and actually made 7/8 appear to be greater. However, despite her drawing, Jane reasoned that the missing piece from 8/9 must be smaller than the missing piece from 7/8, and therefore 8/9 was greater. Even after constructing a complete strategy-argument, Jane expressed doubt, so the interviewer invited her to explore the idea further. She created her own example, using 1/2 and 2/3, which bolstered her confidence in this new way of comparing fractions.

The two participants who did not construct a complete strategy-argument involving complements during the interview conspicuously ignored piece size in their reasoning. Both focused on the number of parts, rather than their size, when thinking in terms of Parts (and otherwise relied on converting to decimals). For example, Kimmy described 5/8 as missing 3 pieces and 5/9 as missing 4 pieces, without making any mention of the size of said pieces. As best we can tell from the data, even with interviewer probing, the size of the pieces did not enter into her reasoning. Like Jane, she noticed that both 7/8 and 8/9 were "missing one piece," but unlike Jane, she was unable to arrive at a complete strategy-argument using complements.

Discussion

We have begun the fine-grained work of identifying strategy-arguments for comparing fractions that are within the ZPCs of some PSTs, depending on their current ways of reasoning about fractions and given low levels of intervention. This finding is encouraging. Our work also reveals substantial diversity in PSTs' reasoning about fraction comparisons—a theme that is underemphasized in the literature. In the absence of documented distinctions, the literature might encourage mathematics teacher educators to treat all PSTs as if they think similarly.

We have shown that certain, nonstandard strategy-arguments, such as Comparing Complements are readily learnable by some PSTs, given their current ways of reasoning. Note that we are not distinguishing PSTs based on supposed ability. Our data do not speak to their mathematical abilities in general, and we do not claim that some of our interviewees were more mathematical capable than others. Instead, we are concerned with how they were thinking about fractions at the beginning of the course, in relation to the progress that they were able to make during the interview. Those PSTs who took the size of the parts into account when reasoning in terms of Parts were able to compare complements with lows levels of interviewer intervention. Those PSTs who consistently ignored the size of the parts did not appear to be readily able to construct Comparing Complements on the day of the interview. However, later on, having first constructed Parts: Denominator Principle, they may have become able to do so. Thus, in making claims about PSTs' ZPCs, these are limited to what was readily learnable *during the interview*.

Conclusion

In our review of the research literature concerning PSTs' knowledge of fractions, we pointed our three problems with the emphasis on negative generalizations. We frame our contributions in response to these problems: (1) Whereas the emphasis on deficiencies runs the risk of promoting low expectations of PSTs' abilities to learn, our approach is concerned with documenting learning and identifying the conditions under which it is readily achievable. (2) Whereas literature that emphasizes deficiencies fails to provide useful information from which to design instruction, our approach identifies PSTs' particular conceptions (in the form of strategy-arguments, in this case) and charts the terrain of viable reorganizations. (3) Whereas generalizations about deficiencies in PSTs' content knowledge fail to distinguish PSTs from one another, our approach focuses on the diversity of reasoning that can be found among PSTs. This research has illuminated our own understanding of PSTs' reasoning about fraction comparisons and has helped to inform instruction in the courses that we teach.

This line of research values PSTs' prior knowledge and identifies desirable mathematical understandings that are readily learnable, given favorable conditions. In this way, we identify PSTs' particular conceptions and chart the terrain of viable reorganizations. Once they move beyond default approaches like converting to a common denominator, we find considerable diversity in the mathematical thinking of PSTs, and we discover what they are ready to learn.

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DIDATIC-MATHEMATICAL KNOWLEDGE AND COMPETENCE OF PROSPECTIVE SECONDARY SCHOOL MATHEMATICS TEACHERS ON LINEAR VARIATION

CONOCIMIENTOS Y COMPETENCIAS DIDÁCTICO-MATEMÁTICOS DE FUTUROS PROFESORES DE MATEMÁTICAS DE SECUNDARIA SOBRE VARIACIÓN LINEAL

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In this paper we describe an educational proposal for prospective secondary school mathematics teachers in Mexico, whose aim was to contribute to the development of their didactic-mathematical knowledge about linear variation through mathematical tasks, the development of the competence of identifying primary mathematical objects and the reflection on hypothetical cases of teaching. The proposal is supported within both the Onto-semiotic Approach (EOS) and the Mathematics Teacher's Didactic-Mathematical Knowledge and Competence model. The proposal follows a design research methodology based on EOS. The results show the complexity related to the development of the competence of identifying primary mathematical objects, they also highlight the influence that competence has on the development of the epistemic and cognitive facets of prospective teachers' didactic-mathematical knowledge on linear variation.

Keywords: Teacher Education - Preservice; Mathematical Knowledge for Teaching; Middle School Education; Precalculus.

Introduction

Linear variation is an important mathematical topic that goes transversely trough the mathematics curriculum in Mexico, it is taught from the fifth year of primary education, becomes a central topic in secondary school (SEP, 2017), and its teaching continues in the upper secondary and higher educational levels.

The teaching of variation is essential because variation is a fundamental notion for the study of physical phenomena that can be observed in nature and found in people's daily experiences. These phenomena have the characteristic of being dynamic, that is, they involve processes that are constantly changing. Therefore, the teaching of variation should support students to make estimations, comparisons, and models that allow to explain the phenomena of change and solve problems demanded by their milieu (García & Ledezma, S / F; Caballero & Cantoral, 2015).

Despite the importance of teaching variation, particularly linear variation, the mathematics curriculum often promotes a static and limited teaching of this mathematical content in secondary school. In this regard, authors such as Bojórquez, Castillo, and Jiménez (2016), Panorkou, Maloney, and Confrey (2016), Thompson and Carlson (2017), and Vasco (2006) highlight that the variational thinking of students is not explicitly considered in the curriculum of mathematics in the elementary and secondary school. Also, they declare that mathematics textbooks are not formulated from a variational point of view and point out the notion of variable magnitude is missing in teaching of mathematics, despite the fact that they constitute a necessary basis for the learning of calculus at later educational levels. Instead, a static teaching of function as a correspondence rule is fostered, without any connection to variation.

Since curricula and textbooks are the primary materials available to secondary school mathematics teachers, and considering these documents foster a limited approach to the teaching of linear variation, it becomes highly important that Mathematics Education provide concrete guidelines to support teachers to broaden their perspective about teaching linear variation.

It is particularly important that, in an early stage, teachers incorporate to their professional practice more enriched strategies for teaching linear variation than the ones suggested in mathematics curriculum. Consequently, it is pertinent to intervene in the initial teacher education, as Godino, Giacomone, Batanero and Font (2017) argue: "mathematics teacher education […] demands attention from the Mathematics Education, since the development of students' basic thinking and mathematical skills depend essentially on this education"(p. 91).

It is noteworthy that there is little research that provides specific orientations on how to strengthen didactic and mathematical teachers knowledge on linear variation in teacher training programs, despite the fact that different researchers have reported difficulties of mathematics teachers and prospective teachers in relation to notions closely linked to linear variation, such as function (Wilhelmi, Godino & Lasa, 2014, Amaya, Pino-fan & Medina, 2016) and proportionality (Balderas, Block & Guerra, 2014).

To address this problem, we designed, implemented, and evaluated a didactic proposal for the teaching of linear variation. We called it *formative proposal* because it is aimed at prospective mathematics teachers. The main goal was to enrich didactic-mathematical knowledge of prospective teachers on linear variation and to initiate them in the development of their competence for identifying primary mathematical objects related to this mathematical notion. The proposal was theoretically supported on both the Onto-semiotic Approach (EOS) and the Mathematics Teacher's Didactic-Mathematical Knowledge and Competence model (DMKC). It was followed a qualitative design research methodology that integrates some of the EOS theoretical tools. The formative proposal included the design of a sequence of didactic activities which included mathematics tasks and several questions and situations to prompt didactical reflections. To design the formative proposal we took into consideration the work of Herrera-Garcia (2020), who characterized several meanings of linear variation pertinent for teaching in secondary school (as a visual representation on the number line, as a graphical representation on the Cartesian plane, numerically as proportional variations of corresponding magnitudes, and as an algebraic formula).

This paper we describe both the stages of the methodology followed to design the formative proposal and the structure of the didactic sequence designed. In addition, we discuss some of the results obtained by its experimentation with prospective secondary school mathematics teachers. We also present part of the analysis of some of the prospective teachers 'productions corresponding to the second activity of the sequence, which allowed us to establish important relationships between the development of the teachers' competence of identification of primary mathematical objects and the development of their didactic-mathematical knowledge.

Theoretical Frame

We support the formative proposal on theoretical tools from the Onto-Semiotic Approach (OSA) to mathematical knowledge and instruction (Godino, Batanero & Font, 2007), as it provides important elements to elaborate an instructional design and methodological tools that allow guide its development and evaluating its implementation. To design the proposal, we used two fundamental theoretical notions (Godino, Batanero & Font, 2008): *institutional meaning* of a mathematical object (the system of mathematical practices shared into an institution to solve a same type of problems) and the typology of *primary mathematical objects*: *situations* (problems, exercises, etc.), *concepts* (given by definitions or descriptions), *languages* (terms, algebraic expressions, graphs, ...), *procedures* (techniques, algorithms, operations), *arguments* (statements to validate or explain) and Propositions (statements about concepts).

Mathematics Teacher's Didactic-Mathematical Knowledge and Competence model (DMKC)

Given the need for theoretical tools to characterize and evaluate the teacher's didactic-mathematical knowledge and skills, the Mathematics Teacher's Didactic-Mathematical Knowledge and Competence model (DMKC) has recently been developed within the EOS (Godino, Giacomone, Font and Pino-fan, 2018). This model emerged as an extension of the Mathematics Teacher's Didactic-Mathematical Knowledge (DMK) model (Godino, 2009) and has been enriched by Godino and collaborators based on the EOS theoretical tools in several investigations (Pino-Fan & Godino, 2015; Pino-Fan, Godino & Font , 2015). In this model, it is considered that the mathematics teacher must have knowledge conserning of mathematical notions they teach (Pino- Fan & Godino, 2015). In addition, the teacher must have a didactic-mathematical (or specialized) knowledge of the different facets or dimensions that intervene in instructional processes: epistemic, ecological, cognitive, affective, mediational, and interactional. In this work, we consider only two facets: epistemic (didactic-mathematical knowledge about mathematics itself) and cognitive (knowledge about how students learn mathematics) (Godino, Giacomone, et al., 2017).

In addition to this, the DMKC model states that the prospective teacher must also develop a series of *didactic-mathematical competences* that allow him to face the problems of teaching mathematics. In particular, in this work we were interested in the competence of *ontosemiotic analysis of mathematical practices*, which, according to Godino, Giacomone, Batanero and Font (2017) consists in the identification of the network of primary mathematical objects and processes involved in mathematical practices, which allows the teacher to understand the progression of learning, manage the processes of institutionalization and evaluate the mathematical skills of their students. It is important to declare that we considered necessary, among all the elements contemplated in the competence mentioned above, the identification of primary mathematical objects in the mathematical practices carried out when addressing the didactic sequence, therefore, we call it in this paper *competence of identifying primary mathematical objects*.

Method

Methodological approach

We followed the methodology for design research proposed by Godino, Rivas, Arteaga, Lasa and Wilhelmi (2014), which integrates elements of both the design-based research methodology and Didactic Engineering within the theoretical tools of EOS. The methodology consisted of four stages; preliminary study (stablishing the institutional meaning of reference on linear variation), design of the didactic trajectory (elaborating didactic sequence by designing mathematical tasks using the primary mathematical objects related to the meanings of linear variation, the creation of GeoGebra applications and the design of tasks to prompt didactic-mathematical reflections), implementation of the didactic trajectory (implementation of the didactic sequence) and retrospective evaluation or analysis (the development of the above described competence and the epistemic and cognitive facets of the DMKC model).

Context and participants

The formative proposal was implemented with nine prospective mathematics secondary school teachers, who were in the eighth semester of Bachelor in a public institution in Mexico that provides programs for secondary school teachers initial education. The experimentatio was carried out in four sessions with a total duration of 15 hours. The designer of the formative proposal also served as the instructor.

Data Collection Instrument

The main instrument to collect the information was the sequence of didactic activities, which is made of five didactic activities with specific tasks for prospective teachers, supported by digital applications with GeoGebra according to the content of each activity. The activities started with a situation of variation in extra-mathematical context, some situations corresponded to linear variation and others did not, in order to help prospective teachers to distinguish when a situation corresponds or not to a case of linear variation. The didactic activities were printed on paper and given to each participant, who were designed with the letter "E" and a number from 1 to 9. The participants were asked to write their answers with different colored pen according to the working modality: black for individual work, red for teamwork and blue for group work.

Structure of the didactic sequence

Each of the activities is organized in three parts, addressing the following aspects: mathematical knowledge, didactic-mathematical knowledge, and the competence of identifying primary mathematical objects. In each of these three parts of the didactic activities different work modalities were considered: individual, team of three people and group discussion.

Part I: *Mathematical work*. This part of the sequence is intended for solving mathematical tasks aimed at enriching the meaning of linear variation of future teachers. For this, situations of linear variation and non-linear variation were proposed in various contexts, such as the relationship between the biological age of a dog and the years lived by him; the relationship between a person's weight (kg) and its height; filling and emptying of cylindrical containers. In these contexts, it was sought that future teachers manage to characterize linear variation based on the proportional relationship between the corresponding variations of two variable magnitudes in different forms of language: dynamic number lines in GeoGebra, Cartesian graphs, tables of values and algebraic expressions.

Part II: *Identification of the primary mathematical objects involved in the mathematical practices developed in Part I*. In the first didactic activity, it was sought that the future teachers express what they understood by three of the primary mathematical objects. To do this, they were asked the following questions: What is a mathematical concept for you?, What is a procedure for you?, What is a property / proposition for you?, adapted from the work of Giacomone (2018). It is important to mention that future teachers were not instructed in the use of EOS, but through group discussions they were guided so that, in a consensual way, they characterized those mathematical objects based on their initial ideas. Having established what is meant by that three mathematical objects, they were asked to identify those used in solving the mathematical tasks of Part I in each of the activities.

Part III: *Analysis of the answers given by hypothetical high school students*. This last part consists of the analysis of answers supposedly provided by students when addressing linear variation problems, with the aim that the prospective teacher analyzed the student's mathematical practice, determined whether or not it is correct and created strategies to guide and provide feedback to the student.

Data analysis and discussion

After the implementation of the five activities that integrated the didactic sequence, we carried out the analysis and interpretation of the answers provided by the prospective teachers. Below, we present some answers corresponding to the first two didactic activities of the sequence, which we interpreted from the perspective given by the theoretical referents chosen.

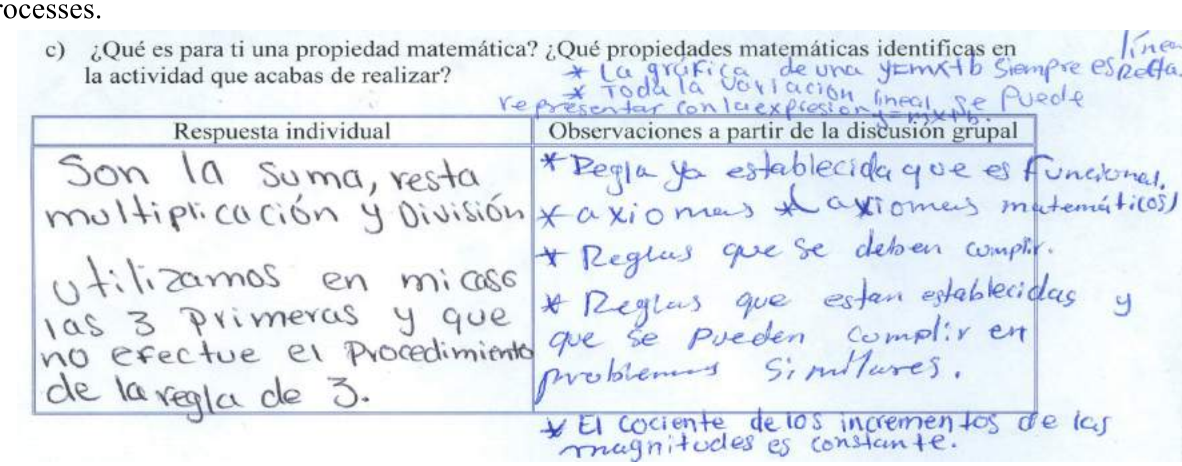
Identification of primary mathematical objects in the mathematical practices

An essential part for developing the competence of identifying primary mathematical objects both in their own and in their students' mathematical practices, is to have a wide perspective about the diversity of mathematical aspects involved in solving a mathematical task, that is, to understand each of the six primary mathematical objects. In this work, due to time constraints, we decided to limit the competence of identifying primary objects to the following three objects: procedures, concepts, and properties.

In the first didactic activity, the future teachers carried out the mathematical tasks of Part I, aimed at expanding their mathematical knowledge on linear variation. Then, in Part II, where future teachers were to explain what a concept, property, and procedure is, the following was found. The future teachers had no problem explaining what a *procedure* is to them and it was relatively easy to identify procedures such as the calculation of basic operations (addition, subtraction, multiplication, and division) in their practices. In the group discussion, the instructor asked questions in order to guide future teachers to recognize other types of procedures, so it is important to highlight that group interaction was essential for the identification of a wider variety of procedures, such as the "rule of three", the clearances of a variable, and the calculation the proportionality constant.

The mathematical object *concept* was a little more difficult to characterize by some teachers. For example, E8 expressed that a mathematical concept "is the problem with which we are working to find a solution", this suggests that for him a concept is the problem to be solved. Furthermore, E8 failed to use the notion he expressed to identify concepts in his own mathematical practices. On the other hand, the future professor E7, defined a concept as: "the meaning of a word" and identified term, equation, and magnitude as concepts in the work carried out in part I. Some of the concepts identified by future teachers individually are slope, proportionality, line, terms, proportionality constant, algebraic expression, and magnitude.

In contrast to the above, it was more difficult for future teachers to explain what a *property* is. For example, E5 expressed that a property "is a mathematical axiom" and E4 wrote that "it is an already established rule that is always functional". Both provided the algebraic expressions "$y = mx + b$" y "$K = \Delta y/\Delta x$" as examples of properties involved in part I of mathematical work, which might suggest that for them the properties have to be expressed algebraically. Other future teachers failed to provide examples of properties. On the other hand, E8 did not explain what a property was, but did mention addition and subtraction as examples. It is important to highlight the enrichment in the answers during the group discussion (this corresponds to the epistemic facet of didactic-mathematical knowledge). In the bottom of the right column (Figure 1), in blue, it can be seen that E8 takes up the ideas of its peers and adds some examples of properties, such as: "The quotient of the magnitudes' increases is constant." This highlights the importance of interactions during mathematical instruction processes.


c) ¿Qué es para ti una propiedad matemática? ¿Qué propiedades matemáticas identificas en la actividad que acabas de realizar?

* La gráfica de una y=mx+b siempre es recta. línea
* Toda la variación lineal se puede representar con la expresión y=mx+b.

Respuesta individual	Observaciones a partir de la discusión grupal
Son la Suma, resta multiplicación y División utilizamos en mi caso las 3 primeras y que no efectue el Procedimiento de la regla de 3.	* Regla ya establecida que es funcional. * axiomas * axiomas matemáticos * Reglas que se deben cumplir. * Reglas que estan establecidas y que se pueden cumplir en problemas similares.

* El cociente de los incrementos de las magnitudes es constante.

Figure 1. Enrichment of E8 answer after group discussion

Explaining what a concept and a property is was a difficult task for future teachers, as has already been reported by other researchers such as Giacomone (2018), Giacomone, Godino, Wilhelmi and Blanco (2018), Burgos, Giacomone, Beltrán-Pellicer and Godino (2017), Burgos, Godino, Giacomone and Beltrán-Pellicer (2018b) and Burgos, Beltrán-Pellicer, Giacomone and Godino (2018a). The recognition of these objects in mathematical practices is a competence that needs time to be developed, partly due to the difficulty of understanding what these primary mathematical objects are. However, the tasks proposed in the didactic sequence were motivating elements to initiate them in the development of that competence.

Gradually, in subsequent activities, future teachers were able to more consistently identify concepts, procedures, and properties in the mathematical practices carried out in the mathematical tasks about linear variation, which also highlights the development of the epistemic facet of their didactic-mathematical knowledge. In Activity 2, which dealt with the relationship between the weight and height of a person, expressed in a numerical table, prospective teachers identified new concepts and new properties of linear variation when creating a Cartesian graph and reflect on some the points obteinded from the table. For example, E5 said: “The slope of two pairs of different points must have the same value for linear variation. The union of the points should form a straight line” (Figure 2).

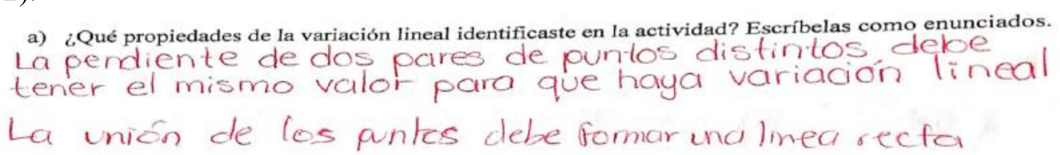

a) ¿Qué propiedades de la variación lineal identificaste en la actividad? Escríbelas como enunciados.
La pendiente de dos pares de puntos distintos debe tener el mismo valor para que haya variación lineal
· La unión de los puntos debe formar una linea recta

Figure 2. E5 and E1 answer during teamwork

Furthermore, prospective teachers showed changes in the type of guidance they would give to students in the hypothetical teaching situations. Initially, their orientations were limited to and inclined towards manipulating algebraic expressions. After the mathematical work and the tasks aimed to identifying primary mathematical objects in Activity 2, they analyzed in more detail the hypothetical answer of a high school student (part III), who made a graph (Figure 3) with data on weight and height provided by the Mexican Social Security Institute, and stated the situation corresponded to a linear variation situation.

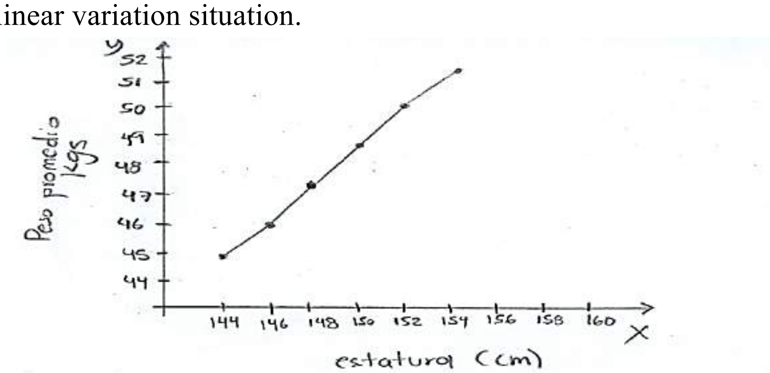
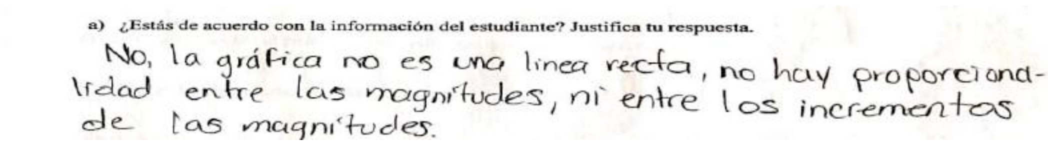


Figure 3. Graph drawn by a hypothetical high school student

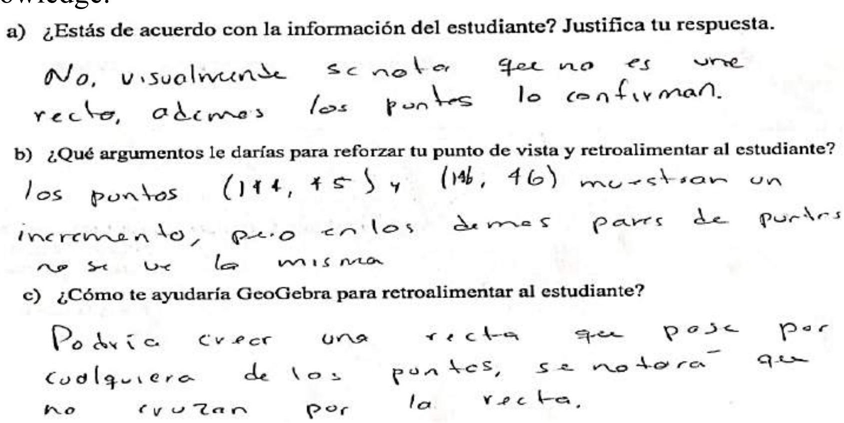
Regarding the student's answer, the future teachers were asked: a) Do you agree with the student's answer?, B) What arguments would you give to reinforce your point of view and give feed back to the student?, And c) How Would GeoGebra help you to provide feedback to the student? In evaluating the student's answer, the future teachers used the primary mathematical objects identified in their own mathematical practices (Figure 2). For example, E1 (Figure 4) analyzed the graphical representation through the properties of linear variation that emerged in part I of this activity. He states that "the graphic is not a line because there is no proportionality between the magnitudes, nor between the magnitudes' increases". E1 evaluated the student's answer and concluded it is not correct; being able to make this assessment corresponds to the epistemic facet of didactic-mathematical knowledge.


a) ¿Estás de acuerdo con la información del estudiante? Justifica tu respuesta.

No, la gráfica no es una linea recta, no hay proporcionalidad entre las magnitudes, ni entre los incrementos de las magnitudes.

Figure 4. The future teacher E1 argues using properties of linear variation

We can find another example of the enrichement of future teachers didactic-mathematical knowledge in the answer of E9 (Figure 4) to the same situation, who first expresses that "visually the graph is not a line". Then, to provide feedback to the student, he proposes to choose pairs of points and identify that there is no constant variation. In addition, he states that GeoGebra would help the student by creating a line that would show that it does not cross all the points. These reflections of E9 on how it would orient a student, show the enrichment of the cognitive facet of its didactic-mathematical knowledge.


a) ¿Estás de acuerdo con la información del estudiante? Justifica tu respuesta.

No, visualmente se nota que no es una recta, ademas los puntos lo confirman.

b) ¿Qué argumentos le darías para reforzar tu punto de vista y retroalimentar al estudiante?

los puntos (144, 45) y (146, 46) muestran un incremento, pero en los demas pares de puntos no se ve la misma

c) ¿Cómo te ayudaría GeoGebra para retroalimentar al estudiante?

Podría crear una recta que pase por cualquiera de los puntos, se notara que no cruzan por la recta.

Figure 5. Prospective teacher E9 provides feedback to the student using GeoGebra

In the examples shown above, it can be identified that future teachers progressed in their professional knowledge, both mathematical and didactic-mathematical, since they provide arguments and orientations (in some cases using GeoGebra) based on different procedures, and properties of linear variation, which allowed them to assess whether or not the situations posed to them and to the hypothetical students correspond to situations of linear variation in different forms of language (algebraic, graphic, numerical and verbal). That is, the formative proposal allowed future teachers to enrich their specialized knowledge of mathematics, since they could identify in their own mathematical practices primary objects related to linear variation, and then, based on them, they could argue why some hypothetical responses of students were incorrect and propose feedback strategies for the student regarding the study of linear variation.

Conclusions

After analyzing the answers of future teachers, we concluded that the competence of identifying primary mathematical objects was a challenging task for them, as documented in studies such as that of Burgos et al. 2017. On the other hand, their answers suggests that they were able to gradually carry out more detailed analyzes of both their mathematical practices and the mathematical practices of the hypothetical students, since they showed a greater diversity of primary mathematical objects in the reflection tasks (parts II and III of the activities).

In the implementation of the activities, time was devoted to the discussion of ideas and the comparison of the answers given, this generated a very rich moment of exchange opinions and allowed prospective teachers to reflect on other mathematical objects that they had not identified by their own, so group discussions allowed them to enrich their initial answers.

A very important aspect that must be highlighted is that, initially, the didactic-mathematical reflections that the prospective teachers generated regarding the tasks set out, used to be very limited and lacking arguments. Subsequently, by working on the activities and the tasks set out to initiate them to develop the competence of identifying primary mathematical objects, they generated more detailed responses, which included the use of primary mathematical objects. This helped them to propose more detailed strategies to guide the students based on argumentation related to properties and procedures of linear variation previously identified in the part I (mathematical work). This highlights the importance of the development of this competence for their teaching practice, since it prompts prospective teachers carry out analysis of mathematical practices that take into account the diversity of primary mathematical objects related to the teaching and learning of a specific mathematical content.

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CONOCIMIENTOS Y COMPETENCIAS DIDÁCTICO-MATEMÁTICOS DE FUTUROS PROFESORES DE MATEMÁTICAS DE SECUNDARIA SOBRE VARIACIÓN LINEAL

DIDATIC-MATHEMATICAL KNOWLEDGE AND COMPETENCE OF PROSPECTIVE SECONDARY SCHOOL MATHEMATICS TEACHERS ON LINEAR VARIATION

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Se describe una propuesta formativa para futuros profesores de secundaria en México, cuyo objetivo fue enriquecer sus conocimientos didáctico-matemáticos sobre variación lineal a través de tareas matemáticas, del desarrollo de la competencia de identificación de objetos matemáticos primarios y de la reflexión sobre casos hipotéticos de enseñanza. La propuesta se enmarca en el Enfoque Ontosemiótico (EOS) y en el modelo de Conocimientos y Competencias Didáctico-Matemáticos del profesor de matemáticas; además, sigue una metodología de investigación de diseño fundamentada en el EOS. El análisis de los resultados muestra la complejidad inherente al desarrollo de la competencia de identificación de objetos matemáticos primarios, pero resalta su impacto en el desarrollo de las facetas epistémica y cognitiva del conocimiento didáctico-matemático sobre variación lineal de los futuros profesores.

Palabras clave: Preparación de Maestros en Formación; Conocimiento matemático para la enseñanza; Educación secundaria; Pre-Cálculo.

Introducción y problemática

El estudio de la variación lineal abarca transversalmente el currículo de matemáticas en México; inicia en quinto año de educación primaria, se vuelve central en la escuela secundaria (SEP, 2017) y está presente en los niveles educativos medio superior y superior.

La enseñanza de la variación es importante porque ésta es una noción fundamental para el estudio de fenómenos físicos que se pueden observar en la naturaleza y que se encuentran en las vivencias cotidianas de las personas. Dichos fenómenos tienen la característica principal de ser dinámicos, es decir, involucran procesos que están en constante cambio. En este sentido, la enseñanza de la variación debe favorecer que los sujetos realicen estimaciones, comparaciones y construyan modelos que permitan explicar fenómenos de cambio y resolver situaciones que demande su entorno (García & Ledezma, S/F; Caballero & Cantoral, 2015).

A pesar de la importancia que tiene la enseñanza de la variación, particularmente la variación lineal, el currículo de matemáticas suele promover un estudio estático y limitado de este contenido matemático en la escuela secundaria. Al respecto, autores como Bojórquez, Castillo y Jiménez (2016), Panorkou, Maloney y Confrey (2016), Thompson y Carlson (2017) y Vasco (2006) destacan que el pensamiento variacional de los estudiantes no se desarrolla explícitamente en los planes de estudio de educación básica y que los libros de texto de matemáticas no están formulados desde un punto de vista variacional. Además, argumentan que no se promueve el estudio de magnitudes variables, a pesar de que constituyen una base necesaria para el estudio del cálculo en niveles educativos posteriores; en su lugar, se realiza un estudio estático de la función como regla de correspondencia, sin relación con la variación.

Dado que los planes de estudio y los libros de texto son los materiales principales que tienen a su disposición los profesores de matemáticas de secundaria, y en estos documentos es limitado el tratamiento propuesto para la variación lineal, es pertinente apoyar a los docentes en el desarrollo de una perspectiva más amplia de la variación lineal que oriente su práctica en el aula, de manera que la enseñanza de este tema no quede restringida a un enfoque estático y algebraico.

Particularmente, es importante fomentar de manera temprana que los profesores incorporen a su práctica docente tratamientos didácticos más enriquecidos para la variación lineal que aquellos planteados en el currículo. En este sentido, es pertinente intervenir en la etapa formativa de los profesores de matemáticas, como afirman Godino, Giacomone, Batanero y Font (2017), "la formación didáctica de los profesores […] reclama atención por parte de la Didáctica de la matemática, pues el desarrollo del pensamiento y de las competencias matemáticas básicas de los alumnos depende, de manera esencial, de dicha formación." (p. 91).

Es importante resaltar que son pocas las investigaciones que orientan de forma concreta sobre cómo fortalecer el conocimiento matemático y didáctico sobre variación lineal en programas de formación de profesores, a pesar de que diferentes investigaciones han reportado dificultades en los profesores o futuros profesores de matemáticas con relación a dos nociones estrechamente vinculadas a la variación lineal, la función (Wilhelmi, Godino & Lasa, 2014, Amaya, Pino-fan & Medina, 2016) y la proporcionalidad (Balderas, Block & Guerra, 2014).

Para atender esta problemática, se diseñó, implementó y evaluó una propuesta didáctica para el estudio de la variación lineal, a la que denominó propuesta formativa por estar orientada a futuros profesores de matemáticas. El objetivo de dicha propuesta fue enriquecer conocimientos didáctico-matemáticos de los futuros profesores sobre variación lineal e iniciarlos el desarrollo de la competencia de identificación de objetos matemáticos primarios relacionados con dicha noción matemática. La propuesta se fundamentó teóricamente en el Enfoque Ontosemiótico (EOS) y en el modelo de Conocimientos y Competencias Didáctico-Matemáticos del profesor de matemáticas (CCDM), y siguió una metodología cualitativa de investigación de diseño que integra las herramientas teóricas del EOS. La propuesta formativa incluyó el diseño de una secuencia de actividades con tareas matemáticas y de reflexión didáctica, para lo cual se caracterizaron diferentes significados de la variación lineal (como representación gráfica en rectas numéricas, como representación gráfica en el plano cartesiano, como representación tabular con variaciones proporcionales y como representación analítica) pertinentes para la educación secundaria. Tales significados se detallan ampliamente en Herrera-Garcia (2020) y ponen de manifiesto la diversidad de propiedades, procedimientos, representaciones, etc., que se pueden estudiar sobre la variación lineal desde un punto de vista variacional, y que es esencial que los profesores de secundaria conozcan para que puedan favorecerlos en el aula.

En este escrito se describirán a grandes rasgos las etapas de la metodología seguida para el diseño de la propuesta formativa y la estructura de la secuencia didáctica diseñada. Además, se discutirán algunos de los resultados obtenidos al implementar la secuencia didáctica con futuros profesores de matemáticas de secundaria, ilustrando con el análisis de las respuestas correspondientes a la segunda actividad de la secuencia, las cuales permiten establecer relaciones importantes entre el desarrollo de la competencia de identificación de objetos matemáticos primarios y el desarrollo de conocimientos didáctico-matemáticos de los futuros profesores.

Referentes teóricos

Para formular, implementar y valorar la propuesta formativa se tomó como referente teórico el Enfoque Ontosemiótico del Conocimiento y la Instrucción Matemáticos, EOS (Godino, Batanero & Font, 2007), pues aporta elementos significativos para elaborar un diseño instruccional y herramientas metodológicas que permiten estructurar su desarrollo y valorar su implementación. Fueron centrales para el desarrollo de la propuesta, por un lado, la noción de *significado institucional* de un objeto matemático, que se entiende como el sistema de prácticas matemáticas compartidas en una institución para resolver un tipo de situaciones problema y, por otro lado, la *tipología de objetos matemáticos primarios* que componen el significado de un objeto matemático (Godino, Batanero & Font, 2008): *situaciones problema* (problemas, ejercicios, tareas, etc.)*, conceptos-definiciones* (introducidos mediante definiciones o descripciones)*, lenguajes* (notaciones, expresiones, gráficos, etc., representados de manera escrita, oral, gestual, gráfica, tabular…)*, procedimientos* (técnicas, algoritmos, operaciones)*, argumentos* (enunciados para validar o explicar) *y proposiciones* (enunciados sobre conceptos).

Modelo de Conocimientos y Competencias Didáctico-Matemáticas del profesor (CCDM)

Ante la necesidad de contar con herramientas teóricas que permitan caracterizar y evaluar los conocimientos y competencias didáctico-matemáticos del profesor, se ha desarrollado recientemente al seno del EOS el Modelo de Conocimientos y Competencias Didáctico-Matemáticos del profesor de matemáticas (CCDM) (Godino, Giacomone, Font y Pino-fan, 2018). Este modelo surge como una ampliación del modelo de Conocimientos Didáctico-Matemáticos del profesor de matemáticas (CDM) (Godino, 2009) y ha sido enriquecido por Godino y colaboradores en diversas investigaciones (Pino-Fan & Godino, 2015; Pino-Fan, Godino y Font, 2015) con base en las herramientas teóricas del EOS. En este modelo se considera que el profesor de matemáticas debe tener conocimiento común y ampliado del contenido, es decir, conocimientos sobre las nociones matemáticas que se estudian en el nivel donde se desempeña y sobre contenidos correspondientes a los niveles posteriores (Pino-Fan & Godino, 2015). Además, el profesor debe tener un conocimiento didáctico-matemático, o especializado, de las distintas facetas o dimensiones que intervienen en el proceso educativo: *epistémica*, *ecológica*, *cognitiva*, *afectiva*, *mediacional* e *interaccional*. En este trabajo se consideraron únicamente las facetas *epistémica* (conocimiento didáctico-matemático sobre las matemáticas mismas) y *cognitiva* (conocimiento sobre la manera como los estudiantes aprenden las matemáticas) (Godino, Giacomone, et al., 2017).

Aunado a esto, el modelo CCDM plantea que el futuro profesor también debe desarrollar una serie de *competencias didáctico-matemáticas* que le permitan hacer frente a los problemas de enseñanza de las matemáticas. En particular, en este trabajo se puso el interés en la *competencia de análisis ontosemiótico de prácticas matemáticas*, que, según Godino, Giacomone, Batanero y Font (2017) consiste en la identificación de la red de objetos matemáticos primarios y procesos intervinientes en las practicas matemáticas, que permite al profesor comprender la progresión de los aprendizajes, gestionar los procesos de institucionalización y evaluar las competencias matemáticas de sus alumnos. Es importante declarar que en este trabajo se consideró únicamente, de entre todos los elementos contemplados en esta competencia, la identificación de objetos matemáticos primarios puestos en juego en las prácticas matemáticas realizadas al abordar la secuencia didáctica diseñada, por ello, se hará referencia en este trabajo a la *competencia de identificación de objetos matemáticos primarios*.

Consideraciones metodológicas

Se siguió la metodología para investigaciones de diseño propuesta por Godino, Rivas, Arteaga, Lasa y Wilhelmi (2014), que retoma elementos de la investigación basada en el diseño y de la Ingeniería Didáctica y los articula con las herramientas teóricas del EOS. La metodología consistió de cuatro etapas; *estudio preliminar* (determinar el significado institucional de referencia sobre variación lineal), *diseño de la trayectoria didáctica* (diseño de tareas matemáticas a partir de la elección de objetos matemáticos propios del significado de variación lineal, la creación de aplicaciones de GeoGebra y el diseño de tareas de reflexión didáctico-matemática que integraron en la secuencia didáctica), *implementación de la trayectoria didáctica* (implementación de la secuencia didáctica) y *evaluación o análisis retrospectivo* (del desarrollo de la competencia descrita y las facetas epistémica y cognitiva del modelo CCDM).

Contexto y participantes

La propuesta formativa se implementó con nueve futuros profesores de octavo semestre de Licenciatura en Educación secundaria con Especialidad en Matemáticas de una institución formadora de profesores en México. Se realizaron cuatro sesiones de trabajo con una duración total aproximada de 15 horas. La diseñadora de la propuesta formativa fungió como instructora.

Instrumento recolección de información

El instrumento principal para recopilar la información fue la secuencia de actividades didácticas, la cual se conformó por cinco actividades didácticas con tareas específicas para los futuros profesores, apoyadas en aplicaciones digitales diseñadas con GeoGebra acordes al contenido de cada actividad. Las actividades iniciaban con una situación problema de variación de contexto extramatemático, algunas eran de variación lineal y otras no, con el propósito de ayudar a los futuros profesores a identificar cuándo una situación problema corresponde o no a un caso de variación lineal. Las actividades didácticas fueron impresas en papel y entregadas a cada participante, a quiénes se designó con la letra "E" y un número del 1 al 9. Se dio la instrucción de escribir las respuestas con pluma de diferente color según la modalidad de trabajo: negro para el trabajo individual, rojo para el trabajo en equipo y azul para el trabajo grupal.

Estructura de las actividades de la secuencia didáctica

Cada una de las actividades se organizó en tres partes, atendiendo los siguientes aspectos: el conocimiento matemático, el conocimiento didáctico-matemático y la competencia de identificación de objetos matemáticos primarios. En cada una de estas tres partes de las actividades didácticas se consideraron diferentes modalidades de trabajo: individual, en equipo de tres personas y discusión grupal.

Parte I: *Trabajo matemático*. Esta parte de la secuencia se destinó a la resolución de tareas matemáticas orientadas al enriquecimiento del significado de variación lineal de los futuros profesores. Para ello, se propusieron situaciones de variación lineal y de variación no lineal en diversos contextos, como la relación entre la edad biológica de un perro y los años vividos; la relación entre el peso (kg) de una persona y su estatura; el llenado y vaciado de recipientes cilíndricos. En estos contextos se buscó que los futuros profesores lograran caracterizar la variación lineal a partir de la relación de proporcionalidad entre las variaciones correspondientes de dos magnitudes variables en diferentes formas de lenguaje: rectas numéricas dinámicas en GeoGebra, gráficas cartesianas, tablas de valores y expresiones algebraicas.

Parte II: *Identificación de los objetos matemáticos primarios involucrados en las prácticas matemáticas* desarrolladas en la parte I. En la primera actividad didáctica se buscó que los futuros profesores expresaran qué entendían por tres de los objetos matemáticos primarios. Para ello, se les plantearon las preguntas siguientes: ¿Qué es para ti un concepto matemático?, ¿qué es para ti un procedimiento?, ¿qué es para ti una propiedad/proposición?, adaptadas del trabajo de Giacomone (2018). Es importante mencionar que a los futuros profesores no se les instruyó en el uso del EOS, sino que a través de discusiones grupales se les guio para que, de manera consensuada, se caracterizaran dichos objetos matemáticos a partir de sus ideas iniciales. Una vez establecido qué se entendería por los tres objetos matemáticos mencionados, se les pidió que identificaran aquellos que intervinieron en la resolución de las tareas matemáticas de la Parte I en cada una de las actividades.

Parte III: *Análisis de respuestas dadas por estudiantes hipotéticos de secundaria.* Esta última parte consistió en el análisis de respuestas supuestamente proporcionadas por estudiantes al abordar situaciones problema de variación lineal, con el objetivo de que el futuro profesor analizara la práctica matemática del estudiante, determinara si era o no correcta y creara estrategias para orientar y retroalimentar al estudiante.

Análisis de datos y discusión de resultados

Tras realizar la implementación de las cinco actividades que conformaron la secuencia didáctica, se realizó el análisis e interpretación de las respuestas obtenidas. A continuación, se presentan algunas respuestas correspondientes a las primeras dos actividades didácticas de la secuencia, las cuales son interpretadas desde los referentes teóricos elegidos.

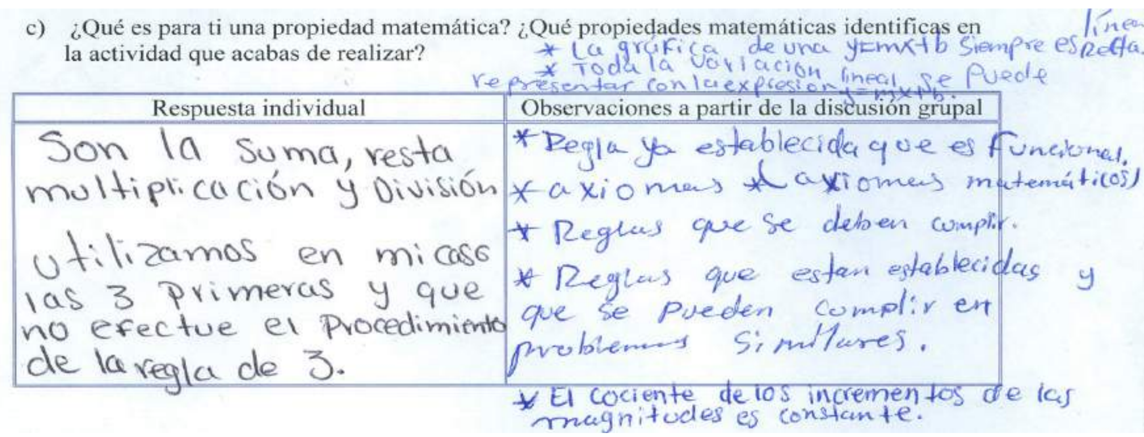
Identificación de objetos matemáticos primarios en sus prácticas matemáticas

Una parte esencial para el desarrollo de la competencia de identificación de objetos matemáticos primarios en las prácticas matemáticas propias o de los estudiantes es tener claridad sobre la diversidad de aspectos matemáticos involucrados al resolver una tarea matemática, es decir, comprender cada uno de los seis objetos matemáticos primarios. En este trabajo, por limitaciones de tiempo, se decidió delimitar la competencia de identificación de objetos primarios a los tres siguientes: procedimientos, conceptos y propiedades.

En la primera actividad didáctica, los futuros profesores realizaron las tareas matemáticas de la Parte I, orientadas a ampliar sus conocimientos matemáticos sobre variación lineal. Luego, en la parte II, donde los futuros profesores debían explicar qué es un concepto, una propiedad y un procedimiento, se encontró lo siguiente. Los futuros profesores no tuvieron problemas para explicar qué es para ellos un procedimiento y fue relativamente sencillo identificar en sus prácticas procedimientos como el cálculo de operaciones básicas (suma, resta, multiplicación y división). En la discusión grupal, la instructora realizó preguntas que guiaran a los futuros profesores al reconocimiento de otros tipos de procedimientos, por lo cual es importante resaltar que la interacción grupal fue fundamental para la identificación de una variedad más amplia de procedimientos, como la regla de tres, los despejes, y el cálculo la constante de proporcionalidad.

El objeto matemático *concepto* fue un poco más difícil de caracterizar por algunos profesores. Por ejemplo, E8 expresó que un concepto matemático "es el problema con el que se está trabajando para darle una solución", lo cual sugiere que para él un concepto es el problema por resolver. Además, E8 no logró emplear la noción expresada para identificar conceptos en sus prácticas matemáticas. Por otro lado, el futuro profesor E7, definió un concepto como: "el significado de una palabra" e identificó término, ecuación y magnitud como conceptos en el trabajo realizado en la parte I. Entre los conceptos identificados por los futuros profesores de manera individual se encuentran: pendiente, proporcionalidad, rectas, términos, constante de proporcionalidad, expresión algebraica y magnitud.

En contraste con lo anterior, para los futuros profesores resultó más complicado explicar qué es una *propiedad*. Por ejemplo, E5 expresó que una propiedad “es un axioma matemático” y E4 escribió que “es una regla ya establecida que es funcional siempre”. Ambos proporcionaron las expresiones algebraicas “$y = mx + b$” y “$K = \Delta y/\Delta x$” como ejemplos de propiedades intervinientes en la parte I de trabajo matemático, lo cual podría sugerir que para ellos las propiedades se expresan de manera algebraica. Otros futuros profesores no lograron proporcionar ejemplos de propiedades. Por otro lado, E8 no explicó que era una propiedad, pero mencionó a la suma y la resta como ejemplos. Es importante resaltar el enriquecimiento que se generó en las respuestas durante la discusión grupal de las respuestas (faceta epistémica del conocimiento didáctico-matemático). En la columna de la derecha (Figura 1), en color azul se puede observar que E8 retoma las ideas de sus compañeros y agrega algunos ejemplos de propiedades, como: “El cociente de los incrementos de las magnitudes es constante”. Esto pone de manifiesto la importancia de las interacciones durante los procesos de instrucción matemática.


c) ¿Qué es para ti una propiedad matemática? ¿Qué propiedades matemáticas identificas en la actividad que acabas de realizar?

* La gráfica de una y=mx+b siempre es recta. línea
* Toda la variación lineal se puede representar con la expresión y=mx+b.

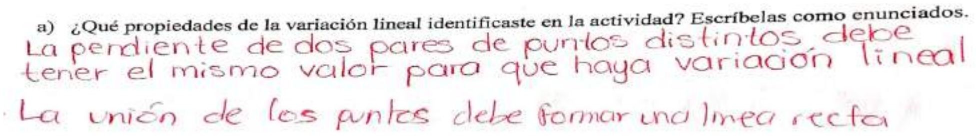
Respuesta individual	Observaciones a partir de la discusión grupal
Son la suma, resta multiplicación y División utilizamos en mi caso las 3 primeras y que no efectue el Procedimiento de la regla de 3.	* Regla ya establecida que es funcional. * axiomas * axiomas matemáticos * Reglas que se deben cumplir. * Reglas que estan establecidas y que se pueden cumplir en problemas similares.

* El cociente de los incrementos de las magnitudes es constante.

Figura 1. Enriquecimiento de la respuesta de E8 tras la discusión grupal

Explicar qué es un concepto y una propiedad fue una tarea difícil para los futuros profesores, como ya ha sido reportado por otros investigadores como Giacomone (2018), Giacomone, Godino, Wilhelmi y Blanco (2018), Burgos, Giacomone, Beltrán-Pellicer y Godino (2017), Burgos, Godino, Giacomone y Beltrán-Pellicer (2018b) y Burgos, Beltrán-Pellicer, Giacomone y Godino (2018a). El reconocimiento de estos objetos en las prácticas matemáticas es una competencia que necesita tiempo para ser desarrollada, en parte por la dificultad de comprender qué son estos objetos. Sin embargo, las tareas propuestas en la secuencia didáctica fueron elementos motivantes para iniciarlos en el desarrollo de su competencia.

Gradualmente, en las actividades posteriores, los futuros profesores lograron identificar de manera más consistente conceptos, procedimientos y propiedades en las prácticas matemáticas realizadas en torno al estudio del tema variación lineal, lo cual pone de manifiesto también el desarrollo de la faceta epistémica de su conocimiento didáctico-matemático. Por ejemplo, en la actividad 2, que trataba sobre la relación entre el peso y la estatura de una persona expresada en una tabla numérica, al crear una gráfica cartesiana los futuros profesores identificaron en sus prácticas matemáticas nuevos conceptos y nuevas propiedades de la variación lineal como las siguientes: “La pendiente de dos pares de puntos distintos debe tener el mismo valor para que haya variación lineal. La unión de los puntos debe formar una línea recta” (Figura 2).


a) ¿Qué propiedades de la variación lineal identificaste en la actividad? Escríbelas como enunciados.

La pendiente de dos pares de puntos distintos debe tener el mismo valor para que haya variación lineal

La unión de los puntos debe formar una linea recta

Figura 2. Respuestas de E5 y E1 durante el trabajo en equipo

Además, los futuros profesores mostraron cambios en el tipo de orientaciones que darían a los estudiantes. Inicialmente, sus orientaciones eran limitadas y se inclinaban a la manipulación de expresiones algebraicas. Después del trabajo matemático y las tareas de identificación de objetos matemáticos primarios en la actividad 2, analizaron de manera más detallada la respuesta hipotética de un estudiante de secundaria (parte III), quien elaboró una gráfica (Figura 3) con datos del peso y la estatura propuestos por el Instituto Mexicano del Seguro Social, y afirmó que se tenía una situación de variación lineal.

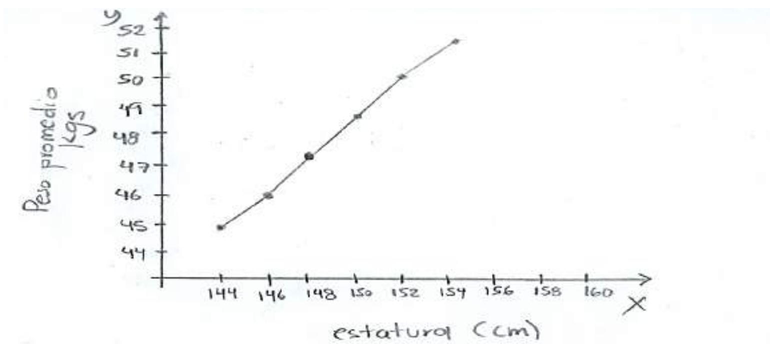
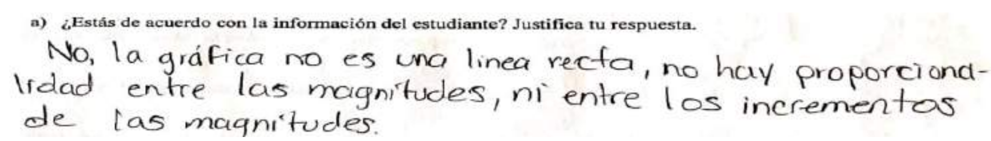


Figura 3. Gráfica trazada por un estudiante hipotético de secundaria

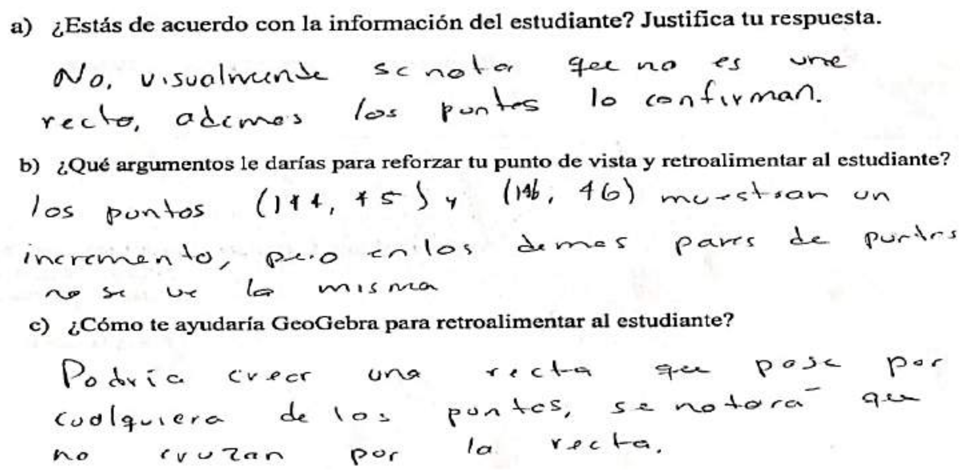
Se les preguntó a los futuros profesores: a) ¿Estás de acuerdo con la respuesta del estudiante?, b) ¿Qué argumentos le darías para reforzar tu punto de vista y retroalimentar al estudiante?, y c) ¿Cómo te ayudaría GeoGebra para retroalimentar al estudiante? Al valorar la respuesta del estudiante, los futuros profesores pusieron en juego los objetos matemáticos primarios identificados en sus prácticas (Figura 2). En la respuesta de E1 (Figura 4) se puede observar fue de su interés analizar la representación gráfica, pues afirma que “no hay proporcionalidad entre las magnitudes, ni entre los incrementos de las magnitudes”, lo cual fue trabajado en la parte I de la actividad. A partir de ese análisis concluye que la respuesta del estudiante no es correcta (hacer esta valoración corresponde a la faceta epistémica del conocimiento didáctico-matemático).


a) ¿Estás de acuerdo con la información del estudiante? Justifica tu respuesta.

No, la gráfica no es una linea recta, no hay proporciona-lidad entre las magnitudes, ni entre los incrementos de las magnitudes.

Figura 4. El futuro profesor E1 argumenta usando propiedades de la variación lineal

Otro ejemplo es el de E9 (Figura 4), quien primero expresa que “visualmente la gráfica no es una recta”. Luego, para retroalimentar al estudiante, propone elegir pares de puntos e identificar que no hay una variación constante. Además, expresa que GeoGebra ayudaría al estudiante mediante la creación de una recta que mostraría que no ésta no cruza por todos los puntos. Estas reflexiones de E9 sobre cómo orientaría a un estudiante, ponen de manifiesto el enriquecimiento de la faceta cognitiva de su conocimiento didáctico-matemático.


a) ¿Estás de acuerdo con la información del estudiante? Justifica tu respuesta.

No, visualmente se nota que no es una recta, además los puntos lo confirman.

b) ¿Qué argumentos le darías para reforzar tu punto de vista y retroalimentar al estudiante?

los puntos (144, 45) y (146, 46) muestran un incremento, pero en los demás pares de puntos no se ve la misma

c) ¿Cómo te ayudaría GeoGebra para retroalimentar al estudiante?

Podría crear una recta que pase por cualquiera de los puntos, se notará que no cruzan por la recta.

Figura 5. Respuesta de E9

En los ejemplos mostrados se puede identificar que los futuros profesores progresaron en sus conocimientos profesionales, tanto matemáticos como didáctico-matemáticos, pues muestran argumentaciones y orientaciones basadas en procedimientos diversos, y propiedades de la variación lineal en diferentes formas de lenguaje, que les permitieron valorar si las situaciones planteadas a ellos y a los estudiantes hipotéticos, corresponden o no a situaciones de variación lineal, en algunos casos utilizando GeoGebra. Es decir, la propuesta formativa permitió que los futuros profesores enriquecieran su conocimiento especializado del contenido de matemáticas, pues podían identificar en sus respuestas objetos primarios propios de la variación lineal, y después, con base a ellos, pudieron argumentar porqué algunas respuestas hipotéticas de los estudiantes estaban incorrectas y proponer estrategias de retroalimentación para el alumno en relación con el estudio de la variación lineal.

Conclusiones

Tras el análisis de las respuestas de los futuros profesores se concluye que la competencia de identificación de objetos matemáticos primarios fue una tarea desafiante para ellos, como se había documentado en trabajos como el de Burgos et al. 2017. Por otro lado, el análisis de sus respuestas sugiere que lograron realizar análisis más finos de sus prácticas matemáticas y de las prácticas matemáticas de los estudiantes hipotéticos, ya que fueron manifestando mayor diversidad de objetos matemáticos primarios en las tareas de reflexión (partes II y III de las actividades).

En la implementación de las actividades se dedicó tiempo a la discusión de ideas y a la comparación de las respuestas dadas, esto generó un momento muy rico de intercambio de opiniones y permitió que los futuros profesores reflexionaran y comentaran sobre otros objetos matemáticos que no habían identificado, pero que con la discusión grupal lograron identificar, lo que permitió enriquecer sus respuestas.

Un aspecto muy importante y que es necesario destacar es que se observó que en un inicio las reflexiones didáctico-matemáticas que generaban respecto a las tareas planteadas, solían ser reflexiones muy limitadas y carentes de argumentos. Posteriormente con el desarrollo de las actividades y con las tareas planteadas para iniciarlos al desarrollo de la competencia de identificación de objetos matemáticos primarios, generaban respuestas más detalladas, que incluían el uso de los objetos matemáticos primarios, esto ayudaba a que las estrategias que proponían para orientar a los estudiantes tuvieran mayor peso en la argumentación basada en el uso de propiedades y procedimientos, identificados previamente en la resolución de las situaciones problema planteados en el diseño de las actividades (trabajo de la parte matemática).

Esto resalta la importancia del desarrollo de esta competencia para su práctica docente, pues genera en los futuros profesores un análisis que toma en cuenta el tipo de objetos matemáticos primarios propios del estudio de algún tema en específico, es decir, funcionó como una herramienta que les permitió observar más que simple detalles, aspectos importantes que consideraron para la enseñanza de la variación lineal.

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WHAT DOES IT MEAN TO BE ME? A PRESERVICE MATHEMATICS TEACHER'S IDENTITY DEVELOPMENT DURING AN EDUCATION ABROAD PROGRAM

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A review of the literature underpinning mathematics education illustrates that in order to achieve an equitable mathematics education, we must consider other methods of preparing mathematics teachers—methods that encourage identity development, specifically cultural awareness and open-mindedness as two key facets of this construct. This study describes and interprets a preservice mathematics teacher's identity development during a semester-long mathematics-focused education abroad program. Findings suggest that we can foster alternative visions of identity—ones that have a better understanding of culture and a greater sense of open-mindedness—through participation in such culturally-rich international programs.

Keywords: Cross-cultural Studies, Equity and Diversity, Teacher Beliefs, Teacher Education-Preservice

Purpose

A review of the literature underpinning mathematics education illustrates that in order to achieve an equitable mathematics education, one that is free of systemic forms of inequality, we must consider other methods of preparing mathematics teachers—methods that encourage identity development, specifically cultural awareness and open-mindedness as two key facets of this construct. For example, the National Council of Supervisors of Mathematics (NCSM) and TODOS: Mathematics for ALL (TODOS) have suggested that mathematics teachers must take a stance that "interrogates and challenges the roles power, privilege, and oppression play in the current unjust system of mathematics education—and in society as a whole" (2016, para.1). In addition, the National Council of Teachers of Mathematics (NCTM) has suggested that in order to promote a culture of access and equity within mathematics education, teachers should be "responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge" (2014, pg. 1). Furthermore, others have noted preservice or in-service mathematics teachers initially dismiss the idea of teaching social justice, believing that it does not belong in mathematics (de Freitas, 2008; Ahlquist, 2001; Weissglass, 2000). As De Freitas has said, "Those frequently heard comments—'I'm just a math guy,' 'I'm one of those people who likes math for the sake of the math only,' 'I'm not one for social justice'—share a particular vision of identity as being a fixed, unmovable, and irresolvable entity" (2008, p. 50), and unfortunately that is a mindset we often see in mathematics.

So, what kind of preparation program do preservice mathematics teachers need? Weissglass has said "any serious attempt to achieve equity in mathematics education must be rooted in an ongoing process of increasing our understanding of how individual prejudices, unaware biases, and systemic societal discrimination affect teaching and learning" (2000, p. 10). Gutstein has asserted that more work needs to be done to alter teachers' personal belief systems built on deficit thinking, specifically when working with diverse children (2000). De Freitas has suggested, "Alternative visions of identity are required" in order to change the fixed, closed mindsets of mathematics teachers and begin to develop a critical mathematics education (2008, p. 49). As Neumayer-Depiper has said, it is not enough to simply develop a set of effective mathematics teaching practices (2009). We must consider other methods of preparing mathematics teachers, methods that encourage identity development. Rather than having a fixed, closed mindset as De Freitas describes, we want preservice mathematics teachers to develop an identity that is more open-minded. We also want our preservice mathematics

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teachers to have a developed sense of their own cultural identity and, as Weissglass says, an increased understanding of how their individual prejudices and unaware biases affect teaching and learning.

Participation in an education abroad program can address these needs and help foster mathematics teachers that are open-minded and have an established sense of their cultural identity. As such, the purpose of this study is to examine how education abroad influences the identities of preservice mathematics teachers in ways that they become more culturally aware and open-minded.

Theoretical Framework

Research has indicated that throughout an education abroad experience, students "challenge their beliefs about the world and its people, develop empathy for and trust in others, learn a significant amount about at least one other culture, and often to their surprise, learn quite a lot about their own culture" (Cushner, 2009, p. 160). Much of this learning comes from being immersed in another culture and having the experience of feeling like a cultural outsider. Merryfield (2000) found that those who left the US and experienced living in another culture "came to understand temporarily what it feels like to live outside of the mainstream…. They became conscious of what happens to identity when people know they don't belong" (p. 439). This reality of feeling like a cultural outsider is a feeling that many mainstream teachers in the US have never experienced, and it leads to a personal understanding of what it is like to be marginalized and stereotyped. This is an impactful experience that facilitates teachers "to become more ethnorelative in their understanding of others, more skilled at crossing cultures, and committed to bringing about change through their work" (Cushner, 2009, p. 165).

Beyond impacting students in these ways, education abroad also has the potential to impact identity on a deeper level. Teacher professional identity is a core aspect of the teaching profession (Sachs, 2005). "It provides a framework for teachers to construct their own ideas of 'how to be', 'how to act' and 'how to understand' their work and their place in society" (p. 15). The development of teacher professional identity is an ongoing process (Beijaard, Meijer & Verloop, 2004) that cannot be forced. Instead, "it is negotiated through experience and the sense that is made of that experience (Sachs, 2005, p. 15). Participation in an education abroad program can impact the identities of preservice mathematics teachers in ways that they become more open-minded and culturally aware, ultimately inspiring a more updated and progressive vision of teaching mathematics. However, little is known regarding the impact of education abroad programs on the identity development of preservice mathematics teachers—specifically their intercultural competencies and open-mindedness as two key facets of this construct. This research explores this timely area of inquiry.

Methods and Data Sources

This is a qualitative case-study of one preservice mathematics teacher's identity development throughout a semester-long education abroad program in England. This program was meant specifically for preservice mathematics teachers (elementary and secondary). Students work at a mathematics education research center, intern in schools, and take mathematics education classes at a university.

This study began the summer prior to their semester abroad and ended shortly after their return to the United States. Three main data collection methods were used: semi-structured interviews, in-country participant observation, and document review. Documents such as student journals, coursework, and the Intercultural Development Inventory (IDI) were used within this study.

The IDI (Hammer, Bennet & Wiseman, 2003) was administered to participants prior to their departure and again upon return to the United States. This assessment places participants on a continuum ranging from "ethnocentric" to "ethnorelative." Ethnocentrism is defined as "the experience of one's own culture as 'central to reality'", in which "the beliefs and behaviors that people receive in their primary socialization are unquestioned: they are experienced as 'just the way things are,'" (Bennett, 2004, p. 62). Ethnorelativism is "the experience of one's own beliefs and behaviors as just one organization of reality among many viable possibilities" (Bennett, 2004, p. 62). There are there categories related to ethnocentrism: Denial, Polarization, and Minimization, and two categories related to ethnorelativism: Acceptance and Adaptation. Teachers with an ethnorelative mindset would be more inclined to engage with students in ways that respect their cultures, backgrounds, and experiences, ultimately meeting NCTM's (2014) call for a responsive mathematics education.

The interviews and in-country observations provided insight into their experiences and day-to-day activities within the program and enabled me to explore and uncover in what ways participants' thinking about culture, mathematics teaching, and open-mindedness evolved throughout the program.

Results

The case presented in this study, whom I will refer to as Ben, showed growth in both cultural awareness and open-mindedness. Ben is a white male who turned 22 years old while participating in this program. Prior to departure, Ben fell in the "ethnocentric" category of the IDI, specifically within minimization (see figure 1). A person within minimization is typically color-blind, "focusing on commonalities and universal values, emphasizing similarities, and holding the belief that all people are fundamentally the same" (Cushner, 2009, p. 156).

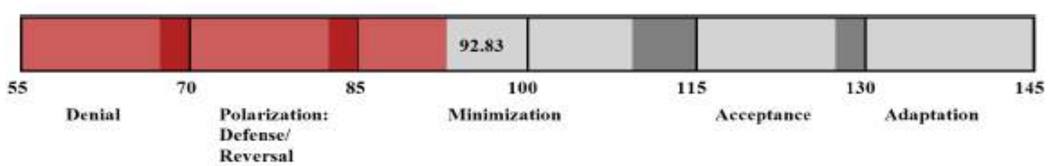

Figure 1: Ben's IDI score at the start of the program

This aligns with some of Ben's comments from the start of the program in which he expressed that he avoided seeing culture because it would prevent him from understanding the individual person at hand. Ben described that he preferred to pay attention to individual (not cultural) differences, saying, "Generalizing cultures, in my opinion, is a bad thing as it takes away the ability for the individual to be themselves."

Ben also had trouble describing his own culture. In our first interview, he described himself as "Caucasian," "Polish," and "Italian," but indicated these weren't identities that he felt connected to. Rather, he said "culturally, it would be more accurate to say my family is a family of helpers over any specific background" because his mom was an occupational therapist and his dad taught in an elementary school. He described that his family was privileged in the sense that they didn't need to identity with "race," or "culture," and that they could identify with something else entirely, like being "helpers."

To Ben, America was too multifaceted to generalize, and he had trouble articulating what it meant to be American. He discussed that there may be "things [people] in the US share, but a majority of those people have vast differences… The culture of the US is close to not being generalizable at all." He was also tentative to label himself as American:

> I guess I consider myself American in the sense that I was raised in America, but I feel like America is too broad of a thing for me to consider myself as…. So, I would say in a manner of speaking, I identify as American because I grew up in America. But I don't really identify with anything that I know could be a generalization of Americans, that I'm aware of.

Throughout the program, Ben wrote journals documenting his beliefs of what qualifies as good mathematics teaching. At the beginning of the program, Ben had little awareness of his own culture, his students' cultures, and how these identities would influence his classroom. When reflecting on what qualifies as good mathematics teaching, Ben spoke of his own views of the subject and what he deems important, saying, for example, that he values conceptual understanding of the subject and views procedures as "tedious," and "a mindless waste of time." However, he never mentioned his students or the experiences and perspectives they may bring to the classroom.

Once Ben arrived in England, he began to notice surface-level cultural differences like driving on the other side of the road, but throughout the program, he began to notice deeper cultural differences such as how the concept of time was viewed in England versus the United States along with contrasts in professional communication styles. He discussed the experience of feeling like an American, a concept he had never thought about previous to this education abroad experience, as he said, "once you open your mouth, people already have all these views of you." He also discussed how he was constantly asked to speak for all of America: "Right off the bat, people were asking me about Trump, school shootings, and violence in America. It was weird being the spokesperson for that." These experiences forced Ben to consider his culture, and he began to notice differences across cultures.

Throughout the program, Ben also began to recognize aspects of American culture. He discussed how America was founded on "sticking it to the man," and "standing up for itself at a time where they felt taken advantage of." He acknowledged that there is a spirit of rebellion in England, saying "there is plenty of public protesting" relating to Brexit, but that he sees this value as being more prevalent in American society. He wrote, "In England, being rebellious is not seen as something valuable," giving as examples, "Students wear uniforms to class, rather than getting to express themselves in different ways," and "They have a formal relationship with their teachers that revolves around the teacher being the head of the classroom." He also described how there was no desire to own a weapon in England, but that America was "founded on the ability to rise up against oppression and rebel," and that "We have a right to bear arms in America so that if we are oppressed, we can take appropriate measures to challenge what is there." He went on to say, "I would argue that the times have changed and we need to reconsider this value," but at the end of the day, the right to bear arms comes from "the spirit of rebellion, a crucial piece of our founding virtues." These types of reflections about cultural differences occurred throughout the entire program.

By the end of the program, Ben's IDI report demonstrated growth (see figure 2). While he still fell within the category of minimization, he was approaching acceptance, an orientation the reflects a recognition and appreciation of cultural differences.

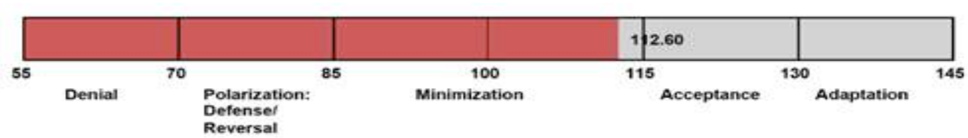

Figure 2: Ben's IDI score at conclusion of the program

Ben also expressed that he was now more interested in culture and that he "keeps cultural and identity in mind, rather than solely the individual." Additionally, he expressed a new understanding of himself and his students, saying:

> I can say my time in Europe has changed what I enjoy about math. I still believe in what I said previously, that the best piece for me is problem solving and the conceptual. I also think that's the most important piece of math. But I think that I've missed something crucial about math. Reflecting on different cultures and understanding of the world has helped me understand some of the disconnect between my students and I back when I was student

> teaching. The truth is, they valued getting a correct answer. They didn't care how. It was the beauty and relief of finishing a problem with a tool they had that pushed them forward. It wasn't their skill, but their ability to use a tool that connected them to the mathematics. Up until studying here and thinking deeply about cultural differences I failed to see some people fundamentally don't feel the way I do.

Throughout the program, Ben began to notice and appreciate cultural difference, ultimately saying:

> Looking at what my students' value, how my students view math, how my students view education, and applying it to my own understanding to grow and change my teaching style overtime is going to be fundamental to my practice…. I will be careful to not push my own view of the mathematics on the students, rather I will shape my strategies and methods to what they enjoy, value, and believe. Over time, after gaining my students trust, I will offer different options to pieces already in place…. Careful reflection on my students and their situations and their founding principles will lead me to become a better, more effective, efficient teacher that can reach out to students in many different ways rather than simply through the mathematics.

By the end of the program, Ben expressed a deeper understanding of his culture and a greater need to pay attention to his students' cultural identities. His descriptions of teaching mathematics had evolved to more closely align with NCTM's call for mathematics teachers that are "responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge" (NCTM, 2014, pg. 1).

Conclusions

Ben's journey throughout this program suggests that through participation in education abroad programming, preservice teachers can become more culturally aware and open-minded. At the start of the program, Ben had little understanding of his own cultural identity, and he avoided noticing cultural difference. His orientation towards cultural difference fell within the category of minimization, reflecting "a tendency to highlight commonalities across cultures that can mask important cultural differences in values, perceptions, and behaviors," (Ben's IDI profile, pg. 6). In addition, when Ben would reflect on his beliefs of what qualifies as good mathematics teaching, he never mentioned his students or the importance of including their perspectives, experiences, and backgrounds into his classroom. Proponents of culturally responsive teaching argue that "Explicit knowledge about cultural diversity is imperative" (Gay, 2002, p. 107) to meet the needs of the diverse student population. As such, Ben was far from achieving this.

However, by the end of the program, he was recognizing that culture influences the experiences, values, beliefs, and perspectives that people have, and he was aware that his students' culture, specifically their experiences and values, were different from his own. He indicated that he now believed his, and his students', cultural identities would influence their experiences in the classroom, and he discussed a desire to incorporate his students' perspectives and values into his teaching. He said that within his teaching, he would take "it slow and not make assumptions" about his students based on his prior experiences, and that he would instead consider his students' prior experiences.

Ben's orientation of minimization is a reflection of the orientations of many teachers across the country. For example, Mahon (2003) studied 155 teachers in the midwestern US, and found that 100% of them fell at minimization or below. This is problematic on many levels. When we avoid noticing culture, it's the dominant culture that is assumed to be "the" culture, and any other cultures may be ignored, or even worse, shut down or demonized. Furthermore, a minimization mindset would limit our ability to create an equitable mathematics education for all students. In order to address issues of racial, cultural, and socioeconomic inequity, we need to see race, culture, and socioeconomic status. Rather than ignoring culture, we want our teachers to acknowledge and include various cultural perspectives within the classroom.

Given Ben's evolution throughout this program, there appears to be potential for shifting the ways in which we prepare mathematics teachers. If we look to NCTM, NCSM, TODOS, and others, we can see calls for addressing issues of equity, access, and social justice in mathematics education. NCTM (2014) has articulated that teachers should be responsive to students' cultures, experiences, and backgrounds. NCSM and TODOS (2016) have suggested that mathematics teachers interrogate the roles of power, privilege, and oppression within mathematics education. At the beginning of this study, Ben was ignoring culture—not recognizing his own culture or the culture of others—making it nearly impossible to achieve the calls from NCTM, NCSM, and TODOS. This study highlights that we can foster alternative visions of identity—ones that have a better understanding of culture and a greater sense of open-mindedness—through participation in such culturally-rich international programs.

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CLASSROOM EVENTS ON PROBLEM SOLVING WITH GEOGEBRA ANTICIPATED BY FUTURE MATHEMATICS TEACHERS

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This paper presents an analysis of the classroom events that a group of future teachers of Secondary Education mathematics identified from their own experience when using GeoGebra to solve problems. The data analyzed are from the written materials by twelve mathematics majors who were studying the Mathematics for Teaching course. The students, in pairs, solved three problems using GeoGebra. They were then asked to posit three events that could arise if their students were to use GeoGebra to solve problems. After analyzing the events presented, they were classified in terms of mathematical reasoning, mathematical creation and techno-mathematical ability.

Key words: Teacher training, problem solving, technology, classroom situations.

Introduction

Teacher training is a complex field of study that ranges from identifying the knowledge required to teach a discipline, to proposing strategies for developing that knowledge. The incorporation of technology into the process of teaching and learning mathematics poses new challenges in teacher training, particularly when defining training programs. What kinds of activities should be conducted during the training period of a mathematics teacher? What technological tools should be used? How should they be used? How does the use of technology influence their initial training?

On the one hand, the appearance of a certain type of technology has expanded the set of tools available to teachers to respond to events or contingencies that occur in the classroom. For example, Rowland and Zazkis (2013) analyze possible actions in response to a hypothetical answer from a student who is asked to provide a fraction between ½ and ¾. The student answers 2/3, stating that for the numerator she chose 2 because it is between 1 and 3, and for the denominator she chose 3 because it is between 2 and 4. One of the options, presented by the authors, for incorporating the student's idea into the classroom proposes representing, with the aid of technology, a geometric situation that can intuitively provide an answer. This would be done by representing the fractions as the slopes of lines that pass through the origin and through a coordinate point (n, d), where n is the numerator and d the denominator of each fraction.

On the other hand, the use of technological tools in the classroom gives rise to certain types of situations that would not appear in another context. Wasserman, Zazkis, Baldinger, Marmur, & Murray (2019) provide an example in which a group of students used a MAPLE command to confirm that a number is prime. When an operation was entered as the input argument (14:2), MAPLE indicated that it was not a prime. This would not have happened if the input had been a number (7). The authors note that this contingency could be exploited to interact with students and discuss the importance of differentiating number sets and conceptualizing multiplication with rational numbers. Hernández, Perdomo-Díaz and Camacho-Machín (2018) present an analogous situation, in this case by using GeoGebra. This program does not provide an answer when the Tangents tool is used, entering as input values two points, one outside the circle and the other on the circumference. This situation offers the opportunity to discuss, with students, questions such as: Why is nothing happening? What important properties related to lines tangent to a circle do these values not consider? That is, the contingency could be leveraged to have a discussion with students on the properties of lines tangent to a circle and how GeoGebra processes them when plotting these lines.

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GeoGebra is one of the most widely used technological tools for teaching and learning mathematics. Its greatest potential lies in the fact that it is a dynamic geometry software (DGS) package that is dynamic enough to analyze questions, conjectures, discover mathematical properties and establish connections between known properties (Jacinto & Carreira, 2017; Sánchez-Muñoz, 2011; Santos-Trigo & Camacho-Machín, 2013). Given this context, teachers must be familiar with the program and have experience using it so as to take advantage of the opportunities that technology offers to gain mathematical knowledge (Camacho-Machín & Santos-Trigo, 2016). Teachers must also possess a certain ability and expertise to deal with the mathematical questions, doubts, and interpretations that arise every day in classrooms (Conner, Wilson, & Kim, 2011). When an event occurs in the classroom, the teacher must decide whether to ignore it, set the issue aside after considering it or try to incorporate it into the class, which in many cases has a certain improvisational component (Rowland & Zazkis, 2013). For Conner, Wilson and Blume (2011), making a good decision requires a certain type of knowledge and skill. It takes "a particular kind of expertise which includes a deep mathematical knowledge that allows them to recognize the opportunity, weigh its merits, and skillfully pursue or dismiss the opportunity" (p. 979).

An analysis of the problem-solving process with GeoGebra that an individual employ can be used to identify interesting situations to propose as activities for training teachers (Camacho-Machín, Perdomo-Díaz, & Hernández, 2019). But it is also interesting to see what situations future teachers imagine could occur when they ask their students to use GeoGebra to solve problems. Anticipating possible events or contingencies that could come up in class is one way to develop teaching skills (Carrillo, 2015). Consequently, analyzing contingencies and anticipating exercises can provide a connection between the training of future mathematics instructors and teaching in Secondary Education. This leads to the question that prompted this research: What kind of classroom situations do future secondary education mathematics teachers anticipate from their own experience using GeoGebra to solve problems?

The work presented here consists of an exploratory study that addresses this question. The research was conducted with a group of mathematics majors who were taking the "Mathematics for Teaching" course. One part of this course consisted of analyzing classroom situations proposed by Heid, Wilson, & Blume (2015), and another part of the course involved doing a Problem-Solving Workshop using GeoGebra. As part of the final activity in this workshop, the students were asked to identify situations that might arise in a secondary education math class in which GeoGebra is utilized to solve problems. The goal of this research is to analyze classroom situations involving the use of GeoGebra to solve problems anticipated by the participants.

Conceptual Framework

The reference for this research is a framework developed from real and hypothetical situations involving secondary education mathematics classes known as Mathematical Understanding for Secondary Teaching (MUST) (Heid, Wilson, & Blume, 2015). This framework considers that the mathematical understanding that an individual need to teach the discipline in high school can be described from three different, closely interrelated, perspectives: mathematical proficiency, mathematical activity and mathematical context. The first perspective focuses on "knowing" mathematics, the second on "being able to do" mathematics, and the third on the ability to "adjust" that knowledge and know-how to secondary education students (Kilpatrick et al., 2015). As these authors point out, this particular understanding of mathematics, typical of a teacher's endeavor, has a dynamic character, since it starts to take shape based on the understanding that a teacher would have as a student, before developing and transforming during the teacher's training and subsequent career.

When analyzing mathematical understanding for teaching from the perspective of mathematical proficiency, the focus is on mastering the content being taught and the ability to make connections between the concepts to be taught and other mathematical content. This perspective thus includes components such as conceptual understanding, proficiency with procedures, strategic competence and flexible reasoning (Kilpatrick, 2015). For Camacho-Machín and Santos-Trigo (2015), the development of these four components is essential for learning that is characterized by constant questioning and problem solving.

From the perspective of mathematical activity, mathematical understanding for teaching comprises the set of specific mathematical actions that an instructor performs while teaching. Zbiek and Heid (2018) advocate for a teaching practice in which the mathematical activity is made explicit, such that the discipline can be extended beyond the content of the subject, consisting of procedures and concepts. The MUST model defines this perspective from three interrelated components (Kilpatrick et al., 2015):

- Mathematical perception: Groups the actions of recognizing and identifying the mathematical characteristics specific to the different structures, the different notations or symbolic forms, as well as the ability to ascertain when a mathematical argument, whether expressed simply or rigorously, is valid, and the ability to connect mathematical ideas with each other (representing ideas in different structures and connecting various concepts) and with the real world (explaining physical problems through mathematics).
- Mathematical reasoning: Groups the observe, conjecture and justify or prove activities by using deductive logic, mathematical properties, regularities and patterns, generalizations of specific cases, restricting properties and extensions to other structures.
- Mathematical creation: Implies the ability to find new paths to express mathematical objects, generate new ones and transform their representation. This is related to choosing representations of objects that highlight their structure, restrictions or properties, when new objects are defined and when they are manipulated by changing their form, but not their representation.

Finally, the mathematical context perspective includes aspects of mathematical understanding that come into play exclusively in the teaching profession, such as recognizing the mathematical nature of students' questions and errors, or recognizing when an argument or solution provided by a student is incomplete or satisfies the conditions of a problem (Kilpatrick, et al., 2015). Among its components, the authors include: synthesizing mathematical ideas, interacting and understanding students' mathematical thinking, knowing and using the curriculum, evaluating students' mathematical knowledge, and reflecting on the mathematics employed in the classroom.

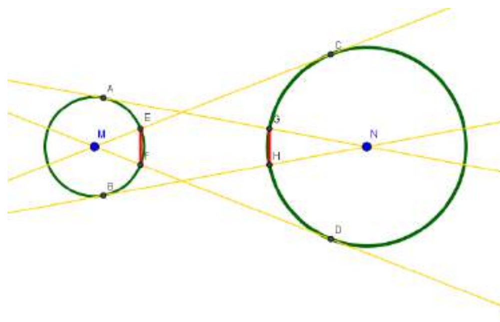
This research combines the mathematical context and mathematical activity perspectives. The former was taken into account in the design of the tasks proposed for the students; specifically, the last component was emphasized by reflecting on the mathematics employed in the classroom, asking students to anticipate *Situations* that could arise in a Secondary Education class where GeoGebra is used to solve problems. For MUST framework, a *Situation* is "a way of capturing classroom practice […] portrays an incident that occurred in the context of teaching secondary mathematics in which some mathematical point is at issue" (p. 4). In each *Situation* a *Prompt* and a set of *Mathematical Foci* can be distinguished. A *Prompt* is something that has occurred or may occur in the context of teaching mathematics, such as a student' question, or a mathematical fact that a student has identify. The *Prompts* proposed by participants were analyzed from the mathematical activity perspective, using their three components as the basis for classifying them.

Methodology

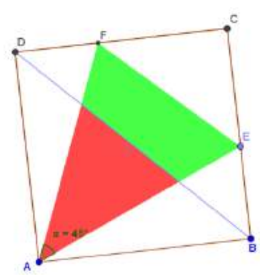
The participants in this research were the students in the "Mathematics for Teaching" course, which is offered as an elective for senior-year mathematics majors at the University of La Laguna (Spain). The main goal of this course is to develop the students' theoretical, practical and instrumental skills associated with the activity of teaching mathematics at the high school and university levels. This includes knowing and using heuristic strategies for solving math problems, as well as technological tools for teaching and learning mathematics.

The data were collected in the 2017-2018 school year over the course of four weeks, in which the students devoted half the class time to analyzing classroom situations designed under the MUST framework (Heid, Wilson & Blume, 2015), and the other half to participating in a GeoGebra Problem Solving Workshop. The workshop consisted of eight, two-hour sessions. In the first five, the students solved the following three problems:

Equal chords. Given two circles with centers M and N, lines are drawn from the center of each that are tangent to the other. The points where the tangents lines intersect the circumferences define two chords, EF and GH. Prove that the length of the chords is the same.



45º angle. Given a square ABCD, draw a 45º angle inside the square, with its vertex at A. This yields two rays that cut the sides opposite A at points E and F (see drawing). Study the relationship between the two parts when triangle AEF is divided by the diagonal BD.



Connect Islands. We want to connect three islands (A, B, and C) with a fiber optic network in a way that uses the least amount of cable. The distances between the islands are 79,322 m (A-B), 64,514 m (A-C) and 95,932 m (B-C). Where should the connection point be located to minimize the amount of cable needed?

In the last three sessions, students focused on identifying and analyzing *Situations*, in the sense of MUST framework, that could happen in a high school mathematics class in which GeoGebra is used to solve the above problems. The data analyzed in this paper are from the report that the students had to submit as a product of those last sessions. The instructions given to the students were:

- Write *Situations* resulting from the use of technology in solving workshop problems.
- In each *Situation* indicate at least three *Prompts*, using GeoGebra to show how they arise.
- Select a *Prompt* for each problem and identify a set of mathematical foci relevant to the prompt.

This activity was intended for students to establish connections between the problem solving activities with GeoGebra that they had carried out in the workshop and the analysis of classroom situations that they had done in the parallel sessions. In this way, the participants moved from a role as students to a role as teachers, taking their own experience using GeoGebra to solve problems as a reference point to reflect on possible situations that may occur in a mathematics classroom where this type of didactic resources.

The course was taken by 18 students, who worked in pairs. For this paper, only the reports from six of the student pairs were selected. The selection criterion was that the students must have handed in all the tasks from the workshop and the study correctly and on time.

The analysis process consisted of identifying what role GeoGebra played in each of the *Prompts* indicated by the participants, and what components of mathematical understanding for teaching that MUST proposes from the perspectives of mathematical activity underlies those prompts. From there, types of class situations were defined that future teachers anticipate from their own experience using GeoGebra to solve problems.

Data analysis

An analysis of the content of the reports prepared by the six students pairs allowed us to identify three types of *Prompts*:

Type 1: The focus is on giving a mathematical explanation to the operation of a GeoGebra tool. These *Prompts* have to do with the development of techno-mathematical ability (Jacinto and Carreira, 2017).

Type 2: *Prompts* that involve justifying or demonstrating a mathematical property that has been observed when making a dynamic construction with GeoGebra.

Type 3: *Prompts* where a conjecture is formulated and GeoGebra is used to build new elements that allow it to be proven or rejected.

The last two types of *Prompts* are closely related to mathematical reasoning and mathematical creation, which are components of mathematical understanding for teaching from the perspective of mathematical activity (Kilpatrick, et al., 2015).

Type 1: The *Prompts* in this category arise directly from the use of tools implemented in GeoGebra and from an understanding of the mathematics on which it is based. Four of the student pairs identified *Prompts* of this type (Table 1). Three of them (P1, P3 and P5) proposed an event related to the use of the Tangents tool that yields an answer they understand; however, it could lead to a classroom situation where a student asks about the steps GeoGebra performed to give that answer. Pair P8 identified an event in which students would question an answer provided by the DGS.

Table 1: Prompts related to the mathematical meaning of actions in GeoGebra

Problem	Topic	Summary of the Prompt	P1	P2	P3	P5	P7	P8
Equal chords	Tangent lines	Can a tangent line to a circle be drawn from an outside point without using the *Tangents* tool?	■	-	■	■	-	-
Equal chords	Tangent lines	Using the *Line* tool, a line tangent to a circle is constructed from an outside point. Why does it seem to satisfy the tangency property when it actually does not?	-	-	-	-	-	■

Type 2: This category includes *Prompts* that relate to the need to prove or demonstrate mathematical conjectures arising from the use of GeoGebra. These hypothetical events related to the action of seeking a justification, whether formal or not, describe a classroom situation in which a secondary education student discovers a property while solving a problem with GeoGebra and asks

the teacher about its veracity or gives an argument that requires formalization. This is the category into which most of the *Prompts* were classified, a total of 10 (Table 2), although all were proposed by three of the student pairs (P1, P3 and P5).

Table 2: Conjectures from Prompts related with mathematical reasoning

Problem	Topic	Summary of the conjecture	P1	P2	P3	P5	P7	P8
Equal chords	Tangent lines	A line tangent to a circle forms an angle of 90° with the radius that connects the center to the point of tangency.	-	-	■	-	-	-
Equal chords	Bisector	The two triangles that are formed by a chord, the two radii it connects and the bisector of these radii are congruent.	■	-	-	■	-	-
Equal chords	Bisector	The bisector of the two lines tangent to a circle from an outside point passes through the center of the circle.	■	-	■	-	-	-
Equal chords	Similarity	Two overlapping triangles (with a common angle) are similar, although they are not in the Thales position.	-	-	-	■	-	-
45° angle	Invariant	Given a family of triangles AEF, where angle A measures 45°, and inscribed in a square ABCD with side l, such that A is on a vertex and E is on side BC, then the height from A of any triangle in the family is constant and measures l.	-	-	■	■	-	-
45° angle	Diagonal of the square	Given a point P on the diagonal that joins two vertices of a square, draw two segments to the other two vertices. This yields four triangles. The heights of the triangles from P match pairwise.	■	-	-	■	-	-
Connect islands	The Fermat point	The Fermat point is the point of intersection of the three circles in which the equilateral triangles used to build it are inscribed.	-	-	-	■	-	-
Connect islands	The Fermat point	The Fermat point, along with three vertices of one of the equilateral triangles used to construct it, form a cyclic quadrilateral.	-	-	■	■	-	-
Connect islands	The Fermat point	The three segments that connect the Fermat point of a triangle with the vertices of the equilateral triangles used to construct it form angles of 120° with one another.	-	-	■	-	-	-
Connect islands	Rotations	Given two points A and B, the triangle formed by A, B and the point resulting from rotating B 60° with respect to A is equilateral.	-	-	■	-	-	-

Type 3: This last category includes the *Prompts* submitted by the student pairs who propose using GeoGebra as a resource to check a previous mathematical idea. Eight *Prompts* of this type were identified (Table 3), presented by four of the pairs (P1, P2, P7 and P8). Unlike the previous category, the formula to describe the *Prompt* starts from a situation in which the high school students have an idea, and then resort to technology to test it. This approach induces a change in the follow-up actions. An existing mathematical idea has to be transferred to a dynamic construction such that the mathematical property to be verified is represented and emphasized. This type of proposal would provide a starting point to develop mathematical creation.

Table 3: Properties from Prompts related to mathematical creation

Problem	Topic	Summary of the mathematical property	P1	P2	P3	P5	P7	P8

Equal chords	Tangent lines	Given two circles and the lines tangent to them from the centers of the other, the chords formed by the tangency points are equal, as are the lines formed by the points of intersection with the secant lines.	-	■	-	-	-	-
Equal chords	Bisector	The bisector of an angle divides any triangle limited by the rays into two congruent triangles.	-	-	-	-	■	-
45° angle	Invariant	The area of a family of polygons inscribed in a triangle is constant.	■	-	-	-	-	-
45° angle	Diagonal of the square	Given two similar triangles, there is a proportionality ratio between their areas.	-	■	-	-	-	-
45° angle	Invariant	Given a rectangle, draw a 45° angle at one of its vertices so as to make a triangle with the points of intersection on the non-contiguous sides. The diagonal of the rectangle divides the triangle into two surfaces with the same area.	-	-	-	-	■	■
Connect islands	The Fermat point	The Fermat point minimizes the sum of the distances to the vertices for any triangle.	■	-	-	-	■	-
Connect islands	The Fermat point	The three segments that join the Fermat point of a triangle with the vertices of the equilateral triangles used to construct it have the same length.	-	■	-	-	-	-
Connect islands	Significant points	In any triangle, the sum of the distances from the vertices to the circumcenter is the minimum possible. (Idem with barycenter, orthocenter, incenter)	-	-	-	-	-	■

This type of *Prompt* includes minor questions that extend or stray from the solution to the original problem. In general, delving into these questions could be useful to segue or connect to other mathematical results. A relevant exercise to prepare for future contingencies is knowing the various branches that originate from a problem and reflecting on potential connections to mathematics. From the point of view of Mathematical Context, the participants showed an ability to reflect on the mathematics of teaching practices, a necessary skill in the classroom (Heid, Wilson, & Blume, 2015).

Final discussion

The main objective of the reflection task proposed to the students in the "Mathematics for Teaching" course was to place them in the teacher's role after having solved a set of problems using GeoGebra. This activity was proposed as a way to anticipate potential contingencies that they would have to deal with in a classroom where GeoGebra is used to solve problems. The goal was to develop their mathematical understanding for teaching from the perspective of the Mathematical Context of Teaching, a component of which is reflecting on the mathematics of teaching practices (Heid, Wilson, & Blume, 2015).

All the student pairs proposed at least one event involving the use of technology for each of the problems. This shows that the participants, in this initial stage of their training as teachers, developed to a certain degree their ability to reflect on the mathematics of teaching practices (Kilpatrick, et al., 2015). However, there are some differences between the pairs. Just P1 has indicated *Prompts* of all three types; two couples indicated just type 1 and type 2 *Prompts* (P3, and P5); one pair gave types 1 and 3 *Prompts* (P8); the other two students pairs' report just include type 3 *Prompts* (P2, and P7). Since all three types of events are situations that can occur in a math class using Geogebra to solve problems, it would be desirable for future teachers to be able to identify, anticipate, and analyze them. The previous results show that this is not something that always happens, which points to the need for training that offers opportunities to reflect and deepen this type of analysis.

The group of proposals categorized as Type 1 included *Prompts* that have to do with classroom contingencies that could be used to interact with students and discuss mathematical meanings of technological facts (Wasserman, et al., 2019). The didactic steps to answer the questions that are proposed as a starting point of the *Prompts* would be the same in a university course as in a Secondary Education classroom.

As concerns the other two types of *Prompts* in our categorization (Types 2 and 3), we observed that the proposals included here were the most numerous. This could be related to the types of situations that each pair encountered while they were solving the problems themselves, during the first three tasks of the GeoGebra Problem-Solving Workshop, which would underscore the close relationship that exists between the different perspectives of Mathematical Understanding (Kilpatrick, et al., 2015).

The inclusion of technology as a tool for mathematical work in the classroom entails a change in the types of situations that a teacher must face in the classroom. This must be accompanied by a change in the types of training activities offered to future teachers of mathematics. The analysis conducted as part of this research shows how tasks involving reflection on one's own experience contribute to the development of mathematical understanding for teaching the discipline, particularly in Secondary Education.

Acknowledgments

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LATINX PARAEDUCATORS LIVED MATHEMATICAL EXPERIENCES

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Mathematics teaching is an inherently social activity, highly dependent on the lived experiences of its members. Drawing on sociocultural perspectives, we explore the mathematical lives of Latinx paraeducators to better understand the ways in which their backgrounds might influence their attitudes toward mathematics and mathematics teaching. Eleven paraeducators participated in the study that used interviews, artifacts, and informal observations to document their lived experiences, particularly in regard to mathematics. The evidence suggests that aspects of their personal narratives, particularly family, home culture, and personal hobbies and interests, had important influences on many of the participants. Implications on the social aspects of mathematics teaching and the need for relevance are provided.

Keywords: Teacher Education – Preservice; Equity and Diversity; Teacher Beliefs; Culturally Relevant Pedagogy

Throughout this century, the percentage of teachers in the United States classified as white has held steady at about 80%, while the percentage of Latinx teachers remains below 10% (Taie & Goldring, 2019). Further, it is projected that by 2026 white students will make up less than half of the U.S. public school student population, with Latinx students nearing 30% (de Brey & colleagues, 2019). Why should this matter, particularly in regard to the teaching and learning of elementary mathematics?

One reason for considering the implications of these changing demographics is the role that teachers' backgrounds have on students and teaching (Glock & Kleen, 2019). For example, Copur-Gencturk, Cimpian, Lubienski, and Thacker (2020) found that teachers displayed the largest negative biases in regard to mathematical ability toward Black and Latinx girls. While explicit biases are more easily seen and measured, implicit biases are less visible but perhaps more harmful to those they marginalize (Greenwald & Banaji, 1995; Harber & colleagues, 2012). Implicit negative biases towards students, in regard to both academic and behavioral factors, tend to exist in teachers that come from differing backgrounds, cultures, and/or races (Glock & Kleen, 2019; Redding, 2019).

A second reason for considering the implications of the above changing demographics involves the role that dialogic interaction plays in mathematics teaching and learning (National Council of Teachers of Mathematics, 2000; National Governors Association, 2010). Mathematics teachers are participants in a dialogic process so, as in any such activity, draw on their personal backgrounds and resources when engaging students in active mathematics learning. It is important to understand the diverse perspectives that can be generated through lived experiences, and how this might impact the ways in which teachers view and approach mathematics instruction.

Frameworks

We view teaching as a social activity grounded in the perspectives and backgrounds of those involved (Vygotsky, 1978; Bruner, 1990). There are numerous perspectives on teaching that build on this idea and attend specifically to issues of equity, race, and power. This study draws from key precepts of *culturally responsive/sustaining pedagogy* (Ladson-Billings, 1995) and the *funds of knowledge* framework (Moll et al, 2005).

Culturally responsive/sustaining pedagogy is a strength-based perspective that celebrates students' home languages and cultures, but also emphasizes active learning and critical, reasoned challenges to the status quo (Ladson-Billings, 1995). While complex in nature, a central instructional premise of culturally responsive pedagogy is to find topics and contexts that are meaningful to a group of learners, and then create learning experiences that bridge these learners' cultural and linguistic heritages with more formal academic knowledge. Establishing meaningful connections to students is a key aspect of effective mathematics instruction, and teacher backgrounds play a significant role in this process (Boaler & Staples, 2008; Gholson & Martin, 2014; Téllez, Moschkovich, & Civil, 2011).

Funds of knowledge refers to an individual's historically accumulated set of abilities, strategies, or bodies of knowledge (Gonzalez et al. 2005; Vélez-Ibáñez and Greenberg 1992). These funds can be recognized by observing "the wider set of activities requiring specific strategic bodies of essential information that households need to maintain their well-being" (Vélez-Ibáñez & Greenberg, 1992, p. 314). In the context of this discussion, we employ the concept of funds of knowledge to encompass both academic and personal background knowledge, accumulated life experiences, skills used to navigate everyday social contexts, and world view(s) structured by broader historically situated sociocultural forces. Just as educators need to recognize the funds of knowledge that K-12 students bring to the school, teacher educator programs need to recognize, and tap into, the diverse social, linguistic, and cultural strengths and assets that teacher candidates bring to their programs.

Each of these theories is premised on building upon students' and families' linguistic and cultural resources and accumulated knowledge. This can support schools and teachers in sustaining students' linguistic and cultural identities and foster a more humanizing perspective of the learning process, including mathematics learning.

Methods

Context

The 11 participants in this study come from various linguistic, cultural, ethnic, and national backgrounds, including Panama, El Salvador, Guatemala, Cuba, Mexico, and the United States. Over half are fully bilingual English and Spanish. All earned a BA in Elementary Education, K-8 state teaching certification, and a K-12 ELL endorsement. In addition, eight pursued a K-12 bilingual endorsement (see Table 1 for more detail).

The participants completed their elementary education certification program while working 30 hours per week in one partnering district in the Pacific Northwest. Participants were selected due to experience and dedication as paraprofessionals working with English learners. Ten are female, over half are first generation college students, nine are Latinx (age range: 23-53), and three have foreign/domestic postsecondary degrees (e.g., dentistry, Spanish, journalism). Each participant also completed numerous requirements for acceptance into their elementary teacher education program at a Research 1 institution.

This preservice program was offered in two locations and served paraeducators located in seven school districts with large English language learning school populations (15%-95%). However, for this study, we focus on one program working in one school district that serves a diverse student population including Spanish speakers (40%), Russian and Ukranian speakers (33%), and smaller percentages of Vietnamese, Hmong, and Tagalog speakers, among others. The elementary and middle schools where the 11 paraeducators worked are located in urban and semi-urban settings. Most teacher education classes were offered face-to-face at the school district location. One course was conducted via videoconferencing and several courses were provided using a hybrid model (a combination of traditional face-to-face and online learning activities). In addition, through the Prior Learning Assessment process, many of the participants earned additional course credits by demonstrating work that matched specific competencies listed in the state standards for new teachers (Morrison & Lightner, 2017).

Table 1: Personal and professional backgrounds of participants

Participant	Languages Spoken	Ethnicity/Race	Origin	Years as a Paraeducator
Meredith	Spanish/English	Latinx	U.S.	1
Janice	Spanish/English	White	U.S.	2.5
Magdalena	Spanish/English	Latinx	Cuba	4
Inez	Spanish/English	Latinx	Panama	14
Javier	Spanish/English	Latinx	Mexico	2.5
Gabriela	Spanish/English	Latinx	Honduras	1.5
Maribel	Spanish/English	Latinx	U.S.	2
Evelyn	Spanish/English	Latinx	U.S.	1
Magda	Spanish/English	Latinx	U.S.	4
Sabrina	Spanish/English	White	U.S.	1
Irene	Spanish/English	Latinx	U.S.	8

Data collection and analysis

Mathematical autobiographies of all 11 participants represent the primary data set. The written stories of the participants' mathematical lives were part of the elementary mathematics methods course taken towards the end of their teacher education program. The evening course was conducted in an abbreviated five-week session while the participants were working in schools as paraprofessionals during the day. The stated purpose of the assignment was for the participants to "become conscious of your mathematical beliefs and the events that may have contributed to their creation." The four specific components of the mathematical autobiography were: 1) self-perceptions about your mathematical abilities and understandings, 2) feelings and attitudes toward mathematics, 3) important events in your mathematical life (in and out of school, with preference given to out-of-school experiences), and 4) where you are now in regard to both mathematics and mathematics teaching. The participants shared their autobiography in oral presentations throughout the semester.

Open, emic coding (Miles, Huberman, & Saldaña, 2015) was conducted on each of the participants' written stories in order to find salient features as identified by the participants. First-cycle codes provided broad categorizations of the kinds of stories being told. Specific codes included age (childhood/adolescence/adult), setting (school/home/community), and location (U.S., home country, other). Second-cycle codes began to delineate the salient features of the stories and included mathematical beliefs/attitudes, mathematical activities (e.g., budgeting), mathematical topics (e.g., numbers, measurement), role of language, and implications on teaching. This thematic analysis led to conclusions about the participants' feelings and attitudes toward mathematics, the role of mathematics in the participants' lives, and how these aspects of the participants' backgrounds were brought to bear on their perspectives on mathematics teaching and learning.

Results

Participants' lived experiences

The narratives of this group of mostly Latinx paraeducators emerged through oral and written reflections of personal language as they practiced culturally responsive teaching and engaged with their students' families and communities, all while enrolled in the elementary mathematics methods course. Recognition of the sociohistorical and political contexts of their own lives, along with the lives of their students, is an important aspect of enacting culturally responsive teaching (Ladson-Billings, 1995). Through their stories, these paraeducators demonstrate great courage and resilience. In many cases, their professional journeys include adapting to a new country or cultural context, learning a new language, facing immigration threats, and resisting pervasive local and national monolingual/monocultural ideologies. Their stories and reflections reveal the importance of listening to students' stories and building on their experiences and trajectories.

Therefore, we focus on the lived experiences of these emerging teachers from their childhood through their adult and professional life. We provide the above overview as background to the salient mathematical features of their lived experiences that constitute the focus of this particular study. The data below build on the above narrative, but provide a much deeper view of the role of mathematics in the life trajectories and eventual professional work of the participants.

Feelings and attitudes toward mathematics

The students exhibited a variety of feelings towards mathematics, including “love,” “love-hate,” “useful,” “neutral,” and “negative.” Many of these feelings stemmed from experiences as a child. For example, while speaking fondly of her times tending to farm animals and baking with her mother, Maribel also recalled several negative school mathematics experiences that framed her overall self-perception:

> I was usually one of those students that would take longer to find solutions to problems. One reason was because I would process what teachers were telling me in English and try to translate it into Spanish in order to understand it. Another reason is because I like to think about the process when trying to find solutions to a problem . . . I don’t remember ever being asked to find more than one solution to a problem by a teacher, or to find different ways to show my thinking. There was only one way, and if it was not how my teacher taught me it was wrong. I believe that this is when my fear of math, and answering questions in front of the class, started.

Many students also reflected on their adult experiences as helping to frame their views of mathematics. Magdalena described her mathematical experiences while making and selling cakes and *bocaditos* (snacks), and Gabriela spoke of her work in the dental field.

Overall, the participants relayed a variety of attitudes toward and self-perceptions of mathematics in their personal narratives. Many of these attitudes and self-perceptions were connected by the participants to particular events in their lived experiences, both as a child and adult. The next section provides more examples of these lived experiences and their importance in shaping their personal and professional lives.

Mathematics in the participants’ lives

The collection of mathematical autobiographies represents a rich set of examples of how personal, family, and cultural background impact the mathematical development of individuals. Javier related numerous examples of this from both his boyhood and adult life. Reflecting on his family heritage and youth, he stated:

> I come from a family of farmers who valued an education. I am the second of nine children and the second to go to college. My family always wanted us to get an education so that as we grew older, we would be able to know how to negotiate prices and know their value. My father sometimes bought and sold cattle and knew the importance of being able to calculate the weight of animals without weighting them and know their worth. As a result, he always encouraged me to study hard and do well in school so that in real life, I would be able to know what something was worth and others would not take advantage of me. When I was around 11 years old, my mom would often send me to buy a few grocery items and even medicine from the pharmacy. It was then that I started applying my math knowledge to buy

> and pay for items. I also started working selling jelly, popsicles, and ice-cream . . . I remember that Sundays were my best days because I would go to the football (soccer) games to sell popsicles and ice-cream. Most Sundays, I would sell around 300 pesos ($13) worth of popsicles and ice-cream. I usually made around 105 pesos ($4.5) from 8:00 am to 3:00 pm.

Javier related additional stories as an adult that involved woodworking, choosing phone plans, and making predictions. All were detailed and grounded in his personal story. Javier's stories reflect a use of mathematics that was grounded in his parents' desire for him to succeed, and in the life activities that dominated his early and adult life. Javier developed an applied view of mathematics throughout his life that, as an educator, translated into an instructional perspective in which mathematics is best learned through the use of applied contexts.

Maribel also had numerous reflections on the importance of her family, setting, and upbringing:

> I was born in a small farm town. Growing up every morning before school my brother and I would feed our farm animals, and make sure they had water for the day . . . Each animal needed a different amount of water in their tanks so the amount of buckets varied depending on the size of the tank, and the animal. Without knowing, I was figuring out volume, by figuring out how many smaller units would be needed to fill a larger unit. This is something that I remembered and used when I started learning about volume in school. I always thought of them as tanks being filled by little buckets.

Maribel's personal relationship with her mother also played a significant role and had an impact on her understanding of fractions:

> My mom would ask me to put ¼ cup of sugar into a mix, or ½ cup of flour into something before stirring. At the time fractions for me were just lines on a cup. As I continued to help my mother I started noticing how each measurement had a relation with the other. For example, if I filled the cup to ½ and then filled the other ½ I would have 1 cup. Or if I was to fill the cup with water to the ¼ line and then add ¼ more the water would add up to the ½ line. This led me to realize that I needed four ¼'s to fill 1 cup. I was around 2nd grade when I started making these relationships with fractions. When I got to 4th grade and started learning about fractions, the numbers looked familiar.

As in the case of Javier, Maribel's hands-on experiences with mathematics have translated into a desire to use both visual and hands-on representations when working with mathematics learners.

Evelyn described her parents as "from Mexico and have elementary and middle school level education. My father was always the one to work as my mother chose to stay home." Evelyn felt a power in mathematics at an early age:

> When it was time for me to start school, I entered Kindergarten with not being able to speak, write or understand English as only Spanish was spoken at home; however, I was able to count to 100 at the end of Kindergarten, and by first grade I was able to add and subtract.

Unfortunately, mathematics became an increasing challenge for Evelyn. In her view, this was because of a lack of support from both teachers and her own parents. These experiences brought on serious professional doubts:

> The things we were learning were beyond me. So since math was too hard for me, there was no way I could become a teacher.

Over time, Evelyn worked to overcome these doubts and recognized several positive mathematical traits in her current life. Her attitude towards mathematics instruction is promising:

> I am working on being more positive with math and am a little nervous about creating a lesson on it, but with supportive educators by my side, I think I will be okay!

Perspectives on mathematics teaching and learning

Despite the variety of personal stories and self-perceptions of mathematics, there was near consensus in regard to perspectives on teaching mathematics. All but one participant talked about allowing students to develop their own ways of thinking and solving problems and to develop understandings through the use of visual and hands-on representations. A variety of reasons were cited, including their own experiences, university mathematics and methods courses, and experiences working with students as paraeducators.

The above examples of Javier and Maribel provide details on two specific ways in which the participants' life stories had direct impact on both their views of mathematics and instructional perspective. Meredith used artistic representations to support the personal narrative of her lived mathematical experiences. As a young girl, she had difficulty engaging in the mathematics learning experiences that were devoid of context or lacked personal meaning. Her self-portrait during this time is quite telling:

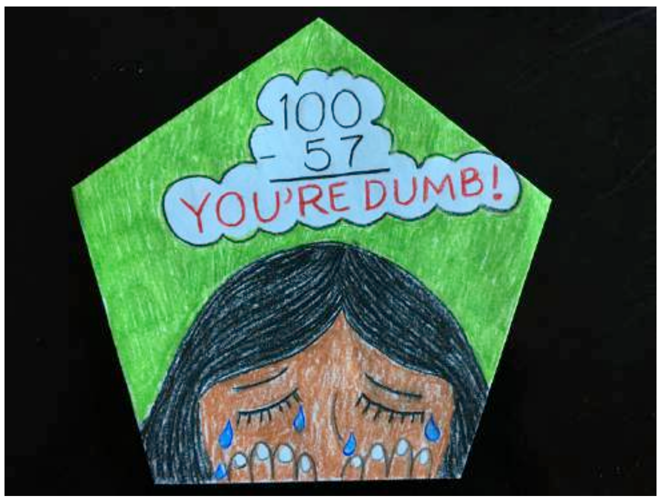

Figure 1: Image of Meredith as a young girl in a mathematics class.

Meredith learned to overcome these challenges, eventually making meaning of complex mathematical ideas in calculus. Her personal goal is to utilize students' home languages and cultures in the core of her mathematics teaching, and to work consistently on making mathematics relevant and personal to her students.

Conclusions

Our study is limited in that it considers only 11 participants who worked as paraeducators and achieved their elementary teacher certification in the same school district. We acknowledge this limitation, but argue that a similar study that would seek to decontextualize either the context or the participants' backgrounds would have other, and perhaps more alarming, methodological issues. The study also draws primarily from one data source when analyzing the participants' attitudes towards mathematics and mathematics instruction, though a secondary and much larger body of data was used to describe the broader socio-cultural backgrounds of the participants. Finally, while we document the participants' views of both mathematics and mathematics teaching, this paper does not address actual classroom practice. Future studies are ongoing to determine the degree to which the participants' views of mathematics instruction are being enacted in their own classrooms.

Our analysis of the stories of these paraprofessionals' lived experiences with mathematics provides a lens into key interactions with their families and other contexts that contributed to their mathematical and professional development as teachers of mathematics. While their views of mathematics differed markedly, nearly all shared the view of teaching mathematics that seeks to generate active learners through the use of contexts, language, and other representations meaningful to students, and that preferably utilize the lived experiences and backgrounds of their students. Each of these principles are consistent with both culturally relevant/sustaining pedagogy as well as funds of knowledge perspectives. Given the changing demographics and anticipated increase in the number of Latinx students over the next ten years, and the importance of caring educators who share students' backgrounds, interests, language, and culture, we hope that more Latinx teachers can enter the teaching profession and develop mathematical perspectives similar to those of the participants in this study.

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MODELING TO UNDERSTAND THE WORLD AROUND US AND OUR PLACE IN IT: IF THE WORLD WERE A VILLAGE

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"The main focus of the day was the "If the World Were a Village" activity/book. I thought it was a really good way to open one's perspective. As an American, I tend to be a bit focused on the US, so to see how much [or how little] of the world is actually represented in my perspective was enlightening."

"Living in the United States … I was surprised that only 5% [of the world population] were from North America"

Increasingly, mathematics is being positioned as a tool to support students' understandings of social (in)justice and their own unique social positioning in the world. To this end, this study analyzes the impact of curricular reform efforts in an elementary mathematics content course. The course focused on fractions and statistics, and the course content was taught through tasks designed to support prospective teachers in understanding and critiquing the world. The authors found that through the course, prospective teachers' content knowledge increased and their knowledge of the world's demographics and social inequities increased.

Keywords: Teacher Education Pre-service, Modelling, Social Justice

Objectives or purposes of the study

The National Council of Mathematics' (NCTM) Catalyzing Change (NCTM, 2018) states that "each and every student should learn the Essential Concepts in order to expand professional opportunities, **understand and critique the world**, and experience the joy, wonder, and beauty of mathematics." (emphasis added, p. 2). If we want students to "function as numerate, critical citizens who are able to use their knowledge in social and political realms, for the betterment of both themselves and society as a whole" (Ernest, 2000, p. 46), we need to teach these students how to understand and critique the world. Some school districts in the United States (for example Seattle Public Schools) are actively working on understanding and critiquing the world in their mathematics classes (Gewertz, 2020).

However, teaching mathematics in a way that enables students to understand and critique the world is challenging. Part of the challenge is due to the tension between focusing on the classical/dominant mathematics goals and on understanding and critiquing the world (Brantlinger, 2013; Gutstein, 2006; Yeh & Otis, 2019). Integrating real-world phenomena into the classroom requires questioning the status quo so as to not reinforce stereotypes (Esmonde, 2014) which may take time away from focusing on the dominant mathematics. This, in turn, may negatively impact student achievement on standardized assessments (Brantlinger, 2013; Chubbuck & Zembylas, 2008).

To address this tension, we selected real-world contexts that would allow both, learning mathematics as well as learning about the world we live in. We situated the mathematics learning in the context of shrinking the population of the world to either 100 people (Smith, 2011) or shrinking the US population down to the size of the class. Our research questions were:

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1. What mathematics do students learn when they engage in mathematics tasks based on shrinking populations to 100 or to class size?
2. What do students learn about the world when they engage in mathematics tasks based on shrinking populations to 100 or to class size?
3. How does working on such tasks affect the students' views of mathematics?

Perspective(s) or theoretical framework

In a teaching mathematics for social justice framework (TMfSJ), a primary goal is for students to become critical members of society who know how to use mathematics to make sense of and possibly change their world (Gutstein, 2003; Raygoza, 2016). TMfSJ tasks involve engaging students in thinking about both the mathematics concepts and social issues relevant to their lives and experiences (Frankenstein, 1983; Gutstein, 2006; Skovsmose, 1994). Therefore, another goal of incorporating social issues into mathematics is for students to both understand mathematics as well as understand and create a more just world (Frankenstein, 2009). To understand and critique the world students often begin by learning about how their own experiences relate to others' experiences in the world in order to understand privilege and injustice. Many prospective teachers (PTs) enter their coursework believing that mathematics is neutral or universal (Greer et al., 2007; Keitel & Vithal, 2008). Mathematics teacher educators need to address the fact that mathematics can never be neutral and no classroom is a neutral space (Frankenstein, 1983; Gutiérrez, 2013; Yeh & Otis, 2019). TMfSJ tasks can be a means of engaging teachers in building their sociopolitical consciousness about the political implications of mathematics, and how math can be leveraged to read and write the world (Gutstein & Peterson, 2005). Gutstein (2003), building on Paulo Freire's (1970) work, distinguishes between reading the world (supporting students in learning about inequities in the world and their own positioning within those inequities) and writing the world (supporting students in developing their own agency to address inequities).

Yet, most PTs have little experience with TMfSJ tasks during their K-12 schooling; therefore, it is important to integrate such tasks into their teacher education courses so they can explore how they may be able to enact TMfSJ tasks in their future K-12 classrooms. This is especially true for content courses so elementary PTs can experience such tasks from a learner's perspective and thus learn to read and write the world (Gutstein. 2006).

In some cases mathematics teacher educators (MTEs) have met resistance from PTs when integrating social justice issues into the mathematics curriculum (Aguirre, 2009; Ensign, 2005; Felton-Koestler et al., 2017; Rodríguez & Kitchen, 2004). However, MTEs have also found that they are able to broaden PTs' perspectives about mathematics and mathematics teaching to include the idea that mathematics could be a tool for social analysis that supports students in understanding the sociopolitical world better (Bartell, 2013; Ensign, 2005; Felton & Koestler, 2015; Felton-Koestler & Koestler, 2017; Leonard & Moore, 2014; Mistele & Spielman, 2009). This aligns with Gutstein's (2003) goal of supporting students in developing their sociopolitical consciousness, and possibly a stronger sense of agency and identity.

Finally, TMfSJ tasks can be designed towards teaching math about, with, and for social justice (Stinson & Wager, 2012). Stinson and Wager define teaching mathematics about social justice as focusing on reading the world. They define teaching mathematics with social justice as enacting equitable pedagogical practices. Finally, they define teaching mathematics for social justice as focusing on both reading and writing the world. Students benefit from all three forms of teaching mathematics about/with/for social justice. Benefits include a view of mathematics as useful and relevant and can potentially develop agency (Gutstein, 2006).

Methods or modes of inquiry:

The study took place in two mathematics content courses for prospective elementary school teachers in the United States, one in 2018 and one in 2019. Content courses are typically taken as prerequisites before students enter their teacher education program. The mathematical content of these courses focused on fractions and statistics.

The 2018 course had 26 PTs and the 2019 course had 7 PTs. The task described below was piloted in the 2018 course and spanned one homework assignment and one class day (2 hours). The task was then refined and implemented in the 2019 course across over two days of instruction (2 hours each). The first part of the task comprised (1) examining www.worldodometers.info and noting how many people are on earth and how some resources are distributed. (2) Shrinking the world population down to 100 people using the book **If the World Were a Village** (Smith, 2011), the movie https://www.youtube.com/watch?v=QrcOdLYBIw0, as well as the website https://www.100people.org/statistics_detailed_statistics.php?section=statistics to make observations about the distribution of the population and resources. (3) Creating of a poster focusing on one or two elements discussed in the book/movie. Students were given private think time, discussion time, and time to make the poster which was to include the following representations: a table, a hundred chart, a number line, and unifix cubes. Students were asked to connect these various representations to support their understanding of the relationships between fractions, decimals and percent. Sample posters can be seen in Figure 1. After the posters were displayed time was spent to discuss the mathematics (connecting decimals, fractions, and percent) as well as what we learned about the context.

In 2018 parts (1) and (2) were assigned as a homework assignment before the first class session, in 2019 they were done in class. In 2019 PTs were asked to estimate percentages for the village such as population, language, age, etc. For example, they were asked “If the world were a village of 100 people, how many would come from North America?” and “would speak English?” In both years part (3) was done in class and in both years, students responded to online survey questions after those three parts:

Online Survey Questions:

- What observations did you make when looking over this website http://www.worldometers.info/?
- What observations did you make when you watched the YouTube video "If the world were a village"?
- What observations did you make when you read the book "if the world were a village"?
- [2018] Please reflect on this homework assignment: What did you learn?
- [2019] What did you learn from today's lesson?
- [2019] What did you learn from today's lesson with respect to mathematics?
- [2019] What did you learn from today's lesson with respect to your understanding of the world?
- How can breaking the population down into 100 people help us better understand the information?
- What math do you think could be addressed with this book in a K-5 classroom?

At the end of the term PTs in each class were asked to respond to the prompt “I used to think math is … now I think math is …”

In 2019 a second day focused on PTs researching information about their hometown and creating a second poster for their hometown and then comparing across the two posters. Data collected included PTs responses to online surveys as well as detailed field notes of all class sessions and copies of all student work. Data analysis began by filling in the field notes with artefacts from class, and reading through survey responses to establish initial themes. Two researchers independently read survey responses and identified themes. The researchers then met to discuss and refine themes in order to create a final codebook of themes, and all data was coded with this codebook. Any disagreements were resolved through discussion. After all survey data was coded, similar themes were collapsed to make larger themes. Themes were collected across all questions rather than analyzing individual questions. These can be seen in Table 1.

Results

With regard to Research Question 1: What mathematics do students learn when they engage in mathematics tasks based on shrinking populations to 100 or to class size?

One major mathematical focus of the task was to make sense of the meaning of percent and connect percent, fractions and decimals. Several PTs commented that before this day they did not realize that percent refers to "per one hundred". Ellie, for example, stated, "I thought it [percent] was just a word," and Amanda wrote in her notes for the day, "Eva explained that percent actually means per hundred, which I never knew." Candy stated that she learned "that if the world was broken down into 100 people, each person would represent a percent. This allows us to conceptualize the population while understanding the real percentages (which can be translated into the actual population)." In addition, PTs connected decimals to fractions with base ten denominators. This is evidenced by Christa's response that "I learned how fractions, decimals and percentages relate to each other and that they all show representations of part of a whole." One way PTs made sense of percent and made connections was by using color purposefully. As shown in the posters below (Figure 1), PTs used colors to connect across representations. Sabine pointed out how the colors helped her: "It was very helpful for me to see with color and the number 100." Jamie reflected on her homework "OMG. As I was completing the homework I realized that a flat partitioned into 100-sized pieces is the same as a hundred chart!!!!! This was a very helpful connection for me."

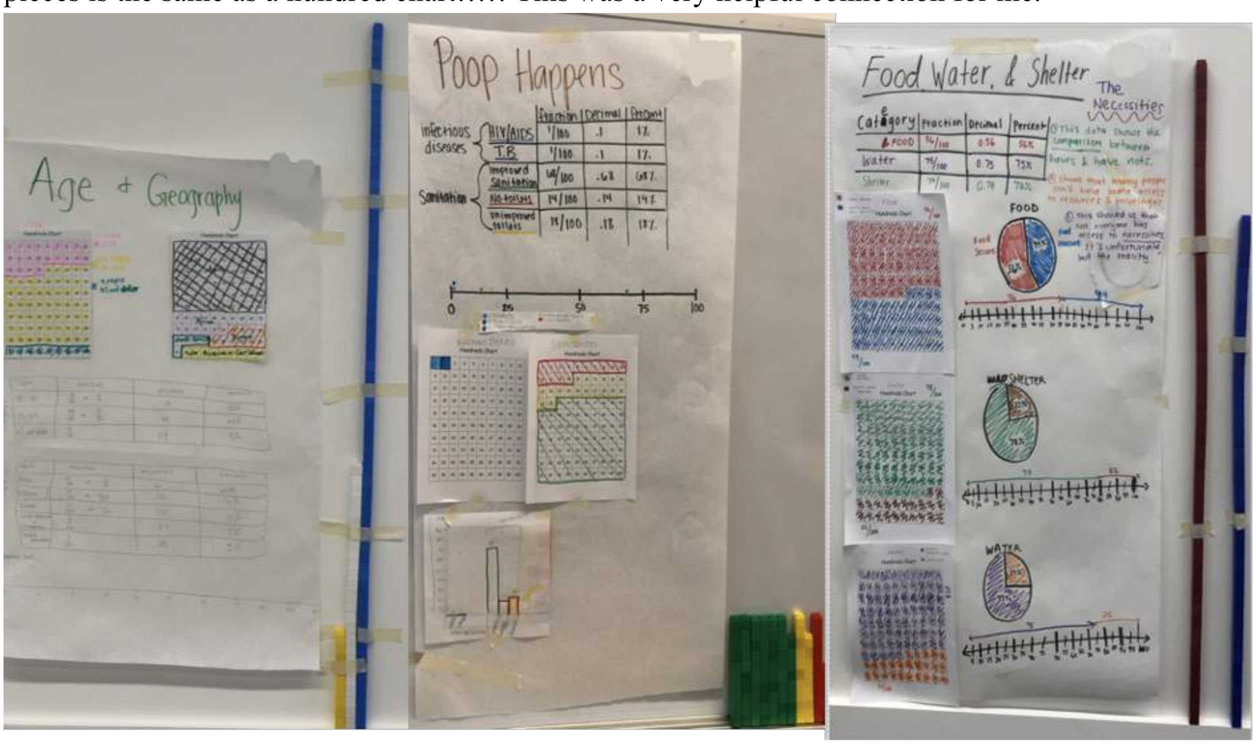

Figure 1: Three posters illustrating connections across representations

In addition to developing meaning for percent, fractions and decimals, scaling the size of the world's population helped the students make sense of the size of our world and the distribution of the world's population. For example, 76% of PTs reported that scaling down the population to 100 helped then understand the world better (see Table 1 for all survey themes). One way this supported their understanding of the world was the ease of visualizing a certain number of people out of 100 rather than billions of people. For example, Walter conveyed that "when numbers are so large the impact is usually lost. The number can be overwhelming and doesn't necessarily relate to us personally. Yet when we break it down to 100 we have a better visualization of what the population would look like. We can visualize 60 people out of 100 or 10 people out of 100. Compare that to billions of people... That is much more difficult to grasp." In this way, PTs were able to make sense of the world by considering a total population of 100 people, which provided a relatable scale.

With regard to Research Question 2: What do students learn about the world when they engage in mathematics tasks based on shrinking populations to 100 or to class size?

Another major focus of the task was to create a space for students to realize that the world is much more complex than they had originally thought, and that the US is not the center (or most of) the world. Seventy percent of the students stated this in the survey. For example, Aimee expressed feeling surprised "to see that a majority of the population came from Asia and not the United States," and Sophie also voiced, "I was surprised to learn about the abundance of other cultures. I think in America the media tends to be very self-centered and produce that as the norm, when in reality looking at the world we are a small fraction of diversity in the world." Additionally, Amanda reflected in her notes that "some of the results were surprising such as that 60% of the population is from Asia and only 5% is from North America."

Table 1: Themes, larger themes bolded, sub-themes not bolded

Theme	2018 N=26	2019 N=7	2018 and 2019 N=33
The world population is large and rising (at a scary rate)	18 (69%)	7 (100%)	25 (76%)
The use of smaller numbers (100 people in the village) was helpful to make the quantities easier to visualize/data easier to visualize	21 (81%)	4 (57%)	25 (76%)
Inequity is larger than I thought	**22 (84%)**	**6 (85%)**	**31 (85%)**
Inequity/inequality is larger than I thought	16 (62%)	3 (43%)	19 (58%)
Food insecurity is much higher than I thought	14 (54%)	5 (71%)	19 (58%)
Education levels are lower than I thought	8 (31%)	3 (43%)	11 (33%)
The world is more complicated/goes beyond the US	**17 (65%)**	**6 (85%)**	**13 (70%)**
World is complicated/diverse	13 (50%)	3 (43%)	16 (48%)
US is not the center of the world	5 (19%)	1 (14%)	6 (18%)
I am a small piece in the world/surprised by the small % of people from NA	2 (8%)	3 (43%)	5 (15%)
I am a small piece in the world/There are so many people in the world	1 (4%)	0 (0%)	1 (3%)
I am/was America centric/ gained perspective	2 (8%)	4 (57%)	6 (18%)
This activity was SHOCKING/jarring/terrifying	**6 (23%)**	**1 (14%)**	**7 (21%)**
Need for action/improving the world	**10 (38%)**	**3 (43%)**	**13 (40%)**
This activity made me realize the need for action	6 (23%)	3 (43%)	9 (27%)
found it helpful to quantify social issues/we can	6 (23%)	0 (0%)	6 (18%)

use number to improve the world			
This activity would be good for elementary students	**8 (31%)**	**1 (14%)**	**9 (27%)**

Thus, scaling the size of the world also helped PTs recognize their inaccurate perceptions of the distribution of people and languages in the world. In 2019 when PTs were asked to make predictions all 7 PTs overestimated how many people would come from North America. While the actual value is around 5, PTs' estimates ranged from 7 to 50 with the mean response being 18. All 7 PTs also overestimated the number of people who speak English. Their estimates ranged from 20 to 80 with a mean of 42 and a median of 40. The actual value is 5 (first language) and 20 (first and second language). Thus, all PTs seemed to overestimate the population of North America as well as the number of English speakers. This activity helped PTs adjust their understanding; Candy, for example stated in her class notes that "learning the proper answer resulted in new understanding of the world. This activity helped me realize that the US isn't the center of the world. Life is very different for people in other countries."

This task helped PTs to grasp the significance of our worlds' population and the current distribution of resources in our world. One of the major themes PTs mentioned across both years was that they learned that the world population is large and rising at an alarming rate (all survey themes are shown in Table 1). Seventy-six percent of all PTs mentioned this theme. Elsie for example stated "There are a lot more people living than I thought," and Sophie stated "The population to me was shocking. When I think about this small realm of people I interact with every day, it blows my mind to picture the world population, let alone this constant rapid increase."

Additionally, almost all PTs (84%) stated that they learned that the inequities are larger than they thought either in general (58%) or by mentioning a specific area such as food insecurity (58%) or education (55%). To give an example, Autumn mentioned general inequities: "I observed a lot of inequalities amongst all the topics explored." Additionally, Christa noticed that "food insecurity was a way larger number than I thought. It is almost half the world's population which is startling to me," and Sabine mentioned both food and education: "The number of people that always have food to eat is very low. I didn't expect it to be that low, it really surprised me. Also, that there are students that don't get an education. It is very sad that they don't get that opportunity."

For some PTs, their new understanding motivated a sense of urgency for action. Even though taking action was not yet an explicit goal of the course, 40% of the PTs mentioned the need to take action or need for improving the world. For example, Marcel stated "It is … important to realize that as Americans, everything is not as we assume it is. We just expect to turn on the faucet and have clean water poured out of it. We pay for tuition and we expect qualified professors. As Americans we must recognize our privilege and use it to help others as much as we can." Gertie said "I learned that there are a lot of things in the world that need our help and attention, and there is a lot of miss-distribution of wealth, food, and resources, and we need to be doing a better job of creating a more sustainable and equitable world.

With regard to Research Question 3: How does working on such tasks affect the students' views of mathematics?

At the end of the term PTs in each class were asked to respond to the prompt "I used to think math is … now I think math is …." Four themes emerged: (a) a shift from math as politically neutral to math as a place for the integration of social issues/social justice [8 PTs in 2018, 2 PTs in 2019], (b) a shift from rom math as boring/uninteresting, to math as interesting/useful [8 PTs in 2018, 4 PTs in 2019], (c) a shift form math as rules and procedures to math as sense-making [12 PTs in 2018, 3 PTs in 2019], and (d) shift from difficult to possible [7 PTs in 2018, 0 PTs in 2019]. Some PTs mentioned more than one theme [9 in 2018, 1 PT in 2019]. Ellie illustrated theme (a) in her response "I used to think math is the most politically neutral subject taught in school. Now I think math is a great place to bring up the issues that are going on in our world while developing a better understanding of the material because of the connections that students can make to their own lives."

Discussion and/or conclusions

Reflecting on the data analyzed for this study with respect to content it seems like PTs learned about the meaning of percent as well as the connection between fractions, decimals, and percent.

Reflecting on the data analyzed for this study with respect to the world, it seems the PTs learned about social justice (reading the world) but not quite yet for social justice (writing the world). PTs seemed to have a better understanding about various characteristics of the world as a whole, which is important for future elementary school teachers because these future teachers need to understand characteristics of the world themselves before they can enact TMfSJ tasks in their own future teaching. Additionally, being aware of their own unique social positions may support more reflective practice with elementary students who differ from them in social position.

The PTs also learned how to use math to make sense of the world. Moving forward, the authors intend to modify the tasks to move more towards learning for social justice. In 2019 the PTs compared what they learned about the world to what they learned about their own local city which allowed them to (a) learn about their local place and compare it to the world, and (b) refine their understanding of how they are situated in the world. Christa, for example, stated "My topic was shelter and that was an underestimate of one in 20 people are homeless [in local city]. … we obviously have issues in [local city]. But then comparing it to the real world numbers. It's still a very privileged place to live. So I think that's important to recognize." The act of comparing their new understandings of the world to the makeup of their local city appeared to make inequity more pressing for PTs. In the future, the authors will continue to develop ways to connect TMfSJ tasks to PTs local reality, e.g., local issues of homelessness, gentrification, in an effort to prompt PTs toward taking action and learning mathematics for social justice.

Reflecting on the data analyzed for this study with respect to the students' views of mathematics we reported on students' views at the end of the term, thus this tasks may only have played part of the reason for shifts, however, all students mentioned shifts in how they viewed math as either not neutral anymore, or more interesting, engaging, focused on sense making, and doable.

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INVESTIGATING ELEMENTARY PRE-SERVICE TEACHERS' CONCEPTIONS OF MATHEMATICAL CREATIVITY

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Research in mathematics education has overlooked creativity in mathematics, partially because of a lack of an accepted definition of mathematical creativity. The present study investigates elementary pre-service teachers' (PSTs') conceptions of creativity in teaching and learning mathematics. Data were collected using observations and semi-structured interviews with nine PSTs and analyzed using thematic analysis. PSTs' conceptions of mathematical creativity included using multiple approaches to solve problems, designing mathematical tasks from scratch, making learning challenging but not impossible, and exercising independence in learning. Implications of these results are applicable to teacher preparation programs, and they suggest a need for more research on the nature of experience(s) that shapes PSTs' conceptions of mathematical creativity and how to develop it.

Keywords: Instructional activities and practices, Affect, Emotion, Beliefs, and Attitudes.

Mathematics educators have argued that "mathematics is a creative, everyday human activity that cannot be built exclusively on rules and routines" (Schram, 1988, p. 8). Yet, according to Silver (1997), cited in Lithner (2008), "students hardly experience mathematics as the highly creative domain it is" (p. 7). If we intend to support students to discover and grow their mathematical talent, a change in broader classroom practices and curriculum materials is necessary, and in order to yield results from this change, creativity in mathematics should be part of educational experience (Mann, 2006). This educational experience should not be limited to students but should be made accessible to teachers who play a critical role in shaping the educational experience of students. Creative teachers are crucial to the development of mathematical creativity in each student through school mathematics education (Lev-Zamir & Leikin, 2011).

For us to nurture teaching with and for creativity by teachers, we must understand their conceptions of creativity (Lev-Zamir & Leikin, 2011). We address this need in the present study by investigating elementary pre-service teachers' (PSTs') conceptions of mathematical creativity. The purpose of the study was to understand how PSTs conceive creativity in the teaching and learning of mathematics. In the following sections, we review related literature and discuss a theoretical perspective that guided the study. We then describe the research methods, results and implications of the findings.

Literature Review

Research in mathematics education has overlooked creativity in mathematics (Haavold, 2018; Haylock, 1987; Leikin, 2009, 2011), partially because of a lack of an accepted definition of mathematical creativity (Mann, 2006). Among the researchers who have studied teachers' conception of creativity specific to mathematics, Bolden, Harries, and Newton's (2010) characterized elementary PSTs as holding a narrow conception of creativity in mathematics. The PSTs' conceptions were largely associated with the use of resources and technology and was bound up with the idea of teaching creatively instead of teaching for creativity. The National Advisory Committee on Creative and Cultural Education (NACCCE) (1999) defined teaching creatively as "teachers using imaginative approaches to make learning more interesting, exciting and effective" (p. 102), and they defined teaching for creativity as teaching that is aimed at developing students' creative thinking and behavior. The former emphasizes creativity in terms of teacher actions, while the latter emphasizes creativity in terms of student reasoning. Bolden et al. (2010) cited literature which indicated that teachers of younger children believe that mathematics is creative but a closer investigation into their beliefs showed that creativity was viewed less in terms of the mathematics itself and more in terms of the creative activities such as construction, art, songs and rhyme, which are availed by mathematics sessions. Complicating the matter, Beghetto's (2007) study with middle and secondary PSTs identified that these teachers can view unique student responses as a potential distraction to classroom teaching, while PSTs from other subjects viewed such responses as worth pursuing. The potential effect of dismissing unique responses by students was that it can hinder the development of creative thinking, even if the teacher is using techniques associated with teaching creatively.

With respect to research investigating PSTs and their awareness of mathematical creativity, Shriki (2010) described the experiences of PSTs in a methods course focused on middle-school geometry where they engaged in activities aimed at cultivating their awareness of mathematical creativity and the complexity of the nature of creativity. She examined creativity by focusing on the value of the process or the product with citations from other researchers. As a process, creativity refers to cognitive abilities, conceptual thinking that involve flexibility, fluency, and originality, and non-algorithmic thinking. As a product, creativity is defined in terms of the novelty or uniqueness of a solution to a problem. Shriki argued that the learning environment and the nature of the assignments were relevant in aiding and growing PSTs' awareness of mathematical creativity and its multifaceted nature. She specifically illustrated that the learning environment provided PSTs with the freedom to work and design their own problems without having to follow certain rules or algorithms, and without fear of having a right or wrong answer. This, in turn, led to PSTs' development of intrinsic motivation, interest and curiosity. PSTs were also encouraged to be reflective about their insights and determine possibilities of generalizing the ideas, which in the end enhanced their mathematical knowledge.

Theoretical Perspectives

Researchers have approached creativity from different perspectives, and there is no general accepted definition of creativity (Haylock, 1987; Mann 2006; Sriraman; 2005). However, most researchers (Haylock, 1987; Lev-Zamir & Leikin, 2011; Leikin, Subotnik, Pitta-Pantazi, Singer & Pelczer, 2013) have adopted Guilford's (1967) characterization of the nature of creative thinking. The common features include fluency, flexibility, and elaboration, all of which fall within the divergent production ability of creative thinking. Fluency pertains to "a matter of retrieval of information from one's memory store," flexibility is "a matter of transformations of information," and elaboration is "a matter of producing implications" (Guilford, 1967, p. 11). Originality is another component of general creativity which Lev-Zamir and Leikin, (2011) defined as characterized by a unique way of thinking and unique products of a mental or artistic activity" (p. 19). These characteristics are mutually related, but they are not required to be present at the same time in order to claim the occurrence of creativity (Lev-Zamir & Leikin, 2011).

At a finer level, some differences exist in researchers' approaches to creativity. Piirto (1999), cited in Lev-Zamir and Leikin, (2011), distinguished between general and specific creativity. He identified general creativity with the application of problem-solving skills used in one field to solve problems in another field, and he connected specific creativity to the logical deductive nature of a particular field. Our study focused on mathematical creativity, which is a specific type of creativity that focuses on mathematics.

To label a behavior as mathematically creative, Haylock (1987) argued that both mathematics and creativity must be clearly present. This implies that for any process or product to be labeled as mathematically creative, it should be valid to the mathematics that was involved in that specific context. Lev-Zamir and Leikin's (2011) later added that defining mathematical creativity in the context of teaching should allude to mathematics, teaching, learning and creativity. An understanding of mathematical creativity was therefore important in this study.

For the present study, we adopt a model for creativity designed by Lev-Zamir and Leikin (2011) to characterize teachers' conceptions of creativity in teaching mathematics. In the model, teacher conceptions are explained in terms of teachers' mathematical content conceptions, which are how teachers view creative mathematical content, and teachers' pedagogical conceptions, which are "their awareness of didactic and psychological aspects of creativity in teaching and learning mathematics" (p. 19). Of the four characteristics of general creativity mentioned previously, their model focuses on flexibility, originality and elaboration because these three are unique to creative teaching. With respect to fluency, the authors consider it a primary indicator of how a teacher is qualified in terms of knowledge and proficiency, rendering it trivial in this model.

In Lev-Zamir and Leikin's model, teacher conceptions of creativity in teaching mathematics are further subcategorized as teacher-directed and student-directed under each of the three components: flexibility, originality and elaboration. Teacher-directed conceptions of creativity are actions by teachers that make them creative and these can be of a mathematical or pedagogical kind. Student-directed conceptions of creativity entail "connecting creativity in teaching mathematics with opportunities provided for the development of students' creativity" (p. 28).

Methods

Context and Participants

The study took place at a university in the southern United States. We recruited nine female pre-service elementary teachers from the early childhood education program, and their participation was voluntary. At the time of study, they were taking a mathematics methods course from either of the two sections taught by two different professors. One of the authors acted as a teacher assistant in both sections. The course was accompanied by a field experience component and it was the first of two courses that students take in the program.

We chose to focus on PSTs at the elementary level because this a critical stage of a child's mathematical development and how teachers are prepared to support them in this development is important. Mathematical concepts are interconnected and having a strong foundation for basic concepts in mathematics is likely to enhance understanding and creativity in learning as students progress to higher levels. We chose this specific course because of the nature of questions that PSTs explored throughout the course. PSTs were expected to reflect more on what mathematics is and what it means to know and do mathematics. These reflections can influence PSTs' beliefs about mathematics, which they are likely to carry on into their teaching together with the experiences they get from the methods course, ultimately impacting how they teach by shaping the approach and attitude of students in mathematics.

The nine participants came from different backgrounds in terms of race which in turn implies different cultures. Three out of the nine participants were PSTs of color and six were white. Of the three PSTs of color, two were born outside the US and moved into the US in their early age. The other was born and raised in the US. It was important to mention this variation in participants as their experiences are likely to inform their conception of mathematical creativity. Moreover, Leikin et al. (2013) indicated that some variables concerned with mathematical creativity depend on culture while other variables are intercultural.

Data Collection

We conducted two observations, one in each of the course sections for a duration of approximately 1.25 hours each. "A major purpose of observation is to see firsthand what is going on rather than simply assume we know" (Patton, 2015, p. 331). Having been familiar with the site, this was a statement that guided our observation in order to avoid uninformed conclusions. Given one of the authors' role in the classroom as a teacher assistant, it was her intention to avoid as much interaction as possible in order to capture majority of the events of the lessons and we therefore assumed the role of an onlooker (Patton, 2015) for the most part of the observations.

We also conducted one individual semi-structured interview (Roulston, 2010) with the nine participants. The semi-structured nature of the interview allowed us to deviate from the order of interview questions, because our interviewees' responses informed the choice and order of questions. For example, we did not have to ask all questions that we had in our protocol when an interviewee responded by also answering a follow-up question(s). We also used probing questions to follow-up on our interviewee's responses and yield more detail and explanations about what our interviewees had said (Roulston, 2010). While probing, in most cases, we used our interviewees' own words to formulate questions. The interview focused on PSTs conceptions of what it means to teach mathematics creatively followed by their conceptions of what it means to learn mathematics creatively.

Data Analysis

To analyze data, we used thematic analysis (Braun & Clarke, 2006) which is a "method for identifying, analyzing, and reporting patterns (themes) within data" (p. 79) or a process that involves looking for patterns within the data and categorizing those patterns according to themes (Fereday & Muir-Cochrane, 2006) cited in Bowen (2009). Thematic analysis uses coding as a strategy (deMarrais & Lapan, 2003). The themes were mainly from the aforementioned framework of Lev-Zamir and Leikin (2011). We analyzed our participants' responses to interview questions and occurrences from the classroom observations in light of flexibility, originality and elaboration. From these categories, we further grouped our findings into the subgroups of teacher-directed and student-directed conceptions of mathematical creativity. Data in the flexibility group included statements about different types of transformation of information in teaching and learning mathematics and varied solution paths to problems that could result from the teacher and/or the student. Data in the originality group entailed PSTs' statements about novel ways of thinking while teaching and learning. Novelty in this case referred to uniqueness from the usual accepted norms and conventions of problem solving in the process of teaching and learning. Data in the elaboration group constituted PSTs' statements about advancing thinking to higher and related levels.

Results

The data presented in this report are from three (Nelly, Paula, and Laura – pseudonyms) out of the nine PSTs. We chose to focus on these three participants here because they provided concise but representative data of the nine participants. We categorized the findings in terms of teacher-directed conceptions of creativity and student-directed conceptions of creativity as explained in the theoretical perspective.

Teacher-Directed Conceptions of Mathematical Creativity

This form of creativity included PSTs' views and actions, both mathematical and pedagogical, that enhance their teaching of mathematics creatively. The different views of PSTs' teacher-directed conceptions of creativity included using multiple approaches to solve problems, designing mathematical tasks from scratch, and making learning challenging but not impossible. Table 1 represents a summary of these conceptions, their description, and their perceived enactment, or the actions that teachers envision to ordain the conceptions.

Table 1: PSTs' Conceptions of Mathematical Creativity in Teaching

Conception	Description	Perceived Enactment
Teacher-Directed Conceptions of Mathematical Creativity		
Use multiple approaches to solve problems	Finding the solution to a mathematical problem using varied solution paths.	Provide support for students, both pedagogical and mathematical to help them think of and use various perspectives when solving problems. This support can be in terms of using supporting and extending moves, and purposeful questioning.
Design mathematical tasks from scratch	Exercise teachers' independence and creativity in determining content and context that will be accessible to students.	Devise contextual problems, use manipulatives, hands-on activities, and games that are appropriate to the goals of the lesson to engage students in thinking about mathematical concepts creatively. Integrate other subjects e.g. English and Science in mathematics lessons.
Make learning challenging but not impossible	Providing challenging problems and situations that will build students' intellectual curiosity and challenge them to think deeply about the problem and its solution.	Teach concepts to enhance sense making by students by not dwelling on algorithms. Extend students' thinking through questioning. Use purposeful questioning to elicit ideas that will help students think for themselves with less input from the teacher. Encourage productive struggle.
Student-Directed Conceptions of Mathematical Creativity		
Exercise independence in learning	Students' ability to develop their own perspective into learning and reasoning independently.	Students solving problems in their own way without being directed on how to do everything. Students putting new perspectives in problem solving.

Use Multiple Approaches to Solve Problems. This conception involved a teacher believing mathematical creativity as teachers' ability to provide support for students, both pedagogical and mathematical, to help them consider multiple perspectives when solving problems. Example quotes from the participants are:

Paula: I think it is super important that kids have different tools coz I don't think people my age and adults were really given anything other than the standard algorithm to solve a problem.

Laura: I think the biggest thing is learning all these different kinds of strategies and knowing that you should encourage it for kids because I feel like usually teachers just want you to stick to a specific strategy.

Both Paula and Laura identified the relationship between a variety of strategies, knowing and flexibility. They emphasized the need to incorporate different approaches to solving problems and making them accessible to students by not restricting them to a specific approach. More generally, their idea fits under the flexibility component of mathematical creativity because shifting perspectives in problem-solving can be considered as a form of transformation of information, say for instance, multiple representations (e.g. visual, symbolic) of a solution.

Design Mathematical Tasks from Scratch. At the time of study, PSTs were working with elementary students once a week during which they designed activities and problems to be worked on by their students, an exercise that they deemed as involving mathematical creativity. Example quotes from the participants are:

Nelly: Making up problems and, like the literature assignment, I felt like that was really creative… We pretend to be elementary schoolers a lot. And so we had to put ourselves in their mindset…. But including games that kids have to problem solve, like problem solving, I think is more, like introduces more creativity into math."

> Laura: So I think we should make math more creative for them (students). So they can still be learning something while, doing a fun activity. Maybe being able to work in groups or making games out of the math,…

Nelly and Laura viewed mathematical creativity as generating tasks that encourage activity while learning. Specifically, they mentioned the need to incorporate games in activities. Creating such activities required them to position themselves like students in order to ensure accessibility of the materials that they would generate.

During classroom observations, PSTs were challenged to make in the moment decisions on how they would support their students' thinking in their placement. Their professors brought up hypothetical scenarios, for example, on student misconceptions and asked PSTs to think of how they would support and/or extend their students' thinking. During our interactions with the PSTs, they also identified the need to integrate science and English in their lesson, what Nelly explained as talk about math not in a math class as being mathematically creative. Generally, we categorized this conception under originality trait of mathematical creativity because the process included considering student's level of understanding and designing tasks that would be accessible to them in terms of context and content, hence requiring specificity and novelty in thinking about the nature of the activities.

Make Learning Challenging but not Impossible. PSTs explained that it is important to put less pressure on students but at the same time maintain their interest and engagement in learning. The following example quotes supported this finding:

> Nelly: creative learning should feel more fun and more challenging but not impossible… teachers should let them (students) figure out things for themselves, instead of just telling."
>
> Laura: But I can see like, why it's better to be more creative because you can put in a whole lot of different perspectives into it. And it doesn't have to be so straightforward.

Nelly emphasized the need for teachers to allow students to make sense of mathematical ideas on their own and Laura also supported this idea by emphasizing the idea of not being straight forward and having students develop their own perspectives into learning. We categorized this conception under the originality component because we portrayed making sense of mathematical concept with less scaffolding as requiring a higher personal cognitive input.

Student-Directed Conceptions of Creativity

PSTs' student-directed conceptions of creativity were closely aligned with their teacher-directed conceptions of creativity. They included students generating their own solutions to mathematical tasks, which we termed as exercising independence in learning.

Exercising Independence in Learning. PSTs viewed students' ability to generate their own and varied solutions and explain their thinking as an indication of mathematical creativity. Example quotes include:

> Nelly: students should be able to do it (math) their own way and not being told exactly how to do everything.
>
> Paula: I would say like being able to come up with like, explain it back to me verbally,…

Both Nelly and Paula address the need for students to own their learning with less input from the teacher and by describing their reasoning. They are allowed the freedom to bring their own perspective into learning and make sense in a manner that best suits their way of making sense. We situate this conception under originality because doing it in your own way and explaining it verbally foster uniqueness in individual thought processes.

Discussion and Conclusion

Findings of the present study indicate that PSTs conceive teaching mathematics creatively as supporting students to use various strategies to solve problems when teaching, developing appropriate mathematical tasks to enhance student activity and engagement while learning, and providing challenging but accessible learning opportunities for students. PSTs conceived learning mathematics creatively as students being independent in learning. These conceptions, teacher-directed and student-directed are intertwined in that, perceived enactments of teacher-directed conceptions of mathematical creativity enhance student-directed conceptions of creativity. For example, when teachers use purposeful questioning to elicit ideas that will help students to think independently and when they support students to use multiple problem solving approaches, teachers can enhance student independence in learning by challenging them to think and bring in self-generated and new perspectives. This observation is partly in line with Lev-Zamir and Leikin's (2011) assertion that defining mathematical creativity in the context of teaching should allude to mathematics, teaching, learning and creativity. Our reason for using partly in line with is because we observed that PSTs commented on, for example, using games and manipulatives to make the learning of mathematics fun, and as an avenue to teaching mathematics creatively. However, the math behind or within the "fun" was not given keen attention. PSTs tended to overlook creativity and mathematics, which was emphasized by Haylock (1987) and Lev-Zamir and Leikin (2011) and attended more to teaching and learning in their descriptions of their conceptions of mathematical creativity. Care should therefore be taken to differentiate between teaching creatively and teaching for creativity (NACCCE, 1999) to ensure that students do not just have fun in class while engaging in games but that they also understand the math behind or within the fun and develop their own creativity.

Connecting to research, results of this study are not unique to mathematical creativity, and thus tie closely to other findings and recommendations from researchers whose focus is not necessarily on mathematical creativity. To begin with, encouraging students to use different approaches to solving problems does not occur naturally if a teacher has not anticipated some of the strategies that students are likely to use. Anticipating student strategies is a key practice for successful orchestration of classroom discussions. Discussions stimulate interaction, an important catalyst to creativity, as the teacher responds to students using assessing and advancing questions, and notices important aspects of student thinking during instruction (Smith & Sherin, 2019). Second, teachers who are open to designing tasks from scratch are likely to demand of the same from their students, by not just providing students with problems to work on, but also challenging them to generate problems that address specific mathematical concepts. This practice demands high cognition and is at the level of doing mathematics (Smith & Stein, 1998). It discourages algorithmic thinking, requires students to comprehend mathematical concepts, processes, and relationships, and demands self-monitoring, only to mention but a few, according to Smith and Stein (1998). These conditions are equally important to cultivating mathematical creativity for both teachers and students. Third, providing students with challenging but not impossible questions has a potential to stimulate students' intellectual curiosity and hence develop their creativity. The points we raise in this discussion do not mean that we should not pay close attention to creativity in mathematics in the field of mathematics education, but rather consider investing in its research as it is a potential contributor to the growth and improvement of teaching and learning of mathematics.

We note that the findings of this study are not a complete representation of what conceptions of mathematical creativity are. Conceptions are informed not just by educational experiences, but by culture and beliefs and therefore this area of research is open to more studies, particularly with a focus on specific types of experiences that shape these conceptions. This will be important in shaping our teacher preparation programs to provide PSTs with those experiences necessary to support their understanding and development of mathematical creativity and that of their students.

We close with noting that a major observation from our literature review was that research in mathematics education has overlooked mathematical creativity due to lack of an accepted definition of mathematical creativity. We argue that we cannot overlook mathematical creativity simply because of lack of coherence in existing definitions of what mathematical creativity is, but we can instead focus on specific conceptions of it and develop its understanding because by doing this, we will not only be focusing on what it is, but also on what it could be, which is not always done in other fields of research with agreed definition of constructs. By doing that we won't limit our understanding on what it is but can explore the what it could be and find connections that will expand our horizon in understanding of the concepts, and possibly impact the field of mathematics education.

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TEACHER EDUCATION (PRE-SERVICE)

BRIEF RESEARCH REPORTS

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THE USE OF MIXED-REALITY SIMULATIONS AS A TOOL FOR PREPARATING PRE-SERVICE TEACHERS AND THEIR PERCEPTIONS AND OPINIONS

USO DE SIMULACION DE REALIDAD-MIXTA COMO HERRAMIENTA EN LA FORMACIÓN DE MAESTROS: PERCEPCIONES Y OPINIONES

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Exposure to technologies such as mixed-reality simulations (MRS) can influence the opinions of pre-service elementary teachers (PSET), as well as the way they see and perceive MRSs as a useful experience in their preparation. Particularly, in the implementation high-leverage practices. The study presented here was carried out in a course of mathematical methods for elementary. The research questions are: What are the PSETs perceptions of the use of MRS as part of their teaching preparation? What are the views of PSET on MRSs as a tool to improve their teaching skills? Results indicate that PSET perceive MRSs as a highly positive, relevant and meaningful tool that supports their learning, as well as a way to improve their high-leverage skills.

Key Words: Technology, Teachers belief, Pre-service teacher preparation.

Introduction

The perception, views, and opinions of pre-service elementary teachers (PSET) regarding their acquisition of teaching skills can be influenced by the integration of technology in their teacher preparation program. Perceptions change when these technologies are linked to practice and are recognized as having several benefits (Gordon, Brayshaw and Gray, 2015; Russell, Bebell, O'Dwyer & O'Connor, 2003). Specifically, we refer to the use of emerging technologies such as Mixed-Reality Simulations (MRS), which are tools intended –but not limited– to provide simulated experiences to pre-service teachers. Engaging with mixed-reality simulations take place in a controlled and safe environment, which has the potential to improve and strengthen teaching skills (Hixon & So, 2009).

This report presents the perceptions, opinions, and points of view of Pre-service Elementary Teachers regarding the utility, benefits, and challenges of using mixed-reality simulations, as part of the teacher training program. Pre-service elementary teachers were exposed to the use of MRSs in an elementary mathematics methods course. The use of MRSs with PSETs was intended to foster high-leverage practices (Ball & Forzani, 2011) such as productive mathematics talk moves (Chapin, O'Connor & Anderson, 2009; Moyer & Milewicz, 2002; Ginsburg, 1997). These productive teaching moves include eliciting, evaluating, and questioning elementary students as they share and explain their solutions to problem solving activities in mathematics. To this end, the following research questions are addressed: What are the perceptions and ideas of pre-service elementary teachers regarding the use of mixed-reality simulations as part of the teacher preparation program? What are the opinions of pre-service elementary teachers on mixed-reality simulations as a tool used to improve their teaching skills, particularly those related to productive mathematical talk move?

Theoretical Perspectives

Insights on The Use of Technology

In order for pre-service teachers have the theoretical understanding of the benefits of being exposed to the use of technologies such as mixed-reality simulations, their perceptions and opinions regarding the usefulness of this must be framed in what Venkatesh, Morris, Davis and Davis (2003) have called the theory of acceptance and perception of the use and exposure to technology. According to Venkatesh et al., the perception that PSETs have about the benefits and utility of the use of technologies can be influenced by their experiences with it in their preparation programs. Venkatesh et al. adopted Davis's (1989) definition of utility as the measure by which an individual believes or perceives that the use of a particular technology (e.g., Mixed-Reality Simulations) improves the work being done, in which case, for the PSETs and the purpose of this study, would be the learning process and the teaching skills fostered through high-leverages practices such as productive mathematics talk moves in problem solving where one seek to elicit and understand a student's thought.

Mixed-Reality Simulations and Teaching Strategies

Mixed-Reality simulations were implemented in the context of a course on mathematics methods for elementary school. High-leverage strategies were integrated into this course (Ball and Forzani, 2011). According to Ball and Forzani, these strategies, are pedagogical skills that directly impact the student's academic performance and that can be implemented at various school levels in a high number of subjects or context. Some of these pedagogical strategies include how to promote an active participation from students, provide positive feedback, how to reveal, understand, and interpret the students' thoughts and ideas through productive mathematical talk moves (Chapin, O'Connor & Anderson, 2009), among others. According to Chapin, O'Connor & Anderson, productive mathematical talk moves (PMTM) refer to all those actions that teachers must carry out during teaching (including moments of group discussion or during an exchange of ideas between a teacher and a student) to better understand the ideas and understandings of the students. Some PMTMs, for example, include asking a student to repeat another student's idea in their own words, or asking a student to go deeper in their explanations, or simply remaining silent to listen to what a student is saying. During mixed-reality simulation sessions, the pre-service teacher practiced how to implement the pedagogical strategies mentioned above. Similarly, Pre-service teachers were able to experience what they could potentially expect when interacting with real students.

What are Mixed-Reality Simulations?

New technologies in education are now implemented to improve the process of teaching and learning. However, little has been done to integrate and expose Pre-service elementary teachers to emerging technologies such as *Mursion*, which are simulation technologies that allow a pre-service teachers to have "repeated practices involving high-risk situations without necessarily risking the loss of valuable resources" (Dieker et al., 2014, p.22).

Mursion is a mixed-reality simulation software where students are avatars and a virtual classroom is simulated. The pre-service elementary teacher has the opportunity to experience situations related to teaching. During a simulation session, a pre-service elementary teacher sees students or avatars in a classroom through a computer–or device– screen. The pre-service elementary teacher initiates the interaction with these avatars in the same way as it would in a classroom. In MRSs, the avatars are controlled by an educational trained specialist. The pre-service teachers are actually in a semi-controlled, structured, and prepared interaction with a specialist. The specialist controls the avatar, and not a computer program as in virtual reality, which allows for greater flexibility to offer the various classroom situations, both expected and unexpected similar to those that occur in a real classroom with a sense of reality in which there is a "real presence", defined by Dieker et al. (2015) as the perception that something real is happening.

Methods

Approximately 166 pre-service elementary teachers have taken the mathematics methods course since fall 2018, of which 92% were women and 8% men. Participants were in their first or second year of the preparation program. All pre-service elementary teachers were required to conduct an interview with an elementary student and assess their understanding when solving a series of mathematics problems. To this end, pre-service elementary teachers were exposed to theories on problem solving. During the course, the PSETs were prepared by analyzing teaching videos, rehearsing play roles as student/teacher, and experiencing three MRS sessions. Each MRS session lasted between 7 and 12 minutes and were video recorded. Each session required the PSETs to implement a different problem. The problems were selected by the researchers following the theoretical framework of cognitive guided instruction problems (Carpenter, Fennema, Franke, Levi & Empson, 2014). MRSs sessions were scheduled three times during the semester to ensure that the theoretical and pedagogical concepts discussed in class were acquired. At the end of the semester, pre-service teachers were asked to complete a survey to measure their perceptions, views, and opinions about the use of mixed-reality simulations.

Instrument

The instrument was developed using items from Hudson, Voytecki and Zhang, (2018), Bousfield, (2017) and Rasimah, Ahmad and Zaman (2011), and adapted for the purposes of the research. The survey consisted of 27 questions with a scale ranging from 1 to 5, where one means completely agree, to a five strongly disagree (see appendix A). Additionally, 4 open-ended questions were included to capture the opinions of the PSETs on what was most positive or negative about the experience with MRSs, the challenges they had while rehearsing with MRSs, and their perceptions of the benefits of being able to be prepared with the use of mixed reality simulations.

Results and Discussion

The implementation of MRSs allows pre-service teachers to participate in virtual classroom environments that simulate real classrooms. In this sense, 94% of participants agree or totally agree that the MRSs simulated a very similar and real experience as classroom, and 97% expressed that they see the practices with the MRS as a very positive experience. For example, participants mentioned that MRSs "provides unexpected experiences that I had to deal with, and help me feel better prepared for the real-world "; "How the avatar speaks and acts, seems like a real children, especially with their responses"; "I liked being able to practice as if it were a real classroom"; "I was able to practice as if I was really in front of the classroom with real students."

The use of MRSs in a teacher preparation program is not intended to displace the actual experience, but rather to enhance the preparation experiences (Peterson-Ahmad, 2018), in a protected, semi-controlled and safe environment (Hatton, Birchfield and Megowan-Romanowicz, 2008). Participants in the study acknowledge the above, for which 97% agree or fully agree that the use of MRSs is an effective way to practice new skills in the classroom, particularly in mathematics, of which 94 and 97%, respectively, expressed feeling more confident and better prepared to teach mathematics effectively or to engage their students in topics that involve discussions of problem solving activities, as they mentioned: "What I liked about the MRSs is that I was able to practice math problems as if I was really teaching it to real students." In this same sense, 88% of the pre-service elementary teachers agree or totally agree that after having been exposed and practiced with MRSs, they feel better prepared to carry out an interview with an elementary student on mathematics topics, where they should evaluate and verify the ideas, thoughts, and understandings using PMTMs.

Pre-service elementary teachers recognized the benefits of been exposed to a cutting-edge technology that is used as an educational tool to improve, enhance, and facilitate the acquisition of educational practices such as high-leverage practices. Several participants mentioned that the experience with MRS “allowed [them] to get the necessary practice before finishing the program and teach real students." Finally, one of the most received opinions by the participants, was that they wish they had the opportunity to have more time to interact with MRSs.

Conclusions

The way in which teachers are prepared is significant, since it is during these times when they acquire the skills and knowledge necessary for their future work as teachers (Vagi, Pivovarova and Miedel Barnard, 2019). Then, its relevant to report the findings of this study, so that other teacher preparation programs know the benefits and potential of integrating emerging technologies such as Mixed-Reality Simulation in their courses and improve the teaching skills of their teachers, since the first day of the program. As shown in the previous section, the perceptions and opinions of the pre-service elementary teacher on the usefulness of MRS are very positive, which would facilitate their integration. Similarly, when the integration of MRSs is seen as positive, efforts and resources allocated to implement simulations would be justified. The use of MRSs in a teacher preparation program, particularly in a math method course, as it was done here, is an innovative way to improve the acquisition of teaching skills by pre-service elementary teachers. During this first phase of the study, it is shown that participants perceive MRS as a useful tool, that helped them improve their teaching skills — particularly with regard to eliciting and questioning students in a classroom or as a teacher-student interview. The well-structured, guided, and planned use of MRSs technologies (Venkatesh et al., 2003) helped the pre-service elementary teacher perceive MRS as a tool intended to improve their pedagogical teaching skills (Ball & Forzani, 2011), and not to replace the real experience, which is a common limitation during the first years in preparation programs.

The results presented here are only the first phase of a larger study that involves analyzing the transcripts of the pre-service elementary teacher after they have interviewed an elementary school student. The results of this study will be analyzed, contrasted and presented in another report in this same forum, meanwhile, more data will continue to be collected on the perceptions, points of view and opinions on the use of mixed-reality simulations in a teacher preparation program.

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APPENDIX A

1. I like to experience with new technologies.
2. I enjoy rehearsing with Mixed-Reality Simulations.
3. I was unhappy when the Mixed-Reality Simulations sessions were over.
4. I would like to repeat the same experience using Mixed-Reality Simulations.
5. I feel that using Mixed-Reality Simulations enhanced my learning experience.
6. I feel better prepared to teach after my Mixed-Reality Simulation session.
7. Using the Mixed-Reality Simulation is an effective way to practice new classroom skills.
8. My Mixed-Reality Simulation session seemed like a real classroom experience.
9. The Mixed-Reality Simulation's students seemed like real elementary students.
10. After practicing my teaching methods using Mixed-Reality Simulations, I am more confident that I can effectively teach mathematics concepts.
11. After the MRS, I feel prepared to conduct a clinical interview to assess and elicit students thoughts and understanding using mathematics productive talk moves.
12. After my Mixed-Reality Simulation sessions, I have more confidence that I can engage students in a discourse about problem-solving activities.
13. I was able to effectively manage the interview during my Mixed-Reality Simulation sessions.

14. After the Mixed-Reality Simulation rehearsal, I feel prepared to orchestrate a group discussion in a classroom to assess understanding using mathematics productive talk moves.
15. After Rehearsing with the Mixed-Reality Simulations, I felt my Clinical interview with an elementary student was conducted effectively.
16. After the Mixed-Reality Simulation sessions, I have more confidence in my ability to manage undesired behaviors in group discussion.
17. After my Mixed-Reality Simulation sessions, I am better prepared to teach lessons involving problem solving.
18. I felt the Mixed-Reality Simulation's interviews were conducted effectively.
19. I felt like I was in a real classroom during the sessions.
20. The Mixed-Reality Simulation sessions prepared me to conduct the clinical interview.
21. The Mixed-Reality Simulation rehearsals helped me to create a plan for the clinical interview with an elementary student.
22. My experience with Mixed-Reality Simulations prepared me to teach.
23. I would like to use Mixed-Reality Simulation to develop my teaching skills in other courses.
24. Reflecting after each Mixed-Reality Simulation helped me to be better prepared for the next rehearsal.
25. Receiving peer feedback after each Mixed-Reality Simulation helped me to reflect on my strengths and weakness in assessing a students' understanding.
26. Receiving peer feedback after each Mixed-Reality Simulation helped me to reflect on my strengths and weakness in conducting a clinical interview.
27. Providing feedback to my peers after each Mixed-Reality Simulation helped me to reflect in my own teaching skills.

Open-Ended questions

1. What did you like the most about the Mixed-Reality Simulation?
2. What did you dislike the most about Mixed-Reality Simulation?
3. What do you consider was the most challenging aspect of interacting in with Mixed-Reality Simulations?
4. Do you believe using Mixed-Reality Simulation was beneficial for your Teacher-Preparation? If yes, why? if not, why?

USO DE SIMULACION DE REALIDAD-MIXTA COMO HERRAMIENTA EN LA FORMACIÓN DE MAESTROS: PERCEPCIONES Y OPINIONES

THE USE OF MIXED-REALITY SIMULATIONS AS A TOOL FOR PREPARATING PRE-SERVICE TEACHERS AND THEIR PERCEPTIONS AND OPINIONS

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La exposición a tecnologías como simulaciones de realidad mixta (SRM) puede influenciar las opiniones de los maestros de primaria en formación (MPF), así como la manera en la estos ven y perciben las SRM como un instrumento útil en su preparación docente. Particularmente, en la implementación de estrategias pedagógicas de enseñanza de alto apoyo académico (EPEAA). El estudio aquí presentado se llevo a cabo en un curso de métodos matemáticos para primaria. Las preguntas de investigación son: ¿Cuáles son las percepciones de los MPF sobre el uso de las SRM como parte de su preparación docente? ¿Cuáles son las opiniones de los MPF sobre las MRS como herramienta para mejorar sus habilidades de enseñanza? Resultados señalan que los MPF perciben las SRM como una herramienta altamente positiva, relevante y significativa que respalda sus aprendizajes, así como una forma de mejorar sus habilidades de EPEAA.

Palabras clave: Tecnología, Creencias de los docentes, Preparación de Maestros en Formación.

Introducción

La percepción, puntos de vista, y opiniones de los maestros de primaria en formación (MPF) sobre cómo la tecnología puede ayudarlos a adquirir nuevos conocimientos y habilidades de enseñanza, pueden verse influenciadas si estas tecnologías se integran como parte de sus programas de preparación. Especialmente, si estas tecnologías están vinculadas a la práctica y se presentan como un instrumento con múltiples beneficios (Gordon, Brayshaw y Gray, 2015; Russell, Bebell, O'Dwyer y O'Connor, 2003). De forma específica, nos referimos al uso de tecnologías emergentes como las simulaciones de realidad mixta (Mixed-Reality Simulations en inglés). Los softwares de simulación de realidad mixta (SRM), son herramientas destinadas a proporcionar a los maestros de primaria en formación experiencias simuladas. Estas experiencias ocurren en un ambiente controlado y seguro, que busca mejorar y fortalecer las habilidades de enseñanza (Hixon & So, 2009).

El reporte que se presenta en esta propuesta muestra cuáles son las percepciones, opiniones y puntos de vista de los maestros de primaria en formación con respecto a la utilidad, beneficios y retos de utilizar simulaciones de realidad mixta, como parte del programa de formación docente. El uso de SRM con los maestros de primaria en formación tuvo la intención de fomentar estrategias pedagógicas de enseñanza de alto apoyo académico (Ball y Forzani, 2011) como los movimientos productivos de conversación matemática (Chapin, O'Connor y Anderson, 2009; Moyer y Milewicz, 2002; Ginsburg, 1997) para obtener, evaluar y cuestionar a los estudiantes mientras comparten y explican sus soluciones a actividades de resolución de problemas en matemáticas. Con este fin, se buscaron responder las siguientes preguntas de investigación: ¿Cuáles son las percepciones e ideas de los MPF sobre el uso de las simulaciones de realidad mixta como parte del programa de formación docente? ¿Cuáles son las opiniones de los MPF sobre las simulaciones de realidad mixta como una herramienta utilizada para mejorar sus habilidades de enseñanza, particularmente las relacionadas a los movimientos productivos de conversación matemática?

Marco Teórico

Perspectivas sobre el uso de la tecnología

Hoy en día, para que los MPF realmente tengan una comprensión teórica de los beneficios de ser expuestos al uso de tecnologías como las simulaciones de realidad mixta, sus percepciones y opiniones con respecto a la utilidad de esta se deben enmarcar en lo que Venkatesh, Morris, Davis y Davis (2003) han llamado la teoría de la aceptación y percepción del uso y la exposición a la tecnología. Según Venkatesh et al., la percepción que tienen los maestros en formación sobre los beneficios y la utilidad del uso de tecnologías pueden mejorarse significativamente cuando se abordan adecuadamente en los programas de preparación. Venkatesh et al. adoptó la definición de utilidad (Davis, 1989) como la medida por la cual un individuo cree o percibe que el uso de una tecnología en particular (p. ej., Simulaciones de Realidad Mixta) mejora el trabajo que se realiza, en cuyo caso — para los maestros de primaria en formación y el propósito de este estudio, sería el proceso de aprendizaje y las habilidades de enseñanza fomentadas a través de estrategias pedagógicas de enseñanza de alto apoyo académico como los movimientos productivos de conversación matemática (MPCM) en la resolución de problemas donde se busque entender y comprender el pensamiento del estudiante.

Estrategias de enseñanza y su relación con las Simulaciones de Realidad Mixta

Las SRM se implementaron en el contexto de un curso de métodos matemáticos para MPF. En este curso se integraron estrategias pedagógicas de enseñanza de alto apoyo académico (Ball y Forzani, 2011). Según Ball y Forzani, estas estrategias, mejor conocidas en inglés como *High-Leverage Practices*, son habilidades pedagógicas que impactan directamente el rendimiento académico del estudiante y que pueden ser implementadas en diversos niveles escolares en un alto número de materias o contenidos. Así mismo, son estrategias que se aprenden a través de la práctica docente y la retroalimentación oportuna y precisa. Algunas de estas estrategias pedagógicas incluyen el cómo tener una participación activa de los estudiantes, dar una retroalimentación positiva, como revelar, comprender e interpretar el pensamiento e ideas de los estudiantes a través de movimientos productivos de conversación matemática, entre otros. Los movimientos productivos de conversación matemática se refieren a todas aquellas acciones que debe llevar a cabo el maestro durante la enseñanza (incluyendo momentos de discusión grupal o durante un intercambio de ideas entre un maestro y un estudiante) para comprender mejor las ideas y entendimientos de los estudiantes acerca del tema que se está enseñando. Algunos MPCM, por ejemplo, incluyen pedir a un estudiante que repita la idea de otro compañero con sus propias palabras, o pedir a un estudiante que profundice en sus explicaciones, o simplemente guardar silencio para escuchar lo que está diciendo un estudiante.Durante las sesiones de SRM, los MPF practicaron como implementar las estrategias pedagógicas mencionadas anteriormente.

Simulaciones de Realidad Mixta

Hoy en día se implementan nuevas tecnologías en educación para mejorar el proceso de enseñanza y aprendizaje. Sin embargo, poco se ha hecho para integrar y exponer a los MPF a tecnologías emergentes como Mursion, que son tecnologías de simulaciones que permiten a los maestros en formación tener "prácticas repetidas que involucran situaciones de alto riesgo sin necesariamente arriesgar la pérdida de recursos valiosos" (Dieker et al., 2014, p.22). Mursion es un software de SRM en donde se simula que los estudiantes (comúnmente conocidos como avatar) están en un aula virtual. El MPF, consecuentemente, tiene la oportunidad de experimentar situaciones relacionadas con la enseñanza en un entorno simulado y seguro. Durante una sesión de simulación, los MPF ven a través de la pantalla de una computadora a los estudiantes o avatar en un salón de clases. El MPF inicia su interacción con estos avatars de la misma manera que lo haría en un salón de clases. De forma tal que lo que parece, a vista del MPF, una interacción con un avatar de computadora, es realmente una interacción semi-controlada, estructurada y preparada con un especialista. El hecho de que sea un especialista el que controla los avatar y no un programa de computación para realidad virtual, da la flexibilidad de poder ofrecer al MPF un infinito número de posibilidades de preparación, en situaciones tan inesperadas, como podría ocurrir en un salón de clases real. Algo importante, es que estos entornos proporcionan al MPF un sentido de la realidad en el que hay un "presente real", definido por Dieker et al. (2015) como la percepción de que algo real está sucediendo.

Métodos

Desde el otoño de 2018 hasta el otoño de 2019, 166 MPF que han tomado el curso de métodos matemáticos para primaria han participado en el estudio, de los cuales 92% han sido mujeres y 8% hombres. Todos los participantes se encontraban en su primer o segundo año del programa de preparación docente. Como parte de los requerimientos del curso, todos MPF debían realizar una entrevista con un estudiante de primaria y evaluar la comprensión de este al resolver una serie problemas de matemáticas. Así, durante el curso MPF fueron expuestos a las teorías de resolución de problemas, y de aprendizaje y enseñanza de métodos matemáticos para primaria, se analizaron videos, se hicieron prácticas y ensayos en clases y se les pidió que tuvieran 3 sesiones de simulación con realidad mixta. Las SRM duraron entre 7 y 12 minutos y fueron video grabadas, y en cada una se usó un problema diferente. Los problemas siguieron el marco teórico de problemas de instrucción cognitiva guiada o mejor conocido en inglés como "*Cognitive Guided Intruction*" de Carpenter, Fennema, Franke, Levi y Empson, (2014). Las sesiones de SRM se distribuyeron durante el semestre para garantizar que los MPF asimilaran los conceptos teóricos y pedagógicos discutidos en clase.

Instrumento

Se usó una encuesta adaptada de Hudson, Voytecki y Zhang, (2018), Bousfield, (2017) y Rasimah, Ahmad y Zaman, (2011). La encuesta comprende 27 preguntas en una escala de 1 a 5 que va desde completamente de acuerdo, a completamente en desacuerdo (ver apéndice A). Además, se incluyeron 4 preguntas abiertas para capturar las opiniones de los MPF sobre lo más positivo o negativo de la experiencia con SRM, los desafíos que tuvieron al ser preparados en su formación a través del SRM y las percepciones de los beneficios de la formación con SRM.

Resultados y Discusión

La implementación de simulaciones de realidad mixta permite a los MPF en formación participar en entornos de aula virtual que simulan aulas reales. En este sentido, el 94% de los maestros en formación están de acuerdo o totalmente de acuerdo en que las sesiones de SRM simulaban una experiencia muy parecida y real a la del aula, y 97% expresaron que ven las prácticas con los SRM como una experiencia muy positiva. Por ejemplo, algunos participantes dijeron:"[Las simulaciones con SRM] proporciona experiencias inesperadas que tuve que solucionar, y me ayudo a sentirme mejor preparado para el mundo real"; "Cómo hablan y actúan los avatar, parecen como niños reales, especialmente con sus respuestas"; “Me gustó poder practicar como si fuera un aula real”; "Pude practicar como si realmente estuviera delante de la salón de clase con estudiantes reales". El uso de SRM en un programa de preparación docente no pretende desplazar la experiencia real, sino más bien mejorar la experiencia de formación de los MPF (Peterson-Ahmad, 2018), en un entorno protegido, semi-controlado y seguro (Hatton, Birchfield y Megowan-Romanowicz, 2008). Los MPF en el estudio reconocen lo anterior, por lo que el 97% están de acuerdo o totalmente de acuerdo en que el uso de SRM es una forma efectiva de practicar nuevas habilidades en el aula, particularmente en matemáticas, de los cuales el 94 y el 97%, respectivamente, expresaron sentirse ahora con más confianza y mejor preparados para enseñar matemáticas de manera efectiva o para involucrar a sus estudiantes en temas que involucren discusiones sobre actividades de resolución de problemas. Por ejemplo, un MPF resaltó lo siguiente: "Lo que me gustó de la SRM es que pude practicar problemas de matemáticas como si realmente lo estuviera enseñando a estudiantes reales". Así, el 88% de los MPF están de acuerdo o totalmente de acuerdo con que después de haber estado expuestos y practicado con SRM, se sienten mejor preparados para llevar a cabo entrevista con un estudiante sobre temas matemáticos donde ellos deban evaluar y verificar las ideas, pensamientos, y comprensiones de sus estudiantes, utilizando MPCM. Otro MPF expreso su opinión en este sentido: “[Usar SRM] fue divertido y atractivo ... las simulaciones me ayudaron a fortalecer mi comprensión sobre los temas que se enseñan en clase, en particular mis habilidades para hablar e interactuar con los estudiantes en clases [es decir, los MPCM] ”.

Por último, los maestros de primaria en formación reconocieron los beneficios de haber estado expuestos a una tecnología de vanguardia que se utiliza como una herramienta educativa para mejorar, acrecentar, y facilitar la adquisición de prácticas educativas como las estrategias pedagógicas de enseñanza de alto apoyo académico, por lo que los MPF mencionan que desearían tener más tiempo para interactuar en con las simulaciones de realidad mixta.

Conclusiones

La forma en que los maestros son preparados es muy relevante, ya que es durante este tiempo cuando adquieren las habilidades y los conocimientos necesarios para su futuro trabajo como docentes (Vagi, Pivovarova y Miedel Barnard, 2019). Así, resulta relevante reportar los hallazgos de este estudio, para que otros programas de preparación de maestros conozcan los beneficios y el potencial de integrar tecnologías emergentes como las simulaciones de realidad mita en sus cursos y mejoren las habilidades de enseñanza de sus maestros de primaria en formación, desde el primer día del de clases. Como se muestra en la sección anterior, las percepciones y opiniones de los MPF sobre la utilidad de las SRM son muy positivas, lo que facilitaría su integración. De igual manera, al ser vista como positiva la integración de SRM, el tiempo, esfuerzo y recursos destinados a implementar las simulaciones estarían acertadamente justificadas. El uso de SRM en un programa de preparación docente, particularmente en un curso de método matemático, es una forma innovadora de mejorar la adquisición de habilidades de enseñanza de los MPF. El uso bien estructurado, guiado, y planificado de la tecnología (Venkatesh et al., 2003) — como se implementan las SRM en este estudio— ayudó a los MPF a percibir a las SRM como una herramienta destinada a mejorar sus habilidades pedagógicas de enseñanza de alto apoyo académico (Ball & Forzani, 2011), y no para sustituir la experiencia real, que es una limitación común durante los primeros años en el que los maestros reciben su formación.

Los resultados presentados aquí son sólo la primera fase de un estudio más amplio que involucra el análisis de las transcripciones de los maestros de primaria en formación una vez que han realizado la entrevista con estudiantes de primaria. Los resultados de este estudio serán analizados, contrastados y presentados en otro informe en este mismo foro, mientras tanto, se seguirán recopilando más datos sobre las percepciones, puntos de vista y opiniones de los MPF sobre el uso de simulaciones de realidad mixta como parte de la preparación de maestros.

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PRE-SERVICE TEACHERS' OPERATIONALIZATION OF COGNITIVE DEMAND ACROSS CONTEXT

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This research uses a framing perspective to examine how pre-service teachers (PSTs) conceptualize cognitive demand when selecting tasks. Our results show that PSTs' operationalizations of cognitive demand are context dependent. Within their methods class, PSTs largely think of cognitive demand in terms of how it promotes understanding of mathematics. When PSTs interact with students, they tend to operationalize cognitive demand as a way to support perceived student disposition and ability, or as a way to determine problem difficulty.

Keywords: Cognition, Learning Theory, Teacher Education – Preservice, Teacher Knowledge

Introduction

Selecting tasks that promote reasoning and problem solving is an important part of mathematics teaching (CAEP Standards, 2020). Research shows that selecting such tasks can be difficult for pre-service teachers (PSTs), but interventions in university methods classes can improve PSTs' ability to choose mathematically rich tasks (Crespo, 2003; Crespo & Sinclair, 2008; Leavy & Hourigan, 2019). Attending to the cognitive demand of tasks is one way to focus PSTs' attention on the mathematical features of tasks (Stein et al., 1996; Stein, Smith, Henningsen & Silver, 2000). *Cognitive demand* (CD) refers "to the kinds of thinking needed to solve tasks" (Stein et al., 2000, p. 3). Low-level tasks rely on applying memorized facts or procedures, requiring little understanding of the underlying mathematical concepts. In contrast, high-level tasks provide for multiple entry points and solution paths, requiring students to engage in meaningful inquiry and problem solving. While tasks of each level of CD support different learning goals, high-level CD tasks are linked to the greatest gains in student learning (Stein et al., 2000). Therefore, it is important that teachers be able to select high CD tasks for instruction.

As teacher educators, we are interested in how PSTs think about task selection in relation to CD as they move from a methods class to their internships. We ask, *how do PSTs operationalize cognitive demand in task selection across contexts?* For the purposes of this study, we consider two contexts: (1) reflecting on CD as students in a methods course that emphasizes rigorous mathematical tasks, and (2) applying CD when teaching middle grades students. Our study adds to the literature of task selection because it considers how PSTs reason about the CD of tasks, which impacts their task selection.

Theoretical Framework

We use the lens of *framing* to explain PSTs' changes in conception of CD across contexts. Framing has been used in science education research to describe how teachers and students understand particular educational contexts, and how that understanding impacts their ideas about knowledge, along with their actions and interactions with others (e.g., Hammer, Elby, Scherr, & Redish, 2005; Elby & Hammer, 2010; Richards et al., 2020). From this perspective, people learn by activating resources, which are "fine-grained knowledge elements" (Elby & Hammer, 2010, p. 410) based on factors such as lived experiences, social interactions, and beliefs. When resources are repeatedly activated together, they form "locally coherent sets" (Elby & Hammer, 2010, p. 413) called frames. In the classroom, these frames give a teacher or a student a sense of "what is going on here"

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

(Hammer et al., 2005), which impacts how they interact with others and the content. A critical feature of this framework is that context determines the resources that people activate. Thus, framing allows for individuals to hold multiple beliefs or perceptions, while identifying which resources are foregrounded during a given activity.

Methods

This brief research report is a secondary analysis relying on a subset of data from a larger study that investigates middle grade mathematics and sciences PSTs' lesson planning behaviors throughout their early field experiences.

Middle grades mathematics methods course

The participants are 10 undergraduate PSTs in a middle grades mathematics and science dual certification program who completed a mediated field experience mathematics methods course, taught by the first author. The field experience component of the course took place at a local middle school, where PSTs selected and implemented tasks with small groups of students. The CD of mathematical tasks was explicitly and regularly addressed in the methods course.

Data sources

The data sources for this brief research report include PSTs' final course papers and the transcripts from two individual semi-structured interviews, conducted by the first author. The final course assignment (Fall 2018) asked PSTs to reflect on how the methods course supported their growth as a learner and doer of mathematics. The first interview was conducted the semester following the methods course (Spring 2019). PSTs were asked to reflect on their process for selecting tasks and preparing lesson plans for the after-school enrichment program. PSTs were probed for whether the CD of tasks played a role in their decision-making. The second interview was conducted a year after the methods course (Fall 2019). PSTs were asked about their teaching internship and to analyze tasks from the casebook authored by Stein, Smith, Henningsen, and Silver (2000).

Analysis

Our research builds on the work of Elby and colleagues (2020) who demonstrated how analysis of written reflections can provide teacher educators with insight on PSTs' framing of classroom activity in a more timely manner than the interaction-analysis techniques traditionally employed with a framing perspective. The authors describe this type of analysis as "framing lite." The data sources were initially coded by the first author for references to CD. Next, we independently looked for trends in PSTs' statements about CD both when they discussed selecting tasks in the abstract and when thinking about task selection in relation to their internships. We then discussed our observed trends and developed a codebook. Each transcript was randomly assigned to two authors who then completed independent coding. After independent coding, we met to examine discrepancies in codes. Once in agreement, we combined the codes into broader categories of the ways that PSTs' discussed CD. We found these categories clustered together based on context and we discussed what PST experiences (e.g. activities in the methods class or interactions with students) might be contributing to these clusters. These clusters became the general "lite" frames that describe PSTs' application of CD.

Findings

We found that PSTs operationalize CD as related to task selection differently depending on context. In situations devoid of K-12 students, like reflecting on their methods class, PSTs largely described CD in terms of math content and student understanding. In this instance, PSTs seemed to frame CD as a feature of mathematical tasks. In contrast, when reflecting on their experiences with real students, PSTs seemed to frame CD as a mediator of perceived student need. PSTs' discussion of CD in relation to task selection shifted toward cultivating student dispositions, attending to perceived student ability, and describing problem difficulty.

Cognitive demand as a feature of mathematical tasks

In their final papers, 7 of 10 PSTs wrote about CD as a significant element of the methods course. PSTs primarily wrote about CD in two ways, (1) a way to categorize tasks and (2) as a way to support students' mathematics learning. PSTs primarily focused on contrasting the features of low and high CD tasks. For example, when reflecting on how to select tasks based on cognitive demand, Briley wrote, "We determined that the former had lower cognitive demand because it only required the memorization of the formula for area, whereas the latter required the application of area and perimeter and an explanation or argument for their thinking." Briley's explanation of CD is congruent with the descriptions provided by Smith and Stein (1998).

When analyzing tasks during the second interview, half of the PSTs linked procedural thinking and application of well-rehearsed algorithms with lower levels of CD. Carson highlighted the difference between students applying a rote procedure and conceptual understanding. He said, "[if] there's a specific way to do it, I don't think that's cognitively high. But when you have to know the whole process and what that process means, then I think that's when it's a cognitively high demanding problem." Mary Jane agreed with Carson, stating that "the fact that you need to sit and think about it, and discuss with others about it shows that there's more cognitive demand that's needed". Both Mary Jane's and Carson's focus is on the mathematical understanding required to solve a high cognitive demand problem.

Cognitive demand as a mediator of perceived student need

When talking about their placements, PSTs still connected CD to task selection in terms of student understanding of mathematics, but it ceased to be their primary focus. Instead, PSTs largely attended to perceived (1) student dispositions, (2) student ability, and (3) task difficulty. For the purposes of this report, we will focus on (1) and (2).

Eight of the ten PSTs linked student disposition to the CD of tasks. According to PSTs, CD impacts students' interest in and willingness to complete the tasks. When reflecting on her field placement, Claire articulated the connection between motivation and tasks. She stated, "I think [cognitive demand] really surfaces, and I think it really ties well into motivation too. Because if you like, if you do get something too easy, like they lose motivation, like in my mentor's class and if you give them something too hard, then they just like, give up because they don't have it." Jessica talked about students' self-efficacy and confidence as considerations for selecting tasks. She stated that selecting high CD tasks "not only promotes a growth mindset, but also lets students know that you believe they can succeed at higher-level tasks."

PSTs addressed their perception of students' ability through task selection and providing instructional supports. Participants discussed the need to find tasks that were not too easy or too hard. In terms of CD, Elizabeth described a task that was the right-fit as "It was the high end where they were challenged, but it was still low enough that they could do it." Every PST made a comment about selecting tasks that were the "right fit" for students. PSTs also discussed providing scaffolds to make tasks more accessible for students. For example, some PSTs articulated the difference between language supports for English language learners and providing mathematical supports. When selecting the right-fit the level of tasks, PSTs also wanted to prevent unproductive struggle, as opposed to looking to create productive struggle. For instance, Grace says, "Or sometimes if it is too cognitively demanding and they're getting frustrated and too flustered I think sometimes it'd be like, helpful to take a break and be like, 'okay, so maybe like, what do you guys remember about this?'" In this case, Grace thought about possible actions if students' struggle became unproductive.

Discussion

The purpose of this study is to understand how PSTs operationalize the concept of CD as they move from university coursework to working with students. Our results show that PSTs apply their conceptions of CD in context dependent ways, which is consistent with a framing perspective (e.g., Elby & Hammer, 2010; Hammer, et al., 2005). Specifically, PSTs' variance in how they operationalize CD across contexts can be understood as the activation of different resources, and subsequently different frames, as they shift from learning formally about CD to applying CD when working with students. We describe PSTs' different frames as: (1) CD as a feature of mathematical tasks and (2) CD as a mediator of perceived student need.

The frame *CD as a feature of mathematical tasks* seems to be activated when PSTs discuss tasks abstractly. PSTs perceive "what is going here" as an assessment of their knowledge of the different levels of CD. Within this frame, activated resources could include PSTs' understanding of course readings and their experiences with selecting and completing tasks in the methods course. In contrast, the frame of *CD as a mediator of perceived student need* is activated in the context of their ongoing work with students. When this frame is activated, PSTs rely more heavily on their experiences with students, rather than the formal definition of CD. Frames are a helpful way of thinking about how PST knowledge builds across contexts, rather than attributing changes in PST behavior to a "washing out" of the teacher preparation program (Richards et al., 2020). We propose that PSTs retained the formal definitions of CD but in practice their framing focused on the perceived needs and dispositions of students.

As teacher educators, we posit that examining PSTs' framing of concepts, even rough frames, can be helpful in supporting PSTs to build new resources and shift their framing (Elby et al., 2020). For example, the framing *CD as a mediator of perceived student need* was supported by PSTs' beliefs that the CD of tasks should be matched to student disposition or perceived student ability. This framing could lead to PSTs choosing tasks of low CD or lowering the CD of tasks during implementation (Stein et al., 2000). PSTs also expressed concern for supporting linguistically and culturally diverse learners' access to high CD tasks. Thus, another implication of this frame is that PSTs' perceived support of students' needs may limit opportunities for students to engage with high CD tasks (de Araujo, 2017). Teacher educators could support PSTs' development of resources and shifts in framing by revisiting the formal definitions of CD and explicitly linking them to instructional practices beyond the initial selection of the task. For example, teacher educators should explicitly model for PSTs how to support students in meeting language objectives without lowering the CD of the mathematics. When linked to student learning and dispositions, these additional experiences may become resources in PSTs' framing of CD as it relates to students' needs.

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REFLECTIVE CONVERSATION AS A MEANS TO DEVELOP KNOWLEDGE IN FUTURE MATHEMATICS TEACHERS

CONVERSACIÓN REFLEXIVA COMO MEDIO PARA EL DESARROLLO DE CONOCIMIENTO EN FUTUROS PROFESORES DE MATEMÁTICAS

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This paper reports that the reflective conversation allows the development of professional knowledge in future mathematics teachers on the basis of questions about the nature and processes of construction of mathematical concepts, from both a mathematical and pedagogical point of view. The teacher of a didactics of mathematics course and ten future teachers participated in this study. The conversations held by them during three sixty-minute sessions were analyzed; the duration of these sessions was determined by the time it took to have the discussion of a question about even numbers and generalization. The type of knowledge developed consisted of recognizing the concept of even numbers as a process of mathematical generalization, andas an ability to be developed in elementary school students.

Keywords: Teacher Knowledge, Teacher Education.

This paper analyzes how reflective conversation helps to develop the knowledge for teaching the concept of mathematical generalization in future teachers. This interest is based on the fact that reflection and integration of content, and pedagogy are two central aspects of this development (Ponte & Chapman, 2016) equally, as is the collective (Arcavi, 2016; Chamoso, Cáceres & Azcárate, 2012; Horn & Little, 2010; Jaworski, 2006; Krainer & Llinares, 2010; Ponte, 2012; Preciado-Babb et al., 2015; Rasmussen, 2016; Rowland & Ruthven 2011; Santagata & Guarino, 2011).

It is now recognized that learning to teach requires the development of different types of knowledge in the teacher. In this sense, the problematization of the mathematical knowledge of the teacher has been the focus in several proposals in mathematics education in order to characterize and model it (e.g. Ball, Thames & Phelps, 2008; Carrillo-Yañez et al., 2018; Krauss, Neubrand, Blum, & Baumert, 2008; Kunter, et al., 2013; Pino-Fan, Godino, & Font, 2018; Shoenfeld, 2011; Llinares, 2012). In fact, a key question is “whether mathematical knowledge in teaching is located ‘in the head’ of the individual teacher or is somehow a social asset, meaningful only in the context of its application” (Rowland & Ruthven, 2011, p. 3).

On the other hand, little is known about the development of knowledge in future teachers (Thanheiser et al., 2014); in this regard it is stated that “it is important for programs to engage prospective teachers in learning opportunities that enable them to reconstruct their initial knowledge and understanding of mathematics teaching. This requires awareness and scrutiny of this prior knowledge. Reflection is a key process for achieving this” (Ponte & Chapman, 2016, p. 283). In this sense, we examine how knowledge of future teachers is developed in context of social interaction in the classroom, because it is a place that offers opportunities to teach, to learn from conversation and to reflect on their own knowledge and experiences (Horn & Little 2010; Kaminski, 2003; Toom, Husu, & Patrikainen, 2015).

One type of knowledge for teaching mathematics is generalization (Demonty, Vlassis, & Fagnant, 2018). Its importance can be seen mainly in two ways: as a way to teach for developing mathematical concepts (Davydov, 1990; Dörfler, 1991) and as an activity for the development of algebraic thinking

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in the study of patterns (Radford, 2014; Warren, Trigueros, & Ursini, 2016; Zazkis & Liljedahl; 2002). Therefore, mathematical generalization was the subject of conversation and reflection.

Conceptual Framework

The ideas of conversational learning of Pask (1976) and Kolb and Kolb (2017) were used and integrated to study how professional knowledge associated with mathematical generalization is developed, as shown in Figure 1. For these authors, learning comes from conversations on a topic that leads to the construction of new meanings and transforms collective experiences into knowledge.

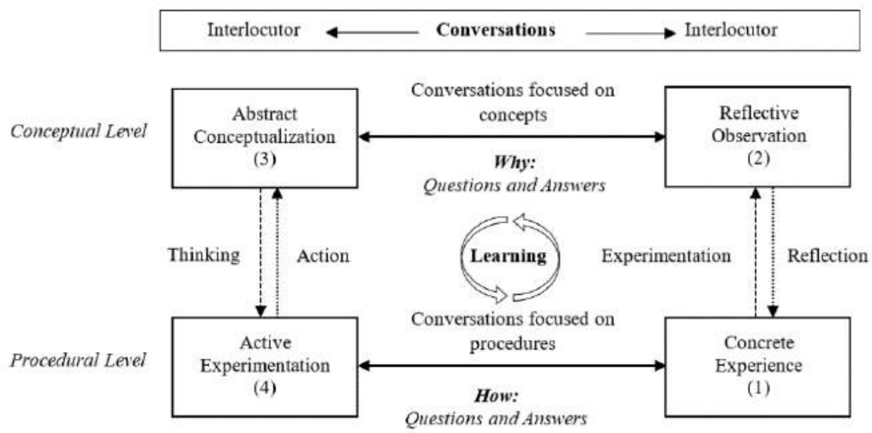

Figure 1: Reflective conversation and Learning based on Pask (1976) and Kolb and Kolb (2017)

We understand knowledge to be what is internalized in conversational interactions and it is relevant to those who share and participate in it (Gee, 2011). In this case, we refer to the specific knowledge of mathematics teachers with a clear orientation to the practical activity of teaching mathematics (Ponte, 2012).

Developing this knowledge from an RC implies the willingness to engage in a dialogue to negotiate meanings, accept questions and discuss the shared ideas of mathematics, its teaching and its learning (Earles, Parrott, & Knight, 2016). The knowledge of future teachers must be associated with their conversational context in such a way that the conversation establishes a space of participation to make explicit their ideas and knowledge related to a topic, making possible the emergence of meaningful relationships. Therefore, conversation allows the expression of thoughts in an open way, and by doing so, it gives way to a reflexive process that influences the meanings, thoughts, and actions of others; that is, it influences the development of knowledge.

Method

The development of knowledge of future teachers in the context of an RC was analyzed with a qualitative-interpretative methodology (Corbin & Strauss, 2008) considering interactions as the focus of the analysis (Kilpatrick, 1988). The analysis was carried out by organizing the data in (i) speaking shifts, and (ii) organizational sequence (Mazeland, 2006).

Seven women (W) and three men (M) from a training program for secondary and high school mathematics teachers at a public university in Mexico participated in this study. They were attending a didactic of mathematics course in their senior year. The participation of the teachers and the students was by invitation.

An open question was posed as a topic of discussion, since learning based on conversations can begin with questions associated to a topic (Pask, 1976) and the demands for solving a problem are a guiding factor for reflection (Dewey, 1938). The question was associated with the property of integer numbers and designed in such away that it could be answered by using the knowledge of arithmetic and of basic algebra and it allowed for the discussion of the idea of generalization, its teaching and its learning. The question was: *What can be said about the result of the multiplication of any two consecutive integer numbers?*

The analysis used the model shown in Figure 1 in the following way: First, the transition between the learning modes of the RC was analyzed based on the identification of the level of conversational interaction (procedural or conceptual levels). In the procedural level, it was considered that conversations are characterized by discussions focused on the construction or use of mathematical procedures, while the focus of the conceptual level is the use of conceptual ideas or theories that answer the posed question. Then, the experiences, reflections, thoughts and acts of the participants during their conversations were analyzed to characterize how it contributes to the development of their knowledge.

Results And Conclusions

Results show that the knowledge developed consisted in conceiving generalization as an important process to explore, explain and validate mathematical results and in recognizing its importance as a necessary mathematical ability to favor the teaching of arithmetic and basic algebra. Furthermore, the participants became aware that many mathematical concepts come from generalizations and raised the idea of the importance of using several representations (arithmetic, algebraic and geometric) to discover and test mathematical results, as happened in this case with the concept of even numbers:

M1: (…) I considered generalization as something of algebraic thinking, and the recognition of patterns and relationships between quantities proper of arithmetic. In this case, we can see a relationship between the quantities that increase from 2 to 4, then to 6, and 2 more units each time. It would be concluded that multiplying two consecutive numbers results in an even number, which is generalizing!

W4: I thought that there could be no generalization in arithmetic (...) but, after reflecting on what we understood as a generalization, as has been in this case: multiplying consecutive integer numbers results in an even number is a generalization; therefore, generalization can be made in arithmetic and algebra. In the sequence [2, 6, 12, 20, 30, …], number four is missing between 2 and 6, and the generalization that comes is that there are multiples of two. This is, an even number of the form $2k$.

RC fostered the development of knowledge to the extent that questions about the nature and construction of the concept of even numbers were made, from both a mathematical and a pedagogical point of view; it also promoted the free and open expression of ideas, answers and counter proposals associated with the initial question. This enhanced the transition between the four learning modes indicated in Figure 1. For example, participants moved from the concrete experience mode (CE) to the reflective observation (RO) mode by questioning and associating numerical relationships between algebraic structures. And they move from abstract conceptualization (AC) to the active experimentation (AE) mode by questioning meanings and looking for explanations and validations of their procedures. In this way, they managed to establish patterns and conceptualize an even number as a mathematical generalization (figure 2):

W2: How can I interpret the expression $2k$ in arithmetic and in algebra? I think that $2k$ can be seen as a whole area or it can represent an even number in arithmetic; in contrast, it may represent a quantity that is changing to double in algebra (...).

M1: Yes, let's say that by constructing a meaning for even numbers (…)

M2: (…) and considering it as the starting point for teaching even numbers.

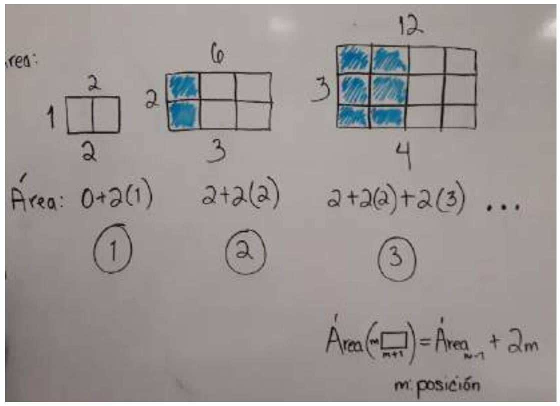

Figure 2: Consecutive multiplication, generalization and even number

We consider that the development of professional knowledge about mathematical generalization from an RC demands an exchange and articulation of ideas based on questioning the mathematical knowledge, its use and the argumentation of procedures and concepts used in the solution of problems, as well as the negotiation of meanings associated with them because this is what allowed the emergence of common understandings about the meaning of generalization and the cognitive demands for its teaching and learning.

These results are consistent with those reported by Demissie (2015), Jaworski (2006), Chamoso, Cáceres and Azcárate (2012) who found that participating in processes of collective inquiry promotes reflective thinking among peers. This study also confirms the results reported by Simoncini, Lasen and Rocco (2014) that a guided dialogue makes it possible for future teachers to obtain better perspectives of their teaching practices, including their thoughts and actions.

We plan to delve into how to incorporate RC in teacher training programs so that it can be a means to develop professional learning, promote reflections in relation to the professional practice, and encourage a shared vision of it (Preciado-Babb et al., 2015; Toom, Husu, & Patrikainen, 2015).

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REAL-TIME COACHING WITH SECONDARY PRESERVICE TEACHERS: THE PRACTICES OF MATHEMATICS TEACHER EDUCATORS

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We examined the coaching practices of three mathematics teacher educators as they engaged in real-time coaching with secondary mathematics preservice teachers. Situated in a novel early field experience and under close supervision, preservice teachers instructed undergraduate students in an introductory mathematics course; teacher educators coached in real time during these teaching episodes. Forty-four preservice teachers participated in this study, resulting in a data corpus of 44 videos of their teaching. Findings indicate that direct coaching was used more than indirect coaching, and pacing was the most prevalent focus of direct coaching.

Keywords: Teacher Education – Preservice; Teacher Educators

Early development of preservice teachers' (PSTs') knowledge and practice for teaching mathematics typically occurs during a methods course and is often accompanied by school-based field experiences. One vexing challenge for the field is a disconnect between what PSTs see and experience in K-12 classrooms and what they learn about effective teaching in on-campus methods courses (Allsopp et al., 2006; Zeichner, 2010). Yet, teacher educators argue that teaching is best learned within, and from, practice (Ball & Cohen, 1999; Zeichner, 2010).

Over the last two years, we have engaged in a project designed to mitigate this challenge - a novel early field experience (EFE) for secondary mathematics PSTs called the University Teaching Experience (UTE; Bieda et al., 2020; Cirillo et al., 2020). In this model, PSTs, while enrolled in a concurrent mathematics methods course, engage in an EFE that takes place in a college-level introductory mathematics course, taught from a problems-based perspective, and under the close supervision of MTE(s). The PSTs work with small groups of students during problem-solving sessions and teach as pairs twice during the semester in a safe, collaborative, and highly mentored context. The MTE *coaches* the teaching PSTs in real-time, providing immediate feedback to the PSTs while in the act of teaching.

Coaching in the UTE resembles coaching that occurs during methods course rehearsals, where the mathematics teacher educator (MTE) can provide feedback in real time, pause the rehearsal for discussion, and/or model certain teaching moves for the PSTs to imitate (Arbaugh et al., 2019; Baldinger, Selling, & Virmani, 2016; Lampert et al., 2013). Although the field is beginning to understand coaching practices during rehearsals (e.g., Lambert et al., 2013), we know little about coaching practices in more authentic approximations of practice (Grossman et al., 2009) like the UTE. Our real-time coaching (RTC) also resembles a model used by Stahl et al. (2018), who investigated the effects of RTC with prospective English teachers through the use of "bud-in-the-ear" technology while PSTs engaged in micro-teaching with peers (although we did not use this technology). Stahl and colleagues found several positive benefits of RTC, including increased confidence in PSTs' knowledge, skills, and capabilities as a teacher, accelerated development of practical skills, and an increasingly discerning and critical position toward professional practice. Hence, RTC appears to be a productive method for MTEs to help PSTs improve their teaching practices and worthy of study. Akin to Lampert et al. (2014), our study was guided by this research question: What was the structure and substance of real-time coaching (RTC) as enacted in the UTE at three sites?

Methods

Data Collection: Participants, Context, and the UTE Teaching Episodes

To address our research question, we analyzed three MTEs' RTC practices at three different university sites. At one site, the UTE took place in Fall 2018; at the other two sites, the UTE took place in Spring 2019. A total of 44 PSTs participated in the UTEs across the three sites, resulting in 44 video recordings of their UTEs. These video recordings captured the interactions between the MTEs and PSTs as they taught in pairs, with each pair teaching twice in a semester.

Data Analysis

We began analysis by identifying RTC episodes in the UTE video recordings. Round one of open-coding (Miles, Huberman, & Saldaña, 2014) RTC episodes identified the *structure* of our RTC (see Table 1). Next, focusing only on the coaching episodes where direct coaching occurred, a list of *substance* codes was generated (see Table 2). The total number of coaching episodes for this study was 258; the number of direct coaching episodes was 227.

Table 1: Structure Codes

Structure Code	Description
Direct Coaching	MTE provides feedback directly to PST. Either MTE or PST may initiate this exchange.
Indirect Coaching	MTE enters the lesson as the teacher and directly addresses mathematics students for the purpose of modeling an instructional move for the PSTs.

Table 2: Direct Coaching Substance Codes

Substance Codes;	Description
Alerting PST to Notice Students	MTE interrupts or alerts the PST to notice a student who is confused or a student with their hand raised, whom the PST otherwise would not have noticed.
Asking for Volunteers	MTE advises PST on (a) how to ask for volunteers, or (b) who to ask to share an answer to the problem.
Asking PST to Expand on Mathematics	MTE requests that PST provides more information on mathematical concepts or examples.
Assisting with Classroom Technology	MTE assists PST with understanding how to use the technology they are attempting to use for their instruction.
Attending to Mathematical Precision	MTE attends to PST's written/oral language/notation, such as terminology, labeling, and mathematical symbols. This is an error in PST's communication, rather than a mathematical error.
Correcting a Mathematical Error	MTE requests that PST corrects a mathematical error. This is a mathematical error, rather than an error in communication.
Getting Math Students' Attention	MTE requests that PST refrains from giving instruction and first gains students' attention.

Getting Students to Write Notes	MTE requests that PST encourages math students to write notes.
Helping PST Understand Mathematics	MTE helps PST better understand a mathematical concept.
Helping PST Understand Student Thinking	MTE helps PST better understand a student's comment, answer, or question.
Making a Pedagogical Suggestion	MTE makes suggestions to PST regarding next steps for monitoring, whole class discussion, or other instructional moves.
Pacing	MTE coaches about speeding up or slowing down the lesson.
Providing Positive Feedback	MTE praises a PST's action.
Raising Voice	MTE asks the PST to speak louder.
Redirecting Instructional Move	PST uses a pedagogical move and the MTE redirects the PST to use a different pedagogical move.
Rephrasing PST's Language	MTE provides a rephrased question or statement for the PST to repeat to students.
Requesting Visual Display – Logistics	MTE requests that PST creates a visual display or makes the display clearer for students (emphasis on logistics; e.g., writing bigger).
Requesting Visual Display - Mathematics	MTE asks that PST adds further information to a visual display of mathematics.
Requesting Visual Representation or Verbal Communication	MTE requests that PST provides a visual representation of spoken mathematics, or to explain verbally an idea that is written.

Findings

The MTEs engaged in about seven times more direct coaching than indirect coaching (88% vs. 12% respectively), indicating that the MTEs made a large majority of their coaching comments directly to the PSTs as opposed to stepping in to teach the mathematics students and modeling teaching practices for PSTs. Examining the substance of direct coaching more closely (see Figure 1), *pacing* was the most prevalent focus of direct coaching, occurring in 24.23% of the direct coaching episodes. MTEs provided feedback to PSTs about both speeding up the lesson and slowing down the lesson. The second most prevalent type of direct coaching was *making a pedagogical suggestion*, occurring in 23.35% of the direct coaching episodes. *Making a pedagogical suggestion* covered a wide range of coaching moves. In general, these were instances where the MTE noticed something in the way mathematics students were interacting or reacting to instruction and had a suggestion for the PST that helped the PST navigate an issue. In contrast, *assisting with classroom technology* and *getting students to write notes* occurred in less than 2% of the direct teaching episodes.

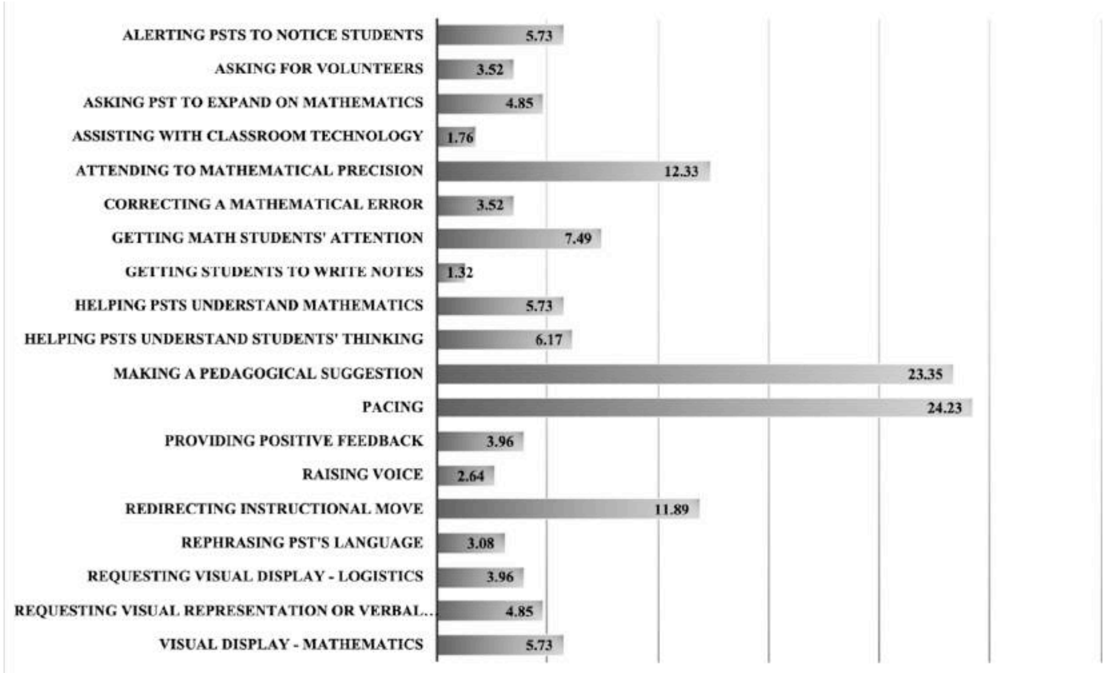

Figure 1: Percentage of Substance of Direct Coaching Episodes (Note: because some episodes were assigned multiple codes, percentages do not total 100%)

Discussion

The findings of this study extend those of Lampert et al. (2013) by providing additional ways that PSTs can be coached during instruction. Further, because PSTs in this study were teaching mathematics students (rather than their methods course peers), this study identified a new structure of coaching: *indirect coaching*. It is interesting to us that in the authentic context of PSTs teaching mathematics to students, the MTEs in this study coached about pacing more than any other focus. This stands in contrast to Lampert et al.'s finding that the MTEs in their study focused on managing time in only 4.3% of interactions between MTE and PST. Another difference between these two studies is in the area of the mathematics as a focus. In Lampert et al.'s study, the MTEs focused on *mathematics* in 11.94% of MTE/PST interactions, with mathematics being defined as "working on and understanding the mathematical content" (p. 233). In our study, while the MTEs focused on *helping PSTs understand mathematics* in 5.25% of coaching episodes, they also coached PSTs in the areas of *attending to mathematical precision* (11.38%), *visual display of mathematics* (5.28%), and *asking PST to expand on mathematics* (4.47%). Our findings show that RTC can occur while PSTs are teaching mathematics students in an authentic context, extending Stahl et al.'s (2019) work. We are left with questions about the impact of real-time coaching on PSTs' growth as mathematics teachers.

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FRAMEWORKS FOR NOTICING IN MATHEMATICS EDUCATION RESEARCH

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Eliciting student thinking and using what is learned of student understanding to inform instruction is critical to effective mathematics teaching. Professional noticing skills assist teachers in identifying, interpreting and responding appropriately to student thinking. Therefore, the development of professional noticing skills in teacher candidates has become a goal of some mathematics teacher education programs. For the purpose of determining whether instruction is assisting in the development of these skills, it is necessary to have a way to measure these skills. This paper is a brief review of how professional noticing has been operationalized in mathematics teacher education research. A search of the ERIC data bases resulted in 405 studies, 89 of which met the criteria for the review. The following results contain a representative subsample of the 89 studies due to space limitations.

Keywords: Teacher Educators, Teacher Education – Preservice, Teacher Education – Inservice/Professional Development, Research Methods

In recent years, there has been an increased focus on using evidence of student thinking to inform and support instruction. In fact, the 2014 National Council of Teachers of Mathematics (NCTM) publication "Principles to Action: Ensuring Mathematical Success for All," recommended "eliciting and using evidence of student thinking" as a mathematics teaching practices to support student learning (NCTM, 2014). As the focus of professional noticing, in general, is to recognize, interpret, and respond to student thinking, noticing aligns well with the aforementioned NCTM mathematics teaching practice. Therefore, the development of professional noticing skills in teachers support student thinking, learning and understanding.

Theoretical Framework

Within recent research, there are three commonly cited definitions of professional noticing. First, van Es and Sherin (2002) define noticing as consisting of "three components: identifying what is important in a teaching situation; making connections between specific events and broader principles of teaching and learning; and using what one knows about the context to reason about a situation". Second, Jacobs, Lamb and Philipp (2010) define noticing as consisting of "three interrelated skills: attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings". Finally, Mason (2002) describes the "discipline of noticing" as consisting of the following techniques: keeping and using a record; developing sensitivities; recognizing choices; preparing to notice at the right moment; and validating with others.

Professional noticing, or simply noticing, has been the focus of much research in the past two decades. Berliner (2001) found well developed noticing skills to be a feature of an expert teacher. Researchers have also stated, novice teachers do not naturally notice important events in a classroom (Jacobs et. al, 2010; van Es and Sherin, 2002; Star and Strickland, 2008). However, research had found noticing skills can be learned (Jacobs et. al, 2010; van Es and Sherin, 2002; Star and Strickland, 2008). There is also evidence to support the development of noticing skills has improved field experiences (Star and Strickland, 2008; Stockero, Rupnow, and Pascoe, 2017).

Structures (e.g. frameworks, scaffolds) can support teachers in the development of noticing skills. The purpose of this paper is to examine noticing frameworks used in mathematics education research to support and measure teacher noticing. The research questions which guided the analysis are: What

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type of frameworks exist to measure professional noticing skills in mathematics teacher education research? How are these frameworks applied (e.g. for measurement, instruction, or both) in mathematics teacher education research?

Methods

I searched ERIC data bases using the search criterion of (notic* AND mathematics) OR ('professional vision' AND mathematics) yielding 405 results. Titles, keywords, and abstracts of the resulting articles were scanned and excluded using the following exclusion criteria (EC). EC 1: The focus of the article was not mathematics. EC 2: The article was not research concerning professional noticing. EC 3: The article was not published in a research journal. EC 4: The article was not regarding K-12 teacher education. EC 5: The article was not a research study. EC 6: The framework in the article was too specific (e.g. Fernandez et. al., 2011 framework for noticing a students' additive/multiplicative thinking in proportional reasoning). Although articles with a specific noticing purpose are important to developing noticing skills in specific instructional areas, these articles were excluded from this paper in the interest of space.

Results

After review, 89 articles were included for analysis. Thirty-two were excluded due to a focus other than mathematics, 224 of the remaining articles were not on professional noticing, 29 were excluded because the article was not published in a research journal, 6 were not K-12 education, 4 articles were not experimental studies, and 21 were excluded for having too specific a focus.

Many researchers use, or adapt, an existing framework to fit their research purposes, although some developed independent frameworks. Researchers used these frameworks in a variety of methods (e.g. to provide to candidates for instructions; to inform the creation of pre and post assessments; to create a coding scheme for qualitative data). Frameworks were used for either instruction, measurement, or both. If a researcher used a framework for both instruction and measurement, sometimes the same framework, occasionally different frameworks. Due to limited space, the following is an abbreviated representative sample of the 89 articles reviewed.

Frameworks

Leatham, Peterson, Stockero and Van Zoest (2015) describes the development of Mathematically Significant Pedagogical Opportunities to Build on Student Thinking (MOST) Framework. MOST is a linear framework developed through an iterative process in mathematics teacher education courses over a four-year period. The framework was developed to identify MOSTs in video captured lessons for analysis. Researchers found the framework to be applicable for instruction and measurement as well. The framework begins with Student Mathematical Thinking: can student mathematics be identified, if so, is there a mathematical point? The second part of the framework involves Mathematical Significance: is the mathematics appropriate, if so, is it central? The final part of the framework is Pedagogical Opportunity: opening, timing? (Leatham et. al., 2015; Stockero et. al., 2014; Stockero et. al., 2017). If the answer to each of the six questions is yes, the interaction is a MOST.

Star and Strickland (2008) developed the 5 Category Observation Framework to identify what teacher candidates noticed from watching a classroom lesson. Star and Strickland created a pre/post assessment consisting of questions related to items and actions in a video the teacher candidates watched. The pre/post assessment was designed to represent each of the five categories and question types evenly. The five categories of the framework include: classroom environment, classroom management, tasks, mathematical content, and communication. The researchers were able to show significant gains in all areas, with the exception of classroom management (which was higher on the pre assessment).

Santagata and colleagues (2007, 2010, 2016) developed and used the Lesson Analysis Framework (LAF). One version of the LAF consisted of three parts: goals/parts of the lesson, student learning, and teaching alternatives; another version of the LAF consisted of four parts: learning goals, analysis of student thinking, effects of teaching, and propose improvements in teaching (Santagata and Yeh, 2016). Researchers use the framework to guide teacher candidate analysis of video.

van Es (2011) introduced the Learning to Notice Framework. The framework consisted of two categories: what teachers notice and how teachers notice. Data in each of these categories are then assigned one of four levels: baseline, mixed, focused, and extended. Statements made concerning some overall aspect of the classroom aspects or interactions, and are evaluative in nature, represent baseline level statements. Mixed level statements are primarily focused on the teacher but begin to mention student thinking and start showing some interpretive qualities. Focused statements primarily concern specific instances of student thinking and began to predict student actions. Finally, the extended level statements began to make connections between teacher response and student thinking.

Sherin and van Es (2009) introduced the Professional Vision Coding Scheme based on their definition of professional vision. The coding scheme consisted of two components: selective attention and knowledge-based reasoning. Selective attention consisted of two dimensions of analysis: actor (student, teacher, other) and topic (management, climate, pedagogy, mathematical thinking). Knowledge-based Reasoning also has two dimensions of analysis: stance (describe, evaluate, interpret) and strategy (restate student ideas, investigate meaning of student idea, generalize and synthesize across student ideas).

Mitchell and Marin (2015) introduced the Structured Analysis Framework based on Mathematical Quality Instruction (MQI). A subset of MQI codes relevant to noticing were chosen to create the framework. Five areas of codes were chosen: lesson or segment structure; teacher mathematical error or imprecision, use of mathematics with students, cognitive demand of task, and student work with mathematics.

Applications of Noticing Frameworks

Noticing frameworks are applied within mathematics teacher education research in a variety of ways. Frameworks are applied in the areas of instruction, measurement, or both. In addition, researchers may use one existing framework for the purpose of instructions and another for measurement. The following are examples of each of these applications of noticing frameworks in mathematics teacher education research.

Instruction. Definitions of noticing (Mason, 2002; Jacobs et. al., 2010; van Es & Sherin, 2002) were the most common influence for instructional design. In fact, Fisher and colleagues (2019) provide an overview of their 3 meeting instructional model which shows alignment with the Jacobs and colleagues (2010) definition of noticing.

Measuring noticing. Jacobs and colleagues (2010) described three skills necessary for the "expertise" of professional noticing: attending to children's strategies, interpreting children's understanding, and responding on the basis of children's understanding. In their research, Jacobs and colleagues developed a coding structure for each of these skills: attending (1) evidence (0) lack of evidence; interpreting and responding (2) robust evidence (1) limited evidence (0) lack of evidence (Jacobs et. al., 2010). This coding structure was then used in their research to code writing prompts to compare the noticing skills of teachers at various levels in their careers (prospective teachers, initial participants, advancing participants and emerging teacher leaders). Many other researchers have adopted this method of measuring noticing skills in their own research (e.g. La Rochelle, Nickerson, and Lamb (2019); Fisher, Thomas, Schack and colleagues (2013, 2018, 2019)).

Another framework found often in mathematics teacher education research on noticing was the learning to notice framework (van Es, 2011). As previously described, the learning to notice framework consists of two categories, with four levels to each category. In the van Es, 2011 study, the researcher employed the framework to measure video club discussions, as a whole, and provide one level for each meeting. It was found the progression through the levels was not linear, especially in levels 2 and 3: meeting 3 was at level 2, meeting 4 was at level 3, meeting 5 was back at 2, meetings 6 and 7 were back at 3, and meeting 8 was back at 2 before jumping to level 4 in meetings 9 and 10. Amador and colleagues (2015, 2016, 2018) modified the learning to notice framework and used it extensively in their work.

Applying multiple frameworks. Amador, Carter and Hudson (2016) is an example applying multiple frameworks. Both Jacobs et. al. (2010) and van Es (2011) are applied in their noticing research. The learning to noticing framework was expanded to include a total of 9 levels: 2 sublevels for each level 1, 2, and 4; and 3 sublevels for level 3. In addition, researchers created a 9 level learning trajectory based on the Jacobs et. al. (2010) definition of noticing: 2 codes for attending, 5 codes for interpreting, and 2 codes for responding. This expansion of their coding scheme also shows influences of the Sherin and van Es (2009) professional vision framework.

Discussion

This paper reported on a small representative subset of the 89 results of a systematic review of the literature regarding operationalizing noticing. As can be seen from the results, the majority of the research on operationalizing professional noticing focuses on measuring noticing employing the Jacobs and colleagues (2010) definition or the van Es (2011) learning to notice framework. In addition, researchers are able to modify these frameworks to suit their research needs.

Of course, whenever a review of the literature is conducted, there are limitations. It is difficult to know, using the search criteria previously mentioned, whether all studies were identified. In addition, due to space limitations, it was not possible to include all studies. For example, in this review studies with specific foci were not included (e. g. curricular noticing).

The main purpose of the paper was to provide researchers and mathematics teacher educators with a summary of frameworks employed to inform instruction and the measurement of noticing. The goal was to assist in mathematics teacher researchers and educators who would like to begin research in professional noticing or developing an instructional unit on professional noticing with a list of common ways to operationalize noticing. Further research to summarize the theory which informed the development of these frameworks is still necessary.

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PROSPECTIVE ELEMENTARY TEACHERS' USE OF CONTEXTUAL KNOWLEDGE WHEN SOLVING PROBLEMATIC WORD PROBLEMS

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This study investigates the extent to which pre-service elementary teachers (PETs) use their contextual knowledge to model and solve eight problems for which the result of the arithmetic operation is problematic, if one takes into consideration the reality of the context. A paper-and-pencil test was administered to 621 PETs enrolled in mathematics content courses. The test included eight experimental items and four buffer items. The findings for a sample of 97 PETs are not very encouraging. The total number of realistic responses varied from 5 to 80 (out of 97 possible for each problem). Overall, the percentage of realistic responses on the eight problematic items was only about 31%.

Arithmetic word problems play an important role in learning mathematics at the elementary school level. There are several practical and theoretical reasons of the inclusion of arithmetic word problems in the elementary curriculum. First, they provide contexts in which students can use their mathematical knowledge so they can develop their problem-solving abilities, an important goal of learning mathematics. Second, word problems provide practice so students can develop their abilities to use their knowledge in real-life situations. Third, word problems serve as motivators so students can see the relevance of the procedures and algorithms learned in school. Fourth, word problems have the potential to provide students with rich contexts and realistic activities in which to ground mathematical concepts and, thus, facilitate the learning of more complex concepts. Finally, word problems provide students with experiences to learn how to use mathematical tools to model aspects of reality, that is, to describe, analyze, and predict the behavior of systems in the real world (Burkhardt, 1994; De Corte, Greer, & Verschaffel, 1996; Verschaffel, Greer, & De Corte, 2000; Verschaffel & De Corte, 1997).

Some critiques (e.g., Gerofsky, 1996; Lave, 1992; Nesher, 1980) argue, however, that the mathematics curriculum fails to achieve these lofty goals because traditional instructional tasks tend to focus on a straightforward application of procedures and computations to solve artificial problems unrelated to the real world. As a result, students tend to approach word problems, more often than desirable, in a superficial and mindless way with little, if any, disposition, to modeling and realistic interpretation. Several pieces of research provide empirical evidence to these claims (Davis, 1989; De Corte & Verschaffel, 1989; Greer, 1993, 1997; Reusser, 1988; Reusser & Stebler, 1997; Schoenfeld, 1991; Silver, Shapiro, & Deutsch, 1993; Verschaffel, 1999; Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Lasure, 1994).

Purpose of the Study

The purpose of the study was to examine prospective elementary teachers' (PETs) reactions and responses to problematic arithmetic word problems for which the solution is not the result of application of the most obvious arithmetic operation suggested by the two numbers given in the problem statement.

As suggested by the research literature, elementary school children tend to ignore the realistic constrains of the context embedded in the statement of the problem, a phenomenon that Schoenfeld (1991) coined "suspension of sense-making." Several critics and researchers argue that children' suspension of sense-making is the result of school practices (Davis, 1989; Greer, 1993; Nesher, 1980;

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

Schoenfeld, 1991; Silver, Shapiro, & Deutsch, 1993). To develop children' disposition to realistic modeling, we must change curriculum and instructional tasks. Since the teacher has an important role in the construction or selection of learning tasks and opportunities, one may argue that researchers and curriculum developers need to understand teachers' reactions and responses to problematic problems.

Theoretical and Empirical Background

Mathematical modeling is the process of representing aspects of reality by mathematical means. In particular, the solution of some physical or real-world problems requires some form of mathematization. That is, the construction of a mathematical model. The complexity of the process of mathematization depends, of course, on the nature of the problem. There are several proposed models of representing reality by mathematical means (e.g., Silver, Shapiro, & Deutsch, 1993; Verschaffel, Greer, & De Corte, 2000), but Silver et al's model (Fig. 1) suffices for our purposes.

According to Silver, Shapiro, and Deutsch's model, a simplified version of the process of mathematical modeling consists of four different stages: understanding of the problem, construction of a model or selection of a mathematical procedure, the execution of the procedure, and the interpretation or evaluation of the outcomes of the procedure.

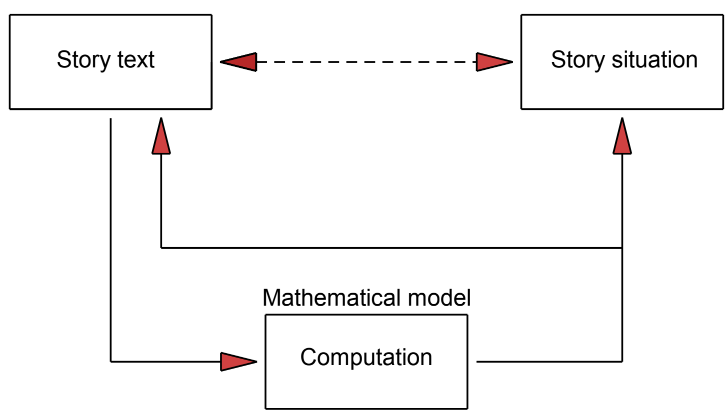

Figure 1: Silver et al.'s (1993) Referential-and-Semantic-Processing Model for Successful Solutions

The first stage of the process of mathematical modeling involves understanding the problem situation embedded in the story text. That is, we need to understand the given or known facts, the unknown information, the superfluous data, and missing information. The second phase involves the construction of a mathematical model or selection of a suitable procedure, operation, or algorithm whose outcome will lead us to the solution of the problem. To perform the second stage of the modeling process successfully, we must understand the mathematical structure of the problem. That is, we must understand the interconnections or relationships among the different types of information related to the solution of the word problem. The third stage of the problem involves mainly performing the computation, procedure, or algorithm either with paper and pencil or using a computational device. Finally, we should interpret and assess the outcome of the mathematical procedure in terms of the realistic context embedded in the story text of the word problem or in terms of the real-world story situation. It is during this step that we need to focus on the meaning of the result of the mathematical model so we can establish the connection between the outcome of the computation and the solution to the real-world story problem. It is during this stage that we need to assess whether our modeling assumptions are realistic or reasonable.

Silver, Shapiro, and Deutsch's model implies that there are three main potential sources of error when solving word problems: lack of understanding of the mathematical structure of the problem, which leads students to select an inappropriate procedure, executing the procedure incorrectly, and failing to interpret or assess the result of the procedure in terms of contextual or everyday-life knowledge. Silver, Shapiro, and Deutsch (1993) examined 195 middle grade students' solution processes and their interpretation of solutions to the following problem:

> The Clearview Little League is going to a Pirates game. There are 540 people, including players, coaches, and parents. They will travel by bus, and each bus holds 40 people. How many buses will they need to get to the game?

Their analysis revealed that 91% of the students selected an appropriate procedure (e.g., long division, repeated multiples, repeated additions, etc.), but only 61% of these students performed it flawlessly (about 56% of the total number of students). Overall, the researchers found that only 43% of the total number of students gave the correct answer (14) to the problem. However, some of these students provided inappropriate interpretations or justifications. For example, one student wrote "14 buses because there's leftover people and if you add a zero you will get 130 buses so you sort of had to estimate. Are we allowed to add zeros?" (p. 124-125). The researchers also reported that about 55% of the students did not get the correct answer because either they did not properly interpret the outcome of the computation or executed incorrectly the procedure. These computational mistakes could have been prevented if students had interpreted their solutions appropriately. Silver, Shapiro, and Deutsch proposed the model displayed in Figure 2 as a graphical representation of unsuccessful solutions that are due to a failure to connect the outcome of the procedure to the real-world context embedded in the story problem.

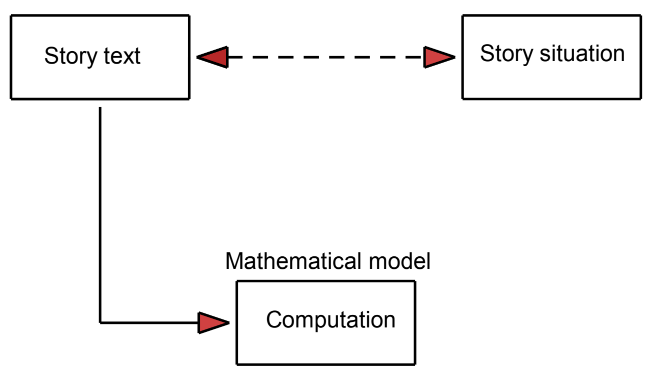

Figure 2: Silver et al.'s (1993) Referential-and-Semantic-Processing Model for Unsuccessful Solutions

Other pieces of research have amply documented elementary school children' improper modeling assumptions when solving problematic arithmetic word problems. Some further examples of the word problems that students have been asked to solve are the following:

1. What will be the temperature of water in a container if you pour 1 liter of water at 80° and 1 liter of water of 40° into it? (Nesher, 1980)
2. John's best time to run 100 m is 17 sec. How long will it take to run 1 km? (Greer, 1993)
3. Lida is making muffins that require 3/8 of a cup of flour each. If she has 10 cups of flour, how many muffins can Lida make? (Contreras & Martínez-Cruz, 2001)
4. In September 1995 the city's youth orchestra had its first concert. In what year will the orchestra have its fifth concert if it holds one concert every year? (Verschaffel, De Corte, & Vierstraete, 1999)

In their study with 75 fifth graders in Flanders, Verschaffel, De Corte, and Lasure (1994) reported that only seven (9%) students provided a realistic and correct response to the temperature problem. Similarly, in the same study, these researchers found that only two (3%) responses included realistic answers or reactions to the running problem. In another study, Contreras and Martínez-Cruz (2001) focused on prospective elementary teachers' solution processes and realistic reactions to the third problem. Their analysis revealed that only 19 (28%) of the participants' responses contained a realistic solution to the problem, but none of the participants made any comments about the problematic nature of the problem.

Verschaffel, De Corte, and Vierstraete (1999) addressed upper elementary school children' difficulties in modeling and solving nonstandard additive word problems involving ordinal numbers. The participants were administered a paper-and-pencil test consisting of 17 word problems, nine of which were experimental items and eight buffer items. The result of the straightforward arithmetic operation yields the correct answer for three of the nine experimental items. An example of such a problem is "In January 1995 a youth orchestra was set up in our city. In what year will the orchestra have its fifth anniversary? However, the solution of the remaining six experimental items is either 1 more or 1 less that the result of the straightforward arithmetic operation of the two given numbers. An example of such a problem is problem 4 stated above. Overall, the researchers found that the percentage of correct responses for each of the six problematic items was less that 25%. An error analysis revealed that 83% of the errors made on these problems were ± errors. In other words, most of the children' errors were due to their interpretation that the result of the addition or subtraction of the two given numbers yielded the correct answer.

Although research has convincingly documented elementary school children' strong tendency to model problematic problem unrealistically, the generalizability of the findings to more mature students, such as prospective elementary teachers, has not been sufficiently investigated. On one hand, since PETs have had even more experiences with traditional school problems, we may argue that there is no reason to expect that prospective elementary teachers would use their contextual knowledge and realistic considerations in their solution processes of problematic word problems. On the other hand, we may claim that PETs may have faced real-world problem situations outside school more often than young children and, having a more developed mathematical knowledge, have a stronger disposition to activate their contextual knowledge when confronted with problematic problems whose realistic solutions require taking into consideration contextual knowledge.

In their 1997 study, Verschaffel, De Corte, and Borghart examined future teachers' responses to seven problematic word problems. The problems were problematic in the sense that they cannot be appropriately modeled and solved by the straightforward application of the suggested arithmetic operation with the two numbers given in the problem statement. The researchers found that the future teachers had a strong tendency to ignore contextual knowledge and realistic considerations when modeling and solving the problematic word problems. In fact, the researchers reported that only 48% of all the responses to the problematic problems could be rated as realistic.

Even thought Verschaffel, De Corte, and Borghart's findings provide some insights into prospective teachers' use of realistic considerations when confronted with problematic word problems, more research is needed to provide a more complete picture about this research area, particularly across different cultures. In the present study, I focus on the extent to which the findings from previous research with pupils and future teachers are generalizable to prospective elementary teachers in the USA.

Methods and Sources of Evidence

The total sample of participants consists of 621 PETs enrolled in different sections of mathematics content courses for elementary teachers at two State Universities in the United States. The present paper reports the results of three groups (97 PETs) for which the analysis has been completed. The PETs had not been previously engaged in any intentional or systematic modeling activities or tasks.

A paper-and-pencil test consisting of eight experimental items and four buffer items was administered to the PETs during regular class instruction. The eight experimental items (Table 1) were problematic in the sense that the outcomes of the arithmetic operations performed with the given numbers in the problem story does not provide the answer to the problem, if one takes into consideration the real-world situation embedded in the contextual problem story. The buffer items, on the other hand, were standard routine problems whose solution is the straightforward result of the operation performed with the given numbers. The experimental items were adapted from Verschaffel and De Corte's (1997) study. An example of a buffer item is "Joel is building a collection of 175 different stamps. He has already collected 107 different stamps. How many more stamps does he need to complete the collection?"

Table 1: The Eight Experimental Items

1. 1175 supporters must be bused to the soccer stadium. Each bus can hold 40 supporters. How many buses are needed? (Carpenter, Lindquist, Matthews & Silver, 1983).
2. 228 tourists want to enjoy a panoramic view from the top of a high building that can be accessed by elevator only. The building has only one elevator with a maximum capacity of 16 persons. How many times must the elevator ascend to get all the tourists on the top of the building? Verschaffel, 1995)
3. At the end of the school year, 50 elementary school children try to obtain their athletics diploma. To receive the athletic diploma they have to succeed in two tests: running 400 m in less than 2 minutes and jumping 0.5 m high. All the children participated in both tests. Nine children failed the running test and 12 failed the jumping test. How many children did not receive their diplomas? (Verschaffel, 1995)
4. Carl and George are classmates. Carl has 9 friends that he wants to invite to his birthday party. On the other side, George has 12 friends that he wants to invite to his birthday party. Since Carl and George have the same birthday, they decide to give a party together. They invite all of their friends. All their friends come to the party. How many friends are there at the party? (Nelissen, 1987)
5. A man wants to have a rope long enough to stretch between two poles 12 m apart, but he has only pieces of rope 1.5 m long. How many of these pieces would he need to tie together to stretch between the poles? (Greer, 1993)
6. Steve has bought 12 planks of 2.5m each. How many 1 m planks can he saw out of these planks? (Kaalen, 1992)
7. Sven's best time to swim the 50 m breaststroke is 54 seconds. How long will it take him to swim the 200 m breaststroke? (Greer, 1993)
8. The flask is being filled from a tap at a constant rate. If the water is 4 cm deep after 10 seconds, how deep will it be after 30 seconds? (This problem was accompanied by a picture of a cone-shaped flask) (Greer, 1993)

After each problem, I have indicated its original source; however, in some cases the numbers were replaced by others.

Students' written responses to the problems were the source of data. Written directions asked students to show all their work to support each of their answers and to write down any questions or concerns they may have about each problem. I recognize that written responses have some intrinsic limitations when compared to verbal protocols. However, written protocols allow researchers to collect data from large samples. Moreover, some researchers (Hall, Kibler, Wenger, & Truxaw, 1989) have validated the use of written responses to infer cognitive processes.

Analysis and Results

Each response to problems 1 and 2 was coded as correct or incorrect. Each response to problems 3-8 was coded as correct, partially correct, or incorrect. Two raters judged every response. A response was judged as correct if it included a realistic numerical answer that estimated or indicated the range of possible solutions and took into account the contextual restraints of the real-world problem situation. A response was judged partially correct if it was incomplete or wrong but included a realistic comment suggesting that the student displayed awareness of the contextual restraints of the real-world problem situation. A response was judged incorrect when it did not suggest any awareness of the contextual restraints of the real-world problem situation. The inter-rater agreement was 99.7%. Table 2 summarizes the results of the analysis.

Table 2: The Number and Percentage of Correct, Partially Correct, and Correct Responses for the 8 Experimental Items

Problem	Number and percent of correct responses	Number and percent of partially correct responses	Number and percent of incorrect responses
1	76 (78.5%)	0 (0%)	21 (21.5%)
2	80 (82.5%)	0 (0%)	17 (17.5%)
3	3 (3%)	16 (16.5%)	78 (80.5%)
4	3 (3%	17 (17.5%)	77 (79.5%)
5	2 (2%)	4 (4%)	91 (94%)
6	24 (24.5%)	1 (1%)	72 (74%)
7	1 (1%)	4 (4%)	92 (95%)
8	0 (0%)	6 (6%)	91 (94%)
Total	189 (24.5%)	48 (6%)	539 (69.5%)
Subtotal	33 (5.5%)	48 (8%)	501 (86%)

As shown in Table 2, PETs' performance on most items was alarmingly poor: The percentage of incorrect responses ranged from a high 95% for item 7 to 17.5% for item 2. Overall, the percentage of realistic responses (correct responses and partially correct responses) on the eight problematic items was only about 30.5%. We should notice, however, that the number of realistic responses was considerable greater for the division problems involving remainders, problems 1 and 2. If we exclude these two problems from the analysis, then the percentage of realistic responses for the remaining six problems is only about 14%.

Discussion and Conclusion

The purpose of the present study was to collect systematically empirical data about the extent to which prospective elementary teachers in the USA activate their contextual knowledge when solving problems whose solution in not the direct result of an arithmetic operation. Using similar problems and methodology as previous studies (e.g., Verschaffel & De Corte, 1997; Verschaffel, De Corte, & Lasure, 1994), a test consisting of eight problematic items and four standard problems was administered to a sample of 621 PETs. The analysis has been completed for 97 PETs (three groups) and it is reported in the present article.

Although previous studies have convincingly demonstrated children' strong tendency to ignore the contextual realities embedded in the story of the problem situation, I was hoping that the findings with prospective elementary teachers would be much more encouraging. After all, prospective elementary teachers are part of a more mature and experienced population and it is reasonable to assume that they have an understanding of the contextual knowledge needed to realistically solve the problems. Therefore, the question of PETs' failure to activate this contextual knowledge needs to be further discussed and investigated. I offer several tentative hypotheses to explain PETs' lack of disposition to model contextual word problems realistically.

First, children and PETs' lack of activation of their contextual knowledge may be due to their constant exposure to traditional and stereotypical school word problems. If this is the case, then this tendency may remain constant or get stronger with additional years of immersion in the mathematical culture of traditional classrooms. Future research is needed to better understand the effects of traditional learning environments on students', including PETs, failure to activate their contextual knowledge to solve problematic problems.

A second possible explanation to understand PETs' tendency to ignore the contextual realities of the situation embedded in the problem story is that they lack enough real-world knowledge of the situational context of the problematic problems. Even though this seems unlikely, follow-up studies should provide empirical data to confirm or refute this hypothesis.

A third explanation may be that PETs approached the problematic problems in an automatic way thinking that they were standard mathematical problems without reflecting on the contextual realities of the problem. Further research is needed to better understand PETs' suspension of sense-making when solving these types of problems.

In conclusion, this study provides, at the very least, some empirical evidence that PETs in the USA lack an initial disposition or reaction to consider the contextual restraints of problems grounded in the real world. Further research is needed to better understand PETs' apparent suspension of sense-making when engaged in solving problems that require realistic interpretations.

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USING STRIP DIAGRAMS TO SUPPORT PROSPECTIVE MIDDLE SCHOOL TEACHERS' EXPLANATIONS FOR FRACTION MULTIPLICATION

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In this report, I describe how prospective middle school teachers created strip diagrams to solve fraction multiplication problems. I analyzed classroom videos from a year-long content course in order to determine what how teachers drew the diagrams and found four critical features of the drawings. I explore how they used the features as they drew and explained their thinking.

Keywords: Teacher Education–Preservice, Rational Numbers, Representations and Visualization

In North America, representations are a critical component in school mathematics (La Secretaría de Educación Pública, 2011; Ontario Ministry of Education, 2005; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Researchers have also emphasized the importance of representations in mathematical thinking (Cuoco, 2001; Janvier, 1987). Both teachers and students use representations to help them solve and make sense of problems (e.g., Lobato et al., 2014), communicate their ideas (Roth & McGinn, 1998), and participate in mathematical activity especially if their language is not the privileged language in the classroom (Turner et al., 2013) However, there are some roadblocks presented in the research on teacher knowledge about representations. Researchers have produced little evidence that teacher preparation programs (both for practicing and prospective teachers) prepare them to successfully integrate representations in the classroom (Stylianou, 2010). In this study, I provide a case demonstrating prospective teachers can sensibly engage in mathematics with representations. In particular, I ask the following questions: *How do prospective teachers draw strip diagrams to solve fraction multiplication problems in class? How do they use the strip diagrams to solve fraction multiplication problems?*

Theoretical Framework

Researchers who have studied representation use in class (Izsák, 2003; Saxe, 2012) have generally agreed to distinguish what is being represented and what is "doing" the representing (cf. von Glasersfeld, 1987). In this study, I refer to representations as observable geometric inscriptions that can be referred or pointed to as the object of discussion (Goldin, 2002). It is this indexical and communicative nature of representations allowing students to explain their thinking and for others to engage in another's way of reasoning. When students create a display to represent their thinking, they also communicate with them. In other words, they tailor their display with an audience in mind (Saxe, 2012) and thus students select salient features to highlight and point when creating and talking with representations. Additionally, I frame representations as culturally and historically rooted. A representation's cultural and historical meaning stems from how communities interact with an inscription over time (Blumer, 1986). For example, a class can ascribe the meaning to the inscription "=" as "execute the arithmetic to the left" if they are continually asked to solve result-unknown problems over time.

Data and Analysis

I analyzed four weeks of instruction from a sequence of two mathematics content courses for prospective middle school teachers (PSMTs) enrolled in a teacher education program. The same instructor taught both courses. The objective of the course was to strengthen the students'

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

mathematical understanding of middle school topics. The 13 PSMTs enrolled in the course were predominantly white women. The students were expected to use a multiplicand-multiplier definition for multiplication, notated by equation N · M = P (Beckmann & Izsák, 2015). In this equation, N denotes the number of base units in one group (the multiplicand), M denotes the number of groups (the multiplier), and P denotes the total number of base units in M groups. The class also defined the fraction a/b as the quantity formed by a parts of size 1/b. They were also expected to explain with drawings rather than memorized algorithms or symbol pushing.

The main data corpus for this study was video and audio-recorded lessons from class. The primary analytical techniques I used were modified from Saxe et al., (2015) and focused on identifying forms and functions of the representations. In this report, I focus on how “coarse forms” were drawn. A coarse form is a set of inscriptions used sequentially. When listening to explanations during discussions, I segmented the drawings based on how the PSMT described the sequence of drawings as indicated by utterances such as “I did this…and then drew this…” (Fig. 1). I then found coarse forms that were similar across all the drawings.

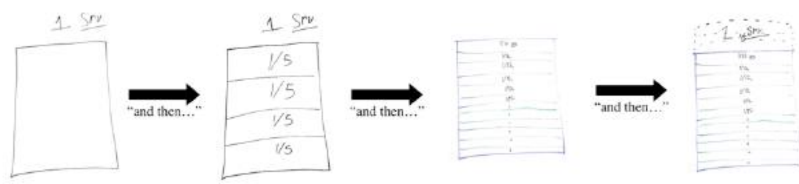
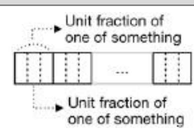
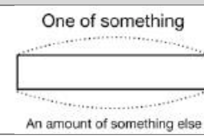
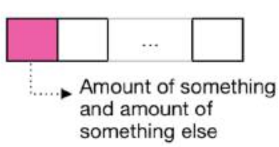
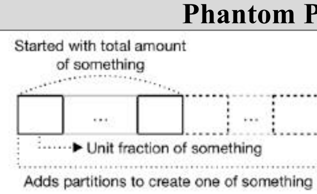


Figure 1: An example of distilling coarse forms

Results

I summarize the four main coarse forms I characterized in Table 1. I then describe how they were used for multiplication problems and illustrating it with student work.

Table 1: Coarse forms characterizing strip diagrams for fraction multiplication

Form	Partitioned Parts	Dual Function of a Strip
Schematic		
Description	Equi-partitioned rectangle with each part partitioned further	A strip, rectangle
Function(s)	Create a particular number of parts	Represent two different quantities where the amount of one quantity is one
	Dual Function of a Part	**Phantom Parts**
Schematic		
Description	One part of an equi-partitioned rectangle	Equi-partitioned rectangle then more parts are added
Function(s)	Part represents an amount of a quantity and another amount of another quantity	Determine the size of a part or partitioned part

Explicitly describing two quantities. As PSMTs drew strip diagrams, they described both the full strip and parts with respect to two quantities, the multiplier and multiplicand. Analytically, I found the Dual Function of a Strip and a Part present in all the strip diagrams. At the beginning of the sequence of lessons, the instructor formally introduced the multiplier-multiplicand definition of multiplication. Throughout the period, the instructor constituted the norm of identifying the group and units in the PSMTs diagrams.

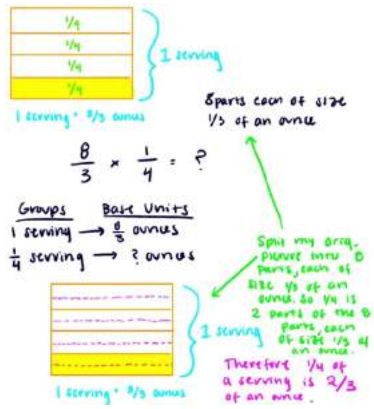


Figure 2: Hannah’s diagram for the Bat Milk Cheese problem

Hannah demonstrated working with two different quantities while solving the Halloween-themed problem, “You had 1/4 of a serving of bat milk cheese. One serving of bat milk cheese is 8/3 ounces. How many ounces of cheese do you have?” She began her drawing by showing one whole serving or one group and showing the size of the group. She wrote out the definition “8 parts each size 1/3 of an ounce.” She then realized she wanted to show eight parts in the strip and noticed she already had four parts. She partitioned each part into two smaller parts to show eight parts. She ended by saying there are two-thirds ounces in one-fourth of a part because there are two parts, each one-third of an ounce in the yellow part indicating one-fourth of a serving.

Determining the number of parts needed. The PSMTs wrestled with the appropriate number of parts required to solve the problems. They thought through the number of parts they created from the multiplicand when it was not divisible by the number of parts they needed.

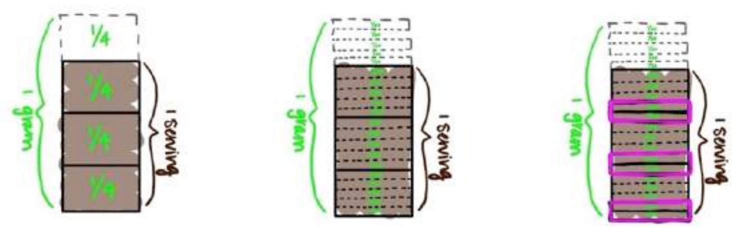


Figure 3: Elizabeth and Jack’s diagram for the Blank Multiplication problem

For instance, Elizabeth and Jack were thinking about the number of parts while working on a multiplication problem, “One serving of ___ is 3/4 ___. You had 2/5 of a serving. How many ___ of ____ did you have?” In this blank problem, the PSMTs were invited to provide their own quantities. During small group discussion, Elizabeth explained that they started with a strip representing one gram partitioned into four and shaded three parts representing three-fourths of a gram or one serving. She wanted to find two-fifths of three-fourths. She partitioned each fourth part five parts then Jack suggested she should “get” two partitioned parts from each parts to get two-fifths of the serving, she highlighted two of the three one-fourth serving partitions. Partitioning of the parts was prevalent in almost all the diagrams created.

Determining the size of a part. The last form, Phantom Parts, emerged towards the end of the sequence of lessons. PSMTs began with a strip representing one group and representing an amount of base units as indicated by the multiplicand. When the multiplier was less than one, the PSMTs they needed to add "Phantom" parts in order to determine the size of the parts. They drew out additional parts to describe the product with respect to the base unit, thus they had to draw a whole base unit to describe the size of the product in terms of base units.

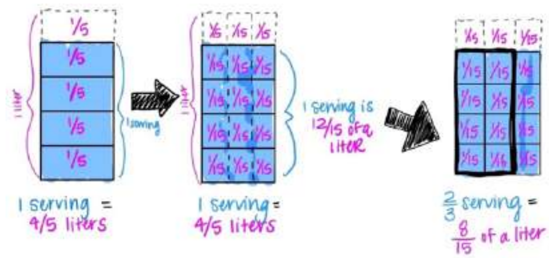


Figure 4: Elizabeth's diagram for the Goblin Goo problem

Consider Elizabeth working on the problem "You had 2/3 of a serving of goblin goo. One serving of goblin goo is 4/5 liters. How many liters of goblin goo do you have?" First, she drew a strip representing both one serving and four-fifths of a liter, similar to Hannah's use of the Dual Function of a Strip in the previous problem. She highlighted four parts to show four-fifths of a liter or one serving as seen in Figure 3. Upon partitioning each fifth into three, she labelled and described each partitioned part as one-fifteenth of a liter. She finally highlighted two-thirds of a serving by highlighting two columns of the partitioned serving to show two-thirds of a serving as eight-fifteenths. While she did not express any reason for changing her diagram from Phantom part to incorporating the Phantom part in the initial strip, the next day while talking to one of the graduate students about this problem, she said, "I think it helps understand how many parts there are of a liter. 'Coz that's why it was confusing to me was putting in in twelfths because that's not twelfths of a liter… You can do much less work if you just understand that there's a pretend liter… just go with liters the whole time. Don't change your wholes last minute."

Discussion and Conclusion

The results of this study provide a characterization of how representations evolve over time. In this case, the forms of strip diagrams evolved. The PSMTs' explanations for multiplication were rooted in two practices—using strip diagrams and a definition of multiplication. Strip diagrams evolved over time to address certain features of both the problem and diagram. By using a quantitative definition for multiplication, they were able to describe parts of the diagram (strips and partitions) with respect to two quantities. Some future steps for both researchers and teachers can be drawn from this report. When analyzing inscriptions, researchers must attend and be explicit about the grain size of the inscription. I have shown how describing coarse forms enabled me to describe continuities and discontinuities between points in time in order to characterize how representations change and potentially teaching opportunities for new forms and functions to emerge. However, although this was helpful for me analytically, such an analysis emerged from the data I had i.e., how these particular PSMTs talked.

Acknowledgments

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PRE-SERVICE TEACHERS' PATTERNS OF QUESTIONING WHILE TUTORING STUDENTS WITH LEARNING DISABILITIES IN ALGEBRA 1

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This study documented changes in the types of questions posed by pre-service teachers (PSTs) who participated in a semester-long professional development (PD) program focused on questioning in algebra. PSTs who participated in the PD—who were conducting 1-1 tutoring for students with learning disabilities during the same time—showed positive changes in the types of questions they posed. PSTs reduced their frequency of closed, leading questions to lead students through solution methods, and they increased their frequency of questions to probe students' thinking, to focus attention on important mathematical ideas, and to establish mathematical relationships.

Keywords: Algebra and Algebraic Thinking, Teacher Education – Preservice, Classroom Discourse, Special Education

The development of positive, productive classroom discourse practices is a challenging component of learning to teach mathematics. Posing questions is one way for teachers to scaffold students' mathematical learning, giving a teacher insight into students' thinking while promoting student autonomy and sense making (Anghileri, 2006). However, posing questions is a complex skill (Hufferd-Ackles et al., 2004; Imm & Stylianou, 2012; Kazemi & Stipek, 2001). For pre-service teachers (PSTs) to pose questions that probe students' thinking, help students focus on important mathematical ideas, and make necessary connections, they likely need consistent, ongoing professional development (PD) and opportunities for reflection. This is especially important for future special educators who may have less training in mathematics content and discipline-specific practices for teaching math.

The purpose of this study was to document changes in the types of questions posed by PSTs who participated in a semester-long PD program focused on questioning in algebra. We worked with four PSTs who were special education majors with a content emphasis in mathematics. We facilitated weekly training sessions, concurrently with which the PSTs tutored students with LD in 1-1 settings. We found that the PSTs became better throughout the PD at asking questions to probe students' thinking and make mathematical ideas explicit.

Research Perspectives

Teacher questioning is one aspect of classroom mathematical activity, and it is difficult to isolate teacher questioning from other aspects of teaching and learning. However, characterizations of the different types of questions that teachers pose—for example, leading (closed) questions, probing questions, or (open) questions to extend students' thinking—have been useful to document features of existing classroom practice (Boaler & Brodie, 2004; Hufferd-Ackles et al., 2004; Moyer & Milewicz, 2002) and to support teachers in improving their questioning practices (Aydogan et al., 2018; Di Teodoro et al., 2012; Piccolo et al., 2008). Boaler and Brodie (2004), for example, established a set of nine categories of teacher questions that shares features with other frameworks. The types of questions that mathematics teachers pose include categories like "leading students through a method," "exploring mathematical meanings," and "orienting and focusing."

Research has shown across multiple contexts that PSTs across grade levels especially struggle with questioning, primarily posing lower-order questions focused on the recall of facts or the steps of a procedure (Diaz et al., 2013; Kaya & Ceviz, 2017; Purdum-Cassidy et al., 2015). It can be difficult

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for PSTs to anticipate how they might pursue extended sequences of questions to probe student thinking (Kilic, 2018; Weston et al., 2018). However, there is some evidence that sustained and supported reflection on the part of PSTs can support improvement in questioning practices, in particular when PSTs work in 1-1 settings with students, and when attention to questioning is integrated with the development of content knowledge and professional noticing (van den Kieboom et al., 2014; Weiland et al., 2014). Thus, it is reasonable to expect that PSTs can benefit from sustained PD in the area of questioning and consistent opportunities to practice questioning in controlled settings.

Data and Methods

Participants

Four PSTs participated in this project for three semesters during a teacher preparation program at a large university. Although the PSTs were special education majors, they had all selected math as their area of focus and were recruited from a mathematics methods course based on their achievement and engagement in the course. In the first year of the project, the PSTs met with the two authors on approximately a weekly basis, for a total of 15 sessions. Most of the time in our sessions was spent developing the PSTs' algebra content knowledge and questioning. Concurrently with our tutor training, PSTs tutored on a weekly basis. The students they tutored were all identified as having LD, and they were enrolled in the first year of a 2-year remedial algebra sequence at a large suburban high school.

Tutor PD

The math content of our PD sessions with the PSTs was primarily related to linear functions and solving systems of equations, because this was the content covered in the associated algebra course for most of the year. Our discussions of questioning proceeded along a trajectory:

- Distinctions between *funneling* sequences (i.e., a sequence of closed questions to lead students towards a solution) and *focusing* sequence (questions that build upon students' contributions and help to focus on key mathematical ideas) (Herbel-Eisenmann & Breyfogle, 2005; Wood, 1998).
- "Buying time" questions, intended for PSTs to slow the pace of conversation and have time to react to student thinking.
- The use of probing questions and how teachers have used probing questions to help students generate correct explanations (e.g., Franke et al., 2009).
- Distinctions between asking and telling, and considerations of when "telling" might be a more appropriate practice than asking closed questions (Baxter & Williams, 2010).

Data Collection and Analysis

In addition to the PD, PSTs provided 1-1 tutoring weekly for approximately 45 minutes per session. Each tutoring session was recorded using a document camera that captured the conversations as any written work produced on the table. We selected three sessions from each tutor to focus our analysis: one session each from the beginning and end of the first year, and one session from the second year of tutoring. For consistency in our sample of data, we selected sessions in which linear functions were the main topic of focus of students' work.

A research assistant transcribed the selected sessions, and we adapted Boaler and Brodie's (2004) categories of questions to code the PSTs' questions across the sessions (Table 1). First, both authors as well as the research assistant coded two transcripts and compared our initial coding. This allowed us to clarify coding criteria. At this stage we also shared our initial coding with an external advisor to get feedback on our coding criteria and improve the validity of our codes. After clarifying our definitions and distinctions between the question types, we each coded two more transcripts to test our reliability. Having achieved sufficient reliability, we divided the remaining transcripts to code. In this final phase, we each highlighted sections that seemed to be edge cases so that we could resolve such cases via discussion and consensus.

Table 1: Types of Teacher Questions, Adapted from Boaler and Brodie (2004)

Question Type	Description
Leading Through a Solution	Questions requiring short answers on the part of the student, generally focused on the steps of a solution.
Exploring Mathematical Meanings	Questions requiring students to establish connections between mathematical ideas.
Probing	Questions that ask students to explain why a particular solution was valid or reasonable.
Orienting and Focusing	Questions that ask students about important elements of a task, to help focus students' attention.
Connecting to Context	Questions about a real-world context, either the given context of a task or introducing a new context.

Results

Table 2 summarizes the PSTs' questions in semester 1 of year 1 in semester 2 of year 1. These two sessions occurred near the very beginning and, respectively, near the very end of our tutor training. (Results from year 2, during which PSTs continued tutoring but no longer received training, will be added prior to PMENA 2020.) Several outcomes are notable in Table 2. First, and most importantly, PSTs substantially reduced their use of "leading" questions with students from the beginning to the end of the tutor training program. Leading questions are those that roughly correspond to initiation (I) questions in an IRE sequence and are generally defined by the fact that they require very short (1-3 word) answers to move a student through a solution procedure.

Table 2: Changes in Tutors' Questions Over Time

	Year 1 Semester 1	Year 1 Semester 2
Leading Through a Solution	86%	68%
Exploring Mathematical Meanings	7%	8%
Probing	4%	15%
Orienting and Focusing	4%	6%
Connecting to Context	0%	2%

With the reduction in "leading" questions, PSTs asked many more questions to explore mathematical meanings, to probe students' thinking, and to focus students on important mathematical ideas. It is notable that, even by the end of the 15-week PD, over two-thirds of PSTs' questions were still leading questions. This is partly explained by the fact that leading questions—by their very definition—typically come in condensed sequences, for example with a tutor posing 3-4 such questions to lead a student through one task or part of a task. Other types of questions are less predictable in nature and create more opportunity for students to direct the discussion.

An Example of PST Questioning

We share one example, from a conversation between Linda (a PST) and a student, Mia, to illustrate the use of questions that went beyond leading questions. This example came from a year 1 semester 2 session. Linda and Mia were together looking at a "worked example," a task in which two linear functions had been graphed (incorrectly) to identify a point of intersection, and Mia needed to determine whether the solution was correct.

Linda: Okay, so did the student graph these correctly? (*leading through a solution*)
Mia: No.
Linda: What did they do wrong? (*probing*)
Mia: For the blue one they went up, well like they went up the y-axis correctly, but they went up-right instead of up-left when they graphed the -3.
Linda: Oh okay so, what did they do with the 3, like what did they do wrong with the 3? (*probing*)
Mia: They went to the right instead of to the left.
Linda: What does it mean when they go to the right? (*exploring mathematical meanings*)
Mia: It is a negative slope.
Linda: When they go to the right? (*leading through a solution*)
Mia: Oh wait no, it's a positive slope when you go to the right.
Linda: Oh okay, so they just forgot the negative.

One phenomenon that is illustrated in this conversation between Linda and Mia was the way in which different questioning types seemed to have a snowball effect. By posing two probing questions, Linda was able to draw enough information out of Mia that she could then use that information to establish more general connections. Just as "leading" questions tend to reproduce themselves, more open-ended questions also beget further open-ended discussion.

Discussion and Conclusion

Prior research has shown how challenging it can be for PSTs to develop productive questioning practices. This work has shown that pre-service special educators can learn to pose more questions that probe and extend the mathematical thinking of students with LD. Even so, leading questions may continue to serve a purpose in 1-1 tutoring sessions as a form of emotional scaffolding (Hord et al., 2018) to maintain students' engagement in their work. While there is not clearly an ideal ratio for the different types of questions tutors should pose during 1-1 work with students, it is promising to see tutors expand their range of questioning in ways that give students opportunities to engage in deep mathematical discussions.

Acknowledgments

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PRE-SERVICE ELEMENTARY TEACHERS NAVIGATING TENSIONS RELATED TO CLASSROOM SOCIAL DYNAMICS THROUGH HYPOTHETICAL TEACHING SCENARIOS

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This study uses hypothetical teaching scenarios as a methodology to explore pre-service teachers' reflective practices about social dynamics in elementary classrooms. Here we unpack tensions that emerged for pre-service teachers as they explored the role of the teacher in responding to a scenario involving third-grade students navigating ideas of mathematical equivalence. In particular, we contrast approaches focused on teacher control with approaches focused on student ownership, attending also to pre-service teachers' emphasis on the individual versus the group. Finally, we share design and methodological implications for the development and use of hypothetical teaching scenarios in teacher education.

Keywords: Teacher Education - Preservice, Elementary School Education, Classroom Discourse, Equity and Diversity

Background and Goals

This research explores how *hypothetical teaching scenarios* (or "case-based scenarios") as a methodology can foster pre-service teachers' reflective practices about social dynamics in classrooms. This paper sits at the intersection of literature on teachers' reflective practices (e.g., Zeichner & Liston, 1996) and case-based scenarios as a research tool for providing authentic windows into the teaching profession (e.g., Sykes & Bird, 1992). Here we explore how scenarios can elicit tensions among pre-service teachers (PSTs) about their views on teaching. One such tension, which has been highlighted in existing literature, centers on classroom approaches focused on teacher control versus student ownership (Stefanou et al., 2004). This tension is knowingly important as new teachers tend to teach in ways similar to how they were taught (Buchmann, 1989), yet mathematics reforms have encouraged a shift toward more student-driven mathematics engagement (National Council of Teachers of Mathematics, 2014). By identifying nuances in this tension through unpacking PSTs' responses to one scenario, we can learn more about how hypothetical teaching scenarios can foster reflective practices in this important area.

A body of work has documented the value of teachers developing *reflective practices* (Zeichner & Liston, 1996), or skills that enable teachers to observe and notice students' social interactions and ideas about content in a classroom. While these skills are important for responding to and building on students' thinking in the moment (Barnhart & van Es, 2015; Jacobs et al., 2010; Stockero, Rupnow, & Pascoe, 2017), learning to notice can be difficult (Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2002), and teachers sometimes focus on their own behaviors at the expense of student thinking (Star & Strickland, 2008). While many approaches exist with related goals (e.g., Nichols, Tippins, & Wieseman, 1997; Rich & Hannafin, 2009), here we explore *hypothetical teaching scenarios* as a mechanism for supporting PSTs' development of reflective practices, particularly related to classroom social dynamics.

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In this work we use one hypothetical teaching scenario to examine PST perspectives on the role of the teacher in addressing issues related to social dynamics in the elementary classroom. Specifically, we explore the tension between traditional approaches in which the teacher maintains control and student-centered approaches in which students drive much of the decision-making and social work of learning (Stefanou, Perencevich, DiCintio, & Turner, 2004). The following two-part research question guides our work: How do PSTs respond to a hypothetical teaching scenario about social dynamics and mathematical equivalence in an elementary classroom? More specifically, what tensions emerge as elementary PSTs engage in reflective practices about mathematics teaching and learning through discussion of the scenario?

Methodology

Case Design

This study builds on the long history of using case-teaching in teacher education (Sykes & Bird, 1992) as a method for supporting teacher candidates in learning a variety of necessary teaching skills and practices for engaging in what Shulman (1992) called "the messy world of practice" (p. xiv). Here we continue the trend of using such cases to capture teaching dilemmas with no clear resolution (Carter, 1999) and to analyze these cases through open-ended qualitative coding of teacher candidates' discussions (Southerland & Gess-Newsome, 1999). In particular, we explore PST discussions that emerged in response to a particular scenario involving three third-grade students working to solve an equivalence problem ($8 + 5 + 4 = 4 + \_\_\_$). The teacher overhears their conversation (Figure 1), in which a hypothetical student ("Pat") tries to get his peers to attend to the location of the equal sign. After reading the scenario, PSTs were prompted to discuss the dynamic among the students: "How do you think the dynamic came about?" and "If you were the teacher, when would you intervene? How? What would you do?"

Rebecca:	I'm not sure what to do. I'm confused. Do I fill in the blank?
Pat:	This is so easy guys! The answer is just 13.
Rebecca:	I don't think it's easy. That was rude Pat.
Gabe:	Plus, I don't think you did it right Pat. I think the answer is 17. Cause 8 plus 5 plus 4 is 17.
Rebecca:	Yeah. That seems smart.
Pat:	You guys are so dumb. You have to pay attention to the equal sign...
Gabe:	Don't act like the boss of us. You always act bossy.
Rebecca:	I think it's 21. Cause I added it all up.
Pat:	If you guys would just listen I could teach you how to do it.
Gabe:	We can figure it out ourselves. Thanks anyway.

Figure 1: Student Dialogue in "Mathematical Equivalence" Teaching Scenario

The dialogue in Figure 1 was adapted from Heyd-Metzuyanim & Sfard (2012) and Langer-Osuna (2011), with a focus on issues of gender in math class interactions. Additionally, the mathematical content for this scenario (equivalence) connects with standards 1.OA.6 and 1.OA.7 in the *Common Core State Standards for Mathematics* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), as well as literature that highlights how noticing the location of the equal sign impacts strategy use and learning outcomes (Alibali, Crooks & McNeil, 2018; Gutiérrez et al., 2018). Further details about the design of this scenario, including the racialized and gendered aspects are discussed in Gutiérrez et al. (2019).

Data Collection and Analysis

Forty-eight elementary PSTs (Mean age = 23.9, *SD* = 4.1) who engaged with the "Mathematical Equivalence" scenario in a math content course for elementary teachers consented to participate. Thirty-three identified as female, two identified as male, and the rest did not report gender identification. The PSTs were randomly assigned to small groups (N=14; 2-4 per group), with 10 minutes to read and discuss the scenario. All discussions were audio recorded and transcribed, and analysis began with an initial process of open coding and data reduction (Saldaña, 2015). The complexity emerging in PSTs' responses related to the role of the teacher in the scenario then led us to identify tensions that emerged around how to address issues related to social dynamics, which we present here along with illustrative examples. Our goal here is to offer interpretations of the tensions we observed and capture the complexity of PSTs' perspectives (Stake, 1995), rather than attempt to systematically characterize all instances of PSTs' reflective practices in the data corpus. By offering rich examples, we aim to demonstrate how this particular hypothetical teaching scenario elicited and promoted critical reflection around crucial topics such as classroom power dynamics.

Findings

Here we examine tensions related to social dynamics that emerged as elementary PSTs discussed the scenario. While some PSTs groups directly grappled with the tensions identified, weighing the benefits and drawbacks of different approaches, other groups explored only one possible pathway. The presence of such wide-ranging approaches across conversations suggests affordances of bringing these approaches into conversation with each other, an implication we take up in our discussion section.

In their conversations, PSTs explored the tension between teacher-led and student-led approaches to addressing issues of power in the mathematics classroom. For example, Group 12 discussed when – and whether – the teacher should intervene in group dynamics. To begin, one PST commented, "I feel like really if I were listening when Rebecca's like that was rude Pat, I feel like I would have been like, Rebecca's right—that is rude, Pat! That's not really like a thing to do because then they're just gonna be wanting to be calling each other out." A second PST noted that "then Rebecca might feel that like you're on her side, like you're picking sides" and questioned, "Or do you kind of just like let them figure it out on their own?" After a brief discussion about lessons learned in another course, the first PST then noted that the latter approach aligns more closely with what the PSTs were taught: "I feel like there's—I mean—you're supposed to encourage them to talk it out." The second teacher concurred and added benefits, saying, "That's what I was thinking. Like let them learn more social skills and stuff."

In this discussion, the PSTs questioned whether they should jump in to let Pat know that his behavior was unacceptable, or whether they should let the students "talk it out" themselves. The PSTs also briefly attended to the goals a teacher might have related to avoiding choosing sides and helping students to learn social skills. This discussion exemplifies the tension between approaches that emphasize teacher control versus student ownership.

A PST in a different group encapsulated this struggle between wanting to take action and letting the students try to handle the situation on their own: "The initial reaction is to, as soon as you hear that negativity, step in. But, sometimes it's better to let that negativity keep going so that they work it out and then you approach at the end and say, 'Okay, how could we have done that better?'" (Group 11) After more discussion, the same PST then elaborated on one challenge of trying to refrain from intervening as a teacher: "It's also hard because in real time, you don't know what comes after each sentence. So you're like, do I stay and wait for this? Do I intervene now? Do I check out another group? I don't know." (Group 11) In both groups, the PSTs acknowledged benefits of letting students "work it out" or "talk it out" and "learn...social skills"; here, however, the PST identified a challenge related to that approach -- that the situation might get worse without teacher intervention ("you don't know what comes [next]") -- identifying one source of tension related to determining the teacher's response.

Within discussions about how to navigate student social dynamics, another issue emerged for PSTs – whether intervention should occur at the individual or group level. In Group 7, a PST started by suggesting that the teacher talk to Pat and "maybe say like, 'Just cause something's easy for you doesn't mean it's easy for everyone else….Can you help explain it to Rebecca?'" Then one of the other PSTs chimed in to suggest, "I think it'd be better to recognize [the] group. You know, I don't think anybody wants to be pulled aside individually." While many other PST groups did not explicitly debate these approaches, disparate ideas were offered ranging from "talk to Pat beforehand" to "go over group work rules [with] the whole class."

Here we see multiple dimensions of how the tension related to the teacher's role in addressing classroom power dynamics played out in PSTs' discussions. PSTs struggled through the issue of when (and if) to jump into conversations and whether intervention should occur at the individual or group level. In both instances they reflected on power dynamics while trying to work out the teacher's role in these interactions.

Discussion

Hypothetical teaching scenarios offer a productive avenue for eliciting tensions experienced by pre-service teachers. We argue that discussion of such scenarios serves multiple purposes: First, the scenarios serve as stimuli that allow PSTs to engage in and build their capacity for reflective practice in which they consider differing approaches to the teacher's role in responding to issues related to social dynamics. Second, hypothetical teaching scenarios can be used as research tools to understand PSTs' perspectives and struggles and to identify points of tension that are ripe for future investigation. Third, these scenarios can serve as teaching tools to elicit different PST viewpoints and then put contrasting perspectives in conversation with each other. While some PST groups in our data corpus explicitly discussed pros and cons of contrasting approaches in a given area, others primarily focused on one potential approach, suggesting benefits of facilitating discussion across groups, an approach we will take in future research. As teacher learning communities often avoid explicit disagreement and discussion of contrasting perspectives (Dobie & Anderson, 2015; Grossman, Wineburg, & Woolworth, 2001), we argue that hypothetical teaching scenarios can serve as a productive resource in teacher education for elevating issues with which PSTs grapple. Furthermore, engaging with such scenarios can help to create a culture of rich discussion around tensions of teaching and learning while also fostering PSTs' reflective practices in ways that can provide opportunities for future learning.

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ERROR PATTERNS IN PROSPECTIVE K-8 TEACHERS' POSING OF MULTI-STEP ADDITION AND SUBTRACTION WORD PROBLEMS

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National and state standards in the US have emphasized the importance of solving and posing word problems in students' mathematics learning for decades. Therefore, it is essential for prospective teachers (PTs) to have the mathematical knowledge necessary to teach these skills to their future students. Unfortunately, little research has investigated how PTs develop problem-posing skills. This study investigated PTs' abilities to pose two-step addition and subtraction word problems in the context of a collegiate teacher education course. The researchers analyzed incorrect problems to identify error patterns among the mistakes made by PTs. By employing thematic qualitative text analysis, the researchers identified eight distinct common error categories. These results can be used to inform teacher education and to adapt tasks and instructional strategies for more effectively helping PTs develop their problem posing abilities.

Keywords: Teacher Education - Preservice, Mathematical Knowledge for Teaching, Instructional Activities and Practices, Number Concepts and Operations

Introduction

The Standards for Preparing Teachers of Mathematics put forth by the Association of Mathematics Teacher Educators (AMTE, 2017) call for beginning teachers of mathematics to "regard doing mathematics as a sense-making activity that promotes perseverance, problem posing, and problem solving. In short, they exemplify the mathematical thinking that will be expected of their students," (p. 9). The standards further indicate that effective mathematics education programs "develop positive dispositions toward mathematics, including persistence and a desire to engage in posing and solving problems," (p. 70). As such, helping prospective teachers (PTs) learn how to pose mathematics word problems should be a goal of teacher preparation programs. Towards this goal, the researchers investigated two research questions:

1. With what frequency are PTs able to write correct, two-step addition and subtraction word problems?
2. What patterns emerge in the errors that arise when PTs write two-step addition and subtraction word problems?

The researchers of this study evaluated addition and subtraction word problems posed by K-8 PTs enrolled in an undergraduate mathematics problem-solving course with the intent of identifying emergent trends of conceptual difficulties. The results of this study can inform mathematics teacher educators (MTEs) in developing targeted and meaningful activities to address PTs' difficulties and support their learning to pose multi-step word problems.

Literature Review

Even though problem posing has been discussed in mathematics education since the 1980s (Brown & Walter, 1983; Kilpatrick, 1987; NCTM, 1989), little has been done to investigate or ensure that teachers are prepared to pose problems to their students. Researchers have dug deeply into the role problem-posing can play in students' mathematical development (Akay & Boz, 2009; Alibali et al., 2009; Sharp & Welder, 2014; Bonotto, 2013; Silver & Cai, 1996; Ticha & Hospesova, 2013), and this work has found that problem-solving skills do not necessarily equate to problem-posing skills.

Furthermore, Alibali and colleagues found that having students write original multiplication word problems "revealed difficulty with the underlying meaning of multiplication" (p. 257). Similarly, Sharp and Welder (2014) found that asking seventh graders to write a division of fractions story problem exposed multiple "areas of limited conceptions that may not have been identified through traditional algorithm-driven assessments" (p. 546).

Since problem posing is an instructional activity that has been shown to benefit student learning by providing instructors insight into students' conceptions, teachers should be prepared to incorporate such activities in their future classrooms. However, to do so, they must first learn how to pose problems themselves. Therefore, it is important that PTs be given opportunities to develop problem-posing skills during their mathematics education preparation.

As MTEs, our attention then turns to how we can efficiently prepare PTs to pose a variety of word problems. The National Research Council tells us that "addition and subtraction are used to relate amounts before and after combining or taking away, to relate amounts in parts and totals, or to say precisely how two amounts compare" (2009, p. 32). Their work and the Common Core State Standards Initiative (NGA & CCSSO, 2010) have highlighted the multiple situations in which addition and subtraction occur by developing a framework for word problems that can be used to categorize them according to their structural differences. This framework produced 14 clearly distinguished categories, which give way to instructional strategies that MTEs can use for guiding PTs in developing problem-posing skills.

Methods

At a tier one research institution in the southern United States, the researchers collected data from PTs enrolled in an undergraduate mathematics problem-solving course that focuses on teaching mathematics through problem solving (Alwarsh, 2018; Bostic et al., 2016; Chapman, 2017; Fi & Denger, 2012). Instructors of this problem-solving course have incorporated a variety of instructional activities and strategies to support PTs in their learning to create original one- and multi-step addition and subtraction word problems that utilize a variety of problem structures. One such task asked PTs to pose four 2-step addition and subtraction problems to match four sets of specified structures (e.g., *change – add to – change unknown and part-part-whole – part unknown*). The data analyzed in this report includes problems posed by PTs in one instructor's course across two semesters. Thirty-seven PTs were enrolled in each semester of the class for a total of 74 PTs. All PTs were enrolled in programs leading to teacher certification in the areas of EC-6 (Generalist) or grades 4-8 mathematics and science, English, or history.

Throughout their coursework, PTs were introduced to the taxonomy of common addition and subtraction situations as identified by the Common Core State Standards (NGA & CCSSO, 2010) as a basis for discussing structural differences between addition and subtraction word problems. After categorizing and solving a variety of one-step word problems, the PTs posed one-step word problems to match each of the 14 possible problem structures. PTs received feedback on the one-step problems they wrote, which were mostly correct. Afterwards, class activities focused on categorizing and solving two-step addition and subtraction word problems. Lastly, PTs were given the aforementioned assignment in which they were instructed to pose four two-step word problems to match given pairs of addition and subtraction problem structures. This assignment resulted in 282 PT-posed, two-step word problems (n=282).

To analyze the 282 word problems, the researchers used thematic qualitative text analysis. First the problems were coded as being correct (n=124) or incorrect (n=158). Next, categories of error patterns identified in the 158 incorrect problems were created both deductively and inductively (Kuckartz, 2014), first at a macro and then micro levels. A temporary category was created based on the number of steps required to solve each incorrect problem, followed by whether the structures of the posed problem matched the assigned structures. As the researchers continued their analyses, they determined whether each new incorrect problem matched an existing category or if a new category was necessary. To test the validity of the devised coding scheme, the researchers re-coded all 282 word problems independently according to their correctness and the categories of error patterns that had emerged. After discussion, agreement was reached for 100% of the analyzed word problems. Lastly, the categories of error patterns were analyzed by the frequency at which they were exhibited by the PTs.

Results

As mentioned, 124 of the 282 word problems correctly provided a scenario that included two situations that matched both of the assigned structures and asked a question that required a two-step calculation utilizing the unknowns from each situation. Fifty-six of the remaining 158 problems correctly posed a valid two-step addition/subtraction question and were only deemed incorrect in this analysis because they simply did not meet the structural criteria of the prompts.

The remaining 102 PT-submitted problems were deemed as having one or more structural errors. The analysis of these errors led to the identification of eight distinct categories of error patterns, dependent upon the number of steps required to solve the problem, the appropriateness of the question(s) asked, and the use of the assigned structures. Due to space restrictions in this report, we will explicate only the most-frequent category of error pattern, problems that only required one step (n=68), and provide examples of the ways in which one-step errors occurred as found in PTs' work. Of the remaining 34 problems, 12 required more than two steps, three required zero steps (as the solutions had been provided within the context of the problems), four required algebra in their solutions, and 15 could not be solved with the given information.

One-step Problems (n=68)

The most common error resulted from PTs who were able to build up two addition/ subtraction structural situations but did not properly utilize the unknown information from the first scenario to form a question that required a two-step calculation (n=68). Sixty-three of these *one-step* problems posed a single question, but the question still failed to connect the two unknowns. These 63 *one-step – one question* problems were further categorized according to whether the PTs used the assigned structures or not. The remaining five *one-step* problems resulted in the posing of two separate questions. Examples of each type are provided below.

One-step – one question – two correct structures (n=31). Thirty-one of the 63 *one-step, one-question* problems built contextual scenarios that correctly matched both of the requested problem structures. However, due to a lack of connection between the two scenarios, the question posed only required one calculation to be solved. For example, one PT submitted the following problem, exhibiting this common error, in response to the second prompt (*change – add to – start unknown; compare – more – bigger unknown*):

> Sarah had some pieces of candy. Four more pieces were given to her, so she had ten pieces of candy total. Amanda had five more pieces of candy than the amount of candy Sarah was given. How many pieces of candy does Amanda have?

This PT provided a scenario that matched the two assigned structures but did not pose a question that would require the solver to utilize the unknown information from the first step as known information in the second step. Specifically, in this example, the unknown information in the first scenario is the number of pieces of candy Sarah starts with, but the second scenario connects the second unknown to the number of pieces Sarah was given. Since this information was provided ("*Four more pieces were given to her*"), the only required step to answer the question is adding four and five to get nine pieces of candy.

One-step – one question – incorrect structure(s) (n=32). The remaining 32 of the 63 *one-step, one-question problems* exhibited the same error as above but were further flawed in the sense that the scenarios posed did not fully match the requested structures. For example, for prompt 3 (*part-part-whole – addend unknown; compare – more – difference unknown*), one PT submitted the problem:

> Sarah has four red shirts and some green shirts. Sarah has two more red shirts than green shirts. How many green shirts does she have?

Again, in this problem, only one calculation is necessary to answer the question posed: four (red shirts) minus two (more red shirts than green shirts) equals two (green shirts), making it a *one-step* problem. However, this problem also exhibits structural issues. The first scenario (*part-part-whole – part unknown)* was not fully developed, as the whole amount was never provided (leaving two pieces of unknown information: the number of green shirts and the total number of shirts). Furthermore, the second scenario created a *compare – more – smaller unknown* situation (when the prompt specified *difference unknown*).

One-step – two questions (n=5). A small subgroup of these *one-step problems* (n=5) contained two independent scenarios and asked two separate questions in an effort to satisfy the two-step prompts. Four of these problems included the correct assigned structures, one did not. The PTs who wrote these problems knew that two steps were necessary to satisfy the given task but showed difficulty in connecting their unknowns into a single question. For example, one PT-submitted the following problem to the third prompt (*part-part-whole – addend unknown; compare – more – difference unknown*):

> Maria has 4 apples and some cherry pies. She has a total of 7 pies. Kara has more pies than Maria. They together have a total of 15 pies. How many cherry pies does Maria have, and how many pies does Kara have?

Discussion

The skill analyzed in this study was the ability of PTs to write addition and subtraction word problems that utilized one unknown in a second scenario to create a two-step problem. The level of difficulty of these problems for solving purposes is quite low, but the necessary skill to create such problems proved quite high with only 44% of the PTs able to correctly formulate a two-step word problem as requested. Alarmingly, a large proportion of the 56% of PTs who were unable to create two-step problems tended toward writing one-step problems with disconnected or incorrectly connected scenarios.

Teachers at every level need to be prepared to create original word problems and support their students in developing problem-posing skills. Our findings are especially concerning given that the participants in this study were being trained to teach mathematics in elementary classrooms yet displayed great difficulty in formulating elementary-level word problems. As we, the researchers, apply the findings to our mathematics problem-solving course, we are using the error pattern framework developed here to facilitate targeted discussions of common errors in the classroom. The findings of this study will inform our development of a task designed to confront common errors head on so that PTs will be more cognizant of effective and ineffective problem-posing strategies. Future research will study the effects of this task.

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CONCEPTUAL AND PROCEDURAL LEARNING IN PRE-SERVICE MATHEMATICS TEACHERS DURING A CONVERSATION

APRENDIZAJE CONCEPTUAL Y PROCEDIMENTAL EN PROFESORES DE MATEMÁTICAS EN FORMACIÓN DURANTE UNA CONVERSACIÓN

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This paper shows a study of the transition from procedural to conceptual learning from a conversational learning approach evidenced by students of pre-service teachers in mathematics in the context of the solution and discussion of a geometrical task. The study was conducted with a group of twelve future teachers and their instructor from a public university. Results found that students are able to move from procedural to conceptual learning when the procedures and understandings of the concepts involved in the task are confronted during the conversation, allowing them to move from the use of the formula of the area to the significance of geometric figures by establishing relationships between their dimensions.

Keywords: Instructional activities and practices, Teacher Education - Preservice, Teacher Knowledge.

In the social construction of knowledge, conversation has an important role in learning since it allows the negotiation of meanings and the emergence of common understandings (Baker, Jensen & Kolb, 2005; Scott, Mortimer & Aguiar, 2006), so learning is conceived as the result of conversations and experiences associated with a topic (Pask, 1976; Vygotsky, 1986; Kolb & Kolb, 2017).

This study is found in the previous approach because it is considered that the promotion of the transition between procedural and conceptual learning in pre-service mathematics teachers is possible through conversation. We are particularly interested in analyzing how to support the transition from procedural to conceptual learning when an instructor speaks with their students about the resolution of a geometrical task.

Authors such as Calcagni and Lago (2018) emphasize the relationship between learning and conversation in classroom interactions and their relationship with educational quality. They point out how the manners of speech in the classroom have implications for the quality of learning of students and propose breaking the typical sequence of conversations in the classroom starting with dialogical conversations. The field of professional development of mathematics teachers also delves into these ideas by exploring the type of learning opportunities that will help to achieve multifaceted and comprehensive conceptualizations in teachers and future teachers (Newton & Poon, 2015; Nagle et al., 2013), as well as the role of dialogue and reflection to generate learning that benefits the transformation of the teaching practice (Aparicio, Sosa, Cabañas & Gómez, 2020; Jaworski, 2006; Saylor & Johnson, 2014).

Procedural and conceptual learning in pre-service teachers in mathematics

Concept and procedure are considered to complement one another in mathematics because their interrelationships transform and expand knowledge (Star, 2005, 2007; Baroody, Feil & Johnson, 2007). Ramsden (1992) proposes that understanding the learning process requires understanding how procedural and conceptual learning lives in the classroom, that is, how a student organizes, proceeds and structures the learning experience and how the student manages to give meaning.

The proposal is to analyze the transition between "know how" (procedural learning) and "know what" (conceptual learning) in pre-service teachers based on the former ideas and the assumption that conversation can be a window for understanding. Understanding such a transition process would shed light on the type of professional learning opportunities that can be achieved by future teachers during conversations among them.

Procedural and Conceptual Learning from the Theory of Conversation

This study is supported by an integrated model between the learning proposal based on the theory of conversation of Pask (1976) and the experiential learning cycle of Kolb and Kolb (2017), as shown in Figure 1. It is recognized that conversational learning transits from a procedural level, characterized by questions and answers focused on how an actual experience is lived (1) and how it is extrapolated (4), to a conceptual level, characterized by questions and answers about the reflection of the experience (2) and its abstraction (3). An initial study reported in Aparicio et al. (2020) shows how conversation and reflection provide opportunities for the development of mathematical and pedagogical knowledge in future teachers when they move between the modes of learning in Figure 1, supported by questioning, exchange and articulation of their procedures and concepts.

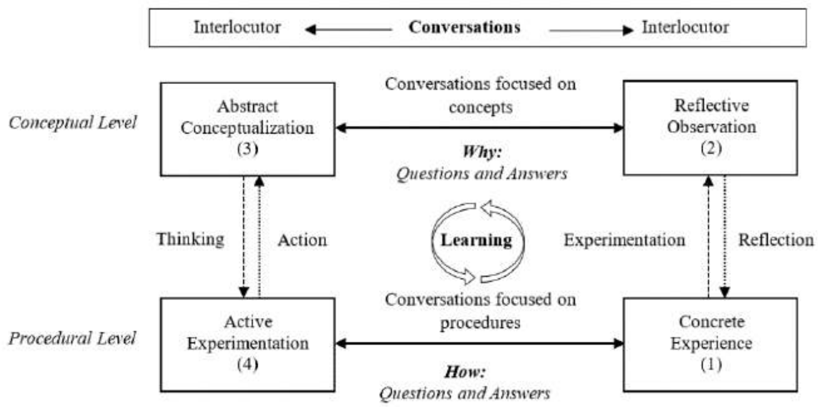

Figure 1: Conversational learning based on the models of Pask (1976) and Kolb and Kolb (2017) [Aparicio et al., 2020]

Method and Analysis

Twelve pre-service mathematics teachers participated in the study when they were taking a course in Didactics of Mathematics in their final year of university training. Two sessions of 90 minutes each were analyzed. During the sessions, the instructor and the students talked about the performance of a geometrical task (see figure 2) which was designed to promote reflection and discussion among the interlocutors about the forms used to solve the task and their arguments. The role of the instructor was to help the students to speak about the "how" and "why" of their procedures and concepts. The sessions were audio taped and transcribed for their analysis, the work done by the students with paper and pencil, and the group conversations conducted with the use of the blackboard were also documented.

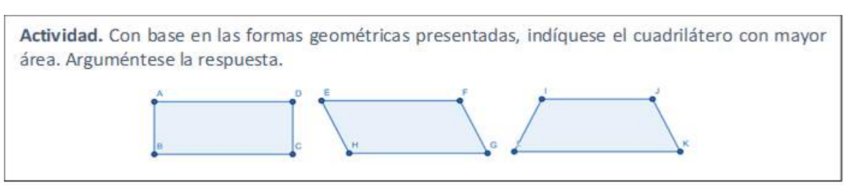

Figure 2: Instrument for data collection

The conversational analysis was carried out in two moments. The first moment examined how the discourse of the instructor and her underlying intent were externalized based on its form and content. The second moment related both aspects of the discourse with the procedural and conceptual learning of the students. The analysis focuses on the identification of references in the statements to techniques, methods, and procedures or to meanings, conceptualizations, and properties of the concepts.

Results

The transition from procedural to conceptual learning was evidenced; however, due to the limited space, this paper only reports the modes of learning through experimentation (1) and learning through reflection (2). Table 1 describes the transition.

Table 1: Relationship between the teaching discourse and conversational learning

		Discourse of the instructor	*Learning modes*
CE (1) Learning by experimentation	Mode	The way to proceed with the task is questioned.	*Procedural:* The spatial reconfiguration is proposed as a technique to solve the task.
	Content	Awareness of how the experience was lived and the selection of the procedure.	*Procedural:* The use of the formula of the area is proposed as part of the procedure
RO (2) Learning through reflection	Mode	a) The geometric content required to solve the task is questioned. b) The reason to use the formula of the area is questioned.	*Conceptual:* The geometrical transformation and the formula of the area are proposed as concepts that allow the task to be solved.
	Content	a) Awareness of how to posit the way in which to proceed with the use of the formula. b) Understandings of the area of the formula of a rectangle are confronted to explain their use in the task.	*Conceptual:* It is recognized that the use of the formula is not enough; it is necessary to signify the geometrical figures by means of the relation between their dimensions.

The transition from procedural to conceptual learning during the conversation of the task has the following sequence. Procedural learning begins with explanations on the technique of spatial reconfiguration which is used to obtain equivalent rectangular forms; subsequently, the formula of the area of a rectangle is used to guarantee the equivalence in the areas of the figures, provided that their base and height measurements are equivalent. This results in a conceptual learning by recognizing and proposing geometric concepts that are the basis for resolving the task; for example, the geometric transformation and the formula of the area. However, the conversation was led to the discussion of the relationships underlying the formula, discussing how to qualify the two-dimensional forms. Therefore, it was recognized that the first thing to do to give meaning to the task and use of the formula during the conversation is to signify the geometric figures by means of the relationship with their dimensions.

The first thing found with regard to the form and content of the conversation that favored such transition was the encouragement to explore ways of proceeding with the tasks. Secondly, opinions are sought on the geometric contents considered essential to its solution. This makes it possible to compare whether the presented concepts give an answer to the task, and finally discuss the understandings of the formula of the area and its contribution to the conceptualization of the area as a quality of flat figures.

Discussion and Conclusion

A contribution of this study is the approach towards a characterization of conversation in the transition between procedural and conceptual learning in pre-service teacher in mathematics. Conversation is characterized, first, by questioning and confronting procedures and concepts used in the resolution of the task. In the second instance, a discussion is held focused on the analysis of the geometric contents that should be moved to solve it, and finally, conversation is led to how to give meaning to the task. The latter opens up a process of negotiation of the proposed ideas and meanings; for example, questions such as: Why is the formula of the area expressed in that way and not in another way? How does it work and what information does it provide on the area concept and its measurement? Why and how is the geometric transformation important for the resolution of the task?

Results found that the characteristics of the conversation allow the participant to freely share their ideas, procedures and understandings of the task and its solution, as well as the contents of area, figure, and measurement, among others. The role of the instructor was as a guide to the reflections on the geometric contents. The conversation is much more reflective when the meanings are shared after being asked for explanations about the reasons why a procedure works and emphasizing the need to consider the meaning of what is done and why it is done.

This type of conversational analysis in the classrooms for training future teachers are necessary to understand with greater precision the limitations and potentialities of conversation in the integration of procedural and conceptual aspects of professional learning.

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CONCEPTUAL AND PROCEDURAL LEARNING IN PRE-SERVICE MATHEMATICS TEACHERS DURING A CONVERSATION

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Se presenta, desde una perspectiva del aprendizaje conversacional, un estudio sobre el tránsito de aprendizajes procedimentales a conceptuales que evidencian profesores de matemáticas en formación al resolver y conversar sobre una tarea geométrica. El estudio se realizó con un grupo de 12 futuros profesores y su instructora de una universidad pública. Se encontró que los estudiantes logran transitar de aprendizajes procedimentales a conceptuales cuando durante la conversación se confrontan procedimientos y entendimientos de los conceptos implicados en la tarea, permitiéndoles pasar del uso de la fórmula del área a la significación de figuras geométricas a partir de establecer relaciones entre sus dimensiones.

Palabras clave: Actividades y Prácticas De Enseñanza, Preparación de Maestros en Formación, Conocimiento del Profesor.

En la construcción social del conocimiento, la conversación tiene un papel importante en los aprendizajes toda vez que permite negociar significados y generar entendimientos comunes (Baker, Jensen y Kolb, 2005; Scott, Mortimer y Aguiar, 2006), de modo que el aprendizaje se concibe como resultado de conversaciones y experiencias relativas a un tópico (Pask, 1976; Vygotsky, 1986; Kolb & Kolb, 2017).

Este estudio se ubica en el enfoque anterior al considerarse que mediante una conversación es posible favorecer la transición entre aprendizajes procedimentales y conceptuales entre futuros profesores de matemáticas. En particular nos interesa analizar cómo apoyar el tránsito del aprendizaje procedimental al conceptual cuando una instructora conversa con sus estudiantes sobre la resolución de una tarea geométrica.

Autores como Calcagni y Lago (2018) destacan la relación entre aprendizaje y conversación en las interacciones del aula y su relación con la calidad educativa. Señalan cómo las formas de hablar en el aula tienen implicaciones en la calidad de aprendizaje de los estudiantes y plantean romper con la secuencia típica de las conversaciones en el aula a partir de conversaciones dialógicas. Estas ideas también se están explorando en el campo del desarrollo profesional del profesor de matemáticas, al indagar sobre el tipo de oportunidades de aprendizaje que ayude a alcanzar conceptualizaciones multifacéticas e integrales en profesores y futuros profesores (Newton & Poon, 2015; Nagle et al., 2013), así como el papel del diálogo y reflexión para generar aprendizajes que beneficien la transformación de la práctica docente (Aparicio, Sosa, Cabañas y Gómez, 2020; Jaworski, 2006; Saylor y Johnson, 2014).

Aprendizaje Procedimental y Conceptual en profesores de Matemáticas en formación

En matemáticas, el concepto y procedimiento son considerados complementarios ya que por medio de sus interrelaciones se logra transformar y ampliar el conocimiento (Star, 2005, 2007; Baroody, Feil & Johnson, 2007). Ramsden (1992) plantea que entender el proceso de aprendizaje requiere entender cómo viven en el aula el aprendizaje procedimental, es decir, cómo el alumno organiza, procede y estructura la experiencia de aprendizaje; y el aprendizaje conceptual, aquello que el alumno logra significar.

De estas ideas y la asunción de que la conversación puede ser una ventana al entendimiento, se propone analizar el tránsito entre el "saber cómo" (aprendizaje procedimental) y el "saber qué" (aprendizaje conceptual) en estudiantes para profesores. Entender tal proceso de transición permitiría esclarecer el tipo de oportunidades de aprendizaje profesional que se pueden lograr en futuros profesores al conversar de una u otra forma con ellos.

Aprendizaje Procedimental y Conceptual desde la Teoría de la Conversación

El estudio se sustenta en un modelo integrado entre la propuesta de aprendizaje basado en la conversación de Pask (1976) y el ciclo de aprendizaje experiencial de Kolb y Kolb (2017), como se muestra en la Figura 1. Se reconoce que el aprendizaje conversacional transita de un nivel procedimental caracterizado por preguntas y respuestas centradas en el cómo se vive una experiencia concreta (1) y cómo se extrapola (4), hacia un nivel conceptual caracterizado por preguntas y respuestas sobre la reflexión de la experiencia (2) y su abstracción (3). Un estudio inicial reportado en Aparicio et al. (2020) se muestra que la conversación y reflexión brindan oportunidades para el desarrollo de conocimiento matemático y pedagógico en futuros profesores si estos transitan entre los modos de aprendizaje de la figura 1, apoyados en el cuestionamiento, intercambio y articulación de sus procedimientos y conceptos.

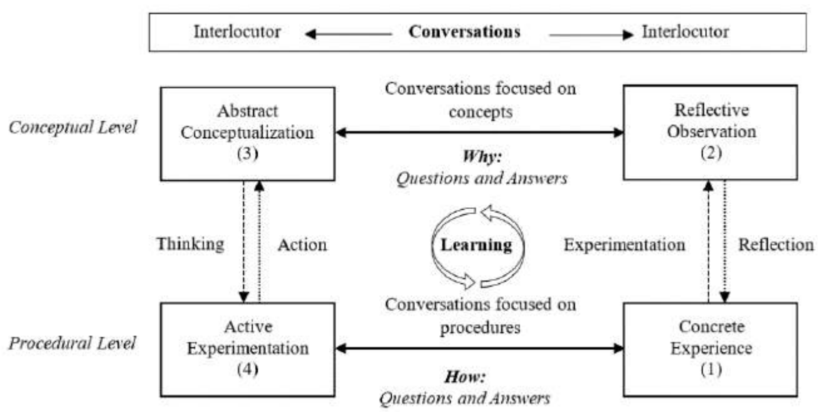

Figura 1: Aprendizaje conversacional basado en los modelos de Pask (1976) y Kolb y Kolb (2017) [Aparicio et al., 2020]

Método y Análisis

Participaron doce futuros profesores de matemáticas que al momento estaban llevando un curso de Didáctica de las Matemáticas en su último año de formación universitaria. Se analizaron dos sesiones de 90 minutos cada una, en las que instructora y estudiantes conversaron sobre la realización de una tarea geométrica (ver figura 2) cuyo diseño buscaba suscitar reflexión y debate entre los interlocutores sobre las formas de resolverla y argumentarla. El papel de la instructora fue apoyar que los estudiantes externaran el "cómo" y "por qué" de sus procedimientos y conceptos. Las sesiones fueron grabadas en audio y transcritas para su análisis, asimismo, se documentó lo realizado por los estudiantes en lápiz y papel y las conversaciones grupales promovidas con el uso de la pizarra.

Actividad. Con base en las formas geométricas presentadas, indíquese el cuadrilátero con mayor área. Arguméntese la respuesta.

Figura 2: Instrumento para la recolección de datos

El análisis conversacional se realizó en dos momentos. En el primero se examinó la manera en que se externalizó el discurso de la instructora, y su intencionalidad subyacente a partir de la forma y el contenido de este. En el segundo, se relacionó ambos aspectos del discurso con el aprendizaje procedimental y conceptual de los estudiantes. Se analizó si lo enunciado aludía a técnicas, métodos, procedimientos, o bien, a significados, conceptualizaciones, propiedades de los conceptos.

Resultados

Se evidenció la transición del aprendizaje procedimental al conceptual, sin embargo, por límite de espacio se reporta lo correspondiente al modo de aprender experimentando (1) y aprender reflexionando (2). En la Tabla 1 se describe la transición.

Tabla 1: Relación entre el discurso instruccional y los aprendizajes conversacionales

		Discurso de la Instructora	*Tipos de Aprendizaje*
CE (1) Aprender Experimentando	Forma	Se cuestiona la manera de proceder ante la tarea.	*Procedimental:* Se propone la reconfiguración espacial como técnica para resolver la tarea.
	Fondo	Se concientiza sobre cómo se vivió la experiencia y la selección del procedimiento.	*Procedimental:* Se plantea la aplicación de la fórmula de área como parte del procedimiento.
RO (2) Aprender Reflexionando	Forma	a) Se cuestiona el contenido geométrico que se requiere para resolver la tarea. b) Se cuestiona la razón del empleo de la fórmula de área.	*Conceptual:* Se plantea a la transformación geométrica y a la fórmula de área como conceptos que permiten dar respuesta a la tarea.
	Fondo	a) Se concientiza sobre cómo argumentan su forma de proceder con el uso de la fórmula. b) Se confrontan entendimientos de la fórmula de área de un rectángulo para explicar su empleo en la tarea.	*Conceptual:* Se reconoce que no es suficiente sólo aplicar la fórmula, sino que se requiere significar las figuras geométricas por medio de la relación entre sus dimensiones.

El tránsito del aprendizaje procedimental al conceptual durante la conversación de la tarea tiene la secuencia siguiente. En lo procedimental se inicia con explicaciones sobre la técnica de reconfiguración espacial con la cual se obtienen formas rectangulares equivalentes en área. Posteriormente se emplea la fórmula del cálculo de área de un rectángulo para asegurar la equivalencia en medidas de área entre ellas, siempre y cuando sus medidas de base y altura sean equivalentes. Lo anterior deriva en aprendizaje conceptual al reconocerse y proponerse conceptos geométricos que son base para dar respuesta a la tarea, por ejemplo, la transformación geométrica y fórmulas de cálculo de área; sin embargo, la conversación fue conducida hacia el debate de las relaciones que subyacen a la fórmula, discutiéndose la forma de cualificar las formas bidimensionales. Debido a ello, se reconoce que, para dar sentido a la tarea y empleo de la fórmula durante la conversación, primeramente, se requiere significar a las figuras geométricas por medio de la relación con sus dimensiones.

Sobre la forma y contenido de la conversación que favorecieron dicho tránsito se detectó que, en primer lugar, se propicia la exploración de las maneras de proceder en la tarea, en segundo lugar, se solicitan opiniones sobre los contenidos geométricos que se consideran esenciales para resolverla. Lo anterior permite confrontar si los conceptos expuestos dan respuesta a la tarea y finalmente, debatir los entendimientos de la fórmula de área y su aporte a la conceptualización del área como una cualidad de las figuras planas.

Discusión y Conclusión

El presente estudio aporta hacia una caracterización de la conversación en el tránsito del aprendizaje procedimental y conceptual de futuros profesores de matemáticas. La conversación se caracteriza, primeramente, por cuestionar y confrontar procedimientos y conceptos usados en la resolución de una tarea. En segunda instancia, se sostiene una discusión centrada en el análisis de los contenidos geométricos que se requieren mover para resolverla y finalmente, la conversación se dirige hacia cómo dotar de sentido a la tarea, en esto último, se abre un proceso de negociación de las ideas y significados propuestos para ello, por ejemplo, se cuestiona ¿Por qué la fórmula del cálculo de área es de esa manera y no de otra?, ¿Cómo funciona y qué información proporciona sobre el concepto área y su medida?, ¿Por qué y cómo la transformación geométrica es importante para la resolución de la tarea?

Se reconoce que las características de la conversación permiten a los participantes compartir libremente sus ideas, procedimientos y entendimientos de la tarea y su resolución, así como de los contenidos de área, figura, medidas, entre otros. La instructora asume un rol de guía para las reflexiones sobre los contenidos geométricos. La conversación es mucho más reflexiva cuando se confrontan significados a partir de solicitarse explicaciones sobre el porqué funciona un procedimiento y se enfatiza la necesidad de considerar el sentido de lo que se hace y porqué se hace.

Continuar este tipo de análisis conversacionales en las aulas de formación de futuros profesores, se considera necesario para entender con mayor precisión, las limitaciones y potencialidades de la conversación en la integración de lo procedimental y conceptual del aprendizaje profesional.

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PROSPECTIVE TEACHERS' STRATEGIES FOR EVALUATING NON-STANDARD ANGULAR MEASUREMENT TOOLS

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Angle measure is pervasive within mathematics curricula from elementary school through higher education. Yet, there is evidence that students and teachers alike experience challenges in quantifying angularity. To promote critical thinking about tools for measuring angles in our geometry courses for prospective elementary and middle-grades teachers, we designed non-standard tools and asked prospective teachers whether these tools would be valid for measuring angles. We present these tasks and our analysis of prospective teachers' justifications regarding the validity of these non-standard tools.

Keywords: Instructional Activities and Practices, Geometry and Geometrical and Spatial Thinking, Quantifying Angularity, Measurement

Measurement is a critical domain of mathematics for students, as well as for prospective teachers (PTs) enrolled in teacher education programs (AMTE, 2017.) Within this domain, angle measure is pervasive within mathematics curricula from elementary school through higher education. Yet, we know from research and teaching that students, as well as prospective and practicing teachers, tend to experience challenges in quantifying angularity (Smith & Barrett, 2017). As mathematics teacher educators, we wanted to help PTs develop productive conceptions of angle measure, so we designed tasks for occasioning conversations about what it means to measure an angle to use in our geometry content courses. In particular, we asked PTs to determine whether several non-standard protractors were valid tools for measuring angles and to provide a justification for their decision. We focused on protractors because, (a) well-prepared beginning teachers are expected to "use measurement tools…[and] are skilled in describing how to select appropriate tools" (AMTE, 2017, p. 78), and (b) U.S. curricula and pedagogy have been critiqued for relying heavily on protractors without sufficiently emphasizing the underlying processes by which a protractor is used to measure angles (Moore, 2012). In this brief report, we present some of these tasks, summarize our analysis of PTs' ways of reasoning about them, and consider implications of these results.

Tasks and Methods

To promote critical thinking about tools for measuring angles in our geometry courses for PTs, we designed a set of five non-standard tools that might be used for measuring angles. Our intention was to design tasks with the potential for encouraging PTs to think about what marks on a protractor might mean and how a tool for measuring angles might be created in the first place, rather than simply taking conventional protractors and the marks upon them as givens. We refer to the tools we created as *funky protractors* (Hardison & Lee, 2020); these tools are the angular analogue of the "strange" and "broken" ruler tasks others have used to occasion reflection on measuring lengths (e.g., Smith, Males, Dietiker, Lee, & Mosier, 2013; Dietiker, Gonulates, & Smith, 2011). For each funky protractor we designed, we altered one or more features to differentiate a conventional protractor from a funky one (e.g., unequally spaced linear or angular intervals between markings, non-standard shape, etc.). We intentionally designed some funky protractors to be valid tools for measuring angles and some to be problematic for measuring angles (from our perspective). Each funky protractor featured points along the boundary numerically labeled in 10° increments as well as one larger point suggesting a position for placing the vertex of an angle to be measured. Here, we focus on two funky

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protractors, which are both valid tools for measuring angles (Figure 1), and our analysis of prospective teachers' responses to these tasks. In this study we address the following research questions: (1) What decisions do prospective teachers make regarding the validity of potential angular measurement tools? (2) What strategies do PTs use to justify these decisions at the beginning and end of the course?

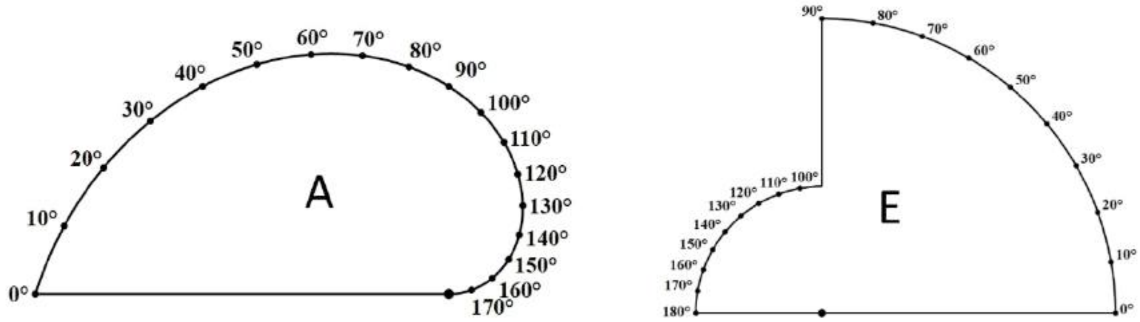

Figure 1: Two Valid Funky Protractors

Participants and Implementation

We implemented the funky protractors tasks with PTs enrolled in three sections of a geometry content course at a large public university; each section was taught by one of the authors. In the second week of the semester, 45 of the PTs evaluated the validity of four funky protractors, including Protractor A (Figure 1), as part of a written homework assignment following a lesson on angle measurement informed by the first author's prior research (Hardison, 2018) and principles of quantitative reasoning (Thompson, 2011). For further details regarding this lesson, see Hardison & Lee (2019). In addition to evaluating the validity of each funky protractor, we asked PTs to explain in writing why each funky protractor was, or was not, a valid tool for measuring angles. After collecting the written responses, we had a whole-class discussion in which PTs discussed their strategies for evaluating the validity of the funky protractors. Fourteen weeks later, 47 of the PTs evaluated the validity of Protractor E (Figure 1) and provided a written justification for their decision as part of their final written exam for the course. We analyzed responses from both the Week 2 homework assignment and final exam.

Analysis

We coded PTs' written responses for each protractor along two dimensions: validity and justification. We first coded whether PTs determined each protractor to be valid, invalid, or if they failed to make a determination of validity (noncommittal). Then the first author used open coding (Strauss & Corbin, 1998) to establish a set of justification codes for characterizing the rationales PTs wrote to support their decisions. When a stable set of codes was established via iterative analyses, all responses were coded independently by both authors and compared; discrepancies were discussed until consensus was reached. A single validity code was assigned to each response; responses could receive multiple justification codes. Descriptions of the justification codes are provided in the findings section below.

Findings

Regarding the validity of protractors, on the Week 2 homework assignment, only two of the 45 PTs (4%) identified Protractor A as a valid protractor; the remaining 43 PTs (96%) concluded Protractor A was invalid. In contrast, on the final exam 77% of PTs identified Protractor E as valid, and 21% of PTs deemed Protractor E invalid (see Table 1).

Table 1: Prevalence of Validity Codes for Protractors A and E

Protractor (Week)	# Valid (%)	# Invalid (%)	# Noncommittal (%)
A (Week 2; n=45)	2 (4%)	43 (96%)	0 (0%)
E (Week 16; n=47)	36 (77%)	10 (21%)	1 (2%)

To support their decisions regarding validity, PTs provided a variety of justifications, from which we abstracted six broad categorizations: *attending to measurable attributes, attending to particular angles, attending to shape features, attending to location, using a standard protractor,* and *other*. We assigned *attending to measurable attributes* to justifications indicating attention to extents of successive instantiations of an attribute; for example, justifications reliant upon checking whether the tool indicated a consistent unit received this code. Furthermore, we established subcodes denoting the quantity indicated: *angularity*, *distance between marks*, and *radial distance to marks*. A response received the *angularity* subcode if the PT attended to angular units (e.g., Figure 2, left) and the *distance between marks* subcode if the PT attended to the distance between marks on the boundary (e.g., Figure 2, right). The *radial distance to marks* subcode was assigned when justifications indicated a PT was considering whether marks along the boundary were equidistant from the suggested vertex position. Although this is equivalent to checking whether the boundary formed a circular arc, PTs were not necessarily aware of this. When the particular quantity could not be inferred, we assigned a fourth subcode: *ambiguous quantity*; for example, this subcode was assigned when justifications referred to "even spacing" without further elaboration or annotations. From our perspective, an appropriate justification involves attending to whether the protractor can be used for counting successive angular units of a specified size (i.e., subcode *angularity*).

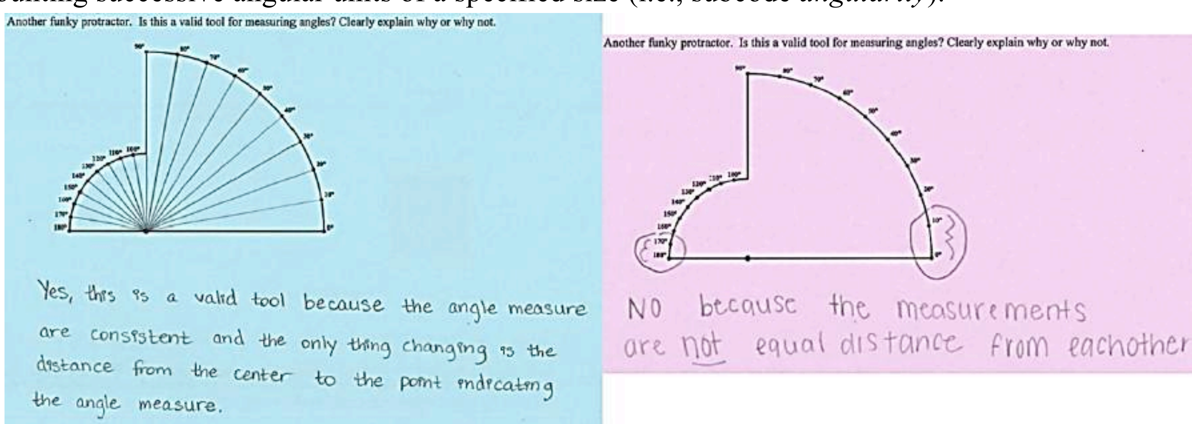

Figure 2: Justifications Coded as Attending to Measurable Attributes: Angularity (Left) and Distance Between Marks (Right)

We assigned *attending to particular angles* if a PT's justification was rooted in the measurements of one or more specified angles. For example, some PTs argued that Protractor E was valid because it was appropriate for measuring right and straight angles without indicating whether the protractor would be appropriate for measuring other, arbitrary angles. Justifications were coded as *attending to shape features* when PTs claimed that protractors were valid or invalid based on the shape of the protractor's boundary or the symmetry of the protractor. Justifications referencing position or location were coded as *attending to location*. For example, some PTs argued Protractor E was invalid because the suggested vertex position was not located on the midpoint of the protractor's straight side. *Using a standard protractor* was assigned to responses indicating that a PT physically superimposed a standard protractor atop a funky protractor to evaluate its validity. Finally, justifications that were unclear or did not fit into any of the aforementioned categories were coded as *other*. Codes for PTs' justifications for each protractor are summarized in Table 2; percentages do not sum to 100% because some responses received multiple justification codes.

Table 2: Prevalence of Justification Codes

Protractor (Week)	Measurable Attributes	Particular Angles	Shape Features	Location	Standard Protractor	Other
A (Week 2; n=45)	24 (53%)	6 (13%)	23 (51%)	4 (9%)	1 (2%)	6 (13%)
E (Week 16; n=47)	30 (64%)	9 (19%)	2 (4%)	4 (9%)	4 (9%)	11 (23%)

Table 3: Prevalence of Measurable Attribute Subcodes

Protractor (Week)	Angularity	Between Marks	Radial Distance	Ambiguous Attribute
A (Week 2; n=45)	1 (2%)	19 (42%)	1 (2%)	4 (9%)
E (Week 16; n=47)	11 (23%)	9 (19%)	0 (0%)	10 (21%)

As shown in Table 2, the majority of PTs (53%) attended to measurable attributes in their justifications for Protractor A. However, as indicated in Table 3, distance between marks along the boundary was the most prevalent attribute indicated in justifications for Protractor A; only one of 45 responses received the *attending to angularity* subcode for Protractor A. The majority of PTs (51%) also attended to shape features, which are irrelevant from our perspective, when evaluating the validity of Protractor A. In contrast, when evaluating Protractor E on the final exam only 4% of PTs attended to shape features. The percentage of justifications coded as attending to measurable attributes increased to 64% for Protractor E with 23% of all responses indicating attending to angularity; additionally, a lower percentage of responses (19%) indicated attending to distance between marks along Protractor E's boundary.

Concluding Remarks

In closing, we are encouraged by the increase in the percentages of PTs giving appropriate validity determinations and attending to angularity over the course of the study, as well as the decrease in the percentage of PTs attending to shape features; however, differences in the design of Protractors A and E merit cautious interpretation regarding these tasks' impact on PTs' content knowledge. More research is needed to understand how to better support PTs' in developing productive quantifications of angularity. From our perspective, the funky protractors activity and the accompanying classroom discussion afforded opportunities for PTs to think critically about the essential features of valid angular measurement tools. It also afforded opportunities for us as instructors to gain insights into PTs' thinking about measuring angles via the features to which they attended in their justifications.

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PRE-SERVICE TEACHERS' DEVELOPING INSTRUCTIONAL STRATEGIES FOR ENGLISH LANGUAGE LEARNERS IN SECONDARY MATHEMATICS

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This study describes the knowledge and skills pre-service teachers (PSTs) identify as important for teaching secondary mathematics to ELLs before and after their credential program. Preliminary results show that most of the PST's (73%) initially described knowledge and skills globally with few specifics on implementation. During follow-up nearly half (45%) of the PSTs described more organized and specific ways to support learning for ELLs. Secondary analysis reveals various approaches to supporting ELLs, including 1) making mathematics accessible, 2) treating everyday language and experiences as resources, and 3) "engag[ing] students in mathematical practice" (Moschkovich, 2013, p. 49).

Keywords: Teacher Education-Preservice, Teaching Tools and Resources, Equity and Diversity

This study is part of a larger research project, Science and Mathematics Teacher Research Initiative (SMTRI), which seeks to explore novice secondary school teachers' beliefs, knowledge, and skills to provide effective mathematics instruction to culturally and linguistically diverse students. Participants in SMTRI are graduates of the Cal Teach program representing five UC campuses. This study is concerned with a subset of data from one campus. Previous research has documented that many teachers feel underprepared to teach mathematics to English language learners (ELLs) (de Araujo, Roberts, Willey, & Zahner, 2018). Therefore, this study identifies the ways pre-service teachers (PSTs) take up and develop ideas and practices for supporting ELLs in secondary mathematics classes. This study explores the research question: in a credential program that emphasizes the integration of content and language instruction, do PSTs ideas about knowledge and skills related to teaching ELLs change? If so, how?

Conceptual Framework

Research-based guidelines for equitable mathematics teaching practices for English Language Learners (Moschkovich, 2013) framed the analysis of the interviews. In particular, we considered the following guidelines because of their alignment with aspects of the SMTRI intervention: Engage students in the eight CCSS for mathematical practice, Keep tasks focused on high cognitive demand, conceptual understanding, and connecting multiple representations, Facilitate students' production of different kinds of reasoning, and Focus on language as a resource for reasoning, sense-making, and communicating with different audiences for different purposes (Moschkovich, 2013).

The study framed the development of teaching practices with a teacher development learning progression that describes a trajectory through four stages and considers the integration of English language development alongside content learning: 0 = *Not Present* (Rule-based or inflexible view of teaching practices), 1 = *Introducing* (global approach to teaching practices), 2 = *Implementing* (organized, planned approach to teaching practices, which includes probing, scaffolding, or connections to students' experience), and 3 = *Elaborating* (teaching practices are flexible and responsive to context) (Adapted from Stoddart, Pinal, Latzke, & Canaday, 2002).

Methods

Participants in this study include 11 PSTs from two cohorts in a single-subject master-credential program for mathematics from one of the five UC campuses. Each participant was interviewed at the beginning and end of their credential program. For this study, the authors solely focused on the sections and questions that were specifically related to ELLs.

Interviews were initially coded using a rubric adapted from the Math Classroom Observation Rubric (MCOR) (see Stoddart, Pinal, Latzke, & Canaday, 2002). The MCOR was developed to characterize mathematics teacher instruction in alignment with the Common Core State Standards. The MCOR characterizes four aspects of teacher practice and pedagogy: 1) mathematics sense-making through applied math/engineering practices, 2) mathematics discourse, 3) English language and literacy development, and 4) contextualized mathematics activity. For each of these categories, the interviews were coded on a scale of 0-3. Differences in scores on pre- and post-interviews were then evaluated to determine changes in participants' responses. Interviews were coded by both authors and scores were calibrated to reach 100% inter-rater agreement. Secondary analysis of the interviews included purposeful selection of four PSTs within three categories: 1) highest overall scores, 2) most improved scores, and 3) scores that decreased. The interviews from these four PSTs served as case studies for more in-depth analysis.

Results

Increased Implementation

During the initial interview, most of the PST's (73%) described knowledge and skills globally with little to no specifics on implementation. For instance, 7 out of 11 PSTs identified using "multiple representations" to support ELLs yet did not explain how or why they would use multiple representations. One PST said, "I think multiple means of representation and having multiple points of access of information at the same time." Another PST similarly explained,

> It's kind of the same supports that you use working with all types of students with IEPs is or just different learning types or abilities. It's the UDL lesson plan with the multiple modes of representation and engagement and expression. You're just trying as many different things as possible to try to activate learning and understanding within all your students. (Transcript, 2020)

During the follow-up, nearly half (45%) of the PSTs described more organized and specific ways to support mathematics learning for ELLs. Out of the eight participants who scored a "1" during their initial interview, two participants increased their scores by one point and two participants increased their scores by two. One participant scored a "2" on their initial interview and improved by one point. One participant's score decreased by one point.

Among participants who described more organized and specific plans for supporting ELLs, general ideas that were discussed initially became more organized and flexible. Illustrating this, one PST mentioned giving students time to practice language during mathematics instruction: "I think group work and discussing among peers is important as well because I think another important part of language development comes from talking and listening [...] so they all have to have that practice of using that academic language." In the follow-up interview, the same PST elaborated upon this idea of "practice" through their descriptions of having ELLs practice in a "safe way" and giving students opportunities to practice discourse in a "low-stakes environment" by using think-pair-shares and reflections. This PST then described, "in a math classroom you are trying to get the math across and not be so worried about that their response isn't in perfect English to you, but that they understand the mathematical concepts that are happening." Other PSTs discussed similar ideas across their pre- and post-interviews (e.g. multiple representations, getting to know the students, supporting practice with math and language, etc.) and the description of these descriptions often became more specific and organized during the follow-up.

Various Approaches to Supporting ELLs

Secondary analysis of PSTs with the most improvement and the highest overall score reveals various approaches to supporting ELLs. Approaches that emerged across the focal cases include 1) making mathematics accessible, 2) treating everyday language and experiences as resources, and 3) "engage students in mathematical practice" (Moschkovich, 2013, p. 49).

Making mathematics accessible. Across the three focal cases, each PST identified ways of making mathematics accessible. Case Study 1 (CS1) discussed getting to know where students came from including their "culture" and "home language" with the rationale that knowing this information supports "students [to] see themselves being involved in the curriculum and being involved in the classroom and that their language is validated and their culture is validated." Other ways CS1 discussed making mathematics accessible included using "discovery learning", supporting "students' understanding of a concept before introducing the term", using a familiar "context", and drawing on "peers" as resources to support learning. Similarly, Case Study 2 (CS2) discussed the idea of facilitating "practice" with mathematics and language. Initially CS2 described "practice" in terms of "talking and listening" and reasoned that students need "to practice their language skills." During the follow-up, CS2 discussed "practice" with more organization and flexibility. CS2 discussed the importance of practicing mathematics and language in a "low stakes environment" with peers and with the guidance of other instructional supports such as "think-pair-shares" and "personal reflection." CS2 stressed the idea that mathematics should be the focus, not correctness of vocabulary, as students are making meaning for concepts. Moreover, CS2 acknowledged that students need time "to process and think."

Treating everyday language and experiences as resources. An approach that consistently emerged across the focal cases included treating everyday language and students' experiences as resources for learning. Illustrating this, Case Study 3 (CS3) described drawing on various perspectives, bringing in context from students' "home lives that relate[s] to the subject matter," and being "flexible" with instruction. Further, CS3 discussed giving students various opportunities to draw on familiar context and their linguistic resources to develop conceptual understanding while supporting increased precision with ideas and terminology. CS3 mentioned, "If students are able to informally talk to each other about concepts and their initial ideas, then they can get more and more precise [...with] explaining, justifying their ideas." CS3 focused on ideas related to contextualizing mathematics activities in a way that reflects the lives and resources that ELLs bring with them to the classroom. Similarly, CS1 elaborated upon the idea that students should use be able to use their home language to support understanding when they described,

> Because mathematics requires students to explain one's thinking a lot of the times, if students bring to the class more proficiency in another language, then using that language to explain mathematical thinking and to process and to think is a really, really great way for them to build conceptual understanding. (Transcript, 2020)

In line with an elaborated view of instructional practices, the focal cases explicitly discussed connections to students' lives and activities and discussed language as a resource.

"Engage students in mathematical practice" (Moschkovich, 2013, p. 49). Two of the three focal cases identified teaching techniques that made explicit connections to supporting students' engagement in mathematical practices. Such techniques included attending to reasoning, eliciting justification and explanations, supporting negotiating, and supporting precision. CS3 described their approach to teaching as trying to uncover the "thinking behind it" as they talked about engaging with ELLs. Specifically, CS3 discussed giving assessments to ELLs where students could "come in and just talk to me and explain verbally or, you know, with a sketch." CS3 talked about creating a classroom culture where students "informally talk to each other about concepts and their initial ideas, then they can get more and more precise." These ideas reflected attending to the reasoning, not the correctness of vocabulary, while supporting students to gradually move towards precision of ideas and vocabulary. In a similar fashion, CS1 described modeling justifications and explanations and supporting students in this practice. As they explained, "you're constructing viable arguments that enforces a lot of communication and when you're communicating about your ideas and when you're defending your answers and you can explain something, that's when you really get a strong grasp on understanding the concept behind things." These two PST discussed instructional strategies that reflect an understanding of the mathematical practices as well as concrete ways of supporting engagement with the practices specific for ELLs.

Discussion

Specific approaches to mathematics instruction for ELLs emerged as a part of the secondary analysis of the focal cases. These approaches include: 1) making mathematics accessible, 2) treating everyday language and experiences as resources, and 3) "engage students in mathematical practice" (Moschkovich, 2013). These approaches align with other works looking at mathematics teaching and ELLs (e.g. Bunch, 2014; Moschkovich, 2013) that can be enacted by monolingual and multilingual teachers. Moschkovich (2013) highlights the importance of supporting mathematical reasoning, conceptual understanding, and discourse and broadening participation for ELLs in mathematics classes. Further, "to support mathematical reasoning, conceptual understanding, and discourse, classroom practices need to provide all students with opportunities to participate in mathematical activities that use multiple resources to do and learn mathematics" (Moschkovich, 2013, p. 46). The approaches that were discussed by the focal cases in this study reflect alignment with these. The consistency of these practices and explicit references to classes and professors throughout the interviews suggest that these PST are taking up and developing the ideas that were presented to them during their credential program. Since improvement was not consistent across PSTs, and one PST scored lower on the follow-up interview, this work also shows that other factors (beyond the scope of this study) may contribute to the ideas PST have about mathematics instruction with ELLs. This has implications for PST education that future teachers may need additional support, outside of the classes they are taking, to fully develop strategies for supporting ELLs in secondary mathematics. PSTs in secondary mathematics would benefit from additional efforts by Cal Teach and credential programs to examine the role of language when learning mathematics and reflect on beliefs about ELs (NASEM, 2018). This study is limited by small sample size and limits inherent to interview data.

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PRE-SERVICE SECONDARY MATHEMATICS TEACHERS' EVOLUTION OF COMMUNALLY AGREED-ON CRITERIA FOR PROOF

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Developing communally agreed-on criteria for proof in a mathematics classroom has been found to empower pre-service secondary mathematics teachers' (PSMTs') learning of proof. To date, we do not know how creating class-based criteria for proof throughout a semester-long course with a focus on secondary school mathematics can promote PSMTs' understanding of proof. In this paper, we reported PSMTs' evolution of what constitutes proof by comparing their initial and revised class-based criteria for proof and investigating their videotaped lessons and video transcripts. Results indicated that PSMTs perceived mathematical values and norms of what counts as proof in their mathematics classroom community as the semester progressed.

Keywords: Teacher Knowledge, Reasoning and Proof, Pre-Service Teacher Education, Instructional Activities and Practices

Despite the importance of proof in school mathematics, pre-service secondary mathematics teachers' difficulties with proof are well-documented (e.g., Ko & Knuth, 2013; Yee, Boyle, Ko, Bleiler-Baxter, 2018). One of the primary challenges pre-secondary school mathematics teachers (PSMTs) face in understanding proof is that they might not perceive or accept mathematical values and norms with respect to learning proof in their classroom community (Dawkins & Weber, 2017). To address this challenge, an emphasis on teaching and learning proof as a social, negotiated, and sense-making process has been found to promote PSMTs' understanding of what constitutes proof and to enhance their ability to evaluate and construct proofs (e.g., Yee et al., 2018). To date, we have only found that Bleiler, Ko, Yee, and Boyle (2015) explicitly shared how undergraduate students in a transition-to-proof course developed, revised, and polished their communal criteria for proof throughout the entire semester. Given that mathematics teachers' knowledge for teaching proof has an impact on their instructional practices (e.g., Bieda, 2010; Stylianou, Blanton, & Knuth, 2009), there is a need to provide insight into how PSMTs' communal criteria for writing mathematical proofs is evolved throughout the semester-long course with an emphasis on secondary school mathematics. More specifically, this study investigated what PSMTs' initial communally agreed-on criteria for proof were and how they evolved throughout the semester.

Theoretical Framework

In the mathematics community, mathematicians actively engage in social practices to negotiate their agreement on the validity of acceptable proofs in the mathematics community (Harel & Sowder 2007). Along with this view, Stylianides's (2007) definition of proof incorporates a focus on the set of statements (i.e., definitions, theorems), the appropriate forms of argumentations, and representations accepted and understood within a particular mathematical community, which shows the general case will be always true without exception. However, research has not adequately investigated how engaging in this type of mathematics classroom community can facilitate evolution of PSMTs' understanding of proof in a course with a focus on secondary school mathematics. To address this research gap, attention must be paid to PSMTs' communally agreed-on criteria for proof in such courses throughout the entire semester.

Methodology

This study was conducted in an elective course focused on proof in secondary school mathematics at a Midwest University in the United States. The instructor (first author) selected, modified, or developed mathematical tasks in the domains of algebra, geometry, and number theory for the course. These three content domains are all vital to and pervasive in secondary school mathematics. In order for PSMTs to take ownership of their communally agreed-on criteria for proof and promote their self-regulation in learning proof throughout the semester, the instructor designed the course following the principle of the before-during-after (BDA) proof instructional sequence (see Ko, Yee, Bleiler, & Boyle, 2016 for detailed information about this three-part proof lesson plan and implementation). There were nine undergraduate students enrolled in the course, and only one PSMT, who transferred from another four-year college, had not taken any of the required proof-intensives courses for his major. The primary sources of data for this paper were the PSMTs' class-developed lists of writing good proofs, as well as the video recordings and their transcripts. The videotaped sections were transcribed by a research assistant and were validated for their accuracy by either the second or the third author.

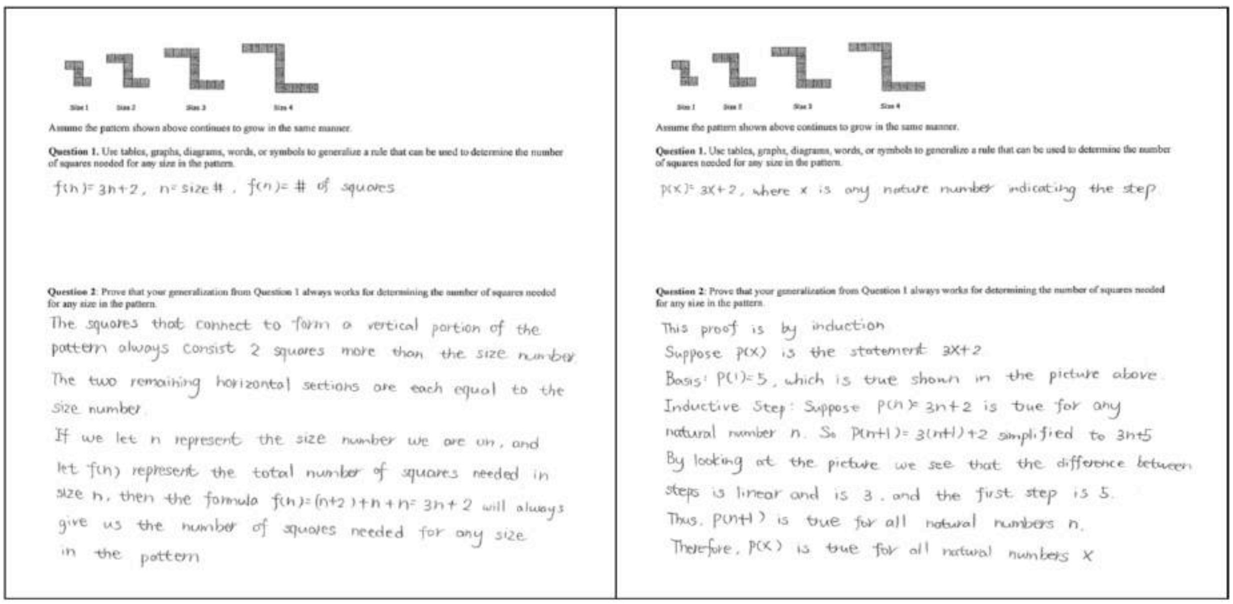

Figure 1: Two Sample Arguments for the Regina's Logo Problem

Results

Followed by the BDA instructional sequence, the PSMTs were asked to evaluate the validity of instructor-selected arguments (see the sample arguments of the Regina's Logo problem adopted from Seago, Mumme, and Branca, (2004) depicted in Figure 1). Also, the instructor served as the representative of the mathematics community to ensure the PSMTs' proposed characteristics for proof were acceptable. Then the whole class determined and ordered that generalization, logical order, correct terminology, clear and precise explanations, and identify given are the five most crucial characteristics of writing good proofs. The PSMTs then discussed their descriptions for each proof criterion as a whole class and came up with an initial list of writing proofs (see Figure 2).

Throughout the entire semester, the PSMTs had two opportunities to modify the initial list of writing good proofs. The first revision happened during the third week of the semester when the whole class did not come to a consensus on the validity of one instructor-chosen argument of the

Sticky Gum problem (Fendel, Resek, Alper, & Fraser, 1996) shown in Figure 3 according to their original proof writing rubrics. For example, Renee suggested that "we need to add something like correct and enough explanations for each step." For instance, Vivian said, "I was thinking maybe under [the] identifying the given [category]." Then the whole class agreed to change the description of the "Identify Given" category as "writing all given information that is pertinent to the proof." During the same time, some PSMTs also suggested that variables and symbols should be added to the "Correct Terminology" category (see Figure 2).

Proof Criterion and Its Order	Description of Each Proof Criterion		
	Initial List (Weeks 1-2)	First Revisited List (Weeks 3-7)	Final List (Weeks 8-15)
1. Generalization	Have to be true for every scenario. No examples.	Have to be true for every scenario. No examples.	Have to be true for every scenario. No examples.
2. Logical Order	Progresses in a sensible manner. Each step needs to build from the previous step(s) and build to the conclusion.	Progresses in a sensible manner. Each step needs to build from the previous step(s) and build to the conclusion.	
2. Logical Order/Clear and Precise Explanations			Progresses in a sensible manner. Each step needs to build from the previous step(s) and build to the conclusion. Correct and sufficient explanations for each step. No unnecessary information.
3. Correct Terminology	Have to use the correct theorems, notation, rules, etc.	Have to use the correct theorems, notation, rules, variables, symbols, etc.	Have to use the correct theorems, notation, rules, variables, symbols, etc.
4. Correct and Precise Explanations	Correct explanations for each step.	Correct and sufficient explanations for each step. No unnecessary information.	
4. Assumption of Knowledge			Only assumed knowledge previously discussed/defined in our course.
5. Identify Given	Writing all the things that we are given at the start.	Writing all given information that is pertinent to the proof.	Writing all given information that is pertinent to the proof.

Figure 2. Initial and Evolution of the Communal Criteria for Proof.

We know that we need to have k number of gumballs, where k represents the number of kids, so that each child has one. We also know that each child wants to have the same color. If we have c number of colors, then worst-case scenario would be that each time we use the machine, we receive a different color gumball. However, after c number of uses, we would have either multiple of at least one color or one of each. Therefore, if we only had one of each color, after the next repetition, we would have more than one in at one color. Following this pattern, we would need to keep using the machine as many times as there are colors multiplied by one less than the total number of kids. This would get us, in the worst-case scenario, one matching gumball for every child except the last one. In fact, if every turn of the machine produced a different color until all colors had been used and repeated this process continuously, we would, at this point, have enough of each color for every kid except one. We would, therefore, only need to use the machine one more time to find a match for the final child. From this information, we can extrapolate that c(k-1) represents the number of times we need to use the machine in order to have a full set of colors for all of the children except for one; adding one more turn would fulfill the request of the stingy children by giving us a full set of one color. Therefore, the expression c(k-1)+1 satisfies the conundrum for any number of children with any number of colors.

Figure 3: A Debated Argument for the Sticky Gum Problem.

As the semester progressed, PSMTs had another opportunity to revisit and modify the proof rubrics. When negotiating the validity of the sample arguments for the statements, "Suppose m, n, and p are positive integers. If m is a factor of n, and m is a factor of p, then m is a factor of $n + p$," some PSMTs pointed out that one of the biggest problems of the first revised proof criteria is clear and logical explanations. Given that not all the PSMTs in this class had completed the required proof-intensive courses for their major, they discussed how much information they should include in each argument to be considered as a proof based on their class rubrics. For example, Ethan explained that their group added the assumption of knowledge category because we "can't just assume that people know things that could have been learned in other classes or in other settings."

After the whole class discussion, all the PSMTs decided to keep the top four and the last characteristics of writing proofs and to combine the other two categories, "Logical Order" and "Clear and Precise Explanations," into one criterion. They also added the fourth criterion pointing out the importance of writing proofs that only used definitions, theorems, or principles that had learned, accepted, and discussed in this class (see Figure 1). The second revised list served as the final version of writing proofs, because all the PSMTs felt that this checklist was sufficient for them to construct and evaluate proofs for the rest of the semester.

Discussion

Throughout the semester the PSMTs did not make a major change on their initial list as seen in Figure 1, concurring with Bleiler et al.'s (2015) finding that instructor-selected sample arguments served as good foundations for students to consider the important characteristics of writing proofs. Another feature of the results is that the PSMTs' communal criteria for proof are consistent with mathematics professors' characteristics of a well-written proof, including logical correctness, clarity, and fluency (Moore, 2016). In addition, the PSMTs recognized that their written proofs should be readable and understood by audiences in their mathematics classroom community as the semester progressed. These two findings reveal that the PSMTs perceived mathematical values and norms of learning proof in their mathematics classroom community through constructing, evaluating, negotiating, and making sense of arguments, which more closely aligns with the practice of mathematicians (Harel & Sowder 2007). Even though PSMTs developed the initial list of writing good proofs and revisited it as the semester progressed, they still negotiated their evaluations for some of the instructor-selected arguments to be considered as proofs or not. Given that PSMTs' communal understanding of what counts as a proof is affected by their instructor of a proof course, comparing how PSMTs and their instructor use their class-developed criteria for proof to evaluate the same arguments can provide more insight into their individual interpretations of each proof criterion within their mathematics classroom community.

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ANALYZING HOW REFLECTIVE DISCUSSIONS IN A CONTENT COURSE INFLUENCE PROSPECTIVE TEACHERS' BELIEFS

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An important challenge in math teacher education is helping prospective teachers (PTs) develop mathematical beliefs that support effective mathematics teaching. A growing body of research has established the potential efficacy of strategies that incorporate reflection with collaborative mathematical problem-solving. However, previous studies tended to collect data only at an individual level and analyze change at an aggregate, whole-class level. This brief research report uses frame analysis of individual, written reflections and small-group and whole-class reflection conversations to provide insight into the reciprocal relationships between individual PT beliefs, small-group interactions, and the whole-class classroom culture in a math content course designed to support PTs in developing productive beliefs about mathematics.

Keywords: Teacher Beliefs, Teacher Education – Preservice, Classroom Discourse

Teachers' beliefs about mathematics teaching and learning can influence their instruction and either support or undermine student opportunities to learn (Conference Board of Mathematical Sciences, 2012; Wilhelm et al., 2017). As a result, mathematics teacher education programs have a long history of trying to develop more productive mathematical beliefs in prospective teachers (PTs) before they begin teaching (Schram et al., 1988). This work remains an ongoing challenge, however, because PT beliefs about mathematics are multi-faceted and based on years of emotionally-charged experiences (Ambrose, 2004; Holm, 2019) and are therefore often resistant to change (Grootenboer, 2008). Shilling-Traina and Stylianides (2013) suggested, therefore, that "it is important that beliefs be explicitly addressed not only in methods courses, but in mathematics courses as well" (p. 404). This report focuses on a content course that was designed to initiate the process of supporting PT belief change with an emphasis on collaborative, small-group problem solving—small groups were consistently identified as important aspects of successful belief interventions (Shilling-Traina & Stylianides, 2013; Szydlik et al., 2003). Based on open-ended reflections, PTs identified groupwork as central to their learning in the course, so it seems likely that group interactions influenced their experience of the class and therefore its effects on their beliefs. This report uses frame analysis to begin to explore the relationships between PTs' small-group interactions and their understanding of mathematics learning in a course designed to influence their beliefs about mathematics.

Literature

Many PTs enter their teaching programs with beliefs that are likely to hinder effective mathematics instruction (Grootenboer, 2008). They tend to hold procedural views of mathematics (Shilling-Traina & Stylianides, 2013) and to be skeptical of the possibility of solving novel problems (Szydlik et al., 2003), though there is significant variation within the PT population, and many PTs hold a mix of beliefs that may align with both more traditional and reform instruction (Ambrose, 2002). Ambrose (2004) identified a number of criteria that could potentially stimulate belief changes in PTs, including emotionally-resonant experiences, chances to reflect on their beliefs and experiences, and participating in a community that embraces such beliefs. Similarly, Szydlik and colleagues (2003) were able to support PT belief change by creating a classroom community that facilitated active problem solving and PT autonomy. The course that is the subject of this brief report followed similar design criteria.

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One challenge in researching change in PT beliefs, however, is that beliefs can be difficult to measure directly. Survey items may be misinterpreted by PTs (Szydlik et al., 2003), and when students are asked directly about their beliefs they may say what they think their instructor wants to hear without truly examining their beliefs or modifying their work with students (Grootenboer, 2008). Surveys and interviews also tend to focus analysis at the level of the individual and fail to address the importance of reciprocal relationships between classroom culture, individual beliefs, and group interactions in the classroom (Cobb, 2000). This brief research report uses frame analysis to begin to examine beliefs at the individual, small-group, and whole-class levels. Frames are the underlying, often implicit, structures of expectations that organize how people understand and react to events, and frame analysis investigates how groups come to a common understanding of a frame or frames that invite a particular type of action or change (Benford & Snow, 2000). Frame analysis can be particularly appropriate for attending to the roles that power and authority play in frame negotiations (Hand et al., 2012), which in the context of collaborative mathematics work often manifests as differences in status—perceived competence, levels of participation, and influence—between group members (Nasir et al., 2014). Frame analysis has been used to explore how particular frames became prevalent in a school community initiating instructional change (Coburn, 2006) and to examine collective belief change about mathematics learning in a collaborative group of teachers (Bannister, 2015). The analysis in this report will focus on identifying frame resonance (Benford & Snow, 2000; Coburn, 2006), which occurs when a frame that one individual offers is taken up and reinforced by others in a group. In particular, this report will investigate the following research question: What relationships can be seen between the frames that resonate in small-group and whole-class reflection conversations, and the frames that PTs use in their individual reflection responses across multiple time periods?

Methods

Context and Participants

The data analyzed in this paper were collected as part of a larger project analyzing PT beliefs about mathematical ability and learning—what Boaler (2016) labeled mathematical mindsets—in a required math content course for PTs in a large, urban, public university in the United States. Aligning with current recommendations to use active learning in content courses for PTs (Litster et al., 2020), the course used "group-worthy problems" to support students in developing their mathematical knowledge and productive beliefs about mathematics and mathematical learning (Nasir et al., 2014). The participants in the current study were drawn from two concurrently offered sections of the course—referred to as Class A and Class B. The author of this report was the instructor of Class B and planned collaboratively with the other instructor so that students in both sections had the same assignments and assessments. There were 65 PTs enrolled between the two sections, and 57 consented to have their classwork analyzed for research purposes.

Data Sources and Analyses

The data used in the current analyses were drawn from reflections that PTs completed in class during weeks 6, 12, and 15 of a semester course—T1, T2, and T3 respectively. The dates aligned with the two midterm exams and the last class session before the final exam for the course. The reflection prompts asked PTs to do the following:

1. Describe a significant or "Aha!" moment from class and explain why it was significant
2. Reflect on their work/participation in the class so far
3. Make or update a goal for themselves
4. Create a plan to move towards their goal
5. Reflect on how their learning in this class might apply to future teaching (only in T3)

PTs responded to the prompts individually and then discussed their memorable moments and goals in small groups. One PT from each group shared their response as part of a whole-class discussion. The small-group and whole-class discussions were audio-recorded. The written, individual reflections were entered into NVivo 12 and group reflections were imported into InqScribe. Individual and whole-class data were analyzed for all consenting participants from both sections, but the small-group analyses focus on the 29 participating students from Class B because a larger proportion of PTs in that class consented to the analysis of their audio data and because their groups for the group reflections were more consistent across timepoints.

The individual reflections were coded for whether they framed important mathematical learning as active, interactive, passive, or unclear. Responses were coded as using an "active" learning frame if the PT described learning through individual problem solving—either in a significant past experience or as part of their goal moving forward. Responses were coded as using a "passive" learning frame if they described learning from listening to an explanation, practicing rote memorization, or taking or reviewing notes. They were coded as using an "interactive" learning frame if the PT framed interaction with others as central to learning, describing a combination of listening, questioning, and problem solving or describing the actions of the group as a whole rather than their actions as an individual. Finally, responses were coded as "unclear" if there was not enough information to tell how the PT was framing learning. Each PT's response included multiple frames, so a given PT could be coded as using multiple types of frames in a given response. Responses were also coded to identify the most common categories of learning goal, which included some goals that were implicitly aligned with active, passive, or interactive frames for mathematical learning. For example, goals focused on participation in class tended to align with an active frame for learning, while goals focused on asking questions or going to office hours tended to imply a passive frame that assumed that the best solution to a challenge is for someone to provide an explanation. (In practice, interactions during office hours were supportive of active learning, but most of the PTs who made office hours attendance their goal did not actually attend them.) The audio-recorded small group reflections were analyzed to identify examples of frame resonance—interactions in which one PT presented a particular frame and other PTs or an instructor endorsed and reaffirmed the frame. These examples of resonant frames were then compared to the data from the individual reflections to look for patterns.

Findings and Implications

Preliminary analyses of these data found that the two classes and the small groups in Class B showed different frequencies and trajectories of particular frames in the individual responses, and many of those differences aligned with examples of frame resonance from the small-group and whole-class discussions from the preceding time periods, especially when those frames were endorsed by high-status individuals. For example, while both classes showed similar distributions of active frames over time, Class A showed a gradual increase in interactive frames, while Class B showed a gradual decrease. Class B's decrease could be traced to some groups having a particularly high frequency in T1 (100% of PTs in Group 3 used an interactive frame in T1) and to other groups (Groups 5 and 6) showing a marked decrease in interactive frames combined with an increase in passive frames. An initial review of those groups' discussions shows that a binary frame of correct versus incorrect strategies resonated particularly strongly in Groups 5 and 6—in contradiction with the course's goal of framing diversity of strategies as desirable, which appeared to resonate with the majority of PTs in both classes.

Another difference between classes was that Class A showed an increase in identifying participation as a focal goal, while in Class B that goal decreased to become almost nonexistent. The contrast may be related to the fact that in Class B's whole-class discussion one of the PTs shared participation as their goal and received minimal acknowledgement from the instructor (the author of this report), in contrast with more enthusiastic responses to other shared goals. While the lack of reinforcement was not intended as a value judgement, the instructor's position of authority within the classroom may have given it unanticipated weight. Frame resonance may have also played a counter-balancing role at the group level: in the two groups in which some PTs did create participation goals in T2 and T3 the conversations in T1 showed strong resonance for that goal. For example, in Group 3 two PTs shared that their goal was to participate more. A high-status PT affirmed the goal as "smart to do" and credited her own understanding of class material as "because I've participated and, like, gone up to the board." She then suggested that her groupmates share in the whole-group discussion to act on their goal.

Finally, both Class A and Class B had relatively low frequencies of goals that focused on grades and passing the class, and Class B's frequency decreased over time, but Group 2's frequency started at roughly twice the class average and stayed roughly constant over time. There are multiple examples in Group 2's discussions that show how strongly a focus on grades resonated with the group members, especially with the PT who seemed to have the highest social and mathematical status within the group. In the group's T1 conversation they encouraged one another to aim for As and Bs rather than just passing the class, which illustrated how the goal became associated with the supportive social relationships within the group. Their T3 conversation also highlighted the ways that larger school frames about the evaluative function of finals week and exams could overshadow the frames and experiences that our course tried to cultivate. In the words of the high-status PT: "So far I'm doing pretty well, but, you know, after this [final] that's probably not the case anymore…It's always the final that's going to break you down, so I'm going to let that go ahead and just be it. So, we're going to pray."

While these analyses are only preliminary, there are some potential implications. The examples all reinforce how important it is for instructors to pay attention to the implications of status during small-group and whole-class discussions, even when the topic is self-reflection rather than mathematical problem-solving (Cobb, 2000; Nasir et al., 2014). Group 2's T3 conversation illustrates why it is so important that PTs are exposed to multiple courses and experiences that support positive belief change over time rather than a single course. It also serves as a reminder that the pressure and structure of finals may make it so that studies that only take measures at the end of courses may underestimate the changes that PTs experience during the courses. Future studies could examine whether this regression is localized to the time around exams or persists after them. Researchers could also examine how the frames around goals and active/autonomous learning that are the focus of this study relate to other PT beliefs about mathematical ability (Boaler, 2016) and math instruction (Ambrose, 2004).

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CAN CONFIDENCE IN MATHEMATICAL AFIRMATIONSINFLUENCE NEGATIVELY IN THE ADVANCE OF THE DISCIPLINE? WHY?

¿LA CONFIANZA EN AFIRMACIONES MATEMÁTICAS PUEDE INFLUIR NEGATIVAMENTE EN EL AVANCE DE LA DISCIPLINA? ¿POR QUÉ?

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The research focuses on the analysis of confidence and doubt in mathematical statements. Based on the Grounded Theory, and on the analysis of a historical case, that of non-Euclidean geometries, and especially on the figure of Saccheri, the question in this document is answered. It is argued that confidence in the truth of Euclidean geometry had as an effect the attribution of qualities to said geometry and to the problems identified in that work. Also, expectations about the solution to those problems and logical, ontological and epistemological commitments were made explicit. It seems that handling these effects of confidence with biases and exaggerations can be negative for the development of mathematical work.

Keyword: Affect, Emotion, Beliefs, and Attitudes; Geometry and Geometrical and Spatial Thinking

Background, objective, and research question

This research focuses on the analysis of states such as the confidence or doubt that people experience around the veracity of mathematical statements (such as postulates, theorems, or results of school assignments, which will be denoted hereinafter as “H”). Rigo (2013) calls these states as “epistemic states of convincement” and represents them as “ESC”. Various studies have shown the presence of these states in all kinds of mathematical practice, both those carried out in school contexts and in professional mathematics (Fischbein, 1982; Harel and Sowder, 1998; Hersh, 1993; Segal, 2000).

In some cases, the security in certain H’s adequately guides the mathematical work, which leads to advances in learning. For example, Inglis, Mejia-Ramos and Simpson (2007) affirm that a considerable reduction in uncertainty allows us to determine when we already have enough tools to carry out a test; the reduction of doubt, works in this case as a force that encourages mathematical activity. In other cases, security can have detrimental effects on the advancement of mathematics learning. For example, in the context of mathematical proofs, Inglis, Mejia-Ramos and Simpson (2007) observed that when a student confidently associated an affirmation based on inductive warrants, he did not have the need later to construct a proof that supported this affirmation.

So, experts have warned that security in mathematical statements can sometimes positively influence the development of learning or disciplinary knowledge, but sometimes that influence can be negative. However, in mathematical education literature -and in that of other disciplinary areas- no systematic conceptual development has been found on the characteristics of this confidence in mathematical statements that account for co-related phenomena, nor, in particular, have they constructed theoretical explanations that allow identifying and understanding the conditions under which the ESC act favorably or unfavorably, at a didactic level. It is considered that it would be necessary to know these conditions in depth, to recreate or inhibit them in school contexts.

To meet these needs, this document suggests a first and provisional answer, based on theoretical explanations, to the question: Why does confidence in mathematical statementsnegatively influence the advancement of the discipline?

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Methodology and methods

The research question posed in the writing demands theoretical explanations associated with the phenomenon of convincement, which go beyond specific descriptions with little clarifying power. However, as already stated, no theories have been constructed that offer such explanations. For this reason, this document does not start from a theoretical framework (it simply does not exist) and its objective is to initiate the development of one. To develop this theory, the qualitative research perspective offered by the Grounded Theory was chosen (GT, Corbin & Strauss, 2015). An analytical tool of GT is “context analysis” (CA). In CA, it is considered that when people act or have some internal experience, they are responding to significant events. Those events are called “conditions”. From a fusion of conditions, and from the internal actions or experiences that they promote, results usually emerge. In CA these results are called “consequences”. Those consequences can stimulate more actions or change their course. The theory thus consists of a set of explanations on how certain actions or internal experiences can be given under a combination of certain conditions, and how certain consequences can arise from these conditions and actions or internal experiences.

This manuscript provides some explanations regarding the ESC phenomena that people experience when doing mathematical work. For this, following the GT, the CA will be used. But here the focus is only on the analysis of the consequences that these ESC have on mathematical work. For this analysis, historical data can be used, which is taken as empirical data (cf. Corbin & Strauss, 2015). This paper analyzes Saccheri's role in the development of non-Euclidean geometries - who is sadly famous for “being a victim of the preconceived notion of his time, that the only possible geometry was Euclidean” (Heath, 1956, p. 211) -, because it illustrates the phenomenon under study. In particular, parts of Saccheri's preface to his work *Euclid vindicated from every blemish* (Saccheri, 2014) are analyzed, because there he explains his ESC and suggest how these ESC negatively influenced his mathematical work. Following the CA, in the analysis the ESC are considered as internal experiences and the consequences of those ESC are identified. Subsequently, these consequences are denoted with concepts and, in the end, those concepts are used to explain how the ESC influenced (at the beginning) the mathematical work of Saccheri.

Report of results: certainty as an obstacle to the advancement of knowledge

Saccheri begins the preface to his work Euclid vindicated from every blemish with the following:

> Of all who have learned mathematics, none can fail to know how great is the excellence and worth of Euclid’s *Elements*. As erudite witnesses here I summon Archimedes, Apollonius, Theodosius, and others almost innumerable, writers on mathematics even to our times, who use Euclid’s *Elements* as foundation long established and wholly unshaken. (Saccheri, 2014, p. 62)

There, the mathematician begins by attributing to Euclid's *Elements* a positive quality (of "excellence") magnified (with the word "great"). Later, he supports the attribution of that quality in the ESC of "wholly unshaken" that various mathematical authorities attributed to the work over a long time. Saccheri (2014) thus continues:

> But this so great celebrity has not prevented many, ancients as well as moderns, and among them distinguished geometers, maintaining they had found certain blemishes in these most beauteous nor ever sufficiently praised *Elements*. Three such flecks they designate. (p. 62)

The attribution of magnified positive qualities to the Euclidean work, according to Saccheri, did not prevent mathematicians from identifying blemishes in it. However, the mathematician appealed to these magnified positive qualities, to interpret these imperfections as minimal negative qualities of the work (i.e. flecks). What consequences arose from this attribution of qualities to the Euclidean work and its imperfections? Saccheri continues:

> The first (fleck) pertains to the definition of parallels and with it the Axiom which in Clavius is the thirteenth of the First Book of the *Elements*, where Euclid says: If a straight line falling on two straight lines, lying in the same plane, make with them two internal angles toward the same parts less than two right angles, these two straight lines infinitely produced toward those parts will meet each other. No one doubts the truth of this Assertion; but solely they accuse Euclid as to it, because he has used for it the name Axiom, as if obviously from the right understanding of its terms alone came conviction. (Saccheri, 2014, p. 62)

There, Saccheri begins by announcing that an imperfection (described by him as "fleck") of the Euclidean work is related to the V Postulate. Then he states that postulate and connects it with an ESC: “No one doubts the truth of this Assertion”. So, for Saccheri, the Fifth Postulate's high degree of commitment to truth value was never questioned. For him, the imperfection was that this statement was given the status of axiom. That is, what he questioned was the strategy used to grant the truth value to the V Postulate. In sum, a first consequence of having attributed magnified positive qualities to the Euclidean work and of having interpreted its imperfections as minimal negative qualities was that, despite finding difficulties in one of its affirmations (the V Postulate), the truth of this affirmation was sustained with a high degree of commitment and only the strategy on which that truth value was based was questioned. Other consequences are disclosed below:

> For some, and these surely the keenest, endeavor to demonstrate the existence of parallel straight lines as so defined, whence they go up to the proof of the debated Assertion as stated in Euclid’s terms, upon which truly from that *Elements* (with some very few exceptions) all geometry rests. But others (not without gross sin against rigorous logic) assume such parallel straight lines, forsooth equidistant, as if given, that thence they may go up to what remains to be proved. (p. 62)

In criticizing the attempts of others to solve the problem of the V Postulate, Saccheri explains his commitment to the Euclidean definition of parallel lines, with the role of the V Postulate as “support of all geometry” or with “rigorous logic”. That Saccheri has made explicit those commitments around the Elements is consistent with the magnified positive qualities that he attributed to the work. Then he announces that he will divide his book into two parts:

> In the First Part will imitate the antique geometers, and … merely undertake without any *petitio principii* clearly to demonstrate the disputed Euclidean Axiom. (p. 65)

There, Saccheri (2014) indicates that the solution to the problem of the V Postulate would be "simple" and that "it would imitate the antique geometers". As a strategy to support the truth of the V Postulate, he intends to demonstrate it without incurring a *petitio principii*.

According to all the aforementioned, how can it be explained that Saccheri ‘reduced’ the problem of the V Postulate to **demonstrate** its veracity? As it was seen, in his preface he started from an ESC of “**wholly unshaken**” associated with the Euclidean work. One consequence of this ESC was to magnify the positive qualities of that work and minimize its imperfections or possible pitfalls. Consistent with these *attributions* and with the very charge of empirical truth that the V postulate had (in finite space), Saccheri maintained his truth and only questioned the strategy on which that truth value was based. In line with this, he raised *expectations* that the solution to the problem would be “simple” and it would suffice to "imitate" the antique geometers. This guided Saccheri's mathematical work, when he proposed as a solution to the problem to support the truth of the V in the strategy of “demonstrating"”(without *petitio principii*). Furthermore, that resolution respects Saccheri's *commitments*, such as following rigorous logic. This set of values, expectations, commitments of all kinds, beliefs, and intentions that are linked to the ESC here is called the *ESC interpretive framework* (IF).

To explain why Saccheri ‘minimized’ the problem of the V Postulate, we analyze how he handled that MI. In the first place, Saccheri accompanied this IF with a magnification of the positive qualities that he attributed to *the Elements* (with words like "great"), a minimization of its imperfections (with words like "fleck") and a simplification of the solution to those imperfections (with " I just commit"). This way of handling IF can be called *exaggeration*. Secondly, the solution to "demonstrate" that Saccheri proposed to solve the problem of the V Postulate, is totally consistent with the IF. It seems that the IF was considered as "rules that should be followed". Under those rules, other options for solving the problem would be discarded. For example, the option of questioning the Fifth Postulate as the only possibility would be ruled out, because it would go against the slogan of "imitating the antique geometers". Thus, the IF acted as a *bias* that guided Saccheri's strategy. According to what was said before, handling the IF with exaggerations and biases may explain why the mathematician trivialized the problem of the V Postulate. And why did Saccheri consider that IF as "rules that should be followed"? Saccheri asserts that the ESC of "wholly unshaken" in *the Elements* was long held by many mathematicians. Thus, questioning what emerged from that ESC would imply questioning centuries of tradition and great authorities, and it seems that neither Saccheri nor any mathematician of his time were willing to face that. Thus, *social consensus* may also explain why Saccheri appears to have "attenuated" the problem of the V Postulate.

Final remarks

We are now in a position to answer the research question, why does confidence in mathematical statements negatively influence the advancement of the discipline? Here it is argued that ESC have effects on mathematical work and decisions. Specifically, evidence has been provided here that ESC can give rise to the attribution of qualities to mathematical works and problems, to expectations on how to solve those problems, to logical, ontological and epistemological commitments, and in short, to an IF. From this IF, the person can guide his mathematical decisions and his disciplinary work, such as, for example, devising a strategy to solve a mathematical problem. It is necessary to clarify that the ESC are not, in essence, favorable or unfavorable for the advancement of mathematical knowledge. What the study shows is that, under certain conditions, they can be harmful. Some of these conditions have to do with how a person handles IF, particularly if it is accompanied by exaggerations, biases, and taking social consensus uncritically.

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¿LA CONFIANZA EN AFIRMACIONES MATEMÁTICAS PUEDE INFLUIR NEGATIVAMENTE EN EL AVANCE DE LA DISCIPLINA? ¿POR QUÉ?

CAN CONFIDENCE IN MATHEMATICAL AFIRMATIONSINFLUENCE NEGATIVELY IN THE ADVANCE OF THE DISCIPLINE? WHY?

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La investigación se centra en el análisis de la confianza o la duda en afirmaciones matemáticas. Con base en la Teoría Fundamentada, y en el análisis de un caso histórico, el de las geometrías no euclidianas, y en especial en la figura de Saccheri, se responde a la pregunta que precede a este documento. Se argumenta que la confianza en la verdad de la geometría euclidiana tuvo como efectos la atribución de cualidades a dicha geometría y a los problemas identificados en esa obra, expectativas sobre la solución a esos problemas y la explicitación de compromisos lógicos, ontológicos y epistemológicos. Parece que, manejar con sesgos y exageraciones esos efectos de la confianza, puede resultar negativo para el desarrollo de trabajo matemático.

Palabras clave: Afecto, emoción, creencias y actitudes; Geometría.

Antecedentes, objetivo y pregunta de investigación

La investigación se centra en el análisis de estados como la confianza o la duda que las personas experimentan alrededor de la veracidad de afirmaciones matemáticas (como postulados, teoremas o resultados de tareas escolares, las que en lo sucesivo se denotarán como 'H'). A esos estados Rigo (2013) los llama estados epistémicos de convencimiento en torno a H y los representa como "eec". Diversas investigaciones han dado cuenta de la presencia de esos estados en todo tipo de práctica matemática, tanto las que se llevan en contextos escolares como en las de la matemática profesional (Fischbein, 1982; Harel y Sowder, 1998; Hersh, 1993; Segal, 2000).

En algunos casos, la seguridad en ciertos H's orienta adecuadamente el trabajo matemático, de lo que se desprenden avances en los aprendizajes. Por ejemplo, Inglis, Mejia-Ramos y Simpson (2007) afirman que una considerable reducción de la incertidumbre permite determinar el momento en el que ya se tienen herramientas suficientes para realizar una prueba; la reducción de la duda, funciona en este caso como una fuerza que incita la actividad matemática. En otros casos, la seguridad puede tener efectos nocivos en el avance de los aprendizajes de las matemáticas. Por ejemplo, en contextos de prueba matemática, Inglis, Mejia-Ramos y Simpson (2007) observaron que cuando un alumno asoció confianza a una afirmación basada en garantías inductivas, no tuvo después necesidad de construir una prueba que soportara dicha afirmación.

Así que los expertos han advertido que la seguridad en afirmaciones matemáticas a veces puede influir positivamente en el desarrollo de los aprendizajes o de los conocimientos disciplinares, pero en ocasiones esa influencia puede ser negativa. Sin embargo, en la literatura de educación matemática -y en la de otras áreas disciplinares- no se ha encontrado un desarrollo conceptual sistemático sobre las características de esa seguridad en afirmaciones matemáticas que dé cuenta de sus orígenes y sus efectos ni, en particular, se han construido explicaciones teóricas que permitan identificar y comprender las condiciones bajo las cuales los eec actúan favorable o desfavorablemente, a nivel didáctico. Se considera que sería necesario conocer a fondo esas condiciones, con el objeto de recrearlas o de inhibirlas en los contextos escolares.

Con el fin de atender a esas necesidades, en este documento se sugiere una primera y provisional respuesta, con base en explicaciones teóricas, a la pregunta ¿Por qué la confianza en afirmaciones matemáticas puede llegar a influir negativamente en el avance de la disciplina?

Metodología y métodos

La pregunta de investigación planteada en el escrito demanda explicaciones teóricas asociadas con el fenómeno del convencimiento, que vayan más allá de descripciones puntuales con escaso poder clarificador. Sin embargo, como ya se dijo, no se han construido teorías que ofrezcan esas explicaciones. Es por esto que en el presente documento no se parte de un marco teórico (simplemente, no existe) y se plantea justo como objetivo iniciar el desarrollo de uno. Para desarrollar esa teoría, se eligió la perspectiva de investigación cualitativa que ofrece la Teoría Fundamentada (TF, Corbin & Strauss, 2015). Una herramienta analítica de la TF es el 'análisis de contexto' (AC). En el AC se considera que cuando las personas actúan o tienen alguna experiencia interna están dando respuesta a sucesos significativos para ellas. Esos eventos se llaman 'condiciones'. De una fusión de condiciones, y de las acciones o experiencias internas que propician, se suelen desprender resultados. En el AC a esos resultados se les llama 'consecuencias'. Esas consecuencias pueden estimular más acciones o cambiar su rumbo. La teoría consiste así en un conjunto de explicaciones sobre cómo ciertas acciones o experiencias internas se pueden dar bajo una combinación de ciertas condiciones, y de cómo a partir de esas condiciones y acciones o experiencias internas, se pueden suscitar ciertas consecuencias.

En este manuscrito se ofrecen algunas explicaciones relativas a los fenómenos de los eec que experimentan las personas cuando realizan trabajo matemático. Para eso, siguiendo la TF, se recurrirá al AC. Pero aquí se centra la atención únicamente en el análisis de las consecuencias que esos eec tienen sobre el trabajo matemático. Para ese análisis, se puede recurrir a datos históricos, que se toman como datos empíricos (cf. Corbin & Strauss, 2015). En este escrito se analiza el papel de Saccheri en el desarrollo de las geometrías no euclidianas -quien es tristemente célebre por "ser víctima de la noción preconcebida de su tiempo, de que la única geometría posible era la euclidiana" (Heath, 1956, p. 211)-, porque ilustra el fenómeno bajo estudio. En particular, se analizan partes del prefacio de Saccheri a su obra *Euclides reivindicado de todo defecto* (Saccheri, 2014), porque ahí él explicita sus eec y deja ver cómo esos eec influyeron negativamente en su trabajo matemático. Siguiendo el AC, en el análisis se consideran a los eec como las experiencias internas y se identifican las consecuencias de esos eec. Posteriormente, esas consecuencias se denotan con conceptos y, al final, se utilizan esos conceptos para explicar cómo los eec influyeron (al inicio) en el trabajo matemático de Saccheri.

Reporte de resultados: la certeza como obstáculo para el avance de conocimientos

Saccheri inicia el prefacio a su obra *Euclides reivindicado de todo defecto* con lo siguiente:

> De todos los que han aprendido matemáticas, ninguno ha dejado de saber cuán grande es la excelencia y el valor de *Los Elementos* de Euclides. Como testigos eruditos aquí, convoco a Arquímedes, Apolonio, Teodosio y otros, casi innumerables, profesionales de matemáticas, incluso de nuestros tiempos, que usan a *Los Elementos* de Euclides como fundamento totalmente incontrovertible establecido desde hace mucho tiempo. (Saccheri, 2014, p. 62)

Ahí, el matemático comienza por atribuir a *Los Elementos* de Euclides una cualidad positiva (de "excelencia") magnificada (con la palabra "grande"). Luego, él soporta la atribución de esa cualidad en el eec de "totalmente incontrovertible" que varias autoridades matemáticas atribuyeron a la obra a lo largo de mucho tiempo. Saccheri (2014) así continúa:

> Pero esta gran celebridad no ha impedido a muchos, tanto antiguos como modernos, y entre ellos distinguidos geómetras, sostener que habían encontrado ciertas imperfecciones en estos Elementos tan bellos y suficientemente elogiados. Ellos designan tres de esos lunares. (p. 62)

La atribución de cualidades positivas magnificadas a la obra euclidiana, según Saccheri, no impidió que los matemáticos identificaran imperfecciones en ella. Sin embargo, el matemático apeló a esas cualidades positivas magnificadas, para interpretar dichas imperfecciones como cualidades negativas mínimas de la obra (i.e. 'lunares'). ¿Qué consecuencias se desprendieron de esa atribución de cualidades a la obra euclidiana y a sus imperfecciones? Saccheri continúa:

> La primera ('mota') se refiere a la definición de paralelas y con ella el Axioma que en Clavius es el decimotercero del Primer Libro de *Los Elementos*, donde Euclides dice: "Si una línea recta que cae sobre otras dos líneas rectas, que yacen en el mismo plano, hace con ellas dos ángulos interiores sobre el mismo lado menores que dos ángulos rectos, estas dos líneas rectas producidas infinitamente hacia esas partes se encontrarán entre sí". Nadie duda de la verdad de esta afirmación; pero únicamente acusan a Euclides en cuanto a ella, porque él ha usado para esa afirmación el nombre de Axioma, como si obviamente solo por la correcta comprensión de sus términos se llegara a la convicción. (Saccheri, 2014, p. 62)

Ahí, Saccheri (2014) comienza por anunciar que una imperfección (calificada por él como "lunar") de la obra euclidiana se relaciona con el V Postulado. Luego, él enuncia ese postulado y conecta con un eec: "Nadie duda de la verdad de esta afirmación". De modo que, para Saccheri, nunca se puso en entredicho el alto grado de compromiso con el valor de verdad del V Postulado. Para él, la imperfección consistía en que a esa afirmación se le dio el estatus de axioma. Es decir, lo que cuestionó, fue la estrategia utilizada para otorgar el valor de verdad al V Postulado. En suma, una primera consecuencia de haber atribuido cualidades positivas magnificadas a la obra euclidiana y de haber interpretado sus imperfecciones como cualidades negativas mínimas fue que, a pesar de hallar dificultades en una de sus afirmaciones (el V Postulado), la verdad de dicha afirmación se sostuvo con un alto grado de compromiso y únicamente se cuestionó la estrategia en la que se basó ese valor de verdad. Otras consecuencias se desvelan a continuación:

> Algunos, y estos seguramente los más entusiastas, se esfuerzan por demostrar la existencia de líneas rectas paralelas, como se definen en *Los Elementos*, de donde salen a la prueba de la afirmación debatida, como se indica en Euclides, y sobre la cual verdaderamente descansan esos *Elementos* (con algunas muy pocas excepciones) y toda la geometría. Pero otros (no sin un gran pecado contra una lógica rigurosa) asumen tales líneas rectas paralelas, como equidistantes, como si de allí pudieran llegar a lo que queda por demostrar. (p. 62)

Al criticar los intentos de otros para resolver el problema del V Postulado, Saccheri explicita su compromiso con la definición euclidiana de rectas paralelas, con el papel del V Postulado como "soporte de toda la geometría" o con la "lógica rigurosa". La explicitación de esos compromisos en torno a los *Elementos* es congruente con las cualidades positivas magnificadas que Saccheri atribuyó a la obra. Luego, él anuncia que dividirá su libro en dos partes:

> En la primera parte imitaré a los geómetras antiguos, y… simplemente me comprometo sin ningún *petitio principii* a demostrar claramente el disputado Axioma Euclidiano. (p. 65)

Ahí, Saccheri (2014) indica que la solución al problema del V Postulado sería "simple" y que "imitaría a los geómetras antiguos". Como estrategia para sustentar la verdad del V Postulado, él se propone demostrarlo sin incurrir en una petición de principio.

De acuerdo a todo lo antes dicho ¿Cómo se puede explicar que Saccheri 'redujera' la problemática del V Postulado a **demostrar** su veracidad? Como se vio, en su prefacio él partió de un eec de **incontrovertible** asociado a la obra euclidiana. Una consecuencia de ese eec fue *magnificar las cualidades* positivas de esa obra y *minimizar sus imperfecciones* o los posibles escollos. De forma congruente con esas atribuciones y, hay que decirlo, con la propia carga de veracidad empírica que tenía el V postulado, Saccheri sostuvo su verdad y solo cuestionó la estrategia en la que se basaba ese valor de verdad. En línea con esto, él se planteó *expectativas* de que la solución al problema sería "simple" y bastaría con "imitar" a los geómetras antiguos. Lo anterior orientó el trabajo matemático de Saccheri, cuando planteó como solución al problema desprender la verdad del V por medio de la estrategia de "demostrar" (sin petición de principio). Además, esa resolución respeta *compromisos* adquiridos por Saccheri, como seguir la lógica rigurosa. A ese conjunto de valores, expectativas, compromisos de todo tipo, planes, creencias e intenciones que se coligan con los eec aquí se llama *marco interpretativo de los eec (MI)*.

Para explicar por qué Saccheri 'minimizó' la problemática del V Postulado, a continuación se analiza cómo él manejó ese MI. En primer lugar, Saccheri acompañó dicho MI de una magnificación de las cualidades positivas que él atribuyo a *Los Elementos* (con palabras como "grande"), de una minimización de sus imperfecciones (con palabras como "lunar") y de una simplificación de la solución a esas imperfecciones (con "simplemente me comprometo"). A esa forma de manejar el MI se le puede llamar *exageración.* En segundo lugar, la solución de "demostrar" que Saccheri se planteó para resolver el problema del V Postulado, es totalmente congruente con el MI. Pareciera que el MI se consideró como "reglas que deberían seguirse". Bajo esas reglas, otras opciones para resolver el problema quedarían descartadas. Quedaría descartada, por ejemplo, la opción de cuestionar al V Postulado como la única posibilidad, porque iría en contra de la consigna de "imitar a los geómetras antiguos". Así, el MI actuó como un *sesgo* que orientó la estrategia de Saccheri. De acuerdo a lo antes dicho, manejar el MI con exageraciones y sesgos puede explicar por qué el matemático trivializó el problema del V Postulado. ¿Y por qué Saccheri consideró a ese MI como "reglas que deberían seguirse"? Saccheri afirma que el eec de incontrovertible en *los Elementos* lo sostuvieron prolongadamente muchos matemáticos. Así, cuestionar aquello que se desprendía de ese eec implicaría cuestionar siglos de tradición y a grandes autoridades, y parece que ni Saccheri ni ningún matemático de su época estaban dispuestos a enfrentar eso. De modo que, el *consenso social* también puede explicar por qué Saccheri parece haber 'atenuado' el problema del V Postulado.

Consideraciones finales

Se está ahora en condiciones de responder a la pregunta de investigación ¿por qué la confianza en afirmaciones matemáticas puede influir negativamente en el avance de la disciplina? Aquí se argumenta que los eec tienen efectos sobre las decisiones y los trabajos matemáticos. Específicamente, aquí se han aportado evidencias de que los eec pueden dar lugar a la atribución de cualidades a obras y problemas matemáticos, a expectativas sobre cómo resolver esos problemas, a compromisos lógicos, ontológicos y epistemológicos, y en suma, a un MI. A partir de ese MI, la persona puede orientar sus decisiones matemáticas y su trabajo disciplinar como, por ejemplo, plantear una estrategia para resolver un problema matemático. Es necesario aclarar que los eec no son, en esencia, favorables o desfavorables para el avance del conocimiento matemático. Lo que muestra el estudio es que, bajo ciertas condiciones, pueden resultar perjudiciales. Algunas de esas condiciones tienen que ver con cómo una persona maneja el MI, en particular si lo acompaña con exageraciones, sesgos y tomando el consenso social de manera acrítica.

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PRESERVICE TEACHERS ANALYSIS OF THE USE OF HISTORY IN MATHEMATICS LESSONS

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A group of preservice mathematics teachers participated in a course on history and technology in mathematics which includes a study of the historical development of mathematical concepts. This study examined the effect of the course on their critique of teaching materials using history. Results show that the participants developed more insightful critiques of teaching materials that integrated history poorly into the development of mathematical concepts.

Keywords: Teacher Education - preservice, Teacher Beliefs, Middle School Education

Introduction

Using the history of mathematics as a way of learning mathematics itself is a well-established part of many mathematics curricula and teacher preparation programs. There are many arguments supporting the place of history of mathematics from enhancing learners' cultural sophistication to helping learner's understanding of concepts. The purpose of this study is to examine changes in pre-service teachers' analysis of mathematics lessons purporting to integrate history into teaching mathematics after they engage in a semester-long course focussed on deploying history as a driver of development of mathematical concepts.

Literature review and relationship to research

The use of history as a lever for teaching mathematics in middle and high school is often relegated to the realm of footnotes and passing mentions of historical facts in textbooks adding historical context to some content (Smestad. Jankvist & Clark, 2014) or use of biography, sometimes in the form of student projects on individual figures, to add historical and/or cultural context to the study of mathematics (see e.g. Tzanakis et al., 2002). Jankvist (2009) referred to this kind of approach as using history as "a motivating factor for students in their learning and study of mathematics" (p. 237) but argued that history can also be used to "support the actual learning of mathematics" (p. 238). Indeed, Siu and Tzanakis (2004) argued for thinking in terms of "integrating history of mathematics" rather than "using history of mathematics."

In the realm of teacher preparation programs Furinghetti's (2007) analysis of the literature identified the following approaches to history of mathematics in teacher preparation programs: (i) "delivering a course in the history of mathematics with the aim of giving a historical background to teachers' mathematics knowledge" (p. 132); (ii) "packages of historical materials focused on mathematical concepts difficult to teach were used to deepen teachers' pedagogical reflection on these concepts" (p. 132); and (iii) attention to the link between history and mathematics teachers' beliefs and attitudes about mathematics. The first of these aligns closely with the use of history noted by Smestad et al. (2014) and Tzanakis et al. (2002). Furinghetti herself adopts a different approach in her program: to use the history of mathematics "as a mediator of knowledge for teaching. The aim was to make the participants reflect on the meaning of mathematical objects through experiencing historical moments of their construction." (p. 133)

The goal of the course in which this study takes place, aligning with Furinghetti, was to provide students with examples of the historical necessity for the development of select concepts on mathematics so that participants would gain a deeper understanding of those topics. The goal of this study is to examine the effect of participating in that course on pre-service mathematics teachers' critique of teaching materials deploying the history of mathematics.

Methods and Methodologies

Setting and Participants

Participants (n = 20) in the study were prospective middle school teachers in a course on history and technology in mathematics which includes a study of the historical development of mathematical concepts. The central assignment for the course was for students to produce a classroom-ready lesson that infuses mathematics history and technology into instruction.

Data Collections Methods

In order to assess the trajectory of the participants' growth in infusing history into lessons in a meaningful way, they were asked to analyze/critique sample teaching materials that were designed to be poor examples of using history of mathematics for concept development.

The first lesson (Open University, 1988) involved students exploring the Fibonacci sequence. They are told at the end that "the sequence discovered by Leonardo of Pisa in 1202 in connection with the breeding of rabbits!" This was intended as a poor example of infusing history into mathematics lessons in a meaningful way in that there are simply a few historical notes added to the lesson and the history is not integral to the development of the concept.

In the second lesson (Penn Museum, n.d.) students are shown pictures of several famous domes and given the vocabulary apex, oculus, diameter, radius, circumference, and area. Students are then given the formulae for calculating diameter, radius, circumference, and area and, given one or two of those values asked to find the others. This was also intended as a poor example of infusing history into mathematics lessons in a meaningful way in the sense of there being a very low cognitive demand with students simply required to plug numbers into formulae with no emphasis on how concepts were developed historically.

Students were asked to respond to the following prompt: "Comment on the quality of these teaching materials in terms of their capacity to engage middle grades students in the history of mathematics. Identify several strengths and several weaknesses of each resource." In addition, participants submitted a brief biography of their experiences with the use of history in teaching mathematics from their own school and college classes.

During the semester participants worked through a series of activities in which the historical necessity of a mathematics concept is demonstrated i.e. exemplar lessons in which history is integrated into mathematics content. These included John Graunt's development of data analysis and the introduction of Rene Descartes coordinate system. In addition, the students worked on developing their own teaching materials that integrate history in mathematics teaching.

At the end of the semester the participants were asked to reread their original critique/analysis of the "bad" lesson plans and respond to the following prompt: "Describe your reaction to your earlier writing. Specifically: How has that reaction been informed by your experiences of the course? Be specific about how the experiences in the course have either reinforced your opinions about the sample lessons or changed your opinion about the sample lessons."

Data Analysis

The participants' original critiques were analysed using the constant comparative method (Glaser & Strauss, 1967) to establish trends in the data and to develop a code book. The code book was then applied to the critiques at the end of the semester with a focus on any changes asserted and the reasons for those changes. In addition, data analysis focused on whether students spoke about a motivation approach or a concept development approach to using history.

Results

Initial analyses/critiques

Of the 20 participants 19 said they would use the domes worksheet (with three saying they would need to add to the worksheet). In contrast, 13 out of 20 said they would use the Fibonacci worksheet with 7 of the 13 saying they would need to add to/modify the worksheet.

The codes associated with the positive aspects noted by the participants were: Integrating history & mathematics (INT), relating to "Real Life" (RL), engaging visual learners (VIS), and crossing disciplines beyond history & mathematics (CD)

Domes worksheet The historical context of the Domes was assessed positively by 13 of the 20 participants for relating the mathematics to "Real Life" (RL). For example, Participant 5 (P5) said "My favorite part of the math domes worksheet is its link to the past through real life examples." P7's view was that the lesson was positive "making the historical knowledge relevant to the student by describing Domes that exist with these measurements all across the world."

The Domes lesson was assessed by 7 participants as being a positive example of integrating mathematics and history (INT) e.g. "I feel like I am walking in a mathematics museum, each page helps me have the connection between the mathematics history" (P6) and "I thought it succeeded in giving a balanced math-history knowledge to the student" (P7).

Other notable positives in the participants critiques were the use of visuals (VIS), mentioned by 6 participants and crossing disciplines (CD) beyond history and mathematics mentioned by 3 participants. It is noteworthy that the positive aspects reported by the participants were always consonant with Jankvist's (2009) category of using history as motivation.

Fibonacci worksheet The Fibonacci worksheet was assessed less positively by the participants. Only one participant assessed the Fibonacci lesson positively for relating to "real life." (RL): "Also, the lesson outline offered great background on Fibonacci, allowing the students to see a real-life relationship and reason for creating such an idea" (P10).

There was an interesting dichotomy in responses coded with INT+/- (Integrating history & mathematics). Several students remarked that the history portion of this lesson was a little "tacked on." For example, "There was only a small short paragraph on the mathematician who created the "Golden Sequence" . . . which in my opinion added no value to the material." (P14) and "The worksheet focused a lot on the mathematics part and less on the history." (P15). However, several participants viewed the lesson positively on the INT code. For example, "It also makes a good connection between the history of the math and the math itself" (P20) and "It allows students to understand how the history relates to the topic that they are learning." (P11)

Use of history to develop mathematical concepts The critiques were also coded, owing to the focus of the activities in the course, for attention to using history to develop mathematical concepts, either providing evidence from the lessons or noting its absence (DEV+/-).

In coding the participants' initial critiques the code DEV was used 6 times. 5 of those were positive assessments of the Fibonacci lesson as using history as a tool for developing concepts. Examples of the positive assessments were "It gave relevance to why the Fibonacci sequence was created by mentioning it was done in connection to rabbits" (P10) and "Understanding . . . where a math equation or concept comes from can be great for student learning. History gives a better understanding when students wonder how a concept was discovered." (P15)

In summary, the participants were positively disposed towards the "bad" lessons. They mostly said they would use the lessons albeit with some modifications. In addition, they tended to note connections to real life as the main positive factor for using history in teaching mathematics.

End of semester analyses/critiques

At the end of the semester the participants were asked to read their critiques of the lesson plans from the beginning of the semester and to react to what they wrote at that time. Analysis of the responses focussed on whether they had changed their mind about the quality of the lessons and what values the participants espoused regarding the use of history in mathematics teaching.

Overall, the participants were better able to recognise the deficiencies of the sample lessons after the course although they mostly continued to say that they would use the materials in their own classrooms. Changes to the participants' thinking were mostly centred on the idea that the use of history, particularly in the Fibonacci lesson, was an add-on and did not contribute to the development of mathematical concepts (DEV). In contrast to the initial critiques where 5 of the 6 uses of the DEV code were positive attributions to the Fibonacci lesson, all 8 uses of the DEV code in the final critiques were negative. For example, one participant noted "the history of the Fibonacci Sequence . . .was added on at the very end . . . This is more of a short "fact," rather than integrated math." (P8) and another noted that "it adds very little to the lesson and would be easy for a teacher to skip over." (P4).

The participants continued to consider relating to "real life" (RL) as an important aspect of mathematics lessons involving the history of mathematics. For example, Participant 5 argued that "students should be able to find domes in their community and complete the new math concept with things in the world around them." Thinking about the class in general as well as the sample lessons one participant remarked "This class helped show me how important it is for math to have real world applications and connect with history." (P2)

One final theme that emerged in the post analysis was the idea of history "humanising" mathematics i.e. making the idea that actual people were involved in the making of mathematics clear. Participant 4 noted that "teaching math history also helps students to see that normal people like them can make big discoveries that change the world" (P4) and Participant 8 noted that "I learned so much about mathematicians. I think this is important to have in our class because students can learn where the math they are learning comes from." (P8). The participants' interest in this was presented more in the sense of motivation for learning rather than for the development of mathematical concepts.

In summary, the participants, while continuing to see positive aspects of the lessons were much more sensitive to the deficiencies of the lessons in terms of the use of history for the development of mathematical concepts.

Conclusion

The use of history in teaching mathematics is commonplace in the preservice teacher mathematics programs. Preservice teachers' prior experiences, and often their experiences in programs, with history tend towards a focus on biography and the use of history as a provider of context and motivation. This study shows the potential for preservice mathematics teachers to begin to view the use of history as a lever in the development of mathematical concepts. This is an important idea for use of history in pre-service mathematics preparation programs. Through participation in a course with exemplars of such mathematical development, the preservice teachers made more insightful critiques of teaching materials that were deficient in this aspect. They continued to consider connections to real life and to mathematicians as an important positive aspect of mathematics lessons using history for its potential to motivate students.

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PRESERVICE TEACHERS' PERCEPTIONS OF A LESSON STUDY CONNECTING MULTICULTURAL LITERATURE WITH CULTURALLY RESPONSIVE MATHEMATICS TEACHING

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This study examines preservice teachers' perceptions about a microteaching lesson study (MLS) that integrated multicultural literature to elicit culturally responsive mathematics teaching (CRMT) during a mathematics methods course. Participation in the MLS' iterative cycle of plan-teach-revise encouraged preservice teachers to develop their pedagogical knowledge, make cultural connections from texts to mathematical concepts, and engage in productive reflection. The MLS also provided preservice teachers with a supportive peer learning community that fostered collaborative learning to improve professional practice around CRMT. Suggestions are shared for how similar practice-based experiences can be used to enhance teacher education with focused practice addressing mathematical thinking, language, culture, and social justice.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices, Teacher Knowledge, Culturally Relevant Pedagogy

Mathematics teachers committed to humanizing pedagogy recognize schools as cultural spaces that draw on students' identities and experiences to leverage mathematical competencies (Yeh & Otis, 2019). When students explore mathematics from global perspectives (e.g., non-Western, indigenous) and applications, they can engage in enriched, meaningful learning that promotes empathy for diverse cultures and fosters problem-solving that transcends borders. One way for students to develop cultural awareness is through reading multicultural literature, which can serve as a mirror to reflect on one's own culture and a window to gain perspective into other's values, beliefs, and traditions. While multicultural literature naturally fits in literacy studies, it can also be used to study mathematics in culturally relevant contexts (Chappell & Thompson, 2000; Leonard, Moore, & Brooks, 2014). Those who wish to bridge multicultural literature into the mathematics classroom must recognize mathematics as a cultural construct and have the opportunity to plan, teach, and reflect on lessons with such texts (Iliev & D'Angelo, 2014; Sletter, 1997). This study reports on preservice teachers in a mathematics methods course who participated in a microteaching lesson study that integrated multicultural literature to make social justice and cultural connections to elementary school mathematics content. The following research question served as motivation for this study: What are preservice teachers' perceptions about a microteaching lesson study using multicultural literature to elicit culturally responsive mathematics teaching during a mathematics methods course?

Theoretical Framework

Reform efforts in teacher education recommend preservice teachers have opportunities to teach mathematics through practice-based experiences that develop pedagogical knowledge and encourage reflective practice (AMTE, 2017). A microteaching lesson study is an example of a practice-based experience that utilizes a simulated teaching environment to reduce teaching complexities, develop pedagogical content knowledge, and elicit reflection from peer and self-assessment. Preservice teachers engaged in a microteaching lesson study benefit from collaborative participation in the iterative cycle of plan-teach-revise in a modified format (see Figure 1) that promotes connections between theory and practice in mathematics education (Fernández, 2005). This study builds on the

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research of microteaching lesson studies and examines how preservice teachers can develop consciousness of culturally responsive mathematics teaching (CRMT) with multicultural literature.

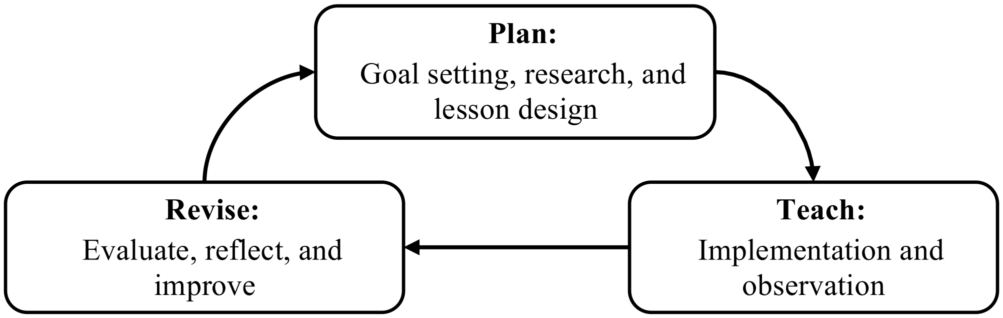

Figure 1: Lesson Study Model of the Iterative Cycle of Plan-Teach-Revise

CRMT can be used to privilege students' cultural and linguistic funds of knowledge, foster meaningful connections to students' prior experiences, and value students' strengths and performance styles to make learning relevant and effective (Gay, 2009). This research used the framework for CRMT coupled with Aguirre and Zavala's (2013) CRMT Lesson Analysis Tool to examine explicit characteristics of CRMT: mathematical thinking, language, culture, and social justice. Framing the inquiry in terms of CRMT aided with organizing the synthesis and examining where efforts were made by preservice teachers to use multicultural literature to elicit CRMT in a simulated teaching environment.

Research Methods

A qualitative case study design was used with multiple data sources. Participants included 16 preservice teachers (14 females and 2 males) enrolled in an elementary mathematics methods course at an urban university in the northeastern United States. In the course, the preservice teachers were introduced to CRMT as the notion of contextualizing mathematics teaching and learning to students' lives (Gay, 2002; Ladson-Billings, 1995a, 1995b). After the preservice teachers familiarized themselves with CRMT with the aid of course readings and a discussion-based review of a sample lesson that was critiqued using Aguirre and Zavala's (2013) CRMT Lesson Analysis Tool, the preservice teachers were assigned a microteaching lesson study that required them to plan, teach, and revise an elementary mathematics lesson with reference to a multicultural text of their choosing.

To help guide the text selection and the design of the activity, the preservice teachers reflected on the ways the text and the lesson: (a) portrayed cultural authenticity, (b) depicted cultural diversity as an asset, and (c) promoted culturally relevant mathematical connections (Harding, Hbaci, Loyd, & Hamilton, 2017). Next, the lessons were critiqued for components of mathematical thinking, language, culture, and social justice as noted in the CRMT Lesson Analysis Tool. Each group was asked to submit their analysis and make any necessary modifications to strengthen the lesson prior to beginning the next phase of the practice-based experience.

After submitting their lessons and planning reflections, the preservice teachers were videotaped teaching their lessons twice to their peers. In round one, two group members taught the lesson while their other two group members and peers observed. Next, the group members reflected on their instruction and revised their lesson. In round two, the group members switched roles, reflected on their instruction, and submitted their final revised lesson. Peers gave constructive feedback throughout the rounds and assessed how the overall experience influenced their perspective of CRMT and their future use of multicultural literature in elementary mathematics instruction. The lessons, videos, reflections, and peer feedback were coded for themes of CRMT using in vivo and descriptive coding techniques (Saldaña, 2016).

Results

Findings revealed that the preservice teachers' perceptions of participating in the microteaching lesson study's iterative cycle of plan-teach-revise encouraged them to develop their pedagogical knowledge, make cultural connections from texts to mathematical concepts, and engage in productive reflection. Overall, the preservice teachers expressed positive experiences engaging in the microteaching lesson study and shared that this particular focus on lesson design with multicultural literature helped them to engage and be reflective in CRMT. They also shared how the practice-based experience provided them with a supportive peer learning community that fostered collaborative learning to improve their professional practice around ways to make social justice and cultural connections to mathematical concepts.

Through strategic lesson design with multicultural literature, the preservice teachers were successfully able to engage their peers in mathematical thinking of various concepts (e.g., measurement, counting, addition, geometry) through the elicitation of meaningful mathematics discussions related to cultural connections. Several preservice teachers used the stories found in the texts to set a context for learning a mathematical concept. For example, one group selected a text about a girl named Sadako and her paper cranes. The story provided a context for students to explore adding paper cranes using various addition strategies (e.g., counting all/on, making tens, friendly numbers, compensation, adding up in chunks). Another group referenced a book about quilt making and created a lesson about how the arrangement of patches on a quilt could generate different dimensions (e.g., 1 x 24, 2 x 12, 3 x 8, 4 x 6). The lesson addressed various mathematical concepts (e.g., rectangular arrays, factors, area, perimeter) and provided opportunities for participants to create their own quilt patch that honors their cultural heritage to add to the classroom quilt. Through the lessons, the preservice teachers were able to elicit ways for participants to self-identify with the mathematics and see themselves as doers of mathematics. A preservice teacher summarizes this nicely in her statement: "In this experience, I was given the opportunity to think outside the box when it came to thinking of different ideas that would interest my students where they would have some ownership over the activity and feel included."

The practice of using multicultural literature to facilitate cultural connections also influenced the preservice teachers' awareness of and confidence in using cultural practices to address issues of social justice and exercise mathematics as an analytical tool to critique societal norms. For example, a preservice teacher shared, "I plan to use multicultural literature in my classroom to help students identify cultural assets and challenge/remove potential barriers." Additionally, several preservice teachers reflected on how they plan to use such texts to advocate for exploring new mathematical concepts (e.g., numerical representations, computations) that may not be present in the dominant mathematics studied. This is evident in a preservice teacher's reflection that said: "I have newfound confidence to incorporate other texts than what may appear in the curriculum to provide alternative perspectives and purposefully incorporate students' native languages in the teaching of mathematical concepts."

The preservice teachers also reflected on their own professional growth to enhance their pedagogical practice given the collaborative nature of the microteaching lesson study. Several preservice teachers noted the amount of effort that goes into planning meaningful lessons. For example, a preservice teacher said: "I now realize that a lot of planning and preparation is required to really give students the best possible mathematics lessons. I now have new tools to help me prepare the best lessons I can for my students." Other preservice teachers reflected on their newfound realization about culture in the classroom. For instance, a preservice teacher said: "Not only did this experience teach me how to create a lesson plan and be aware of the academic needs of my students (or peers), but it also taught me how to be aware of their backgrounds and how culture plays a huge role in all subjects." Similarly, another preservice teacher commented on how her appreciation has grown for using literature to shed light on how people engage with mathematics outside of the mainstream, Westernized perspective. She stated: "By incorporating multicultural literature, mathematics can become relatable and mathematics can create bridges that connect cultures and people. I now realize how much it can elevate a lesson." A comparable statement made by another preservice teacher noted: "Previously, I had not considered how to integrate multicultural content into mathematics instruction. Yet, incorporating multicultural mathematics literature into instruction can foster greater acknowledgement and appreciation of students' cultural identities, which facilitates better learning environments in our classrooms and schools."

Discussion

The significance of this work is to build on the research of microteaching lesson studies and examine how preservice teachers' perceptions engaging in such a practice-based experience provides insight into how preservice teachers develop pedagogical knowledge and benefit from reflective practice. This study was unique in that the microteaching lesson study connected multicultural literacy with CRMT. Participation in this experience benefited the preservice teachers in that they were able to collaborate with one another and use constructive peer feedback to guide improvements in the planning, implementation, and reflection of their lessons. The preservice teachers were tasked with creating a learning environment that empowered students to see and engage with mathematics across cultures. Multicultural literature can serve as a conversation starter to introduce new mathematical ideas and widen students' horizons. The findings from this work signify that efforts must be made in teacher preparation to implement innovative practices (e.g., microteaching lesson studies, multicultural literature) that elicit CRMT to improve the quality of mathematics instruction for all students.

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CONCEPTUAL SYSTEMS WHEN IMPLEMENTING MODEL-ELICITING ACTIVITIES

SISTEMAS CONCEPTUALES AL IMPLEMENTAR ACTIVIDADES PROVOCADORAS DE MODELOS

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In this article we analyze the results of a study focused on promoting the development of conceptual systems of teachers, associated with the teaching and learning of the exponential function when implementing Model-Eliciting Activities [MEA] in the classroom. For the theoretical framework we used the models and modeling perspective [MMP]. The methodology was qualitative, and a multilevel approach (researcher, teacher, and students) was used. The results show how the conceptual system of a teacher was modified, extended, and refined. First, the teacher focused on the instructional dimension, then he expanded, and refined to include the mathematical and historical content dimensions.

Keywords: Preparation of teachers in training, Mathematical Knowledge for Teaching, Modeling, Pre-calculation.

There is a need to carry out research related to the professional development of mathematics teachers (Doerr, 2004; Kieran, 2007; Jung, 2013; Sevinc & Lesh, 2018). According to Clark and Lesh (2003), Doerr and Lesh (2003), Sevinc and Lesh (2018) the MMP allows the description of the teachers' conceptual system related to the teaching and learning of mathematics and how this system evolves. The research question discussed in this article is: How does the teachers' conceptual system –related to the process of teaching and learning of the exponential function– evolve as they design, implement and evaluate MEAs?

Theoretical Framework

The Models and Modeling Perspective mentions that learning is a process of developing conceptual systems, which continually change, modify, extend, and refine during the student's interactions with their environment (teachers and peers) and when solving problems (Lesh, 2010). The theoretical framework used in this document is the MMP (Doerr, 2016; Lesh, 2010; Lesh & Doerr, 2003), which suggests the use of multi-level methodologies to support the development of teacher training programs. The MMP makes it possible to provide a context for students to develop models when carrying out the MEAs; at the same time, teachers are immersed in environments where the development of models is encouraged by interpreting, explaining, and predicting student behavior when modelling MEAs (Sevinc & Lesh, 2018). The interaction among students, teachers, and researchers is considered important to promote the development of knowledge, and to understand the evolution of teachers' knowledge based on the interpretation they make about how individuals reveal, test, refine, and review their knowledge and skills (Clark & Lesh, 2003). In other words, the teacher influences the construction of knowledge by students and vice versa. Something similar occurs with researchers; they influence teachers and, at the same time, are affected by them.

Doerr and Lesh (2003) propose to design didactic sequences (model development sequences) to support the development of conceptual systems. Sequences include Model Eliciting Activities [MEA], Model Exploration Activities [MXA], and Model Adaptation Activities [MAA] to promote the student to manipulate, share, modify, and reuse models to build, describe, explain, manipulate, predict or control mathematically significant systems (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Helping teachers understand the MMP and how to use it to promote mathematics learning is an interesting and complex task.

Clark and Lesh (2003) point out that knowing the nature of teacher understanding involves understanding at least three dimensions such as: "psychological connections, instructional connections, and historical connections" (p. 159). That is, how ideas develop in the minds of children and adults, how the development of ideas can be supported by the use of available curriculum materials, and the circumstances under which students develop an idea in a historical way. Therefore, the description of the teacher's conceptual system can be done through the use of dimensions such as: the teacher's mathematical knowledge dimension [DM], the instructional dimension [DI] and the historical dimension of the students' development of knowledge [DH] (Clark & Lesh, 2003). These dimensions are used to describe the teacher's conceptual system about teaching and learning of the exponential function and its evolution, or modification, extension and refinement during the process of designing, implementing, and evaluating a didactic sequence based on the MMP.

Methodology

The methodology was qualitative. The "teacher training under the models and modeling perspective" scheme was developed over 18 months, under a multilevel perspective (researcher, teacher and students) (Doerr & Lesh, 2003). The participants in this study were two teachers, but the process will only be exemplified with one of them. The data sources were: activities designed by the teacher, students' documents, audios of the interactions –student-student, teacher-student, researcher-teacher– and the students' interpretations of the created models when carrying out activities. Three stages are described, each characterized by: design, implementation and evaluation of a didactic sequence (MEA-MXA-MAA). The evolution of the teachers' conceptual system related to the teaching and learning of the exponential function is interpreted through the dimensions observed by Clark and Lesh (2003): a) DI, b) DM, and c) DH.

Results and Discussion

First stage (MEA Design-Implementation-Evaluation Cycle)

i) MEA design (researcher-teacher). The researcher encouraged the teacher to reflect on the following dimensions. a) DM: analysis of the concept of exponential function; b) DI: revision of textbooks, bibliography of the MMP, and use of technology. The teacher built an MEA in the population growth context. The underlying knowledge was the exponential function. Previously, the teacher participated as an observer in the implementation of an MEA.

ii) First implementation (teacher-students). Five students participated, which were grouped into two teams. a) DM: One of the teams included tabular, graphic, and verbal representations in their models. The other team included only tabular and verbal representations. The teacher and the students, during the group discussion, wrote the algebraic function $P(t) = P_i(1 + r)^t$ associated with the situation. b) DI: The design and implementation of the MEA enabled students to use calculators and spreadsheets to solve the problem situation. c) DH: The teacher focused on the students' interest generated by the activity and on the final representations they presented. He did not observe the process of building the model.

iii) Evaluation of the first implementation (researcher-teacher). The interaction between the teacher and the researcher allowed the interpretation and analysis of how the concept of exponential function was constructed, modified, extended and refined (DM) by the students, when carrying out the MEA (DH). Based on the review of audios, the letters of the students, and the six design principles of the MEAs, the teacher analyzed the activity (Vargas-Alejo & Montero-Moguel, 2019). The support that the researcher provided to the teacher was essential to modify the MEA before the next implementation, as well as the role of the teacher in the classroom (DI).

Second stage (Design-Implementation-Evaluation Cycle of MEA-MXA)

i) Design of MEA and MXA (researcher-teacher). The teacher's conceptual system evolved in the three dimensions. a) DM: The teacher proposed that students should perform an MXA to deepen the mathematical concepts of variation, covariation, base, exponent, ordered to the origin, growth rate, graph translation, increasing and decreasing function that emerged when carrying out the MEA. b) DI: The researcher proposed to include NetLogo –multi-agent programmable modeling environment– in the MXA. The teacher, supported by an expert researcher, modified one of the activities of the software library; he also included a GeoGebra applet in the sequence. c) DH: The modifications made by the teacher were based on the analysis of how the students created, modified, and expanded their conceptual system in the first implementation.

ii) Second implementation (teacher-students). Ten students were grouped into four teams a) DM: the four teams included tabular, graphical, algebraic and verbal representations in the models. The teacher validated all the representations, but observed that some teams did not differentiate constant growth from exponential growth, therefore, during the group discussion, he intervened through questions to clarify the difference. b) DI: The activity with NetLogo and the GeoGebra applet allowed the students to dive deeper into the concepts associated with the exponential function. c) DH: The teacher extended his interpretation and explained, in a more detailed way, the process of model construction by the students.

iii) Interaction of the researcher and the teacher after the second implementation (Researcher-Teacher). The teacher described how the students went through different cycles when they carried out the MEA. Then, he pointed out how in each cycle the students expanded and refined their knowledge of the linear function and, later, the exponential function. Finally, he mentioned how the MXA made it possible to go deeper into the mathematical concepts immersed in the models, even when it was necessary to institutionalize them (DM and DH). Based on the review of "the MEA quality assessment guide" (Lesh, 2010, p. 32) the teacher analyzed the audios and letters of the students. He characterized the models constructed by the students, with or without technology (DI) and distinguished between models that require direction, models that require extension or refinement, situated models, and comparable and reusable models.

Third stage (Design-Implementation-Evaluation Cycle of MEA-MXA-MAA)

i) Design of MAA (researcher-teacher). a) DM: The teacher, supported by the researcher, designed the MAA in the context of energy saving and investments where the concepts of linear function and exponential function are underlying. He gave meaning to the design of a model development sequence composed of MEA, MXA and MAA, to promote the development of the exponential function concept. b) DI: He designed the sequence based on Lesh et al. (2003). c) DH: The teacher enabled the transition through different modeling cycles, and encouraged students to review, deepen and expand their mathematical knowledge related to the exponential function.

ii) Third implementation (teacher-students). Ten students participated and were divided into four teams a) DM: All students construct models that include: the linear function to describe the savings situation, the exponential function to analyze investments, and a diversity of verbal, tabular, graphical and algebraic representations. b) DI: The models created to solve the problem situation included in the MAA were based on the knowledge developed using NetLogo and GeoGebra (MXA). c) DH: The teacher explained how the knowledge about the exponential function learned by the students when carrying out the MEA allowed them to carry out the MAA. The models were characterized as shareable and reusable.

iii) Interaction of the researcher and the teacher after the third implementation (researcher-teacher). The teacher described the different modeling cycles developed by the students when solving the model development sequence and analyzed how they evolved during the realization of the MEA, MXA and MAA (DH). He used the levels of covariation (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) and "the MEA quality assessment guide" (Lesh, 2010, p. 32) to generate a classification of the models built, in terms of the use of linear and exponential relationships (DM); the understanding of the exponential function and differentiation of the linear function; as well as the use given to the functions to interpret and predict the situation.

Conclusions

How does the teachers' conceptual system –related to the process of teaching and learning of the exponential function– evolve as they design, implement and evaluate MEAs? The evolution of the conceptual system was as follows: the emphasis was placed first on DI, then on DM, and finally, the three dimensions, DI, DM and DH, became relevant. In the first stage, the teacher focused on the description of representations created with and without the support of technology. He described whether or not the students proposed the exponential function in its algebraic form. He emphasized the final product and not the process of construction of the exponential function concept, which he did in the second stage. In the third stage, the teacher was able to describe how the students developed mathematical ideas, modified them, adapted them and refined them throughout the process of carrying out the sequence. This allowed him to realize that in the process of learning the exponential function, students can first use the linear function by assuming a constant variation and then build knowledge about the exponential function.

The MXA allowed the students to go deeper into concepts. This was observed when they carried out the MAA, as students were able to use the concepts of exponential and linear function to solve a different activity, in a different context. They learned to differentiate and use both functions to describe the MAA. The multilevel interaction between students, teacher and researcher, during each phase of the different stages was fundamental for the evolution of the conceptual system of the teacher related to the process of teaching and learning of the exponential function. Although it is true that there is no linearity in the development of conceptual systems, this study identified that the design, implementation and analysis of each activity contributed to the modification, expansion and refinement of the teacher's conceptual system.

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SISTEMAS CONCEPTUALES AL IMPLEMENTAR ACTIVIDADES PROVOCADORAS DE MODELOS

CONCEPTUAL SYSTEMS WHEN IMPLEMENTING MODEL-ELICITING ACTIVITIES

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En este artículo se analizan resultados de una investigación centrada en promover el desarrollo de sistemas conceptuales de los profesores, asociados a la enseñanza y aprendizaje de la función exponencial al implementar Actividades Provocadoras de Modelos (MEAs por sus siglas en inglés: Model-Eliciting Activities) en el aula. El marco teórico fue la perspectiva de modelos y modelación [PMM]. La metodología fue cualitativa, se usó un enfoque multinivel (investigador, profesor y estudiantes). A partir de los resultados, se observó cómo el sistema conceptual se modificó, amplió y refinó. Primero se centró en la dimensión instruccional, enseguida se amplió y refinó para incluir la dimensión de contenido matemático y dimensión histórica.

Palabras Clave: Preparación de maestros en formación, Conocimiento Matemático para la Enseñanza, Modelación, Pre-cálculo.

Existe necesidad por realizar investigaciones relacionadas con el desarrollo profesional de los docentes de matemáticas (Doerr, 2004; Kieran, 2007; Jung, 2013; Sevinc & Lesh, 2018). De acuerdo con Clark y Lesh (2003), Doerr y Lesh (2003), Sevinc y Lesh (2018) la PMM permite describir el sistema conceptual de los profesores sobre la enseñanza y aprendizaje de las matemáticas y cómo este sistema evoluciona. La pregunta de investigación que se discute en este artículo es: ¿Cómo evoluciona el sistema conceptual de los docentes sobre la enseñanza y aprendizaje de la función exponencial al diseñar, implementar y evaluar MEAs?

Marco Teórico

En la PMM se menciona que aprender matemáticas es un proceso de desarrollo de sistemas conceptuales, que cambian de manera continua, se modifican, extienden o amplían y refinan a partir de las interacciones del estudiante con su entorno (los profesores y compañeros) y al resolver problemas (Lesh, 2010). El marco teórico utilizado en este documento es la PMM (Doerr, 2016; Lesh, 2010; Lesh & Doerr, 2003), la cual sugiere el uso de metodologías multinivel para apoyar el desarrollo de programas de formación docente. Este enfoque posibilita proveer un contexto para que los estudiantes desarrollen modelos al resolver MEAs; de manera simultánea, los profesores se ven inmersos en ambientes donde se propicia el desarrollo de modelos al interpretar, explicar y predecir el comportamiento de los estudiantes al modelar las MEAs (Sevinc & Lesh, 2018). Se considera importante la interacción entre estudiantes, profesores e investigadores para propiciar el desarrollo de conocimiento, además permite entender la evolución del conocimiento de los profesores a partir de la interpretación que hacen acerca de cómo los individuos revelan, prueban, refinan y revisan sus conocimientos y habilidades (Clark & Lesh, 2003). Es decir, el profesor influye en la construcción de conocimiento por los estudiantes y viceversa. Lo mismo ocurre con los investigadores, quienes influyen en los docentes y se ven afectados por los mismos.

Doerr y Lesh (2003) proponen la estructuración de secuencias didácticas para apoyar el desarrollo de sistemas conceptuales. Las secuencias incluyen Actividades Provocadoras de Modelos [MEA], Actividades de Exploración de Modelos [MXA] y Actividades de Adaptación de Modelos [MAA] para promover que el alumno manipule, comparta, modifique y reutilice modelos, para construir, describir, explicar, manipular, predecir o controlar sistemas matemáticamente significativos (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Apoyar que los profesores comprendan la PMM y la utilicen para propiciar el aprendizaje de las matemáticas es una tarea interesante y compleja.

Clark y Lesh (2003) señalan que conocer la naturaleza de la comprensión del profesor implica entender al menos ciertas dimensiones, como: “conexiones psicológicas, instruccionales e históricas” (p. 159); es decir, cómo se desarrollan las ideas en la mente de los niños y adultos, cómo puede apoyarse el desarrollo de ideas mediante el uso de materiales curriculares disponibles, y las circunstancias bajo las cuales se desarrolló una idea en los estudiantes, de manera histórica. Por lo tanto, la descripción del sistema conceptual de los docentes se puede hacer mediante el uso de dimensiones tales como: dimensión de conocimiento matemático del docente [DM], dimensión instruccional [DI] y dimensión histórica, sobre el desarrollo de conocimiento por los estudiantes [DH] (Clark & Lesh, 2003). Con estas dimensiones, se describe en este documento el sistema conceptual de un profesor sobre la enseñanza y aprendizaje de la función exponencial y su evolución o bien modificación, ampliación y refinamiento durante el proceso de diseño, implementación y evaluación de una secuencia didáctica fundamentada en la PMM.

Metodología

La metodología fue cualitativa. El esquema de formación de docentes bajo la PMM se desarrolló a lo largo de 18 meses, bajo una perspectiva multinivel (investigador, profesor y alumnos) (Doerr & Lesh, 2003). Los participantes en este estudio fueron dos docentes, pero sólo se ejemplificará el proceso con uno de ellos. Las fuentes de datos fueron: actividades diseñadas por los docentes, documentos de los estudiantes, audios de las interacciones estudiantes-estudiantes, docente-estudiantes e investigador-docente y los reportes de las interpretaciones a los modelos construidos por los estudiantes al realizar actividades. Se describen tres etapas, cada una caracterizada por: diseño, implementación y evaluación de una secuencia didáctica (MEA-MXA-MAA). La evolución del sistema conceptual de los docentes sobre la enseñanza y aprendizaje de la función exponencial se describe a través de las dimensiones observadas por Clark y Lesh (2003): a) DI, b) DM y c) DH.

Resultados y Análisis de Resultados

Primera etapa (Ciclo Diseño-Implementación-Evaluación de MEA)

i) Diseño de MEA (investigador- profesor). El investigador propició reflexión del docente en las dimensiones: a) DM: análisis del concepto de función exponencial; b) DI: revisión de libros de texto, de bibliografía de la PMM y uso de tecnología en la MEA. El docente construyó una MEA en el contexto de crecimiento poblacional. El conocimiento subyacente fue la función exponencial. Previamente, participó como observador en la implementación de una MEA.

ii) Primera implementación (profesor- alumnos). Participaron cinco alumnos, agrupados en dos equipos. a) DM: Uno de los equipos incluyó en sus modelos representaciones tabular, gráfica y verbal y el otro equipo, sólo tabular y verbal. El docente y los alumnos, durante la discusión grupal, construyeron la función algebraica $P(t) = P_i(1 + r)^t$asociada a la situación. Se basaron en los modelos desarrollados en la clase. b) DI: El diseño e implementación de la MEA, contempló que los alumnos pudieran utilizar calculadoras y hojas de cálculo de Excel para que resolvieran la situación problema. c) DH: El profesor centró su atención en el interés de los alumnos generado al realizar la actividad y en las representaciones finales presentadas por los alumnos. No observó cómo fue el proceso de construcción del modelo.

iii) Evaluación de la primera implementación (investigador- profesor). La interacción entre el docente y el investigador permitió interpretar y analizar la forma como el concepto función exponencial fue construido, modificado, ampliado y refinado (DM) por los estudiantes, al realizar la actividad (DH). Con base en los seis principios de diseño de las MEAs, la revisión de audios y las cartas de los estudiantes, el docente analizó la actividad, publicada por Vargas-Alejo & Montero-Moguel (2019). El apoyo del investigador fue fundamental para modificar la MEA antes de una siguiente implementación, así como las estrategias instruccionales (DI).

Segunda etapa (Ciclo Diseño-Implementación-Evaluación de MEA-MXA)

i) Diseño de MEA y MXA (investigador- profesor). El sistema conceptual del docente evolucionó en las tres dimensiones. a) DM: El docente propuso que los alumnos deberían realizar una MXA para profundizar en los conceptos matemáticos –variación, covariación, base, exponente, ordenada al origen, tasa de crecimiento, traslación de gráficas, función creciente y decreciente– que emergieron al realizar la MEA. b) DI: El investigador propuso incluir NetLogo –el entorno de modelación programable multi-agente– en la MXA. El docente apoyado por un investigador experto modificó una de las actividades de la biblioteca del software; incluyó, además, en su secuencia un applet con GeoGebra. c) DH: Las modificaciones que realizó el docente se fundamentaron en el análisis de cómo los estudiantes habían construido, modificado y ampliado su sistema conceptual en la primera implementación.

ii) Segunda implementación (Profesor- Alumnos). Diez estudiantes fueron agrupados en cuatro equipos. a) DM: los cuatro equipos incluyeron representaciones tabulares, gráficas, algebraicas y verbales en los modelos. El docente validó todas las representaciones, pero observó que algunos equipos no diferenciaban el crecimiento constante de un crecimiento exponencial, por lo tanto, durante la discusión grupal, intervino mediante preguntas para que se aclarara la diferencia. b) DI: La actividad con NetLogo y el applet de GeoGebra permitieron que los estudiantes pudieran profundizar en los conceptos asociados a la función exponencial. c) DH: El profesor extendió su interpretación y explicó, de una manera más detallada, el proceso de construcción de modelos por los estudiantes.

iii) Interacción del investigador y el docente después de la segunda implementación (Investigador- Profesor). El docente describió cómo los alumnos transitaron por diferentes ciclos cuando realizaron la MEA, señaló cómo en cada ciclo los alumnos ampliaron y refinaron su conocimiento sobre la función lineal y, posteriormente, la función exponencial. Finalmente, mencionó cómo la MXA posibilitó la profundización en los conceptos matemáticos inmersos en los modelos, aún cuando se requirió institucionalizarlos (DM y DH). Con base en la revisión de “la guía de evaluación de calidad de MEA” (Lesh, 2010, p. 32) el docente analizó los audios y las cartas de los estudiantes. Caracterizó los modelos construidos por los estudiantes, con o sin tecnología (DI) para distinguir entre los modelos que requieren dirección, modelos que requieren extensión o refinamiento, modelos situados, y modelos compartibles y reutilizables.

Tercera etapa (Ciclo Diseño-Implementación-Evaluación de MEA-MXA-MAA)

i) Diseño de MAA (investigador- profesor). a) DM: El docente, apoyado por el investigador, diseñó la MAA en el contexto de ahorro de energía e inversiones donde subyacen los conceptos de función lineal y función exponencial. Dio significado al diseño de una secuencia de desarrollo de modelos compuesta por MEA, MXA y MAA, para propiciar el desarrollo del concepto de función exponencial. b) DI: Él diseñó la secuencia con base en Lesh et al. (2003). c) DH: El docente posibilitó la transición por diferentes ciclos de modelación, y fomentó que los alumnos revisaran, profundizaran y ampliaran su conocimiento matemático relacionado con la función exponencial.

ii) Tercera implementación (Profesor- Alumnos). Participaron diez alumnos agrupados en cuatro equipos a) DM: Todos los alumnos construyeron modelos que incluyeron: la función lineal para describir la situación del ahorro, la función exponencial para analizar las inversiones, y una diversidad de representaciones verbal, tabular, gráfica y algebraica . b) DI: Los modelos creados para resolver la situación problema incluida en la MAA, se basaron en el conocimiento desarrollado al usar NetLogo y GeoGebra (MXA). c) DH: El profesor explicó cómo el conocimiento sobre la función exponencial, aprendido por los estudiantes al realizar la MEA, les permitió realizar la MAA. Los modelos fueron caracterizados como compartibles y reutilizables.

iii) Interacción del investigador y el docente después de la tercera implementación (Investigador- Profesor). El docente describió los diferentes ciclos de modelación desarrollados por los alumnos al resolver la secuencia de desarrollo de modelos y analizó cómo ellos evolucionaron durante la realización de la MEA, MXA y MAA (DH). Utilizó los niveles de covariación (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002) y “la guía de evaluación de calidad de MEA” (Lesh, 2010, p. 32) para generar una clasificación de los modelos construidos, en términos del uso de relaciones lineales y exponenciales (DM); de la comprensión de la función exponencial y diferenciación de la función lineal; así como del uso que se les dio a las funciones para interpretar y predecir la situación (MAA).

Conclusiones

¿Cómo evoluciona el sistema conceptual sobre el proceso de enseñanza y aprendizaje de la función exponencial de los docentes al diseñar, implementar y evaluar MEAs? El mayor énfasis se puso primero en la DI, enseguida en la DM, y finalmente, las tres dimensiones, DI, DM y DH, tomaron relevancia.

En la primera etapa el docente se concentró en describir las representaciones construidas con o sin el apoyo de la tecnología. Describió si los alumnos propusieron o no la función exponencial en su forma algebraica en su modelo. Puso énfasis en el producto final y no en el proceso de construcción del concepto de función exponencial, lo cual hizo en la segunda etapa. En la tercera etapa, el docente logró describir cómo los estudiantes desarrollaron ideas matemáticas, las modificaron, adaptaron y refinaron durante todo el proceso de realización de la secuencia. Ello le permitió concluir que, en el proceso de aprendizaje de la función exponencial, los estudiantes podrían usar primero la función lineal al suponer una variación constante y, posteriormente, construir la función exponencial.

La MXA permitió que los estudiantes lograran profundizar en conceptos matemáticos relacionados con la función exponencial. Esto se vio reflejado al realizar la MAA, ya que los estudiantes pudieron usar los conceptos sobre función exponencial y lineal para realizar una actividad distinta, en diferente contexto. Aprendieron a diferenciar y utilizar ambas funciones para describir la MAA. La interacción multinivel entre los estudiantes, profesor e investigador, durante cada una de las fases de las distintas etapas fue fundamental para que evolucionara el sistema conceptual del docente relacionado con el proceso de enseñanza y aprendizaje de la función exponencial. Si bien es cierto que no existe linealidad en el desarrollo de sistemas conceptuales, en este estudio se identificó que el diseño, implementación y análisis de cada actividad contribuyó para que el sistema conceptual del docente se modificara, ampliara y refinara.

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EXPLICIT TEACHING OF QUESTIONING IN MATH METHODS COURSE: PRESERVICE TEACHERS' ATTEMPTS TO ASK PROBING QUESTIONS

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After an explicit unit of core activities on questioning, preservice teachers (PTs) completed an assignment to select a problem-solving task, anticipate student solutions, and plan probing questions. After analyzing PTs' work, we discovered that, although most PTs planned probing questions, many also planned questions focused on information or procedures. Next steps include exposing PTs to probing questions focused on meanings, context, or representations.

Keywords: Instructional activities and practices, Teacher Education- Preservice

According to National Council of Teachers of Mathematics (NCTM, 2014), effective teaching of mathematics requires asking "purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships" (p. 35). Although teacher questioning has long been viewed as a critical component of mathematics teaching, rigorous and challenging mathematics standards (Common Core State Standards for Mathematics [CCSSM], 2010) and NCTM's (2014) release of effective mathematics teaching practices brought attention back to teacher questioning in the U.S. Especially, given that teachers in the U.S. ask fewer probing questions that support the deep levels of student understanding than teachers in other high-achieving countries (Stigler & Hiebert, 2009a,b), it is critically important to put emphasis on the practice of asking purposeful questions in teacher preparation programs. The main reasons for asking purposeful questions are to surface the student's understanding and reasoning, to probe student thinking, and to gather more information related to the student's understanding of their problem-solving strategies, key mathematical ideas, and meaning inherent in representations (Huinker & Bill, 2017). In an attempt to foster preservice teachers' (PTs) ability to use such purposeful probing questions, we contemplated that an explicit teaching of questioning was needed, and we planned and implemented a series of core activities in a methods course for teaching mathematics in the elementary and middle grades. After PTs were explicitly taught questioning through the series of core activities, they completed an assignment in which they selected a problem-solving task, generated different anticipated student solutions to the task, and planned probing thinking questions for each of their anticipated student solutions. We analyzed PTs' planned questions in the assignment, in order to determine how our explicit teaching of questioning through a series of core activities helped them cultivate good questioning skills, and to identify next steps that we would either begin to address or continue to implement in our methods courses. The guiding question for this study is: What types of questions do elementary PTs propose in order to probe the thinking of their students?

Theoretical Framework

Teacher questioning in mathematics has been investigated in many studies with a focus on inservice teachers (e.g., Boaler & Brodie, 2004; Franke et al., 2009; Kawanaka & Stigler, 1999; Martino & Maher, 1999; Myhill & Dunkin, 2005; Shahrill, 2013; Sahin & Kulm, 2008; Wimer, et al., 2001); however, fewer studies have focused on PTs' questioning (e.g., Akkoç, 2015; Cakmak, 2009; Hähkiöniemi, 2017; Hollebrands & Lee, 2016; Moyer & Milewicz, 2002; Weiland, et al., 2014). Akkoc, Hollebrands and Lee, and Hähkiöniemi investigated PTs' questioning in relation to technology, in computerized environments, or with the use of dynamic software. Moyer and

Milewicz observed PTs' assessment interviews with students, and found that they often asked questions with single answers in rapid succession, rather than probing for student thinking. Weiland et al.'s case study observed two PTs' assessment interviews in their field study; found an increase in their competent follow-up questions; and suggest that PTs can develop their questioning skills through rich field experiences. Cakmak reported that PTs often view the role of questioning as simply a way to motivate students or get their attention.

Many of the studies have produced classifications for probing questions in mathematics. Sahin and Kulm classified teacher probing questions as ask students to (1) explain/elaborate their thinking, (2) use prior knowledge/apply, (3) justify/prove. Kawanaka and Stigler classified questions as requesting (1) analysis/synthesis/conjecture/evaluation, (2) how to proceed in solving a problem, (3) the methods that were used to solve a problem, (4) the reasons why something is true/why something works/why something is done, and (5) other information. Boaler and Brodie's classification was: (1) exploring mathematical meanings and/or relationships, (2) probing, getting students to explain their thinking, and (3) extending thinking. Moyer and Milewicz categorized PTs' questioning strategies as questioning of (1) only incorrect responses, (2) non-specific questioning that did not acknowledge an individual child's responses, and (3) competent questioning that attended to a child's responses and probed for more information. Hähkiöniemi (2017) divided PTs probing questions into categories as probing method, reasoning, cause, meaning, argument, extension, and unfocused probing.

In this study, we aim to explore the nature of elementary PTs' probing questions as a result of explicit teaching of questioning skills through a series of core activities, and in relation to implementing problem-solving tasks. Rather than using an existing classification, we explored the question types that emerged from our PTs, which would complement the previous studies, help us better understand the ways our elementary PTs attempt to ask probing questions, and aid us in redesigning our explicit teaching of questioning module based on the emerging types.

Methods

The sample consisted of 115 PTs in five sections of an elementary/middle math methods course for a mix of elementary and special education majors. In an attempt to foster PTs' ability to use purposeful probing questions, we designed and implemented an explicit questioning module involving a series of core activities, as recommended by Morrissey, et al. (2019).

First, PTs studied the four types of questions as gathering information, probing thinking, making the math visible, encouraging reflection and justification (NCTM, 2014), and the idea of assessing and advancing questions (Huinker & Bill, 2017) through descriptions, examples, and discussion. Then, they watched and analyzed classroom teaching episodes that demonstrate the teacher's use of purposeful questions, and examined excerpts from lesson transcripts for the types of questions posed by the teachers. They categorized teacher's questions by four types and also identified whether each question was assessing or advancing. Next, they examined lesson plans that include planned purposeful questions during implementation of problem-solving tasks. These lesson plans included teacher's planned questions for a variety of anticipated student solutions to a mathematical problem involving various degrees of mathematical sophistication and understanding. Last, they studied four different student works to a given problem-solving task, and created three (two assessing and one advancing type) purposeful probing questions for each of the four student work samples. Finally, they completed an assignment in which they selected a problem-solving task, generated a number of different anticipated student solutions to the task, and planned probing thinking questions for each of their anticipated student solutions. The assignment prompted PTs to ask questions about the anticipated student work in order to help students deepen their understanding, to probe into their reasoning, and/or to have them evaluate and judge their own work. We analyzed PTs' planned questions in order to explore PTs' competency in asking probing questions, as a result of explicit teaching of questioning through a series of core activities. We used an open coding strategy (Strauss & Corbin, 1990) to categorize PTs' probing questions. One researcher analyzed the questions, and proposed initial categories with descriptions. Two other researchers reviewed a subgroup of the data, and agreed with the categories, and suggested revisions for the descriptions. Then, the three researchers, in collaboration, revised the descriptions to make them more subtle and accurate. After finalizing the categories and descriptions, one researcher coded the data; another researcher reviewed the codes and indicated their disagreements with the coding. The discrepancies were discussed, and the codes were revised, repeatedly, until 100% agreement was achieved.

Six categories of questions emerged: (1) focusing on information and/or procedures, which ask students to identify numbers, shapes, key words, and/or procedures to use to solve a problem; (2) focusing on context, which ask students to identify information given in a problem, think in the context of the problem, or connect their solution back to problem context; (3) focusing on meanings, which ask students to explain why something was done, interpret the meaning of given information, explain their reasoning, or interpret the meaning of the answer in terms of the context; (4) focusing on representations, which ask students to use different ways to represent a problem situation, such as draw a picture, write an equation, use manipulatives, etc., ask students about their current representation, or ask students to justify their representations; (5) general questions with no focus, which ask students to explain how they approached solving a problem with no reference to the context of a problem, or ask students general questions about the concepts involved in a problem - overall, this type of question can be used with any other problem students are asked to solve; and (6) other, in which the PT will show, explain, talk through, remind, give an alternative problem –in general, questions are not directed at students. The emerging categories showed that PTs were able to ask probing questions that focus on context, meanings, or representations. Yet, some question types that focus solely on information and procedures, and do not probe into student thinking also emerged, along with general questions and other comments/questions unrelated to students' current work and understanding.

Results and Discussion

All but eight PTs proposed more than one type of question. The assignment required PTs to only ask questions that probe into student reasoning in their current solution, and that help them deepen their understanding. However, 88% of the PTs asked at least one question related to information and procedures, which do not consider the underlying conceptual understanding of the student, and do not probe into student thinking as required. It is worrisome that the majority of the PTs believed that questions related to information and procedures are probing thinking questions. For example, "Why did you solve for the smaller numbers instead of doing it as a large number?" is a question that is not linked to underlying meanings, problem context, or meanings inherent to representations. Frequently, the questions in this category focused on numbers, procedures, or notations, as in "Why did you use an equal sign instead of any other sign?" and "Could you use any other number and get the same result?"

Most (72%) of the PTs asked questions that focus on the meanings. This is an important component of questioning, as requiring students to explain the meaning of their solutions serves to reveal underlying gaps in students' conceptual understanding. Sample questions coded in this category are when PTs asked questions about the meanings of the numbers or operations involved in the solution, such as "What does 159 represent [in your solution]?" "Why do we add 1 to Fred's equation?" "Why did you use multiplication?" and "What does the graph tell you?"

About 35% of the PTs asked questions that focus on the representations, which also require students to explain their reasoning as they justify their representations. These types of questions provide the strong base of conceptual understanding that is necessary for building procedural fluency (NCTM, 2014). Sample questions coded in this category are: "How did you show in your picture that Johnny gave away to Tina?" "How could you represent this with pictures?" "What if they added two more bow colors, how would your drawing and equation change?" and "How could you show your circles using a number sentence?"

Although making connections between the solution and the problem context is also an important part of mathematics proficiency, as recommended in the CCSSM, only 14% of the PTs planned questions asking students to relate their solution to the problem context. For example: "What does your equation have to do with the lights Garrett wants to buy?" and "What relationship does doubling the savings have to do with the price of the shirts [in the problem]?"

Thirty-four percent of the PTs planned general questions with no relationship to the context of the problem, which did not assist students in evaluating their solutions and moving forward in solving a problem. For example, "How did you get your answer?" or "What strategy did you use to solve this problem?" were coded as general questions, because they are not specific to the student's current understanding, and can be asked for any type of student work. When PTs ask how students got their answers, students in most cases would simply go through the steps of their solutions. Another most frequent general question was, "Have you thought of another way this could be done?" which is not related to a student's work and understanding. Other general questions asked about general facts without connection to a student's work or problem, such as "How many minutes are there in an hour?" "What is the formula for area?" etc.

In a few cases, PTs either planned to show, explain, or talk through the problem and solution, instead of asking probing questions, or planned to tell students to check their work. Further analyses of question types in the future will consider variability across content and grade level.

Implications

The high percentage of PTs who planned to ask probing questions indicated that explicit teaching of questioning through core activities helped cultivate PTs questioning skills. Nevertheless, the low percentage of PTs who proposed questions relating student solutions to problem context will be addressed in future methods courses, as authors focus specifically on planning questions that move students forward and are related to meaning, representations, and context of solutions rather than general or procedural questions. The emerged categories of probing questions, as focusing on meanings, context, or representations, will be used as a guide for potential types of probing questions that can be asked specifically to probe into students' conceptual understanding. PTs will be asked to analyze teaching episodes and lesson plans in relation to these three probing question categories and create assessing questions focusing on meanings, context, and representations. The other two categories, general questions or focus on information/procedures, will be used as non-examples of probing thinking questions. PTs will also be asked to analyze teaching episodes and lesson transcripts to look for these two categories. The five categories of PTs' questions, and three categories of PTs' probing questions, complement the existing question classifications in the reviewed literature. In particular, the emerged categories provided subcategories for a combination of Boaler and Brodie's and Moyer and Milewicz's classifications.

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SUPPORTING FRACTIONS AS MEASURES IN AN ONLINE MATHEMATICS METHODS COURSE

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This brief report describes the initial results of The Fractions as Measures for Prospective Elementary School Teachers (PTs) Study. Research shows that PTs' conceptions of fractions on a part-whole understanding may be problematic when teaching children about improper fractions. We created an instructional lesson sequence for PTs focused on using unitizing, iterating, and partitioning to think of fractions as measures in multiple situations. Initial results from this ongoing study indicate that PTs' manipulations of unpartitioned rod manipulatives supported both their construction of more powerful fractions schemes and their ability to verbally justify their reasoning.

Keywords: Teacher Education-Preservice, Instructional Activities and Practices, Rational Numbers

The Standards for Preparing Teachers of Mathematics state that beginning elementary school teachers should have a strong foundation in fractions, including the idea that "fractions have multiple interpretations, including part-whole relationships, measures, quotients, ratios, and operators" (AMTE, 2017). Although the conception of a fraction as a *measure* (Kieren, 1980; Lamon, 2007) is emphasized early in the Common Core State Standards for Mathematics (NGA/CCSSO, 2010), many students and prospective teachers (PTs) remain focused on *part-whole* meanings of fractions (DeWolf & Vosniadou, 2015; Newton, 2008; Norton & Wilkins, 2010; Olanoff, Lo, & Tobias, 2014). In this paper, we present an instructional sequence in an online mathematics methods course for supporting PTs' construction of conceptions of fractions as measures. Our aim of this study is to describe how PTs reason with fractions in this online environment, with the goal of informing design modifications to support measurement conceptions.

Theoretical Framework

We adopt a constructivist epistemology in thinking about PTs' fractions meanings as the product of their organizing mental structures (*schemes*) to fit their experiences (von Glasersfeld, 1995). The construct of *scheme* refers to the way researchers model how individuals operate mentally in service of a goal. A scheme consists of three parts – recognition of a situation, operations (mental actions), and an expected outcome. Individuals' schemes become established as they become refined and generalized through their use, via processes of assimilation and accommodation (Piaget, 1970). When a scheme is interiorized, the situation, operations, and anticipated result are experienced altogether as a unified and connected structure (a concept) that can itself be operated upon (Hackenberg, 2010; Piaget, 1970).

We focus on four specific schemes pertaining to fractions – the part-whole scheme (PWS), the measurement scheme for unit fraction (MSUF), the measurement scheme for proper fractions (MSPF), and the generalized measurement scheme for fractions (GMSF) (Steffe & Olive, 2010; Wilkins & Norton, 2018). The PWS involves *partitioning* a whole into discrete pieces that can be disembedded (removed from the whole without modifying the whole) and double-counted to form a numerosity of part(s) within a numerosity of a whole. The distinction between the PWS and the measurement fractions schemes is the individual's ability to iterate fractional units (which is absent from PWS). The MSUF builds upon the PWS as the individual conceives of the *size* of a disembedded unit fraction and its relation to the size of the unpartitioned whole (i.e., that *equipartitioning* an amount of 1 into *n* parts and *iterating* that amount *n* times results in the size of 1). The MSPF extends this notion to the size of a composite (but proper) fraction. An individual with a GMSF understands the size of an (im)proper fraction (*m/n*) as the result of coordinating mental operations to include partitioning the size of ‘1’, disembedding a unit fractional size (1/*n*), and iterating the disembedded fractional unit *m* times.

In order for an individual’s fraction scheme to become interiorized as a fraction concept, his or her fraction scheme must be *reversible*. For instance, an individual with a reversible GMSF could reverse his or her ways of operating to determine the size of ‘1’ from a given improper fraction size. Reversing the MSPF involves forming the size of ‘1’ from a given (composite) proper fraction size, and reversing the MSUF involves forming the size of ‘1’ from a given unit fraction size. Reversing the PWS involves forming the numerosity of the whole from a given proper fraction (e.g., reasoning that if three parts represents the fraction 3/7, then the whole must be 7 parts). Wilkins and Norton (2018) explain PWS requires partitioning and disembedding, whereas MSUF also requires iterating. By engaging in both partitioning and iterating, individuals are constructing actions with inverse relationships between each, which also promote reversibility (e.g., partitioning undoes iterating and vice versa when these actions are composed).

Methods

Participants, Context, and Instructional Sequence

We began with a pilot of six PTs enrolled in a face-to-face mathematics methods course to investigate PTs’ understandings of linear fraction tasks (Boyce & Moss, 2017). We provided the PTs with unpartitioned rods to help them carry out iterating and partitioning operations for different tasks with fractions. PTs video recorded themselves discussing how to solve particular tasks and showed how they used the rods to help them make sense of each task.

The positive results from the pilot study encouraged us to create an instructional lesson sequence for unitizing, iterating, and partitioning in the linear representation of fractions and expand data collection to an online, asynchronous, undergraduate mathematics methods course for elementary school teachers (*n*=80). This methods course is delivered 100% online for 15 weeks. PTs are typically enrolled in their junior year and one year from a student teaching experience. They complete readings, watch lectures and supplemental videos of whole class instruction, participate in a group discussion, and take a quiz each week. The course content is teaching and learning rational numbers and proportional reasoning.

We designed and implemented the instructional lesson sequence for PTs. The lessons begin with reading the first two chapters of *Fractions into Practice*: *Grades 3-5* (Chval, Lannin, & Jones, 2013). Chapter 1 provides examples of children’s work with analyses of children’s understanding of partitioning, fair shares, and the meaning of fractions. Chapter 2 provides an overview of developing children’s understanding of the meaning of the unit in a fraction. PTs engage in an activity in which they use various pattern blocks to think about partitioning into equal parts. Then, we introduce fractions as measures and the roles of unitizing, iterating, and partitioning in forming sizes consistent with standard and discretized representations (i.e., linear). PTs also engage in thinking about why it is essential for students to identify the unit and recognize its connection with a fractional part. This is accessible to PTs who have not yet constructed fractions as measures. We then foster reflection on iterating and partitioning by asking PTs to complete a series of activities using rods that increase in sophistication and difficulty in terms of thinking of fractions as measures (Baroody, Baroody, & Coslick, 1998, p. 9-16). These tasks encourage use of partitioning, iterating, and unitizing with fractions in order to arrive at an accurate answer. The tasks from the culminating activity, which, use similar wording and form of items from Norton and Wilkins (2010), are listed below:

- If the red rod is 1/5, what rod represents 1/2?
- If the light-green rod is 1/4, what rod represents 1/2?
- If the purple rod is 2/3, what rod represents 1/2?
- If the purple rod is 2/3, what rod represents 7/6?
- If the light-green rod is 1/2, what rod represents 9/6?

We conjectured that PTs might describe reasoning with a MSUF scheme to solve tasks 1 and 2, a MSPF scheme to solve task 3, and a GMSF scheme to solve task 5. We randomly assigned PTs in discussion forums to one of the five aforementioned tasks and asked them to make a video showing their solution(s). PTs posted the video to a discussion forum and discussed the different ways of making sense of their group's assigned task.

Data Collection and Analysis

The design research approach (Cobb, Confrey, diSessa, Lehrere, & Schauble, 2003; Collins, Joseph, & Bielzaczyc, 2004; Kelly, 2003) was used to investigate how PTs reasoned in an online setting and employed to study and understand the means of supporting and organizing student learning of fraction tasks presented in the online course. The framework of design research allowed us to engineer the learning environment, systematically study what takes place, and make adjustments to the curriculum (Cobb et al. 2003; Collins et al., 2004; Kelly, 2003).

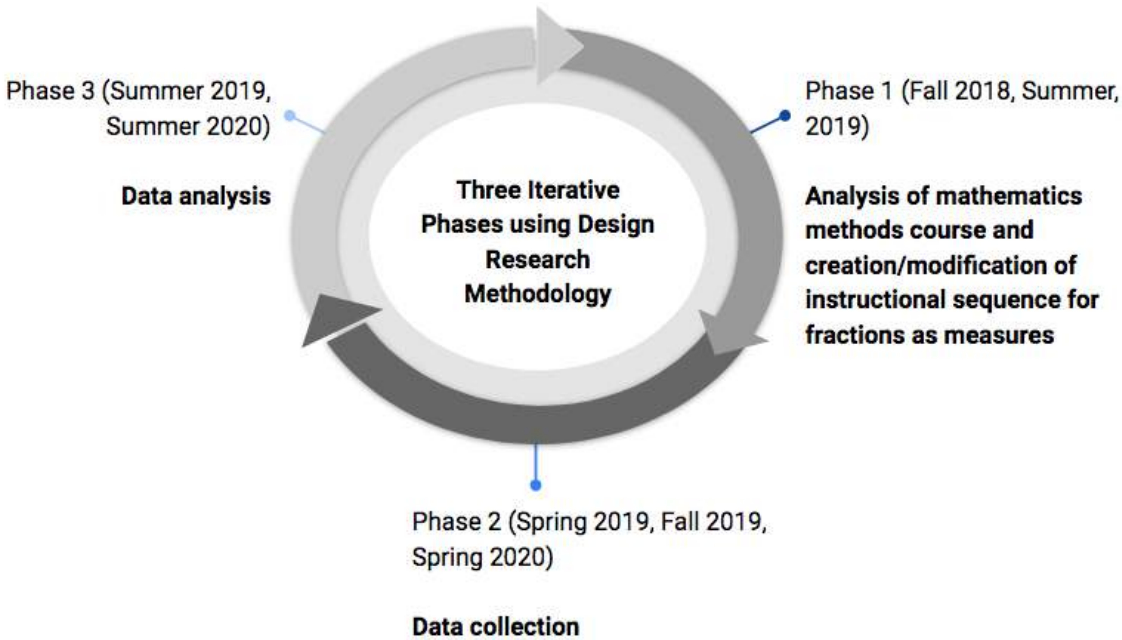

Figure 1: Iterative Phases of Data Collection and Analysis

The data consisted of videos of PTs verbally reflecting on how they solved fraction tasks using rods and PTs' online discussions contrasting their reasoning and their peers' reasoning on these tasks. First, a graduate assistant blinded and renamed the video files and blinded the online discussions so that the five raters could not identify PTs' names. Two researchers identified repeating themes in PTs' videos, while acknowledging unique instances (Corbin & Strauss, 2014). Themes were organized according to "the unit and the whole", "reasoning with arithmetic operations, "alternative methods", and "orientation of the video". The two researchers created a list of twenty questions based on the themes and six researchers quantitatively coded (1: evidenced, 0: not evidenced) in pairs. Following the coding, pairs compared codes to determine inter-rater reliability. Using what we learned from the data analysis, we refined the pedagogy of the online course as well as the content. Figure 1 outlines the phases of our data collection and analysis.

Results

The videos showed that the unpartitioned rods helped PTs verbalize their constructed unit and show how their partitioning and iterating relates to particular fraction reasoning. As PTs began working with concrete manipulative rods, we found that they were able to develop physical actions that were both reversible and composible (e.g. iterating and partitioning). By acting on length models in this way, PTs had to iterate and partition. Before this, many PTs commented that they had used linear representations, but usually in the form of a number line. Thus, conceptualizing fractions with bars using rods was new to them.

Conclusion and Significance to the Field

Results from the study showed that tasks similar to those in the sequence tested provide opportunities for PTs to progress from PWS toward measurement fraction scheme development. By anticipating their own strategies and solutions with (im)proper fractions, PTs will better be able to consider their own students' reasoning strategies more systematically and critically determine (in)effective instructional moves when teaching fractions. By providing PTs means to visualize the unit and engage in reversible actions, and verbally reflect on their thinking (with their video explanations) we posit they might interiorize measurement schemes and be able to anticipate solutions more fluently. Findings from this study could inform (1) (in)effective instructional sequences and progressions that PTs experience when constructing fractions as measures, and (2) affordances and constraints of online asynchronous learning environments for the development of these instructional sequences.

We are currently in Phase 2 of Data Collection and are including new data in the form of drawings, paired with the videos in online and face-to-face mathematics methods courses. We are refining the list of questions based on the themes used to analyze the videos so that other instructors that collect similar data can use the protocol to determine fraction schemes.

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PREPARING TO ELICIT STUDENT THINKING: SUPPORTING PST QUESTIONING IN AN UNIVERSITY TEACHING EXPERIENCE

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This study examines how preservice teachers (PSTs) plan for and enact questions that elicit student thinking during an early field experience. We analyzed teaching videos and their corresponding lesson plans from 17 PSTs over 34 lessons in a field experience in a freshman-level university mathematics class. Our findings show PSTs tended to use three types of questioning sequences when teaching, with the quality of questioning in those sequences linked to the quality of planned questions. Findings described here discuss the implications for supporting PSTs' lesson planning during early field experiences.

Keywords: Teacher Education – Preservice; Teacher Educators; Classroom Discourse

Being able to elicit and respond to student thinking is a core practice of ambitious teaching (Gotwals & Birmingham, 2016). In order to elicit student thinking, teachers need to: (1) select a task that affords opportunities for eliciting, (2) anticipate student thinking, (3) know the learning goal and assess students' proximity to the goal, and (4) plan questions to deepen student understanding (Boerst et al., 2011; Orr et al., 2020; Shaughnessy et al., 2019; Sleep & Boerst, 2012; TeachingWorks Resource Library, 2020). Each of these components of eliciting student thinking is complex. The complexity of this practice makes it difficult for novice teachers, whether being utilized through simulations or in face to face interactions with students (Shaughnessy & Boerst, 2018). One-way math teacher educators (MTEs) help preservice teachers (PSTs) prepare for this complex practice is through lesson planning. In this brief research report, we share findings from a project that explored the following questions: *During the UTE, to what extent did PSTs enact questions as planned? How did the quality of planned questions correspond to the quality of questions as enacted?*

Methods

This project analyzed data collected for a larger project studying the effects of the *University Teaching Experience* (UTE) model for secondary mathematics PST learning across three different teacher preparation programs[1]. In the UTE, PSTs teach an entry-level undergraduate mathematics course while taking their first methods course (Bieda et al., 2019). PSTs plan, enact, and reflect on a series of lessons while being supported by MTEs. The MTEs support the PSTs through providing feedback on lesson plans, in-the-moment coaching and leading post-lesson debriefs. In addition to the support of their MTEs, the PSTs are also supported by a mentor teacher who is the course instructor for the mathematics course. Throughout the course of the UTE semester, PSTs teach at least two lessons and observe other PSTs teach while working with mathematics students in groups. In this brief report, we will only present findings from data collected at our university site.

Data Collection

During the implementation of the UTE in fall 2018, there were 17 PSTs. The PSTs planned and taught lessons in pairs, resulting in seven pairs and one group of three. The PSTs taught two lessons over the course of the semester in a college algebra course. The first lesson was roughly half the class period (~40 minutes). The second lesson was the entire class period (~ 80 minutes). Lesson enactments were captured using Swivl robot video-recording (www.swivl.com) and audio from the class sessions was later transcribed. PSTs completed lesson plans for each session following a modified version of the Think Through a Lesson Protocol (TTLP) (Smith et al., 2008) and shared with MTEs for feedback through Google Docs.

Data Analysis

In order to understand the relationship between the quality of questions enacted as compared to the quality of questions planned, all the questions in the lesson plans and the transcribed enacted lessons were coded based on the Instructional Quality Assessment Academic Rigor (AR) for Teachers' Questions rubric (Boston, 2012). This tool was selected because reliability and validity has been established (Boston & Wolf, 2006). For any question written in the plan or asked of students during the lesson, it was assigned one of 6 question types: probing, exploring mathematical meaning and relationships, generating discussion, procedural or factual, other mathematical, and nonmathematical (Boston, 2012). Examples of each type from our data are represented in Table 1.

Question Type	Example
Probing	Why does this work?
Exploring mathematical meaning and relationships	Why do you think that would work, to switch P and M, given what we've been doing with the other inverses?
Generating Discussion	Looking at these examples here, take a second, look at them, and think, which of these relations or graphs are functions?
Procedural or Factual	Is it invertible?
Other Mathematical	Any questions on how we did those few steps?
Nonmathematical	Why would technology affect the prison rate?

Table 1: Examples of Question Types

For the enacted lessons, we narrowed the data to examine only situations where PSTs engaged in sequences (sustained questioning) involving AR questions, thus excluding any questions that were primarily procedural or factual, other mathematical, and non-mathematical. We segmented lesson transcripts into questioning sequences by determining when the PST asked the initial question to elicit student thinking around a specific question. Then, we identified a sequence end when the PSTs moved to another topic. We analyzed these sequences and generated three categories to describe the patterns of questioning in these sequences. Afterwards, we compared the enacted sequences to the portion of the lesson plan with corresponding content. We looked for patterns in how the questioning sequences evolved depending upon whether PSTs had asked questions that were planned or unplanned. In the section below, we share our findings about the patterns that surfaced related to describing relationships between planned and enacted questions.

Findings

We discovered that PSTs utilize three distinct questioning sequences when they asked questions involving Academic Rigor (AR). In the first type, PSTs maintained AR throughout the sequence. In this type, PSTs began with an AR question, the students responded, and the PSTs continued to ask AR questions throughout the sequence. The second type of questioning sequences involved PSTs reducing the AR. In this type, the PSTs started with an AR question, the students did not respond, and the PSTs lowered the AR of the questions for the remainder of the sequence. The third type –the "hook" method – emerged when PSTs began with a non-AR question, which "hooked" the students to respond, and then the PSTs raised the AR of the questions for the remainder of the sequence.

Given our research focus, we wanted to explore the relationship between questions as planned and the emergence of these different types of questioning sequences. When looking at the total amount of enacted questions, as sorted by questioning sequence, we found the most common questioning sequence was the "hook" method followed by the scenario where the AR is maintained. In further analysis of mapping enacted questions on to the lesson plan, we found PSTs enactment involved more planned questions for sequences where the AR is maintained when compared to the other two questioning sequences (see Figure 1).

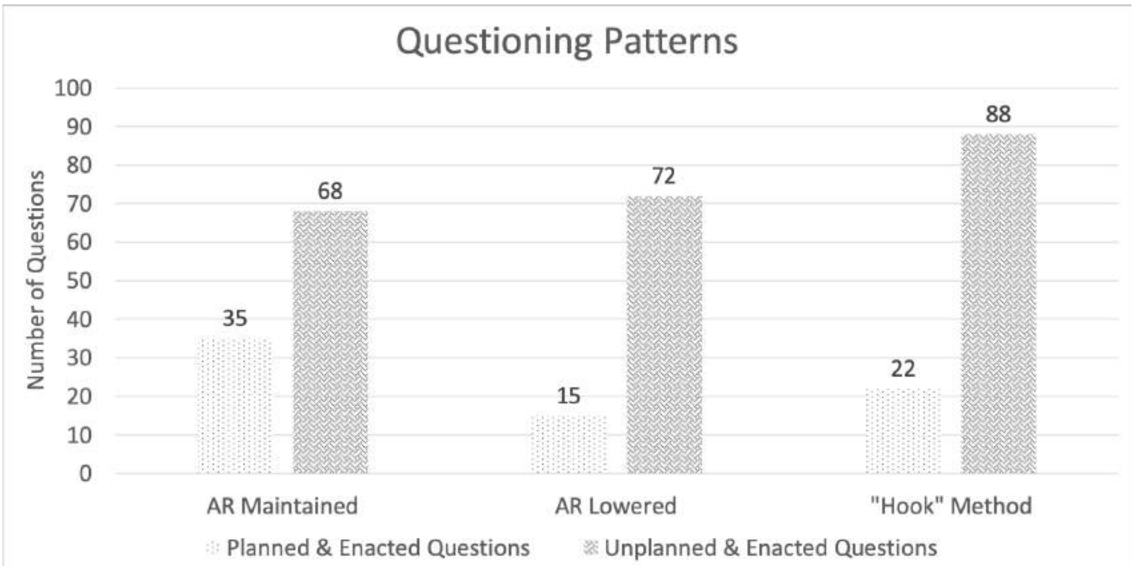

Figure 1: Planned vs Unplanned Questions in Enactment

Note in Figure 1 that situations where PSTs lowered the academic rigor involved the fewest number of planned questions. To investigate whether planning high-quality questions correlated with a greater number of AR questions in enactment, we also investigated the quality of enacted questions during parts of the lesson where AR questions had been planned (but not enacted). We hypothesized that planning AR questions would ultimately support higher-quality questioning sequences during the parts of the lesson the questions were being employed, even though the planned questions were not asked. Our findings are represented in Figure 2.

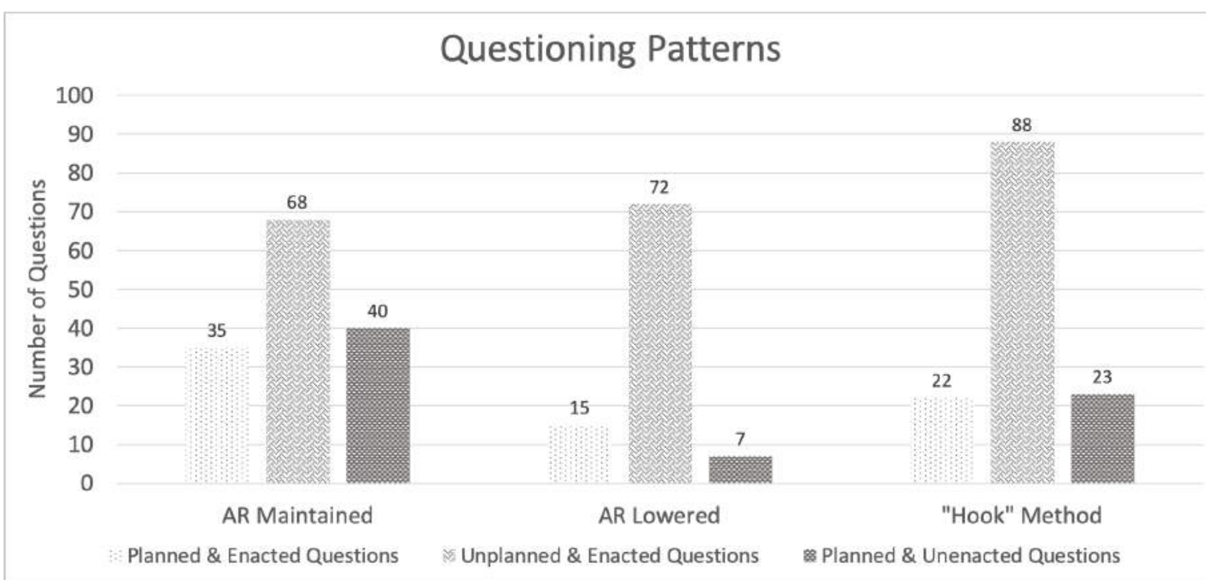

Figure 2: Planned & Unenacted Questions vs Enacted Questions

Through this investigation we found, in situations where AR is maintained, PSTs not only asked the most planned questions, but they also had the most planned and unenacted questions for these sequences. We also found the reverse was true. In situations where the AR was lowered, PSTs not only asked the least amount of planned questions, but they also had the least amount of planned and unenacted questions (fewer questions overall) suggesting that less attention to those situations in planning affected the quality of their eliciting of student thinking during enactment

Discussion and Conclusion

Through engaging in the UTE experience, PSTs are given opportunities to plan for and elicit student thinking through questioning while being supported by MTEs. These findings suggest that even with support, PSTs find themselves in classroom situations they had not anticipated and are unsure of how to respond. The situations tend to arise when students have gaps in prior knowledge or engage in the task in an unanticipated way. This is to be expected, as PSTs classroom inexperience often means that they face difficulties with anticipating students' responses (Arbaugh et al., 2019; Taylan, 2018). Improving the quality of PSTs' questioning must involve not only helping them to anticipate student responses, which typically improves with more classroom experience, but also how to draw on high AR questioning when the situation does not unfold as planned. MTEs can provide this support through encouraging PSTs to consider follow up questions, as well as, how the questions align with the learning goals. Giving additional attention to the alignment between the questions and learning goals may result in (the inevitable) unplanned questioning sequences that more likely maintain AR.

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PROSPECTIVE TEACHERS' INTERPRETATIONS OF MATHEMATICAL REASONING

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Calls for teaching school mathematics with a focus on mathematical reasoning (MR) are included in curricular documents across the world, but little is known how prospective teachers (PSTs) understand MR. In this paper, we report on a study in which we engaged 24 PSTs preparing to teach grades 1-8 in analyzing a series of student-generated arguments for evidence of student reasoning with a focus on student-provided justifications. We examined PSTs' interpretations of MR prior to and after instruction. Our results showed that PSTs interpreted MR broadly in terms of student thinking, validating thinking, problem-solving, connecting ideas, or sense-making. Some PSTs also interpreted MR as evidence of student understanding or described MR in terms of strategies teachers use to support students' reasoning skills. We discuss changes in PSTs' interpretations of MR after instruction.

Keywords: Teacher Beliefs, Reasoning and Proof, Teacher Education - Preservice

Framing of the Study

Developing students' mathematical reasoning (MR) skills is the desired goal of school mathematics education (Australian Curriculum, Assessment and Reporting Authority, 2015; Department for Education, 2014; National Council of Teachers of Mathematics [NCTM], 2000, 2009; National Governors Association Center for Best Practices, & Council of Chief State School Officers, 2010). However, little is known about how teachers, including prospective teachers' (PSTs'), interpret MR. To date, only a handful of studies documented how practicing and PSTs make sense of and understand MR (Clarke et al., 2012; Herbert et al., 2015). Herbert et al. (2015) shared that Australian and Canadian elementary practicing teachers have broad and ambiguous perceptions of MR. As such, they interpreted MR as thinking, communicating thinking, problem-solving, validating thinking, forming conjectures, using logical arguments to validate conjectures or connecting aspects of mathematics.

In this paper, we describe instructional intervention designed to heighten elementary PSTs' attention to students' reasoning in the context of generating mathematical arguments. Our study draws on the variation theory of learning (Lo, 2012), which highlights the importance of providing learners with multiple experiences with a given phenomenon, to generate a wide range of opportunities that help learners attend to and make sense of different features of that phenomenon. In our work with PSTs, we draw on the variation theory to purposefully bring PSTs' attention to elementary students' MR, particularly different ways in which students might reason to justify while generating mathematical arguments. Our goal was to answer the following research questions: (1) How do PSTs interpret MR in the context of elementary school mathematics classrooms? And, (2) How does engaging PSTs in analyzing elementary students' arguments for evidence of MR impacts PSTs' views on MR?

Method

Participants and Study Context

Participants were 24 PSTs enrolled in a semester-long mathematics content course for elementary and middle grades education majors, Algebra and Geometry for Teachers. The course was designed to support PSTs' conceptual understanding of mathematical ideas essential to elementary and middle

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school mathematics curriculum. Instructional emphasis was placed on understanding, interpreting, and assessing students' MR about fundamental mathematics concepts in the K-8 school mathematics.

Drawing on descriptions of elementary students' reasoning provided by the NRICH team at the University of Cambridge (see https://nrich.maths.org/11336) and the variation theory of learning, we designed the Student Reasoning Assessment Tool (SRAT) (see Table 1) to bring PSTs' attention to different justifying actions in the context of student-generated arguments and to help PSTs develop reasoning assessment skills. Along with the SRAT, we also created a set of class activities that we used to engage PSTs in analyzing MR evident in elementary students' written arguments.

Table 1: Student Reasoning Assessment Tool (SRAT)

Levels	Descriptions of elementary students' reasoning levels
L0	Student tells what he or she did
L1	Student attempts to provide some reasoning (not necessarily relevant, complete, or valid) for what he or she did
L2	Student provides a chain of reasoning, which is incomplete, insufficient, or invalid, to support the assertion
L3	Student provides a chain of acceptable valid reasoning in support of the assertion; the argument is at best partial
L4	Student provides an exhaustive acceptable chain of valid reasoning in support of the assertion; the argument can be accepted as proof

Prior to class intervention, we asked PSTs to share in writing their own interpretations of MR. They were also given a set of student-generated arguments and asked to analyze these arguments for evidence of student reasoning with attention to student-provided justifications. During the intervention, using the SRAT, individually and in small groups, PSTs examined a wide collection of sample arguments for evidence of student reasoning. They were also asked to anticipate how elementary students could reason and communicate their mathematical reasoning in different problem contexts. After the intervention, we asked PSTs to revisit their initial descriptions of MR.

Data and Data Analysis

We analyzed PSTs' written responses to two journals, which each PST completed at the beginning and end of the semester and in which they reflected on the meaning of MR. The journal prompts were intentionally open-ended to avoid leading PSTs in any specific direction that could suggest interpretations of MR. The prompts were as follows:

- Journal 1: Think about yourself as a mathematics teacher. When you hear the term mathematical reasoning, what does it mean to you? In the best possible way, describe your understanding of this term. Explain how mathematical reasoning might look.
- Journal 2: Building on your learning in this class, define mathematical reasoning. Did your understanding of mathematical reasoning change when comparing to how you interpreted it at the beginning of the semester? If yes, explain why. If no, explain why not.

We first coded the data with analytic codes derived from the existing literature on teachers' perceptions of MR (e.g., Davis & Osler, 2013; Herbert et al., 2015). We then conducted the inductive analysis to identify any additional themes within our PSTs' responses. We continued comparing and contrasting the identified themes until we established final definitions of codes, which then were applied to our data. Finally, we tabulated code frequencies to identify any patterns in our PSTs' interpretations of MR.

Results

PSTs' Interpretations of MR

Our PSTs interpreted MR in multiple ways, and that overall, they used two lenses while discussing the term MR. Most frequently, PSTs interpreted MR from the perspective of a learner (see Table 2). Some PSTs also viewed MR from the perspective of a teacher. PSTs with the learner perspective saw MR as the process that describes how students think, validate (justify), make sense, solve problems, or connect mathematical ideas. PSTs with the teacher's perspective viewed MR as data (products) that give teachers evidence of students' understanding of mathematical concepts, or as strategies with which teachers engage students in MR in the mathematics classroom.

Table 2: PSTs' Views of MR

View	Interpretation of MR	The number of PSTs* with the view (n, %)	
		Semester beginning	Semester end
Student-Centered	Thinking	6 (25%)	10 (42%)
	Validating thinking	18 (75%)	22 (92%)
	Sense-making	13 (54%)	17 (71%)
	Problem-solving	8 (33%)	9 (38%)
	Connecting mathematical ideas	9 (38%)	14 (58%)
Teacher-Centered	Evidence of students' understanding	2 (8%)	7 (29%)
	Teacher support for students' reasoning	3 (13%)	3 (13%)

Note. The total number of participants, n = 24. The categories are not mutually exclusive. Most PSTs shared multiple views.

Student-centered Views of MR

With a focus on an individual student, PSTs most frequently viewed MR as validating thinking. They emphasized justifying actions as representative of MR. PSTs also discussed modes of representations (e.g., verbal, written, or pictorial forms) that students might use to validate or explain their reasoning. They focused on the role that reasoning plays in supporting the growth of one's mathematical understanding by describing that while students reason about mathematics, they learn and develop a deeper understanding of mathematical concepts or problem-solving strategies. PSTs also viewed MR as specific aspects of the problem-solving process, the entire process of problem-solving, or decision-making in problem-solving situations. Some PSTs described MR as one's thinking about mathematics, mathematical problems, or specific problem-solving strategies.

Teacher-centered Views of MR

PSTs interpreted MR as evidence of student learning and articulated that teachers use students' reasoning as a resource for making instructional decisions. PSTs also described that by paying attention to students' reasoning, teachers identify the needs of students with diverse mathematical abilities or levels of understanding.

Changes in PSTs' Interpretations after Class Intervention

We observed two changes in PSTs' interpretations of MR while comparing their views from the beginning to the end of the semester. (1) Change in the breadth of interpretations (17 PSTs, 70%). After the class intervention, many of the PSTs augmented their initial interpretations and included additional perspectives on the meaning of MR, which they did not initially consider. On average, after the intervention, most PSTs gained awareness of one to three additional interpretations of MR. (2) Change in the depth of interpretations (14 PSTs, 58%). PSTs' views of MR after the class intervention remained largely consistent with their initial views. However, their interpretations of MR were more nuanced and included more precise descriptions of reasoning actions. While describing justifying prior to the class intervention, for example, many of the PSTs interpreted justifying broadly as explaining why. After the class intervention, PSTs discussed justifying with attention to specific attributes of justifications such as logic, generality, or modes of representations that a student might use to justify a mathematical statement.

Conclusion and Discussion

In our work with PSTs, we positioned them to analyze students' MR as future mathematics teachers (see presented earlier journal prompts). Our data revealed two perspectives that PSTs used as their lenses while describing MR: student-centered and teacher-centered. With a focus on each of these perspectives, PSTs interpreted MR in a broad sense. Within their student-centered interpretations, PSTs described MR as thinking, validating thinking, sense-making, problem-solving, or connecting mathematical ideas. Within their teacher-centered interpretations, PSTs interpreted MR as evidence of student learning that helps teachers make instructional decisions or as a pedagogical practice that teachers use to engage students in reasoning and encourage their mathematical thinking. PSTs' broad interpretations of MR might not be surprising since reasoning, problem-solving, sense-making, mathematical thinking are all intertwined and often viewed as interconnected practices that support one another (NCTM, 2009; Kilpatrick et al., 2001).

Our results also revealed that classroom activities that exposed PSTs to a large sample of students' work with a focus on student reasoning about justifications increased PSTs' awareness of specific justifying actions. While sharing their views of MR, PSTs have begun to provide more nuanced and precise descriptions of reasoning actions related to justifying. Supporting PSTs in building a comprehensive vision of MR should include efforts of helping them make a shift from a broad understanding of MR as thinking, sense-making, problem-solving, or connecting mathematical ideas to seeing these aspects of reasoning in terms of more specific and tangible reasoning actions. Loong and colleagues (2013) argued that teachers who do not have a strong understanding of specific reasoning actions might likely be ineffective in promoting MR in their classrooms.

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RESOURCES THAT PRESERVICE AND INSERVICE TEACHERS OFFER IN COLLABORATIVE ANALYSIS OF STUDENT THINKING

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This study examines a professional development (PD) program, set in a summer mathematics program for middle grades students with a research-based curriculum, where preservice and inservice teachers collaborated in interpreting and responding to student thinking. We investigated the resources that participants contributed to this collaboration, and the opportunities the non-traditional PD setting afforded for the sharing of these resources. Our embedded case study consisted of two classes, where participants taught and then engaged in video-recorded debriefing sessions each day. Their discussions focused on what they noticed in class and how they responded in the moment or anticipated responding in future lessons. We find that participants' observations from class catalyzed the sharing of resources, both from the program and from outside experiences, that contributed to the analysis of student thinking.

Keywords: Teacher Knowledge; Instructional Activities and Practices

Professional development (PD) focused on practice offers prospective and practicing teachers opportunities to learn and examine the real work of teaching (Ball & Forzani, 2009; Grossman et al., 2009a). Representations of practice (Grossman et al., 2009) captured in video episodes, student work, transcripts, and narratives are often used by researchers as effective tools to help unpack and decompose practice (e.g. Sherin & van Es, 2005; Kazemi & Franke, 2004; Silver et al., 2007; Oslund, 2016). Lacking in these studies, however, are opportunities to interact or be present within the dynamics of the context. Experiences such as lesson study (Lewis, Perry, & Hurd, 2009) provide direct access to the classroom with demonstration lessons that serve as representations of practice, followed by debrief opportunities for the decomposition of practice.

Little (2003) posits that improvement of teaching and learning is strengthened when teachers collaboratively engage in analysis of teaching practice. When representations of practice (Grossman et al., 2009a) are contextualized in a common work environment, they become resources for teacher learning. Researchers have noted differences in what expert, beginning, and novice teachers perceive and understand of classroom events (Berliner, 2001). This skill of professional noticing is an important component of decomposing core practices of teaching (Jacobs, Lamb, & Philipp, 2010). Our work examines a PD where preservice and inservice teachers taught together in an informal summer mathematics program and engaged in discussion sessions about the student thinking and teaching practices they noticed. We hypothesize that a "neutral" setting provides a productive space for negotiation of norms and practices across distinct cultures in which these groups are immersed: the preparation programs of preservice teachers, and inservice teachers' schools. Additionally, the program uses a research-based curriculum in which students construct understandings of pre-algebra concepts while using representations to explore problems and communicate their reasoning. We hypothesize that this curriculum creates opportunities for the analysis of students' mathematical thinking.

In this study we focus on two research questions:

1. What resources do preservice and inservice teachers contribute when discussing how to respond to students' mathematical thinking in a non-traditional setting?
2. What opportunities for the analysis of student thinking and the decomposition of teaching practice does a non-traditional collaborative teaching setting afford?

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Methodology and Theoretical Framework

Our study is an embedded case study (Yin, 2009) that took place in an informal summer mathematics program for students in grades 3-8 in the United States. In the two-week program, students took courses that explored pre-algebra concepts. Each class was taught by a "master teacher" (MT), an inservice teacher with multiple years of experience in the program, and assisted by inservice teachers seeking professional development (PDT), undergraduate preservice teachers (PST), and undergraduate "fellows" (F). The focus of our study is on the "teachers" at this summer program. The cases consisted of two classes collectively taught by two master teachers, three inservice PD teachers, six preservice teachers, and two undergraduate fellows.

During the program, inservice teachers and undergraduates taught together each morning, then convened after class in small groups, with each group consisting of the teacher and assistants for one class. In the meetings, participants responded individually to written reflection prompts, then engaged in videotaped group discussions about what they had observed in the day's class. We transcribed these discussions and divided the transcripts into conversational "episodes," with each episode focused on a unified theme or observation. We coded an episode as a *rich episode* if it included sustained discussion of students' observed mathematical thinking.

To address the first research question, we analyzed these episodes and identified how participants interpreted or suggested responding to student thinking. We also identified resources that various participants contributed to the discussion in each episode. In conceptualizing *resources* we use *mathematical knowledge for teaching* (MKT, see for example Ball, Thames, & Phelps, 2008) as an organizing framework. MKT consists of mathematical and pedagogical knowledge used in the work of teaching, comprised of components such as *specialized content knowledge*, *knowledge of content and students*, and *knowledge of content and curriculum*. Rather than attempt to sort resources into discrete MKT domains, we use these domains to orient our awareness of intellectual resources (Little, 2003) that participants use to make sense of student thinking, weigh possible responses, and understand the curricular context in which this thinking takes place. We take a grounded approach (Strauss & Corbin, 1997) in our analysis of resources, identifying and categorizing instances in which participants share knowledge and experience.

To address the second research question, we coded for instances in which participants made reference to features of the non-traditional context of the collaboration, either comparing this context with other mathematics teaching settings, or describing affordances or challenges of the context itself (such as opportunities for students to develop ideas prior to the presentation of formal procedures, or the fact that the program is not bound by the school calendar). We analyzed these instances for evidence that the non-traditional teaching context afforded opportunities for participants to make explicit their beliefs and dispositions about mathematics teaching (whether in their current non-traditional setting or in other more conventional settings) or their reasoning in interpreting or choosing how to respond to student thinking.

Results

In this section we describe some preliminary results of our analysis of the resources that teacher participants in the program offered in their discussions, and of the opportunities that the shared teaching context afforded for the collaborative analysis of student thinking.

Types of Resources Offered by Teacher Participants

In addressing the first research question, we offer three examples of categories of resources that have emerged from our data.

Awareness of alternative algorithms: a resource offered by PSTs. In Episode 3X_ 505, a group consisting of one MT, two PDTs, two PSTs, and two Fs discussed a fellow's observation of a student who lacked access to an activity on the Pythagorean theorem because he did not know how to multiply numbers by hand. The group discussed different resources that might help the student. A PDT mentioned two students who had shared their own algorithms for multiplication to the whole class. Two PSTs indicated that they had learned about alternative algorithms, and the pedagogical value of these, in a content course for elementary teachers. Even though neither PST described any algorithms for multiplication other than those described from class, their awareness of the existence and value of multiple algorithms helped to frame the discussion of students' knowledge and use of non-standard algorithms in an asset-oriented way.

Knowledge of models and metaphors for operations: a resource offered by PDTs. In Episode 1Y_506, one MT, two PDTs, and one F brainstormed strategies for helping students understand integer subtraction; the fellow had observed some difficulty with the idea of subtracting an integer by adding its opposite. The participants launched into a discussion of possible ways to make the connection between subtraction and addition concrete for students:

PDT.155: You can think like banking terms I owe you an orange.

F.145: I think they kind of understood that a little bit when that was brought up …

MT.152: Yeah, and what if someone owes you, like we talked about bigger numbers, let's say that you have $25.00 … your brother had borrowed 20 of your dollars … Well, what if he paid you back – well, that's not subtracting …

PDT.180: Because we're subtracting –

MT.152: Do you see what I'm wanting? I want to subtract a negative to make it – I mean it is going back … Well, subtracting 20 – oh, I don't know. Okay, so I don't know. Anyway, think about that.

PDT.810: And he already owes you money. It would be like you're subtracting from his deficit, right?

The PDTs' contributions, rooted in pedagogical knowledge of common metaphors for the concept of integer subtraction (Quigley, 2011), added to the group's opportunities to develop a coherent explanation for why subtracting an integer is equivalent to adding its opposite.

Knowledge of the program curriculum: a resource offered by MTs. In Episode 1Y_203, one MT, one PDT, and one F discussed students' efforts to determine distances between numbers on the number line prior to the introduction of any formal procedure for integer subtraction. All three participants (MT, PDT, and F) commented that one student had shown evidence of beginning to realize that the distance from a negative integer to a positive integer could be found by adding the absolute values of the two numbers, but that she did not apply this principle consistently. The MT commented, "She doesn't have the rules down yet, which we're not even to yet … so, it's pretty cool that she's already getting it." The MT's comment referenced her curricular knowledge that formal procedures for subtracting integers and solving distance problems would come later in the summer course. This allowed her to frame the student's developing fluency as a stage of mathematical discovery rather than as a deficit.

Affordances of the Non-Traditional Teaching Context

Our analysis suggests that the shared context affords at least two distinct types of opportunities for in-depth analysis of students' mathematical thinking and of teaching practice.

The first type of opportunity emerges from the curricular context for the shared representation of practice. The program curriculum provides students with opportunities to use visual and tactile representations to explore problems, and postpones the introduction of formal procedures. As a result, teacher participants sometimes have opportunities to describe students' conceptual thinking about pre-algebra ideas without the backdrop of a normative procedure against which student performance might be compared. We see this in the example of Episode 1Y_203, in which the absence of a formal procedure for finding distances on the number line afforded an opportunity for teacher participants to notice students' improvised methods for calculating distance, such as counting unit intervals on the number line and adding absolute values in the case in which the two endpoints are on opposite sides of the origin.

The second type of opportunity emerges from the contrast between the participants' shared teaching context and other contexts in which these participants had been mathematics teachers or learners. We find that the contrast among these teaching settings can provide the stimulus for a participant to make explicit their beliefs about mathematics teaching and learning. For example, in Episode 3X_505, in which teacher participants discussed a student who had difficulty with multiplication, several undergraduate students (PST and F) expressed surprise that such a student had been placed in an upper-level course in the summer program. They noted that the MT of their morning class (not present in the discussion) helped the student access the activity by providing a multiplication chart, but also stated that if they encountered such a student in a traditional setting, they would want to divert the student to remedial instruction. This afforded the MT an opportunity to provide a counterpoint for this deficit framing of the student, asserting that the activity's aim was to lead students to discover the Pythagorean theorem, not to assess multiplication, and that such an accommodation did not significantly undercut the mathematics.

Discussion and Next Steps

Our initial findings suggest that instances of non-standard student thinking serve to activate the unique knowledge resources of preservice and inservice teachers and offer opportunities for negotiations of teaching beliefs and norms across teaching cultures. Because the non-traditional structure and curriculum of the shared teaching context creates disequilibrium with the course-based preparation of PSTs and the more traditional teaching experiences of inservice teachers, participants from both constituencies are encouraged to make explicit their own initial conceptions of students' thinking, mathematical dispositions, and attitudes (Figure 1).

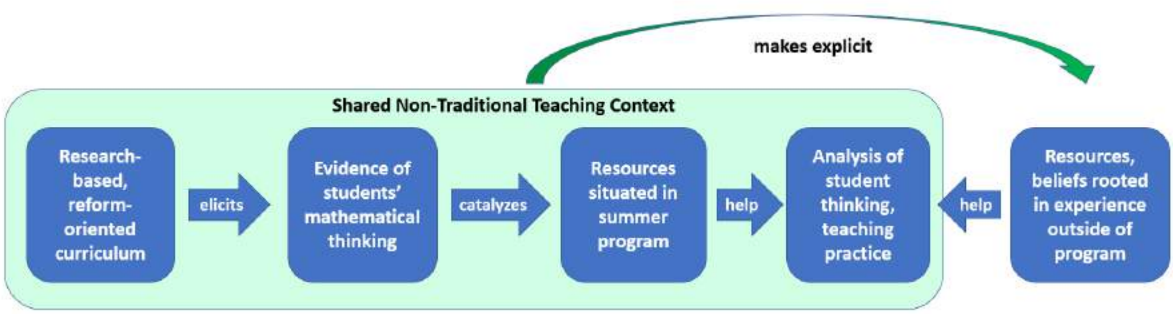

Figure 1: Model for the role of a shared non-traditional teaching context in PD.

Next steps for our study include a more thorough synthesis of the types of knowledge resources that participants contribute to the collaborative analysis of student thinking, along with an analysis of the degree to which these resources are differentiated by participant type (MT, PDT, PST, F). We also plan to develop a more robust coding scheme for assessing the role of the non-traditional setting in activating (or muting) the external resources, such as preservice teacher experiences and knowledge of traditional curricula, that participants bring into the program.

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PROSPECTIVE ELEMENTARY TEACHERS' CONTENT KNOWLEDGE OF DECIMAL MAGNITUDE AND PLACE VALUE

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Research suggests that robust mathematical knowledge for teaching is essential for high quality instruction and learning, and yet studies also reveal that prospective teachers (PTs) may not have had sufficient opportunities to develop this knowledge. Decimal understanding is one particular area of difficulty for elementary students and PTs alike, but few studies have focused on characterizing PTs' decimal understanding. In this study we examine 28 PTs' ability to create and explain models for comparing decimals, following instruction on this topic. We find that participants are able to effectively use models to identify and reason about the larger of two decimals, but that they struggle to articulate underlying mathematical ideas such as the role of place value in decimal magnitude or connections among decimal models.

Keywords: Teacher Education – Preservice, Mathematical Knowledge for Teaching, Rational Numbers

Research suggests that teachers of mathematics must have substantial knowledge of the content that they will teach and of appropriate ways to do so; these together are known as mathematical knowledge for teaching (MKT; Ball, Thames, & Phelps, 2008). Unfortunately, some studies also suggest that prospective elementary teachers' (PTs') MKT is still developing; areas for growth span topics such as fractions (e.g., Van Steenbrugge et al., 2014), geometry (e.g., Aslan-Tutak & Adams, 2015), decimals (e.g., Stacey et al., 2001; Widjaja et al., 2008) and more (Hill, 2010). Although decimal concepts have been shown to be difficult for learners, fewer studies focus on PTs' knowledge of decimals (Kastberg & Morton, 2014) than fractions (Olanoff et al., 2014) or whole numbers and operations (Thanheiser et al., 2014). This study attempts to contribute to and update the small body of literature on PTs' knowledge of decimal magnitude and place value. Based on PTs' responses to two open-ended tasks, we describe the models and strategies they use to make sense of and compare two decimal quantities. Further, we analyze PTs' written explanations of their models and strategies, and the underlying mathematical ideas that they identify as important.

Background & Theory

Here, we introduce some recommendations for supporting PTs' MKT and studies which give images of it, then highlight a gap in this literature. We describe how the MKT framework bounds our study by defining what is visible in the data, and discuss *conceptual understanding* and how this strand of mathematical proficiency figures in our analysis.

Decimals in Elementary Mathematics and Teacher Preparation

Elementary mathematics standards span many topics, including *number concepts*. Number concepts pertain to the structure of the base ten system and its extension to decimal quantities. Number concepts are important because of the ways in which they undergird foundational elementary mathematics such as counting and operations (Association of Mathematics Teacher Educators, 2017; Conference Board of the Mathematical Sciences (CBMS), 2012). Knowledge of decimals, specifically, is also expected of students. In grades 4 and 5 alone, the Common Core State Standards call for students to order, compare, and model decimals (National Governors Association Center for Best Practices, Council of Chief State School Officers, 2010).

Because of their presence in elementary school mathematics, policy documents call for coursework for PTs to also focus on number concepts, including decimals (CBMS, 2012). This call is amplified by empirical research suggesting that children and PTs alike struggle with decimal understanding (e.g., Muir & Livy, 2012, Steinle & Stacey, 1998). Stacey and colleagues (2001) found that some overgeneralizations common among children are much rarer among PTs, but that other misconceptions persist from childhood through to adult populations. Despite knowledge that sense-making around decimals is challenging for PTs, few studies have characterized their understanding. In Kastberg and Morton's (2014) literature review, they identified just three studies since 1998 which attended specifically to PTs' decimal understanding. Broader inclusion criteria uncover a few additional studies with relevant findings, but PTs' decimal understanding and learning processes remain under-researched. The literature does not effectively characterize PTs' understanding of the magnitudes of decimal quantities or explore their reasoning related to comparing decimals.

It is concerning that research on PTs' decimal MKT is so scarce. Without robust images of PT knowledge, mathematics teacher educators may be without the information needed to support PTs' growth. PTs must have deep understanding of decimal concepts and procedures, since this topic is prevalent in upper elementary curriculum, and since their MKT is known to contribute significantly to quality teaching and learning (Hill et al., 2005, 2008). We turn our focus to a brief elaboration on the construct of MKT generally, and a look at how it informs this study.

Mathematical Knowledge for Teaching

MKT includes *subject matter knowledge* and *pedagogical content knowledge* (Ball, Thames, & Phelps, 2008). The former is knowledge which a teacher must have about the mathematics itself, including knowledge which is common among adults, as well as specialized knowledge, needed primarily or exclusively by teachers. Pedagogical content knowledge focuses on the teaching of mathematics. Our participants had the opportunity to demonstrate understanding of the relative sizes of two decimals, and knowledge of how tools for modeling decimals relate to concepts about decimal place value. These are examples of subject matter knowledge; PTs in this study did not have the opportunity to display pedagogical content knowledge.

Conceptual Understanding as Part of Mathematical Proficiency

Given the importance of teachers having robust MKT, policy documents recommend that prospective elementary teachers engage in substantial coursework focusing specifically on elementary mathematics (CBMS, 2012). Studying these concepts at an appropriately deep level for adults who are future educators involves a higher level of connection-making between mathematical ideas than would be expected of elementary students, in part because robust MKT includes knowledge of the *connectedness* of mathematical ideas within and across grade levels. Furthermore, this connection-making is important because it is characteristic of *conceptual understanding*, one of the five "Strands of Mathematical Proficiency" outlined by the National Research Council (2001). Mathematical proficiency requires learners to have *well-connected* knowledge of concepts within a larger body of mathematical knowledge. This conceptual knowledge is, by nature, "rich in relationships" (Hiebert & Lefevre, 1986, p. 3). In our study, we attend to the mathematical connections that participants in our data set do or do not make.

Research Questions

In this study, we pose two main research questions. Following instruction on elementary decimal concepts, (1) what models and other strategies do PTs use to compare two decimal quantities, and to what extent do they use these models appropriately and successfully? (2) What is the nature and quality of PTs' understanding of decimal place value and magnitude, as evidenced by their writing about comparing two decimal quantities?

Methods

Data for this study was collected in the context of the Elementary Mathematics Project (EMP), an NSF-funded project which designs and conducts research surrounding curriculum for use in content courses for prospective elementary teachers. One of EMP's seven instructional units is *Number Concepts*, which focuses on the consistencies of place value structure from large to small (decimal) numbers, as well as on modeling numbers. Regarding decimals, PTs using this curriculum have the opportunity to learn about decimal place value; area, linear, and other models; comparing decimals by place value or by looking at same-sized pieces; and more.

Participants in this pilot study are 28 PTs from two different institutions. Site A was a public community college in the Northwest of the United States. The course instructor at Site A has a doctorate in math education, but her appointment is in the mathematics department; she taught 17 of the 28 participants. Site B was a private four-year college in the Midwest. The instructor has a master's degree in math education but is also housed in a mathematics department. Socio-demographic data was not collected from participants, however, the student body of undergraduate teacher education programs tends to be primarily female, and roughly 19-22 years of age. All participants used the EMP Number Concepts unit, then completed an eight-item, open-ended post-test, designed by the EMP team. We analyzed this item:

> As a future teacher, you may encounter a student who is having difficulty determining which of two decimal values is greater. For example, 0.4 and 0.32.
>
> a) Provide a model that would help a student to think about the sizes of 0.4 and 0.32.
>
> b) Explain how your model would help a student compare these two quantities and which important mathematical ideas it addresses.

We analyzed responses by first open coding all elements of PTs' drawings, writing, and symbols. We did this by examining whether the participant explicitly and correctly identified the larger value, what model they provided and how it was labeled, and what they wrote about. This resulted in 32 codes which together captured PTs' choices of model and the content of their explanations. 100% consensus was achieved between two coders, after discussion.

Preliminary Results

Promising findings from our preliminary analysis include the fact that the majority of PTs explicitly identified the correct quantity as larger (n=20, 71%) and were able to provide one or more models that was accurate and useful for comparing (n=25, 89%). Seven responses were unclear as to whether 0.4 or 0.32 was larger, but only one was explicitly incorrect. Ten PTs provided decimal squares only as a model for comparison, nine provided number lines only, and six gave both. (It is unsurprising that these models were most common, in that these were two of the most prominent models in the EMP curriculum.) Two of the three PTs remaining used place value charts. This reveals that, generally, PTs are able to compare decimals, and to create and interpret models to aid in comparison, following instruction.

Data from this study also uncovered three primary challenges and areas for growth for PTs. First, we found that it seems to be more difficult to use a number line than decimal squares for the purposes of understanding decimal magnitude and relative magnitude. Most decimal squares were proportional and well-labelled, showing the size of each of the two decimal values, relative to a whole, and to each other. PTs' number lines were also generally proportional (80% of 15 number lines) and showed how hundredths could be created by partitioning tenths into tenths (67%). However, many number lines were truncated (67%), often beginning at 0.3, which limited their ability to communicate the magnitude of each of the individual decimal quantities. Furthermore, two of the 15 number lines were partitioned into elevenths instead of tenths, and two were drawn or interpreted "backwards" (smaller numbers to the right); comparable challenges did not emerge for decimal squares. Finally, explicit interpretation of the number lines was rare, just 27% described how their number line should be read and understood, while 63% of PTs who drew decimal squares explained how to interpret their model.

Second, we found that PTs' explanations did not always attend to relevant connections or reasoning, specifically surrounding the representation of 0.4 as 0.40. Of the 28 responses, 11 PTs stated or showed that 0.4 is equivalent to 0.40, or that it is appropriate to "add a zero" to the end of a decimal number. However, only five of these 11 PTs explained why this is true or useful. Though several PTs highlighted this equivalence or stated this "trick" for re-representing the quantity, less than half of those who did so attempted to justify the equivalence, or explain why a learner might find it easier to think of four tenths as forty hundredths. This leaves us uncertain as to the depth of understanding achieved by some of these PTs.

Finally, we found that references to the importance of place value were conspicuously rare. Although PTs had been asked to "Explain how your model would help a student compare these two quantities and which important mathematical ideas it addresses," less than half of the participants mentioned place value as one of these important mathematical ideas. This was highly surprising to us, as we conceive of place value as *the most* important mathematical idea undergirding these models and comparisons.

Discussion

Above, we highlighted the importance of well-connected *conceptual understanding* for both students and teachers. Our findings suggest that, while PTs have notable strengths for completing decimal tasks and using relevant tools to do so, they are less likely to articulate underlying mathematical connections. For example, few PTs in our study connected their models to decimal place value concepts, or their strategies for comparison to reasoning and justification for those strategies. This calls into question whether they have sufficient conceptual understanding to contribute to robust MKT.

A clear vision of PTs' skills and knowledge related to decimal concepts and procedures is useful and necessary for mathematics teacher educators, as they are charged with developing curriculum for use in teacher preparation coursework. We suggest that characterizations such as we have provided here may support these teacher educators in understanding PTs' strengths and needs, a first step in making changes to improve teacher preparation courses.

Next steps for this study include continuing analysis of a larger set of tests from the corpus of EMP data. The 28 tests in this study were selected as a pilot sample, but represent only about 10% of the PT participants who took the EMP unit test during this phase of data collection. We also hope to analyze corresponding pre-tests, to better understand the growth which may have happened as a result of engagement with instruction around decimal concepts. In addition to going broader, we also hope to go deeper by re-examining the types of claims that PTs made about place value in particular and exploring possible connections between these claims and PTs' chosen models. This will empower us to create more robust characterizations of PTs' knowledge of decimal place value concepts and examine the ways in which models may facilitate or demonstrate knowledge development.

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TEACHERS CANDIDATES' IMPLEMENTATIONS OF EQUITABLE MATHEMATICS TEACHING PRACTICES: AN EXAMINATION OF DIVERGENT PATHS

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In this multiple-case study, we track the diverging paths of two teacher candidates enrolled in an undergraduate elementary mathematics methods course as they developed their understanding of equitable teaching practices that were central to their course learning objectives. The cases were purposefully selected based upon our previous finding that, by the end of the course, Mary and Rose were extreme opposites in terms of their implementation of equitable mathematics teaching practices they attributed to the course. Teacher preparation programs are designed to help beginning teachers develop the skills to teach content equitably to diverse learners. Thus, methods instructors must consider the ways that teacher candidates use their knowledge and skills in their teaching.

Keywords: Instructional activities/practices, equity and diversity, teacher education, pre-service teachers, mathematics education

Amid the complex debates about the nature and purposes of teacher preparation, a critical question pervades: How do we prepare mathematics teachers to enact equitable teaching practices? The authors seek to understand their elementary mathematics methods teacher candidates' understanding and implementation of equitable mathematics teaching practices that are centered in their elementary mathematics methods course by looking deeply at two cases over the course of the semester. One teacher candidate, Mary (pseudonym), implemented equitable mathematics teaching practices that she attributed to learning in the course (via her end of course survey) with 100% fidelity for extended periods of time in her final project video. Another teacher candidate, Rose (pseudonym), implemented a few of the practices. The differences observed in Mary and Rose's final course project videos led us to wonder how two students in the same course and section could finish with such dramatic differences in their level of understanding and implementation of equitable teaching practices.

Related Literature and Framework

This study is rooted in socio-cultural theories that argue that: knowledge is developed and transmitted through social contexts, culture plays a fundamental role in cognition, and knowledge is dependent upon human interactions with each other and the world around us (Crotty, 1998).

Individuals bring their culture to the learning environment, and their culture and knowledge are constructed and reconstructed in the moment-to-moment interactions in the learning environment. With this perspective, classroom contexts provide a space where culture is produced and possibly changed (Nasir & Hand, 2006).

Although our definition of equity and equitable mathematics teaching continues to evolve, in principle it focuses on deliberate efforts to interrupt systems of oppression, harm, racism, and violence as they show up in schools, classrooms, and teaching practices particularly in ways that ensure students have access to learning ambitious mathematics content and learn mathematics in empowering environments that support their development of a positive mathematics identity.

Course Description and Design[1]

The mathematics methods course which catalyzed this study is designed to equip teacher candidates with the instructional skills to develop classroom cultures that enable students from diverse backgrounds to fully participate and learn using a practice-based model[2]. Course resources, targeted reading assignments, and focused class sessions exposed teacher candidates (TCs) to issues of status, equity issues that arise in mathematics classrooms, and effective teaching practices that help mitigate them for diverse learners, including Native American, Latinx and, Black boys.

To understand equity issues affecting Native American, Latinx and, Black boys in mathematics classrooms, our teacher candidates reviewed frameworks and publications by the Smithsonian National Museum (2019) and the National Indian Education Association (2019) to explore local Native American communities, artifacts, and curricular resources; they learned about emergent bilingual Latinx (EBL) supports including allowing students to use their first language, incorporating sentence starters and sentence frames within anchor documents, and attending to the language demands of mathematics lessons (Aguirre et al., 2012; Ahn et al., 2011; Bresser, 2003; Khisty, 2002; Torres-Velasquez & Lobo, 2005, & the Board of Regents of the University of Wisconsin System's WIDA English Development Standards, 2020); and they incorporated supports for mathematics learning and identity development for Black boys (Berry, 2004; Jett et al., 2015).

Methods

This multiple case, follow up study is designed to expand our knowledge of factors that might have influenced Mary and Rose's alignment to and/or divergence from the equitable mathematics teaching practices centrally featured in our elementary mathematics methods course (Seawright & Gerring, 2008). The following research questions guided our study:

1. In what ways do Mary and Rose's elementary mathematics coursework demonstrate their alignment with or divergence from the equitable mathematics teaching practices that were explicitly taught in their elementary mathematics methods course?
2. How does the work produced by Mary and Rose compare to each other, as students enrolled in the same course section who were both placed in internship classrooms with culturally and linguistically diverse students?
3. What might explain this alignment and/or divergence?

Study Participants

This study took place at a large, predominantly white, research-intensive, 4-year university. The two cases were a subset from teacher candidates enrolled in one of three elementary mathematics methods courses during the fall semester of their senior year ($n=55$). Our prior study incorporated a funneling sampling sequence (Erikson, 1986) where nine cases (two of which were Mary and Rose, who are both white and female) were selected for further examination of equitable teaching practices. In that study, Mary had the highest degree of implementation fidelity over an extended period of time, and Rose had the lowest. Thus, we selected these two divergent cases for our current study.

[1] Some of the work and ideas represented in this course, including the title and structure of the major projects are drawn from Math Methods Planning Group at the University of Michigan (under the direction of Dr. Deborah Ball (https://deborahloewenbergball.com).

[2] This course was developed by Dr. Imani Goffney as a part of her NSF grant, Mathematical Knowledge for Equitable Teaching, Award No. 1725551.

Data Sources and Equitable Mathematics Practices

Written materials submitted by Mary and Rose for 3 course projects, assigned at the beginning, middle, and end of the course, provided opportunities to demonstrate their understanding of equitable mathematics teaching practices in their internships and are the primary data sources for this study. The first project was the Student Thinking Project (STP). TCs prepared 2- 3 grade appropriate mathematics tasks and a series of probing questions to elicit student thinking with 2 students selected from their internship. TCs analyzed their own video recordings to explain student selection, and useful strategies in raising and equalizing their students' status. The second project was the Circulating and End of Class Check Project (EOCC). TCs video recorded themselves circulating while students worked independently. They practiced observing, responding to questions, and probing and intervening when appropriate. The final project was Leading a Whole Class Discussion Project (WCD). For this project, TCs video-recorded themselves teaching a mathematics lesson that incorporated a whole class discussion and an end of class check. For their analysis, they identified a 3-5-minute video segment highlighting one equitable mathematics practice covered in the course. They explained their practice selection and identified areas for improvement.

Analytical Techniques

Our qualitative data analysis (Saldaña, 2015) began with deductive coding of Mary and Rose's written projects to identify evidence of equitable teaching practices that were the central focus of the course. Two members of the research team independently coded each of the participants' written submissions for all three projects. Codes were labeled and mapped to the course. Then, the five members of the research team met to review initial codes for each case to ensure inter-rater reliability and to confer on preliminary findings. Next, we drafted analytical memos for each participant. In the second phase of analysis, we independently examined our coded data to identify evidence of each participant's alignment with, or divergence from, the equitable mathematics teaching practices that were the focus of the course. We also examined the relationship between the participants' degree of alignment with course goals across the term. Coded phrases, such as "repositioning a low status student," were secondarily assigned one of two codes: "aligned," as in this teaching practice or description was similar to the equitable teaching practices featured in the course, or "divergent," not aligned with the equitable teaching practices featured in the course. Secondary codes were examined by two members of the research team to ensure at least 80% inter-rater reliability. Once at least 80% inter-rater reliability was confirmed, secondary codes were integrated with our analytical memos to develop the participants' case narratives. Our findings are described in the following section.

Findings and Summary

Mary and Rose began the course with differing levels of understanding of equitable mathematics teaching. Both Rose and Mary's STPs incorporated asset-based language to describe their students and both cases discussed their desire to learn more about their students' thinking through this project. However, 95% of Mary's STP was aligned with the equitable teaching practices taught in the course, while only 61% of Rose's STP was aligned, indicating that she grappled with implementing equitable teaching practices. The most compelling data was the EOCC data. Mary's EOCC and WCD data was 100% aligned with equitable teaching course goals. Rose's EOCC data showed a decrease in alignment with course goals (50%) and a movement toward procedural mathematics instruction. Rose's divergent trend increased on her WCD. She described supporting her students' "math smarts" by acknowledging their correct solutions. Differences in their understanding and implementation of equitable teaching were evident from the first project (STP) to the EOCC.

Tables 1-3: Percentage of Alignment with Course Goals by Participant and Course Project

	Table 1: Student Thinking Project
Rose: 61% aligned	*I think that in order to equalize status issues in the classroom, it is important that students work in a variety of groups. It is important that students are in groups with students as the same abilities as them, but it also equally as important that they work with a variety of students. This way, students are learning from their peers and they are all seen as equals. If students are always in the same group they may realize they are grouped by ability. However, if every once in a while students are working and sharing with a variety of people in the classroom, they will also benefit from others that are on a different level.*
Mary 95% aligned	*To neutralize the status of the groups, I would ask the children to stay within their table groups when discussing answers so one child doesn't always go up to the same person for discussions. Additionally, I would ask students to help their peers by teaching them a skill, not doing it for them. I would want to emphasize group success and the idea that unless everyone is successful, then no one in the group is fully successful. I would set roles within the group so that the children could each have a task to perform.*
	Table 2: Circulating & EOCC Project
Rose: 50% aligned	Exit ticket: Does the problem 6789+987 require regrouping? How do you know? *I hope to learn if students know when it is appropriate to regroup, and if they can explain what regrouping is.*
Mary: 100% aligned	Exit ticket: Miss Smith is preparing materials for the table groups. She puts 4 worksheets in each table group bin. There are 5 table groups. Write an equation or draw a picture to illustrate this problem. *By giving this prompt, I hope to develop a deeper understanding of the students' skills at drawing arrays. The children have been working on this skill for 2 days. However, through completing this project, I noticed that the children still have some confusion about arrays and while they may get the problems correct, they might not always be confident about their answers. This leads me to believe that there could be some confusion about the concept, and I would like to learn more about where this confusion comes from so that I can re-teach the confusing parts.*
	Table 3: Leading a Whole Class Discussion
Rose: 6% aligned	*I chose the first aspect, purposefully using questions to elicit, probe, and connect students' mathematical ideas. I did this during the lesson when I was doing the first problem with my class. After they decided what the single digit number was, and they decided what number was in the ones place, they knew they had to multiply. The numbers were 5 times 2, and with the traditional algorithm this requires regrouping. However, I asked the students "If I wrote the number 10 down instead of regrouping, what method would I be doing?" Students responded by saying that that method is partial products. In this lesson, I also addressed students' math smarts by reaffirming them when they solved problems correctly*
Mary: 100% aligned	*The reason why I chose this skill is because it is important for students to be able to participate in whole class discussions. First, it minimizes status issues because if all students are able to participate in a respectful class discussion, each child's affective filter will naturally be lowered to (hopefully) a level where they are able to participate in the class activity since there will be an environment of respect in the classroom. I chose to practice this skill on this particular day because money is something that almost every child has seen before, but many children don't understand the value of it. So, everyone will be able to contribute to the conversation and it will be engaging because it is a topic that they are probably curious about.*

Discussion and Conclusion

How can two teacher candidates who receive the same content in the same class, with the same instructor, have dramatically different outcomes? One explanation may be mentoring. In her final course project, Mary describes classroom norms like using a positive behavior intervention system, and she reflects on failing to implement two talk moves due to minimal modeling from her mentor. However, Rose does not reflect on existing classroom norms or practices. Therefore, we wonder whether her mentor modeled equitable mathematics instruction and whether Rose may have been constrained by structures in her internship. Another possible explanation is that Rose's personal experiences or beliefs may have prevented her from implementing equitable teaching strategies. We know that our experiences shape our beliefs, biases, and identities. It is possible that Rose's personal beliefs and identities did not align with our course teachings. We know that changes in beliefs occur if equity is deliberately and explicitly implemented throughout teacher preparation. These cases suggest that teacher preparation programs should develop interventions to help TCs demonstrate proficiency for teaching mathematics in equitable ways. Finally, this tells us that novice teachers, even graduates from the same program, will have a wide range of understanding and skills relating to equitable mathematics teaching, so it is essential for school districts to provide induction support.

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EXPLORING SHIFTS IN A STUDENT'S GRAPHICAL SHAPE THINKING

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In this report, we present results from semi-structured clinical interviews with a preservice secondary teacher which were conducted prior to and after a teaching experiment intended to support the student in developing emergent graphical thinking. We illustrate how the student engaged in static shape thinking in the pre-clinical interview to describe relationships represented graphically. In the post-clinical interview, the student used both static and emergent reasoning to describe relationships. Hence, we provide an empirical example of a student developing more sophisticated graphing meanings while underscoring the importance of probing students' shape-based thinking.

Keywords: Algebra and Algebraic Thinking, Preservice Teacher Education

Researchers have shown that graphs represent information and relationships in ways that are difficult to express in other forms (e.g., Arcavi, 2003) and provide insights into students' thinking and learning (e.g., Moore et al., 2014). However, students and teachers experience persistent difficulties creating and interpreting graphs (e.g., Clement, 1989; Leinhardt et al., 1990). For instance, students often treat graphs as literal representations of a situation (e.g., interpreting a time-speed graph of a biker as the bikers' traveled path). The research examining students' graphing meanings indicates common instructional approaches do not provide students sustained opportunities to develop meaningful ways of representing relationships between covarying quantities. These failings may stem from the fact that covariational reasoning is generally absent in U.S. school curricula (Thompson & Carlson, 2017). Hence, in this paper we leverage Moore and Thompson's (2015) construct of graphical shape thinking to explore the research question: Can (and if so how can) a student whose meanings for interpreting graphs are constrained to shape-based thinking reorganize her meanings to include emergent thinking?

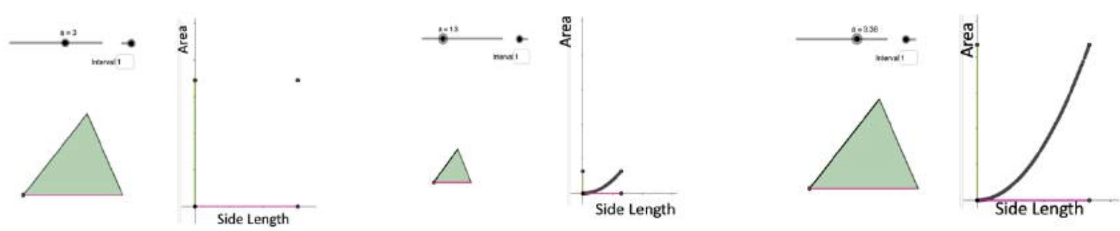
Methods, Participants, and Analysis

This report is situated in a larger teacher experiment (Steffe & Thompson, 2000) that sought to examine two preservice teachers (hereafter students) developing meanings for quadratic and exponential relationships via their covariational reasoning. Here we focus on one student, Josie (pseudonym). Josie was enrolled in a secondary mathematics teacher education program at a large university in the northeast U.S. and had completed a calculus sequence. We present data collected during the clinical interviews prior to and after the teaching experiment to provide insights into Josie's mathematics at the outset of the study and to explore shifts in her meanings at the end of the study. Two members of the research team were present at each interview and each session was video and audio recorded. In order to analyze the data, we used generative and convergent approach (Clement, 2000) in combination with conceptual analysis (Thompson, 2008). With the goal of characterizing Josie's meanings, we used an iterative approach to construct viable models of her meanings and ways of reasoning. During retrospective analysis, we re-watched all interview and teaching sessions to identify instances that provided insights into Josie's static and emergent shape thinking, which we used to develop initial models of Josie's mathematics. We compared these models to researcher notes taken during on-going analysis. When evidence contradicted our initial models, we made new conjectures, including the possibility of shifts in Josie's meanings, and refined our models with the new conjectures in mind. This process resulted in viable characterizations of Josie's mathematics.

Theoretical Perspective: Graphical Shape Thinking

Extending previous characterizations of students' quantitative and covariational reasoning (see Thompson & Carlson, 2017), Moore and colleagues (Moore & Thompson, 2015; Moore, 2016) described two types of *graphical shape thinking* students leverage when constructing or interpreting graphs. Moore and Thompson (2015) characterized static graphical shape thinking as entailing "actions based on perceptual cues and the perceptual shape of a graph" (p. 784). Static shape thinking may include *associations* between the shape of the graph, function name, and analytic rules. For example, a student may associate a parabolic graph (or "U-shape") with the term "quadratic" and a rule of the form "$y = \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$". While such associations likely have been productive for a student as she addressed tasks in school mathematics (e.g., shifting 'parent' functions), these associations may not support students when addressing a novel task (e.g., determining a relationship from a data set) or representation (e.g., the polar coordinate system).

Whereas, static shape thinking involves treating a graph as an object, Moore and Thompson (2015) described emergent thinking as a student conceptualizing "a graph *simultaneously* as what is made (a trace) and how it is made (covariation)" (p.784). A student thinking emergently conceives a graph as an in-progress trace representing two covarying quantities magnitudes or values. For example, consider the *Growing Triangle Task* which shows a scalene growing triangle (https://bit.ly/2BjdEKZ). Students are asked to represent the relationship between the triangle's base (in pink) and area (in green). To reason emergently, the student must first construct a coordinate system that represents each quantity on an axis and understand a point in this coordinate system as simultaneously representing both quantities' magnitudes (Figure 4a). The student can then imagine how this point will move in the coordinate system as the triangle's area and side length vary; reasoning emergently entails understanding the graph of the relationship as being produced by the trace of this point as the quantities covary (Figure 1b/c).



(a) (b) (c)

Figure 4. The Growing Triangle Task

Results

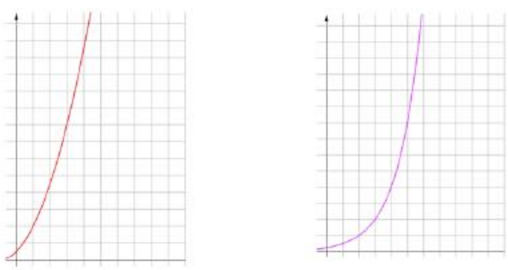
In both clinical interviews, we provided Josie two graphs representing quadratic and exponential relationships and asked her to identify the relationship represented by each graph (Figure 5a/b). We hoped to gain insight into Josie's meanings for graphs and her ways of identifying relationships. We also provided tables (e.g., Figure 2c) of values representing quadratic and exponential relationships to explore the connections between Josie's meanings for these relationships in different representations. We note during the teaching episodes, we engaged the students in tasks designed to support their reasoning covariationally and emergently (e.g., the *Ferris Wheel Task* as described by Carlson & Moore, 2012; Moore, 2014). While Josie's activities addressing these tasks was critical to her reorganizing her graphing meanings, we focus on data from the pre- and post-clinical interviews to highlight shifts in her graphical shape thinking for brevity's sake.

Pre-Clinical Interview

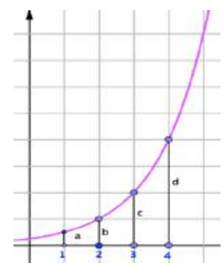
When asked to identify the relationship represented by the graph in Figure 5a, Josie claimed:

> It looks like half a parabola, I would say x^2… but I would have to see the other half, definitely to clarify. But it looks like it is going to come back up (*motions as if making a curve to the left of the graph with her hands*)...yeah...It [the graph] doesn't look like, it's, it's… to me if it [the graph] is exponential, they start close to kind of here (*showing the intersection of the axes*), they are closer to zero and shoot up ....so, to me it's [the graph in Figure 2a] half a parabola.

For the graph in Figure 5b, Josie explained, "this one looks like an exponential growth… it started from something very close to zero and then increases very fast. Yeah, that's what I am thinking, it's an exponential growth." We infer Josie leveraged static shape thinking as she determined the relationship represented by each graph. Specifically, she focused on the shape of the graph (e.g. "half a parabola", "started… close to zero and then increases very fast") which she associated with a functional class and analytic rules (e.g. x^2, "exponential growth").



x	y
1	3
2	12
3	27
4	48
5	75



(a) (b) (c) (d)

Figure 5: A graph representing (a) quadratic and (b) exponential relationship, (c) Table of values representing quadratic relationship, (d) a recreation of Josie's explanation

Although Josie's meanings grounded in static shape thinking supported her in correctly describing that Figure 2a and b represented a quadratic and exponential relationship, respectively, her meanings did not entail a way to describe the same types of relationships represented in a table. For example, after determining several consecutive slopes to determine the relationship in Figure 5c was non-linear, Josie noted "All the y's are multiples of three and it is rapidly increasing, maybe it could be an exponential growth." We infer that Josie's meanings for determining a relationship from a table only supported her in determining if a relationship was linear or non-linear; one possible explanation for this is that her meanings for non-linear relationships (e.g., quadratic, exponential) entailed mostly shape-based associations.

Post-Clinical Interview

After having multiple opportunities to construct and graphically represent covariational relationships in the teaching experiment, we engaged Josie in the post-clinical interview 6 weeks after the last teaching episode. Addressing the same problems in Figure 2, she initially engaged in static shape thinking as she relied on visual properties of the curve. However, in this case Josie was able to unpack her thinking via her covariational and emergent reasoning to make claims about the relationships represented by each graph and table. For example, for Figure 5a, Josie first responded, "[the graph] looks quadratic… it is increasing and something like that is either exponential or quadratic or cubic." She further described, "it looks like half a parabola" and making hand motions as if drawing a parabolic curve to the left of the vertical axis explained, "then it would be a parabola and it would be a quadratic." However, Josie then claimed, "let's see if the amounts of amounts of change are the same" and determined two points, (1,2) and (3,8), on the curve, and calculated the differences in the y-coordinates as 6. She said, "if I knew the next one point, I could check if the amounts of amounts of change are same and it would be quadratic" and concluded, "but it looks like a quadratic and I am going to say quadratic." Hence, Josie's meanings now entailed that a graph represents quadratic change if the amounts of change of amounts of change are constant, a defining characteristic of quadratic change.

As a second example of Josie unpacking her static shape thinking, consider her response when determining the relationship represented by the graph in Figure 5b. She first claimed the relationship as exponential because the graph "is starting off slow and then shoots up." After this, she imagined points on the horizontal axis and motioned as if drawing vertical distances from these points to the graph (see Figure 5d for a recreation of her hand motions) and described, "for equal changes in the x's", and motioning her fingers on imaginary segments as seen in Figure 5d "like from 1 to 2 we are not increasing... maybe a half, but 2 to 3 we are increasing maybe by a one, 3 to 4 we are increasing by a two maybe, that is a little more and so on and so forth, and then it increases by more. So, definitely I am going to say it is exponential." In each case, rather than being constrained by making shape-based associations, Josie's meanings for interpreting graphs (and tables, like Figure 5c) now included being able to unpack a graph in terms of the relationship between covarying quantities (e.g., constant second differences for quadratic, increasing by amounts that themselves increase by a factor of two for exponential).

Discussion

In this report, we characterize a student's static and emergent shape thinking before and after a teaching experiment designed to support her in developing meanings for graphs as emergent traces. Addressing our research question, the student was able to reorganize her shape-based meanings for interpreting graphs to meanings that entailed interpreting graphs according to the underlying covariational relationships they represented. We conjecture the numerous opportunities Josie had to reason about and represent relationships between covarying quantities in the teaching experiment supported her in moving beyond static shape thinking.

Second, and addressing the "and if so how" part of the research question, we note how Josie's meanings for interpreting graphs included elements that appeared static in the post-interview. For instance, she continued to use shape-based associations, which is not surprising as these associations can still be useful in determining the relationship represented by a graph. However, as she justified her choice, Josie was able to unpack the graph in terms of the relationship represented by the covarying quantities, which is indicative of her emergent shape thinking. Hence, Josie's activity highlights how a student may still use shape-based meanings while being able to unpack these meanings in terms of the underlying relationship. This underscores the importance of researchers carefully attending to students' graphing meanings. A student engaging in shape-based activity does not mean they are constrained to such reasoning; it is important to examine if the student can unpack their thinking further.

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EXPLORING PRESERVICE TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE OF TEACHING FRACTIONS

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Teachers' pedagogical content knowledge (PCK) influences their instruction and, by consequence, their children's opportunities to learn better. Within the domain of fractions, items assessing PCK are nested within larger assessments, with little explicit focus on the PCK domain. We report on the development and initial validity argument for a PCK for Fractions assessment that assesses preservice teachers' (PSTs') knowledge of students' fractional reasoning. Results suggest the assessment can differentiate between PSTs of different levels in their teacher education program, and that items appear to assess the intended construct. Implications for future study, and for how PCK may develop among PSTs is discussed.

Keywords: Teacher Knowledge; Number Concepts and Operations

Teachers' professional knowledge influences their instruction and, as a result, their students' mathematical learning (Hill et al., 2008a; Hill et al., 2008b). Within mathematics education, such professional knowledge is described by Ball et al. (2008) as Mathematical Knowledge for Teaching (MKT), with a pragmatic distinction between content knowledge (CK) and pedagogical content knowledge (PCK). Although both domains are considered essential, there are relatively few measures of teachers' PCK (Copur-Gencturk et al., 2019; Hill et al., 2008a). Copur-Gencturk et al. (2019) note that many items designed to assess PCK may be better described as CK items both at a theoretical and statistical level. We conjecture that one potential reason for such misalignment of PCK items is due to the nature of the overarching assessments themselves. Rather, by designing an MKT assessment that measures both CK and PCK, there is a risk of unintentionally overemphasizing elements of CK and underrepresenting elements of PCK (Copur-Gencturk et al., 2019).

In the current paper, we report on efforts to design and pilot an assessment of preservice teachers' (PSTs) PCK for teaching fractions in grades 3-5. Current measures of PCK for fractions tend to situate such items as part of an overarching MKT assessment (Depaepe et al., 2015; Kazemi & Rafiepour, 2018). Such assessments tend to distinguish CK and PCK items and may align certain items with key elements of PCK. However, many designed PCK items are open responses, which are time-consuming to code. Our own efforts focus on designing closed-response items to allow for less time-consuming coding for practitioners who may use the assessment. Additionally, we chose to initially focus on one element of PCK, knowledge of students' conceptions, and reasonings about fractions. Our intentional narrowed focus allows for a more concerted effort to examine the nature of this subconstruct, as well as aspects of item design that may inform development of measures for additional aspects of PCK. Thus, the purpose of this study is to construct an initial validity argument for a measure of PSTs' PCK for elementary children's reasoning about fractions.

Theoretical Framework

Developing professional knowledge is an essential component of teacher education programs (AMTE, 2017). Such knowledge is distinct from content knowledge and general pedagogical knowledge and it is called pedagogical content knowledge (PCK) (Shulman, 1986). PCK represents essential knowledge for teaching to facilitate student learning of the subject matter. Ball, Hill, and colleagues extended Shulman's work to articulate a framework for Mathematical Knowledge for

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Teaching (MKT) (Hill et al., 2008b). Extending Ball et al.'s (2008) framework, different scholars have examined teachers CK and PCK on various grounds. Herbst and Kosko (2014) developed an instrument to assess high school teachers' mathematical knowledge for teaching geometry (MKT-G). Khakasa & Berger (2016) applied six domains of MKT to categorize secondary school teachers' mathematical knowledge based on their interpretations of open-ended tasks. They found both the amount of experience and quality of experiences affect teachers' MKT (Herbst & Kosko, 2014; Khakasa & Berger, 2016).

Of particular interest in the present study are those analyses of teachers' MKT for fractions. Depaepe and colleagues (2015) compared CK and PCK on rational numbers among secondary and elementary preservice teachers. They found that CK items are generally easier for PSTs to answer correctly than PCK items. Additionally, PSTs' CK scores were considered low, despite taking a course related to teaching rational numbers. Although secondary PSTs significantly outperformed elementary PSTs on CK for rational numbers, there was no observable difference in how both groups scored on PCK for rational numbers. This is an interesting finding, considering that PSTs' CK and PCK scores are positively correlated (Depaepe et al., 2015; Kazemi & Raflepour, 2018). A common assumption in the literature on teacher knowledge is that strong CK is required to have strong PCK (Izsák et al., 2019; Shulman, 1986). However, in comparing professional development approaches for elementary PSTs, Trobst et al. (2018) found that focusing on enhancing CK of PSTs was less effective than focusing specifically on PCK. Collectively, these findings suggest that teachers' PCK for fractions is related to CK for fractions, but growth in PCK stems from a focus on contexts related to the teaching and learning of fractions.

Method

Sample & Measure

Participants included 58 preservice teachers enrolled in a teacher education program in a Midwestern U.S. university. Participants included 47 early childhood education majors (grades Preschool to 3rd, with optional 4^{th} & 5^{th} grade endorsement) and 11 middle childhood education majors (grades 4-9). Each licensure program includes two mathematics methods course and participants were solicited from each course across both programs (31 juniors; 27 seniors).

An initial version of the PCK assessment included 20 questions, which were subjected to cognitive interviews with two experienced elementary math coaches (Karabenick et al., 2007). In the cognitive interviews, we asked two expert teachers to interpret each item and explain their rationale for their responses. Following cognitive interviews, we analyzed responses related to each item. Some items were interpreted as intended and remained unchanged, while others were revised or removed based on participants' responses. The revised PCK assessment included 15 questions; 9 multiple-choice, 5 multiple-response. Figure 1 illustrates an example question including four items (where an item is counted as a singular response, a question may group one or more items). The question was inspired by literature describing children's fraction learning progressions (Battista, 2012; Hackenberg et al., 2016). Each image represents a specific task of teaching that illustrates a scenario in the form of student work (assessing the reasoning of students' shading 2/3 of 12). Items in the assessment followed a similar structure of assessing children's reasoning based on descriptions from the research literature.

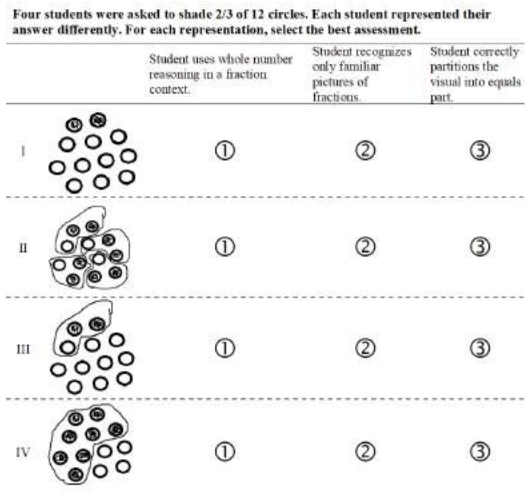

Figure 1: The Example of Item for Measuring PCK-Fractions

Analysis and Results

Analysis of data followed recommendations from the *Standards for Educational and Psychological Testing* (AERA et al., 2014), which states that survey/assessment development should integrate various sources of evidence across multiple studies to construct a validity argument. We collected validity evidence for response processes and test content. Validity evidence of *response processes* focuses on whether participants' responses to our items aligns with the intended theoretical design of the item. In this paper, we conducted a classical item analysis to examine the internal reliability of both the assessment and the items. We also used cognitive interview data to inform decisions on whether certain items should be retained or removed when conducting the item analysis. Evidence for *test content* focuses on how an assessment represents the content and whether scores can be interpreted as intended (AERA et al., 2014). Since our PCK assessment seeks to measure the effect of teacher education initiatives or interventions, we used an independent t-test to compare PCK scores of juniors and seniors as one example of such evidence (considering progress in a teacher education program as the intervention).

The initial item analysis of 30 items resulted in a Cronbach's alpha coefficient of .284. Although the customary threshold for Cronbach's alpha for piloted assessments is typically at or near .70 (Nunnally & Bernstein, 1994), many pilots of successfully validated PCK assessments have tended to report initial Cronbach's alpha coefficients above .60, but somewhat below .70 (e.g., Depaepe et al., 2015; Herbst & Kosko, 2012). Nevertheless, the initial model's reported alpha coefficient was below accepted norms. For each item, we examined point-biserial correlations as an indicator for potential removal. Point-biserial coefficients correlate an item's score (0 or 1) with the total score of the assessment, providing an index of potential fit for the assessment (Crocker & Algina, 2006). It is customary to identify items with point-biserial coefficients below the accepted norm of .30, and consider any interview data, aspects of face validity, etc. before determining whether an item should be removed. Items are removed one-at-a-time and the Cronbach's alpha and point-biserial coefficients are recalculated again.

Our final item analysis model yielded a Cronbach's alpha coefficient of .640. While the majority of remaining items had point-biserial coefficients at or near .30, we chose to retain a subset of items that had lower coefficients (~.20). We retained these items for several key reasons. First, initial pilots of assessments often include smaller samples and may not represent the variance in responses of participants from a larger, more representative sample. As our assessment included PSTs from a single university and no inservice teachers, we believe our sample is unlikely to be representative. Second, although point-biserial and Cronbach's alpha coefficients are useful psychometric indicators, they are but one part of the validity argument for developed measures (Wolf & Smith, 2007). Cognitive interview data for these particular items indicated that they were both interpreted in the manner they were intended and that both responding math coaches considered such content as normative (i.e., these were demonstrated student actions they had seen or seen something similar). Thus, the resulting PCK assessment includes 17 items nested in 7 questions (M=9.48, SD=3.04; *Range* = 4 to 15). Item difficulty ranged from .28 (28% of items answered correctly) to .93 (93% answered correctly), suggesting a wide range in difficulty.

Next, we used an independent samples t-test to compare PCK scores of juniors and seniors in our sample. Results were statistically significant ($t = 2.23$, $df = 56$, $p = .03$), suggesting that PSTs in the sample who were enrolled in their second mathematics methods course had higher scores (M=10.34) than their counterparts enrolled in the first mathematics methods course (M=8.62). This result was still statistically significant for early childhood majors when middle childhood majors were removed from the sample ($t = 1.97$, $df = 45$, $p = .05$), with a similar difference in scores between juniors (M=8.31) and seniors (M=10.00). These results suggest that the PCK assessment distinguishes between PSTs who are earlier and later in their teacher education program.

Discussion

Prior scholars' efforts to construct items assessing PCK are typically open-response and part of MKT measures covering both CK and PCK domains (Depaepe et al., 2015; Kazemi & Rafiepour, 2018; Trobst et al., 2018). Contrasting prior approaches, we sought to develop items exclusively focused on the PCK domain, with particular attention to the knowledge of students' conceptions and reasoning about fractions. The purpose of this study was to construct an initial validity argument for an assessment of PSTs' PCK. Thus, we reported results of a pilot on our assessment that included items on knowledge of students' conceptions and reasonings about fractions, with validity evidence supporting test content and response processes. Results from psychometric analysis, as well as evidence from previously conducted cognitive interviews, supports the claim that our assessment measures teachers' PCK for fractions. Findings from the independent t-test support the claim that our assessment can measure growth due to teacher education initiatives. Although preliminary, the evidence presented in this paper provides a useful baseline for an initial validity argument. Future study is needed to both verify these preliminary findings and to examine other features of validity for such assessments.

Acknowledgments

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TEACHER EDUCATION (PRE-SERVICE):

POSTERS

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FUTURE TEACHERS OF BASIC GEOMETRY: CROSSROADS BETWEEN THE KNOWLEDGE OF TWO SCHOOL CULTURES OF MATHEMATICS

FUTUROS DOCENTES DE GEOMETRÍA BÁSICA: ENCRUCIJADA ENTRE SABERES DE DOS CULTURAS ESCOLARES DE LAS MATEMÁTICAS

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This qualitative research had the purpose of identifying the crossroads between the school cultures of secondary school and teacher training institutions in their training practicum. With the teacher's specialized Theory of Knowledge, geometry activities were analyzed. We conclude that geometry in secondary school is linked to shape and measure, while teachers in training struggle to access deductive reasoning. However, the crossroad between geometric reasoning and secondary school is weak, because the mathematical knowledge of the content is not robust.

Keywords: School culture, geometry teaching, cognition, teacher training

Crossroads Between Mathematical Cultures

Basic education schools and teacher training institutions have their own mathematical cultures. In the case of geometry, is the formation of knowledge about its teaching achieved at the crossroad of mathematical cultures?

The phenomenon of crossroads had already been addressed by Engeström, Engeström and Kärkäinen (1995), in order to explain institutional cognition and the construction of symbolic ceilings in cultural communities. For the analysis of the empirical data recovered from the observation of the practicum of future secondary school mathematics teachers, this research uses the Mathematics Teacher's Specialized Knowledge (MTSK) (Carrillo et al., 2018), as it addresses an analytical model of the teacher's knowledge in an integral way for all its dimensions. Furthermore, it is a methodological tool that helps analyze the practices and knowledge of future mathematics teachers, and identifies the construction at the crossroad.

Conclusion: crossroads as a condition of teacher training

The professional practice of the future teacher goes beyond the classroom, as it is a favorable context for the exploration of specialized knowledge. In addition, it allows for the prominence of the figure of the student as a knowing subject to decrease, which gives way for the process of learning itself to rise as the protagonist of a teaching activity.

The analysis of the initial training context in which this research was developed allowed us to observe important aspects about the future teacher's knowledge linked directly to the transmission process of mathematical knowledge. This allowed us to recognize knowledge within the practice of the future teacher, which was reflected in practical aspects such as the design of tasks. However, the crossroad between such reasoning and secondary school is not consistent, because the mathematical knowledge of the content is weak.

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FUTUROS DOCENTES DE GEOMETRÍA BÁSICA: ENCRUCIJADA ENTRE SABERES DE DOS CULTURAS ESCOLARES DE LAS MATEMÁTICAS

FUTURE TEACHERS OF BASIC GEOMETRY: CROSSROADS BETWEEN THE KNOWLEDGE OF TWO SCHOOL CULTURES OF MATHEMATICS

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Esta investigación de corte cualitativo tuvo el propósito de identificar el cruce de fronteras entre las culturas escolares de la escuela secundaria y de la institución formadora de docentes, en el prácticum de la formación. Con la Teoría del Conocimiento especializado del profesor, se analizaron actividades de geometría. Se concluye que la geometría en la secundaria está ligada a la figura y la medida, en tanto que los estudiantes para maestro luchan por acceder al razonamiento deductivo. Pero el cruce de frontera del razonamiento geométrico a la escuela secundaria es débil, porque el conocimiento matemático del contenido no es robusto.

Palabras clave: Culturas escolares, enseñanza de la geometría, cognición, formación docente

Los cruces de fronteras entre culturas matemáticas

La escuela de educación básica y la institución formadora de docentes tienen sus propias culturas matemáticas. En el caso de la geometría ¿La conformación de conocimientos sobre su enseñanza se logra en el cruce de fronteras de las culturas de las matemáticas?

El fenómeno de cruce de fronteras ya había sido abordado por Engeström, Engeström Y Kärkäinen (1995), para explicar la cognición institucional y la construcción de techos simbólicos en comunidades culturales. Para el análisis del referente empírico recuperado de la observación del prácticum de futuros maestros de matemáticas de secundaria, esta investigación apunta al Conocimiento Especializado del Profesor de Matemáticas (MTSK) (Carrillo *et al*., 2018), pues aborda un modelo analítico del conocimiento del profesor de manera integral en todas sus dimensiones, y es una herramienta metodológica para analizar las prácticas y el conocimiento del futuro profesor de matemáticas, e identificar la construcción en el cruce de fronteras.

Conclusión: el cruce de fronteras como condición de la formación docente

La práctica profesional del futuro profesor es una actividad que va más allá del aula, es un buen contexto para la exploración del conocimiento especializado, además de perseguir la idea de mostrar menos protagonismo a la figura del estudiante como sujeto cognoscente y más al propio proceso de aprendizaje como protagonista de una actividad docente.

El análisis del contexto de formación inicial en la que se desarrolló esta investigación nos permitió observar aspectos importantes acerca de los conocimientos del futuro profesor ligados directamente a los procesos de transmisión de conocimiento matemático; nos permitió reconocer conocimientos dentro de la práctica del futuro docente, que se reflejaron en aspectos prácticos como el diseño de tareas. Pero el cruce de frontera de dicho razonamiento a la escuela secundaria no es consistente, porque el conocimiento matemático del contenido es débil.

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DEVELOPING PROSPECTIVE TEACHERS' REPRESENTATIONAL FLUENCY OF WHOLE NUMBER MULTIPLICATION USING ARRAY REPRESENTATIONS

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Keywords: Number Concepts and Operations, Representational Fluency, Teacher Knowledge

Representational fluency, the ability to make "connections among mathematical representations to deepen understanding of mathematics concepts and procedures" is an important aspect of building problem-solving skills (NCTM, 2014, p.10). From our experience teaching mathematics content courses for prospective elementary teachers (PTs), we have found that PTs often struggle to make connections between different models of multiplication, which is supported by prior research (Lo, et al., 2008). To attend to this issue, we designed a series of multiplication tasks (see, Olanoff et al. 2018). In this poster we present findings from our second round of implementation with the goal of deepening PTs' specialized content knowledge of multiplication by fostering representational fluency and connecting representations to sense-making procedures.

Our sequence of tasks started with presenting PTs with an 29 x 23 array grid, followed by subsequent tasks. Our goals for these tasks were for PTs to 1) Utilize the array model to understand that a product can be found by decomposing the array into different regions and combining those regions, 2) Use the strategies of summing and combining in this model to develop the partial products algorithm, 3) Understand the connection between the standard US multiplication algorithm as related to the array model, and (4) Recognize the distributive property as a driving force behind the partial products algorithm. Our analysis showed that we were successful at achieving some of our goals, but that the tasks would require modifications in order to meet others. For example, we found that PTs were successful with breaking the array into different amounts and summing them, but many used chunking that was inefficient, as they did not actually make multiplication easier. In subsequent tasks the PTs were also found to focus on the total number of squares (as 667) in a base-10 representation of the array rather than the connection to the original 29 x 23 grid. In addition, the majority of PTs were unable to create symbolic notations that matched with the way they used the base-10 blocks, indicating that in spite of success with prior tasks they struggled to make the connection between the array model and the standard or partial products algorithms.

Overall, through reflecting on our implementation we learned that we needed to re-consider ways to help PTs focus more explicitly on the two numbers being multiplied in an array, rather than only the total number of units. Additionally, we needed to identify ways to better support them in relating symbolic representations of multiplication to array models, *specifically* to understand how the distributive property manifests itself in the symbolic algorithms (both US standard and partial products) and in the array representations. In our poster, we will share our task sequence, show sample PT strategies and examples that indicated a success in representational fluency and places where PTs struggled.

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THE POWER OF CARING AND FUNDS OF KNOWLEDGE IN TEACHER LEARNING

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Keywords: Culturally Relevant Pedagogy, Teacher Knowledge, Teacher Education - Preservice

This study honors the premise that teaching mathematics meaningfully for diverse learners includes developing *dispositions* and *practices* that draw on children's "cultural, linguistic, and community-based knowledge" (Turner, Drake, McDuffie, Aguirre, Bartell, & Foote, 2012; p. 68). We propose that supporting practicing and prospective K-6 mathematics teachers (PMTs) as *learners* who utilize their *own* "cultural, linguistic, and community-based knowledge" can authenticate the potential of such opportunities in PMTs' future work with students. In our poster, we describe such a learning experience for Maya (a practicing, pre-school teacher) and Zoe (a poster author, preservice mathematics teacher, and project researcher), two PMTs in a graduate mathematics education course who were tasked with 3D designing and printing a manipulative that would be used with a child in a problem-solving interview.

We utilize *funds of knowledge* to capture the value of "historically accumulated and culturally developed bodies of knowledge and skills" that are deemed essential for human functioning and well-being (Moll, Amanti, Neff, & Gonzalez, 1992; p. 132). Noddings (2010) illuminates how a *relational sense of caring* that is "receptive" to "what the cared-for is feeling" (p. 2) can create an authentic space for sharing funds of knowledge during learning, which occurred when Zoe embraced Maya's experiences growing up in the Dominican Republic (DR).

This case is part of a larger teacher education study to help understand PMTs' knowledge development as they Make manipulatives (blinded). Data includes videos from design sessions and written assignments, *and* the tools, because a manipulative's design reflects the intentions and understandings of the Maker(s) (Pratt & Noss, 2010). Purposeful sampling (Patton, 2002) for an exploratory case study (Yin, 2009) helped us analyze how funds of knowledge connected to the design and use of Maya and Zoe's tool, called *No Más Caídas* (*No More Spills*).

Maya and Zoe hoped No Más Caídas would make counting playful—a trait they viewed as fading from K-6 activities. As Maya articulated anxieties over the intricacies of the design process, Zoe invited her to share her experiences learning mathematics as a child in the DR. At first, Maya was reluctant, deeming her lived experiences as irrelevant, but as the PMTs engaged in and out of class, a *confianza* (mutual trust) (González, Moll, & Amanti, 2005) developed that informed the design of their tool and their learning. For example, they opted for marbles as their counting objects, to connect to the everyday objects children use to count in the DR (like beans and rocks). Also, Maya invited her pre-school age daughter for the final interview with their tool, integrating the funds-of-knowledge focus on family (Moll, et al., 1992). To her delight, Maya discovered her daughter's mathematical capabilities with the tool exceeded her expectations, writing *"trabajando con números más grande a los que ella estaba acostumbrada, más su alegría y dedicación al usar nuestra herramienta favorecio nuestra entrevista."*

Providing PMTs funds-of-knowledge learning opportunities can offer a transitioning vehicle between the teacher education setting and the PMTs' own classrooms. In our poster, we share these and additional funds of knowledge from Maya and Zoe's experiences, relaying their power in the PMTs' learning, and promoting PME-NA's goal to support the "ample diversity of ways of teaching and learning mathematics" (PME-NA, 2020).

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PROSPECTIVE TEACHERS' AFFORDANCES AND CHALLENGES OF SEEING STUDENTS' MATHEMATICAL STRENGTHS

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Educational policy and national reform movements in mathematics education emphasize the importance of drawing on students' resources, fostering students' identities, and attending to power dynamics during instruction (Nasir & de Royston, 2013; NCTM, 2008). This poster shares initial findings from a cross-institutional study of 223 prospective teachers (PTs) enrolled in elementary mathematics methods courses with a focus on how to see students' strengths.

The research question was: *What affordances and challenges arose for PTs as they reflected upon and attempted implementation of pedagogical strategies focused on seeing students' strengths?* Across seven institutions, mathematics teacher educators (MTEs) chose Skinner and colleagues' (2019) *Learning to See Students' Mathematical Strengths* as a common text for their courses because it highlights strengths-based teaching and offers five strategies PTs could enact in their varied PreK-6 field experiences. The text's strategies are (1) trusting students with complex tasks, (2) randomly grouping students, (3) having conversations about smartness in math, (4) noticing power and privilege, and (5) using critical friends to challenge and support you. The strategies were used as a framework to examine specific ways PTs consider and build upon students' mathematical strengths.

Data sources included PTs' written responses to (a) pre-reading questions, (b) post-reading reflection prompts, and (c) reflective questions at the end of the semester. Post-reading prompts included questions that asked PTs to identify which strategy seemed easiest to implement in practice and which might be most difficult. While PTs were not required to implement any of the strategies in their field experiences, at semester end they were asked to specify if they had attempted any strategies and why, and to identify which strategy they might focus on next and why. Data was examined by the MTEs across student, across prompt, and across course, and then cross-intuitionally for prominent themes in PTs' responses.

Initial findings across institutions indicate that trusting students with challenging tasks was perceived as the second-most difficult strategy for PTs to implement yet was also the most attempted strategy during field experiences. Noticing power and privilege had the lowest occurrence of all the strategies, and examination of corresponding qualitative data revealed that PTs may not feel ready or equipped to do this work. Of note is that only one of the 223 PTs responded that they would not consider using any of these strategies in their future practice. The poster explores how PTs discussed each of these strategies, and potential pathways to leverage the affordances of a common text across institutions to support PTs' seeing strengths.

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THE UTE MODEL: ANIMATING PRE-SERVICE TEACHERS' VISIONS FOR STUDENT ENGAGEMENT

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Study Overview and Methods

The UTE model affords secondary mathematics PSTs with the chance to combine their first methods course with an early field experience in a first-year undergraduate mathematics course, learning about teaching strategies while attempting to implement them in the classroom (Author et al., 2019). During the UTE, PSTs plan, execute, and receive feedback as they teach a series of lessons in the undergraduate mathematics course while being supported by mentor teacher educators. PSTs participated in pre- and post-UTE interviews that followed Munter's (2014) protocol for assessing PST's vision for high-quality mathematics instruction (VHQMI). These interviews allow for insights into the experience of these PSTs and reveal evolutions in their shifting beliefs about the role of the teacher in the classroom, the use of mathematical tasks, the nature of classroom discourse, and the level of student engagement. All interviews were transcribed and then analyzed using Munter's (2014) rubric as a guide.

Results

While findings across all four VHQMI categories have been noteworthy, of particular interest to this study has been PST responses that fall into Munter's 'student engagement' category. Codes in this category refer to "non-content-specific characterizations of student behavior", making this category a way to capture PST thoughts that describe a generic vision for the classroom that lack sufficient specificity regarding the role of the teacher, the nature of classroom discourse, or the use of mathematical tasks. Tracking the presence or absence of these generic responses has been helpful in revealing the places where PST visions gain specificity, shifting from the ambiguous to the explicit, from broad sweeping claims to detailed articulations of classroom practice. For example, consider the comparison of a PST's pre-interview answer: "If [the students] are engaged in instruction, I think is a big indicator if they are actually grasping the concept" to the same PST's post-interview answer: "I would pay attention to the types of questions [the teachers] are asking students and how that's eliciting responses." This shift in thinking may suggest an early field experience can promote high-leverage teacher questions as a concrete, specific means of enacting a previously generic vision for student engagement. This study tracks these movements from the generic to the specific, looking for insights into the development of PST thought that might inform our understanding of teacher preparation.

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SEE MATH (SUPPORT AND ENRICHMENT EXPERIENCES IN MATHEMATICS) PROGRAM

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Teacher candidates (TCs) can learn how to teach complex skills when teacher education programs (namely methods courses and field experiences) provide clear and purposeful experiences (Grossman et al., 2009). The following poster describes one such innovation, a complementary field experience called SEE Math (Support and Enrichment Experiences in Math) that is paired with a mathematics methods course. SEE Math aims to support TCs as they learn to teach mathematics through problem solving while promoting equity over multiple iterations with a single child.

The SEE Math program is an extension of the TEACH Math module called "Learning Case Study Module" (Drake et al., 2015; Turner et al., 2012). In the TEACH Math module, TCs conduct a series of interviews with a child about their funds of knowledge and mathematical knowledge through a series of Cognitively Guided Instruction (Carpenter et al., 1999; Carpenter et al., 2015) tasks. The Learning Case Study module is one way in which TC can learn how to elicit children's thinking and make small adjustments to existing curricula in ways that are relevant to children and their lives.

During our eight-week program of SEE Math, TCs are also paired with an elementary-aged student in a case-study setting. TCs learn to craft and implement tasks that promote problem-solving in the context of a case study of a child's thinking while they collect and analyze student data to inform future instructional moves. For example, the TCs conduct a Getting to Know You Interview (as is outlined in the TEACH Math module) and create a Venn Diagram about the connections and individual interests of the TC and the child. The bulk of the SEE Math activities support TCs to leverage existing curricula (such as the book by Kazemi & Hintz, 2014) to adapt tasks and create new tasks in ways that is relevant to their child based on the first interview.

There are multiple culminating outcomes of the program. For the children, SEE Math culminates in a final experience where children and TCs engineer a tower or a catapult out of normal materials found in a home. For the TCs, their experience with SEE Math Program culminates in a mock parent-teacher conference that they conduct with their elementary mathematics teacher at the conclusion of the semester. Examples of student work in the poster will show how SEE Math builds on the TEACH Math module and offers TCs an opportunity to focus on the nuances of children's strengths rather than traditional measure of achievement and skill. In addition to the theoretical foundations of SEE Math, we also intend to include examples of work from the TCs' case studies and experiences from TCs, students, and parents who participated in the program.

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PRE-SERVICE ELEMENTARY EDUCATION TEACHERS' SOLUTIONS FOR WORD PROBLEMS: USING STRIP DIAGRAM VS. ALGEBRAIC APPROACH

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Creating, connecting, and translating multiple representation are "important cognitive processes that lead students to develop robust mathematical understandings" (Huntley, Marcus, Kahan, & Miller, 2007, p. 117). These cognitive processes are also considered to be crucial elements of preservice teachers' (PSTs') pedagogical content knowledge (Dreher, Kuntze, & Lerman, 2016).

In order to investigate PSTs' cognitive processes regarding multiple representations, we collected data from 73 PSTs, who enrolled in a mathematics content course for elementary education majors in Spring 2019. We analyzed PSTs' solutions to an assessment task following seven weeks of instruction related to the use of strip diagrams, double number lines, and algebraic equations to solve problems involving ratio and proportional relationships and word problems (Beckmann, 2014). The PSTs were asked to determine the total number of cookies Bonnie baked when given information about the cookie types (e.g., 1/3 of the cookies were chocolate chip, 1/6 were peanut butter, 1/6 were oatmeal raisin, and 24 were cinnamon) in two ways: using a strip diagram and writing and solving an algebraic equation. We used an error analysis technique (Radatz, 1979) to sort and interpret the responses based on fluency with strip diagram and algebraic solutions. The PSTs who exhibited complete reasoning were able to use both representations and the PSTs who exhibited incomplete reasoning were unable to use at least one of the representations. The preliminary analysis of solutions revealed the following themes in the PSTs' strategies (Table 1).

Table 1: PSTs' Strategies for Solving the Cookie Problem

	Complete Reasoning	Incomplete Reasoning
Strip Diagram	Partitioned a single bar based and labeled parts based on cookie type and fractional amount or quantity	-Drew multiple bars to represent each fractional amount -Did not consider fractional amounts as parts of the same unit -Misinterpreted the unit of fraction
Algebraic Solution	Defined variable as total amount of cookies wrote expressions based on type of cookie in terms of the fraction of the whole	Assumed sum of given amounts equaled the number stated in the problem Considered fractional amounts as numbers rather than representing parts of the unit

Students' identification of the unit of a fraction is related to unitizing mental process (Lamon, 2012). The findings of our analysis indicate that PSTs' unitizing mental processes is a determining factor in their use of representations to solve word problems. No matter which representation the PSTs used, the way they considered the unit or sub-unit amount was the foundational step in their solutions. These results suggest that further studies investigating PSTs' understanding of unitizing related processes within representations can help us prepare instruction that aligns with their understanding path for the mathematical concepts.

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DESIGNING PEDAGOGIES OF PRACTICE FOR A CRITICAL PRACTICE-BASED TEACHER EDUCATION

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Keywords: Teacher Education – Preservice, Teacher Educators.

The current moment of practice-based teacher education (PBTE) has set out to develop a shared language in the field (Grossman & McDonald, 2008), disrupt the assumption that learning to teach does not require sustained learning (Ball & Cohen, 1999), establish teacher education as an "agent of professional countersocialization" (Ball & Cohen, 1999, p. 6), restructure teacher education's organizational dichotomy between theory and practice (Grossman, Hammerness, & McDonald, 2009), and to do so by learning in, from, and through practice (Ball & Cohen, 1999).

Since the initial call for a turn toward PBTE (Ball & Cohen, 1999), much of the scholarly attention has been on the development of and research on various pedagogies of practice (Grossman et al., 2009) – the vehicles by which practice-based actually becomes practice-based. Reflective of and responding to Grossman, Compton, and colleagues' (2009) finding of the lack of approximations of practice within teacher education, contemporary PBTE researchers and teacher educators intensely focus on the design, facilitation, and outcome of approximations.

Despite this focus, little attention has been paid to the nature of the simplification (Grossman & McDonald, 2008) within approximations of practice. Critiques of the current moment of PBTE have included these simplifications and their ties to a technocratic view of teaching and teacher preparation (Zeichner, 2012). While it may be necessary for these pedagogies to expose TCs to context less complex and authentic than a classroom, what have we lost in the process? Have we compromised preparing teacher candidates to respond to students in adaptive (Hatano & Inagaki, 1986) and culturally responsive ways (Gay, 2002)? Have we marginalized the "social, cultural, political, and situated dimensions of teachers' practices" (Philip et al., 2018, p. 9) and inadvertently worked against education's ultimate goals of equity and justice?

In order to turn the gaze of PBTE toward goals of equity and justice, PBTE must (re)evaluate the structures we have built, with pedagogies of practice being a main focus. Some PBTE scholars (Dutro & Cartun, 2016; Kavanagh, 2017; Kavanagh & Danielson, 2020) have begun to explore the ways in which core practices can become social justice oriented, possible theoretical considerations for such endeavors, and the design and facilitation of pedagogies of practice to serve social justice purposes. Despite these efforts, must is left to accomplish in developing critical practice-based teacher education pedagogies.

In order to further establish a framework of critical PBTE, this poster presentation will provide an initial proposal in the design of various pedagogies of practice through the representation, decomposition, and approximation of discretionary spaces (Ball, 2018). Through discretionary spaces, teacher educators and teacher candidates will investigate moments where teachers have the "discretion either to reproduce unjust and inequitable social patterns or to interrupt those patterns through their embodied activity in the classroom" (Kavanagh & Danielson, 2020, p. 71). These spaces are inherently tied to systems of oppression and inquiry such as race, class, gender, and language. Calling upon these spaces centers the design and investigation of pedagogies at the intersection of the relationship between practice and improvisation and the social, historical, and cultural complexities of teaching.

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DOCUMENTING ADAPTIVE EXPERTISE THROUGH THE EVOLVING USE OF AN ENACTMENT TOOL

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While orchestrating a whole-class discussion, a teacher draws upon a variety of "moves" (Boerst et al., 2011) to maintain the structure and goals of the discussion. In this paper, we focus on the orienting move of asking students to restate what a peer has said as a means to have an idea made public and mark it as a worthwhile part of the discussion (Chapin et al., 2013. We follow Ghousseini and colleagues (2015) by conceptualizing such moves as *enactment tools*, which "translate abstract conceptual tasks into more concrete steps and objectives" (p. 462), and considering the context, steps, and goals surrounding the move. To facilitate and document the development of adaptive expertise (Hatano & Inagaki, 1986), researchers have centered the use of enactment tools in coached rehearsals (e.g., Ghousseini et al., 2015), leaving more to know about how tool use transitions into more complex settings, such as student teaching. In this work, we address the following research questions: In what ways has a teacher candidate's (TC's) use of the restating tool evolved over time? In what ways has the purpose and goals associated with the restating tool changed?

We focus on one TC ("Diana") who regularly and explicitly "took up" this move in a secondary mathematics methods course and in her student teaching. She used the restating tool in coached rehearsals (e.g., Campbell et al., 2020), scripting tasks (e.g., Baldinger et al., 2018; Campbell et al., 2019), classroom videos from student teaching, and reflected on her use of the tool in interviews. To focus on Diana's evolving use of the restating tool, instances of the tool's use were identified in the data and paired with rationale and contextualization Diana provided for using the tool. Instances underwent open coding and analytic memo writing (Miles et al., 2014), which focused on: (1) the sequence of how the dialogue unfolded, (2) consistencies and changes in the restating tool, and (3) the purposes and goals associated with the tool's use.

Initial findings illustrate Diana's adaptive use of the enactment tool, as well as the purposes associated with its use. Over the course of a year and across contexts, Diana adapted the tool and her enacted sequence based on contextual, mathematical, and social purposes. These purposes included highlighting an important mathematical idea, orienting students to an idea, and positioning students productively in the classroom. Contextual and situational factors across contexts and time also contributed to nuances in the tool's use and purposes in enactment.

We found that TCs can develop adaptive expertise through opportunities to be responsive to students' social and mathematical needs. Such development can be documented by attending to changes in enactment tools—specifically how changes in the sequence and associated goals relate to contextual factors. These findings have implications for the contextualization and authenticity of the design of approximations of practice (Grossman et al., 2009) in teacher education and in research on TC development.

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EVALUATION OF A SOFTWARE SOLUTION FOR REFRESHING PRE-SERVICE TEACHERS' MATHEMATICS CONTENT knowledge

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Keywords: Mathematical Knowledge for Teaching, Teacher Education – Preservice, Teacher Knowledge

Mathematics has a privileged place in the school curricula. One of the primary concerns of teacher education institutions is to prepare future teachers to take a leadership role in school mathematics education. Wahyu, Mahmudi, and Murdanu (2018) argued that pre-service teachers should have sufficient knowledge of both mathematics and pedagogy to be successful in their teaching career. However, Bowie, Venkat, and Askew (2019) indicated a need for student teachers to revisit primary school mathematics that provides a deep understanding of key mathematical concepts in order to be better prepared for future teaching careers. This poster draws from our 2-year Consecutive B.Ed. program in which students are required to complete an online elevatemymath (EMM) before their mathematics methods course. The EMM was designed as self-paced modules where pre-service teachers first complete a pretest, followed by modules and the post-test. This software solution was implemented in response to elementary pre-service teachers' perceptions about their level of mathematical preparedness and challenge of split attention in attempting to (re)learn elementary mathematics content alongside learning the specific mathematical (pedagogical) knowledge and classroom practices for effective teaching.

We were interested in the perceived value of the online refresher course in pre-service elementary teachers' perceptions of their mathematical preparedness and competence for their methods course in the program. We asked: How do pre-service teachers perceive the value of a software solution for refreshing their mathematics content knowledge? In what ways the software solution was helpful in engaging pre-service teachers in the methods course? and, to what extent was the software solution beneficial for the pre-service teachers in preparing them to teach elementary mathematics? Data was collected using an online questionnaire focusing on the perceived value of the EMM refresher course, pre-service teachers' perception of preparedness for engagement in the methods course, and their perception of preparedness to teach mathematics in the first year. The questions were a mix of quantitative and qualitative open text responses.

Data were analyzed based on emergent themes related to perceived value and perception of preparedness and the percentages of responses. The numerical and free-response (qualitative) data (anonymous) from online surveys (n = 204) over a 2-year period shows a variation in pre-service teachers' perceived value of EMM refresher course and perception of preparedness to teach elementary mathematics. Findings indicate that there was a considerable increase in the depth of knowledge and degree of connectedness of elementary school mathematical concepts after the refresher course. More than 50% of the pre-service teachers felt that the refresher course contributed to positive growth and better prepared them with planning and assessing a mathematics class. Some of the pre-service teachers did express that having a refresher course helped them to some extent during their first teaching practice and the refresher course was a good value for money. For example, one pre-service teacher expressed that "the refresher course helped in revising some concepts, recognizing some areas that needs improvement. Knowing my strengths helped me in moving forward."

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PROFESSIONAL TEACHER NOTICING AS EMBODIED ACTIVITY

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There is growing evidence to suggest that construction of knowledge is an embodied activity (Alibali & Nathan, 2012; Barsalou, 1999; Gallese & Lakoff, 2005). We conjecture that embodied cognition is a useful theoretical lens for explaining teachers' professional noticing. Evidence from eye-tracking studies suggest that experienced secondary teachers process visual events in a recorded classroom more quickly, while preservice teachers (PSTs) scan the room more frequently and with fewer fixations on particular students or events (van den Bogert et al., 2014). More recently, Kosko et al. (2019) observed that differences in where PSTs attended when watching a 360 video of an elementary math lesson coincided with differences in written descriptions of their noticing. 360 video records in a spherical direction, and PSTs can move their head to determine where in the classroom they attend. Given such evidence, we sought to examine further examine the relationship between where PSTs attend in watching 360 video and what they describe in written noticings. To this end, the purpose of this study is to report preliminary evidence supporting a theory of professional noticing as embodied activity.

Participants in this study included four elementary preservice teachers (PSTs) at the beginning of their teacher education coursework (sophomores). Using Oculus Go headsets to record their viewing session, participants watched a 360 video of third-grade students informally explored the Commutative Property. After viewing the video, PSTs wrote what they noticed to be significant moments for the teaching or learning of mathematics. We used Systemic Functional Linguistics (SFL) to examine PSTs' written noticings. SFL examines how grammar functions to convey meaning (Eggins, 2004). In this analysis, we examined how transitive processes conveyed experiential meaning of referents in the grammar. We then examined how this experiential meaning was related to recordings of where PSTs attended in the video. One PST wrote "The teacher **was moving** all around the classroom and he [teacher] **was asking** students to work together…" The first bolded words signify material processes, which corresponded to the PST moving their head to attend to the teacher walking around the classroom at various points. The second bolded words signify a verbal process. Interestingly, the PST did not always visually track the teacher when he was asking questions. Contrasting this example, a second PST, who consistently used material processes but no verbal processes in their written noticings, always looked at the teacher when the teacher was talking. Analysis of these patterns is ongoing, but initial findings suggest PSTs' attending manifests in both auditory and visual means, which are represented in how meaning is conveyed in their written noticings.

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DISTINCTIONS IN PRESERVICE TEACHERS' ASSET-BASED NOTICINGS OF MIDDLE SCHOOL STUDENTS' MATHEMATICAL STRENGTHS

DISTINCIONES EN AVISOS BASADOS EN ACTIVOS DE PROFESORES DE PRESERVIO DE FORTALEZAS MATEMÁTICAS DE ESTUDIANTES DE ESCUELA INTERMEDIA

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As mathematics teacher educators (MTEs), we design methods courses to "provide candidates with tools and frameworks to support a more asset- and resource-based instructional approach focused on students' strengths in learning" (AMTE, 2019, p. 35). Through an asset-based orientation, MTEs can foster preservice teachers (PSTs) ability to view every student as a doer of mathematics, thereby recognizing that *all* students have mathematical strengths (Bannister et al., 2018; Featherstone et al, 2011; Jilk, 2016). PSTs who develop more robust orientations about what it means to do mathematics and by whom, are more likely to question and disrupt any socially-learned deficit orientations they may have about diverse learners (see Celedón-Pattichis et al., 2018). Countering and replacing these orientations among PSTs with cultural and mathematical *asset-based orientations* will require MTEs to better understand how PSTs understand and notice mathematical strengths.

Complex Instruction (CI, Cohen & Lotan, 1997) is an asset-based pedagogical framework, grounded in the recognition that each and every student brings varied and different mathematical strengths and statuses to the classroom. The framework recognizes that during group work, peer's assign competences to one another, impacting who contributes to the groups' thinking and who learns mathematics. Often, the mathematical strengths of a "low-status" student may be ignored or dismissed. CI defines techniques for teachers to disrupt these socially-influenced biases. To enact these techniques, however, PSTs must believe and be able to recognize mathematical strengths in every student.

Our work seeks to answer the following research question, *What distinctions in the quality of mathematical strengths do PSTs notice during a group-worthy task?* To do so, we draw upon the research on teacher noticing aligned with Sherin's (2001) notion of professional vision as the ability to notice and interpret significant features of classroom interactions. Four cohorts of PSTs enrolled in our different teacher preparation programs during their junior or senior methods course engaged in three key activities to learn to consider students' mathematical strengths: (1) read and respond to a CI paper; (2) name strengths in peers after completing a group-worthy task together; and (3) implement the same task with a group of 4–6 middle school students to identify mathematical strengths for every student. Data from PSTs' class artifacts, group recordings, reflection papers across the two sites were analyzed using both holistic and descriptive coding (Saldaña, 2016).

Results indicated that PSTs welcomed the invitation to learn about students' mathematical strengths and were able to identify them in most middle schoolers. Yet, PSTs' noticed qualitatively different types of mathematical and behavioral strengths. In this poster, we present the distinct types of strength-noticing patterns among the PSTs, and their movement towards asset-orientations. Results will be useful for MTEs and further analyses of PSTs' dispositions.

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MATHEMATICS IS EVERYWHERE: INTERSECTION OF PST PERCEPTIONS AND NON-MATHEMATICS-EDUCATION FACULTY PERCEPTIONS AND OBSERVABLE ACTIONS

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Mathematics traditionally has been taught as a discrete set of content areas related to numbers and operations, algebra, geometry, measurement, and data analysis and probability. And, while the National Council of Teachers of Mathematics [NCTM] (2000) has stressed the need for students to make connections between these mathematical content areas, to other disciplines, and to the real world, these connections have been weak. These weak connections are consequential. When students of mathematics do not make these connections, they are limited in their development of mathematical thinking. They may be able to attain procedural fluency, but they are unable to continuously develop conceptual understanding, necessary for the understanding of more abstract mathematical concepts. Without such connections, they do not develop the strategic competence and adaptive reasoning necessary to problem solve successfully. Too, these weak connections perpetuate long-standing cultural beliefs that mathematics is irrelevant to other content areas and/or to the real world.

Within teacher-preparation programs, instruction is frequently given in a discrete fashion in which pre-service teachers [PSTs] receive mathematics instruction from one professor in one class, social-studies instruction from one professor in another class, and so on. This lack of integration among disciplinary methods reinforces the lack of connection that is encouraged in the NCTM and Common Core [CC] process standards for children to become fully proficient in their mathematical thinking. While structurally this discrete division of content appears overt, my suspicion was that most of the non-mathematics-education professors were integrating mathematical practices in their instruction. This assumption guided the study and was confirmed in the triangulated analysis of a three-part, data-collection process of (1) 180 PSTs' alignment of the CC Standards of Mathematical Practices [MSPs] with instructional examples from their non-mathematics education courses (Spring '14 to Spring '18); (2) 19 non-mathematics-teacher educators' survey data on how they use the MSPs in their instruction; and (3) 11 full-class, video-taped observations of two social-studies, two science, two English-and-language-arts, one creative-arts, and one physical-education/health teacher educators who were surveyed (Spring 2018). While PSTs and their non-mathematics education professors initially held exclusive views of their definitions of mathematics and mathematical thinking, project results reveal that even in non-mathematics-focused courses, most of the MSPs are being reinforced.

Overall, the objective of this project is to provide evidence that mathematical thinking occurs everywhere, despite beliefs about its discrete nature. Educators, no matter their disciplinary expertise, can strengthen students' mathematical thinking in meaningful ways. Through a more united front in helping students develop their mathematical thinking, we can strengthen the connections students make between mathematical content areas, other disciplines, and the real world. Only in making these connections will students be able to attain all five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive dispositions (NRC, 2001).

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CREATIVE MATHEMATICAL REASONING AND CONTENT AS AN EVALUATIVE FRAMEWORK FOR PRESERVICE TEACHER EXPERIENCES

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Our research focused on developing a profound understanding of fundamental mathematics (PUFM; Ma, 1999) for Preservice Secondary Mathematics Teachers (PSTs). We considered content and reasoning ability. Our research questions were: (a) To what extent does working on a quadratic exploration task engage preservice secondary mathematics students in components of creative mathematical reasoning (CMR; Lithner, 2008)? (b) Which mathematics concepts do preservice secondary mathematics students draw upon while engaged in the task?

Theoretical Perspectives

We used Lithner's (2008) classification of CMR and imitative reasoning (IR) to describe student reasoning. CMR includes a *novel* reasoning sequence, makes use of *plausible* strategies, and has a *mathematical foundation*. Lithner described IR as the "opposite" (p. 256) of CMR.

Methods

In a mathematics content course for third- and fourth-year PSTs focused on the roles of technology in the teaching and learning of mathematics (Cullen, Hertel, & Nickels, 2020), we asked students to explore the effects on the path of the vertex as each parameter in the quadratic standard form, $y = ax^2 + bx + c$, was varied. We video recorded class sessions, coded for CMR and IR (Lithner, 2008), and identification of secondary mathematics curricular concepts.

Results

Throughout the exploration we identified students engaged in CMR with concepts from secondary mathematics. For example, Jared reasoned about the concept of slope and linearity while reasoning about the path traced by the vertex as b was varied. Jared's reasoning was *novel* because he asked himself why the path was linear. Jared's strategy—to purposefully adjust parameter sliders, one at a time—was *plausible* because it allowed him to draw conclusions about the effects of those parameters. Jared's conclusion that the slope depended on b was based on a *mathematical foundation* of what is meant by dependent. Thus, we concluded that Jared's reasoning was an example of CMR that involved consideration of secondary-level mathematical content (e.g., linearity, quadratics, rate of change, loci of points) at a profound level.

Discussion and Conclusions

As we reflect on our PSTs' engagement with the Exploring Quadratics task (Cullen, Hertel, & Nickels, 2020), we learned that the task kept PSTs engaged in CMR (Lithner, 2008) throughout the multi-day exploration. Likewise, the content areas which they drew upon were pertinent to their developing subject matter knowledge (Shulman, 1986) and, because concepts were debated in a way that focused on meaning, rather than from an algorithmic approach, the activity seemed to be supporting the development of PSTs' PUFM (Ma, 1999). As a result, we suggest that analyzing tasks for PST populations looking for CMR as well as in-depth engagement with mathematical content linked to future teaching assignments may serve as a framework for identifying appropriate tasks.

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TEXTBOOK USE OF CHILDREN'S THINKING TO SUPPORT PROSPECTIVE ELEMENTARY TEACHERS' GEOMETRIC UNDERSTANDING

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Keywords: Post-Secondary Education, Geometry and Geometrical and Spatial Thinking, Teacher Education - Preservice

Researchers have suggested that one way to motivate and support prospective elementary teachers' (PTs) mathematical understanding is through the use of authentic examples of children's mathematical thinking (e.g., *Circles of Caring*, Philipp, 2008). Philipp notes that some PTs may care more about children as whole beings than they care about mathematics. Therefore, integrating how the content directly relates to the teaching and learning of children can offer a way to leverage the care PTs have for children to motivate PTs to care about the mathematics.

Ball and her colleagues (2008) have identified the ability to analyze children's mathematical thinking as a valuable component of Specialized Content Knowledge (SCK), or knowledge unique to teachers of mathematics. However, Max and Amstutz (2019) found that activities related to the content domains of Geometry and Measurement & Data (Conference Board of Mathematical Sciences, 2012) provided fewer opportunities for PTs to develop their SCK.

Therefore, the goal of this study is to investigate the intersection of the Geometry and Measurement & Data content domains with examples of children's mathematical thinking in textbooks currently used in content courses for PTs. For this investigation we focused our analysis on the top three textbooks that US mathematics teacher educators recently reported using (Max & Newton, 2017): Beckmann (2018), Sowder et al. (2017), and Billstein et al. (2020). This poster will report findings and provide examples of children's thinking being utilized in the study of two-dimensional geometric concepts (e.g., shapes, polygons, angles) and measurement (e.g., length, angle size, area). Initial textbook analysis involved identifying instances relating content to the teaching and learning of children and noting the ways in which these instances were being used to support PTs' development of SCK. For example, some samples illustrated children's work in which they had applied a non-traditional method and asked PTs to analyze the validity of the child's thinking.

All three textbooks included practice exercises at the end of some sections that attached names to sample thinking, at times referencing the names as students or by grade level. However, Beckmann (2018) and Sowder et al. (2017) actively used examples of children's thinking throughout their lessons to support PTs' development of content knowledge, specifically SCK. Additional references to children were found in mentions of elementary concepts, content standards, and research conducted with children, prompting consideration of whether these types of connections to the teaching and learning of children might also serve as motivation for PTs.

Future analysis will continue to investigate the ways textbooks used in content courses for PTs reference children and their mathematical thinking as well as the potential impact of these instances on motivating PTs' development of SCK. By revealing and highlighting the integration of connections to the teaching and learning of children in content courses designed for PTs, we hope to support mathematics teacher educators in creating classroom cultures that can leverage the care PTs have for children to motivate PTs to deepen their mathematical understanding in ways that will support the learning of their future students.

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USING MULTIPLE STRATEGIES TASKS TO EXPLORE PRE-SERVICE TEACHERS' PERSISTENCE

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This study explored influential factors that affect pre-service teachers' (PSTs) persistence in mathematics learning during professional development (PD). As a part of a larger study, this study was guided by concepts of persistence and mindset - "the core beliefs students have about learning and the change that learners may or may not be conscious of." (Dweck, 2006; p.6) Persistence is an action when students continue to engage in a mathematical task despite facing challenges (Boaler & Staples, 2008). Studies on persistence have focused on students' traits (Cobb, Gresalfi, & Hodge, 2009; Grant & Sonnentag, 2010; Rayneri, Gerber, and Wiley, 2006), while factors that affect students' development of such traits have not been studied.

In mindset interventions, MS has been used as a problem-solving approach, but few have studied how MS fosters a growth mindset (Lynch & Star, 2014). Lynch and Star developed instructions to encourage MS in the student-teacher dialogue. Multiple Strategies (MS) intervention encourages students to use more than one method to solve a math task (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). This includes teachers' ability to assess which solutions that students came up with should be discussed to encourage students for further inquiry (Stein, Engle, Smith, & Hughes, 2008). However, if teachers are not careful, the instructional can become a "show-and-tell" (Ball, Lubienski, & Mewborn, 2001), and subsequently hinder students' persistence. This led to the research questions of the study: *How do PSTs choose to persist when problem-solving with challenging tasks? How MS-based PD impacts PSTs' persistence in challenging mathematics tasks?*

The study participants are a convenient sample of pre-service K-12 teachers who attended the PD, delivering the MS intervention, an opportunity offered by the College of Education at a large mid-western university. Surveys were used to measure PSTs' to measure mindset (Levy, Stroessner, & Dweck, 1998), and persistence (Duckworth, Peterson, Matthews & Kelly, 2007). Among the PD attendees, PSTs attended at least four sessions out of five total PD sessions. 12 PSTs met this requirement Further qualitative analysis was done among the 12 and six PSTs agreed to participate in follow-up PD interviews.

Analyses of PSTs' cases helped understand PSTs persistence views on the task can be affected by knowing the instructional practice of MS was available through working with peers in a collaborative learning environment. PSTs also appreciated the opportunity to work on challenging tasks with a reminder of the availability of MS during the PD. It was challenging to understand the relationship between PSTs' persistence levels and their views on success or failure, as all demonstrated a high persistence level. Further studies with PSTs with varied persistence will reveal this relationship.

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ADVANCING UNDERSTANDING OF EQUITY WITH CASE STUDY DILEMMAS: LESSONS FROM PRESERVICE TEACHERS

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Keywords: Teacher Education-Preservice, Instructional Activities and Practices

Recent reform in teacher education looks to understand and support equity initiatives in mathematics education (AMTE, 2017; NCTM, 2000, 2014). These efforts encourage preservice teachers (PTs) to engage in authentic classroom situations that explore equity-based practices by connecting theory and practice (Chao, Murray, & Gutiérrez, 2014; Ching, 2014). PTs can use case studies to reflect on their own teaching experiences and position themselves in scenarios in which they may not be familiar (Redman & Redman, 2007). This study reports on PTs in mathematics methods courses participating in a series of case study dilemmas designed by the researchers to elicit conversations of equity in mathematics education. We examined the following research question: How do PTs engage with case study dilemmas in mathematics methods courses to advance their understanding of equity in teaching mathematics?

We conducted a qualitative case study design that used multiple data sources and referenced Gutiérrez's (2007) equity framework that describes equity as a complex notion in terms of access, achievement, identity, and power. Participants included 43 PTs enrolled in three mathematics methods courses across two universities in the United States. Both courses were structured to introduce equity on the first day of class and address equity-based teaching practices throughout the semester with course readings, activities, and discussions. In planning for the methods courses, we created two case study dilemmas to facilitate equity discussions focused on identifying and challenging assumptions, biases, and stereotypes as well as exploring equity through a lens of fairness. The first dilemma prompted the PTs to examine equity in terms of access and identity by exploring a teacher's response to receiving a new student in her classroom that did not look like her or the rest of her students. The second dilemma, focused on access and achievement, analyzed a teacher's intent to have equitable expectations in the context of a zero-tolerance homework policy. Pre- and post-surveys were collected from the PTs to gain insight into their changed perspectives on equity. The surveys and transcripts of the recorded discussions were coded using in vivo and descriptive coding techniques (Saldaña, 2016).

Participation in the case study dilemmas encouraged the PTs to reflect on their understanding of equity and develop their pedagogical knowledge of equity-based teaching practices. The first dilemma prompted the PTs to examine a teacher's assumptions toward a student. The PTs identified how the assumptions invited marginality and reaffirmed mathematics identity. They also discussed how the assumptions impacted the student's opportunity to learn ambitious mathematics and strategized ways teachers can be proactive in learning students' needs and cultural backgrounds. In the second dilemma, PTs debated what it means to have equitable expectations and how teachers can leverage multiple mathematical competencies using a variety of resources and assessments. Overall, the PTs commented on how they benefited from using the case study dilemmas to make them more aware of their biases and better understand what equity looks like in the mathematics classroom. Recommendations will be shared for using similar instructional activities to equip PTs with the knowledge and skills needed to embed equity-based practices in their professional practice.

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RELATIONSHIP BETWEEN NOT KNOWING AND SUCCESSFUL PROBLEM SOLVING AMONG PRE-SERVICE SECONDARY MATHEMATICS TEACHERS

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Conceptual Perspective and Research Questions

Mason and Spence (1999) state that "awareness of knowing and of not knowing is crucial to successful mathematical thinking" (p. 147). Synthesizing few studies on the topic of not knowing, one may conclude that not knowing is a step to understanding, carrying an important value in learning because from it knowing can follow. In our previous study, we found that students have difficulty externalizing not knowing while solving reasoning tasks (Author, 2019). Taking it further, in this study we are examining the following *research questions*: does the complexity of task relate to students' externalization of not knowing? to what extent students' not knowing is associated with successful problem solving?

Methodology

This study employed quantitative methodology. Pre-service secondary mathematics teachers (N=116) enrolled in a math methods course were selected for the study. The problem solving protocol was used to collect student written work while solving connected algebraic reasoning tasks. The protocol consisted of two instructions: a) solve the given task, and b) describe what you are not-knowing while solving the task. The tasks were designed based on the same concept of weighted average in numerical (task 1), semi-abstract (task 2), and abstract (task 3) contexts. Students' demographics data was also collected including grade point averages in discipline-specific coursework (M-GPA) as well as in pedagogy-related coursework (P-GPA). Each task was graded using the following levels: 1) no solution provided, 2) incorrect solution, 3) partially correct solution, and 4) correct solution. Along with this, students' externalization of not knowing while solving each task was rated using the following levels: 1) ignorance, 2) deflection, 3) non-relational not knowing, and 4) relational not knowing. The data was analyzed using descriptive statistics.

Results

In response to the research question 1, we found that complexity of the task relates to the level of students' externalization of not knowing. More specifically, if the task is too easy (task 1), the correlation between correctness of task and externalization of not knowing is negative $r=-.01$ ($p>.05$). As complexity of the task rises from numerical to semi-abstract level (task 2), the correlation becomes practically significant ($r=.16, p<.10$). However, as the task becomes more complex (task 3) the correlation coefficient decreases in value and significance ($r=.03, p>.05$). Another observation revealed significant correlation between students' externalization of not knowing for tasks 2 and 3 ($r=.53, p<.01$). In response to the research question 2, the study findings showed significant correlation between students' overall successful problem solving on all three tasks and not knowing expressed while solving the tasks ($r=.37, p<.01$). Moreover, students' discipline specific M-GPA was significantly related to their overall problem solving performance ($r=.35, p<.01$) whereas the pedagogy-related P-GPA was significantly associated with students' externalization of not knowing ($r=.28, p<.01$).

The study results might serve as a stepping-stone to further research not knowing, as it may directly link to more effective and efficient student learning.

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FOSTERING STUDENT TEACHERS' SPATIAL REASONING: THE ROBOTICS MARS CHALLENGE

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Keywords: Technology, Instructional Activities and Practice, Teacher knowledge.

Spatial reasoning has been identified as a key element not only for learning mathematics, but also other fields related to science, engineering and technology (Gold, et al. 2018; Julià & Antolì, 20016, 2018). Research in this venue has identified some issues, including gender disparities in spatial reasoning abilities and their impact on the gender gap in STEM achievement (Lauer et al., 2019). These differences, however, can be reduced with targeted training (De Castell et al., 2019; Laurer et al., 2019). Such training can start at early years through robotics tasks (Francis et al., 2016; Francis et al., 2017). Teacher education program can be informed by such research results, increasing their focus on spatial reasoning and robotics.

Since 2014, the University of Calgary required all student teachers to complete the course STEM Education which has an emphasis on innovation and transdisciplinary (Preciado et al., 2016). The course involves robotics through exploration and design using Lego EV3 and WeDo kits. We conducted a preliminary study analyzing 20 student teachers' narratives, corresponding to a component of the course, with the purpose of identifying the elements of spatial reasoning involved in the task through the lens of the students.

The literature on spatial reasoning encompasses diverse perspectives including definitions of spatial reasoning (Davis et al., 2015; Ramful, et al. 2016; Zwartjes et al, 2019), spatial skills and spatial habits of mind (Kim & Bednarz, 2013), as well as the framework provided by Francis et al. (2017), developed from utterances of 19 experts in different fields on spatial thinking. We considered this variety of perspectives on spatial reasoning to conduct a deductive thematic analysis (Braun & Clarke, 2006) on the narratives from students' Mars Challenge robotic task (Francis et al., 2019) which requires students to work as a team to build and program a robot that moves different objects to designated areas.

Findings and discussion

From the students' narration on the Mar Challenge regarding the robotics tasks, seven significant aspects of spatial reasoning were identified: Visualization, 2D-3D reasoning, construction process, pattern recognitions, transformation (rotation), scaling, and the design process involved imagining. From this analysis, we can conclude that the task has potential to develop spatial reasoning skills for student teachers, with a potential impact on their future students. Such approach has also the potential to both reduce the gender disparities regarding spatial reasoning through the engagement in robotics tasks and address the need for more people to consider STEM career paths.

The narratives evidenced some impact of the course on student teachers' spatial reasoning in the Mars Robotics task. However, the sample size does not allow a generalization of the results to other students in the program. Therefore, there is a need for the development of a research design that addresses these limitations to explore the impact of the course on students' spatial abilities and describe their learning processes.

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TEACHER EDUCATION (IN-SERVICE) / PROFESSIONAL DEVELOPMENT

RESEARCH REPORTS

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METHODOLOGY FOR THE DESIGN OF DIDACTIC SEQUENCES FOR SECONDARY MATHEMATICS IN A TECHNOLOGICAL CONTEXT

METODOLOGÍA PARA EL DISEÑO DE SECUENCIAS DIDÁCTICAS PARA MATEMÁTICA DE SECUNDARIA EN UN CONTEXTO TECNOLÓGICO

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In this paper a didactic proposal is presented which consists of a methodology aimed at secondary level mathematics teachers, the aim of this proposal is to provide the teacher with a tool to design teaching sequences in order to take them to the classroom; These designs have the particularity of incorporating digital technology for the development of the sequence and its implementation. The methodology was developed based on the articulation of: the didactic structure of Díaz-Barriga, the teaching method ACODESA of Hitt and the curricular developments of Taba. The methodology has been tested with a group of 11 teachers in a course-workshop of 40 hours, the results obtained were three didactic sequences elaborated by three teams of teachers in a technological context (using Geogebra), where it is perceived that it is possible to design using this methodological proposal, however, teachers presented some difficulties during the process of articulation with technology.

Keywords: Teaching activities and practices, Secondary education, Technology, Teacher training

Introduction

The curriculum of basic education in Mexico points out the importance of teachers being involved in the design of activities or didactic sequences, as part of their practice; however, this task can be complicated for teachers, since their working conditions and prevailing teaching practices have limited their practice to the reproduction of textbook activities, away from the design and planning of teaching.

Careful analysis shows that curricula and study guides do not provide sufficient guidance in order for teachers to design teaching activities. This explains the low production of didactic sequences designed by teachers. Taking into account the absence of specific methodological recommendations for the design of didactic mathematical sequences, the following question arose: how do you structure and evaluate a methodological proposal for the design of didactic sequences of the subject of mathematics from a problem situation with the support of Geogebra software aimed at secondary school teachers? The following specific objectives were derived from this:

1. Establish a didactic structure for the design of sequences.
2. Characterize the type of problem situations, which will be the starting point for each sequence.
3. Determine the role that Geogebra will play in the design of didactic sequences.
4. Develop the methodology and experiment with it by designing didactic sequences.
5. Structure a course-workshop for mathematics teachers to evaluate the methodology.

Theoretical References

1. Elements to structure a didactic sequence

Within our work, the didactic structure of Díaz-Barriga (2013) is used, since it is an appropriate way to organize activities. Also, it rescues some interesting points that should be incorporated in the design. However, the aspects considered by Díaz-Barriga have been modified within this methodological proposal due to the fact that they were adapted to the needs of the mathematical discipline.

2. Teaching method ACODESA

ACODESA was used for the purpose of organizing classroom management and mathematical knowledge. This teaching method proposed by Hitt and Cortés (2009) is divided into 5 stages that take into consideration individual work, teamwork, classroom debate, self-reflection and institutionalization. This method proposes a specific way of carrying out the teaching process and the learning process, highlighting the role that the student should play within the classroom and that the teacher should play through the use of problem situations, where the pupil's task is specified in each of its stages, paying special attention to the representations that are generated during the development of mathematical activity.

3. Methodology to plan a learning unit

Taba (1962) proposes a set of four stages to elaborate what she calls a "teaching-learning unit", of which we have only taken and adapted the stage of generalization, which is considered a crucial stage in the mathematical discipline. This stage is not reflected in other proposals, but here it has been considered important in order to involve the student in activities that allow him to generalize the mathematical concepts discussed in a sequence.

Articulation of the theoretical references used

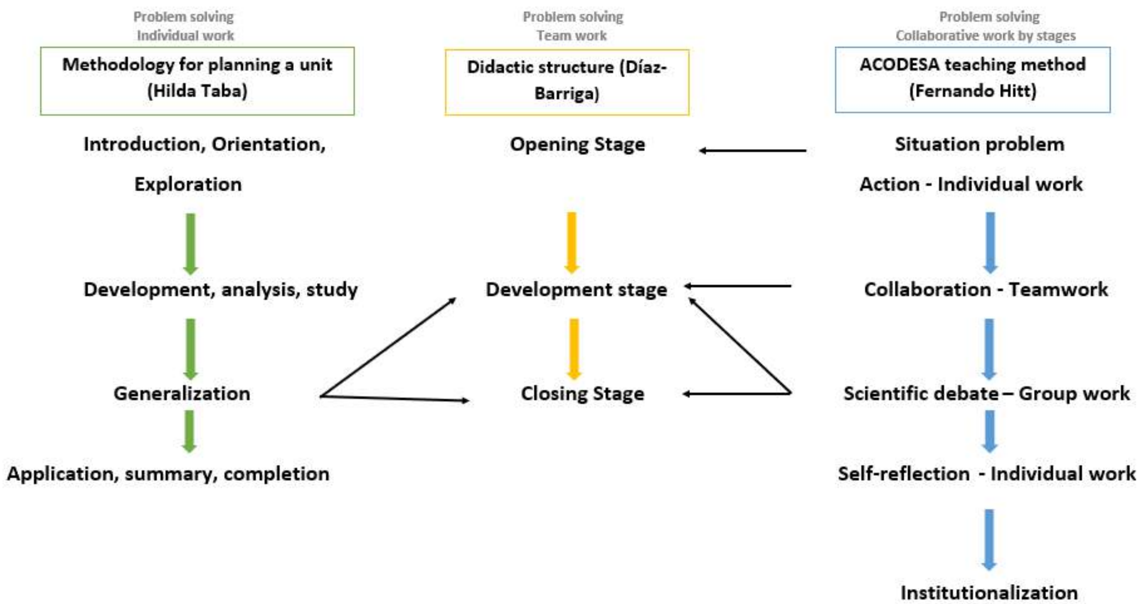


Figure 1: Diagram of the theoretical articulation. Source: own elaboration

Methodological actions

The methodology is qualitative and consists of the following actions.

Phases	Methodological actions
1.- Actions related to specific objective 1	Research of the contributions towards the design of didactic sequences. Characterization of the didactic structure in which the sequence will be broken down, based on the analysis of several didactic structures.

2.- Actions related to specific objective 2	Characterization and elaboration of the problem situations, based on the revision of books, articles and scientific journals related to mathematics.
3. Actions related to specific objective 3	Establishment of the use of Geogebra for the didactic sequence, starting from the review of contributions where technology is used as an educational resource.
4. Actions related to specific objective 4	Elaboration of the design methodology for didactic sequences, based on the articulation of the theoretical references used. Design of didactic sequences based on the methodology developed. Analysis of the didactic sequences designed, taking as reference the characteristics presented by the methodology and analyzing its correspondence with it.
5. Actions related to specific objective 5	Elaboration of the didactic-mathematical reflections, with the purpose of identifying the characteristics that describe the design methodology. Structuring of a course-workshop aimed at mathematics teachers to test the methodology and designs of didactic sequences elaborated. Evaluation of the design methodology, based on the analysis of the products produced in the workshop implemented.
6. Actions related to the general objective	Evaluation of the design methodology based on the general analysis of the workshop.

Table 1: Methodological actions. Source: own elaboration

Characteristics of the methodological proposal

Once the theoretical references were articulated, a methodological proposal was defined with the following characteristics: it is based on a problem situation that does not allude to the mathematics that will be used, contextualized to show the application, meaning and utility of mathematics.

The methodology specifies the objective of each of the elements of the didactic structure (opening, development and closing); the activities and questions in the opening stage are oriented towards the understanding of the problem situation. The development stage involves the mathematical procedures necessary for the resolution of the problem situation. Where it is considered relevant, a stage of generalization is started, with the purpose of extending the applications of mathematical concepts at the development stage. Finally, at the closing stage, the mathematical concepts that have emerged during the resolution of the situation and during the generalization are institutionalized and formalized.

The use of technology is incorporated into the methodology with the aim of linking various contents and putting into play the mathematical thought of the student. Geogebra software is considered to be a powerful tool to reach the proposed goal, since this technology offers the possibility to enrich the discussion on concepts, to diversify problems or exercises and to explore various solution strategies. It also allows the student to explore and visualize the meaning of the relationships between mathematical objects.

The methodology includes specific recommendations for incorporating the use of digital technology, specifically Geogebra, for each stage of the teaching structure. In the opening stage Geogebra is used to build simulations of the problem situation, allowing the exploration of this situation qualitatively. At the stage of development and generalization Geogebra is used to model the problem situation without the context in which it was posed and also to model the generalizations arising during the process of resolution of the situation.

In the closing stage the software is used as a tool to justify the mathematical results arising during the development of the sequence or to propose different solutions strategies to those employed.

The following is an outline of the characteristics described in this proposal:

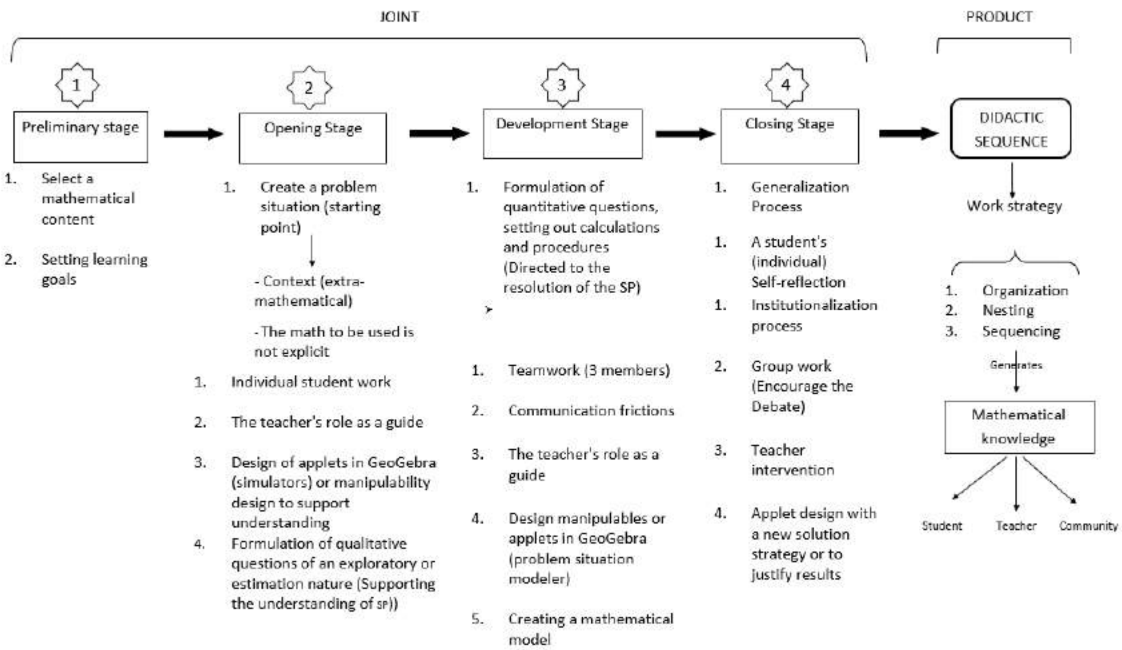


Figure 2: Outline of the methodological proposal developed. Source: own elaboration

Design of didactic sequences using the methodological proposal

Based on the elaborated methodological proposal, three designs of didactic sequences were made, with the intention of testing the methodology and applying them in a course-workshop for teachers. These designs were analyzed a priori to identify if the sequence was able to respond to the characteristics that were established in the methodological proposal elaborated. Once the design of the didactic sequences was completed, the program of a course-workshop for teachers was developed in which the designed sequences were analyzed as a starting point.

Implementation of a course-workshop

In order to involve teachers in the design of didactic sequences, a course-workshop was developed in which 11 mathematics teachers participated, with a duration of 40 hours. The purpose of the workshop was to analyze whether it was possible to design didactic sequences, which would present the characteristics that integrate this methodology. To carry out this workshop-course, three designs of didactic sequences were used, one of them is shown above this section and didactic-mathematical reflections were prepared with the intention that the teacher will identify and become familiar with the elements of the methodological proposal, through reflection on the work carried out in the resolution of the sequences. A bank of problem situations was prepared as well, with the aim of facilitating the task of the participants in the transformation of the situations for the purposes of the present study.

The following activities were carried out: (a) Addressing and analyzing didactic sequences previously designed with the proposed methodology; b) analyzing the proposed methodology based on the didactic sequences discussed and c) designing didactic sequences based on the proposed methodology.

Analysis

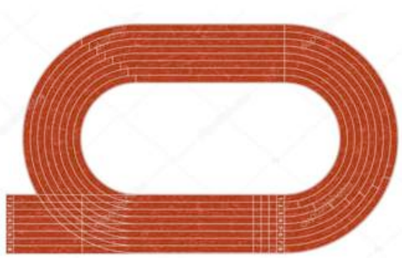
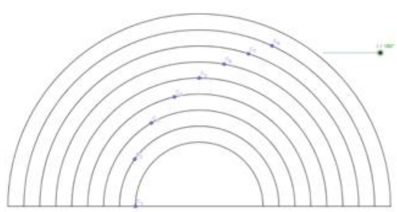
The analyses are described below. The tools to collect information were: paper productions of teachers, audio recordings of a work team, field diary (observations during sequence design), video recording of the presentations of the sequences designed and an evaluation format for participants.

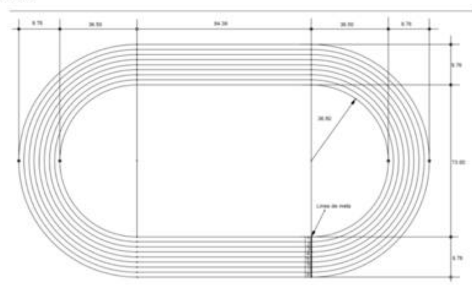
Analysis of didactic sequences designed by teachers

The focus was on the products developed by the teachers, to evaluate how they take into account the characteristics of the proposed methodology in the designs built, as well as identifying the purposes they pursue in each of the activities, this with the aim of analyzing the correspondence between the design perspective that the teacher has and the ideas proposed by the instructors in the course-workshop.

A general analysis was made of the way in which the elements of the product are contrasted with the methodology, the coherence between the activities developed with the purposes of the didactic structure, the way in which they suggest organizing work in the classroom and the role of the student and teacher, the use teachers assign to technology in the sequence designed and the level of mathematics used in the product developed by the teacher.

The following is an analysis of a didactic sequence designed by a team of teachers.

Didactic sequence	**Product analysis**
Opening Stage	
Didactic Sequence: The Track **Opening** Figure 1 shows a plane with the official measurements in meters that must have an athletic track for 400-meter flat races. The track has 8 numbered lanes from 1 to 8, we will call here runner 1 the athlete who runs down lane 1, runner 2 to which he runs down lane 2 and so on.  **Individual work** 1. Will all players run the same distance if everyone starts from the finish line and their finish line is also the finish line? Explain your answer. 2. Applet1 shows that it would happen at the first corner of the track if all players exit the finish line at the same time and assuming they all run at the same speed.  3. Write with your words what you observe. 4. Explain why runner 1 is the first to travel the curve if everyone advances at the same speed. **Teamwork** 5. Why do runners, at the start of a race, stand in a staggered row?	The teacher selected a problem situation from the problem bank, however, they adapt the problem situation according to the content they want to promote. An ideal situation would be that the teacher would first be aware of a content and from there conceive a problem situation (Hitt and Quiroz, 2009; Soto, Hitt and Quiroz, 2019). Within the problem situation, the text is devoted to providing information about the image included in such a situation. However, no context is defined that would lead to the formulation of a problem. Motivation is one of the most important points in teaching mathematics (Gravemeijer and Doorman, 1999). By not properly elaborating the problem situation, teachers do not analyze in greater depth what the present mathematics will be. They set out very general questions, rather than asking questions that need to be more in depth, leading to a specific mathematical theme. They requested technical and mathematical support from the instructors in order to carry out their construction, since they had the idea of how the construction should be, but they did not know how to make it in Geogebra. It was not clear to the teachers that the

	construction of applet requires mathematical relationships in the software to be able to capture the intended idea in the activity (Rabardel, 1995). The applet succeeds in playing the role of simulator of the problem situation. The teacher needs to have a greater interaction with the use of software to develop their potential in mathematical tasks. The teacher would have to go through a process of "instrumental genesis" as authors (Rabardel, 1995; Guin and Trouche, 1999) have pointed out.
Stage of development	
Development **Teamwork**  5. Describe the trajectory each runner performs. 6. How would you get the distance a player runs in corners? 7. Calculate the distance that player 1 travels when taking a full turn. 8. Expose to the group as they gained the distance traveled by Player 1. **Individual work** 9. As the width of each lane has an official measurement of 1.22 m, that will be the distance that will separate Corridors 1 and 2. Calculates the distance corridor 2 will travel, if part of the finish line and its finish line is also the finish line. 10. If the start line of Corridor 1 is the finish line, where should the start line of Corridor 2 be located for Corridors 1 and 2 to travel the same distance to reach the finish line? 11. Indicates where each of the other runners (runners 3, 4, 5, 6, 7 and 8) should be located at the start of the race so that everyone travels the same distance when reaching the finish line. 12. I put your findings before the group	The development stage rescues the purposes set out in the proposed methodology. The teachers managed to identify when to introduce the different working modalities with respect to the student's learning purpose. The types of questions show an attempt to involve other mathematical content such as proportionality, but teachers rule out introducing this content and focus on the perimeter. They include teamwork as support to carry out complicated tasks of the didactic sequence, group dialogue to communicate the different representations of the strategies of each team, as well as self-reflection to reinforce what was learned individually from practice (Hitt, Saboya and Cortés, 2017). It does not describe the role that the teacher will have when carrying out these different working modalities in the classroom (Hitt, Saboya and Cortés, 2017). The design does not present an applet within the development stage, because the teachers presented difficulties in making it within the software due to their lack of knowledge about its use and how to represent the image of the track (Rabardel, 1995; Guin and Trouche, 1999; Soto, Hitt and Quiroz 2019).

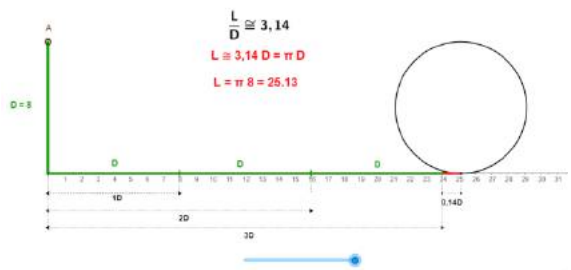
Closing Stage	
Closing **Individual work** 1. You will be given the drawing of a circumference and its midpoint on a white sheet. Follow the instructions to discover a procedure for calculating the length of a circumference from them. -Plots the diameter of the circumference. -Cut at least four pieces of thread whose length is the measurement of the diameter. -With those pieces of thread surround the circumference, How many pieces of thread did you need to surround the circumference? 2. From what is done expresses a procedure to find the length of the circumference. 3. Look carefully at Applet 2 and answer the following: is the procedure you developed equivalent to the one shown on the Applet? 	The use of manipulatives is promoted, which motivates the student to carry out the activity and allows for the activation of his imagination and his mathematical reasoning (according to the teaching method ACODESA). Generalization is promoted by establishing a formula for calculating the perimeter of the circle. The applet serves to show the results of the activity carried out with the manipulatives. The institutionalization process was a complicated task for teachers since the closure is limited to discovering geometric formulas rather than emphasizing the formalization of the mathematical concepts used (Hitt and Cortés, 2009). This gives us the indication that the teachers do not manage to abstract the mathematics that is broken down in the didactic sequence in order to formalize it.

Table 2: Analysis of the sequence prepared by the team 1. Source: own elaboration.

Analysis of the course-workshop taught

Once the course-workshop was completed, the teachers' perceptions of the methodological elements, the correspondence between the product and the proposed methodology and the didactic sequences elaborated by the teachers were analyzed. From this analysis, he obtained a general idea of the aspects in which it was necessary to devote more time and why, and what the advantages and disadvantages were of having taught the workshop course to teachers in such conditions.

Results and conclusions

The teachers were able to adapt to the proposed methodology based on the experience they gained in dealing with and analyzing the didactic sequences by contrasting them with the methodology; the didactic-mathematical reflections on previously designed sequences allowed them to identify the characteristics of the methodology used for the design.

The workshop course has shown that teachers can apply this methodology to design their own sequences when working in collaboration with other teachers. However, they faced difficulties in institutionalizing the mathematics involved in the sequence and in constructing the applets they proposed in Geogebra.

The problem situations proved to be a challenge for the teacher, it is recommended in a second edition of the course, to open a wider space dedicated to the formulation of problem situations by the participants of the course, on the mathematical topics of their greatest interest.

The methodology was structured by articulating theoretical elements. This was evaluated through the implementation of a course-workshop in which participants managed to design didactic sequences.

The theoretical elements taken from Díaz-Barriga and Taba served as the basis for the design of the didactic structure, however, it was necessary to adapt them according to the specifications that the discipline calls for.

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METODOLOGÍA PARA EL DISEÑO DE SECUENCIAS DIDÁCTICAS PARA MATEMÁTICA DE SECUNDARIA EN UN CONTEXTO TECNÓLOGICO

METHODOLOGY FOR THE DESIGN OF DIDACTIC SEQUENCES FOR SECONDARY MATHEMATICS IN A TECHNOLOGICAL CONTEXT

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En este trabajo se presenta una propuesta didáctica la cual consiste en una metodología dirigida a docentes de matemáticas de nivel secundaria, el objetivo de dicha propuesta es brindarle al docente una herramienta para elaborar diseños de secuencias didácticas con el fin de llevarlos al aula de clase; estos diseños tienen la particularidad de incorporar tecnología digital para la elaboración de la secuencia y para la implementación de esta. La metodología fue elaborada con base en la articulación de: la estructura didáctica de Díaz- Barriga, el método de enseñanza ACODESA de Hitt y los desarrollos curriculares de Taba. La metodología se ha puesto a prueba con un grupo de 11 docentes en un curso-taller de 40 horas, los resultados obtenidos fueron tres secuencias didácticas elaboradas por tres equipos de docentes en un contexto tecnológico (usando GeoGebra), donde se percibe que es posible realizar diseños aplicando esta propuesta metodológica, sin embargo, los docentes presentaron algunas dificultades durante el proceso de articulación con la tecnología.

Palabras clave: Actividades y prácticas de enseñanza, Educación Secundaria, Tecnología, Capacitación docente

Introducción

El currículo de educación básica en México señala la importancia de que el docente se involucre en el diseño de actividades o secuencias didácticas, como parte de su práctica, sin embargo, esta tarea puede resultar complicada para el docente, puesto que sus condiciones laborales y las practicas docentes predominantes han limitado su práctica a la reproducción de actividades de los libros de texto, alejándolos del diseño y la planeación de la enseñanza.

Un análisis cuidadoso nos muestra que los planes y programas de estudio no proporcionan las orientaciones suficientes para que el docente pueda diseñar actividades de enseñanza. Se explica así la escasa producción de secuencias didácticas diseñadas por los docentes. Tomando en cuenta la ausencia de recomendaciones metodológicas específicas para el diseño de secuencias didácticas matemáticas nuestro objetivo es estructurar y valorar una propuesta metodológica que permita a los docentes de matemáticas diseñar secuencias didácticas con apoyo del software GeoGebra, teniendo como objetivos específicos, los siguientes:

1. Establecer una estructura didáctica para el diseño de las secuencias.
2. Caracterizar el tipo de situaciones problema que serán el punto de partida para cada secuencia.
3. Determinar el papel que jugará GeoGebra dentro del diseño de secuencias didácticas.
4. Elaborar la metodología y experimentarla diseñando secuencias didácticas.
5. Estructurar un curso-taller dirigido a docentes de matemáticas para valorar la metodología.

Referentes teóricos

1. Elementos para estructurar una secuencia didáctica

Dentro de este trabajo se utiliza la estructura didáctica de (Díaz-Barriga, 2013), dado que es una forma apropiada para organizar las actividades y además rescata algunos puntos interesantes por incorporar en el diseño. Sin embargo, los aspectos que considera Díaz-Barriga se han modificado dentro de esta propuesta metodológica por el hecho de que se adaptaron a las necesidades de la disciplina matemática.

2. Método de enseñanza ACODESA

ACODESA se utilizó con el propósito de organizar la gestión dentro del aula y el conocimiento matemático. Este método de enseñanza propuesto por (Hitt y Cortés, 2009) se divide en 5 etapas que toman en consideración el trabajo individual, trabajo en equipo, debate en el aula, la auto-reflexión y la institucionalización. Este método propone una manera específica de cómo llevar a cabo el proceso de enseñanza y el de aprendizaje, resaltando el papel que el alumno debe jugar dentro del aula y el que debe realizar el docente mediante el uso de situaciones problema, donde la tarea del alumno es especifica en cada una de sus etapas, poniendo especial atención a las representaciones que se generan durante el desarrollo de la actividad matemática.

3. Metodología para planificar una unidad de aprendizaje

Taba (1962) propone un conjunto de cuatro etapas para elaborar lo que ella llama una "unidad de enseñanza-aprendizaje", solamente hemos tomado y adaptado, la etapa de *generalización*, la cual se considera una etapa crucial en la disciplina matemática. Esta etapa no se ve reflejada en otras propuestas, pero aquí se ha considerado importante para involucrar al alumno en actividades que le permitan generalizar los conceptos matemáticos discutidos en una secuencia.

Articulación de los referentes teóricos utilizados

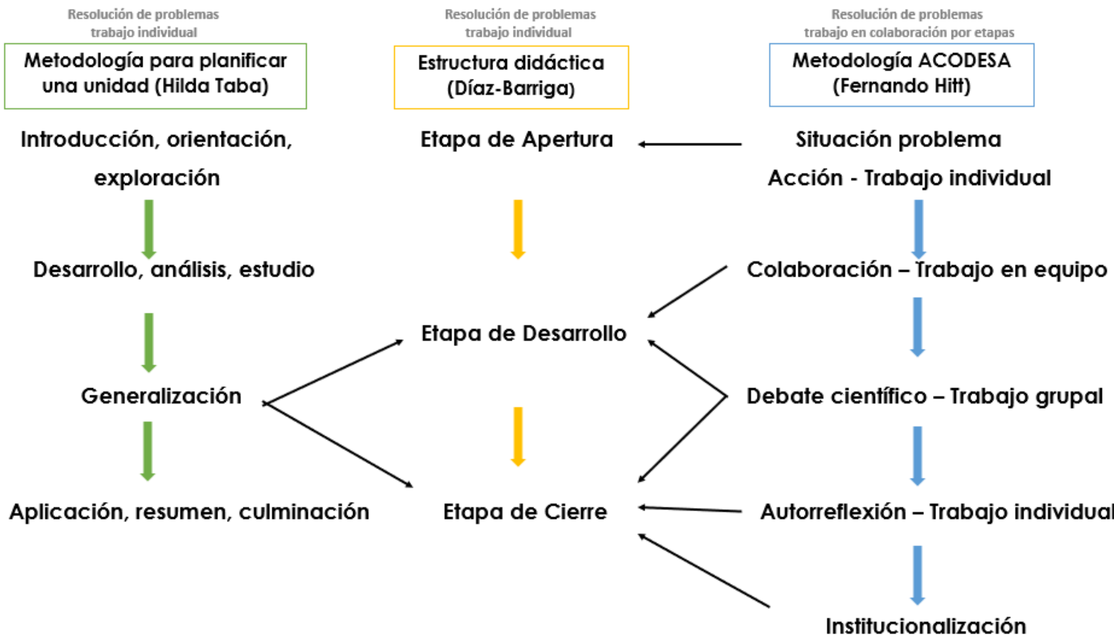


Figura 1: Esquema de la articulación teórica. Fuente: elaboración propia

Acciones metodológicas

La metodología es de carácter cualitativo y está constituida por las siguientes acciones.

Fases	Acciones metodológicas
1.- Acciones relacionadas con el objetivo específico 1	Investigación de las aportaciones hacia el diseño de secuencias didácticas. Caracterización de la estructura didáctica en la que se desglosará la secuencia, a partir del análisis de varias estructuras didácticas.
2.- Acciones relacionadas con el objetivo específico 2	Caracterización y elaboración de las situaciones problema, a partir de la revisión de libros, artículos y revistas científicas relacionadas con la matemática.
3. Acciones relacionadas con el objetivo específico 3	Establecimiento del uso de GeoGebra para la secuencia didáctica, a partir de la revisión de aportaciones donde se utilice como recurso educativo la tecnología.
4. Acciones relacionadas con el objetivo específico 4	Elaboración de la metodología de diseño para secuencias didácticas, tomando como base la articulación de los referentes teóricos utilizados. Diseño de secuencias didácticas basándose en la metodología elaborada. Análisis de las secuencias didácticas diseñadas, tomando como referencia las características que presenta la metodología y analizando su correspondencia con esta.
5. Acciones relacionadas con el objetivo específico 5	Elaboración de las reflexiones didáctico – matemáticas, con el propósito de que el docente identifique las características que describe la metodología de diseño.

	Estructuración de un curso-taller dirigido a profesores de matemáticas para probar la metodología y diseños de secuencias didácticas elaborados. Valoración de la metodología de diseño, a partir del análisis de los productos elaborados en el curso-taller implementado.
6. Acciones relacionadas con el objetivo general	Valoración de la metodología de diseño a partir del análisis general del curso-taller.

Tabla 1: Acciones metodológicas. Fuente: elaboración propia

Características de la propuesta metodológica elaborada

Una vez realizada la articulación de los referentes teóricos, se ha dado forma a una propuesta metodológica que presenta las siguientes características: se parte de una situación problema que no hace alusión a la matemática que será utilizada, contextualizada en la medida de lo posible para mostrar la aplicación, significado y utilidad de la matemática.

La metodología especifica el objetivo de cada uno de los elementos de la estructura didáctica (apertura, desarrollo y cierre); las actividades y preguntas en la apertura están orientadas hacia la comprensión de la situación problema, en el desarrollo se ponen en juego los procedimientos matemáticos necesarios para la resolución de la situación problema; cuando se considera pertinente se abre una etapa de generalización cuyo propósito es la ampliación de las aplicaciones de los conceptos matemáticos en la etapa de desarrollo y por último en el cierre se institucionalizan y formalizan los conceptos matemáticos que han emergido durante la resolución de la situación y durante la generalización.

Se incorpora el uso de tecnología dentro de la metodología con el objetivo de vincular varios contenidos y poner en juego el pensamiento matemático del alumno. Se considera que el software GeoGebra es una herramienta potente para llegar al objetivo propuesto, ya que esta tecnología brinda la posibilidad de enriquecer la discusión sobre los conceptos, de diversificar los problemas o ejercicios y de explorar varias estrategias de solución. Permite además que el alumno explore y visualice el significado de las relaciones entre los objetos matemáticos.

La metodología incluye recomendaciones específicas para incorporar el uso de tecnología digital, específicamente GeoGebra, para cada una de las etapas de la estructura didáctica. En la etapa de apertura GeoGebra se utiliza para construir simulaciones de la situación problema, que permitan explorar esta situación de manera cualitativa, en la etapa de desarrollo y generalización GeoGebra se usa para modelar la situación problema desprovista del contexto en el que se planteó y para modelar también las generalizaciones surgidas durante el proceso de resolución de la situación. En la etapa de cierre el software se usa como herramienta para justificar los resultados matemáticos surgidos durante el desarrollo de la secuencia o bien para plantear estrategias de solución diferentes a las empleadas.

Diseño de secuencias didácticas aplicando la propuesta metodológica

Con base en la propuesta metodológica elaborada se realizaron tres diseños de secuencias didácticas, con la intención de poner a prueba la metodología y aplicarlas en un curso-taller para docentes. Estos diseños fueron analizados a priori para identificar si la secuencia lograba responder a las características que se establecieron en la propuesta metodológica elaborada. Una vez concluido el diseño de las secuencias didácticas se elaboró el programa de un curso-taller para docentes en el cual las secuencias diseñadas se analizaron como punto de partida.

Implementación de un curso-taller

Con el propósito de involucrar a los docentes en el diseño de secuencias didácticas, se elaboró un curso-taller en el cual participaron 11 docentes de matemáticas, con una duración de 40 horas. La finalidad del curso-taller fue analizar si era posible elaborar diseños de secuencias didácticas, los cuales presentaran las características que integran esta metodología. Para llevar a cabo este curso-taller, se utilizaron tres diseños de secuencias didácticas, una de ellas se muestra arriba de este apartado y se elaboraron reflexiones didáctico – matemáticas con la intención de que el docente identificará y se familiarizará con los elementos de la propuesta metodológica, a través de la reflexión del trabajo realizado en la resolución de las secuencias.; así como también se elaboró un banco de situaciones problema, con el propósito de facilitar la tarea a los participantes en la transformación de las situaciones para los fines del presente estudio.

Las actividades realizadas fueron las siguientes: a) Abordar y analizar secuencias didácticas previamente diseñadas con la metodología propuesta, b) analizar la metodología propuesta tomando como referencia las secuencias didácticas discutidas y c) diseñar secuencias didácticas a partir de la metodología propuesta.

Análisis

A continuación, se describen los análisis que se realizaron. Las herramientas para recolectar la información fueron: producciones en papel de los docentes, grabaciones de audio a un equipo de trabajo, diario de campo (observaciones durante el diseño de secuencias), grabación de video a las exposiciones de las secuencias diseñadas y un formato de evaluación para los participantes.

Análisis de las secuencias didácticas diseñadas por los docentes

Se tomaron como foco de atención los productos elaborados por los docentes, para evaluar la forma en que toman en cuenta las características de la metodología propuesta en los diseños construidos, así como también se identificaron los propósitos que persiguen en cada una de las actividades, esto con la finalidad de analizar la correspondencia entre la perspectiva de diseño que tiene el docente y las ideas propuestas por los instructores en el curso-taller.

Se analizó de manera general la forma en que se contrastan los elementos del producto con la metodología, la coherencia entre las actividades elaboradas con los propósitos de la estructura didáctica, la forma en que sugieren organizar el trabajo en el aula y el papel que presenta el alumno y docente, el uso que le asignan los docentes a la tecnología en la secuencia diseñada y el nivel de matemática que se utiliza en el producto elaborado por el docente.

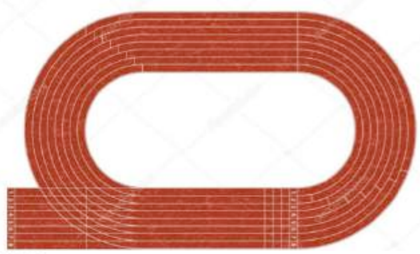
A continuación, se muestra el análisis de una secuencia didáctica diseñada por un equipo de docentes.

<table>
<tr><th>Secuencia didáctica</th><th>Análisis del producto</th></tr>
<tr><th colspan="2">Etapa de Apertura</th></tr>
<tr><td>

Secuencia Didáctica: La pista

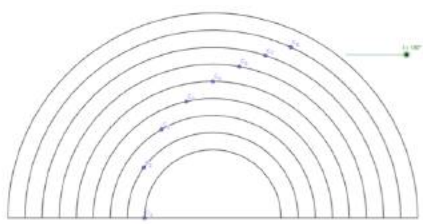
Apertura

En la figura 1 se muestra un plano con las medidas oficiales en metros que debe tener una pista de atletismo para carreras de 400 metros planos. La pista tiene 8 carriles numerados del 1 al 8, llamaremos aquí corredor 1 al atleta que corre por el carril 1, corredor 2 al que corre por el carril 2 y así sucesivamente.



Trabajo individual

1. ¿Correrán todos los jugadores la misma distancia si todos parten de la línea de meta y su línea de llegada es también la línea de meta? Explica tu respuesta.
2. El applet1 muestra que sucedería en la primera curva de la pista si todos los jugadores salen de la línea de meta al mismo tiempo y suponiendo que todos corren a la misma velocidad.



3. Escribe con tus palabras lo que observas.
4. Explica por qué el corredor 1 es el primero en recorrer la curva si todos avanzan a la misma velocidad.

Trabajo en equipos

5. ¿Por qué los corredores, al inicio de una carrera, se ubican en una fila escalonada?

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El docente seleccionó una situación problema del banco de situaciones problema, sin embargo, adaptan la situación problema acorde al contenido que desean promover. Una situación ideal sería que primero el docente fuera consciente de un contenido y a partir de ahí concebir una situación problema (Hitt y Quiroz, 2009; Soto, Hitt y Quiroz, 2019). Dentro de la situación problema, el texto se dedica a dar información acerca de la imagen incluida en tal situación. Sin embargo, no se define un contexto que dé pie a la formulación de un problema. La motivación es uno de los puntos más importantes en la enseñanza de las matemáticas (Gravemeijer y Doorman, 1999). Al no elaborar apropiadamente la situación problema, los docentes no analizan con mayor profundidad cuál será la matemática presente. Establecen preguntas muy generales, en lugar de plantear cuestionamientos que necesiten de mayor profundidad, que vayan conduciendo hacia una temática matemática específica. Solicitaron apoyo técnico y matemático de los instructores para realizar la construcción, dado que ellos tenían la idea de cómo podría ser la construcción, pero desconocían como elaborarla en GeoGebra.

Los docentes no tenían claro que la construcción de un applet requiere establecer relaciones matemáticas en el software para poder plasmar la idea pretendida en la actividad (Rabardel, 1995). El applet logra cumplir con el papel de simulador de la situación problema.

El docente necesita tener una mayor interacción con el uso de softwares para desarrollar su potencial en las tareas matemáticas. Precisamente el docente tendría que pasar por un proceso de "*génesis instrumental"* como lo han señalado autores como (Rabardel, 1995; Guin y Trouche, 1999).

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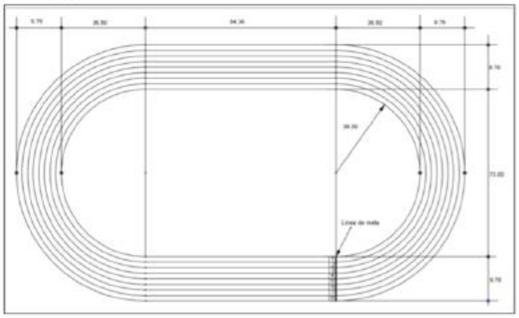
<tr><th colspan="2">Etapa de Desarrollo</th></tr>
<tr><td>
Desarrollo
Trabajo en equipos

5. Describan la trayectoria que realiza cada corredor.
6. ¿Cómo obtendrían la distancia que corre un jugador en las curvas?
7. Calculen la distancia que recorre el jugador 1 al dar una vuelta completa.
8. Expongan ante el grupo como obtuvieron la distancia recorrida por el jugador 1.
Trabajo individual
9. Como el ancho de cada carril tiene una medida oficial de 1.22 m, ésa será la distancia que separará a los Corredores 1 y 2. Calcula la distancia que recorrerá el Corredor 2, si parte de la línea de meta y su línea de llegada es también la línea de meta.
10. Si la línea de salida del Corredor 1 es la línea de meta, ¿dónde deberá ubicarse la línea de salida del Corredor 2 para que los Corredores 1 y 2 recorran la misma distancia para llegar a la meta?
11. Indica dónde deberán ubicarse, al inicio de la carrera, cada uno de los otros corredores (corredores 3, 4, 5, 6, 7 y 8) para que todos recorran la misma distancia al llegar a la meta.
12. Expón ante el grupo tus conclusiones
</td><td>
La etapa de desarrollo rescata los propósitos que se establecen en la metodología propuesta. Los docentes lograron identificar en qué momento introducir las diferentes modalidades de trabajo con respecto al propósito de aprendizaje del alumno.
Los tipos de preguntas muestran un intento de involucrar otros contenidos matemáticos como la proporcionalidad, pero los docentes descartan introducir este contenido y se enfocan en el perímetro.
Incluyen el trabajo en equipo como apoyo para llevar a cabo tareas complicadas de la secuencia didáctica, y al mismo tiempo el diálogo grupal para comunicar las diferentes representaciones de las estrategias de cada equipo, así como la autorreflexión para reforzar lo aprendido de manera individual a partir de la práctica (Hitt, Saboya y Cortés, 2017).
No se describe el papel que el docente tendrá al momento de llevar a cabo en el aula estas diferentes modalidades de trabajo (Hitt, Saboya y Cortés, 2017).
El diseño no presenta un applet dentro de la etapa de desarrollo, dado que los docentes presentaron dificultades para poder realizarlo dentro del software por falta de conocimiento acerca de su uso y de cómo representar la imagen de la pista (Rabardel, 1995; Guin y Trouche, 1999; Soto, Hitt y Quiroz 2019).
</td></tr>
<tr><th colspan="2">Etapa de Cierre</th></tr>
<tr><td>
Cierre
Trabajo individual
1. Se te entregará el dibujo de una circunferencia y su punto medio en una hoja blanca.
Sigue las instrucciones para que a partir de ellas descubras un procedimiento para calcular la longitud de una circunferencia.
-Traza el diámetro de la circunferencia.
-Corta al menos cuatro trozos de hilo cuya longitud sea la medida del diámetro.
-Con esos trozos de hilo rodea la circunferencia,
¿Cuántos trozos de hilo necesitaste para rodear completamente la circunferencia?
2. A partir de lo realizado expresa un procedimiento para encontrar la longitud de la circunferencia.
3. Observa cuidadosamente el Applet 2 y responde lo siguiente: ¿el procedimiento que desarrollaste es equivalente al mostrado en el Applet?
</td><td>
Se promueve el uso de manipulatives, lo cual motiva al alumno a realizar la actividad y permite activar su imaginación y su razonamiento matemático (de acuerdo con el método de enseñanza ACODESA).
Se promueve la generalización a partir del establecimiento de una fórmula que permita calcular el perímetro del círculo. El applet sirve para mostrar los resultados de la actividad realizada con los manipulatives.
El proceso de institucionalización fue una tarea complicada para los docentes dado que el cierre se limita a descubrir fórmulas geométricas en lugar de enfatizar la formalización de los conceptos matemáticos utilizados (Hitt y Cortés, 2009). Esto nos indica que los docentes no logran abstraer la
</td></tr>
</table>

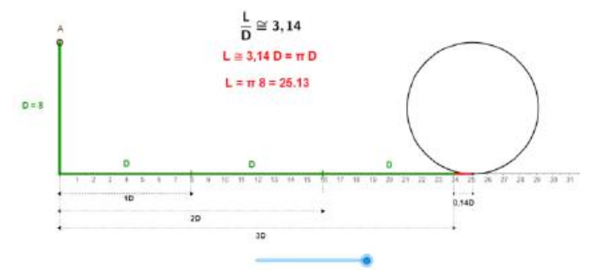
	matemática que se desglosa en la secuencia didáctica para formalizarla.

Tabla 2: análisis de la secuencia elaborada por el equipo 1. Fuente: elaboración propia.

Análisis del curso-taller impartido

Una vez culminado el curso-taller, se analizaron las percepciones de los profesores sobre los elementos metodológicos, la correspondencia entre el producto y la metodología propuesta y las secuencias didácticas elaboradas por los profesores. A partir de este análisis obtuvo una idea general sobre cuáles fueron los aspectos en los cuales se requirió dedicarle más tiempo y por qué, y cuáles fueron las ventajas y desventajas de haber impartido en tales condiciones el curso-taller a los docentes.

Resultados y conclusiones

Los docentes lograron adaptarse a la metodología propuesta a partir de la experiencia que obtuvieron al abordar y analizar las secuencias didácticas contrastándolas con la metodología; las reflexiones didáctico-matemáticas sobre las secuencias previamente diseñadas les permitieron identificar las características de la metodología empleada para el diseño.

El curso-taller ha puesto en evidencia que los docentes pueden aplicar esta metodología para diseñar sus propias secuencias cuando trabajan en colaboración con otros docentes, aunque han enfrentado dificultades principalmente a la hora de institucionalizar la matemática involucrada en la secuencia y al construir en GeoGebra los applets propuestos por ellos mismos.

Las situaciones problema resultaron ser un reto para el docente, se recomienda en una segunda edición del curso, se abra un espacio más amplio dedicado a la formulación de situaciones problema por parte de los asistentes al curso, sobre los temas matemáticos de su mayor interés.

La metodología se logró estructurar articulando elementos teóricos, esta fue valorada a través de la implementación de un curso – taller en el que los participantes lograron diseñar secuencias didácticas.

Los elementos teóricos tomados de Díaz-Barriga y Taba sirvieron como base para el diseño de la estructura didáctica, sin embargo, fue necesario adaptarlos de acuerdo con las especificaciones que la disciplina necesita.

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COACHES' AND TEACHERS' NOTICING THROUGH ANNOTATIONS: EXPLORING ANALYTIC STANCE ACROSS COACHING CYCLES

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We share results of a study on the analytic stances of coaches' and teachers' as they annotated key moments from classroom video of the teacher's lessons. In the analysis, emphasis was on the analytic stances of the coaches and how their annotations related to trends in teachers' annotations. Findings indicate differences in how coaches and teachers noticed across the coaching cycles, suggesting the annotations were influenced by the interactions between the coaches and teachers and the teachers' perceptions of coaching process. As a result of our analysis, we characterized one coach as having a high ratio of questions to suggestions, another as having annotations coded as interpretation, another as having more evaluations and suggestions, and the fourth as having asked more questions. Some teachers mirrored the analytic stance of their coach over time and other teachers shifted their analytic stance in ways that suggest they were responsive to their coach's analytic stance.

Keywords: Middle School Education, Teacher Education, Technology, Annotation, Video

Noticing is a crucial practice for teachers attempting to engage in responsive or ambitious teaching. Noticing is defined as attending to students' thinking, interpreting that thinking, making decisions in response to what they have noticed, and making connections to broader principles of teaching and learning (Jacobs, Lamb, & Philipp, 2010; van Es & Sherin, 2008). Researchers have noticed differences in how novice and expert teachers notice students' mathematical thinking; experts are more likely to notice relevant aspects of student thinking, while novices initially focus on superficial aspects of instruction (van Es & Sherin, 2008; Walkoe, 2015). However, novices improve their noticing as a result of interventions or training (e.g. Huang & Li, 2012; van Es & Sherin, 2008).

Researchers have documented the impact of a number of interventions on teachers' ability to notice student thinking, but coaching as an intervention to influence teacher noticing has not been adequately explored. Coaching is an emerging form of professional development; researchers highlight the impact of coaching on teacher learning and students' mathematical understanding (Gibbons & Cobb, 2017; Kraft, Blazar, & Hogan, 2018). Multiple models for coaching have been used in mathematics education, including instructional coaching (Knight, 2006), student-centered coaching (Sweeney, 2010) and content-focused coaching (West & Staub, 2003). Content-focused coaching emphasizes mathematics content learning goals as an outcome of coaching cycles. In a content-focused coaching model, teachers and coaches meet to plan lesson that is then collaboratively taught. Following the lesson, the coach and teacher meet to debrief the lesson, with attention to the mathematical learning goals of the lesson. Recently, we adapted the content-focused coaching model for an online context (see Author, 2019). We included an asynchronous annotation component in which the coach and teacher recorded written reflections tied to specific moments in the video of the teacher implementing the lesson. Engaging educators in annotating is not new in mathematics education, but annotating video through a coaching cycle is new.

Non-coaching professional development contexts show how interventions influence annotations. Stockero, Rupnow, and Pascoe (2017) engaged secondary prospective teachers in field experience

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interventions focused on noticing and analyzing video to mark moments that were mathematically important. Similarly, as part of a video club, Walkoe (2015) had teachers tag video where they noticed interesting student algebraic thinking. Findings showed evidence that the annotation process, as part of a larger professional development project, impacted teachers' noticing of students' algebraic thinking. Based on these findings, and our prior experience with annotations (Author, 2019), we engaged coaches and teachers in the process of annotating as part of online content-focused coaching cycles. We focused primarily on the analytic stances of the coaches' annotations, with a secondary focus on the teachers' noticing to illustrate the relationship between the frequency of particular analytic stances for coach-teacher pairs. Van Es and Sherin (2010) define analytic stance as an aspect of noticing to describe *how* noticing occurs. The analytic stance is the way one approaches and analyzes practice through noticing and the process through which they communicate noticing. We answered the following research questions: 1) What analytic stances do coaches assume as they annotate? 2) How do teachers' analytic stances relate to coaches' analytic stances across coaching cycles?

Method

Using a cohort model, we engaged nine coaches and twenty-eight teachers teachers in an intensive two-year professional development model that focused on supporting teachers to engage in ambitious, responsive instruction. For this study, we focus on four coaches and five teachers in the first cohort because they completed four coaching cycles. The four coaches all had experience with a variation of face-to-face content-focused coaching articulated by West and Staub (2003). We explored the analytic stances of the four coaches; we looked for variation in the coaches' stances and the ways that variation had an impact on the nature and evolution of how the teachers reflected on their lessons. We also explored the analytic stances of the teachers in the annotation process to better understand the associative relationships between the annotations of the coaches and teachers.

Participants

Each coach (Alvarez, Lowrey, Bishop, Riess) was highly knowledgeable about mathematics education and mathematics teacher education, with extensive experiences leading professional development opportunities for mathematics teachers. Three of the coaches had more than a decade of mathematics coaching experience. Each coach was partnered with a middle grades mathematics teacher for up to four coaching cycles. Alvarez, Lowrey, and Riess all coached one teacher and Bishop coached two teachers.

Data Collection

Teachers in the project took part in a professional learning model that included three components: a course based on the *5 Practices for Orchestrating Mathematics Discussion*, (Smith & Stein, 2011); demonstration lessons that we termed *teaching labs* (similar to studio model or lesson study, e.g. Fernandez & Yoshida, 2004; Higgins, 2013; TDG, 2010); and online content-focused coaching cycles. During the annotation process, the coach and teacher were each asked to annotate the lesson video of the teacher's own implementation of the collaboratity planed lesson, with the teacher always annotating before the coach. The following prompt was provided to teachers:

> Add your comments, questions, and thoughts to the video segment in Swivl at any points in the video that might be interesting to discuss further. For example, were there any moments that surprised you? (i.e., misconceptions that emerged, strategies that you did not anticipate, struggles/challenges, or any "Ah-ha" moments) Were there particular instances that showed evidence of student thinking? Is there something that you see as you watch the lesson that relates to the goal you set for this coaching cycle?

The coaches were not given specific instructions for how to annotate the videos. The goal for this study was to understand what the teacher and coach noticed from the video, particularly because the video was the coaches' first view of the lesson, as they were not present in person during the lesson. To enter the comments, the coach or teacher paused the video and typed their comments, which were then synced to the video with time-stamps. This allowed the teacher or coach to watch the video and comment on specific moments. The purpose of the coaches' annotations were to spur dialogue for the lesson debrief meeting, so we note that some comments were intended as conversation catalysts and were influenced by what the coach wanted to discuss with the teacher. The unit of analysis for this study was the annotations from the coaches (n=328) and teachers (n=213) as they took part in coaching cycles across a two-year span. The coaches annotations accounted for 60.6% of the total annotations (n=541).

Data Analysis

We considered each annotation a separate data unit for analysis. Given the focus on understanding noticing, we created a codebook with four main categories: subject (who), specificity (general or specific; coded as specific if there is some connection in the annotation) analytic stance (how noticing was communicated), and content (see Figure 1). For the purposes of this paper, we focus on analytic stance. We based our articulation of analytic stance largely on the work of Sherin and van Es (2008) and van Es (2011). Their list of stances included tag, describe, evaluate, interpret, suggest, and question. Based on the literature, we considered the codes of *tag* and *describe* to reflect less advanced noticing and *evaluate* and *interpret* to reflect more advanced noticing because of the attempt to assign a value judgment (evaluate) or provide some meaning (interpret). The code of interpret was assigned when the annotation included an inference to make meaning (e.g. Sherin & van Es, 2009). We consider *suggest* and *question* to be less advanced forms of noticing than evaluate and interpret because the content of suggestions and questions do not necessarily center on something that happened or could be noticed. The code for *tag* was only used in the absence of any other code for analytic stance.

To analyze the annotation data, three researchers met initially and coded a subset of the annotations, representing approximately 10% of the total data set. In this process, the codebook was refined to its current status (Figure 1). Following the finalization of the codebook, the three researchers analyzed another 10% of the data together to ensure consistency with coding. After several rounds of coding to ensure reliability, pair coding commenced with two researchers independently coding all annotations from a given coach-teacher coaching cycle. We calculated Kappa for each coaching pair and the two researchers met to reconcile differences in codes, resulting in final codes for each coach-teacher pair for each coaching cycle. Kappa ranged from 0.63 to 0.70, indicating good to excellent reliability (Landis & Koch, 1977). Following the assignment of codes, we conducted frequency counts related to all codes for the coaches and teachers across coaching cycles. We then conducted frequency counts for the coaches as a group.

Subject			
Teacher		Students(s)	
Teacher		A Student	Group of Students (2+)	Whole Class
Specificity			
General	Specific	General	Specific
Focus on General Teaching Strategies, Pedagogy, Content of a Lesson, Context of a Problem	Focus on a connection between the teacher and another person(s) or aspect (e.g. pedagogical strategies, interactions with others, or students' thinking)	Focus on general actions, aspects, ideas, topics, lesson structure or contextual features (e.g. physical environment, home environment, disposition, behavior, etc.)	Focus on connection between a student and another person(s) or aspect (e.g. his/her mathematical thinking, his/her interactions with others, or specific teaching strategies)

Analytic Stance (How)					
Tag	Describe	Evaluate	Interpret	Suggest	Question
Identify a moment; absence of other codes	Recount or explain event or interaction	Statement with value judgement	Attach or attempt to make meaning; make sense	Offer a suggestion	Pose a question directly to another person watching video

Content							
Lesson Goals			Contextualizing				
Instructional Practice	Math Content	Coaching	School/Classroom Context	Math Content or Curriculum	Students(s)		
Lesson Content							
Introduction/Launch	Task Description & Sequencing	Questions/Questioning	Summary	Assessment	Adaptions	Discourse	
Lesson Features							
Instructional Needs	Misconceptions/ Challenges	Anticipating	Selecting/Sequencing	Connecting	Participation Structure	Next Steps	Student Strategy
Technology							
Technical Logistics/Challenges	Referencing Swivl/Annotations/Video	Use of Technology					

Figure 1. Codebook for Subject, Specificity, Analytic Stance, and Content of annotations.

Results

The analytic stance varied across the coaches. Collectively, the coaches' annotations were coded as evaluation 23.8% of the time, interpretation 21.3% of the time, suggestion 30.2% of the time, and question 42.4% of the time. Table 1 reports the percentages for each analytic stance code for each coach. The codes were not exclusive, as each annotation was coded with as many analytic stance codes as applied.

Table 1. Analytic stance of coaches

	Evaluate	Interpret	Suggest	Question
Alvarez	14.9%	7.5%	10.5%	49.3%
Lowrey	17.6%	48.3%	22.0%	40.7%
Bishop	39.0%	12.7%	46.6%	27.1%
Riess	11.5%	11.5%	32.7%	71.2%

Analysis across coaches reveals differences in the frequency with which we applied the codes of evaluation, interpretation, suggestions, and questions. Alvarez, Lowrey, and Riess were coded as evaluative in approximately 15% of the annotations. In contrast, Bishop was coded as evaluative in 39% of the annotations. Coaches also varied the extent to which they were coded as interpretative. Lowrey was coded as interpretative in 48.3% of the annotations compared to 7.5%, 12.7% and 11.5% for Alvarez, Bishop, and Riess, respectively. There were also differences in the application of the suggestion code, with Alvarez' annotations coded least frequently and Bishop most frequently. Riess's annotations were coded as questions far more than the other coaches. Given the variation in analytic stance across the coaches, we provide profiles of each of the four coaches to illustrate the differences in how coaches annotated, based on what they noticed. We then consider coaches' analytic stances in relation to trends in the analytic stances of the teachers they coached.

Alvarez: High Questioner to Suggestion Ratio

Alvarez had the highest question to suggestion ratio of any of the coaches. By the third and fourth coaching cycles, Alvarez posed questions in nearly 70% of the annotations she wrote. As an example, in one annotation, she wrote:

> How did you decide who to call on? This question gets at the idea of always paying attention to the instructional strategies laid out in the 5 Practices. So even at a "micro share out level", how do you decide who to call on and why? (I think about this a lot in my own practice and there are a variety of answers! The key is to make those decisions as often as possible to support students' thinking and understanding!)

In this example, Alvarez included both a question and a suggestion in the same annotation. She used questioning to prompt the teacher to consider certain aspects of practice and then suggested that the teacher be purposeful when deciding who to call on during discussions.

Over the coaching cycles, the annotations of the teacher Alvarez worked with were increasingly coded as interpretation, although the interpretation code was still infrequently applied. During the first and second coaching cycle, the teacher did not make any interpretations; however, during the third and fourth coaching cycle, 13% of the teacher's annotations included an interpretation. In the third coaching cycle, the teacher wrote, "Students were saying that the dependent variable is being multiplied every time, but they meant that the previous y-value is multiplied by the growth factor to get the next value." In this example, the teacher interpreted by providing an explanation about what students were saying to make meaning from the experience. The data from the annotations matched data from our analysis of the pre- and post-lesson debrief meeting transcripts for conversations between the coach and teacher, in which Alvarez was more likely to elicit information from the teacher than other coaches (Authors, 2019).

Lowrey: High Interpretive

Lowrey's annotations were coded as interpretation 48.3% of the time, a much higher frequency than other coaches. Lowrey also posed questions to the teacher in many of her annotations (40.7%). During the first coaching cycle, Lowrey annotations were coded as interpretation 60.4% of the time. As an example of an interpretation, Lowrey wrote, "How is the independent think time then moving into group talk working for your students? It seems natural for the students to work in this way and I was wondering your take on how it supports student learning." The example was coded as interpretive because Lowrey referenced students having think time and then made meaning of the situation, noting that the process seemed "natural" for the students. In the interpretation, Lowrey described how she made sense of the teaching move with respect to the students. In addition to frequent interpretations, Lowrey asked questions, with the prevalence of questions increasing across the coaching cycles. She included questions such as, "Moving to the back of the room puts the focus on the problem rather than on you. Was this an intentional move? Is it a typical move for you?"

The increased frequency of Lowrey asking questions across the coaching cycles coincided with a decrease in the frequency of annotations of the teacher with whom Lowrey worked. During the first coaching cycle, the teacher wrote 44 annotations. During the second coaching cycle, the teacher wrote three annotations. During this same coaching cycle, Lowrey there was an increase in the number of annotations coded as questions. This increase in asking questions, such as, "Do these partners see the connections between their ideas? Are they working independently or as a partner group?" was evident in the third coaching cycle as well, when the teacher wrote only four annotations. The questions Lowrey wrote often asked about specific aspects of the lesson that may not have been obvious from the lesson video. The increase in Lowrey posing questions suggests Lowrey may have been responding to the teacher's lack of annotations. This raises further questions about the purpose behind Lowrey's increase in questioning and the intentionality of Lowrey's decisions in response to the teacher. Perhaps Lowrey posed questions to the teacher to encourage the teacher to increase the frequency of annotations. Lowrey and the teacher were not able to complete a fourth coaching cycle because of a lack of participation from the teacher.

Bishop: High Evaluation and Suggestion

Bishop's annotations were more frequently coded as evaluations and suggestions. Bishop's annotations were coded as evaluation 39% of the time. As an example, she made statements such as, "I like how you gave these students advance warning that they would be sharing with the whole class. This helps them prepare and feel more comfortable doing so." In this example, Bishop evaluated the teacher's decision to tell students they would be presenting to the whole class. She initiated the annotation with a statement of what she liked and followed up by including text about why that teaching move was important. In addition to including evaluative comments in the annotations, Bishop provided suggestions to the teachers much more often than the other coaches, with 46.6% of the annotations coded as suggestion. The following is a suggestion Bishop provided:

> I noticed that earlier in the video, you read the problem to the class. I was wondering if you might consider using a literacy strategy to introduce the problem to the class. There is a lot of information to deal with in this problem. So, I was wondering if a literacy strategy designed to focus students on all the important information might result in more students (during the individual think time) incorporating the tax into their thinking.

In this example, Bishop made the suggestion to use a literacy strategy to support students. In contrast to Alvarez and Lowrey, Bishop's rate of asking questions was much lower. She asked questions in approximately one-quarter of the annotations.

The annotations of the two teachers with whom Bishop worked both had high numbers of annotations coded as evaluation, with 23.8% of their combined annotations containing some type of evaluation across the four coaching cycles, as compared with approximately 10% of the annotations of the other teachers. At the end of the four coaching cycles, the annotations of both teachers were coded as evaluation at a similar frequency as Bishop, suggesting that the teachers may have followed Bishop's lead to view the lesson videos with an evaluative perspective. Additionally, both teachers and Bishop evaluated the students as well as the teacher. As an example of one teacher providing and evaluation of herself, Parsons wrote, "That was a very thorough explanation!" Bishop regularly included evaluations of the teacher, such as, "Another great move! 'What do you like about ___ answer?' A great assessing question." In this example, Bishop evaluated the assessment question that Parsons included. Evaluations focused on students included text such as, "I loved how they got right into the discussion." Across the four coaching cycles, evaluative comments were frequent for both Bishop and the teachers with whom she worked, which may suggest similarities in how the coach and teachers perceived the annotation process. Analysis of Bishop's coaching debrief meetings show consistent patterns; there was a high percent of conversation dedicated to evaluative comments and direct suggestions for teachers.

Riess: High Questioner

Riess's annotations were coded as questions at a much higher rate than any of the other coaches, more than 70%. Questions included "Do we know if they really understand what the words independent and dependent mean? How could you check for understanding here?" The frequent use of questions was evident across all of the coaching cycles. During the second coaching cycle, Riess included a question in every single annotation she wrote (n=10). The other coaching cycles included frequent questions as well. In addition to posing questions, Riess provided a suggestion to teachers in

32.7% of the annotations. This rate of providing suggestions was not all that different from other coaches, but was the second most common analytic stance associated with Riess's annotations. The frequent use of questioning distinguished Riess from the other coaches.

Interestingly, the two most notable trends across the coaching cycle for the teacher with whom Riess worked related to suggestions and questions. Across the coaching cycles, the teacher reduced the number of suggestions she provided for herself through the annotations and increased the frequency of questions she asked to Riess. Initially, the teacher included comments to suggest to herself what she should have done pedagogically, as compared to what actually transpired. In the first and second coaching cycle, she included suggestions 30.0% and 50.0% of the time. Suggestions included, "Missed connection about 13 and Baker's Dozen; could have asked more questions." In this example, the teacher provided an evaluation and then suggested to herself that she should have made a connection for students and then asked for questions. By the fourth coaching cycle, the teacher did not include any suggestions to herself in the annotations. Instead, in the third and fourth coaching cycles, the frequency of posing questions increased. Questions were written directly to the coach, as a way to ask for input from the coach, most commonly about instructional moves. As an example, the teacher wrote, "When should I have gotten the students more involved? For them to ask clarifying questions or have them restate task expectations?" The teacher sought direct input from the coach on student participation. This type of interaction points to the function of the annotations as a way to initiate conversation prior to the debrief meetings that followed the annotation process. We conjecture that the teacher may have started to write questions to the coach that she wanted answered during the debrief meeting, when the two of them would discuss the lesson.

Discussion and Implications

Annotations were an intermediary for communication that occurred after the teacher's lesson implementation and before the debrief meeting between the coach and teacher as part of online content-focused coaching cycles. Substantive differences existed in how coaches annotated aspects of instruction and learning, meaning how they approached and analyzed practice. Additionally, results show an associative relationship between the analytic stances of coaches and teachers. The analytic stances of the coaches and teachers illustrate aspects of the coach-teacher relationship. The following elaborates on the trends in noticing, with conjectures grounded in research literature to explain the results.

First, the findings of this study highlight the perceived relationship between the coach and teacher, from the teacher perspective. Data from Alvarez and her teacher show stark differences in the analytic stance, with the coach having a high question to suggestion ratio and an absence of the teacher posing direct questions to the coach in any coaching cycle. We conjecture that the teacher had perceived roles for the participants in the coaching process (i.e. coach and teacher) and may have assumed it was the coach's responsibility to pose questions to the teacher, not vice versa. As another example of a perceived relationship, we found evidence in other coaching interactions to suggest that teachers began to annotate in ways similar to coach, in a process similar to enculturation. The situated perspective of the teacher (e.g. Lave & Wenger, 1991), in relationship to the coaching cycles, appears to have influenced the analytic stance of the annotations.

Second, we conjecture that over the coaching cycles, the shift in some teachers' annotations may have been the result of thinking about how the annotation process coordinated with the coaching debrief meetings. Over time, the teacher working with Riess began to ask more questions, directly posed to the coach. It is possible that these questions were intended to be reminding prompts to support the synchronous debrief meeting that would follow the annotation process; over time, we believe the teachers may have recognized the relationship between the constituent parts of the content-focused coaching cycle (i.e. planning meeting, lesson implementation, annotations, debrief meeting) in a way that supported increased integration (i.e. annotations were discussed during the debrief meeting). Further analysis of the coaching debriefing meetings, in coordination with the annotations will allow us to draw substantive claims about this interaction.

Third, we conjecture that coaches used the annotations to be responsive to the teachers beyond the video lesson and debrief process. In the case of Lowrey, the teacher began to show signs of decreased participation in the online professional development project. In turn, Lowrey increased the number of questions she posed directly to the teacher, as a way to elicit reactions from the teacher. It is possible that Lowrey's questions were intended to encourage increased the teacher's participation in the coaching cycles. Interviewing the coaches about their rationales for their annotations will help us understand their annotations.

The focus on analytic stance for both coaches and teachers is important to understand *how* individuals notice within the context of online coaching cycles. Researchers have emphasized the importance of noticing to enact ambitious instruction (Jacobs et al., 2010; van Es & Sherin, 2008). Many researchers have focused on the noticing of particular participant groups, such as prospective teachers (e.g. Roth McDuffie et al., 2014; Schack et al., 2013) or practicing teachers (e.g. van Es & Sherin, 2008), but few have focused on the noticing of individuals with different experience levels as they interact with the same representations of practice. Researchers have shown notable differences in expert and novice noticing (e.g. Huang & Li, 2012); we contend that our study provides data on the interactions of different participants in a way that highlights the importance of coaching to support teachers. Researches have shown that coaching impacts teacher practice (e.g. Gibbons & Cobb, 2017; Kraft et al., 2018; Sailors & Price, 2015), but knowing exactly how coaches approach noticing provides a clearer understanding of how to support both coaches and teachers. As evidenced in the data, the four coaches analytically approached their noticing in very different ways. These findings raise questions about the intentionality with which particular coaches are assigned to work with particular teachers. Perhaps some teachers would respond better to the practices of some coaches over others. Knowing these various trends in coaches' analytic stances and knowing how the teachers interacted with the coaches provides increased understanding for professional developers as they consider coach-teacher pairs and how to support coaches. We encourage others to be aware of the differences in coaches' noticing and recognize the associative relationship that may develop between a particular coach and teacher based on how the two analytically notice.

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ARTICULATING EFFECTIVE MIDDLE GRADES INSTRUCTIONAL PRACTICES IN A TEACHER-RESEARCHER ALLIANCE

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One of the most intransigent problems in mathematics education is the culturally-influenced divide between classroom practice and educational research. This paper describes our explicit attempt to bridge that divide by translating research on instructional practices linked to improving students' mathematics achievement into a brief guide outlining constructs, features, strategies, routines, and tools for use in a teacher-researcher alliance. We outline the design and development process, share the guide itself, and summarize data addressing the utility of the guide for a research and professional development project in which 100 U.S. Grades 6-8 teachers are collaborating to improve middle grades modeling and problem solving achievement.

Keywords: instructional activities and practices, teacher education - inservice / professional development, middle school education

The persistent, culturally-situated divide between educational research and teaching practice in school mathematics is well documented (Cai *et al*., 2017). In addition to vastly different contexts and goals, one reason for the teacher-researcher divide arises from communication - researchers and teachers rarely interact as colleagues, researchers typically disseminate findings in ways that foreground abstractions of teaching and learning, and teachers often seek out specific, situated tools for their everyday practice (Labaree, 2003). One area of common ground centers around shared goals among teachers and researchers to generate information about "what works" in particular contexts in hopes they may inform educators in other contexts (Krainer, 2014). An emerging model for nurturing that common ground is to establish a *Teacher-Researcher Alliance for Investigating Learning* [TRAIL] (Koichu & Pinto, 2018). While the TRAIL format addresses many of the challenges of teacher-researcher collaboration, there are few examples in the literature, and none addressing research aimed at investigating instructional methods for improving student mathematics achievement. Recently, we have engaged in an intentional effort to build a U.S. teacher-researcher alliance centered around investigating and articulating effective instructional routines to promote modeling and problem solving achievement among Grades 6-8 students. One of the first efforts within our alliance has been to create a 2-page instructional practices guide that communicates findings from research in ways that support teachers' translation into practice.

The framework we have developed is organized around the constructs of *Explicit Attention to Concepts* (EAC) and *Student Opportunities to Struggle* (SOS) (Hiebert & Grouws, 2007). Hiebert and Grouws (2007) identified EAC and SOS as broad clusters of instructional methods which researchers have linked to increases in student achievement. In this paper, we describe efforts to form a teacher-researcher alliance to further articulate features, strategies, and routines for EAC and SOS instructional practices, with the specific aim of supporting teachers' implementation of these constructs in their classrooms. We emphasize our methods for translating research findings into actionable practices, especially our engagement with 100 teachers as partners in investigations the effectiveness of the associated instructional routines for improving student achievement.

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Perspective(s) or Theoretical Framework

The driving motivation for this research project is an optimistic belief in the capacity of teachers and researchers to collaborate for improving student achievement. Broadly, large-scale analyses suggest teacher factors account for about 30% of the variation in student's mathematics performance, second only to student factors - which account for about 50% of variation - and exceeding all other remaining identified factors combined (Hattie, 2003). In addition, literature suggests mathematics teachers can (and do) serve as co-producers of relevant professional knowledge with researchers, while directly improving outcomes for their students and affecting positive changes in their local contexts (Kieran, Krainer, & Shaughnessy, 2012). Locally, our prior work with hundreds of teachers through a university-based professional development center has led our team of researchers to view mathematics teachers as having rich, varied expertise, with pragmatic insights from adapting and enacting curriculum in their schools. For this project, we leaned into that perspective by seeking a way to situate our research within a broader effort to bridge cultures through a mutually-valuable partnership centered around shared goals for improving student achievement.

Teacher-Researcher Alliance

Historically, relationships between university researchers and teachers have been asymmetrical; teachers are positioned as in need of the knowledge that researchers provide, with little acknowledgement of the value the experiential knowledge of teachers (Gore & Gitlin, 2004). Ironically, the knowledge produced by researchers often does not have the practical and contextual information that teachers find useful for their practice (Gore & Gitlin, 2004; Krainer, 2014). To bridge the cultures of teaching and research, we must recognize different ways of knowing and view relations as symmetries rather than hierarchies (Krainer, 2014). Central to this perspective is a view of teaching as an ongoing process of experimentation in which teachers naturally engage in regular testing of often informal hypotheses about student's abilities, the effects of instructional activities, and learning outcomes (Cobb, 2000). Researchers can play a role in that experimentation, helping to coordinate activities, gather evidence for drawing inferences, and plan for implementation of teacher-led interventions.

In particular, we conceptualize this project through the five features in the Teacher-Researcher Alliance for Investigating Learning (TRAIL) theoretical framework for scalable partnerships between educational researchers and teachers (adapted from Koichu & Pinto, 2018):

- **Professional Growth** - through participation, teachers enhance their educational research competencies, researchers build their knowledge and abilities to engage in classrooms.
- **Authenticity** - teachers engage in substantive research around questions drawn from real problems of practice, researchers match methodology to existing school systems.
- **Shared Agency** - mechanisms are established so teachers and researchers can each advance individual needs and goals, with room for personal expression and creativity.
- **Choice** - the partnership includes a network of projects, run simultaneously, so that teachers can select from a menu of options for participation.
- **Creating and Using Knowledge** - opportunities for determining "what works" flows from both teaching and research; practical knowledge is co-created.

EAC & SOS Instructional Practices

Nearly all mathematics professional development programs are designed to improve student learning by attempting to affect teachers' knowledge, beliefs, and instructional practices. However, student achievement is a distal goal for programs primarily focused on teachers, and there is limited research demonstrating even modest effects of professional development programs on student achievement (Gersten *et al.*, 2014; Kennedy, 2016). To design professional development with the greatest potential to positively impact instruction and student achievement, our project has focused on instructional strategies identified in research literature as most likely to improve student learning. Hiebert & Grouws' (2007) synthesis of research on instructional strategies with evidence for improving students' mathematical learning has been our primary touchstone.

Hiebert and Grouws identified two constructs underlying instructional practices supporting conceptual understanding (defined as "the mental connections among mathematical facts, procedures, and ideas", p. 382) with research evidence indicating positive effects across study design, teaching formats, and contexts (p. 387):

- **Explicit attention to concepts** (EAC) - Teachers and students explicitly discuss mathematical concepts and make connections among concepts, facts, and procedures through activities such as questioning, discussing, comparing, and relating.
- **Student opportunity to struggle** (SOS) - Students engage in productive struggle with important mathematical ideas through sense-making around comprehensible problems that require them to"figure something out that is not immediately apparent".

EAC can be seen as a more externally mediated approach in which the teacher ensures concepts and connections are made public and clear to students. In contrast, SOS is focused on experiences that engage students in developing understandings through their own sense-making activity. Recently, Stein, Correnti, Moore, Russell, and Kelly (2017) found group means on achievement measures were significantly higher for students of teachers who self-reported a preference for, as well as demonstrated through video-recorded instruction, instructional practices centered around EAC and SOS. Students whose teachers aligned with EAC alone performed significantly better than students of teachers aligned with SOS alone, who in turn performed better than students of teachers aligned with neither element. Additionally, several studies have shown that SOS positively impacts student achievement, particularly when it precedes EAC practices (Kapur, 2014; Loehr, Fyfe, & Rittle-Johnson, 2014; Schwartz, Chase, Oppezzo, & Chin, 2011).

Methods

The goal of this project was to establish a teacher-researcher alliance (with TRAIL features) in order to articulate instructional practices for the purposes of an extended research project in the context of professional development. To put the research in context, we next provide (a) a brief overview of the project, (b) a description of the project team members who developed the framework, (c) a summary of our process for developing a framework related to the EAC and SOS constructs, and (d) a brief description of the associated data collection and analysis.

Project Overview

The heart of this project is a group of 100 Grades 6-8 teachers across 45 schools and 23 districts working in an area spanning approximately 200 miles of a U.S. state with low population density and a strong tradition of local control in education. Funded by a multi-year federal research grant to investigate methods for improving middle grades mathematics achievement, the researchers recruited the teachers by obtaining approvals from their respective district administrators to invite Grades 6-8 mathematics teachers to participate in a 3-year research-professional development partnership. The professional development (PD) involves (a) three module meetings (15 hours total) for collaborative development of the EAC and SOS framework with opportunities for classroom implementation between each session, followed by (b) three week-long summer institutes (one each summer) for planning teacher-led classroom studies of EAC and SOS instructional routines, and (c) embedded classroom support provided by an experienced, dedicated instructional support team (the PD Team).

The PD Team plays a pivotal role in our teacher-researcher alliance by bringing together personnel to bridge the research-practice divide through developing and implementing PD to support teachers' implementation of EAC and SOS instructional practices in their classrooms. The PD team includes a math professor, a math education professor, three full-time mathematics instruction specialists (similar to coaches), a postdoctoral researcher, and a graduate student. The team has extensive expertise and knowledge in mathematics education and professional development. Four of the PD Team members have taught mathematics at the secondary level in the local area for between seven and 16 years, three PD Team members have worked as math coach/specialists for between five and eight years, and four PD Team members have masters or doctoral degrees in mathematics education.

Development of an EAC-SOS Guide

In collaboration with the researchers and teachers, the PD Team led the development of a 2-page **EAC-SOS Guide**. The PD Team met weekly for three months to expand and interpret the conceptual and research foundations of EAC and SOS instructional practices, with a primary purpose of communicating research findings in ways deemed relevant and useful among teacher participants. Using Hiebert & Grouws' (2007) and Stein et al. (2017) as initial resources, the PD Team unpacked the research concepts and associated studies in the context of situated instructional practices. The central challenge of the development work was to communicate instructional routines under investigation by the researchers in ways that maintain fidelity to the research supporting EAC and SOS as effective for promoting students' mathematics achievement while clarifying distinguishing features and levels of specificity that are necessary for teachers to translate the research to their day-to-day instruction. Eventually, the EAC-SOS Guide came to include separate pages for EAC and SOS as **constructs** of instructional practices with robust research evidence supporting positive effects on development of mathematics students' conceptual understanding. For each construct (identified by a distinguishing color and icon), the guide lists three **features** of mathematics instruction characterized by the respective constructs, as well as four **strategies** teachers can engage in during classroom instruction and two **routines** per strategy selected by the researchers to be further investigated through clinical cross-over trials in the teachers' classrooms (see Figure 1 for the design template). Based on teachers' feedback on early drafts, each strategy was supplemented by a short list of instructional tools which may be well-suited to implementation of the associated routines.

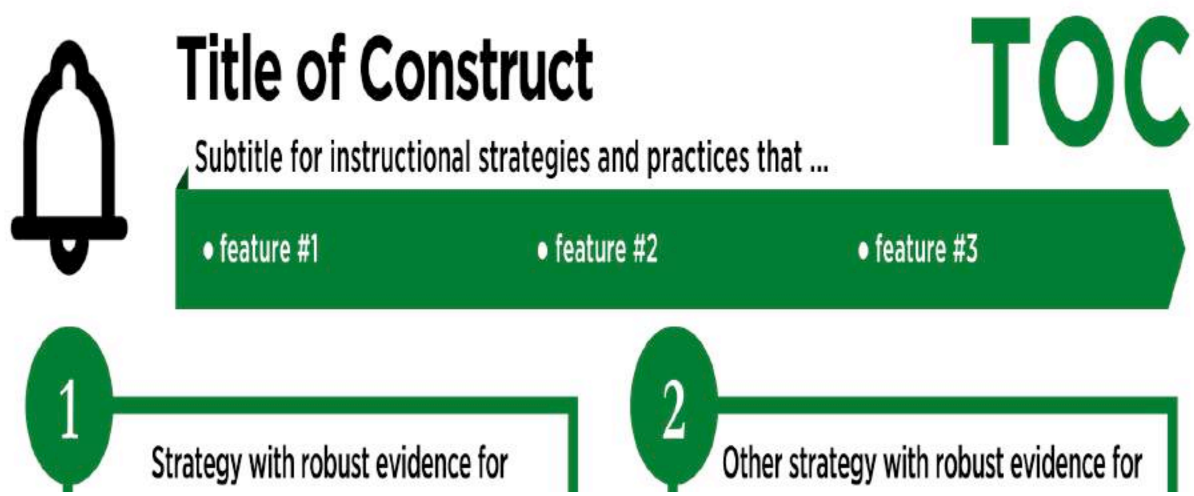

Figure 1. Design Template for the EAC-SOS Guide, with features, strategies, routines, and tools.

Data Sources

We used the PD modules to evaluate and refine the articulation of instructional practices in the EAC-SOS Guide. Participating teachers completed a Teaching Context Survey (adapted from Stein *et al*., 2017), addressing their beliefs and current practices surrounding EAC & SOS instruction, as well as curricular formats, school characteristics, instructional content, and related factors needed to estimate effects of instructional interventions on student achievement across teachers' individual contexts. During the first PD module, teachers previewed the Guide, recommended changes to better support implementation, and rated their level of familiarity and experience with the 8 strategies listed in the guides. Teachers each also selected one of the 8 strategies they would like to try first in their classrooms, and completed a "Stop Light" reflection at the next professional development meeting to communicate the challenges they encountered (red light), ways in which the PD Team can support implementation (yellow light), and positive outcomes they saw in their classroom practice (green light). In the Results section below, we present the final EAC-SOS Guide and summarize the teachers' strategy selections.

Results

The EAC-SOS Guide (see Figure 2, or http://bit.ly/eac-sos-guide) is the primary result of our collaboration among teachers and researchers to articulate instructional constructs, features, strategies, routines, and tools supporting research into the improvement of student mathematics achievement. Following the first PD Module, 94 teachers selected a routine to try in their classroom. More teachers selected an SOS routine (59%) instead of an EAC routine (41%). Teachers' rationales for their choice of an SOS or EAC routine are summarized in Table 1.

Table 1. Counts of rationale provided for teachers' selected routines, by construct.

	Supporting Student Thinking	Fit with Curriculum	Improving Teaching Skills	Fit with Content	Other (Collaboration, General Interest)	*Total*
EAC	15	10	2	9	3	*39*
SOS	24	8	16	6	1	*55*
Total	*39*	*18*	*18*	*15*	*4*	*94*

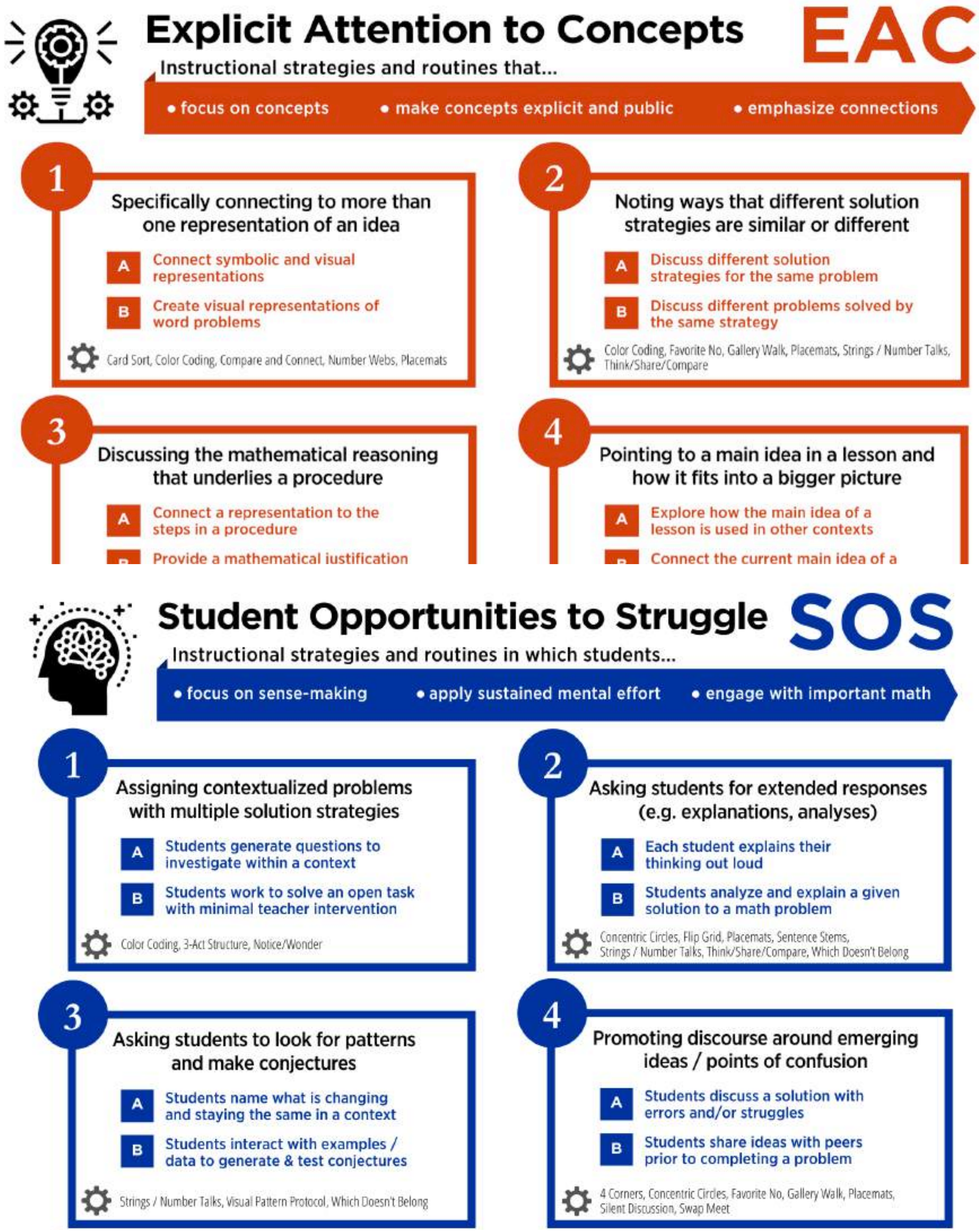

Figure 2. EAC-SOC Guide to Instructional Practices for Improving Math Achievement

Table 2 shows the routines selected by teachers for initial testing, together with exemplar statements related to why that particular routine was selected. (The examples were selected based on a combination of frequency of occurrence of 'why' reasoning across multiple responses in conjunction with those that seem well tied to the routine itself.).

Table 2. Routines selected by participating teachers, with example rationales they provided.

Routine	Frequency	Example Whys?
EAC1A: Connect symbolic and visual representations	7.4% (7)	...Many of my students are procedural, but they actually don't understand the nature of the problem and what it's saying.
EAC1B: Create visual representations of word problems	10.5% (10)	My students have been taught a lot of procedures and do not always understand what it looks like...
EAC2A: Discuss different solution strategies for the same problem	10.5% (10)	Students will use various methods to solve systems of equations and I would like to have them compare and contrast the different strategies.
EAC2B: Discuss different problems solved by the same strategy	0.0% (0)	*No exemplars*
EAC3A: Connect a representation to the steps in a procedure	4.3%(4)	I'm hoping that an explicit connection to steps in a procedure to a context will help with understanding systems of equations.
EAC3B: Provide a mathematical justification for the steps in a procedure	3.2% (3)	...it is important for my students to be able to explain how they work through a problem and also collaborate with peers...
EAC4A: Explore how the main idea of a lesson is used in other contexts	3.2% (3)	Because this will relate to what I am currently teaching students. Relate percents to fractions
EAC4B: Connect the current main idea of a lesson to a prior math concept	5.3% (5)	I will be starting out percent lessons next week...I want to focus on making connections to what they did with ratios & proportions.
SOS1A: Students generate questions to investigate within a context	2.1% (2)	I always ask them questions but they rarely ask each other questions. I feel like we all need to improve in this.
SOS1B: Students work to solve an open task with minimal teacher intervention	8.5% (8)	I want to see the productive struggle with my students and see how the handle it.
SOS2A: Each student explain their thinking out loud	10.5% (10)	I think understanding becomes more clear and cemented in our brains when we can express them in words.
SOS2B: Students analyze and explain a given solution to a math problem	2.1% (2)	Teaching Intervention math my students often just want to say the answer instead of explain how they arrived at that answer.
SOS3A: Students name what is changing and staying the same in a context	5.3% (5)	I want to incorporate more "non-teacher directed" entry point activities and I feel that this is a place to start...
SOS3B: Students interact with examples or data to generate and test conjectures	4.3%(4)	...I want my students to interact with data to understand how adding/subtracting numbers from the data will affect the mean and median.
SOS4A: Students discuss a solution with errors and/or struggles	14.9% (14)	I want students to get more comfortable with analyzing others answers and share ideas
SOS4B: Students share ideas with peers prior to completing a problem	8.5% (8)	...Students can bounce and gain ideas that they may not have thought of or get reinforcement on an idea they had.

Discussion

The primary outcome of this research is the EAC-SOS Guide. The time and resources supporting the design and development of the document - especially the associated efforts to situate the development within a teacher-researcher alliance - indicate great potential value in the document to support efforts to address the culturally-entrenched challenges of merging research and practice in the context of professional development aimed at improving mathematics achievements in the middle grades. In addition to the direct input teachers had in the development of the Guide itself, teachers' initial selections of routines to try in their classrooms also provides positive indications that both EAC and SOS constructs are appealing to practicing teachers interested in better understanding and leveraging their students' thinking, implementing their curriculum, improving their repertoire of teaching methods, adapting their instruction to the mathematical content, and collaborating with peers.

In addition to practical uses for the EAC-SOS Guide in professional development and research settings, we encourage colleagues to consider transferring our conceptual framework, especially the TRAIL model for collaboration between teachers and researchers and emphasis on articulating research findings in practical terms, to future projects. We view the results reported in this paper as provisional, and intend to further refine and articulate the constructs, features, strategies, routines, and tools by creating a modern website using similar development methods (e.g., selection preferences, challenges, affordances, supports, evidence for positive effects). In addition, we look forward to conducting classroom research with our teacher partners, and are hopeful the associated research findings will clarify the contexts under which the instructional routines are especially promising for classroom implementation. We welcome collaborators interested in extending that work.

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STUDYING A SYNCHRONOUS ONLINE COURSE USING A COMMUNITY OF INQUIRY FRAMEWORK

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We studied two iterations of an online course provided to rural mathematics teachers. The online courses, which involved primarily synchronous activity, emphasized high-leverage discourse practices. We applied a community of inquiry framework, which emphasizes deep intellectual work, and its three tenets: cognitive presence, social presence, and teaching presence. We adapted the framework by creating a category on content-related interactions and by using mediating processes from our conjecture maps (e.g., Sandoval, 2014) to characterize cognitive presence. The adapted framework allowed us to notice substantive differences between the course iterations, especially in relation to teaching presence and cognitive presence. The implications of the study are that the framework helps us gauge the efficacy of synchronous online interactions and to better gauge goals for future iterations of the course.

Online platforms and learning environments are emerging as important contexts for teachers' professional development (Johnson et al., 2018; Keengwe & Kang, 2012; Means et al., 2009), and thus as sites of research. The online context provides access to professional development to teachers who may be geographically distant from conventional professional development providers, such as institutions of higher education, and from critical masses of colleagues. Because this trend is likely to grow, it is important to conceptualize ways to research online professional development, specifically within mathematics education. Many online learning projects have asynchronous environments. We argue that online *synchronous* learning environments also have the potential to provide impactful professional learning experiences for teachers, as they incorporate features of face-to-face environments. However, there has been inadequate research on the efficacy of features within synchronous online environments. In this study, we explore one component of a three-part online professional development model for middle grades mathematics teachers in rural contexts. We apply a *community of inquiry* framework (Garrison, Anderson, & Archer, 2000) because of its assumptions about engaging participants in demanding intellectual work, and connect the framework to the design literature, specifically *design conjectures* (Sandoval, 2014). The mediating process component of design conjectures provides a way to characterize and analyze the cognitive presence in a community of inquiry, which has been an outstanding methodological problem in studying online contexts (Akyol & Garrison, 2011).

The purpose of this study is to examine the quality of the learning environment we created in a synchronous online course. The community of inquiry framework allowed us to examine the characteristics of the teaching, cognitive, and social presences. Prior research has shown that confirmed that online environments can establish social presence (e.g. Whiteside, Dikkers, & Swan, 2017). However, it has been less clear how content-related features of online courses are evident, particularly interactivity related to subject matter, cognitive presence, and aspects of teaching presence. In this study, we use the community of inquiry framework to explore these features within and across two cohorts of teacher participants who took the same online course.

Online Professional Development Course

We designed a three-part online professional development model with the goal of providing rural mathematics teachers access to high quality professional development. We originally designed and implemented the three components in face-to-face formats, which we then iteratively transformed into fully online versions for the purposes of this project. Our project utilized a series of synchronous online experiences, which departs from the typical asynchronous nature of much of the current online professional development, educational coursework, and virtual teacher communities. The three parts of the model included online course modules, demonstration lessons, and online coaching.

The online course modules emphasized discourse practices that orient teachers toward high-leverage discourse practices to facilitate mathematically productive classroom discussions (Smith & Stein, 2011). These discourse practices are catalyzed by five practices emphasized in the course, entitled *Orchestrating Mathematical Discussions* (OMD), anticipating, monitoring, selecting, sequencing, and connecting. The modules also emphasize key aspects of lesson planning, such as goal-setting, in addition to having teachers solve and discuss high-cognitive demand tasks. The specific goals of the modules were to: develop awareness of specific teacher and student discourse moves that facilitate productive mathematical discussions; to understand the role of high cognitive demand tasks in eliciting a variety of approaches worthy of group discussions; and to further develop participants' mathematical knowledge, particularly the rich connections around big mathematical ideas (Ball, 1991; Ma, 1999). The modules involved a combination of synchronous and asynchronous work to minimize the amount of time teachers met together virtually (Robinson, Kilgore, & Warren, 2017). This minimized logistical challenges and maintained a high degree of teacher effort and attention due to the shortened synchronous time. Hrastinski (2008) found that synchronous and asynchronous components complement each other.

We conducted the OMD course in Zoom, a video conferencing platform, which allowed for synchronous whole class and small group interactions. In addition, we simultaneously used Google Docs and Google Draw, which allowed participants to collectively develop and share artifacts, including approaches to mathematical problems. The instructor presented challenging tasks to the participants, who then worked in virtual breakout rooms to create a document that they shared with the other groups. They talked to each other, worked simultaneously on the shared document, and used the chat window to communicate in the virtual space. The course instructor listened to and participated in these group discussions to facilitate the group work.

Framework

We draw primarily from the community of inquiry framework (Garrison et al., 2000), that identifies three components of online learning environments: cognitive presence, teaching presence, and social presence. Garrison and Cleveland-Innes (2005) define a community of inquiry in terms of deep learning that extends beyond simple interactions, stating that a community of inquiry is a place where "ideas can be explored and critiqued; and where the process of critical inquiry can be scaffolded and modeled" (p. 134). Below, we describe each of three components of online learning environments: teaching presence, cognitive presence, and social presence.

Teaching Presence

Teaching presence entails three aspects of teaching: interactive instruction, design, and direct instruction. We highlight the interactive aspects of teaching, following Garrison and Cleveland-Innes (2005), who state that "if students are to reach a high level of critical thinking and knowledge construction, the interaction or discourse must be structured and cohesive" (p. 134). Structuring interaction in productive ways includes explicitly articulating the designed features of the learning environment and directing students via explanation, scaffolding, or evaluative feedback. Anderson, Rourke, Garrison, and Archer (2001) describe these three roles of the instructor as designer, facilitator, and subject matter expert. In our perspective, the most critical role of the instructor is to *facilitate* productive interactions with and between the participants, as this is most likely to elicit and advance thinking related to the goals of the course. This is different from the social interactions described below, which are not necessarily related to course goals but are productive for building community. Articulating the design of the learning environment creates expectations and opportunities for participants; however, as learners engage with design, the instructor must insert themselves into the online interactions (via feedback, explanation, and scaffolding) in order to support students understand expectations and develop their thinking.

Cognitive Presence

Garrison, Anderson, and Archer (2001) describe *cognitive presence* as "the extent to which learners are able to construct and confirm meaning through sustained reflection and discourse" (p. 11). Cognitive presence is synonymous with critical engagement with content; consequently, developing analytic tools to characterize cognitive presence must include the intellectual practices deemed essential to the learning goals. In past research, cognitive presence has been the least analytically developed dimension of community of inquiry (Akyol & Garrison, 2011), in part because processes and content that determine cognitive presence are specific to a given discipline and domain.

In order to characterize cognitive presence, we turned to the design literature to identify the essential processes related to our learning goals. We utilized *conjecture maps* to operationalize cognitive presence analytically. According to Sandoval (2014), "*conjecture mapping* is a means of specifying theoretically salient features of a learning environment design and mapping out how they are predicted to work together to produce desired outcomes" (p. 19) and is intended to reify the conjectures regarding the learning environment and how they interact to promote learning. There are four main elements to a conjecture map. The first element involves *high-level conjectures* about how the learning context supports learning. Researchers then operationalize these conjectures in the *embodiment* of the learning design, the second element, by articulating tasks, participant structures, and so forth that provide opportunities for learners to engage with content. In the third element, researchers conjecture how the design of the learning environment (embodiment) generates *mediating processes* that produce desired *outcomes.* Mediating processes occur *within* the learning environment and potentially lead to the outcomes that may occur *outside of* the learning environment, such as enacting high-leverage discourse practices in mathematics classrooms. To analyze cognitive presence, we turn to the third element, the mediating processes, as they were the intended targets of the designed learning environment.

Designers of a learning environment articulate mediating processes to reflect the desired practices, and associated cognitive work, that should result from the design of the learning environment. Thus, observations of mediating processes focus on the interactions between learners and the learning environment. To articulate the mediating processes, the project team reflected on the goals of the project, the goals of the course, and specific aspects of the learning environment, to generate four mediating processes, described in the methods section.

Social Presence

Garrison et al. (2000) describe *social presence* as the ability for participants to project themselves and to establish personal and purposeful relationships. Rourke, Anderson, Garrison, and Archer (2001) state that the three main components of social presence are affective responses, interactive responses, and cohesive responses. As described in our analysis below, we separated out content-related interaction from the other two categories and primarily used affective and cohesive responses to define social presence.

Research Focus

We used the community of inquiry framework to focus on how the synchronous online environment affected the teaching presence and the opportunities for the teacher participants to engage with the content directly and with each other around the content. We studied two cohorts of teachers who participated in the same online course; this allowed us to compare the two course enactments to better characterize teaching and cognitive presence within and across cohorts. As noted above, the ability of designers to facilitate social presence is sufficiently documented, and we, too, found that the teacher participants engaged in friendly banter around their lives and jobs. We conjecture that this social presence facilitated interaction around content, but we do not explicitly explore that conjecture in this paper. Our research questions were:

1. What aspects of teaching presence were evident in the two online courses, especially in terms of interactive teaching and direct instruction?
2. How did this teaching presence differ across the two cohorts, and how were these differences related to teacher participation?
3. To what extent did the participants in the two cohorts engage in mediating processes (evidence of cognitive presence)?
4. To what extent did the participants in the two cohorts engage with each other around the content (level of content-related interactions)?

Methods

Data Collection

We video recorded the OMD sessions, six for each cohort, twelve in total, using screen capture technology. For each session, we video recorded the host computer as well as each of the breakout rooms, creating three to five video files for each session. We used Panopto so that we could record the Zoom window and simultaneously the Google Docs the groups were creating. We transcribed all of the breakout rooms and the subsequent whole class discussions, omitting the introductory whole class segments in which the instructor outlined the task goals and expectations. This reduced the frequency of the design aspect of teaching, which was not emphasized for this study. Overall, there were 24 episodes for Cohort 1 and 45 for Cohort 2. These differences reflect data collection issues we encountered in Cohort 1 and less participation from Cohort 1 in the OMD course relative to Cohort 2.

Data Analysis

We coded the transcripts turn-by-turn, assigning as many codes as were relevant to a given turn across the three presences. After a few rounds of consensus coding, we independently coded transcripts, with pairwise kappas between 0.45 and 0.54, considered moderate agreement (Landis & Koch, 1977). Below, we provide more details related to the codes we applied for each of the presences.

Cognitive presence. The coding team for cognitive presence included members of the project team who were most familiar with the course design, including two instructors and one designer, in addition to the two lead researchers. To analyze cognitive presence, we coded all turns for the presence of the following four mediating processes we had previously developed in the conjecture mapping process (Sandoval, 2014):

- Explaining how features of a lesson/task design influence opportunity to engage in mathematical thinking;

- Explaining mathematics in the task in ways that make connections;
- Explaining anticipated or observed strategies or misconceptions for a given task; and
- Explaining how teaching moves impact access to mathematical thinking, participant frameworks, student authority, or formative assessment.

Teaching presence. We coded for the three forms of teaching presence, though we minimized the emphasis on the design aspect of teaching. For interactive teaching, we adapted the facilitating discourse category from Anderson et al. (2001). Our codes included: *identifies areas of agreement/ disagreement*; s*eeks to reach consensus*; *encourages (acknowledges, reinforces)*; e*licits contributions; presses or probes*; and r*edirects discussion*. For direct teaching, we adapted the direct instruction category and our codes included: *explains content*; *focuses or funnels*; *summarizes or provides feedback*; and *responds to technical concerns*. The coding team consisted of one of the two lead researchers and four doctoral students.

Social presence. Adapting the work of Rourke et al. (2001), we used the three main categories of affective responses, content-related interaction, and cohesive responses, each of which had sub-categories that we modified to take into account that we were working in a synchronous video-based environment. For example, in the content-related interaction category, we revised *continuing a thread* to *building from or extending another participant's response* and we revised *quoting from other messages* to *repeating or paraphrasing another participant's response*. The coding team was the same as for teaching presence.

Second phase of analysis. In the second phase of our analysis, we reorganized and reduced the codes. We pulled out the two codes for content-related interaction - *building from or extending another participant's response* and *refers to or paraphrases another participant's response* – because we felt these codes involved interaction around content and were not purely social in nature. That left the affective and cohesive categories to represent social presence. We collapsed all codes in the cognitive presence category into one count, and did the same for social presence and for each of the three aspects of teaching presence – design, interactive teaching, and direct teaching. We also added a category for *technical*, which involved all turns related to resolving technical issues in the Zoom and Google environments. We then collected all the totals of the reorganized and collapsed categories across each cohort.

Results

We present the results initially by looking at cross-cohort comparisons, which allowed us to look at patterns within cohorts and then patterns within cohorts. The comparisons look at different distributions and rates within and across categories for both instructors and teacher participants. The purpose of presenting the cross-cohort comparisons is not to evaluate the instructors or teacher participants, but to highlight the ways the community of inquiry framework provides insights into the dynamics and efficacy of the online courses. We noticed three distinct trends. First, we noticed a much more prominent teacher presence in the Cohort 1 course. Second, we noticed that participants in Cohort 2 had much higher rates of cognitive and social presence, as well higher rates of turns involving technical issues. Third, we noticed that the teacher participant turns were more evenly distributed in Cohort 2 than Cohort 1. We discuss each of this in detail below.

Teaching Presence

The Cohort 1 instructors had six times as many turns coded as interaction (building from or referring to other participants' contributions) and roughly three times as many turns coded as interactive and direct teaching, with similar rates of design codes (see Table 1). There are some possible reasons why these differences occurred. One reason is that the instructor with the most prominent presence in the Cohort 1 implementation had a different teaching style than the instructor with the most prominent presence in Cohort 2. This is borne out in the analysis of the coaching cycles that were another component of the professional development project, in which the primary instructor in Cohort 1 had considerably higher rates of direct assistance (explanations, suggestions) than the other coaches. A second reason is that the number of participants in Cohort 2 was 50% higher per session (roughly three more participants per session in Cohort 2), perhaps allowing the instructor to offload some of the work onto the participants. For example, the *interactive presence* and *interactive teaching* categories for Cohort 1 indicate that the instructor was much more active in engaging participants with their contributions, and referring to and paraphrasing those contributions to make them objects of discussion. Perhaps this was because the participants did not engage each other with their ideas to the satisfaction of the instructor and she felt compelled to intervene. At any rate, these substantive differences likely had an impact on the learning opportunities for the participants.

Table 1: Teaching Presence across Cohorts

Instructors	Interactive Response	Interactive Teaching	Design	Direct Teaching
Cohort 1	41	82	38	99
Cohort 2	7	28	28	30

Participant Rates across Categories

The teacher-participants in Cohort 2 contributed nearly twice as many turns in three categories: cognitive presence; social presence; and technical concerns, with roughly equal numbers of turns coded as content-related interaction. This may have been in part because there were on average 50% more participants in each session relative to Cohort 1, but this does not account for all of the difference nor does it account for the fact that the content-related interaction was roughly equal. In the two sessions in which there were the most participants in Cohort 1 (8 of the 11), roughly 80% of the overall turns coded as interactive response and social presence occurred, and over 50% of the cognitive presence. This suggests that for this group, having more participants was associated with higher levels of interaction, both content-related and socially-related. Importantly, Cohort 2 participants engaged in the mediating processes at much higher rates, which we hope eventually to tie to the project outcomes and other data sources in the project.

Table 2: Participant Rates across Cohorts

Teacher Participants	Cognitive Presence	Content-related interaction	Social Presence	Technical Concerns
Cohort 1	101	104	185	77
Cohort 2	189	108	406	194

Distribution of Turns across Participants

We noted that contributions from teacher participants were more equitably distributed in Cohort 2 than Cohort 1 with respect to cognitive presence. The top four contributors in Cohort 1 accounted for 61% of the cognitive presence codes, while the top four contributors in Cohort 2 accounted for 48% of the cognitive presence codes. Although this may be related to the fact that there were more 50% participants per session on average for Cohort 2, one of the participants in Cohort 1 who attended all of the sessions had very few turns coded as cognitive presence. This participant similarly had low levels of interaction with her coach in the coaching cycles, which suggests that she was having difficulties engaging with key processes emphasized in the project. Nevertheless, the distribution indicates that learning opportunities, and evidence of potential learning, were not ideally distributed in Cohort 1.

Two Cases of Participants

We turn to two cases of teachers, one from Cohort 1 and one from Cohort 2 to provide nuance to our findings. The participant from Cohort 1, Dixon, had the second highest number of codes applied from his cohort (70), with the highest number of cognitive presence and social presence codes. Much of the turns coded as social presence were related to expressing vulnerabilities related to his teaching, while much of the turns coded as cognitive demand explained the impact of teachers' actions, often including detailed examples from his class. The participant from Cohort 2, Fleming, had a below average number of contributions for that cohort (67), with above average for turns coded as cognitive presence (24) and content-related interactions (16), and below average for turns coded as social presence (20). Fleming's contributions were mostly related to content, though she occasionally joined in the social banter. Most of her cognitive presence turns involved describing mathematical strategies. These two cases demonstrate differences in the content and nature of participation. Dixon was introspective with respect to his teaching, describing situations he faced, often in a self-deprecating manner; however, he did not often describe or analyze mathematical strategies. Fleming, by contrast, talked more frequently about mathematical strategies, her own and those she anticipated seeing in students, and relatively less frequently engaged in social banter or talked about her teaching.

Discussion

A goal for this study was to explore how a synchronous online environment could engage mathematics teachers in demanding intellectual processes that could help them grow professionally. We employed a community of inquiry framework to research the social and intellectual vibrancy of the environment and the ability of instructors to productively structure interactions around the goals of the course. The framework allowed us to see substantive differences between instructors and between participation rates of two cohorts enrolled in the same online course. It also allowed us to conjecture about the relationship between the instructors' actions and the ways the teacher participants interacted with each other and with the content.

The results provide insights into how the online courses achieved content-related purposes related to interaction (recognizing and building from the contributions of others) and cognitive presence (engaging in the mediating processes). Participants in both cohorts engaged in forms of social and cognitive presence, albeit in ways that differed substantively. Notably, the instructors in Cohort 1 were far more active in structuring and modeling content-related interaction, in explaining content, and in providing feedback to participants. Conversely, the participants in Cohort 2 contributed in considerably higher rates with respect to cognitive and social presence. We conjecture that these differences occurred because Cohort 2 had on average 50% higher attendance rates and because the most active instructor in Cohort 1 had a style oriented toward direct instruction, which was consistent with results related to her coaching style (Authors, date). We will attend to these patterns closely as we finalize our analysis for the same two cohorts in the second online course. We also note that the two cases of teacher-participants discussed above demonstrate how participants in the same learning environment engage differently with the content and with the social features of the environment. These results highlight the ability to research dynamics between instructors and participants in a synchronous online environment and to consider how differences in individuals contribute to those dynamics.

Implications

We focus on three implications. First, we feel this study is a step toward formulating a version of the community of inquiry framework suited for synchronous online professional development in mathematics education. Three key adaptations we made to the framework reflect issues highlighted in the mathematics education literature: productive content-related interactions, characteristics of cognitive presence that reflect learning processes for mathematic teachers, and a stronger focus on non-design aspects of teaching. The category of content-related interactions is comprised of high-leverage practices with respect to developing collective knowledge within a community. Related to the second adaptation, using mediating processes to characterize cognitive presence provided us meaningful ways to connect our conjectures about our learning environment and the practices we hoped they would facilitate. A third adaptation involved focusing on interactive teaching and direct instruction as the two teaching components of greatest interest to research a synchronous online environment. These two aspects of teaching account for the ways instructors intervene to structure interactions and to focus those interactions on content in synchronous moments. The online learning environment afforded us a comprehensive picture of these actions, as all teacher-participant interactions took place in an environment that was video-recoded. From this study, we can reach a tentative conclusion that online synchronous platforms can be sites of teacher learning and research.

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"I MUST BE A GLUTTON FOR PUNISHMENT": TEACHERS' EMOTIONS RELATED TO VIDEORECORDING OF MATHEMATICS INSTRUCTION

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Along with the rapid development of technology, videorecording teachers' lessons for professional development or educational research have become commonplace. Although the benefits of the use of videorecording has been well documented, few studies have attended to teachers' affective experiences in relation to videorecording. In this study, we examined teachers' emotions to being videorecorded and watching their videos. We found that all teachers experienced negative emotions related to videorecording at the initial stage of the PD but these negative experiences faded over five coaching cycles. Despite the negative emotions all teachers found the use of video tremendously beneficial to their own professional development.

Keywords: Emotions, Videorecording, Mathematical coaching, Professional development

The Role of Video in Professional Development

The value of videorecording in teacher education and professional development has been widely acknowledged (Gaudin & Chaliès, 2015; Marsh & Mitchell, 2014). Having access to video of one's own instruction allows teachers to re-watch the teaching episodes and analyze the aspects of their teaching that go beyond just recall (Roller, 2016). By providing teachers with a clear picture of what their teaching looks like, videos provide essential information about their practices that allows teachers to craft a professional vision oriented towards developing necessary skills critical for appropriately noticing and attending to students' academic needs (Sherin 2004; Sherin et al. 2011). Having continuous experience with watching videos enhances teachers' abilities to observe, identify, and interpret classroom actions (Coffey, 2014; Krammer et al., 2006; Sherin & van Es, 2009; Star & Strickland, 2008). Gaudin and Chaliès (2015) pointed out that, "viewing a classroom video engages [the] teacher in a complex activity that elicits cognitive, emotional, and motivational processes" (p. 46).

Chan et al., (2018) suggests that videos engage teachers in retrospection about their own practices, while minimizing the "cognitive and emotional involvement they experience while teaching" (Chan et al., 2018, p.193). However, others (e.g., Kleinknecht & Schneider, 2013) suggest that teachers' do engage emotionally and motivationally when watching videos, but this engagement is higher while watching the videos of unknown teachers. Although having opportunities to watch and reflect on instruction is generally useful for teachers' professional development, there is added value in engaging in this practice around one's own teaching (Zhang et al., 2011). Gradual exposure to similar experiences helps teachers get accustomed to recognizing elements in their existing practices that need improvement (Borko et al., 2008). However, despite the benefit of reflecting on your own instructional video, researchers found that the teachers usually feel uncomfortable with being filmed or being watched by others (Borko et al., 2008; Lasagabaster & Sierra, 2011; Sherin & Han, 2004). Hence, public viewing of teachers' videos requires building a safe and friendly community of support. As pointed out by Borko et al. (2008, p. 422), "to be willing to take such a risk, teachers must feel a part of a safe and supportive professional environment. They also should feel confident that showing their videos will provide learning opportunities for themselves and their colleagues, and that the atmosphere will be one of productive discourse." Although watching videos can elicit a range of negative emotions, they provide evidence of the complexity of classroom practice and make student thinking visible (Barnhart & van Es, 2015; Santagata & Yeh, 2013), thereby motivating teachers to continue with effective practices and innovative to address issue they observe that deter positive student outcomes (Siry & Martin, 2014; Sun & van Es, 2015).

Teaching is emotional work (Hargreaves, 2000; Nias, 1996, Sutton & Wheatley, 2003). Studies of teachers' emotions has increased in recent years because of their influence on teachers' motivation and subsequently their behaviours (Mesquita et al., 1997). Teachers' negative emotions (e.g., anger and frustration) (Emmer, 1994) have been found to negatively influence teachers' focus and attention, thereby reducing their intrinsic motivation to teach (Ryan & Deci, 2000; Trigwell, 2002). On the other hand, emotions, like joy and satisfaction, assist in generating effective ideas and strategies (Sutton & Wheatley, 2003). As such, emotions are deeply connected to teachers' cognitive and psychological processes which influences their instructional outcomes. In this regard, exploring teachers' emotional experiences related to videorecording – including being videotaped and watching the videos – will provide insight into the range of emotions teachers' experience in relation to videorecording and the reasons underlying these emotions. A significant number of studies have documented the positive influences of video on teachers' professional growth, however, these findings of this study will provide insight into the extent to which elementary teachers may experience negative emotions related to videorecording; the extent to which elementary teachers experience positive emotions related to videorecording; what triggers these emotions; and, whether or not, they may influence what and how a teacher instructs. We focus specifically on answering the following research questions:

1. How do elementary teachers describe their emotions in relation to videorecording their instruction and watching their videos? How do they describe these emotions?
2. What reasons do teachers describe for their emotions related to videorecording their mathematics instruction?

Method

Participants

The participants included seven elementary (grades K-6) teachers working across three different schools within the same district. Table 1 shows demographic data on the teachers.

Table 1. Demographic data on participants

Teacher	Position	Gender	Grade Level	#of years of teaching
Bill	Special education teacher	Male	3rd	5
Sandra	Special education teacher	Female	Kindergarten	6
Laura	Elementary grades teacher	Female	4th	9
Anthony	6^{th} grade math teacher	Male	6th	15
Wilma	Elementary grade teacher	Female	2nd	10
Katie	Kindergarten teacher	Female	1st	19
Jessica	Elementary grade teacher	Female	2nd	5

Holistic Individualized Coaching (HIC)

This study was situated in a larger study that unfolded over a year. Participants were involved in a year-long professional development (PD) program, that involved a coaching model called Holistic Individualized Coaching (HIC) (Author, 2019). Holistic Individualized Coaching (HIC) is a coaching model designed to attend to the multi-dimensional aspects of teaching with the goal of advancing teachers’ instructional practices and overall professional well-being. For an academic year, the participants engaged in five cycles (5) of coaching that focused on enhancing their mathematical knowledge for teaching (MKT), shaping productive mathematics-specific beliefs, developing emotional regulation strategies, and promoting efficacy calibration. It involves five steps: (i) a pre-coaching discussion of a lesson to be coached, (ii) development of a content-specific mini teacher profile, (iii) third, pre-lesson support, (iv) in-class coaching where the instruction is videotaped, and (v) the post-coaching conversation which focused on data from the videotaped lesson. The teacher and coach watched the videorecorded lesson before the post-coaching calls. Pre-coaching and post-coaching conversations were audio-recorded.

Data Sources

Audio-recordings of post-coaching conversations. The post-coaching conversations provided data about participants’ mathematical knowledge for teaching, emotions, efficacy, and their teacher role during the lesson. Regarding emotions we specifically asked teachers to describe their emotions related to the video-recording. We also probed to determine the underlying reasons for the emotion. For this study we specifically focused on data related to the teachers’ emotions about videorecording. Each of seven teachers engaged in five coaching cycles that had one post coaching conversation where they talked about emotions related to videorecording. In total, there were 35 possible instances where teachers described their emotions and the reasons underlying these emotions.

Analyses and Findings

(1) What emotions do elementary teachers describe in relation to having their instruction videorecorded and watching their instructional videos? How do they describe these emotions?

To answer this research question, we focused on the emotion they stated, in relation to being videotaped and watching their own instructional videos and used thematical analysis (Braun & Clarke, 2006) to determine their emotion categories. To value the participants experiences, we used their own words. In instances where the word or term used was not a documented emotion, we determined by the category of emotion by analyzing the context in which it was stated. Of the 35 possible experiences (5 coaching cycles x 7 teachers), teachers reported emotions on 30 experiences. For five instances the teachers did not state an emotion related to videorecording. Figure 1 shows the words and phrases teachers used to describe their emotions related to videorecording – including being videotaped and watching their instructional videos.

Teacher	Coaching Sessions				
	Post-coaching 1	Post-coaching 2	Post-coaching 3	Post-coaching 4	Post-coaching 5
Kate	Pretty good/Ok/not anxious Neutral	Not too anxious Negative	Feel good/Curious Positive	Usually don't care Neutral	No emotional reaction/Pretty comfortable Neutral
Wilma	Unbothered Neutral	More comfortable Positive	N/A	Fine Neutral	Struggled/Relief Mixed-C
Bill	Uncomfortable Negative	n/a	A little bit of anxiety Negative	Pretty calm/Never excited Neutral	Easier to watch Negative
Laura	Positive outlook Positive	Nervous Negative	Avoided watching/Pleased Mixed -C	Not anxious anymore Neutral	Frustration Negative
Antwon	Felt good/Anxious Mixed -C	N/A	Anxious/useful Mixed-C	Reflective Neutral	Appreciative Positive
Sandra	Nervous/Uncomfortable /Better Mixed-C	Enjoyment/better Positive	N/a	Dislike/Fine Mixed-C	Comfortable Neutral
Jessica	Anxiety/Relief Mixed-C	Intimidating/self-conscious/Comfort/ More calm/less nervous/less anxious Mixed -C	N/a	Excited/Anxious/Comfort Mixed-C	Comfort Positive

Figure 1. Teachers' descriptions of their emotions related to videorecording

Very few of the emotions stated aligned with the discrete emotions described in the literature, such as enjoyment and anxiety (Barrett, Gendron & Huang, 2014; Frenzel, 2004). In particular, there was one instance of enjoyment (which was mixed); no instances of anger or excitement; and, no instances of pride, shame, or guilt. Teachers tended to describe their emotions in non-typical ways, such as (a) in reference to a previously felt emotion, "easier to watch" or "feeling better" and (b) using a negative emotion in a positive way, for example "not bothered" or "not anxious anymore". We also observed that teachers talked about their experiences with multiple emotions. For example, when talking about the video-recording experience, Jessica described three emotions, "excited-anxious-comfort". We labeled teachers' experience of multiple emotions as mixed emotions. Figure 2 shows the categorizations of the emotions.

Teacher	Emotions			
				Mixed
	Positive	Negative	Neutral	Complex
Katie	Feel good/Curious	Not too anxious	Pretty good/Ok/not anxious Usually don't care Pretty comfortable	
Wilma	More comfortable		Unbothered Fine	Struggled/Relief
Bill		Uncomfortable A little bit of anxiety Easier to watch	Pretty calm/Never excited	
Laura	Positive outlook	Nervous Frustration	Not anxious anymore	Avoided watching/Pleased
Antwon	Appreciative		Reflective	Felt good/Anxious Anxious/useful
Sandra	Enjoyment/better		Comfortable	Nervous/ Uncomfortable/Better Dislike/Fine
Jessica	Comfort			Intimidating/self-conscious/Comfort/ More calm/less nervous/less anxious Excited/Anxious/Comfort Anxiety/Relief
Total	6	6	9	9

Figure 2. Teachers' emotions about videorecording organized by category

We observed that when teachers experienced mixed emotions, they were a combination of both negative, positive and neutral emotions; for example, “anxiety-comfort-intimidating” [Jessica, coaching cycle 2]. This was in contrast to prior work on teachers’ emotional experiences related to teaching (e.g. Authors, under review) where teachers also experienced what we referred to as mixed-positive, a combination of multiple positive emotions; and, mixed-negative, a combination of multiple negative emotions. Jessica’s videorecording experience during the second round of coaching provides an example of the mixed-complex emotional experience. She stated,

> …when it started, it was, you know, just videorecording myself, it was a little intimidating. And then having people watch it. You weren't-- you weren't sure, because it's like, OK. What are they going to think?... I believe in what you say, that you're not in here to judge what's kind of going on. You're in here to help with-- or, you know…. Because, I mean, I'm not saying I'm not a little self-conscious when somebody is in here watching and listening, but I-- it's helped me work through that, because I don't want to be. When somebody comes in my room, I don't want to have to change my way of teaching or be nervous or things like that. So, it's helped me in that aspect, also. [So I’m] a little bit more calm, less nervous, less anxious.

In the excerpt Jessica described her initial feelings of intimidation about the videorecording, concerned about whether she would meet expectations. She talked about feeling a bit self-conscious about being watched but she felt comfortable knowing that the coach was in the classroom to provide support, not to judge. She had been actively working on regulating her emotions so she was relatively calmer and less nervous about being videotaped.

Along with neutral emotions, mixed-complex were the most predominantly felt emotions related to videorecording stated by teachers in nine instances each. Teachers also experienced positive and negative emotions with similar frequency – six instances each. Teachers appeared to become more neutral or positive about videorecording over time or repeated exposure, although they didn’t end the year having positive emotions. This may be due to the reasons underlying the emotion – why they were feeling these emotions. We describe these reasons next.

(2) What reasons do teachers describe for their emotions related to videorecording their mathematics instruction?

To answer the second research question, we identified teachers’ responses related to their emotions about videorecording and the reasons underlying these emotions. We coded each of these reasons to reflect what elicited the emotion. We then looked across the codes and organized them in themes that captured the core aspects of teachers’ videorecording experiences including: (i) teacher self-related, (ii) student-related, (iii) teaching-related, (iv) coach-related, and (v) videorecording related. Table 2 shows types of emotions (row) in relation to reasons (column).

Table 2. Reasons Underlying Emotions

Emotions	Reasons				
	Teacher (Self)-related	**Student-related**	**Teaching-related**	**Coach-related**	**Videorecording-related**
Positive	Knowing teacher self Appreciation of the entire process	Opportunity to enhance learning opportunities for students	Opportunity for reflecting on and improving instructional practices Objective	Conversation around the video is helpful	Video captures crucial moments in teaching not accessible in memory

			evaluation of teaching		
Negative	Discomfort with watching self on video (hearing voice and seeing self)	Makes students’ thinking visible	Unsatisfied with lesson Discomfort with watching instruction process	Pressure to perform for coach Ensure they meet coach’s expectations	Video reveals the unknown about students’ thinking, behavior and teaching
Neutral	Objective evaluation of teaching	Provides window into students’ thinking	Opportunity for reflecting on and improving instructional practices	Will not be judged by coach Comfort with conversation around the video	Comfort with video in the classroom over time Video does not record personal thoughts
Mixed-Complex	Reasons including a combination of those described from each of the above categories				

Positive. Reasons for positive emotions were aligned with a mastery goal orientation perspective (Elliot, 2005) where teachers saw the videorecording experience as a way to learn more and improve their instruction in ways that would benefit students’ learning (Maher et al, 2014). Sandra and Anthony’s statements capture the teachers’ experiences well.

> Teachers don't get this. You do have to watch yourself when you were student teaching, but then you had no experience so you couldn't really use it very well. Whereas now I've been teaching only seven years, but it's nice to see myself and what I'm doing. I really enjoyed that part of it and it's great. There's nothing I don't like about it. Because you've [the coach] provided me with ideas and you also tell me some things I can work on or things that you like. Just the feedback is great. I really liked it. [Sandra, Coaching Cycle #2]

> I don't want to mess up the concept, as far as presenting it to the students. Number one, I don't want myself as a professional to look bad. And then, number two, I don't want the students to get the wrong information, or for me to tell students something that's inaccurate. And so the video documents that. If I said something that was inaccurate or if I taught the students something that was wrong, it's documented. [Anthony, Coaching Cycle #5]

Teachers appreciated the videorecording as it allowed them to see who they were as teachers and how that was enacted in the classroom. They allowed the teachers to see critical moments in their teaching that were not accessible through memory and with enough detail that would be useful. Watching the videos also allowed for meaningful conversation around their teaching that was data-driven. One central reason was that being able to watch their teaching themselves, teachers did not have to rely on the feedback of others, or interpretations from their own memory, but allowed for a more objective self-evaluation of their teaching.

Negative. Teachers experienced a range of negative emotions related to videorecording including, “uncomfortable”, “a bit of anxiety”, “nervousness”, “frustration”, and “more comfortable”, in relation to being videotaped, the anticipation of watching the video, and watching the video. Most teachers experienced discomfort in seeing and hearing themselves on video and felt anxious thinking about what would possibly be revealed on the video that they did not observe during instruction. Sandra, for instance, reported her sense of discomfort to seeing herself teaching on the video:

> Yeah, and then hearing your voice, maybe no one likes to hear their voices. I'm so high-pitched, I'm like really, no wonder they don't listen to me. I don't have that voice, down and deep [Sandra, Coaching Cycle #1]

Wilma, Anthony, and Jessica also discussed the presence of the camera seemed to set both teachers and students in performance mode – making them more conscious of their actions and perhaps diminishing authenticity. Willa discussed her feeling to "perform" in front of camera:

> Instead of probably conducting my class in a way that maybe I typically would, it's like I got to speed this through to the end because by gosh, this is what she's [the coach] here to see... It's still--I'm getting more comfortable with having the camera there, but I still feel like I'm performing versus just teaching somewhat. [Wilma, Coaching Cycle #2]

Neutral. Several teachers reported neutral emotions such as "calm", "not bothered", "not anxious", and "comfortable", as they did not feel they were being judged about their teaching (e.g., Katie). In recognizing that his calibration between his thoughts about the lesson and the actual lesson was misaligned, Bill had feelings of calm about the videotaping process.

> I was pretty calm today. Yeah, I think when I think the lesson goes terribly, I watch the video and go, oh, it wasn't as bad as I thought. And when I think the lesson goes really well, I watch the video, and I was like, oh, it wasn't as good as I thought either. It's always somewhere in the middle of what I actually perceived it to be. [Bill, Coaching cycle #4]
>
> I mean, does anybody really like the way they look on video? I don't know...Yeah. So I didn't even think about that. Yeah, I don't really mind being recorded because I don't think that anybody will ever see that in the way that they would judge me. Like only teachers or people involved in education would ever look at that-- which makes it way more comfortable [Katie, Coaching Cycle #5]

Mixed-Complex. Mixed-complex emotions include a combination of positive, negative, and/or neutral emotions in relation to the videorecording experience. As shown in the quote below. Sandra felt initially nervous about watching herself on video but seeing her instruction made her feel great because it turned out well. She also expressed some generalized discomfort about watching herself on video. Sandra's range of emotions reflects the complexity of calibration and evaluation processes before and during watching videos.

> So I was nervous, and then so I was able to watch it by myself, which was great. but yeah, I was really nervous at first. And being that you never see me teach, you don't want to look like you don't know what you're doing. It turned out good. I personally like the whole process, because when you have a regular evaluation in the classroom-- well, you have lesson plans, and you turn it in, but you never really talk about it. you never know what day they are going to show up anyways. I guess I just felt like I was more prepared doing this this way, and it made me really think it through. Really doing it, like when you said, what activities will you have? What are you expecting from them? And then watching, which is just very uncomfortable, for anybody watching it, and seeing yourself in the video. [Sandra, Coaching Cycle #1]

Emotions about Watching Video vs. being Videotaped. One interesting point to note is that five of the seven teachers distinguished their emotions about watching videos and being videotaped. They expressed 23 instances within three major types of emotions about being videotaped: positive emotions (6 instances, e.g. "I think it's [videorecording] one of the best things going on" Anthony C4); neutral emotions (12 instances, e.g. "I was okay being videotaped…comfortable, Sandra C5); and negative emotions (5 instances, e.g., I feel like during my lesson especially when you guys are there, when it's being videotaped, I'm paying extremely close attention to things that, you know, every little detail that is going on, Jessica C4). Reasons underlying these negative emotions include the first time of being videorecorded, perceiving videorecordings as performances, and the anxiety of meeting instructional expectation. These concerns and negative emotions were not reported in emotions related to watching videos.

Discussion

One key finding was that the emotions teachers described about videorecordings do not align with the kinds of discrete emotions described in the literature. Instead of using the emotion words such as “anger”, which is one of the most reported emotions in prior studies (e.g. Sutton & Wheatley, 2003), or “anxiety”, teachers used words and phrases like “avoided watching” or “intimidating” to explain the negative emotion they felt about videorecording. Moreover, teachers’ descriptions of their emotions about videorecording did not mirror the emotions related to teaching described in the literature. For example, the teachers used words such as “better, feel good, felt better” more in expressing their positive emotions around videorecording, while the most commonly used words for positive emotions found in the literature in relation to teaching are “enjoyment” and “pride” (Frenzel, 2014).

While there was greater consistency across teachers in moving towards neutral and positive after cycle 3, some of our teachers did not get “used to” the videorecording over 5 cycles even though they appreciated the benefits of it. Although they became more comfortable or positive over time, most of the teachers did not feel positive emotions by coaching cycle 5. For example, Bill expressed that he found watching the recordings of his instruction “easier to watch” meaning he still did not feel comfortable about watching the videorecordings. Borko and colleagues suggest that gradual exposure to similar experiences helps teachers to get accustomed with recognition of the elements in their existing practices that need improvement (Borko et al., 2008). This suggests some teachers need more experience with the videorecording so their respective emotions are neutral or positive.

One important take-away for us as professional developers and researchers is that although teachers experienced negative emotions, they still found the experience valuable and they wanted to continue the program because of the perceived benefits. This implies that teachers have professional capacity to prioritize their learning and growth, even if it comes with negative emotions. Researchers and professional developers should pay close attention to teachers’ emotions when videorecording is utilized, so that teachers’ negative emotions are addressed in a way to maximize their learning. Another point to consider is the relative nature of emotions when multiple PD sessions are used over time. As shown in our data, teachers often reported their emotions in the current video recording in comparison to their earlier video-recordings. Thus, professional developers need to listen and attend to teachers’ emotions carefully from the onset of their collaborative work to create a rapport, engendering trust and showing genuine interest in supporting teachers in their work. In so doing, we can more effectively utilize available tools and resources, like videorecording, to foster teachers’ continued engagement in professional development.

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RECONCILING LOCAL CONTEXTS AND EXTERNAL CONCEPTUAL RESOURCES IN MATHEMATICS TEACHERS' COLLABORATIVE SENSEMAKING

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Mathematics teachers develop understandings about instruction across multiple settings, such as classrooms, workshops, and professional learning communities. When teacher teams collaborate, their prior teaching and learning experiences meaningfully inform their sensemaking. However, current research does not explicitly link teacher conversations and these multiple settings for learning. In this study, we seek to understand secondary mathematics teachers' collaborative learning in schools as part of broader teacher learning ecologies. Using discourse analysis and a comparative case study design, we examine how two teacher teams' conversations recruit external conceptual resources to support the development of their collective pedagogical judgment. In particular, these external resources offered the teams rich representations of practice and productive framings of teaching problems.

Keywords: Teacher Education - Inservice / Professional Development, Communication, Teaching Tools and Resources, Learning Theory.

In a recent review of research on teacher collaborative discourse, Lefstein and colleagues (2019) noted that researchers in this nascent field rarely attend to broader contexts of teacher conversations; "rather, they primarily focus on the immediate context of the setting or intervention" (p. 6). This stands in sharp contrast to calls to develop theory and methods to study teacher learning as distributed across contexts (e.g., Kazemi & Hubbard, 2008). For example, consider Borko's (2004) AERA presidential address:

> For teachers, learning occurs in many different aspects of practice, including their classrooms, their school communities, and professional development courses or workshops. It can occur in a brief hallway conversation with a colleague, or after school when counseling a troubled child. To understand teacher learning, we must study it within these multiple contexts, taking into account both the individual teacher-learners and the social systems in which they are participants. (p. 4)

Borko's call resonates with the PME-NA Conference's theme of "manifestations across different cultures, places and contexts," pointing to the need to develop tools to look across contexts for making sense of learning. It also aligns with our own experience as teacher educators (Buenrostro & Ehrenfeld, 2019; Marshall & Horn, under review) and our overall goal to develop better ways to design, facilitate, and analyze teacher collaboration toward improved mathematics instruction.

For the past four years, we have been supporting teacher learning through conversations with mathematics teacher teams in their schools. In these conversations, we noticed that teachers often build their ideas on what we refer to as external conceptual resources. We define external conceptual resources as frameworks, tools, and concepts that transcend different teacher learning environments, which are external to the local school context and echo voices external to the local team. Examples of such resources include teaching practices from workshops, curricula from previous schools, or teaching strategies introduced at a conference.

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We have multiple motivations for looking at mathematics teacher learning across contexts. First, calls to study teacher learning across multiple contexts (e.g., Borko, 2004; Kazemi & Hubbard, 2008; Lefstein et al., 2019) coincide with recent moves in the learning sciences away from seeing learning as tied to specific places to learning that is distributed across environments — multiple settings for individuals' sensemaking that are sometimes referred to as learning ecologies (Barron & Bell, 2016). When they are invoked, external conceptual resources often echo voices from these other contexts; by attending to them in our analysis, we offer a method to study teacher conversations as they are embedded in broader learning ecologies. Second, we think there is practical value in analyzing the particular role of external conceptual resources in mathematics teachers' collaborative sensemaking; understanding how teachers recruit these resources in their sensemaking can illuminate productive ways to design things like professional development, curricular tools, and analytic frameworks for teaching with the goal of supporting their productive use in schools and classrooms. Thus, our main goal in this study is to better understand how external conceptual resources contribute to mathematics teacher collaborative sensemaking.

Theoretical Framework: A Situative View on Teaching and Teacher-Learning

We take a situative perspective on mathematics teachers' learning which supports investigations into activity systems and contexts that shape meanings (Borko, 2004; Greeno, 1998; Horn & Kane, 2015). Teachers constantly make sense of pedagogies and educational reforms in the context of their classrooms and amidst wider socio-historical forces (Coburn, 2001; Horn & Little, 2010). For example, Marshall & Horn (under review) found teachers' uptake of learning from professional development (PD) workshops is largely influenced by the goals and demands in their local teaching situations. As mathematics teacher educators, we aim to design environments that present teachers with opportunities to learn (OTLs) which, in turn, will support their instruction.

To study mathematics teachers' OTLs in conversations about instruction, we follow Horn & Kane (2015) who emphasize how rich conceptual resources support rich OTLs. In previous analyses of conceptual resources in teacher conversations, researchers attend to teachers' representations of practice and productive frames for problems of practice as being consequential in how teachers make sense of and improve instruction (Bannister, 2018; Brasel et al., 2016; Hall & Horn, 2012; Horn & Kane, 2015; Vedder-Weiss et al., 2018). Representational infrastructures are "technologies, ways of talking, and materials that support how people engage with conceptual practices in their activity" (Hall & Jurow, 2015, p. 174). Representations of practice are a part of these infrastructures that make different aspects of teaching more or less visible (Little, 2003), and they are critical for considering alternative ways of working in the future (Hall & Horn, 2012). Framing is a discursive process by which meanings are generated by participants to imply what are relevant and legitimate ways of understanding and discussing a situation (Goffman, 1974). Vedder-Weiss and colleagues (2018) argue that productive, collaborative framing of problems of practice links teaching, learning, and subject-matter, creates opportunities to rethink practices, and positions teachers as having the power, authority and responsibility to cope with the problem. Bannister (2018) underscores that looking at collaborative problem frames is a useful analytic tool both in identifying problems (diagnostic frames) of practice and in suggesting solutions (prognostic frames). In this study we contribute to the research of conceptual resources in mathematics teacher conversations by investigating how, through teachers' references, external conceptual resources have the potential to support both rich representations and productive framing of practice, which in turn can provide the collaborative team with rich learning opportunities.

In our analysis of teacher learning, we draw on Horn's (2020) conception of teacher learning as the development of pedagogical judgment to inform our research question: How do external conceptual resources contribute to teachers' pedagogical judgment? In our work, pedagogical judgment consists of three interrelated but analytically distinct components: (1) pedagogical action: choices teachers make, intentional or not; (2) pedagogical reasoning: different interpretations and rationales supporting actions; and (3) pedagogical responsibility: teachers' sense of ethical or situational obligations. In this analysis, we look for evidence of teachers' efforts towards the alignment of these three components during their development of pedagogical judgement.

Data and Methods

Research Context and Data

To investigate the role of external resources in mathematics teachers' collaborative sensemaking, we use data from a larger research-practice partnership between our research team and a professional development organization (PDO) for secondary mathematics teachers, where we designed a system for video-based feedback on teachers' instruction. We facilitated and filmed 33 cycles of classroom observations followed by a lesson-debrief conversations with teacher teams in their schools, organized around videos of teachers' classrooms. The primary goal was to use the classroom video to elicit, engage, and develop secondary math teachers' pedagogical judgment.

Data Analysis

Phase 1: Conceptualizing teacher conversations as part of a larger teacher learning ecology and selecting cases. During our analysis of the lesson-debrief conversations, we noticed that teachers often built their ideas on the external conceptual resources we defined earlier. Our primary units of analysis were episodes of pedagogical reasoning (EPRs), which are segments of conversation participants reason about an issue of instruction (Horn, 2007). We created an inventory of EPRs in which we noted what, when, where, and by whom such resources were referenced. In building this inventory of EPRs, we started to better understand how the debrief conversations are discursively connected to other settings in the larger teacher learning ecologies (see Figure 1). For this paper, we chose to analyze cases around two conceptual resources from contexts often mentioned in teacher conversations: experiences in previous schools and workshops. We selected these two cases from Noether High School and Rees Middle School (see Table 1, all names of schools and teachers are pseudonyms) because they provide exceptionally illuminating examples of ways teachers reason with and about external conceptual resources in their respective local contexts.

Table 1: School Context Summary

School	Student Demographics	Debrief Participants
Noether High School	60% Latinx, 15% African/African American, 15% Asian/Asian American, 10% White, 5% Filipinx	Teachers: Brad (filmed), Marisa and Greg. Researchers: Lani and Nadav
Rees Middle School	80% Latinx, 5% Asian, 5% White, 5% Filipinx, 5% African/African American	Teachers: Ezio (filmed) and Veronica. Researchers: Patty, Lani and Nadav

Phase 2: Analyzing the intersection of local contexts and external conceptual resources. Using interaction analysis methods (Jordan & Henderson, 1995), we looked closely at videos of each of the two debrief episodes. To pursue our overarching research question, we asked a number of sub questions that allowed us to understand the meanings the teachers were negotiating and (re)constructing; these questions included: How are the resources referenced in teachers' conversations? How are they taken up (possibly with transformations) by other participants (Goodwin, 2018)? How do they support reasoning about teachers' goals and responsibilities (Horn,

2020)? How do they offer (or not) adequate representations of practice (Hall & Horn, 2012; Little, 2003) and productive problem framings (Bannister, 2018; Vedder-Weiss et al., 2018)?

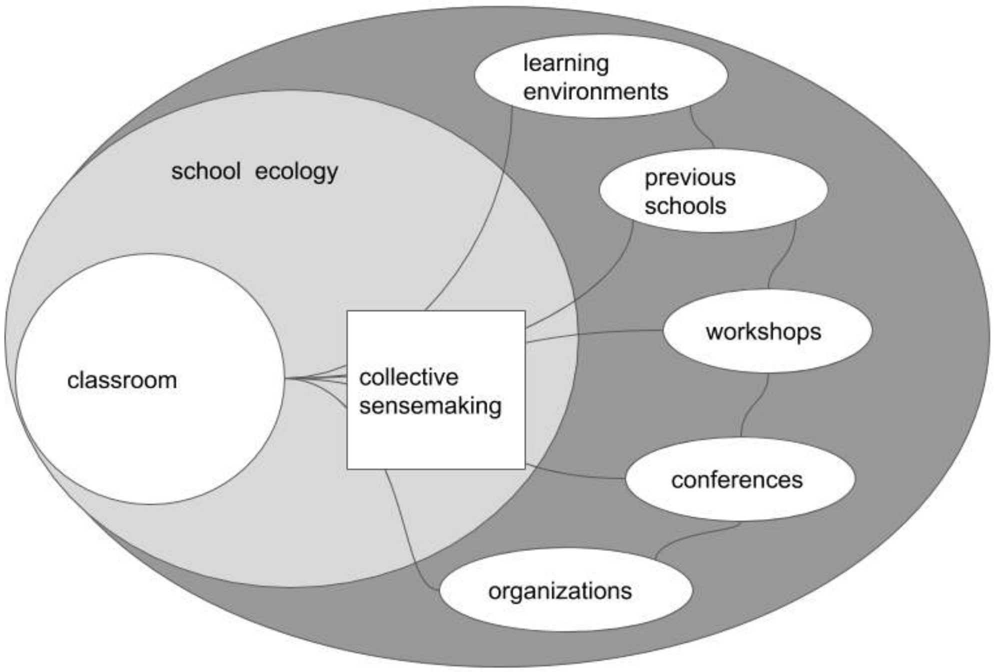

Figure 1: A Schematic Representation of Collective Sensemaking as Part of Teacher Learning Ecologies

Findings: Honing Pedagogical Judgment by Reconciling Local Contexts and External Conceptual Resources

In the focal debrief conversations, external conceptual resources supported the development of teachers' pedagogical judgment by affording richer representational infrastructure and more productive problem frames, which in turn increased alignment among teachers' pedagogical action, reasoning, and responsibility. We illustrate this process in the following cases, where teachers collectively reconciled local problems of practice by mobilizing external resources for their sensemaking. Table 2 summarizes the two cases.

Table 2: Case Studies Summary

External Conceptual Resource	Recruited From...	Local Context	Afforded...
College Preparatory Math (CPM) curriculum	Marisa's previous school	Noether debrief conversation May 2018	representation of teaching as establishing learning environment. framing of the problem of students' mathematical agency around processes of building classroom culture.

Random Grouping and Purposeful Grouping as instructional practices	Park City Mathematics Institute (PCMI) workshop and Kagan training	Rees debrief conversation December 2017	representation of teaching as entailing judgments between contrasting structures. framing the problem of labeling students by underscoring local school context and teacher agency.

EPR 1: Representation and Framing of the Problem of Leveraging Students Agency

The first case is of a debrief conversation at Noether High School. After filming Brad's Algebra 1 classroom, we returned to debrief the lesson with him and his school-based team consisting of Brad, Marisa, and Greg. We introduced clips from this lesson to discuss Brad's topic of interest which was the way he provides feedback to students. The first clip featured Brad giving a group of students a strong cue about how to proceed with the problem. Following Brad's comment "I wonder if there's a better question that I could have asked" the research team and teachers in the debrief then suggested alternative interactions that would promote more independent student thinking. For example, Greg suggested "why do you think that?" and Marisa suggested "how did you figure that out?" as alternative responses. Important to our analysis is that at this point, the conversation was focused on illustrations of teacher-student interactions, and that the team was motivated by a shared responsibility to leverage students' mathematical agency.

Problem-based curriculum as an external resource. After some comments about the content of the task, Marisa referenced her experience of teaching a problem-based curriculum, College Preparatory Math (CPM), in her previous school:

> I'm looking forward to hmm possibly next year having a problem-based curric- well, I don't know if we're going to teach geometry- what we're going to teach, but if we do teach geometry, if we could possibly use the CPM curriculum because one of the reasons I like that particular curriculum is because it's in groups from day one. Every day students are working in groups. That's the culture of the classroom that's built up. So, they're creating the meaning from doing the problems and developing the mathematics by the problem solving that they're doing and the teacher's kind of just there facilitating and you're walking around the whole time asking the questions and guiding if they need it, but it's all coming from the students all the time.

Marisa describes her experience in a classroom where students have mathematical agency and see themselves and their peers as resources of mathematical knowledge. She attributes it to the problem-based curriculum, but more specifically to the structures (such as working every day in groups) that contribute to a collaborative classroom culture. Marisa then contrasted this experience with their current classrooms' situation, where they are "going kind of back and forth" between direct instruction and groupwork. She perceived that students in their current classrooms get frustrated when teachers don't hand over answers, as in her narration of students saying—"well, just tell me, just tell me what it is." The episode ended with others acknowledging that the problem involves the disruption of traditional classroom norms, and that even with a problem-based curriculum, students are always "going to struggle the first time" (Greg).

Representation and framing. In the Noether team's debrief, the teachers discussed the focal teacher, Brad's, groupwork facilitation and ways to leverage students' agency and communication. The representation of teaching made visible in this episode began with a focus on teacher-student interactions. Then, Marisa shared her previous experience of teaching the CPM curriculum. Drawing on the external conceptual resource of CPM, Marisa supplemented this representation of teaching with aspects of designing a learning environment and reframed the problem of practice from the particularities of the teaching interactions (micro) to the construction of a certain classroom culture (macro). We see the new frame as productive in that it moved the conversation away from the problem frame of leveraging students' agency as transient and technical and yet still positioned the teachers as having the power to address it.

Pedagogical judgment. Implicit in this reframing is Marisa's reasoning that to leverage students' mathematical agency, teachers' differing responses is not enough; teachers need to engage in a more macro pedagogical action, in this case setting collaborative classroom norms from the first day. This frame offered the team images of new actions that support students' mathematical agency. We argue that Marisa's experience in her previous school with the CPM curriculum was a meaningful external resource for learning in this conversation; CPM was not used as a resource to reason about CPM. Teachers in conversation were not trying to teach CPM as a new curriculum; rather, they reasoned with CPM about groupwork facilitation, towards a better alignment of the team's pedagogical actions with their pedagogical responsibilities.

EPR 2: Representation and Framing of the Problem of Labeling Students

The second case comes from a debrief conversation at Rees Middle School. We filmed Ezio's classroom and returned after a few days to debrief the lesson with him and his colleague Veronica. We introduced clips from this lesson to discuss Ezio's topic of interest which was group dynamics. Ezio's lesson objective was to help students distinguish between linear and nonlinear equations, and he structured his 90 minutes lesson around two group tasks. Students were randomly assigned to groups of three. After listening to some student conversations in the debrief, Ezio became concerned that not all students were contributing to their group conversations, and we discussed whether providing more structure might have helped students work more productively together. For example, Ezio mentioned a structure where the student holding the marker can't talk, and Veronica wondered if group roles would have been helpful.

Random and purposeful grouping as external resources. At this point, a member of the research team (Patty) prompted the teachers to elaborate on their understanding of group dynamics. Ezio responded by initially distancing himself from the practice of random grouping used at PCMI: "at least in PCMI, I really did not agree with the random grouping." Veronica pressed him on this, saying, "But you are doing random grouping." He confirmed that he was:

Ezio:	Yeah. I let the computer pick it out. I've been trying it out. We got Kagan training
Patty:	mm-hmm
Ezio:	a couple years ago and at least what they said made sense, where it's purposeful—
Veronica:	Purposeful grouping.
Ezio:	Yeah, like a high low—
Patty:	Yep.
Ezio:	There was a structure to everything. and uh—
Patty:	so you're wondering if—
Ezio:	I was wondering, I didn't agree with PCMI but I wanted to try it out to see.
Patty:	Okay

Ezio repeated twice how he first did not agree with random grouping at PCMI. He contrasted random grouping with purposeful grouping, which was introduced to him a few years before and made sense to him. However, Ezio then narrated how random grouping became a resource for him to experiment with. He continued by connecting his experience to the local school context and to an institutional practice he saw as inequitable (tracking):

Ezio:	I know where my bias is but—
Patty:	Yeah.

Ezio: Let's try it out. and I do— one thing I do fear... (1.5-second pause) so like what's bothering me in this school, we do— ok unofficially, unofficially we track kids.
Patty: Yeah.
Veronica: Officially.
Ezio: No, unofficially.
Patty: Yeah, well, we talked about this, right? last time.
Ezio: so I've had kids tell me, "Oh, we're in the dumb class." They know; they already have that label.
Lani: Is this one of the groups of kids that is in the "dumb class"?
Ezio: No, no.
Veronica: No.
Lani: Okay.
Veronica: ...like in a Kagan, when you purposefully group, the kids automatically know.
Lani: Yeah.
Patty: Yeah, which they do.
Veronica: Whereas if you randomly group, they don't know.
Lani: Right.
Ezio: Right, so I don't want to subconsciously be telling kids, "Oh, I think you're awesome."
Patty: Yeah, yeah.
Ezio: That's the one thing I did like about the random grouping.

Ezio contrasted random and purposeful grouping twice: First, when introduced to them; while he did not agree with random grouping, purposeful grouping made sense to him. Second, when evaluating their contribution to his sense of pedagogical responsibility; even though purposeful grouping made sense to him, it functioned to reproduce what was "bothering in this school." In contrast, even though he did not agree with random grouping for its lack of structure, the one thing that he did like about it was that it disrupted the institutional process of labeling kids.

Representation and framing. An aspect of teaching that becomes visible in Ezio's representation of teaching is that teachers are constantly exposed to many different, sometimes even contrasting instructional practices that they need to compare, contrast, and reason about. This representation, supported by the reference of the two external conceptual resources of random and purposeful grouping, afforded a framing of the problem of labeling kids that (1) underscores the local school context as Ezio experiencing it; and (2) positions the teachers as having "the power, authority and responsibility to cope with the problem" (Vedder-Weiss et al., 2018).

Pedagogical judgment. Ezio used the story of his engagement with random grouping, first at PCMI and then in the ongoing work of teaching, to reason about his pedagogical actions, and to stress their connection to his sense of responsibility to disrupt what he saw as unethical tracking in Rees Middle School. Specifically, random grouping could ensure that students did not perceive themselves to be "lower" or "higher" than others with regard to mathematical placement or ability. Puncturing the conversation at multiple points, random and purposeful grouping served as resources to support Ezio and Veronica's reasoning about their instruction. Ezio described himself as a reluctant user of random grouping while carefully articulating the reasons to use the practice. By doing that, Ezio made visible pedagogical judgments as an ongoing aspect of his practice, and the team consolidated connections between their actions and their shared sense of responsibility.

Discussion

We presented cases of mathematics teachers collectively reconciling local contexts and external resources. Teachers are exposed to many different, sometimes even contrasting, conceptions of good teaching (Britzman, 2012) in their larger learning ecologies, and external resources often aid in their sensemaking with and about these conceptions. By attending to these external resources, we underscore a unique role of collaborative sensemaking opportunities in the overall teacher learning ecologies. Specifically, we build on previous situated research of teacher conversations to offer a discursive mechanism by which external conceptual resources contribute to teachers' pedagogical judgment: by affording richer representational infrastructure and more productive problem frames, teachers were able to bring their pedagogical responsibility, actions, and reasoning into closer alignment.

As a field, we came to acknowledge the importance of what the PME-NA Conference theme describes as "manifestations across different cultures, places and contexts." We see this work on external conceptual resources as a step towards developing better ways to design, facilitate and analyze mathematics teacher learning across contexts.

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EXAMINING HOW TEACHERS ENACT THE SUGGESTIONS OF A COACH: CRITIQUE OF A METHODOLOGY

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In this methodology paper, we present a methodology for characterizing how teachers use coaches' suggestions. We identified suggestions from planning conversations and explored the extent to which the teachers implemented the suggestions in enacted lessons. The planning conversations took place within online content-focused coaching cycles. A primary challenge confronting content-focused coaches when working one-on-one with teachers is finding a productive balance between giving suggestions and inquiring into teachers' practices through reflective questioning. This paper articulates a process for identifying suggestions made by a coach during a planning conversation and an analytic process for examining how a teacher takes up the suggestion during lesson implementation. We discuss the methodological challenges we encountered and tradeoffs in our decisions related to low- and high-inference claims.

Keywords: Inservice Teacher Education/Professional Development, Research Methods, Coaching

Coaching as a form of professional development is a promising practice (Campbell & Griffin, 2017; Ellington, Whitenack, & Edwards, 2017). Within mathematics education, content-focused coaching (e.g. West & Cameron, 2013) is a common model. Content-focused coaching involves iterative cycles in which a coach works one-on-one with a teacher, with a focus on students' mathematical learning goals. Each coaching cycle contains three sequential components: a pre-conference discussion to plan a lesson; a collaboratively taught lesson; and a post-conference discussion to debrief the lesson (Bengo, 2016; West & Staub, 2003).

Research on coaching has highlighted two competing stances for how coaches talk with teachers: *reflective* or *directive* (Deussen, Coskie, Robinson, & Autio, 2007; Ippolito, 2010; Sailors & Price, 2015). Coaches using a *reflective* stance emphasize collaborative inquiry in which the coach elicits ideas from the teacher; these ideas become the basis of the coach-teacher discussion (Ippolito, 2010). Coaching moves associated with a reflective stance include probing questions and low-inference, non-evaluative observations as means to catalyze teacher thinking (Costa & Garmston, 2016). In contrast, a *directive* coaching stance involves the use of suggestions and evaluative feedback (Ippolito, 2010). The challenge in content-focused coaching is to find the right balance between when to provide a teacher with direct assistance in the form of a suggestion and when to employ an inquiry stance (West & Staub, 2003). It is crucial for researchers within mathematics education to explore these stances and their impact on teacher learning and uptake of new practices.

Despite the importance of mathematics coaches strategically choosing appropriate actions when working with teachers, little is known about how mathematics coaches using a content-focused coaching model interact with teachers (Gibbons & Cobb, 2016). Furthermore, little research exists on how direct assistance from a coach during a coaching cycle supports a teacher to implement a lesson. As a first step in addressing this gap, this paper outlines a methodology for analyzing how teachers take up a coach's suggestions when planning and enacting lessons. Specifically, this study is guided by the question: How do we characterize the extent to which a teacher uses the suggestions of a coach during the implementation of a lesson? This methodology would help mathematics educators better understand the impact of a coach's suggestions on the pedagogical actions of a teacher and serve as a first step towards the larger inquiry of how the discursive actions of a coach impact teachers' practices.

Methodological Processes Applied to Coaching Conversations

To identify the suggestions made by the coach during the collaborative planning conversations, we used results from a broader analysis of the coaching conversations. These results came from a broader study that analyzed planning and debriefing conversations within content-focused coaching cycles from four coaches paired with eight teachers over a period of two years. In that broader analysis, we focused on the discursive moves of mathematics coaches during coaching cycles. We developed a codebook to analyze the transcripts of planning and debriefing discussions between coaches and teachers; this codebook characterized the discursive moves of the coaches and teachers as well as the content of the conversations. The section of the codebook that focused on the discursive moves of the coach was comprised of five broad categories; including suggestions (see Figure 1). We defined a *suggestion* as a statement from the coach recommending an action for the teacher.

Coaching Stance	Discursive Move	Description	Example
Reflective	Invitation	Statement or question that invites the teacher to reflect or respond	“What might be some strategies we could use to increase student participation?”
	Description	Statements that share a direct observation and do not contain inference, interpretation, judgement or opinion	“I noticed that during the turn-and-talk, Alex did not say anything to his partner.”
Directive	Suggestion	Statement that recommends an action	“I think we should use a turn-and-talk prior to the whole class discussion.”
	Explanation	Statements that provide an interpretation or rationale of an event, interaction, or mathematical idea	“Turn-and-talk is a powerful strategy to use prior to a whole class discussion because it increases student participation.”
	Evaluation	Statements that offer praise or critique	“I think it was a great idea to use a turn-and-talk.”

Figure 1: Excerpt from the larger codebook focusing on coaching discursive moves

We parsed the transcripts of the planning and debriefing conversations into stanzas, which included a coach’s statement and the participant’s response, as well as text needed for context (Saldaña, 2013). This broader data set included the analysis of n = 1719 stanzas from 41 transcripts of coaching conversations. We coded stanzas in pairwise teams after a lengthy calibration process that involved five researchers. We met via video conferencing software, Zoom, to reconcile disagreements. Kappas ranged from 0.39 to 0.65, considered moderate to strong reliability (Landis & Koch, 1977).

The following is an excerpt from a coach’s comments that was coded as a *suggestion*:

> One of the really nice moves you can do if the group shares a thought about something, and it’s somewhat ambiguous, is you can turn to the class and say, “Can someone else use their own words to explain what Dave is saying?”

In this comment, the coach recommended the teacher prompt students to paraphrase a peer’s explanation as a means to increase student participation in classroom discussions.

Piloted Version of Data Analysis Process for Coaches’ Suggestions

The purpose of this paper is to detail a methodological process for identifying suggestions made by coaches and characterizing how those suggestions were taken up by the teachers during lesson implementation. We wanted to understand how teachers incorporated the suggestions coaches provided during planning meetings into their teaching. We describe our current analytic attempts to highlight the methodological affordances and challenges of the work and to gain insight from others in the mathematics education community. Knowing the extent to which teachers follow coaches' suggestions in their teaching is important information for coaches as they plan to support teachers.

Coding Suggestions from the Coaching Transcripts

We collected all stanzas previously identified as involving a coach's suggestion. If a stanza contained multiple distinct suggestions (i.e. more than one action was recommended by the coach), each suggestion was placed into a different row in the spreadsheet where we tracked the suggestions, with the goal of distilling a coach's suggestions to the smallest granular size. This process converted the unit of analysis from a stanza to an individual suggestion. For example, a coach made the following statement to the teacher during a planning conversation:

> Whereas, if you really want them to be able to understand the formula, you're going to be asking different questions about, where did this come from? What do we know about volume? What does volume mean? Those kinds of more probing questions as they're working or as they're thinking about it, and as you're launching. Then, what I'm also thinking about is your ticket out the door idea. This idea of, do you want to do some checking in with students in terms of their understanding about volume related to the cylinder and the cone, either before they leave you Monday, or possibly Tuesday, so that you get a sense of, beyond just your questioning and asking each individual group, would you want to have—would it be helpful to have some kind of—some documentation to look back at in terms of students' understanding? I'm just thinking of a little, mini half-sheet, or something, if it would be, again, helpful to figure out where kids are in their thinking.

We coded this statement as two distinct suggestions. First, the coach suggested that the teacher ask different questions about the concept of volume. Then, the coach provided suggestions about the ticket out of the door. These two suggestions were listed separately.

Next, two coders individually coded each of the coach's suggestions for the content of the suggestion using the codebook (see Figure 2).

Content								
Lesson Goals	Instructional Practice	Math Content	Coaching					
Contextualizing	School/Classroom Context	Math Content or Curriculum	Students(s)					
Lesson Content	Introduction/Launch	Task Description & Sequencing	Questions/Questioning	Summary	Assessment	Adaptions	Discourse	
Lesson Features	Instructional Needs	Misconceptions/ Challenges	Anticipating	Selecting/Sequencing	Connecting	Participation Structure	Next Steps	Student Strategy
Technology	Technical Logistics/Challenges	Referencing Swivl/Annotations/Video	Use of Technology					

Figure 2: Content codebook, with categories in purple and codes in yellow.

Each coder created a concise statement capturing (a) the *general nature of the suggestion* and (b) the *specific action in the suggestion* recommended by the coach (see Figure 3). The coders then met to reconcile and reach consensus for their content codes and created statements. For example, during a planning conversation, a coach said:

> Kids are always surprised that getting one of each happens so much more often than getting two of the same. Your experimental data is definitely going to show that. What if you asked them, "If you toss a coin twice, what are the things that could happen?" You ask them that at the beginning, just to get an idea of where they are.

We applied the content codes of *question*/*questioning*, *assessment*, and *introduction*/*launch* because the coach recommended the teacher ask a question during the launch phase of the lesson to assess student thinking prior to beginning a task (see Figure 3). The general nature of the suggestion was written as: "Ask students questions during launch to assess their understanding." The specific action suggested by the coach was captured by the coders through the statement, "Ask the question 'If you toss a coin twice, what are the things that could happen?' during the launch to assess student understanding."

Stanza Excerpt	Content Codes	General Nature of the Suggestion	Specific Action in the Suggestion
Kids are always surprised that getting one of each happens so much more often than getting two of the same. Your experimental data is definitely going to show that. What if you asked them if you toss a coin twice, what are the things that could happen? You ask them that at the beginning, just to get an idea of where they are.	Assessment, Question/ Questioning, Introduction/ Launch	Ask students questions during launch to assess their understanding	Ask the question "If you toss a coin twice, what are the things that could happen?" during the launch to assess student understanding.

Figure 3: Excerpt from the coding spreadsheet

If there was not a specific action within the suggestion, the *specific action in the suggestion* column was left blank. Additionally, if the suggestion was deemed to not be observable within the lesson video, we coded the suggestion as *not observable*. For example, if the coach suggested an action for the teacher in future lessons beyond the current coaching cycle, the suggestion would be *not observable* during the lesson video.

Coding Enactments of Suggestions

The coders independently watched the video of the implemented lesson to identify how the teacher followed the general and specific suggestions from the coach. For each suggestion marked in the spreadsheet generated from analyzing the coaching transcripts, the coders worked on two levels. First, they considered if the general nature of the suggestion was *present* or *not present* in the lesson. Second, they considered the extent to which the specific action of the suggestion was taken up by the teacher. To code the specific action, they chose between the following codes that represented a continuum of uptake: *not present*, *partially adhere*, *mostly adhere*, or *fully adhere* (see Table 1).

Table 1: Coding Scheme for a Teacher's Enactment of a Specific Suggestion

Code	Description	Example (connected to Figure 2)
Not present	The teacher did not enact any part of the coach's suggestion during the lesson.	The teacher did not ask any questions during the lesson launch.
Partially present	The teacher enacted only a single part of the coach's suggestion during the lesson.	The teacher asked a question during the lesson launch but the question does not relate to assessing understanding.

Mostly adhere	The teacher enacted multiple, but not all, parts of the coach's suggestion during the lesson.	The teacher asked a question during the lesson launch to assess understanding but used a question worded differently the question suggested by the coach.
Fully adhere	The teacher enacted all parts of the coach's suggestion during the lesson.	The teacher asked the exact question suggested by the coach during the lesson launch.

In cases where the teacher did not have the opportunity to enact a suggestion from the coach, the instance was coded as *no opportunity*. For example, if the coach suggested the use of a specific teacher talk move during a whole-class summary discussion of a task (e.g. Smith & Stein, 2011) but the class period ended before the teacher was able to begin the summary discussion, we coded that as the teacher not having the opportunity to enact the coach's suggestion. The code *no opportunity* is different than *not observable* in that *no opportunity* was based on a suggestion that could be observable but was beyond the scope or inclusion of the implemented lesson due to constraints such as time or classroom happenings in the moment of the lesson. Figure 4 contains a flowchart summarizing this analysis process.

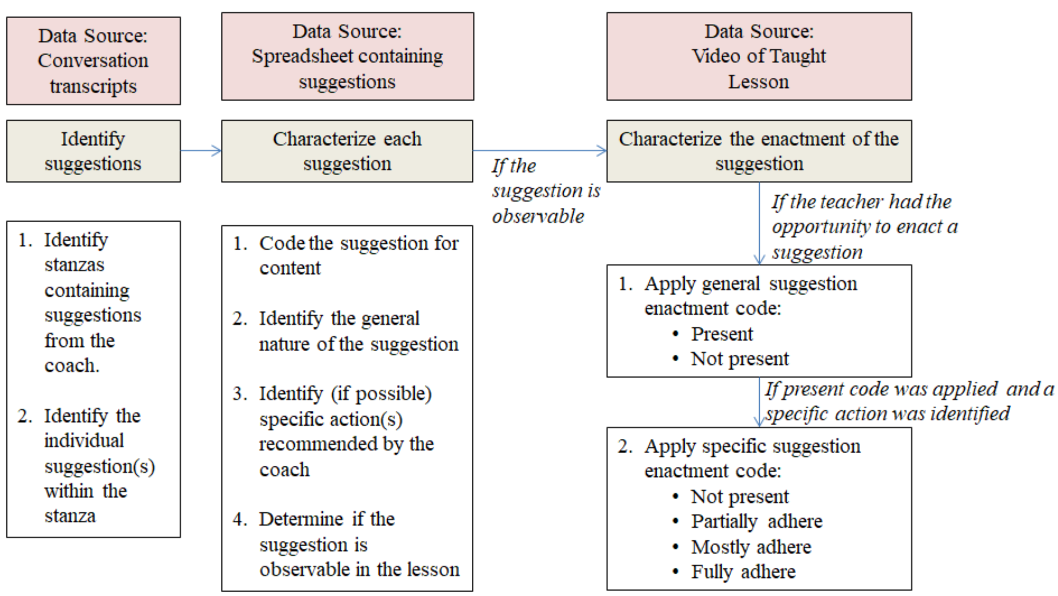

Figure 4: Flowchart of analysis process

Methodological Affordances and Challenges

In this section we describe the methodological affordances of our analysis process as well as the challenges we encountered when: a) identifying suggestions given by the coach within conversation transcripts; b) categorizing a coach's suggestions based on how the suggestion invited action from the teacher; and c) characterizing the extent to which teachers took up a suggestion when teaching a lesson. We discuss these affordances and challenges within each step of our analysis process to highlight methodological issues when characterizing direct assistance from coaches and the impact of that direct assistance on teachers' instructional practices.

Identifying Suggestions

The first challenge we encountered was establishing a reliable coding process for determining when coaches made suggestions. The coaches used a range of discursive moves when recommending an action to a teacher (Gillespie, Amador, & Choppin, 2019), making the identification of suggestions problematic. For example, a coach said:

> Another idea for your launch is, again, I don't know how I feel about this, but is to give them a pool with side length five, and actually have them work on their—or do an independent time of—and explaining the tiling. You've got this patio going around the outside of the five by five square and how many tiles would it take for that patio.

This statement was coded as a *suggestion* because the coach recommended a task and participation structure for the teacher to use during the introduction of a lesson. A similar call for action was also expressed by a coach through the question, "What if you did just one example, not of a one by one, but a five by five or something?" Even though this discursive move is a question, the recommendation of modifying an activity is embedded within the question. Thus, this statement was also coded as a *suggestion*. We coded discursive moves as *suggestions* if the move contained an explicit recommendation for action or a clear, implicit recommendation embedded within a conversational move. This decision allowed the coding process to capture a wider variety of suggestive moves from coaches. However, because this decision required the researchers to analyze conversational moves for the underlying intent, the coding entailed some inference of intent. Besides acknowledging the inferential nature of some of the coding, we have struggled with how to methodologically characterize the nature of our inferences. In summary, accurately identifying suggestions from a coach requires analyzing discourse for both explicit and implicit recommendations directed at the teacher; analyzing for implicit intent, however, causes the coding process to become increasingly inferential.

Characterizing Suggestions

The second challenge in developing the analytic process involved characterizing the actionable nature of the suggestions. Initially, we realized that many of the suggestions were broadly stated, providing considerable latitude in how teachers might interpret and use them. For example, during a planning conversation, a coach said, "You probably want to look for students that have done the different strategies so that you know who you want to share how they did it, how they organized it." In this instance, the coach recommended the teacher monitor student thinking in order to select students to share different strategies during a summary discussion, but did not offer specific guidance as to which strategies to select. In the absence of a specific action, we coded the suggestion as a general suggestion and then coded for whether or not that general action was *present* or *not present* during the teaching of the lesson. However, the general nature of the suggestion made it more difficult to consider the tangible impact of the coach's suggestions. We found other suggestions to be specific; these suggestions provided exact language describing an action a teacher should use during the lesson. For example, a coach said, "You could even ask a question about—there's eight options there. 'If you play it 40 times, how many times would you expect so-and-so to win, and how many times would you expect the other guy to win?'" In this example, the coach recommended the teacher ask questions and provided two specific questions for the teacher to use. Suggestions that were specific were easier to identify in the enacted lesson and code but also represented a more localized impact on practice. Our coding scheme allowed us to distinguish between general and specific suggestions, but we were able to make more definitive decisions for specific suggestions. This poses a dilemma: our coding of uptake may only capture highly tangible evidence of impact while missing potentially more powerful and broader impact from the more general suggestions. Our challenge can be summarized in the following way: general suggestions have the potential to convey broader pedagogical principles than more specific ones, but entail greater inference when coding.

Characterizing Take Up of Suggestions

A third challenge was determining the extent to which a teacher enacted the precise details in a specific coach suggestion. We used a continuum to code the enactments. Coding along a continuum for the presence of specific details in a specific suggestion entailed qualitative characterization of the suggestion. Our continuum had these four characterizations, from low to high presence of a suggestion: a) no part of the suggestion, b) a single part of the suggestion, c) multiple parts of the suggestion, or d) the suggestion in its entirety. As a result, we created the codes *not present*, *partially adhere*, *mostly adhere*, or *fully adhere* to use when coding the teacher's enactment of specific suggestions (i.e. Table 1). As an example of coding a teacher's enactment of a suggestion using this continuum, during a planning conversation a coach said:

> They could even try tossing it three times. You could make it into a situation where a couple kids are playing a game. One kid wins if all three of the tosses match. One kid wins if only two of them match, and is it a fair game idea?

This was coded as a specific suggestion because the coach recommended the teacher use an activity and provided specific context and questions to use in the activity. During the lesson, the teacher facilitated an activity in which students flipped three coins but did not frame the activity as a game that may or may not be fair. Because only a single part of the specific suggestion was enacted by the teacher, the suggestion was coded as *partially present*. This qualitative characterization allowed us to describe accurately the extent to which the teacher enacted the suggestion but posed challenges related to the reliability of our coding.

To limit inference and potentially support more reliable coding, our coding continuum did not evaluate the effectiveness of the enacted suggestion; if a specific suggestion was evident in the lesson, we did not then consider whether or not the enactment was productive. So, while the teacher carried out the suggestion, the way that happened, and the implications of what happened subsequently in the lesson, may not have been aligned with the coach's intent. Avoiding evaluating the productiveness of the enactment of a suggestion has implications for the ways we consider the impact of a coach's suggestion on teacher practice, but using a low-inference coding scheme had its advantages. Although the process was anecdotally deemed reliable by coders during our first round of coding, additional calibration will be needed to reliably code the enactment of specific suggestions using the four leveled codes. In summary, characterizing the ways teachers enact a specific coach suggestion requires the use of a continuum, posing challenges for reliability. This challenge can be partially mitigated by limiting focus to the ways the teacher followed the explicit language of the coach's instruction, not considering intent or effectiveness; yet this limits the extent to which the data can be used to make claims about the impact of the coach on productive practice.

Discussion

Coaching is an increasingly popular professional development practice in mathematics education; however, more needs to be known about how coaches interact with teachers (Gibbons & Cobb, 2016). We presented the methodological challenges and opportunities in characterizing how the *suggestions* of a coach impact the practice of a teacher. Focusing on coaches' suggestions is one start to more fully understanding how coaching influences teaching practice. Researchers should continue to examine how the actions of a coach impact the practice of a teacher. More specifically, because coaches use discourse as a primary tool to engage teachers (Costa & Garmston, 2016; Heineke, 2013), we anticipate similar challenges exist in identifying coaching discursive moves beyond suggestions (e.g. invitational or evaluative). Knowing how discursive moves (i.e. Figure 1, Deussen, Coskie, Robinson, & Autio, 2007; Ippolito, 2010; Sailors & Price, 2015) impact the development of teachers would be beneficial for knowing how to support teachers through coaching. Thus, the specific challenges discussed within this paper relate to more global challenges that will be encountered during any analysis of how the discursive actions of a coach impact the practice of a teacher.

By illuminating the challenges and opportunities learned through our work, we aim to support future researchers by emphasizing the complexity of analyzing the relationship between the discursive moves of a coach and a teacher's practice. For example, when identifying a coach's suggestions within varied discursive moves in a full coaching conversation, we had to consider language that explicitly and implicitly communicated a recommendation to capture the nuanced ways the coach provided suggestions. Coding is this way required us to develop rules that reliably connected the language of a coach to their intent to provide a suggestion. These coding decisions introduced inference. However, not considering intent through implicit language would have constrained our ability to identify instances in which coaches leveraged their expertise to share actionable ideas with teachers for use in a lesson. We anticipate similar challenges will be encountered in identifying other discursive moves of coaches.

The second challenge was characterizing the suggestions in ways that considered the implications on a teacher's enactment of the suggestion. We created two classifications, *general nature* and *specific action*, but found general suggestions can promote broader pedagogical principles than more specific ones, but require greater inference when coding. We posit this challenge will also apply to coding schemes attempting to capture the impact of other coaching discursive moves. For example, a coach using broad or general invitational moves may provide the teacher with more latitude to reflect, but analysis may require higher levels of inference.

Our third challenge was limiting inference when coding for the enactment of specific suggestions; accomplished in part by not considering the effectiveness of the enacted suggestion. Low-inference coding in this context has advantages but disregarding effective enactment prohibits the study from claims about how the actions of a coach connects to changes in the quality of a teacher's practice. In other words, our current process will allow us to make low-inference claims about how teachers used the suggestions of the coach but prevent us from making claims about how the suggestions influenced the quality of the lesson. Discussion of these challenges invites future mathematics education researchers to consider methods to overcome these obstacles.

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CHANGES IN PRACTICING SECONDARY TEACHERS' PROFESSIONAL NOTICING OVER A LONG-TERM PROFESSIONAL DEVELOPMENT PROGRAM

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Much of the research on the development of professional noticing expertise has focused on prospective teachers. We contend that we must investigate practicing teachers as well, and in particular practicing secondary teachers, because they bring with them years of teaching experience and are situated in unique contexts. Hence we studied the longitudinal growth of the professional-noticing expertise of a group of practicing secondary teachers (N=10) as they progressed through a 5-year professional development (PD) about being responsive to students' mathematical thinking. Results indicated that the first half of the PD supported their interpreting and deciding-how-to-respond skills, and the second half of the PD supported their attending skills, which were already strong even before the PD. We compare these results with the activities that occurred in the PD and discuss implications for future research and PD programs.

Keywords: Teacher Education - Inservice / Professional Development, Algebra and Algebraic Thinking

Introduction

Professional noticing of students' mathematical thinking is a specific type of teacher noticing expertise, and it occurs when the teacher notices a student's mathematical strategy during instruction (Jacobs & Spangler, 2017). In that moment, the teacher (a) attends to the details of the student's strategy, (b) interprets the student's mathematical understanding, and (c) decides how to respond to the student on the basis of the student's mathematical understandings (Jacobs, Lamb, & Philipp, 2010). These three component-skills occur simultaneously, are interrelated, and are interdependent. Additionally, this noticing expertise distinguishes itself from other types of noticing expertise that only include the component-skills of attending and interpreting, or that focus on issues other than the student's mathematical thinking (e.g. issues of equity, representations, or funds of knowledge; Dreher & Kuntze, 2015; Hand, 2012; McDuffie et al., 2014). It is important to understand the development of teachers' professional noticing expertise because research has shown that teachers who attend to student thinking support student achievement (e.g. Boaler & Staples, 2008; Jacobs, Franke, Carpenter, Levi, & Battey, 2007) and learn from their practice (Sowder, 2007; Wilson & Berne, 1999).

Teachers can differ in their noticing patterns depending on their experiences, backgrounds, and education (Santagata, Zannoni, & Stigler, 2007; Miller & Zhou, 2007). Professional development (PD) on noticing children's thinking can improve teacher noticing (Sherin & Han, 2004; van Es & Sherin, 2010) and has been shown to extend to the classroom (Sherin & van Es, 2009) and support teacher learning after the PD is complete (Franke, Carpenter, Levi, & Fennema, 2001). With respect to professional noticing of students' mathematical thinking, much of the research has focused on prospective teachers' development (e.g. Fernández, Llinarez, & Valls, 2012; Fisher, Thomas, Schack, Jong, & Tassel, 2018; Monson, Krupa, Lesseig, & Casey, 2018; Simpson & Haltiwanger, 2016; Tyminski, Land, Drake, Zambak, & Simpson, 2014). Few have analyzed the development of

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professional noticing expertise among practicing teachers (e.g., Jacobs et al., 2010; LaRochelle, 2018). Practicing teachers differ in important ways from prospective teachers because they bring with them years of teaching experience to PD activities and are situated in contexts that may or may not be conducive to what they learn in the PD (Levin, Hammer, & Coffey, 2009). Hence, it is important for the field to study the growth of this expertise in practicing teachers as well.

Practicing Teachers' Professional Noticing of Students' Mathematical Thinking Expertise

We found two studies that document the growth of professional noticing expertise among practicing teachers. Jacobs and her colleagues (2010) showed us that for practicing primary teachers, teaching experience alone may not adequately support robust professional noticing expertise, and that long-term, sustained PD may be necessary to develop these skills. They found significant positive differences across all three groups of practicing teachers for each component-skill of professional noticing, with attending reaching a ceiling level after 2 years. Hence, for practicing primary teachers, there is a clear need for sustained PD in order to develop this important expertise.

LaRochelle (2018) conducted an analysis similar to Jacobs et al. (2010), comparing the professional noticing expertise of prospective secondary teachers, experienced secondary teachers, and experienced secondary teachers who had completed 4 years of long-term, sustained PD about responding to students' mathematical thinking. Secondary teachers differ in important ways from primary teachers because they experience different school structures (Blatchford, Bassett, & Brown, 2011; Ferguson & Fraser, 1998) and exhibit different conceptions about mathematics, students, and teaching (Weinstein, 1989). However, LaRochelle's findings did indicate that for many practicing secondary teachers, teaching experience may not adequately support this expertise. In particular, the experienced secondary teachers exhibited similar professional noticing skills to the prospective secondary teachers, whereas the experienced secondary teachers with four years of sustained PD exhibited stronger professional noticing skills than the experienced secondary teachers.

However, it is unclear from LaRochelle's (2018) study what trajectories of development might exist for practicing secondary teachers, and what activities might support this development. Hence, we build on his study by documenting the longitudinal growth of a group of practicing secondary teachers as they progressed through a 5-year PD program that focused on being responsive to students' mathematical thinking. Specifically, we measure the growth that we saw in each component-skill of professional noticing at various points in time during the long-term PD. Consequently, we answer the following question: What changes in experienced secondary teachers' professional noticing expertise can be seen across 5 years of sustained PD about being responsive to students' mathematical thinking? Answers to this question allowed us to compare the growth we saw with the activities that occurred during the PD, and we share implications of these results in our discussion section.

Methods

Participants

Initially, we selected a cohort of 32 master teachers (16 mathematics and 16 science) for a five-year fellowship through a highly competitive application process that included analyzing student work, video clips of teaching and an interview (Nickerson et al., 2018). In addition to evidence of student-centered teaching, we sought teachers who had a disposition as a learner. When they began, the 16 mathematics teachers (who are the focus of this paper) had 2 – 30 years of teaching experience, with an average of 13 years. All teachers came from high-needs school districts in the south-western region of the United States. Due to attrition (e.g., moving, changing content areas, and the like), we have longitudinal data for 10 of the teachers.

Mathematics Teacher PD

Here we describe the nature of the PD over the five years of the PD program. We shifted the PD activities over the five years, foregrounding some activities in the first few years and others in the subsequent two years. The focus of the PD in the first few years was on further development of content knowledge and pedagogy. In the latter years, the PD was more explicitly focused on developing teacher leaders. We describe the nature of these activities over time.

In the first year, teachers watched and discussed videotapes of teaching and were introduced to the *professional noticing* skills of attending to, interpreting, and deciding how to respond to students' mathematical thinking. The PD began with a teacher educator interviewing a student to illustrate the challenges and affordances inherent in one-on-one interviews. Teachers discussed how to conduct an interview, including the importance of wait time, questioning, and avoiding directing students to a particular strategy or answer; and what one might learn from an interview, such as how students approach problems, what one can learn from incorrect responses and how those are often steeped with kernels of understanding, students' affect, and so on. The teachers, working in pairs, then interviewed secondary students.

During subsequent PD sessions, the teachers engaged in many other activities that allowed them to discuss students' content-specific ideas and how to build on those ideas. For example, in year 1 teachers solved pattern-generalization problems in multiple ways and made connections among the solutions, and discussions of student thinking naturally occurred during these activities. Teachers also discussed interviews that they conducted with students at their schools and brought artifacts of student work across a variety of content areas to the PD sessions.

In addition to activities that focused on individual students, teachers also learned about complex instruction (Cohen, 1994) and discussed issues related to facilitating group work. During the third year, teachers, accompanied by teacher educators, traveled to school sites to observe each other teach and then debrief and reflect on the experience. Teachers also coached each other in team-teaching a group of middle school students. Discussions of students' ideas were always framed using the Professional Noticing Framework.

Toward the end of year 3 and throughout year 4, we began a more explicit focus supporting teachers in learning how to lead PD. Activities included selecting artifacts and rehearsing situations they may face in their leadership practice. In year 5, they engaged in lesson study, and the lesson debriefs in the lesson study were explicitly structured to attend to and interpret student ideas before discussing how to modify the lesson.

The PD activities described above align with what the literature has shown to be productive activities for supporting teachers' professional noticing expertise (Fisher et al., 2018; Jacobs et al., 2010; Monson et al., 2018). In general, these studies have shown that activities such as doing mathematics together, learning about students' mathematical thinking through frameworks and research articles, *decomposing* (Grossman et al., 2009) the practice of professional noticing, and practicing the component-skills of professional noticing with written artifacts, video artifacts, and in one-on-one interviews with students can support teachers' professional noticing skills.

Data Collection

In May, 2013, we collected (Y0) baseline data; we repeated data collection in year 3 (Y3) and again in Year 5 (Y5). As part of the data collection, participants analyzed and responded to prompts about the Pattern Generalization Video, a video-recording of a class of middle school students engaged in a figural pattern generalization task with beams and rods (National Center for Research in Mathematics and Science Education, 2003).

Pattern Generalization Video. In the Pattern Generalization Video, students are considering a set of rods that are being connected in a way that creates a beam (see Figure 1). The rods are connected to form triangles, and the length of the beam is the number of rods that form the bottom. Students begin by finding a recursive pattern based on beams of lengths 1, 2, 3, and 4. Then, students are challenged to find an explicit rule for finding the number of rods for a beam of any length. Near the end of the video, two groups of students share. The first group sees the number of rods that make up the top of the beam (L-1 rods), the number of rods that make up the middle of the beam (2L rods), and the number of rods that make up the bottom of the beam (L rods). They recognize that the total is the sum of these three sections, and write L + (L-1) + (2L) = total. The second group to present deconstructed the beam in a different way; instead, they see a set of 4 connected rods, a triangle with a rod on top, and they iterate this pattern from left to right until they reach the last set of 4 rods, where they have to erase the last rod on top, and write the equation, total = 4L - 1. Participants have opportunities to notice ideas such as quantitative reasoning, use of algebraic symbols, meanings for operations and symbols, and other generalization concepts (Jurdak & El Mouhayar, 2014; Lannin, Barker, & Townsend, 2006).

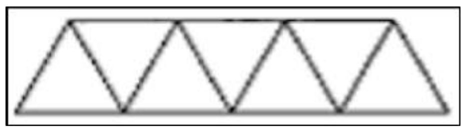

Figure 1. A beam of length 4 has 15 rods.

Prompts. After watching the video, the participants responded to the prompts listed below. The prompts were adapted from Jacobs and her colleagues (2010), and are related to the three professional noticing component-skills:

1. (*Attending*) Describe in detail what the two groups of students who presented at the board did in response to the task.
2. (*Interpreting*) What did you learn about these students' mathematical understandings?
3. (*Deciding how to Respond*) Pretend you are the teacher of these students. What problem(s) might you pose next, and why?

Data Analysis

Analysis procedures followed those of Jacobs et al. (2010). Three researchers independently coded the blinded responses for each of the component skills and then discussed codes to resolve any differences. For attending responses, researchers analyzed the extent to which participants attended to the details of the students' strategies. This coding involved identifying the important details of each student's strategy and counting the number of details each participant shared. For interpreting responses, researchers analyzed the extent to which participants interpreted the students' understandings, which involved looking for evidence that participants interpreted specific and nuanced understandings that were consistent with the students' work and with the research on students' pattern generalization skills (e.g. Jurdak & El Mouhayar, 2014). For the deciding-how-to-respond responses, researchers analyzed the extent to which participants based decisions on the students' mathematical understandings. This analysis involved looking for evidence that participants connected their decisions to the students' mathematical ideas, provided specific problems and/or number choices in the problems they selected, provided a rationale that was consistent with the student work and the research on students' pattern generalization skills, and provided evidence of anticipating or building on the students' thinking. Ultimately, each of these component skills was assigned a score of 0, 1, or 2.

Findings

In Table 1, we provide the mean scores of the ten participants, at three time points, in response to the Pattern Generalization Task, described above. Attending scores started high, remained stable at Y3, and increased to a ceiling level in Y5. Interpreting scores were modest at Y0, grew in Y3, and then remained stable. Means for deciding how to respond started low (most participants provided lack of evidence for deciding how to respond on the basis of students' mathematical ideas), then rose relatively dramatically by Y3, and remained stable.

Table 1: Mean Scores in the three component skills for Y0, Y3, Y5

	Attending			Interpreting			Deciding how to Respond		
Year	Y0	Y3	Y5	Y0	Y3	Y5	Y0	Y3	Y5
Mean	1.4	1.4	1.9	0.7	1.0	0.9	0.3	0.9	0.9

Examples

In this section, we share examples of two participants who exhibited significant improvements in scores on Attending to the details of students' strategies and on Deciding How to Respond from Year 0 to Year 5.

Adam. In Year 0, Adam's responses provide insight into both the relative importance of students' ideas and the detail with which they were shared (see Table 2). For example, in Year 0, Adam recognized that both groups of students connected their formulas to the model, but did not provide details of the strategy. For example, beyond saying that each student used a formula, the actual formula and the students' explanations were not shared. The descriptions were so general that one would be hard-pressed to be able to recreate the strategy, or be able to use what was shared to plan next steps for instruction. In contrast, in Year 5, Adam was able to share the specific formulas that each student used, and provide details about how students connected their formulas to their physical representation of the pattern. This response signifies a change in response from Years 0 to Year 5, not only in the number of details provided but, also implicitly, in the value that Adam placed on the students' strategies. We conjecture that because Adam valued students' ideas and had learned more about students' thinking about pattern generalization, he paid close attention to them and was thus better able to recall those details.

Table 2. Adam's Responses to Attending to the Details, Year 0 to Year 5

Year 0	*Lack of Evidence* *Student 1:* I do not remember. I'm going to assume Tristan is the last girl who presented. The students presented their formula algebraically. Next, they connected the symbols with their model. The students who were part of the group were there for support. Before they presented, they were engaged in the discussion. *Student 2:* She also presented her formula and connected the symbols with the model they had built.
Year 5	*Robust Evidence* *Student 1:* She explained her formula L+(L-1)+2L= total by connecting each part of the formula to the beam diagram. She explained the bottom of the beam was "L", the top of the beam was the length of the bottom minus the one rod and the total middle rods was 2 times the length of the base of the beam. *Student 2:* She also explained her formula which was 4L-1. The teacher asked her to explain how [the formula] was connected to the diagrams. She pointed at the rods beam diagram showing how the rods enclose 4 complete triangles but at the end you are missing one rod to complete the figure. [draws two figures and notates each as follows: 4(1)-1 and 4(2)-1].

Ella. We share the example of Ella to highlight changes from Year 0 to Year 5 in deciding how to respond (see Table 3). First, in Year 0, the problem Ella selected serves a funneling function toward a correct answer (Andrews & Bandemer, 2018) when Ella wanted her students to answer a question about whether the two formulas are, in Ella's words, "the same." Then, in the rationale, Ella appears to use directive language by sharing that she wanted students to *see that* the generalizations are equivalent, rather than, for example, *explore whether* the generalizations are equivalent. Her language "see that" speaks to funneling toward a correct answer rather than an openness to exploring and building on students' ideas or anticipating other student responses. In contrast, in Year 5, Ella suggests four questions, two that are specific to Tristian's and Beverly's formulas, and two that reflect an openness to learning about students' ideas. Two of the questions that Ella posed are specific and specifically related to the students' previous approaches. Further, her question, "What do your results tell you about both generalized formulas?" coupled with her rationale about questioning reflects a stance that values students' ideas and provides students with opportunities to reflect on the connections between the two formulas, rather than funneling students toward a single response. These kinds of changes reflect the sorts of responses we would hope to see and that teachers can eventually enact with their students in the classroom.

Table 3. Ella's Responses in Deciding How to Respond, *Year 0 to Year 5*

Year 0	*Lack of Evidence* Problem or Problems: I may ask students if these 2 formulas are the same. Rationale: The purpose being for students to *see that* even though the problems were deconstructed and generalized differently that the end result, the generalizations are equal. [italics added]
Year 5	*Robust Evidence* Problem or Problems: Can we use Tristan's equation to find the number of rods in a beam of length 10? What about Beverly's? How do the answers compare? What do your results tell you about both generalized formulas? Rationale: This questioning would hopefully lead students to the idea that these 2 generalized expressions are equivalent.

Change in Responses. Looking across both sets of responses provides the reader with the opportunity to see how responses for attending and deciding-how-to-respond changed over time. In both cases, the responses became more detailed and reflected a valuing of and curiosity about students' ideas. This orientation is one that we know is generative. That is, teachers who are attuned to and curious about their students' ideas continue to grow in their mathematics teaching practice long after PD has ended (Franke et al., 2001), and so improved professional noticing is an outcome we seek not only for the specific expertise that teachers develop, but also for their orientation toward students that continues to aid their own learning for years to come.

Discussion: The Development of Practicing Secondary Teachers' Professional Noticing Expertise and Related PD Activities

In our study we found growth from Y0 to Y3 in the Interpreting and Deciding How to Respond component-skills. In the first three years, the teacher educators spent much time engaging teachers in activities and discussions around individual students' ideas, including interviews, analyzing video and written artifacts, and observing each other teach with a protocol that focused on the students' thinking. As other studies have shown (e.g. Jacobs et al., 2010; Monson et al., 2018), these experiences provided many of our teachers with opportunities to learn about and build on student thinking in a rich way and supported them to develop a disposition to build on student thinking, both of which may have helped them to develop their interpreting and deciding-how-to-respond skills. We believe that the opportunities to consider the research on student thinking in pattern generalization and investigate a single student's (or small group of students') ways of reasoning allowed many of our teachers to develop their understanding of students' learning trajectories within the domain of pattern generalizations, which we posit is an important component of developing one's interpreting and deciding-how-to-respond skills (Nickerson, Lamb, & LaRochelle, 2017).

However, our results indicate that many teachers' interpreting and deciding skills had room for improvement. This differs from Jacobs et al.'s (2010) study of primary teachers, wherein most had demonstrated robust professional noticing skills after 4+ years of PD. As Nickerson et al. (2017) point out, there is a difference between the student thinking frameworks for secondary mathematics and for primary mathematics, in that the student thinking frameworks for primary mathematics are much more explicit and well-connected than those available for secondary mathematics. It is likely that well-connected and detailed learning trajectories at the secondary level could help teacher educators further support teachers' professional noticing expertise. For example, we noticed that the video artifact we selected showed students constructing explicit symbolic generalizations that also made connections to the figure, which is a high level of sophistication of generalization skills (Jurdak & El Mouhayar, 2014; Lannin et al., 2006). During the PD activities, teachers talked about some of the earlier stages of students' generalization skills, such as recursive thinking. However, what fruitful directions should teachers pursue after students demonstrate sophisticated generalization skills?

With respect to the attending component-skill, we did not see growth from Y0 to Y3. This also differs from Jacobs et al.'s (2010) study of primary teachers, wherein the primary teachers reached a ceiling level of attending skills after 2 years. However, we noticed that many of the secondary teachers in our study exhibited strong attending skills prior to engaging in the PD. Our teachers were specially selected from a large pool of applications to participate in the 5-year PD program, which may have contributed to our results (Nickerson et al., 2018).

From Y3 to Y5, we saw growth in teachers' attending skills, but not their interpreting and deciding-how-to-respond skills. During these years, teachers focused on issues of coaching and becoming leaders in their respective teaching communities. In year 5, they engaged in a year-long lesson study activity in which they collaboratively planned, taught, re-taught, and debriefed about a lesson. We hoped that lesson study would become a PD structure that they could bring to their respective teaching communities that focused other teachers' attention on student thinking. During discussions about becoming a teacher-leader, we maintained a focus on being responsive to student thinking because instruction that attends to student thinking has many positive benefits for both students and teachers (e.g. Franke et al., 2001; Jacobs et al., 2007). Hence, student thinking was still an important component, but it may have been back-grounded by the other discussions regarding being a teacher-leader. We wondered if the growth in attending may have resulted from the intense focus on attending to evidence of student thinking, in both the lesson study and the discussions of coaching. During lesson study, teachers were required to gather as much evidence of student thinking as possible for the debrief and lesson modification, which may have given them more opportunities to practice attending to student thinking and supported in them a belief that one should attend to student thinking.

These results differ from studies of prospective teachers, which found significant improvements in all three component-skills in much shorter amounts of time (e.g. Fisher et al., 2018; Monson et al., 2018). However, there are many factors that make comparing our results to the results reported in their studies tenuous. First, it is unclear to what extent the surrounding contexts support or constrain teachers' professional noticing skills. Practicing teachers do not often have the luxury of attending a PD on a weekly basis (as do prospective teachers), and the classroom contexts in which they work may or may not be conducive to supporting this expertise (Levin et al., 2009). We imagine that the competing obligations of practicing teachers (and especially our emerging teacher leaders) may have influenced the development of their professional noticing expertise. Second, it is unclear how the selections of artifacts influence the results of these studies. In particular, we wondered if we had overly challenged our teachers by selecting an artifact that exhibited sophisticated generalization skills (Nickerson et al., 2017). Third, we recognize that the field does not have an agreed-upon standard for high quality professional noticing skills. For example, while Monson et al. (2018) used the term "emerging ability" to represent the highest levels of noticing in their study, Fisher et al. (2018) and Jacobs et al. (2010) used the terms "robust evidence" to represent the highest levels of noticing. How do the responses that received the highest scores compare across each study? What we can say for certain is that significant growth was documented in each of these studies. However, comparing the amount of growth with other studies is less obvious.

In this study, we documented the longitudinal changes of a group of practicing secondary teachers as they progressed through a long-term PD about being responsive to student thinking and becoming teacher-leaders. Our results indicated that different PD activities supported each component-skill differentially. Specifically, the activities in the first half of the PD that supported discussions about individual student thinking and how to build on student thinking seem to have supported our teachers' interpreting and deciding-how-to-respond skills. In addition, the activities in the last half of the PD that supported discussions about coaching others and using evidence of student thinking seem to have supported our teachers' already-strong attending skills. These results have implications for both researchers and teacher educators. In particular, our study can inform teacher educators about the nature of the activities that contribute to various components of professional noticing through explicit or implicit practice. Additionally, our study provides researchers with insight about (a) artifacts they might use to measure professional noticing and (b) rubrics for categorizing responses, as well as factors that may support or inhibit practicing secondary teachers' development of professional noticing expertise.

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AN EXPLORATION OF MATHEMATICS TEACHER LEADERS IN PME-NA PROCEEDINGS FROM 1984-2019

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School districts across the United States are turning to mathematics teacher leaders (MTLs) to support the teaching and learning of mathematics. And yet, what does research seem to say about MTLs? In this paper, we report findings from an exploration of PME-NA proceedings between 1984 and 2019 to examine the role of MTLs. In particular, we examine the following: historical MTL submission trends and the extent to which these trends are coupled with the implementation of national MTL events; broad methodological trends; as well as the ways in which MTLS are positioned. Our findings indicate that future research requires explicitly describing MTLs' roles within systems of professional development to better understand their impact on practice and learning.

Keywords: Teacher Education – Inservice/Professional Development, Instructional Leadership

Introduction

Mathematics reformers have long called for improved learning opportunities for all students across preK-12 classrooms in the United States. Although instruction that promotes mathematics as sensemaking and problem solving has been recommended (Cobb et al., 2018), this shift dramatically differs from the way many classroom teachers once learned and taught math (Hiebert, 1999). Thus, local school systems are left to determine how to create conditions that support changes in teachers' instruction (Hopkins et al., 2013). To address this challenge, many schools are hiring mathematics coaches, or as we will refer to in this paper – *mathematics teacher leaders (MTLs)* – as they embody key features of effective professional development (Gibbons & Cobb, 2017). Indeed, MTLs have become a popular professional development fixture in United States schools (Fennell, 2017).

Given the rapidity with which MTL positions have spread, there is an urgency which requires the field of mathematics education to better understand the research surrounding effective MTL implementation. The overarching aim of this paper, then, is to explore and examine the research related to MTLs in the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA) proceedings between 1984 and 2019. In doing so, our purpose is to discover patterns, commonalities, and trends across the PME-NA proceedings that will inform future research directions, while simultaneously deepening our own understanding of how MTLs are positioned across varied contexts. We hope to illuminate this important work so that other mathematics education researchers can build upon and advance policy and practice surrounding MTLs as there are many schools without a formalized position. We note that the research presented within this paper is only a small slice of the work our team has initiated in which we are examining MTLs' positionality within educational research studies as a whole.

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Guiding Literature

Roles and Responsibilities of MTLs

Using knowledge of best practice regarding professional development, many school districts have created MTL positions to meet the demands of high stakes accountability and to support teachers in their work to provide quality mathematics instruction for each and every student (McGatha & Rogelman, 2017). MTLs are expected to provide professional development within the context of teaching and learning versus the traditional "sit and get" approach to professional development. AMTE provides a broad working definition of a MTL which identifies these individuals as "…teachers, teacher leaders, or coaches who are responsible for supporting effective mathematics instruction and student learning at the classroom, school, district, or state levels" (2013, p.1). To narrow the definition further, we define MTLs as a district- or school-based support person whose knowledge and expertise in mathematics content, pedagogy, and children's learning trajectories assists teachers with their content, pedagogy, and understanding of children's learning trajectories (Campbell & Malkus, 2013).

Even with the definition, the roles and responsibilities of MTLs are complex and ever evolving. Because the term MTL has different and distinct interpretations depending on location, the work of a MTL is diverse and spans across contexts to include administrative tasks, instructional tasks, professional development tasks, and data analysis. To align with our definition, we focus on the type of support MTLs provide. Depending on location and need, there are different models of support that MTLs provide: individual or teacher pair (Barlow et al., 2014), working with teacher groups (Elliott et al., 2009; Lesseig et al., 2016), whole school-level (Campbell et al., 2013; Felux & Snowdy, 2006; McGatha & Rigelman, 2017). Individual and pair support activities could include observations with coaching cycles and modeling lessons. Small group support could include assisting with professional learning communities or team meetings. Whole school support could include trainings and professional development sessions. Additionally, some MTLs are expected to provide support in the form of student intervention; this type of support could model intervention instruction for teachers as a form of professional development. In order to provide this support, MTLs need the necessary expertise in mathematics content and pedagogy and must exhibit key leadership skills in working with adult learners (AMTE, 2013; NCTM, 2012).

Key Historical Events

The use of MTLs to support the teaching and learning of mathematics is not a new call to action. Drawing upon Fennell's (2017) work discussing MTL policy recommendations from a historical perspective, we note key events that have influenced the development and use of MTLs across the United States.

The 1970s saw the emergence of projects that focused on creating positions for MTLs (e.g., *Developing Mathematics Enthusiasts* project, Fennell, 1978). Across these projects, school-level MTLs were identified and employed as mentors to provide content-specific support to other teachers and school stakeholders. In the decade that followed, Fennell notes three key events which underscored the importance of these newly created school-based MTL positions: (1) the National Council of Teachers of Mathematics (NCTM) recommended state certification endorsement for elementary mathematics specialists, (2) John Dossey, the acting president of NCTM during this time frame, published a call for mathematics specialists (Dossey, 1984), and (3) the National Research Council's (NRC) *Everybody Counts* (1989) report expressed the need for elementary mathematics specialists. The combination of these three national events within such a short timeframe highlights the urgency and significance of implementing highly competent and prepared MTLs within schools.

The 1990s were marked by a lull in policy and events related to MTLs. However, this trend came to a halt in the early 2000s with a resurgence of MTL policies and events. Fennell (2017) highlights nine key milestones (p. 6) during this time, including the NCTM's *Principles and Standards for School Mathematics* (2000), the NRCs *Adding it Up* (2001), and the National Mathematics Advisory Panel (2008) documents all making recommendations related to the use of MTLs to support the teaching and learning of mathematics. Additionally, during this time, national legislation centering on No Child Left Behind (2001) and followed by the Every Student Succeeds Act (2015) prompted schools and districts to create MTL positions as a way to address the push for assessment and accountability in mathematics.

Moving into the 2010s, the call for MTLs continued to advance. A major event in 2010 was the release of the *Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentials and Degree Programs* from the Association of Mathematics Teacher Educators (AMTE; revised in 2013). In that same year, a joint position statement calling for all elementary schools to have access to mathematics specialists was released by AMTE, the Association of State Supervisors of Mathematics (ASSM), the National Council of Supervisors of Mathematics (NCSM), and NCTM. Building upon AMTE's standards, NCTM/CAEP released the Elementary Mathematics Specialist Standards (NCTM, 2012), with both sets of standards used by many programs across the country.

Policy recommendations and key events do not stop here. As we progress through the 2010s, we see MTLs referenced in Linda Gojak's NCTM president message (2013), as well as an AMTE research conference focused on elementary mathematics specialists in 2015. When looked at as a whole, this timeline overview shows the call for MTLs has persisted for decades and continues to be at the forefront of research, policy, and organizational recommendations.

Research Questions

The overarching purpose of this paper is to explore PME-NA submissions[1] that center on mathematics teacher leaders (MTLs) during the years 1984-2019. Specifically, we ask the following three research questions:

1. What are the historical trends for PME-NA MTL submissions between the years of 1984 and 2019 and to what extent do these trends align with the implementation of key MTL events and/or policies?
2. What methodological trends are observed across PME-NA submissions between 1984 and 2019?
3. How are MTLs positioned across PME-NA submissions between 1984 and 2019?

Method

Below, we outline the procedure that was systematically used to integrate the research related to MTLs across PME-NA proceedings in years 1984-2019. Additionally, we describe the methodological parameters of our data identification and analysis.

Data Identification

We initiated our exploration by conducting a comprehensive search of PME-NA proceedings between 1984 and 2019. In targeting this date range, we drew upon the key MTL events as identified by Fennell (2017). The lower date range was identified due to the published call for elementary mathematics specialists (Dossey, 1984) mentioned in Fennell's (2017) MTL milestones. The upper date range was identified as 2019 as this was the last year for which PME-NA proceedings were available. We note that the PME-NA proceedings provided us with a data source that was both entirely focused on mathematics and peer-reviewed.

[1] When we use the word submission, we are referring to submissions that appear in published PME-NA proceedings, and may include any of the following: research reports, brief research reports, posters, working groups, and/or plenaries.

To ensure our search maintained high recall and precision (Sadelowski & Barroso, 2007), we used search terms beyond mathematics specialist, mathematics coach and mathematics teacher leader to capture the nuanced ways in which MTLs might be positioned in the proceedings. These terms included: interventionist, response to intervention specialist, resource teacher, instructional coach, Title I Step-Up Mathematics, mathematics lead, and mentor. Initially, we used the search function to apply each of these terms to each published PME-NA proceeding between 1984-2019[2]. In doing so, we noticed that our search brought up many submissions that were not directly related to our central research questions. Thus, we made the decision to only include submissions in which the search terms appeared in the title, abstract, and/or keywords[3]. If the search terms appeared in the body of the submission, but did not appear in the title, abstract, and/or keywords, then that submission was excluded. To examine relevancy across all paper modalities, our collection included papers, research briefs, posters, working groups and plenaries. We also included all methodologies. Last, due to our interest in preK-12 education, we eliminated those entries that emphasized undergraduate mathematics education or faculty studies. Ultimately, our search resulted in 109 unique submissions.

Data Analysis

The submissions in this analysis were coded in several distinct ways. We first completed counts to determine the frequency of submissions for each year, and also looked at spikes and declines in year-to-year submissions (Research Question 1). Next, during our analysis of the methods sections, we applied the following coding scheme for methodology (Research Question 2): qualitative (QUAL), quantitative (QUANT), mixed methods (MIXED), and other (OTHER[4]). Our last coding scheme was also applied while reading the methods section and centered on how the MTL was positioned within the submission (Research Question 3): School-Based Coach, Researcher, Pre-Service Teacher, Pre-Service Teacher Mentor Teacher, Teacher Leader, Mentor Teacher, or Mentor of Students. All codes were mutually exclusive.

Findings

We now present the findings for each of our three research questions in the space that follows.

Research Question 1: Overall Trends

We first explore overall PME-NA MTL submission trends between the years of 1984-2019. As illustrated in Figure 1 below, there are several trends we wish to highlight.

[2] We were unable to include the years 1994, 1993, 1992, 1988, 1986, and 1984 because the search function on those proceedings did not work.

[3] For posters, we only searched for our terms in the titles and/or keywords because those submissions did not have abstracts.

[4] Submissions coded as Other included Plenaries and Working Groups as these submissions did not have a methodology.

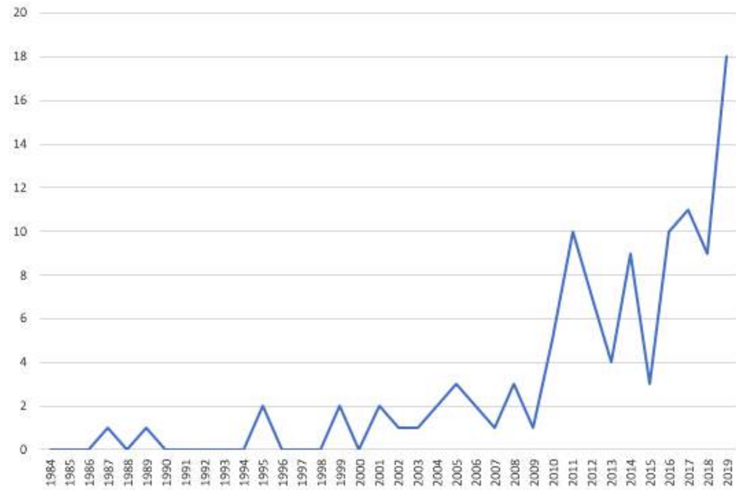

Figure 1: Overall PME-NA MTL Submission Trends from 1984-2019

Overall, there has been an increasing trend in PME-NA MTL submissions between 1984-2019. Starting in 1984, when Dossey published the call for elementary mathematics specialists, there were zero MTL submissions. In 2019, after the implementation of 22 different events for elementary mathematics specialists (Fennell, 2017), there were 18 MTL submissions. This finding indicates the, overall, increased focus on research involving MTLs.

An interesting trend is also noted when comparing year-to-year submission patterns with the key events identified by Fennell (2017). That is, we see the largest spikes in MTL submissions between the years 2010-2011 (increase of 5 MTL submissions), 2013-2014 (increase of 5 MTL submissions), 2015-2016 (increase of 7 MTL submissions), and 2018-2019 (increase of 9 MTL submissions), and these trends are tightly coupled with key MTL events outlined in Table 1. In other words, the largest MTL submission spikes appear to follow the implementation of key MTL national events. Furthermore, we see somewhat pronounced decreasing trends in MTL submissions between the years of 2011-2012 (decrease of 3 submissions), 2012-2013 (decrease of 3 submissions), and 2014-2015 (decrease of 6 submissions). For the most part, these trends are coupled with the absence of key MTL national events. For example, the absence of key MTL events in 2011 and 2014 might help explain the decrease in MTL PME-NA submissions between the years 2011-2012 and 2014-2015.

Table 1: Select MTL Events from Fennell (2017)

Year	Events
2010	AMTE released *Standards for Elementary Mathematics Specialists*
2013	NCTM President's Message from Linda M. Gojak: *It's Elementary! Rethinking the Role of the Elementary Classroom Teacher*
2015	NCTM Research Brief *The Impact of Mathematics Coaching on Teachers and Students* (McGatha, Davis, Stokes) AMTE EMSs Research Conference Every Student Succeeds Act

Research Question 2: Methodological Trends

As previously mentioned, we read the methods section for each of the 109 MTL submissions to better understand broad methodological trends for PME-NA MTL submissions between 1984-2019. Overall, 68% (n = 67) of the MTL submissions were coded as QUAL, 23% (n = 22) were coded as MIXED, and 9% (n = 9) were coded as QUANT. Hence, most submissions involved qualitative investigations, while quantitative investigations surfaced less frequently. Furthermore, preliminary analysis indicates variability in the quality and types of research questions posed, and further analysis is required within each methodological approach.

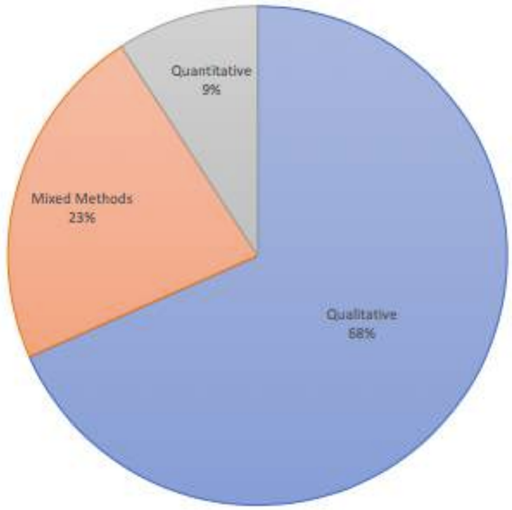

Figure 2: Methodological PME-NA MTL Submission Trends from 1984-2019

Research Question 3: Positioning of MTL

We also analyzed the nuanced ways in which MTLs were positioned in PME-NA submissions from 1984-2019. Across our data set of 109 submissions, MTLs were positioned in seven different ways. In Table 2 below, we provide a description of each of the ways in which MTLs were positioned, as well as a count for each. We note that, overall, MTLs were most frequently positioned as a School-Based Coach (n=44), followed by Researcher (n=22), and then Pre-Service Teacher Mentor Teacher (n=16). Other roles, such as Pre-Service Teacher (n=3) and Mentor of Students (n=2) less frequently emerged.

Table 2: MTL Positioning in PME-NA Submissions from 1984-2019

Positioning	Count	Description
School-Based Coach	44	The MTL is released from their classroom teaching position and is charged with supporting teaching and learning across one or more schools.
Researcher	22	The MTL is a university researcher, faculty member, and/or staff member who serves as a coach/mentor to others in various contexts.
Pre-Service Mentor Teacher	16	The MTL is a classroom teacher that mentors or supervises pre-service teachers at a preK-12 school.
Teacher Leader	12	The MTL is a classroom teacher who receives professional development to become a teacher leader without mention of any supervisory role(s).

Mentor Teacher	5	The MTL is a classroom teacher who mentors their peers/colleagues/in-service teachers, and serves in a supervisory/mentoring role.
Pre-Service Teacher	3	The MTL is a pre-service teacher who provides peer-feedback to other pre-service teachers in the context of a methods course (e.g., rehearsals, reform-based lessons, etc.).
Mentor of Students	2	The MTL is a classroom teacher who mentors preK-12 students.

Discussion and Implications

The overarching purpose of this paper was to initiate the integration of research on MTLs across PME-NA proceedings in years 1984-2019. In the space that follows, we summarize the main findings for each of our three research questions, and also discuss implications.

Overall, we observed an increasing trend in the number of PME-NA MTL submissions during our identified time frame. Furthermore, we observed coupling between year-to-year MTL submission spikes or declines and the presence or absence of national MTL events. That is, large spikes in year-to-year MTL submissions were coupled with the implementation of a national MTL event, while declines in year-to-year MTL submissions were coupled with the lack of national MTL event. This seems to indicate that MTL policies and events at the national level are actively shaping MTL research agendas and publication. That is, in the presence of national MTL policies and events, MTL research is occurring. Conversely, in the absence of national MTL policies and events, there is less attention to MTL research. Whether related to national events or not, there is a general upward trend in MTL research and interest in the mathematics education community in this research.

Regarding methodological trends, our analysis indicated that most of the MTL PME-NA submissions involved qualitative methods, while mixed and quantitative methods were less prevalent. We have several hypotheses to help explain this trend. First, it is possible that researchers seem to be most interested in asking research questions about MTLs that can best be answered using qualitative methods. For example, early stage, qualitative studies are needed to understand MTLs’ work before implementation/impact studies at a large scale can evaluate roles related to variables like student achievement. However, this is – perhaps – too easy of an explanation, and there is likely more going on here. Second, it is possible that the research community lacks quantitative measures and instruments that can validly and reliably be used to document the nuanced work of MTLs. While there is some research that has begun to explore this hypothesis (Harbour, Livers, & Hjalmarson, 2019), more is needed. Third, and relatedly, impact and/or influence is so difficult to measure because there are many confounding variables and multiple levels to MTLs’ work. Ultimately, this makes it rather challenging to tease apart MTLs’ unique impact. Thus, future research should focus on MTLs as part of the system of professional development in the school/district to understand their impact on both teachers' practice and students' learning.

Last, in exploring the ways in which MTLs are positioned across PME-NA proceedings, we identified seven different categories that ranged from MTL as School-Based Coach to MTL as Pre-Service Teacher. This speaks to the wide-spread variation in the ways in which researchers refer to individuals in this position. Although prior research has already suggested that the field lacks a common definition for MTLs (Baker et al., 2017; National Mathematics Advisory Panel [NMAP], 2008), our study adds further evidence in support of this trend. Hence, future research should be developed to explore the characterization of MTL work and practice in order to compare different implementation models, better describe MTL roles within schools/districts and their work with both teachers and students, and further develop MTL knowledge and skills to better support preparation programs and other ongoing professional learning experiences.

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TEACHER EDUCATION (IN-SERVICE) / PROFESSIONAL DEVELOPMENT:

BRIEF RESEARCH REPORTS

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A COLLABORATIVE PROFESSIONAL DEVELOPMENT MODEL: PROMOTING SHIFTS IN CLASSROOM PRACTICE

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To ensure conceptual learning in mathematics, teachers must shift many aspects of their instructional practices. We report on a two-year endeavor using a collaborative and responsive professional development model to help elementary school teachers enact seven shifts in classroom practice. We share evidence of teachers addressing the instructional shifts and discuss the promise of the approach used for those interested in co-constructing collaborations between and among universities and school districts.

Keywords: In-service Teacher Education; Professional Development; Instructional Leadership; Instructional Activities and Practices

Purpose

We report on a two-year professional development (PD) program designed to improve teachers' knowledge of mathematics, strengthen their pedagogical skills, and foster collaboration to reflect on practice and improve teaching. Guided by the Leading for Mathematical Proficiency Framework (Bay-Williams et al., 2014), this PD supported teachers' instructional shifts to better implement the Standards for Mathematical Practice (CCSSO, 2010). As part of the PD team, we (authors) implemented cycles of a "responsive and emergent" curriculum (Confrey & Lachance, 2000, p. 244) to ascertain teachers' thinking and address their needs. We offered multiple, collaborative opportunities for participants to revisit and reflect upon their teaching practices, as they aimed to implement at least some of McGatha and Bay-William's (2013) seven instructional shifts (see Table 1).

Figure 1: Shifts in Classroom Practice (Bay-Williams et al., 2014, p. 24)

Shift 1	From same instruction toward differentiated instruction
Shift 2	From students working individually toward community of learners
Shift3	From mathematical authority coming from the teacher or textbook toward mathematical authority coming from sound student reasoning
Shift 4	From teacher demonstrating 'how to' toward teacher communicating 'expectations' for learning
Shift 5	From content taught in isolation toward content connected to prior knowledge
Shift 6	From focus on correct answer toward focus on explanation and understanding
Shift 7	From mathematics-made-easy for students toward engaging students in productive struggle

These seven shifts support teachers in creating a classroom culture where students are active participants in the learning process, namely: differentiating instruction, having students work as a community of learners, affirming mathematical authority comes from sound student reasoning, communicating clear expectations for learning, connecting content to students' lived experiences and prior knowledge, focusing on explanation and understanding, and engaging students in productive struggle. Our research question was: In what ways has the collaborative and responsive professional development model impacted teachers' shifts in classroom practice?

Theoretical Underpinnings

Change theory (Fullan, 2006) highlights the significance of stakeholders' democratic participation in a continuous and deliberate process to support a shift from existing towards new practices. Underwood (2015) acknowledged the contribution of professional learning communities to encourage teachers to re-think, re-learn, and re-engage with others as they articulate and process the meaning(s) of new practices. The PD team also realized the importance of building trust, acknowledging teachers' opinions and needs, and being willing to rethink our own practices. Therefore, we designed an iterative process, what we call the Collaborative and Responsive Professional Development (CRPD) model, to continuously and deliberately build trust, hear/see teachers' ways of thinking and knowing, and utilize this information to design meaningful experiences for the teacher participants.

Methods

Participants and Setting

The CRPD model emerged from a two-year partnership between a higher education institution and two school corporations. Participants were 60 elementary teachers (Grades K-6) from eight schools. The year-round PD support included: summer workshops, two full-day workshops (one in the Fall and one in the Spring), and four after-school workshops (two in the Fall and two in the Spring).

In this partnership, the PD team included two mathematics educators, one mathematician, one math coach, and two mathematics education graduate students. The full-time math coach was an experienced master elementary and middle school mathematics teacher. This arrangement of the PD model bridged the gaps between the PD team and the teacher participants as the mathematics coach proactively brought the teachers' voices to the planning sessions and helped to translate the projects' goals into meaningful and relevant learning activities for the teachers.

During PD sessions, the teachers were encouraged to work with others from the same grade level across schools and same-school participants from different grade levels so that they can benefit from the experiences of others and continue their interactions during the school year as a community of learners (Wenger, 1998, 2000).

Data Sources and Analysis

Data sources included an initial teacher inventory of classroom practices and teachers' annual self-reports on their instructional practices, self-assessed mathematics proficiency, and their experiences in professional development. The annual self-reports were a deliberate effort to understand the goals teachers had set for making shifts in their practice, capture their thinking about any shifts they had accomplished, and ascertain what shifts they still sought to make. The data comes from teachers' responses to the prompt: 'Have you noticed any changes in your math classroom, your students, or yourself? Could you describe them or share some specific examples?' Analyzing cases for which we had all data for two consecutive years (N = 20).

We used thematic coding analysis (Braun & Clarke, 2006) for coding the teachers' written responses with the seven shifts as the main themes. First, two authors collaborated to code each teacher's response according to whether and how it provided evidence of any of the seven shifts in classroom practice (Figure 1). To ensure a consistent understanding of the coding process, first, both coders worked together to code the first five cases from both years' data. Then, once an established coding system was developed each author separately coded the remaining teachers' responses. Those two authors came back together to discuss their respective codes and reconcile any differences. Those reconciled codes were shared with the other two authors, any differences were discussed, and again, reconciled to get agreement on 100% of the cases.

Results

To capture a holistic picture of how the teachers claimed to have shifted their classroom practices, we examined the distribution of the percentage of teachers whose responses referred to a specific shift (Figure 2).

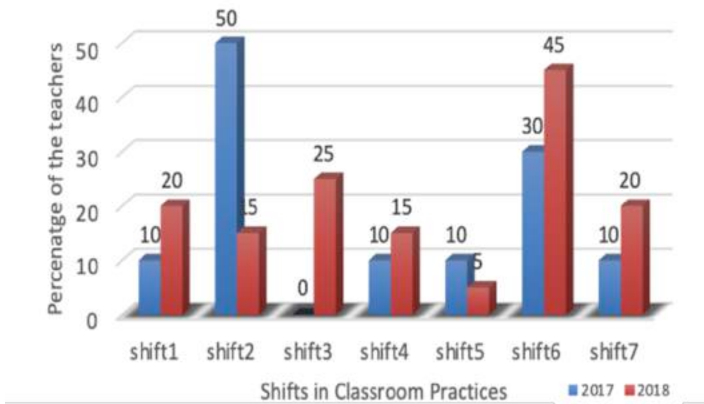

Figure 2: Distribution of Teachers' Self-Reported Shifts in 2017 and 2018

Over the two years, this group of teachers had shifted their instruction in each of the seven ways described by McGatha and Bay-William's (2013). However, two of the shifts (Shifts 2 and 6) stood out in our analysis, as they were mentioned substantially more often by the teachers. Shift 2, towards creating a community of learners, was mentioned by 50% of the teachers in year 1. Shift 6, towards a focus on explanation and understanding, was mentioned by 30% of the teachers in year 1 and 45% in year 2.

Regarding Shift 2, one teacher mentioned, "My students love when it is math time and how they get to share what they did to solve the problem because more than likely it is different than their neighbor's ideas!" (Teacher#6, 2017). Teachers' responses reflected that being engaged in the activities around collaboration influenced their learners' listening, comprehending, accepting, and critiquing multiple ways of mathematical reasoning and thinking. Regarding Shift 6 teachers reported, "Students are stating their claims as 'I agree with…. because' or 'I disagree with …... because' (Teacher#7, 2018), which illustrates students recognizing the significance of using reasons to validate mathematical arguments.

Evidence for Shift 1, toward differentiated instruction, came from many teachers endorsing Math Workshop and Number Talks, as they realized that those instructional practices assisted them in recognizing specific strengths and gaps in their learners' mathematical thinking and guided them in designing appropriate next step instructional activities. While in year1 none of the teachers mentioned Shift 3, toward mathematical authority coming from sound student reasoning, teachers realized that by sharing their mathematical authority with other collaborators in learning, they could positively contribute to their students' learning. For example, one of the teachers stated:

> During math, the kids are the ones that do the teaching. I serve as a guide. Instead of me making sure everybody is doing everything absolutely correct, I am able to sit back and let the kids make mistakes and explore and that's awesome (Teacher#18, 2018).

For Shift 4, toward teachers communicating expectations for learning, teachers started to see that transparency of the instructional objectives and flexibility in accepting various entry or exit points from the learners helped develop a sense of ownership in their students. One teacher mentioned, "mathematics is … more about the process and thinking rather than the right answer" (Teacher#2, 2018). Only a small number of teachers referred to Shift 5, toward content connected to prior knowledge, which might suggest that any change in this shift requires a longer period of instruction and training. Regarding Shift 7, engaging students in productive struggle, teachers realized that changes in their teaching style could help their students' perseverance. One of the teachers shared a success story of her student who did not like math in the early grades, however, made significant gains and contributed often to on-going mathematical class discussions (Teacher#21, 2018).

When examining the distribution of the percentage of the teachers whose responses referred to a specific shift (Figure 2), we found that in general teachers' reports of enacting a specific shift in teaching increased for five of the seven shifts. While there was a decrease in the number of reports of addressing Shift 2, we do not think this necessarily means that teachers were not giving attention to this shift as work on the other shifts implicitly shows they are giving attention to this one. A similar pattern was observed for Shift 5, content connected to prior knowledge. We assume that initially the teachers might have been focusing more on their pedagogical orientation rather than on curricular materials while employing new instructional techniques in their teaching. However, we believe that with the passage of time subtle differences in their content and curricular knowledge will also be visible.

One limitation of this study is that the data sources are self-reports of teachers' perceptions of change in their instructional practices. We tried to minimize the impact of using self-reports by corroborating classroom shifts with the mathematics coach's observations in the teachers' classrooms.

Discussion and Implications

Research has shown that many reform efforts result in existing classroom practices remaining unchanged because it is difficult to shift mathematics understandings, attitudes, and experiences (Ball, 1996; Tzur et al., 2001). Researchers also offer that "professional development that is embedded in daily classroom practices of teachers in which there is a continuous loop of observation, feedback, and discussion in order to sustain learning" (Underwood, 2015, p. 26) helps in developing new capacities to sustain changes in instruction gradually. We are encouraged by the fact that teachers in this PD created achievable learning targets for their students by attending to their needs, interpreting their understandings, and creating opportunities to develop as mathematical doers and thinkers. We think that the shifts reported by the teachers will gradually lead them towards implementing the mathematical practices in their classrooms (Kilpatrick et al., 2001; National Council of Teachers of Mathematics, 2000).

We think this study provides some evidence of the feasibility of facilitating shifts in teachers' practices when professional development is centered on teachers' needs and engages the teachers in ways that we expect them to teach their students. The opportunities afforded by the processes of the Collaborative and Responsive Professional Development (CRPD) model show the potential of promoting teachers' effective shifts in mathematics teaching.

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SELF-EFFICACY AND THE KERNEL OF CONTENT KNOWLEDGE

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In this study, when mathematics teachers were provided professional development pertaining to physics, their physics content knowledge improved, as did their self-efficacy in teaching mathematics through physics. This analysis reveals how these two improvements interacted and how that interaction changed over time. Also, this study examines what components of self-efficacy were influenced. These results have practical significance for STEM professional development design and implementation, while revealing theoretically significant nuances in the development of teacher knowledge. Intriguingly, self-efficacy gains were correlated with content knowledge gains, but only in the content knowledge that was retained over a longer period of time, suggesting that teachers' content knowledge may have a kernel, or core, that is more correlated with affects and beliefs, such as self-efficacy.

Keywords: Teacher Knowledge, Beliefs

Studies suggest that integrated approaches to teaching STEM (Johnson, 2013; NRC 2002), weaving together science, technology, engineering, and/or mathematics, improves student achievement (Becker & Park, 2011), reflects the nature of STEM professions (Wang et al., 2011), enables deeper understanding (NRC, 2012), and highlights mathematical relevance (GAIMME, 2016). However, integration of insular disciplines brings new needs to teacher education because the diverse knowledge needed to teach integrated STEM is not prevalent in the teacher workforce (Roehrig et al, 2012). Although both the American Academy of the Arts and Sciences (Pallas, Neumann, & Campbell, 2017), and the National Academy of Sciences (2012) recommend creating teaching resources for the integration of STEM disciplines, new resources are insufficient by themselves. Professional development (PD) providers should strive to equip practicing teachers with the content knowledge (CK) and self-efficacy (SE) to effectively teach integrated STEM content.

Teachers' SE (Bandura, 1997) has been shown to correlate with teachers' CK (Swackhamer, 2009), but how might this correlation differ in an area outside a teacher's specialization, such as physics CK with math teachers? In this study, we examine PD that supports physics-based and inquiry-based math teaching, by analyzing teachers' CK and SE. Data from 20 in-service math teachers informs the following questions, for both "short term" (after a 1-week summer workshop) and "long term" (after 4 monthly post-workshop meetings):

1. Based on pre-post data, does inquiry-based PD influence CK about physics-based math, or SE for teaching math through physics, or components thereof?
2. Do pre-post differences in these CK and SE variables correlate with each other?

Theoretical Perspectives

SE pertains to certainty about one's abilities (Bandura, 1997). While some studies have linked teachers' SE to students' achievement and motivation (Caprara, et al., 2006; Skaalvik & Skaalvik, 2007), longitudinal studies have revealed that correlations between teaching SE and instructional quality are not purely causal or consequential (Holzburger, Philipp, & Kunter, 2013). Although some studies have found that continued and objective-focused PD improves SE (Brinkerhoff, 2006), little research has indicated how PD influences teachers' SE in a subject outside their expertise. Examining effectiveness of inquiry-based PD, prior research shows mixed results, sometimes improving and sometimes worsening SE for teaching science (Avery & Meyer, 2012). Few instruments measure SE

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for teaching math and science in an integrated way. Mobley's (2015) SE for teaching integrated STEM uses a 3-factor model, with a *social* factor, including motivating students, a *personal* factor, including developing new knowledge, and a *material* factor relating to access to tools.

With Common Core came an emphasis on mathematical modeling (CCSS, SMP). However, many practicing teachers had minimal training in mathematical modeling and in the sciences that utilize modeling. While many teacher training programs adapted to include more modeling coursework, PD remained essential for practicing teachers. This landscape accentuates the importance of studies such as this, in which math teachers are supported in the learning of physics or other STEM content. In this study, CK was measured by multiple-choice items similar to math questions found on a physics Advanced Placement test. We consider CK to be similar to Shulman's (1986) *subject matter content knowledge*, and because of the interdisciplinary nature of this study, is related to Ball's (1993) *horizon knowledge*, which implies awareness of how math content spans the curriculum. Considering subdomains of subject matter knowledge (Ball, Thames, & Phelps, 2008), we suspect these subdomains may interact differently with non-cognitive variables, such as affect and belief, and may persist differently over time.

Methods

Professional Development Workshop

The grant-funded workshop, titled *Let's Get Physical! Teaching Mathematics through the Lens of Physics*, included 32.5 hours over 5 consecutive summer days, followed by 4 monthly 1-hour meetings during the following fall semester. The inquiry-based PD highlighted the themes: (a) integration of math and physics and (b) student motivation.

The grant provided each teacher's school with physics lab equipment, including Vernier physics packages, Logger Pro software, spring kits, current probes, circuit boards, refraction blocks, lasers, track systems, and iPads. During days 1-4, the teachers completed 2-3 physics experiments each day and discussed pedagogical topics related to student motivation. On day 5, a mathematics and physics panel of faculty and graduate students made presentations about applied topics and current research. During the 4 follow-up meetings, conducted through video-conferencing, the participating teachers shared lesson ideas and experiences with one another.

The physics labs in the workshop, available online (Author2 & Author1, 2017), were inquiry-based and aligned to standards in middle school math, Algebra I & II, and geometry. In one lab, teachers modeled the behavior of live insects to learn about displacement, velocity, and geometry. In another, teachers dropped coffee filters and modeled their fall to learn about drag, logarithms, and graphical methods. Other labs involved basketballs, toy cars, lasers, and circuits.

Participants and Recruitment

University faculty and administrators from local schools recruited applicants through meetings and emails. Teachers of middle school math, Algebra I & II, and geometry were encouraged to apply. Twenty math teachers, from 5 school systems, were selected. Most held bachelor's degrees in mathematics, 14 held graduate degrees in education, 2 held master's degrees in mathematics, and 2 held master's degrees in physics. Six participants were male, and 14 were female. Because administrators participated in recruitment, more teachers with leadership qualities may have been more likely to apply. Because this workshop was marketed as *Let's Get Physical! Teaching Mathematics through the Lens of Physics*, teachers with more interest and knowledge in physics may have been more likely to apply.

Data Collection and Analysis

Using instruments described in Table 1, data was collected from 20 in-service teachers before and after the week-long summer workshop, and also after 4 monthly post-workshop meetings.

Table 1: Descriptions of Instruments

Construct	Length	Scale	Cronbach Alpha
Self-Efficacy (SE)	8 items	0 = *Certainly I am not capable.* 10 = *Certainly I am capable.*	.96
Content Knowledge (CK)	5 items	0 = incorrect, 1 = correct	.59

We framed the SE inventory using Mobley's (2015) 3 factors - social, personal, and material. Per Bandura's (2006) advice for maintaining content validity, all items were phrased as capability statements, and caution was taken to avoid confusion with self-worth or locus of control. Also, to refine our instrument, we piloted it at a STEM education conference.

Items for the *social* sub-scale of SE say, *I am capable of…*

- leading my students in conducting physics labs in such an effective way that all of my students are motivated to learn math.
- anticipating and preventing likely student errors while conducting physics labs.
- coordinating a superior cross-curricular math lesson with a science teacher at my school.

Items for the *personal* sub-scale of SE say, *I am capable of…*

- making meaningful connections between physics and mathematical concepts.
- revising a physics lesson plan to make it appropriate for my mathematics classroom.
- responding immediately if a student asks me how a math homework problem is related to physics.

Items for the *material* sub-scale of SE say, *I am capable of…*

- finding related physics-based examples, no matter what mathematical concept I am planning to teach.
- teaching students to use technology and equipment to do physics labs, without technical difficulties.

We used paired *t*-tests to detect significantly non-zero pre-to-post differences, and we used regression analysis to determine statistically significant correlations between those differences.

Results

CK in physics-based mathematics improved over the course of the 1-week workshop. Post-test CK scores (M=3.20, SD=1.28) exceeded pre-test scores (M=1.70, SD=1.30). However, some of this acquired CK was impermanent. Four months later, when re-tested, the gains in CK (M=2.37, SD=1.64) were no longer significantly different from pre-test scores. See Table 2.

Table 2: Overview of Results

Research Question	After 1-Week Workshop	After 4 Monthly Meetings
Did the PD influence CK?	$t(19)=4.94$*	$t(18)=1.79$
Did the PD influence SE?	$t(19)=6.10$*	$t(18)=5.57$*
Did CK and SE gains correlate?	$r(18)=.022$	$r(17)=.515$*

*Significant at the .05 level

Regarding SE, however, the benefits did not fade. The teachers showed significant improvement in SE, both in the short-term and in the long-term. Short-term (M=6.69, SD=1.79) and long-term (M=6.91, SD=2.00) post-workshop SE ratings significantly exceeded those pre-workshop (M=4.46, SD=2.06), and also significantly improved in each SE subscale.

Regression analysis was used to test if the gains in SE or its subscales significantly correlated with participants' content knowledge gains. In the short-term, gains in CK and SE were not significantly correlated, and none of the SE subscale gains significantly correlated with CK. However, in the long-term, CK and SE gains did correlate, $r(17)=.515$. For 2 SE sub-scales, the correlation was significant as well. See Table 3.

Table 3: Correlation Tests in Gains after 4 Monthly Post-Workshop Meetings

	correlation with CK gains
self-efficacy (SE) gains	$r(17)=.515$*
SE *social* subscale gains	$r(17)=.514$*
SE *personal* subscale gains	$r(17)=.511$*
SE *material* subscale gains	$r(17)=.374$

*Correlation is significant at the .05 level.

Discussion

The *Improving Teacher Quality* (ITQ) grant program from the U.S. Department of Education, which provided the funding for this PD project, has been de-funded at the federal level. In the face of funding limitations, local and state education agencies are planning various strategies for supporting STEM education. With integrated STEM initiatives, teachers are more frequently expected to collaborate across disciplines and teach content peripheral to their areas of expertise. As math curricula adapt to ever-changing technology, the need for cross-disciplinary PD will increase, and integration of math with computer science, biology, engineering, and data science should be deliberately implemented, with effects on both short-term and long-term CK and SE gains examined. This study suggests that as future PD is provided, implementing an inquiry-based approach will improve the overall effectiveness of these supports. In addition, PD should attend to personal, material, and social concerns about teaching mathematics.

When future studies examine correlations between CK gains and SE gains, the findings of this study should be considered in research design. Our results suggest that short-term studies may not reveal connections that would be apparent in longer-term studies. Future theoretical research about types of CK should also consider changes over time. Because our short-term CK and SE gains were not correlated, but our long-term gains were, we suspect that there was a kernel, or core, of CK that persisted longer, and that this CK kernel was more likely to have influenced teaching practice, since it was correlated to pedagogical SE. Also, one might suspect that teachers who chose to use certain physics-based lessons in their classes in the fall might have retained certain parts of CK, and thus, teaching practice might be influencing both SE and CK. Thus, instead of viewing CK as a substance that can be acquired and then retained, this study substantiates a more complex model of knowledge, one in which teachers participate in a process of using their content knowledge, reminiscent of Sfard's (1998) participation-acquisition framework. Future studies should examine how teaching practice influences CK, and how teachers decide to use inquiry-based lessons in math classes.

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CONTRIBUTING FACTORS TO SECONDARY MATHEMATICS TEACHERS' PROFESSIONAL IDENTITY

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Our study explores contributing factors informing secondary mathematics teachers' professional identity. Data from five semi-structured interviews were evaluated using the provisional coding method. Results indicate mathematics identity, beliefs about teaching, and beliefs about mathematics all play an integral role in the ways teachers discuss their professional identity with some differences found between teachers' level of experience. This work informs the field by expanding on an existing framework to deepen our understanding of professional identity.

Keywords: Affect, Emotion, Beliefs, and Attitudes; Teacher Beliefs

Research into teachers' professional identity aims to understand the interplay between the social and individual perspectives to identify the thoughts, influences, and impacts they have on a teacher's image of self (Beijaard et.al, 2004). A teachers' professional identity can have a large impact on their persistence in their profession (Hong, 2010). Teachers' view of themselves and their experiences act as motivators for their beliefs, actions, and future goals, which in turn affect their commitment, teaching quality and decision making (Hong et al., 2017). Exploring teacher professional identity has the potential to shed light on the high attrition rate of teachers in the field, factors that may support or inhibit teacher growth, and factors that may link to teacher practices and decisions related to their profession.

This study draws on research identifying specific components as important to teachers' professional identity. For example, Canrinus et al. (2012) asked teachers about their job satisfaction, self-efficacy, occupational commitment, and change in level of motivation as a way of exploring their professional identity. Further, we explored components discussed in prior research alongside teachers' content specific identity (mathematics identity) to better understand how these identities may overlap and inform one another. The research question informing our study is: how do secondary mathematics teachers describe their professional teaching and mathematics identity through the lens of prior research?

Methods

Participants

Table 1: Participant Information

Participant Pseudonym	Degree	Years of Experience	Position
Lilly	Bachelors	1	Primarily teaches grades 6-8 within an elementary school setting
Mary	Bachelors	1	Primarily teaches grades 7-8 within an elementary school setting
Bailey	Masters	5	Primarily teaches grades 9-12, focused on Algebra II, within an high school setting
Eva	Masters	16	Primarily teaches grades 6-8 within an elementary school setting, a Nationally Board Certified Teacher

Anisha	Masters	15	Primarily teaches grades 9-12, focused on Algebra I, within a high school setting

While this study included 36 secondary mathematics and science teachers, for the purpose of this paper, we included 5 secondary mathematics teachers. These teachers were in the first summer of their first year participating in a multi-district professional development grant and had varied levels of experience (see Table 1). These participants were all female and classified themselves as Caucasian, Non-Hispanic.

Data Collection and Analysis

Semi-structured interviews were conducted with each of the participants. In order to capture teachers' professional identity, questions about their self-perceptions (e.g., how would you describe yourself as a math teacher), how others viewed them (e.g., how do you think your administrators view you as a math teacher), motivation for going into the profession (e.g., why did you become a teacher), and future self (e.g., if you exited the field of teaching today, how would people describe the legacy that you left behind) were asked.

The provisional coding method was conducted to code interviews, which entails beginning with a set of a priori codes that draw from prior literature (Saldaña's, 2015). The a priori codes we used were based on five constructs explored in Hong's (2010) article: emotion, commitment, value, micropolitics, and self-efficacy. Hong (2010) included an additional a priori code, knowledge/beliefs, which we did not initially include as it was anticipated that mathematics identity would capture some of these ideas. However, we did end up including some additional codes related to beliefs during the coding process. In addition to these codes, we created a list of a priori codes based on four factors identified in prior research related to mathematics identity (Cribbs et al., 2015): interest, recognition, competence, and performance. After the initial round of coding, the additions to the original list were discussed and a consensus was met on a new list of codes. Further detail about these codes will be provided in the results.

Results

Four overarching themes emerged: professional identity, mathematics identity, beliefs about teaching, and beliefs about mathematics. Aspects of professional identity and other influential factors - taking into account the complexity of construct – will be discussed.

Professional Identity

There were initially five a priori codes (themes) used in exploring professional identity: emotion, commitment, value, micropolitics, and self-efficacy. These themes were further broken into sub-themes, creating a set of 11 themes.

Emotion. Out of the 5 participants, only one participant had evidence of the theme emotion, which connects specifically to stress, burnout, and well-being. Bailey described an incident that was "so rough that I almost decided not to continue teaching." However, it was evident through this interview as well as the other interviews how interconnected many of the themes were. For example, when discussing her struggles with the profession, Bailey indicated that district expectations (micropolitics - structures and support) were a primary reason for the tension she was experiencing in her position. It was only when she moved to a different teaching position that these tensions resolved and her persistence in teaching (commitment) was evident.

Commitment. Statements related to the theme commitment were evident in each of the participant interviews. For example, when responding to a question on how the profession was viewed by those outside of it, Lilly indicated that the perception of math being difficult "motivates" her. In all but one case, Bailey, commitment was discussed with reference to outside perceptions (community or society) of the profession and in terms of countering perceptions or motivating a sense of commitment due to these perceptions.

Value. Within the value theme, utility was evident in two of the interviews where teachers indicated a "calling" related to them pursuing the profession. Interest was evident in four out of the five interviews, with teachers indicating enjoyment in teaching and even connecting the "love" for teaching with caring about kids. Importance was only evident in two interviews with statements relating to making a "difference every day," particularly for their students.

Micropolitics. This theme was evident at a much higher frequency than the previous themes through the three sub-themes (decision making, status, and structures and supports). Decision making was only evident in two of the interviews with Bailey indicating a lack of autonomy as a professional due to requirements by the district ("they took away zoom math and that was something that the district pushed heavily for us to have students use which we didn't always agree with…"). The other sub-themes were evident in all five interviews. With reference to status, both Lilly and Bailey positioned themselves as novice teachers. Other comments indicated differing levels of status within the larger community (others outside of school or society) such as teaching perceived as "not very good" by others or having an elevated status because "you know, just working with that level of students." Finally, teachers had varied levels of support and structures in place as evident in their comments. However, comments seemed to indicate perceived support from administration across all interviews.

Self-efficacy. Self-efficacy was the final theme evident under professional development and included four sub-themes. The first sub-theme was classroom management and was evident in all of the interviews. Comments by Eva and Anisha indicated a high level of efficacy at being a "good classroom manager" compared with the less experienced teachers who expressed comments such as "my classroom management is not the greatest right now." This finding is not surprising given the varying levels of experience. Student engagement was evident in three of the five interviews, with two teachers noting struggles with engaging students with comments such as "I'm guilty of often times kind of being a boring teacher" but also indicating that they viewed engagement as important and necessary for effective teaching. Instructional strategies, much like engagement, focused on challenges and strengths of the teachers. Nearly every teacher mentioned struggling to connect the real world with the math content. Overall, strengths for strategies focused on collaborating and working with students. Finally, the general sub-theme for self-efficacy focused on statements about being effective with teaching, but often without enough specificity to know what that meant to the teacher.

Mathematics Identity

Mathematics identity included the sub-themes interest, performance, competence, and recognition.

Interest. Three of the five teachers indicated interest and/or enjoyment of mathematics as a subject area. All these comments related to why they decided to teach mathematics. For some of the teachers, interest in the content area seemed to connect to their teaching, but this was not the case for other teachers who either discussed these ideas separately or did not discuss interest in the content or interest in teaching.

Recognition. Two of the teachers specifically discussed being recognized in mathematics. One teacher discussed this through teacher recognition when she was a student, and the other teacher discussed her role in helping others, being positioned/recognized as knowing mathematics.

Competence. Three of the five teachers discussed experiences related to their competence with mathematics, with statements such as "it was something that I got." As with interest, most of the comments related to reasons for why the teacher decided to teach mathematics. However, two teachers indicated that students knowing that they [the teacher] understood the content was something they wanted their students to know or felt that they knew about them.

Performance. Three of the five teachers also discussed performance with mathematics. All these comments related directly to their rationale for choosing to teach mathematics.

Beliefs about Teaching

Beliefs about teaching was a theme that emerged in our second round of coding. Four of the five teacher interviews provided evidence of their teaching beliefs. Two of the teachers (Lilly and Bailey) seemed to be trying to reconcile what they thought effective teaching looked like or what they had hoped it would be like with their current practice. Other ideas such as being an enthusiastic teacher, learning through problem solving and different strategies, and students being actively engaged were discussed by the teachers.

Beliefs about Mathematics

Beliefs about mathematics was a theme that also emerged in our second round of coding for two of the four teachers. Lilly's responses seemed to indicate a belief that mathematics is applicable to the real world and that all students have a capacity to learn mathematics. Conversely, Bailey's comments seemed to indicate that students were either a "math person" or not a "math person", such as stating that "he is not a math kid" and "you either love it [math] or you hate it [math]."

Discussion

Findings support the inclusion of the constructs explored in Hong's (2010) study, but also support the inclusion of additional factors that seemed to play a role in teachers' professional identity as evident in the interviews we conducted. Figure 2 provides an overview of themes by participant to help convey some of these patterns with the size of bubbles aligned with the frequency of the code.

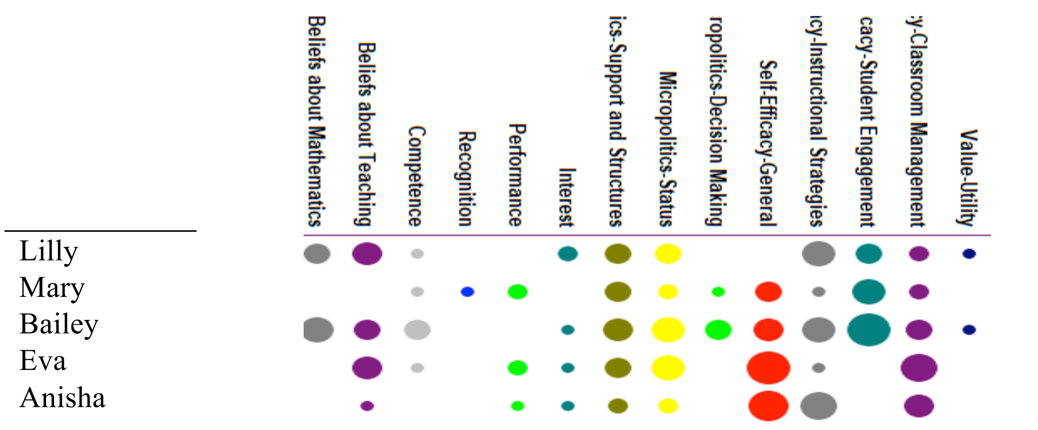

Figure 2: Trends Based on Frequency of Theme and Participant

Although there is value in considering teachers' professional identity individually, exploring the construct in relation to other factors helps to provide a more complete picture of how teachers may see themselves within the larger community of educators.

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MATHEMATICS TEACHERS' PERCEPTIONS OF THEIR INSTRUCTION FOR MULTILINGUAL LEARNERS THROUGH PROFESSIONAL DEVELOPMENT EXPERIENCES

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As the demand to challenge and attend to multilingual learners has increased, teachers have not received adequate professional development to combat biases and perceptions implicitly engrained throughout the education system, especially in mathematics classrooms. This study implemented a studio day professional development cycle with inservice teachers who worked with multilingual students in Math 1 classrooms. This study examined teachers' initial perceptions of multilingual learners and their understanding how to prepare for, challenge, and support multilingual learners. Teachers reported that, while their previous learning experiences around multilingual learners and mathematics were limited, this professional development opportunity allowed them to extend beyond simply attending to vocabulary to consider how to access text in rich ways to engage their students in more meaningful learning.

Keywords: Equity and Diversity; Teacher Education – Inservice/Professional Development; Marginalized Communities; Affect, Emotion, Beliefs, and Attitudes

Multilingual learners are among the fastest growing student populations in U.S. schools (National Clearinghouse for English Language Acquisition, 2009). The increase of multilingual learners is not an isolated phenomenon. Each state has experienced an increase in this population (Cheuk, 2016; Lee & Buxton, 2013), and it is expected that multilingual learners will make up 25% of the students in K-12 settings by 2025 (National Education Association, 2005). Despite the current numbers and projections, problematic trends regarding teacher preparation and practices have persisted in relation to multilingual learners, including deficit-based thinking models among teachers (de Araujo et al., 2016; McLeman & Fernandes, 2012; Pettit, 2011). In an attempt to address such deficit-based thinking, this study used a "studio days" model (Von Esch & Kavanagh, 2017) to introduce teachers to instructional mathematical routines that could engage multilingual learners in rich mathematics content (Kelemanik et al., 2016). This study sought to answer the following research question: How did mathematics teachers' perceptions of multilingual learners and their understanding of how to challenge and support multilingual learners evolve as teachers engaged in professional development experiences?

Theoretical Framework

This study is organized around two complementary theoretical ideas—instructional mathematical routines and key principles of reform-based instruction for multilingual learners. Both are meant to help teachers think about ways to engage multilingual learners with content in meaningful ways.

Instructional Mathematical Routines

Instructional mathematical routines are intended to support students to engage productively with content, providing them with tools that they can grow familiar with and return to regularly so as to solve cognitively demanding mathematics tasks (Kelemanik et al., 2016). Routines allow students to focus on their learning, because they provide structured ways for students to make sense of rich, challenging mathematics; build important mathematical thinking habits; and provide more students with access to important mathematics. Zwiers et al. (2017) developed routines specifically for multilingual learners to help teachers amplify, assess, and develop these students' language in

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mathematics classrooms. As such, multilingual learner student engagement in these mathematical instructional routines can help them develop mathematics thinking and language simultaneously.

Key Principles of Reform-Based Instruction for Multilingual Learners

Five key principles of reform-based instructional practices for multilingual learners in mathematics classrooms also guided this work (Roberts & Bianchini, 2019). These principles grounded our work with teachers, the conversations we had with teacher participants about the teaching and learning of mathematics for multilingual learners, and the analysis of data collected. In this paper, we focus on our fourth principle, *identifying academic language demands and supports for multilingual learners* (Aguirre & Bunch, 2012; Lyon et al., 2016). This principle focuses on the language demands in the tasks teachers provide and asks teachers to implement appropriate supports so that all students can read disciplinary texts, share their ideas and reasons in whole class and small group discussions, and communicate mathematics information in writing.

Methods

Our study was situated in one school district in Central California that included a substantial number of multilingual students. Teachers participating in this ongoing study were engaged in a two-year professional learning program organized around mathematics "studio days" for multilingual learners (Von Esch & Kavanagh, 2017), in which teachers developed and studied a single lesson focused around one instructional routine and one mathematics language principle during each cycle. The findings of this paper come from the first studio day cycle. Using Von Esch and Kavanagh's professional development model of studio days, we created a cycle of three professional development meetings for our participants. The studio day cycle of interest paired the instructional mathematical routine "Three Reads" (Kelemanik et al., 2016), which provides students access to rich text, with the principle *academic language demands and supports.*

Participants

Nine mathematics teachers from three high schools and their district mathematics instructional leader (who helped facilitate the professional development) participated in this study, with four teachers serving as our focal teachers. Of the nine teacher participants, five were female and four were male. Seven were White/Caucasian, one was Latinx, and one was Asian American. One teacher was in her first year, three had 1-4 years of experience, and four had 10-19 years of experience. One teacher was bilingual (Polish), and the rest were monolingual English-speakers. At Ash High School, the percentage of multilingual learners was 4.5%; at Birch High School, 6.0%; and at Cedar High School, 9.4%. Teachers noted that they had both multilingual and reclassified students in their classes.

Data Collection

We conducted two individual semi-structured interviews (Glesne, 2011) with the four focal teacher participants and the district mathematics instructional leader to understand how teachers supported students, especially multilingual students, in accessing mathematical content. More specifically, the pre-interview explored teachers' perceptions of multilingual learners, what they did to support multilingual learners in their classrooms, and their prior experience with preservice or professional development tailored towards challenging and supporting multilingual learners in mathematics. Following the first studio day cycle, in the post-interview, we asked participants about how they supported multilingual students to access text and to attend to academic language demands. We also asked about their experiences with the studio day cycle.

Data Analysis

We qualitatively analyzed both interviews across all five interviewed participants to identify key pieces of talk related to the following: (1) teacher preparation and professional development; and (2) attitudes towards their preparedness for challenging and supporting multilingual learners. We compared participants' responses across the two interviews. We drew on fieldnotes from studio days and classroom observations as ancillary data to triangulate our findings. Throughout the data collection and analytic process, we wrote analytic memos to develop our ideas, test our conjectures, and track our research processes (Yin, 2016).

Findings

Our first set of findings examines teachers' baseline perceptions of multilingual learners and professional learning experiences related to multilingual learners. The second set focuses on teachers' evolving perceptions related to multilingual learners and their preparedness in challenging and supporting multilingual learners in their mathematics classrooms following their participation in the studio day cycle.

Multilingual Learners – Initial Perceptions

All four teachers reported having large Spanish-speaking multilingual learner populations in their Math 1 classes. All four also used strategies in their classes to support their multilingual learners. Specifically, while many students were reclassified throughout the district, the teachers reported that they still implemented academic language support for these students in similar ways to students officially classified as "English learners." For example, Ms. Lacrosse reported that over 50% of students in her typical classroom were classified officially as "English learners." Her approach to supporting multilingual learners in communicating their reasoning was the following: "Even if they don't have a lot of academic knowledge, they do have at least enough to be able to express it at the level they are at and be exposed to others who are richer [in linguistic ability]." One of Mr. Huerta's supports was "rewr[iting] the text to make it simpler." Ms. Parker went further, rewriting two to three of her lessons each week to adapt the text for multilingual learners.

Professional Learning Experiences with Multilingual Learners – Initial Perceptions

Of the four focal teachers participating in this study, three mentioned having extensive experience with multilingual learners through prior teaching assignments, one shared that this was their first school year working with multilingual learners on a full-time basis. Only one of the four focal teachers mentioned receiving any explicit training related to multilingual learners during their teacher education program. The other three focal teachers stated that their preservice teacher education did not include any class or support around multilingual learners in mathematics specifically. All four focal teachers discussed a lack of district-mandated professional learning experiences related to multilingual learners, including during the implementation of a new curriculum three years prior. While Mrs. Hope, the district mathematics instructional leader, did share that she provided supplemental support through her typical interactions and coaching time with the teachers, it appeared this was the only professional development experience regarding multilingual learners and mathematics the district provided.

Perceptions of Multilingual Learners – Post Studio Day 1 Cycle

Through their studio day cycle participation, focal teachers reported shifting their focus for multilingual learners from singular vocabulary words for a given lesson to allowing multilingual learners to access and participate more fully through the implementation of the "Three Reads" mathematics language routine. Ms. Lacrosse discussed that her perception of supports for multilingual learners changed from focusing solely on vocabulary or specific problems to providing access to key mathematical ideas: "It's about reaching out and providing more support so that they can access the material, reading, writing." Focal teachers also discussed that student engagement with the text, including for multilingual learners, increased through the use of the "Three Reads" mathematics language routine. Mr. Huerta brought up the importance the routine had in his class as it was being implemented more frequently, signaling to the students that the process of engaging with the text was important. With multilingual learners engaging with the text at a higher frequency and at a higher cognitive and linguistic level, teacher perceptions of their multilingual learners appeared to be challenged. Still, Mrs. Hope, the district math instructional leader, reiterated the need for a future focus on communicating reasoning while using academic language.

Professional Learning Experiences with Multilingual Learners – Post Studio Day 1 Cycle

Teachers reported the studio day cycle was an effective opportunity to analyze the teaching practices they implemented as they related to academic language, especially with regards to multilingual learners. Ms. Lacrosse discussed both the benefit of having other teachers provide feedback on her teaching and the benefit of watching other teachers implement the same lesson. Mr. Huerta shared that, even though he and another teacher at his school site did not teach the same lesson during the studio day, he was still able to adapt his teaching practice after observing his colleague. All focal teachers explained that a key benefit was being able to interact with and work with other teachers at their school sites and within the district. Mr. Ming reflected on the value of building a community of critical educators on his campus through observing other teachers and reflecting on how those teachers may be working with multilingual learners: He found useful "the whole coming together, drafting of lessons together, then executing the lesson, and then debriefing on it." Mr. Huerta echoed this sentiment, saying, "I would say the other beneficial thing… being able to see the math studio day happen at all three campuses together and having the collaboration of teachers that were in different schools within our district."

Discussion and Conclusions

We found that teachers' perceptions of multilingual learners shifted from holding beliefs that multilingual students could not access text-heavy curriculum at the same level as non-multilingual learners to realizing that they could challenge and support multilingual learners using mathematics language routines. The mathematics language routine of focus, "Three Reads," allowed teachers to engage their multilingual students in more rich content and language learning beyond simply reviewing math vocabulary terms. Teachers were able to provide their students with practice in accessing text and thus in reading, writing, and talking about math in richer ways. Teachers valued this professional learning opportunity and the chance to interact with fellow Math 1 teachers as a way to improve their mathematics teaching practices related to multilingual learners. Our work suggests the need for future professional development and research efforts to focus on other important aspects of academic language in mathematics, including communicating reasoning.

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A FRAMEWORK FOR THE FACILITATION OF ONLINE PROFESSIONAL DEVELOPMENT TO SUPPORT INSTRUCTIONAL CHANGE

UN MARCO PARA LA FACILITACIÓN DEL DESARROLLO PROFESIONAL EN LÍNEA PARA APOYAR EL CAMBIO INSTRUCCIONAL

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In this research report we discuss the development of a framework regarding the facilitation of online professional development geared at supporting instructional change at the undergraduate level. The research in undergraduate mathematics education includes various large-scale projects aimed to support individuals or departments in reforming their instruction to align with recommendations from professional organizations and existing mathematics education research standards. One area that needs attention is the use of online synchronous environments to match faculty across the world and form collaborations to support the inclusion of student-centered activities in their mathematics classrooms. This research report discusses the actions that facilitators take in these environments and lays the groundwork for the use of this framework in our and other contexts going forward.

Keywords: Post-Secondary Education; Teacher Education – Inservice / Professional Development; Systemic Change

Instructional shifts towards student-centered pedagogies are taking place throughout North America within mathematics departments. This change is oftentimes centered around individual faculty (e.g., Author, 2019; Speer & Wagner, 2009) but also from the perspective of larger groups of faculty (e.g., Author, under review; Hayward & Laursen, 2016) or even departments at-large (e.g., Apkarian & Reinholz, 2019; Laursen, 2016; Reinholz & Apkarian, 2018). Notably, professional communities also call for this instructional reform (Mathematical Association of America [MAA], 2015). The research in undergraduate mathematics education community has embarked upon numerous large-scale research projects to investigate how to support instructional change (e.g., Author, under review; Kuster et al., 2016), namely to make instruction more student-centered. Additionally, this community has engaged in large scale projects to support departments in improving instruction and student outcomes (e.g., Association of Public & Land-Grant Universities [APLU], 2016).

Our multi-institute collaborative grant, BLINDED, is one such project in which we aimed to not only support mathematicians in reforming their instruction with various support models but to research those support models' impact on the mathematicians and their communities. Our support model consisted of instructional materials (both for the student and faculty), a summer workshop, and online professional development, which we classified as an online working group (OWG). The OWG offered an opportunity for faculty to collaborate on their instruction through a lesson study model (Demir et al. 2013) in online synchronous environments. In this OWG, participants engaged in lesson studies on multiple units of Inquiry-Oriented (IO) materials (Rasmussen & Kwon, 2007) by doing the mathematical tasks from those units, anticipating student thinking that could arise from those units, filming and then subsequently bringing video clips from that instruction to the OWG to share and discuss.

In previous research, we have discussed the development and usage of a framework to categorize and understand the conversation that occurs when OWGs are discussing the sharing of instructional video as a means to support their instructional change (Author, under review). However, the next step that emerged from that work was to analyze the role that the facilitator played in that OWG. Given the importance of the role of a facilitator in professional development settings (van Es et al. 2014), our next research steps were to develop a framework to categorize and understand how facilitation occurs of these OWGs. In this endeavor, we sought to understand the facilitation of OWGs when facilitators initiate discussions about the mathematical content of novel IO curricular materials. We will discuss the development of a framework to understand the facilitation of these OWGs. The research question for this research report is: *What actions do facilitators take within online working groups focused on doing and understanding the mathematical content of novel IO curricular materials?*

Methods

Research Setting and Data Collection

Data from this research report comes from a large NSF funded project, BLINDED. BLINDED recruited mathematicians in 2015-2017 who were interested in changing their instruction to be more student-centered and specifically use one of the IO curricula: differential equations, linear algebra, or abstract algebra. During the first year of the project, the three Principal Investigators led their respective OWG. In subsequent years, the project team was able to double the number of OWGs that could be facilitated by recruiting the previous year's participants to lead their own OWG. Consequently, in 2016 and 2017, 4 facilitators, who were previously participants, each led their own OWG. The development of our framework comes from these 4 individuals' OWGs. Each OWG was screen recorded using QuickTime and all OWGs were transcribed. Each of these OWGs consisted of 3-4 participants. As this analysis focuses on when the facilitators were leading discussion on doing the mathematics from the novel IO curricula, this yielded 14 transcripts for analysis.

Data Analysis

The creation of the framework followed an iterative process of revision and refinement via individual open coding and comparison between the researchers (Creswell & Poth, 2017). Altogether, 14 transcriptions of videos were investigated, coded and compared by at least two researchers in each iteration. During the first iteration, we analyzed two video transcriptions and proposed descriptors for the action that the facilitator took. In crafting our descriptors, we consulted the work from van Es and colleagues (2014) to look for common threads. In their work they focused on developing a framework on *how* facilitators could use in-the-moment moves to support productive discussion while viewing video of instruction (van Es et al., 2014). We then convened and compared our suggestions for each of the corresponding facilitator's actions, by grouping similar descriptions in one category and assigning that category a code. For instance, the expressions *chose participant to start, called on participant* and *called on a participant to share their thoughts* were grouped under *asked participant to share their mathematical work* and assigned the code SHARE; *brought experience from the classroom to the conversation, related it back to what students would do* and *tried to make sense of why students have made mistakes in the past* were coded as PAST to indicate that the facilitator reported on what students have done in the past. This process generated a first draft of the codebook.

Following this step, the remaining twelve video transcriptions were assigned to two researchers each. Every pair individually coded their assigned portions and then came together to compare their results and agree on one code per statement. Then, the three coders convened to discuss the overall results. We saw the need to distinguish between *what* the facilitator is claiming or asking, and *how* they were doing it. In particular, we focused on the type of statement that was being made (imperative, interrogative, exclamatory, and declarative).

This led to the second iteration of coding for the same initial two transcripts, where each statement was assigned a *what* and *how* code. We subsequently reconvened to compare individual results. Additional codes were suggested, initial ones were redacted and eventually we noticed a commonality between some codes which allowed us to create the elements of the framework, the *facilitation (how)* and *conversation (what)* themes. Within the facilitation theme, we generated five categories and two actions that pertain to each category. Additionally, we realized that only imperative and interrogative statements were meaningful in certain actions that the facilitator made under the *gathering* and *verifying* categories.

Results: Facilitation Framework

Figure 1 is the framework for facilitation of online professional development. The framework contains two overarching elements: facilitation and conversation. Facilitation is the element of the framework that would transcend the context of the OWG. The conversation categories emerged from our previous research as well as this analysis and would be different if this framework was applied in different contexts. While we believe the conversation codes could serve as a starting point for other groups, the nature of the content under consideration will largely determine these categories.

Framework Elements	Categories	Action	Description
Facilitation	Progressing Session	Transition	Transitioning to a new or different mathematical task/subtask
		Think	Giving individual think time
	Gathering**	Individual	Asking for a specific individual to share or contribute
		Open	Opening the floor to the group for a response
	Verifying**	Elaborate	Asking for an elaboration or clarification from an individual
		Restate	Inquiring about a participant's contribution by restating that contribution
	Contributing	Provide	Providing input, advice, or insight
		Restate	Restating a participant contribution
	Supporting the Group	Encourage	Encouraging statement
		Acknowledge	Acknowledging or thanking participant(s) for contribution(s)
Conversation	Content	Pedagogy	Teacher instructional moves or ideas about teacher instructional moves
		Goal	Learning goals of task/subtask/lesson
		Time	Timing information about lesson
		Past	Previous students' mathematical reasoning and/or work
		Anticipate	Anticipation of students' mathematical reasoning and/or work
		Mathematics	Mathematical content or mathematical technology

	SubCode	Description
**Subcode for Gathering and Verifying	Imperative	Makes a request or gives a command of some sort
	Interrogative	Asks a question, inquires for more information

Figure 1: Framework for Facilitation on Online Professional Development

Here, we focus on the categories within the facilitation element of the framework. There are five categories of actions that our facilitators did in online working groups. Namely, they Progressed the session, Gathered information, Verified information, Contributed their own thoughts to the session, or Supported the group. Each category yielded two actions, or codes. For example, under the Gathering category, we find two actions: Individual and Open. Individual was a code used to describe when the facilitator was asking a specific individual to share their thoughts whereas Open was used to code for when a facilitator asked for any volunteer to share their thoughts. While both actions concern gathering information, they are clearly two distinct actions a facilitator can take during an OWG. It is worth noting that Restate is an action under the Verifying and Contributing categories as a facilitator would Restate for different purposes. For example, a facilitator would Restate a participant claim with the (implied) intention being to inquire about a participant's contribution. That is, the facilitator would Restate what the participant said for the purposes of having that participant expound on what they had just said. Whereas, a facilitator would also Restate a contribution, potentially in paraphrased ways, as a means to contribute to the conversation with no (implied) intention of getting a response from the original commenter of that statement.

Another important aspect of the framework to note is the inclusion of the subcodes for Gathering and Verifying. All four of the actions under those two categories are about the facilitator doing something that desired a response from someone, whether that be a specific Individual, Opening the floor to a question, asking for an Elaboration, or Restating for the purposes of further explanation. However, in our analyses, it became clear that there were always two different ways to achieve those goals. We used terms from the field of linguistics. Namely, imperative requests are ones in the form of a sort of command; whereas, interrogative requests are ones that ask for more information. For instance, a facilitator would call upon a specific Individual to share or contribute by making a request or giving an Imperative command (e.g., "Participant, tell me what you were thinking about.") This would contrast with the same action, Individual, but could have been asked in the form of an Interrogative question: "Participant, what mathematical theorem led you to that conclusion?" We treated instances such as these as both falling under the action of asking a specific Individual to contribute, but the means the facilitator went about that were different. This was the case for the Gathering and Verifying codes so for all of those codes they received the subcode of either Imperative or Interrogative.

Some coded examples of this are:

Facilitator - Imperative: "So, keep going with that [line of thought] Participant."

Facilitator - Interrogative: "So, I, we are talking about … the Sudoku property and each symmetry appears at most once and each symmetry appears at least once. Is the hint here, so what are students going to approach and how are they gonna approach this question?"

Conclusion

Through our iterative coding process, we developed a framework that captures the actions facilitators take in overseeing online professional developments. These categories including Supporting the group, Progressing the session, Contributing to the discussion, Gathering information, and Verifying information. Part of our framework, within the Gathering and Verifying categories, also notes the different ways in which facilitators can gather or verify information. Namely, we differentiated between imperative requests (e.g., "participant, tell me what you think about that") and interrogative ones (e.g., "participant, do you agree with the other participant's claim?"). Our future work will consist of in-depth case studies of each facilitator to enhance this understanding. Importantly, as noted, the framework we developed shares many similarities to that of van Es and colleagues (2014). The implications from this are important. Characteristics and actions of facilitators are key to understanding how facilitation occurs and how facilitation techniques can be trained and learned, while the content may be salient.

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INFLUENCES ON EARLY-CAREER MATHEMATICS' TEACHERS VISION OF TEACHING WITH TECHNOLOGY: A LONGITUDINAL STUDY

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This paper reports on a longitudinal study of mathematics teachers' development of a vision of teaching with technology where we document professional events and activities that point to continued evolution and devolution of those beliefs. We extend earlier work and ask participants to reflect on the experiences they have had as early career teachers, and how they have influenced their beliefs since graduation. We find that there are significant opportunities for professional learning after graduation, and recommend continued development of graduate-level coursework that is technology-dependent. We also find that the use of Desmos is particularly influential in changing beliefs about the role of technology.

Keywords: Instructional Vision; Technology; Teacher Beliefs; High School Education;

Background

Preservice secondary mathematics teachers (PSMTs) at Miami University take a required mathematics course, where they revisit their own learning of secondary mathematics and investigate concepts by way of problem solving with various technological tools. In previous work, we sought to understand the impact of this course on future teaching practices. We defined *vision of teaching with technology* as an imagined state wherein PSMTs are able to translate their technological beliefs into principles on which they will base future instructional decisions and practice (Cox & Harper, 2016). We found that as a result of participation in this course, PSMTs develop a vision of teaching with technology that is better aligned with that expressed in modern policy documents (e.g., NCTM, 2000, 2014, 2016). We also found that PSMTs draw heavily on their index of personal experiences to illustrate their visions and that descriptions of curricular experiences were central of what PSMTs referred to as "responsible use of technology" (Cox & Harper, 2016).

Drawing on the work of Phillip (2007) and Ertmer (2006), and Pajares (1992), we have delineated technological beliefs as separate from general pedagogical beliefs (Cox & Harper, 2016). We define teachers' technological beliefs as understandings, premises or propositions about the role(s) technology plays in instruction. Thus, when we refer to technological beliefs, we mean those beliefs concerning the role technology plays in mathematics instruction.

We know that inservice teachers seem more likely than PSMTs to perceive the cognitive benefits of technology beyond motivation and fun. Some teachers recognize its power to visualize difficult concepts or meet the needs of diverse learners (e.g., Wachira, Keengwe, & Onchwari, 2008), and inservice teachers seem more likely than PSMTs to believe technology has value beyond a computational device or answer checker (e.g., Wachira & Keengwe, 2011).

Kagan (1992) noted that if a teacher education or professional development program is to be successful at promoting belief change among teachers, "it must require them to make their preexisting personal beliefs explicit; it must challenge the adequacy of those beliefs; and it must give novices extended opportunities to examine, elaborate, and integrate new information into their existing belief systems" (p.77).

After making the PSMTs' preexisting personal beliefs explicit in their vision of teaching with technology, we wanted to examine and document key professional events and activities that point to both continued evolution and devolution of their technology beliefs. This paper reports on a longitudinal study of a mathematics teachers' development of a vision of teaching with technology. Since beliefs and practice are dialectic (Thompson, 1992), it is likely that an individual's vision has been impacted by their teaching and other experiences. Moreover, given the importance of indexing personal experiences when articulating a vision, we wanted to document professional events and activities that point to continued evolution and devolution of mathematics teachers' vision of teaching with technology. We sought to answer the research question, what seminal teaching experiences impact an individual's vision of teaching with technology?

Methodology

Participants in this study are recent graduates (2013-2017) of Miami University who have been teaching secondary (6-12) mathematics for at least two years. Supported by our alumni office and social media connections, we identified a pool of 80 possible alumni. Within that pool, 44 had participated in an earlier phase of this research (Cox & Harper, 2016). Those for whom we had collected earlier writing samples documenting their vision of teaching with technology were sent the *Teaching with Technology–Longitudinal* online survey. The Longitudinal survey was sent to 44 individuals, of which 9 responded (response rate of 20.5%). One of the nine responded that they were not currently teaching mathematics. The remainder of the pool (n=36) were sent the *Teaching with Technology* online survey. Of the 36, thirteen responded (response rate of 36%), all of whom reported that they are currently teaching mathematics.

In the first part of both surveys, we invited participants to narrate their experiences with teaching mathematics with technology. Rather than ask for the titles of specific courses that they recently taught, we asked them to imagine a recent course and then identify the mathematics taught within that course by broad discipline. For instance, a participant might report teaching a course that is 75% algebra, 10% statistics, and 15% trigonometry. Then, they are asked to report the types of technology they used. We chose not to list specific brands of technology, and instead asked for the technology genre. For instance, we asked about Dynamic Geometry Software (DGS), rather than specifying GeoGebra or Desmos.

The *Teaching with Technology–Longitudinal* and the *Teaching with Technology* surveys were conducted digitally using Qualtrix software. Both versions of the survey include two lines of inquiry. First, we asked participants to describe their beliefs about the role technology plays in mathematics teaching. Participants in previous phases of research were familiar with this question having answered something similar at the conclusion of a required mathematics technology course in their undergraduate program. In those cases, we phrased the questions on the survey differently. We provided them with their original *vision of teaching with technology* statements written at the conclusion of the mathematics technology course, and invited them to identify passages that still represent their thinking, as well as passages about which they now think differently. Second, we asked participants to identify experiences such as graduate study, professional development, or classroom episodes; as well as the impact (or lack thereof) the experiences have had on their developing vision of teaching with technology.

Results

We hypothesized that there could be many potential influences on teachers' vision for teaching with technology once they left their teacher preparation program. We chose to focus our questions on three: graduate education, professional development workshops and colleagues. Due to space restrictions, we will expand on only one of these, professional development. Then, we will take a longitudinal look at how these teachers' beliefs have evolved.

Professional Development

Of the 21 participants, fifteen did not report having participated in any professional development (PD) workshops related to the teaching of 6-12 mathematics with technology. Here, we focus on the six who did. Of those six, five were specific about the focus of the PD. Four of the six experienced math-specific professional development. This took two forms. Some of the math-specific technology PD described took the form of individual or department-based PD focused exclusively on Desmos. Other teachers experienced PD at state conferences where they attended sessions about incorporating technology into their classroom planning and teaching. One commented about the benefit of conference-based PD, "I was able to gain insight about other teachers' experiences with new technology and collaborate with other math teachers to learn more about the availability and access of technology for students in other districts."

In addition to math-specific PD, two teachers received other technology-focused PD, but reported that it was oriented toward classroom management or non-mathematical applications such as Kahoot, Google Classroom, or other apps.

Additionally, we asked participants to tell us whether this activity influenced their vision of teaching with technology, or impact how they incorporated technology into their teaching. All felt that they had felt influenced, however the way that they described the influence was different for those experiencing math-specific and non-math-specific professional development. For those who reported non-math specific PD, the influences they expressed were more oriented toward pedagogical shifts in assessment, differentiation, and project-based learning. Those who experienced math-specific PD reported that they now incorporated specific technologies into lesson planning and teaching. These teachers were better at generating content using GeoGebra and Desmos, and felt more confident about how to incorporate it more regularly in their teaching. In all cases, the influences were most often described as improved technical knowledge and in no cases did the descriptions indicate a change in TPCK (Mishra & Koehler, 2006).

A Longitudinal Look

We now narrow our view to just those eight participants who were a part of our earlier study, where they wrote statements about their vision of teaching with technology. For these eight participants, we have additional data wherein they respond to these statements from their current perspective. We chose to categorize these responses as either *renunciations* or *amplifications* of their earlier beliefs. We asked participants to identify up to six passages in their original sample and describe why they either continued or stopped believing something specific and related to teaching mathematics with technology. Not all participants chose to either renounce or amplify a passage, and some teachers chose more than one passage to address.

Renunciation. Three participants identified portions of their previous writing that they wanted to renounce. All three participants no longer believed that students would use technology as a distractor or to "goof off" during class, and came to see it as more valuable than before. "As a teacher I have found that sometimes this free exploration with technology allows for students to make deep connections and allows for them to think freely and oftentimes more critically." One participant went further and renounced their earlier beliefs about technology as indiscriminately good. They wanted to make a more clear distinction between those technologies that simply make teaching easier and those that impact mathematical learning.

Amplification. There were seven times that participants identified portions of their previous writing to amplify. From these passages, we find that early career experiences serve to amplify the need to incorporate digital technologies into education in general. Participants still believe that the educational needs of students change over time and reflects the increased presence of digital technologies in the workforce. As one participant noted, "this passage resonates with me, because I know that I am in a profession where I will have to be continually changing the way I teach. ...our job is to prepare them, and technology allows us teachers to give our students more appropriate learning experiences.

There were two participants who chose to amplify statements that were more specific to mathematics instruction. The first of these focused on the potential of technology to go beyond the application of learned procedure and address the fundamental "why" behind the algorithms. The other participant amplified the role of technology in helping students be more reflective learners of algebra when they use technology to get quick and early feedback.

Discussion

Looking across the three spheres of influence: Additional Coursework, Professional Development and Colleagues, it is clear to us that early career teachers continue to have opportunities to learn about technology for teaching. With more than half pursuing graduate degrees, it would be a good idea to continue to develop graduate-level content courses that incorporate technology for teaching mathematics. Similarly, it is clear that mathematics-specific professional development can influence teachers to utilize new technologies in lesson planning and instruction. Developing stand-alone experiences that can be enacted during department meetings or building-wide PD time might be another way to influence teacher beliefs about the role of technology in mathematics instruction.

Based on specific responses, we recommend incorporating Desmos into professional experiences for early-career teachers. Whether in graduate classes, PD, or in conversations with colleagues, participants are talking about and learning about Desmos. Needing to learn about a technology that students will be using on standardized assessments seemed to be a strong motivator for individual learning, but also district-wide professional development planning. Further, when Desmos is used in graduate-level mathematics curriculum, teachers find that exposure useful and influential.

Conclusion

Our original findings indicated that our mathematics problem solving with technology course had a short-term impact on preservice teacher beliefs about teaching mathematics with technology. This study documents not only the longevity of that belief change, but also the identification of significant experiences in the lives of new teachers that either act to amplify or disrupt those changes over time. This study also positions new teachers as *people from whom we can learn*. We feel that studies that position mathematics teachers as knowledgeable professionals are especially needed at a time where teachers are often deprofessionalized and undervalued.

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MATHEMATICS SPECIALISTS AND TEACHER LEADERS: AN ONGOING QUALITATIVE SYNTHESIS

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Mathematics specialists are sometimes known as math coaches, mathematics teacher leaders or other titles. The definition in this paper is a facilitator or leader of teachers focused on professional development in mathematics. The focus of this qualitative synthesis of the literature is to investigate how this role has been studied, defined, and investigated. This preliminary analysis has documented the research methods used in studies, examined the focus of studies, and raises questions about the different types of teacher leadership that may exist.

Keywords: Elementary School Education, Instructional Leadership, Teacher Education - Inservice / Professional Development, Research Methods

The term mathematics specialists were first introduced as a concept in an editorial in *Teaching Children Mathematics* (Dossey, 1984). However, research about mathematics specialists was dormant for a few decades following that call to action until the early 2000s. McGatha and Rigelman (2017) offer a framework that organizes terminology for the different roles that might fall under the umbrella term of mathematics specialist. One set of responsibilities involve the mathematics specialists as professional development facilitator or leader of teachers. A second set of responsibilities involves their work teaching mathematics to students either as a teacher whose primary content to teach is mathematics or as a teacher who might work with small groups of students who need focused instruction in mathematics. We frame these as "responsibilities" because many mathematics specialists have multiple types of responsibilities. For example, the mathematics resource teacher might work with small groups of students in need of extra mathematics support but could also be called on to facilitate lesson study with grade-level teams of teachers. The collection of roles and responsibilities is more like a terrain of options than a set of discrete categories.

Recommendations for professional development for teachers consistently point to needing ongoing, school-based support for mathematics (Darling-Hammond, Hyler, & Gardner, 2017; Woulfin & Rigby, 2017). Models such as lesson study, professional learning communities (PLC), math labs or individual coaching often include a facilitator, coach or specialist at some stage. For instance, a math coach might be assigned at a school to work with all first-grade teachers in a PLC to provide additional knowledge and expertise in mathematics. Their role in that PLC might vary over time and the math coach may work with a different grade-level PLC the following year. However, that role is under-investigated in the research, but studies are emerging about the knowledge and skills required. As such, this paper presents early findings and a preliminary report to raise questions from existing research about mathematics specialists.

Purpose

The purpose of our work is to synthesize existing research and develop understanding of the positioning of mathematics specialists and teacher leaders in the research. Our focus in this analysis is on the mathematics specialists as supporting teacher professional development and not on their work as teachers. We ground our definition in Campbell and Malkus (2013) in recognizing that a mathematics specialist is an on-site support person whose uses knowledge and expertise in mathematics content, pedagogy, and children's learning trajectories to assist teachers with their content, pedagogy, and understanding of children's learning trajectories. While many mathematics specialists are primarily teachers of mathematics, the role should be of interest to mathematics education because of its potential to provide school-based professional development. In this study, we seek to identify patterns and trends in research about mathematics specialists in schools. Our goal is to both create a framework for their roles and responsibilities and to describe how that research has been conducted to gather evidence regarding the complexity of the roles and responsibilities of mathematics specialists. Our purpose is to recommend future directions for research about mathematics specialists and to synthesize what is already known.

Methodology

We have begun working through the stages of qualitative synthesis suggested by Thunder and Berry (2016). The first step was to create criteria for the studies, identify the databases and develop a list of search terms. Each member of the team was responsible for selecting a database for the set of search terms. Then, we aggregated the list across those databases. Our first step was to eliminate pieces that were not articles (e.g., book reviews) and then to eliminate irrelevant articles (e.g., from fields outside education, focused on athletic coaching). For relevance, we included articles focused on mathematics and education. We coded articles that were clearly about mathematics specialists as MC, ADMIN for those regarding school leaders, PD for papers about professional development, and PST for articles about pre-service teachers; some articles were coded with multiple codes. In the portion of the review presented here, we have also pared down the list to identify those articles that mention math specialists (or related terms) in the title or the abstract, and thus coded MC. These articles have the greatest potential to provide insight about mathematics specialists. Later work may investigate how math specialists might appear in other parts of the publications.

Results

Overall, we can see that there has been an increasing trend in research that has been published about mathematics coaches and specialists between our target years of 1991-2018 (see Figure 1). Furthermore, we see the largest spikes in year-to-year publications between 2015-2016 (+7), 2002-2003 (+5), 2009-2010 (+5), and 2016-2017 (+5). Some of these spikes seem to be related to NCTM's release of Research Briefs in 2009 and 2015.

Wanting to better understand broad methodological trends for our 192 publications about mathematics coaches and specialists, we read through the abstracts and methods sections to identify the methods used in each study. As illustrated in Table 1, 72 studies utilized qualitative methods, 48 used mixed methods, and 23 employed quantitative methods. Furthermore, we identified 34 articles that had been published in practitioner journals, and given that they did not have discernable methods, we did not code for methods in this group. Last, we identified 14 items, including book chapters, editorials, and literature reviews that – similar to the practitioner pieces – did not have a discernible method. Thus, we simply coded these as Other. At this point, we are still including them in our database because we are interested in portrayals of the role of the mathematics specialists and different perspectives on the topic. So, while they may not be empirical research, they may still have something to offer in learning more about the work in practice and how it has changed over time.

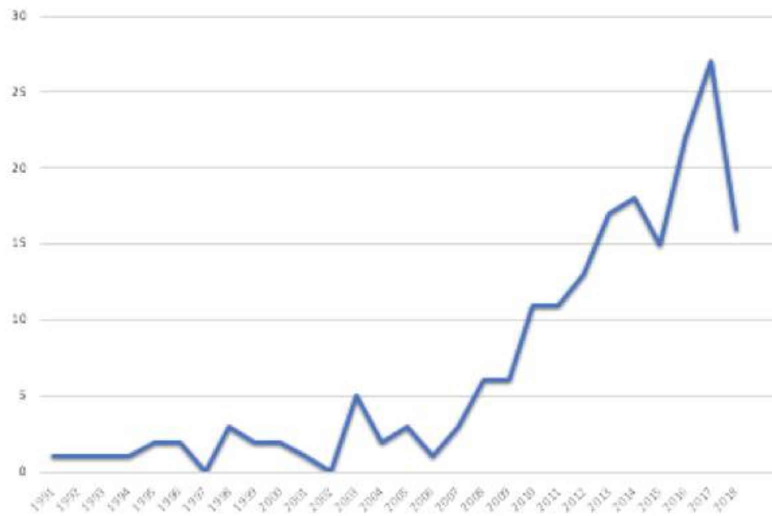

Figure 1. Overall Trends from 1991-2018

Last, and as previously mentioned, we read through the abstracts and assigned at least one of the following codes based on what the article was about: MC (Mathematics Coach), PD (Professional Development), PST (Pre-service Teacher), ADMIN (Administrator). Frequencies for each individual category and overlapping categories can be found above in Table 1.

Table 1. Methodology and Abstract Topic Frequency

Method	**Frequency**
Qualitative	72
Mixed	48
Practitioner	34
Quantitative	23
Other	14
Combination of Codes	**Frequency**
MC (only)	95
MC, PD	57
MC, ADMIN	19
MC, PD, ADMIN	8
MC, PST	8
MC, PD, PST	5
MC, PD, ADMIN, PST	1

Discussion

We are encouraged by the number of studies focused specifically on mathematics specialists and the increasing focus on their work, knowledge, and roles as distinct from other roles in schools. It is not surprising that research about them is likely intertwined with studies of teachers and administrators as their work is designed to include these groups. We are also encouraged to see the variety of methodological approaches that have been used as this will provide a richer and more nuanced understanding of the different types of work mathematics specialists do and its impacts on teaching and learning mathematics.

In narrowing a list of thousands of articles down to a shorter list, we encountered questions about what to eliminate and what articles to keep for our review. Some eliminations were clear (e.g., when the study is in a different field than education), but some questions have been more complicated. Two groups of questions include:(a) determining what makes a study have enough mathematics education to warrant further investigation and (b) considering mentoring pre-service teachers.

The first question we needed to consider was: *What is enough mathematics for a study to be about mathematics teaching and learning?* This may not seem like a complex question, but is complicated when attempting a large synthesis study. The first aspect is when mathematics achievement is used as a student outcome but the abstract does not include discussions of mathematics-focused interventions. Such studies may provide other academic supports for students, but are not focused on mathematics specifically. The second type of ambiguity is when it is not clear the professional development is focused on teaching in mathematics context. We felt only having a mathematics outcome variable was insufficient for inclusion when the intervention was focused on other aspects of teaching and learning. The second question we needed to consider was: *How similar or different the role of mentor teacher is from the role of a math coach?* A collection of studies that emerged focused on mentors of pre-service teachers. We have not yet answered the question posed. A math coach also does one-to-one work with teachers (e.g., co-planning lessons, observations, co-teaching), but mentoring a pre-service teacher may have different features. Both may fall under the broad category of mentoring, but we are not sure yet if mentoring pre-service teachers needs to be analyzed independently from other types of coaching or peer mentoring among teachers.

In addition to the two questions, we also note considerations in regard to abstracts. The first consideration involves the term mathematics specialists being included in the abstract. We have narrowed our list of articles down to 192 that, based on the abstracts, are investigating some aspect of math coaching work. However, there are hundreds more that do not mention that role in the abstract. This supports our claim at this point that mathematics specialists continue to be "hidden players " (Hjalmarson & Baker, 2020) in the research about mathematics specialists. "Hidden" in the present synthesis means that in studies of professional development, the role of the person who might be facilitating the professional development continues to be unmentioned or vague. The second consideration involves the need for more comprehensive, clear, or structured abstracts (Kelly & Yin, 2007) that describe the major aspects of studies (e.g., questions, research design, participants). Some journals already require such abstracts (e.g., *Journal of Engineering Education*). In terms of stakeholders or participants in teacher professional development studies, abstracts could include more about the facilitators of such experiences.

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TOWARDS REFLECTION ON THE PRACTICE IN A LEARNING COMMUNITY OF HIGHER-LEVEL MATHEMATICS TEACHERS

HACIA LA REFLEXIÓN SOBRE LA PRÁCTICA EN UNA COMUNIDAD DE APRENDIZAJE DE PROFESORES DE MATEMÁTICAS DE NIVEL MEDIO SUPERIOR

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In this report, the authors describe the creation of a learning community for mathematics teachers with the purpose to improve their knowledge about mathematics and its teaching. It also includes a description of how these teachers have structured reflection sessions on class planning, learning activity trials, analysis of teacher performances, and how to improve the design of the teaching sequence. An important result of the first 'Teaching-Reflection Cycle' carried out within the community is that learning about pedagogical content knowledge can be promoted.

Keywords: Teacher's knowledge, reflection on practice, teaching of functions.

Shulman's influential framework (1986, 1987) on the components of effective teachers' knowledge helped stimulate research to improve teaching in many subjects. In particular, for the teaching of mathematics, there are two key investigations. Ball, Thames, & Phelps (2008) extended Shulman's categories for the teaching of mathematics and to build a framework of mathematical knowledge for teaching (MKT). Rowland, Huckstep, & Thwaites (2005) adopted another approach also based on Shulman's work, they built a framework known as The Knowledge Quartet (KQ) focusing on the teacher's mathematics knowledge that emerges within the classroom.

High school teachers frequently do not receive pre-service pedagogical training. In Mexico, teachers of this educational level, start teaching without any type of instruction. They get their pedagogical content knowledge in practice. Courses are often provided. However, the rationale of the lectures is not well planned and sometimes topics are disconnected from the curriculum that teachers follow in their schools (Sánchez & Huchim, 2015; Sosa & Ribeiro, 2014).

Researchers have sustained that mathematical knowledge for teaching could improve the teaching of mathematics (Askew et al., 1997; Rowland et al., 2000).

Considering the aforesaid situation, a group of mathematics teachers –from a Mexican high school– have decided to create a learning community with the aim to ameliorate their teaching by improving their knowledge for teaching mathematics.

For the authors, the 'Learning Community' is a group with a shared identity, characteristics, or purposes. The learning is the objective that brings members together and gives meaning to the community for different purposes (Valdés, Pilz, Rivero, Machado & Walder, 2014, p. 56). A learning community provides: (1) a framework for teachers to learn and develop knowledge together, (2) opportunities to take advantage of teachers' tacit knowledge and make it public to be shared and criticized, and (3) support to be more than technicians who implement others' ideas to become thinkers, researchers and conceptualizers (Elbaz, 1983; Schön, 1983; Wood, 2007).

The purpose of this report is to inform the process of the creation of a learning community; how teachers used reflection to design, apply, and discuss teaching activities and how evidence of teachers gaining knowledge emerges during this process.

Mathematics Teachers' Learning Community

A group of 12 mathematics teachers from a high school level, who works on the same educational center, participated in an academic call to improve their employment status. Aware that they lack the necessary knowledge for teaching, those teachers created a learning community as a means to increase their knowledge and to obtain a promotion. However, during the first period of activities carried out in this community, only 5 teachers continue working together.

To structure the community, teachers used the method employed in the Keli Lesson Study, part of China's professional development program Xingdong Jiaoyu (Huang & Shimizu, 2016, p. 395). The program consists of two stages: practice and reflection. Like the Chinese approach, the community members also focus on professional development through reflective processes that center on how mathematical knowledge manifests itself in practice, and thus teachers gradually transform their practice. An important aspect of the learning community is that there is no institutional intervention, as it is a self-managed initiative of the teachers.

Schön (1983) differentiates two types of reflection that can occur and determine professional knowledge of an individual: reflection in action and reflection on action. Reflection in action is the process of monitoring and adapting behavior in context, while reflection on action is the process of evaluating what has already been done.

Community in Action

The structure of the activities carried out within the community that will be referred to as 'Teaching-Reflection Cycle' is composed of 4 stages:

1. A teacher designs or proposes a Hypothetical Learning Trajectory (HLT) (Simon, 1995) to be revisited by the other members and to plan its trial in the classroom.
2. A member of the community tries out the HLT, and another colleague video-records it.
3. Community members analyze the video and reflect on the teaching activity carried out in the classroom by their colleagues. To think about the teaching process, the teachers use a protocol structured by the researcher (who is a member of the community). Teachers also make suggestions to modify the HLT to apply it later on. All the community sessions are recorded on video and the totality constitutes research data.
4. A process of individual insight reflection is performed.

Activities carried out in a Teaching-Reflection Cycle related to a minimum area problem

1. During the first meeting for the aforementioned cycle, teachers agreed to apply an HLT based on the quadrilateral area problem (see Figure 1) proposed by Lola, a member of the community. She knew the problem from a master's course she took and has implemented it in her classroom several times. The members of the community agreed to deal with it in Mario and Tadeo's groups, but they switch activities. Mario taught the HLT and Tadeo recorded the session in the video.
2. In a second meeting, members of the community watched that video, analyzed and discussed Mario's intervention, and reflected on the teaching and learning processes carried out. They also gave Tadeo's ideas for the next trial.
3. Tadeo taught the HLT in Mario's classroom and Mario video-recorded the trial.
4. In the course of the third meeting, teachers watched the second video, discussed and reflected on both interventions, and reached conclusions concerning how other teachers can use the HLT in their classrooms.
5. Both teachers made an individual insight reflection about what the whole process contributed to their own teaching practices.

Stages 1) and 2) retrieving Schön (1983) are the moments of reflection in action in which teachers analyze and discuss the relevance of the HLT and its implementation. Stages 3) and 4) is the moment of reflection on action, in which teachers evaluate what has been done to make necessary changes considered for future applications.

The Minimum Quadrilateral Area Problem

The HLT of the quadrilateral area problem, included in Figure 1, was designed with the next characteristics:

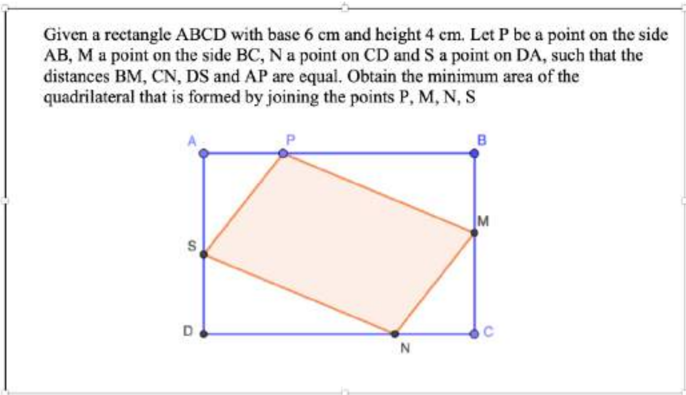

Figure 1: Problem: Compute the minimum area of a quadrilateral

1. Learning goal: Solve the geometry problem by proposing a quadratic function model. In the solving process, students will observe tabular, graphic, and algebraic representations of the quadratic function and learn how to find its minimum.

Learning activities will be applied in a two-hour session as follows: (1) Translate word problem to geometric language, (2) Area computation, (3) Tabulating task, (4) Graphing, (5) Generalization, (6) Identification of the minimum area value.

2. The hypothesis of the learning process: At the end of the activities, students should learn three representations of the quadratic function that models the problem: tabular, graphical, and algebraic, and find the minimum area that solves the problem.

Teachers' Knowledge

Members of the learning community ask themselves what they should know about knowledge for teaching mathematics to be effective teachers. For that purpose, a member of the community introduced the Knowledge Quartet (KQ) proposed by Rowland et al. (2005) as a theoretical framework for the analysis and reflection processes of their teaching activities. From the perspective of KQ, the knowledge and beliefs that are evidenced in the teaching of mathematics can be typified by means of four dimensions: (1) Foundation, (2) Transformation, (3) Connection, and (4) Contingency.

Analysis of Activities within the Learning Community

The KQ framework is also used as a tool for analyzing the 'Teaching-Reflection Cycle' related to the minimum quadrilateral area problem. For the coding of KQ dimensions, the analysis is based on Rowland, Turner & Thwaites (2014, pp. 319–320) where the contributing codes for each dimension are found. Five videos were analyzed using MAXQDA software to transcribe and for coding.

KQ categories that emerge in an episode of the Teaching-Reflection Cycle

The thoughts that appeared in the reflection process about how Mario taught the HLT were incorporated in Tadeo's teaching. This can be seen as a way to build up practical knowledge for teaching quadratic functions. An example of this statement can be appreciated in the next episode.

In Mario's class, a student passed to the blackboard to solve the problem, assuming that the points P, M, N, S were situated in the midpoint of each corresponding side of the rectangle ABCD. This is considered a situation of deviation from the agenda in the teaching process, an aspect that was not expected, so it is a Contingency (a dimension of the KQ framework). Mario dedicated a long period of time to make the students think about the data of the problem in a different way.

A student in Tadeo´s class, while teaching the HLT makes the same interpretation regarding the midpoints, and drew a rhombus inscribed in the rectangle. Tadeo knew in advance that this could happen and manages to lead students to make the most general interpretation considering the problem's conditions.

Tadeo: Well, but ... is everyone clear about what the problem asks you to calculate?
Student 1: The rhombus area.
Tadeo: Well, here I think, that in the wording there is a part that we are not considering because we put the points P, M, N, and S as he said (referring to student 1), you placed it in the midpoint– [pointing P, M, N, and S for each side of the ABCD rectangle] – (See Figure 1).
Student 1: Ah! M and S are not midpoints.
Student 2: Because it had to be the same distance from A to P and from B to M.
Tadeo: Exactly, you already saw that among the conditions of the problem there is a part that says that the distance from B to M, from C to N, from D to S is the same that from A to P. Did you consider that information?

This is an example of the Foundation Dimension of the KQ Framework. The Contingency situation for Mario became a Foundation situation for Tadeo, due to the previous reflection on the first teaching trial within the community.

Discussion

This is the first 'Teaching-Reflection Cycle' of the community, of several more cycles which are being analyzed as part of a broader research work that aims to observe how these cycles can enrich the practical knowledge of the teacher. This methodology of data analysis supported by the KQ framework can be seen as an appropriate frame for analyzing what happens within the community to show how knowledge is produced.

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THE EVOLUTION OF TEACHERS' CONCEPTIONS OF TEACHING MATHEMATICAL MODELING THROUGH PARTICIPATION IN LESSON STUDY

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Mathematical modeling is an important mathematical practice, yet it is a relatively new idea in school mathematics. Limited resources such as lack of teacher preparation have contributed to challenges of teaching modeling. Thus, professional development, such as lesson study, a continuous improvement approach to teaching, might support teachers in implementing modeling. For this study, three secondary teachers with varying levels of experience participated in two cycles of lesson study on mathematical modeling. The teachers were interviewed about their conceptions of teaching mathematical modeling before and after the lesson study. Analysis of the interview transcripts revealed that the teachers' conceptions of teaching modeling evolved in ways that indicated the teachers learned pedagogical strategies, realized further benefits of teaching modeling, and refined their instructional focus for teaching mathematical modeling.

Keywords: Modeling, Teacher Education – Inservice / Professional Development

Policy documents have called for the incorporation of mathematical modeling into the curriculum (e.g., NCTM 2000, 1989; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Yet, research has indicated that teachers may have challenges with teaching this important mathematical process that is becoming increasingly more prominent in science, technology, engineering, and mathematics fields (Cirillo, Pelesko, Felton-Koestler, & Rubel, 2016). The challenges of teaching modeling include limited experience, teachers' dispositions and lack of resources (e.g., Meyer, 2015; Newton, Madea, Alexander, & Senk, 2014). These challenges are unfortunate because mathematical modeling can provide students with opportunities to engage in ill-structured problems about authentic real-world situations that are unlike typical textbook tasks. Thus, researchers have recommended that teachers receive professional development (PD), such as lesson study, on implementing mathematical modeling (Turner et al., 2014).

In previous studies lesson study has supported teachers to improve their teaching (e.g., Lewis, 2016; Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012). Lesson study is an iterative process that includes curriculum study, lesson planning, teaching and observing, and reflecting/debriefing. Lesson study could support the teaching of mathematical modeling because lesson study employs a variety of tools and resources that influence improvement of teaching (see e.g., Lewis, C., 2016; Takahashi & McDougal, 2016.) Hence, this study sought to understand: What are teachers' conceptions of teaching mathematical modeling after participating in lesson study on mathematical modeling?

What is Mathematical Modeling?

There are many descriptions of mathematical modeling, but there is no agreed-upon definition. Hence, for the purpose of this study, a working definition of *mathematical modeling* is inspired by Cirillo, Pelesko, Felton-Koestler, and Rubel's (2016) description of the features of mathematical modeling: Mathematical modeling is an iterative process that authentically connects to the real world. It is used to explain phenomena in the real world and/or make predictions about the future behavior of a system in the real world. Mathematical modeling requires creativity and making choices, assumptions, and decisions, and can have multiple approaches and solutions.

Challenges of Teaching Mathematical Modeling

A review of literature revealed multiple challenges for teaching mathematical modeling. For one, preservice and inservice secondary mathematics teachers reported a lack of self-efficacy with respect to pedagogical strategies for teaching mathematical modeling, specifically because of the ill-structured nature of modeling activities (e.g., Kuntze, Siller, & Vogl, 2013; Chan, 2013). Other researchers observed that teachers have struggled with anticipating multiple student responses for mathematical modeling activities. This pedagogical challenge led to other obstacles with respect to classroom management, the handling of multiple student approaches, and the facilitation of whole-class discussions (e.g., Pereira de Oliveria & Barbosa, 2013; Borromeo Ferri & Blum, 2013). These challenges indicate that teachers need further preparation to teach mathematical modeling.

A need for teacher preparation is particularly true when considering the US context. Evidence of insufficient teacher preparation was revealed through a survey of education programs (n = 72). Of the respondents, only 15% of the programs required a mathematical modeling course for preservice secondary teachers (Newton, Madea, Alexander, & Senk, 2014). This finding could indicate that novice secondary teachers may not implement modeling as recommended (see e.g., CCSSM). Gould (2013) found, in a national survey of inservice teachers (n = 274) from 35 states that teachers held misconceptions about mathematical modeling. While Gould's (2013) results indicate that teachers may lack content knowledge for teaching modeling, teachers may also hold beliefs that impact their willingness to teach mathematical modeling. As an example, Anhalt, Cortez, and Bennett (2018) found that after completing a modeling activity, preservice teachers acknowledged that implementing modeling could provide many opportunities for students to engage in rigorous mathematics, rich discussions, and develop multiple solution approaches. At the same time, those teachers were concerned that mathematical modeling activities would be too complex for most secondary students. These types of beliefs may hinder teachers from implementing modeling activities. Moreover, inservice teachers have reported limited resources, such as time and access to quality mathematical modeling curriculum materials prevented them from implementing mathematical modeling regularly, if at all (Gould, 2013; Huson, 2016). To address these challenges researchers have recommended lesson study as a means of professional development to support the teaching of mathematical modeling.

Using Lesson Study to Address Challenges of Teaching Mathematical Modeling

Through the process of lesson study, teachers have opportunities to learn from each other while collaboratively: planning, observing teaching, and debriefing to improve lesson plans (Lewis et al., 2009). Previous studies have found that lesson study provides opportunities for teachers to improve content knowledge, learn pedagogical strategies, and focus on student thinking (e.g., Murata et al., 2012; Lewis et al., 2009). This focus on student thinking can influence teacher learning (e.g., Suh & Seshaiyer, 2014). For instance, teachers observed by Inoue (2011) drew on anticipated student responses to facilitate discussions and support students' engagement with mathematical reasoning. Additionally, Cajkler et al. (2015) observed how teachers' beliefs changed after reflecting on their observations of student thinking for students who typically had minimal classroom participation. These aforementioned aspects of lesson study are likely to support teachers in the implementation of open-ended modeling activities.

Methods

Setting and Participants

Three secondary teachers who were teaching in a diverse vocational high school in the mid-Atlantic region of the United States were recruited based on their interest in improving their teaching of mathematical modeling. Loren, a second-year teacher, earned her bachelor's degree in secondary mathematics education, and as part of her degree program, completed one course on mathematical modeling for secondary teachers. Anne, who had previously been an engineer, had six years of teaching experience. Anne held a bachelor's degree in electrical engineering and a master's degree in curriculum and instruction. Karen had 21 years of experience teaching and had earned a bachelor's degree in computer information systems and a master's degree in education. Karen had participated in professional development on mathematical modeling, and she had some experience teaching modeling.

Lesson Study on Mathematical Modeling. The teachers participated in two lesson study cycles containing the following activities: curriculum study, lesson planning, teaching and observing, and debriefing. During the curriculum study, the researcher introduced mathematical modeling, and the teachers explored curriculum materials related to modeling. For instance, the teachers explored tasks such as those provided in the *Mathematical Modeling Handbook* (Gould, Murray, & Sanfratello, 2012). Next, the researcher facilitated planning meetings and guided teachers in completing an annotated lesson plan, similar to a common format used in Japanese lesson study (e.g., Gorman et al., 2010; Lewis & Hurd, 2011). The lesson plan template contained cells for learning goals, anticipated student responses, planned instructor actions, and rationale for tasks. To complete the lesson study cycles, each teacher enacted the lesson while the lesson study team observed. Then, the team met to debrief the lesson enactments after the first enactment (i.e., Loren's) and the third enactment (i.e., Karen's). During the debrief sessions, the researcher executed a debrief protocol to support revision of the lesson plan based on evidence of student thinking collected by the teachers while observing each other's enactments.

Interview Protocol

Interviews were conducted before and after the lesson study, regarding the teachers' conceptions of teaching mathematical modeling. The questions provided teachers with opportunities to share their conceptions of teaching mathematical modeling. For example, one of the questions was: "Are you currently teaching mathematical modeling, or have you ever taught mathematical modeling? Describe your teaching approach to mathematical modeling (e.g., frequency, resources for tasks, aspects of the modeling cycle addressed)." Then follow-up questions were asked about teaching approaches to modeling, including the benefits and challenges of teaching mathematical modeling. Prior to the lesson study, if the teacher was not currently teaching modeling, then the teacher was asked why modeling was not being taught, and to describe any hypothetical benefits and challenges of teaching modeling.

Data Analysis

Data for the study consisted of audio-recordings and transcripts of two interviews per teacher for a total of six interviews. Once the audio data were transcribed the transcripts were uploaded to Dedoose (2016), web-based qualitative data analysis software. The transcripts were analyzed using a constant comparative approach (e.g., Strauss, 1987; Hatch, 2002). Initial deductive codes were developed using themes that emerged in the literature with respect to the working definition of modeling as well as benefits and challenges of teaching modeling. Then the coding dictionary was revised further as inductive codes emerged from the data. Themes were organized into analytic memos to inform the findings. The analysis of the data revealed three cases about the teachers' conceptions of teaching mathematical modeling: Loren learned pedagogical strategies for teaching mathematical modeling; Anne realized the benefits of teaching mathematical modeling; and Karen focused on shifting her classroom culture.

Findings

Case 1 – Loren: Learned Pedagogical Strategies for Teaching Mathematical Modeling

Before the Lesson Study.

In the pre-lesson study interview, Loren shared that she had not taught authentic mathematical modeling yet, but she expressed several potential benefits and her concerns for challenges of teaching mathematical modeling. For example, she mentioned that teaching mathematical modeling "makes mathematical modeling more interesting to students" and that students could have opportunities to be creative and to see multiple approaches to a modeling activity. Although Loren acknowledged potential benefits of teaching modeling, she also mentioned that modeling activities could be "complicated" and that students could get "easily frustrated" while engaging in modeling. She seemed to think that the complexity of modeling activities could be an obstacle for students. Even though Loren had not been teaching modeling, she indicated that she attempted to provide opportunities to experience similar benefits through her teaching of word problems.

As mentioned earlier, Loren had limited experience with teaching mathematical modeling. She described her approach to teaching modeling as implementing "word problems." When reflecting on her experience teaching modeling she said:

> The only experience that I've really had, and I don't even know if you would classify it as modeling is just like drawing pictures and setting up scenarios with word problems…I'm thinking of when I student taught geometry, and we would do those drawings with the ladder leaning on the house, and I've taught students ways to show mathematical situations, but I don't know if it's truly modeling.

Here, Loren acknowledged that her use of word problems was not quite modeling even though students had opportunities to apply multiple mathematical representations in their solutions. Loren also conveyed enthusiasm for participating in lesson study and improving her teaching of mathematical modeling so that she could genuinely engage her students in the benefits of teaching mathematical modeling.

After the Lesson Study. After participating in lesson study, Loren's observations of students in her classroom and her two colleagues' classrooms influenced her conceptions of the benefits of teaching mathematical modeling. While she mentioned several benefits in her initial interview, Loren added that when the students engaged in modeling during the lesson study, she observed them engage in mathematics that was "valuable in the real world" and "really relevant." She observed that teaching modeling was "a lot more rigorous for them than just teaching them how to do procedures." Loren also recognized how the modeling tasks provided opportunities for students to collaborate with their classmates. Prior to the lesson study, Loren was concerned that mathematical modeling activities would be too complex. However, after focusing on student thinking during lesson study, she saw how students could persevere and collaborate to produce multiple solution pathways for mathematical modeling activities.

Loren's new experience with teaching mathematical modeling also exposed some worthwhile challenges of teaching modeling. Loren found it challenging to support students with "group roles" while they collaborated on the modeling tasks. She also worried about "tutoring [students] too much." Thus, to avoid too much "telling," Loren said she relied on the collaboratively planned lesson which was annotated with possible student responses and questions to ask students. This type of lesson planning seemed to influence Loren's teaching approach to mathematical modeling as she indicated in the transcript below.

> It [lesson planning] really showed me how valuable that is, and I like so enjoyed like going into [the lesson] knowing, what I wanted to say, what I didn't want to say, and what I thought the students

> would say. I planned timing and things like that, but the lesson plan that we did went in depth which was great. So, I think it'll change the way I teach. Not only teaching modeling but just teaching in general because it showed me the importance of lesson planning.

As a new teacher, the lesson planning activities and focus on student thinking during lesson study influenced her approach to teaching mathematical modeling. While Loren expressed some concerns about the challenges of teaching mathematical modeling, she also seemed to indicate that her new knowledge of pedagogical strategies would continue to support her teaching.

Case 2 – Anne: Realized the Benefits of Teaching Mathematical Modeling Before the Lesson Study.

Similarly, to Loren, Anne mentioned that she had not quite taught mathematical modeling before, but she acknowledged potential benefits and challenges of teaching mathematical modeling. Anne, conveyed that mathematical modeling could advance students' mathematical thinking and that teaching modeling was "teaching them more the approach to solving problems rather than just rote memorization of solving problems, different representations and ways to solve things, but mostly just to get them thinking more analytically rather than plugging things into a formula." While Anne recognized that modeling could provide students opportunities to engage in complex mathematics, she also indicated that students could become frustrated with ill-structured modeling tasks. Anne noted that when she had implemented open-ended mathematical tasks her "biggest challenge" was "letting them [students] struggle." She also mentioned that the students were uneasy with tasks that could have multiple student responses and not being "spoon-fed, step-by-step" instructions for solving the tasks. Anne seemed to think that struggling with mathematics would be beneficial for students, but she also mentioned her frustration with the lack of time for including open-ended tasks within her mandated curriculum when she said "It's just sometimes it's hard with so much to get through. There's definitely room for more."

After the Lesson Study. After the lesson study, Anne still maintained her perceived benefits of teaching mathematical modeling. She indicated that she saw students in her classroom, and her two colleagues' classrooms "get better at problem solving." She also thought those skills would translate to other areas outside of the math classroom. A new conception that Anne mentioned was based on her observations of students. She saw that engaging in mathematical modeling gives students opportunities to "collaborate with other students and things like that rather than just being instructed directly." Anne's evolved conceptions about her students were notable. After the lesson study and implementation of mathematical modeling, she discussed how the intentional focus on student thinking allowed her to observe "skills in students that [she] wouldn't have seen in a traditional way of teaching them." More notably, she had the following to say about one of her students in particular:

> One of my students, like I got more out of him from this activity than I have the whole semester. I was able to see his thought process and things like that, that I had never seen before because I guess really, I'm always looking at a paper and what he's writing, and he's not a big sharer in class. So, it was really hard for me to see. But then when I saw what he was doing I was like, wow, he's really, really working on this. Like really his, the way his mind was working, was very much different than what I had thought.

Anne's discussion about her observations of this particular student indicated that her initial beliefs, regarding students' abilities to engage with mathematical modeling, had evolved to see that students could persevere through modeling activities. As a result, Anne mentioned that she was including more modeling tasks and asking more open-ended questions throughout her lessons. Through her participation in the lesson study, Anne's conceptions evolved so that she had a deeper realization of the benefits of teaching mathematical modeling.

Case 3 – Karen: Focused on Shifting Classroom Culture

Before the Lesson Study. As the most experienced teacher, Karen, was well-aware of many benefits and challenges of teaching mathematical modeling prior to the lesson study. When describing the benefits of modeling, she conveyed her implementation of modeling had evolved. For instance, she mentioned how students struggled with the messiness of modeling tasks at first, but then "eventually they started to deeply think about it and contribute to each other's ideas and bounce ideas off of each other and reference each other's input as a class discussion." Because of her previous experience with teaching modeling, unlike her two colleagues, she had a clear vision for implementing modeling. Although Karen recognized many benefits to teaching modeling, she also noted challenges with teaching modeling. Karen's primary concerns about teaching mathematical modeling were with regard to her available resources. She indicated that when it comes to teaching mathematical modeling, "the primary constraint is the lack of great tasks and the lack of time." Karen shared that she implemented a modeling activity within each of her units, but she was looking forward to the opportunity to collaborate with colleagues to further improve her teaching of modeling. Although Karen's conceptions of teaching modeling did not evolve as drastically as her other two colleagues' conceptions, she still found value in the lesson study and her conceptions of teaching modeling emerged to a different teaching focus.

After the Lesson Study. As hypothesized by the researcher, many of Karen's conceptions about the benefits and challenges of teaching mathematical modeling did not change much after teaching the lesson study. Rather than observing noticeable changes in her conceptions, the researcher noted that Karen's focus after the lesson study had moved to cultural aspects of teaching mathematical modeling. For example, Karen noted that she appreciated the level of student engagement she observed during the lesson enactments. She also spoke about her evolved focus on cultivating a classroom culture that would support the teaching of mathematical modeling. Specifically, she said she was challenged with:

> Creating that environment where they want to do this type of math is an ongoing challenge of mine…just getting [students] to be comfortable with being uncomfortable is what I try and get to tell them. I am becoming more and more aware of the importance of establishing appropriate culture, and it doesn't matter what the task is or how wonderful my task is. If I can't get the students to buy what I'm selling, it's not, it's just not going to have the impact that I wanted to have. I need to have these kids believing that they can model.

It seemed as though Karen was less focused on her earlier challenges with limited resources for teaching modeling, and more focused on her teaching approach. She also noted other benefits of participating in the lesson study.

During the post-lesson study interview, Karen expressed that she found the lesson study to be beneficial for multiple reasons. First of all, even though she was the most experienced teacher, she found that the collaborative nature of lesson study supported her to anticipate student thinking for the implemented lessons. Consequently, she communicated that she would like to continue to collaborate with teachers in her school to improve her teaching of mathematical modeling. More specifically, Karen said she appreciated the " ability to plan with someone else and anticipate [student thinking], regardless if it's modeling or not modeling, but I wish I had that with my modeling tasks. I know my modeling tasks would be improved if I could do that." Even though Karen was a veteran teacher, she indicated an aspiration to collaborate with her colleagues in the future to improve her teaching of mathematical modeling.

Reflections Across the Cases

Before the Lesson Study. Prior to the lesson study, all three of the teachers acknowledged that teaching mathematical modeling could have many benefits for students but also challenges for students and teachers (see Table 1). Loren, who had engaged with modeling as a student, and Karen, who had been teaching modeling, both recognized that teaching mathematical modeling could appeal to student interest. Anne also indicated that engaging in modeling would provide more rigorous problem-solving opportunities than typical textbook texts usually provide. Yet, at the same time, Loren and Anne were concerned that the complexity of modeling could discourage some students. Notably, Karen did not focus on challenges for students, but she expressed challenges she faced as a teacher with finding authentic modeling tasks and time to plan and implement mathematical modeling. At the beginning of the study, as expected, the teachers were approaching the teaching of mathematical modeling in various ways, as Loren and Anne had not quite implemented open-ended modeling tasks. When considering, the teachers' early conceptions of teaching modeling, their evolved conceptions after the lesson study suggested that the lesson study had a positive impact on their conceptions of teaching mathematical modeling.

Table 1:Teachers' Conceptions of Teaching Mathematical Modeling

		Benefits	**Challenges**	**Teaching Approach**
Loren	Before	• Student interest • Creativity • Multiple approaches	• Complexity of modeling	• Word-Problems
	After	• Relevant math • Rigorous math • Student collaboration	• Time • Curriculum	• Focus on Student Thinking • Group Roles
Anne	Before	• Problem-solving	• Complexity of modeling	• Word-problems
	After	• Student collaboration • Access for all students • Focus on student thinking	• Culture Shift	• Increased modeling tasks • Open-ended questions
Karen	Before	• Student interest • Student collaboration	• Time • Curriculum	• Modeling in unit plans • Adapting tasks
	After	• Rigor and relevancy • Student collaboration	• Culture Shift	• Collegial Collaboration • Culture Shift

After the Lesson Study. After the lesson study, the teachers' expressions conveyed evidence that indicated their conceptions of teaching mathematical modeling had evolved. As an example, Loren and Anne's concerns about the complexity of modeling tasks being too challenging for students had shifted. Instead, they expressed how teaching modeling provided opportunities to focus on student thinking and observe how all of their students could engage in rigorous mathematics. Similar to Karen, before the lesson study, Loren now conveyed curriculum and time as being her main challenges to implementing modeling. Likewise, Anne's new perceived challenges were aligned with Karen's challenges of creating a classroom culture for teaching modeling. Additionally, Loren and Anne had other conceptions after the lesson study about the benefits of modeling that were similar to Karen's conceptions before the study. For example, before the lesson study, Karen indicated that modeling could appeal to student interest and promote student collaboration; whereas, Loren and Anne did not mention those benefits until after the lesson study. When considering the cross-case findings, it is apparent that the evolution of the teachers' conceptions about teaching mathematical modeling was supported by participating in the activities of lesson study. More importantly, one might hypothesize that Loren's and Anne's conceptions of teaching mathematical modeling progressed more expediently than if Loren and Anne had attempted to implement mathematical modeling on their own or through conventional PD.

Discussion and Conclusions

The teachers' conceptions about teaching mathematical modeling after participating in lesson study indicated that lesson study can, given the right conditions, support the teaching of mathematical modeling. For one, the teachers indicated that participating in lesson study provided opportunities to explore curricular resources and learn new pedagogical strategies. They also had time to collaborate with colleagues to improve mathematical modeling lessons. This finding is important as teachers in previous studies have indicated that they had limited resources for teaching mathematical modeling such as curriculum and time (Gould, 2013; Huson, 2016). The teachers in this study also focused on how their implementation of mathematical modeling provided multiple benefits for students. The teachers spoke about how the lesson study provided opportunities to engage with student thinking in ways that supported students' engagement in the modeling process. This finding is contrary to previous studies where teachers struggled with pedagogical skills needed for teaching modeling such as facilitating multiple student responses (e.g., Kuntz, Siller & Vogl, 2013; Pereira de Oliveria & Barbosa, 2013; Borromeo Ferri & Blum, 2013). Another notable finding from this study was that the teachers observed how all students were capable of engaging in rigorous mathematics. In contrast, teachers in previous studies expressed concerns about students' abilities to engage in complex mathematical modeling tasks (e.g., Anhalt, Cortez, & Bennet, 2018). These cases present compelling evidence for the use of lesson study on mathematical modeling to support teachers' conceptions of teaching mathematical modeling. Further research is needed to understand how lesson study on mathematical modeling can be employed in different contexts to achieve improvements in teachers' content and pedagogical knowledge with respect to modeling.

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CHARACTERIZING FEEDBACK GIVEN AMONG MATHEMATICS TEACHERS: CLASSROOM OBSERVATIONS

CARACTERIZANDO LOS COMENTARIOS ENTRE MAESTROS DE MATEMÁTICAS: OBSERVACIONES EN EL SALÓN DE CLASE

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The objective of this study is to analyze the nature of feedback given among 58 middle school mathematics teachers participating in a targeted professional development program. As part of the professional development, teachers participated in instructional rounds in which they worked in groups of five or six to observe and give each other feedback on classroom visits. The feedback was written on forms during the observations and discussed during debrief meetings after the observations. This paper characterizes the feedback written by teachers as they observed their colleagues teaching. The preliminary results show that teachers' written feedback was largely descriptive and focused on instructional, rather than mathematical, elements of the lesson.

Keywords: Teacher Education – Inservice / Professional Development, Systemic Change, Instructional Vision, Middle School Education

The purpose of this research is to characterize the feedback middle school mathematics teachers provide to their peers as part of Instructional Rounds (IR). Instructional Rounds have been proposed as an alternative to the periodic, short-term professional development (PD) workshops that are typically held for a few days during the school year or summer (e.g., Goodwin et al., 2015; Teitel, 2015). They involve a collaborative effort among teachers as they observe each other in the classroom and learn from their collective expertise (City et al., 2009). Instructional Rounds exemplify other features of PD programs that have been shown to have an impact on teachers' practice. For example, they take place in the context of schools (Mewborn & Huberty, 2004; Quick et al., 2009) and encourage teachers to collaborate and problem solve as they reflect upon their experiences of teaching (Hawley & Valli, 2000). Specifically, the research question guiding this research is: *How can the feedback teachers give to one another as part of Instructional Rounds be characterized?*

Theoretical Framework

This study is grounded in the premise that IRs are one way in which teachers can learn and improve their practice as they share their expertise and reflect on their own practice with their peers (Kennedy et al., 2011). However, there is a dearth of research on the types of feedback teachers provide to one another on classroom observations. Scheeler et al. (2004) conducted a meta-analysis of research feedback and found that of the 208 teachers included in the meta-analysis, only 9 teachers were inservice teachers. The nature of feedback university supervisors provide to preservice teachers has shown that immediate feedback following a teaching episode (Cornelius & Nagro, 2014) or using bug-in-ear technology during live teaching (Scheeler et al., 2006) can lead to change in practice.

With respect to inservice teacher education, the recent emergence of video clubs for teachers has provided opportunities to study teachers' observation (e.g., Star & Strickland, 2008) and noticing (e.g., Sherin & van Es, 2005) skills. These studies have described the results of the implementation of video clubs (e.g., Beiseigel, 2018; Wallin & Amador, 2018), or have analyzed teachers' responses to viewing rich clips of classroom episodes (van Es & Sherin, 2008). However, more research is needed to understand how teachers provide feedback to their peers and to categorize and describe the nature of their feedback.

Methods

Context

The participants of this study were 58 middle school mathematics teachers from 7 local school districts. The teachers participated in a three-year PD program consisting of a two-week intensive Summer Institute, followed by four days of follow-up PD during the academic year. In the final year of the program teachers participated in IRs, starting with norms development and team-building during the Summer Institute, and involving peer classroom visits and feedback cycles during the school year. Teachers worked in teams of five or six throughout the IR process.

During the PD teachers participated in targeted activities to help them understand the importance of using one another as instructional resources and to practice giving meaningful feedback. After reading an article on the teachers implementing IRs in schools (Troen & Boles, 2014) and discussing the differences in peer feedback and the standard evaluation measures, teachers practiced giving feedback on a classroom video of a teacher not in the program. Then, teachers gave a short model lesson and received feedback from their peers in the audience. In this way, they practiced giving and receiving feedback in a safe space with their teams.

During the school year, every team traveled together to observe each of their teammates' classrooms. The visits included a pre-observation meeting where the observed teacher described their mathematics and instructional goals for the lesson, a classroom observation, and a post-lesson debrief where the observers provided feedback to the teacher.

Data Collection and Analysis

The data for this study came from the observation forms teachers completed during their classroom visits. To record their thoughts for the debrief sessions, each observer was provided with a form with a section for "Mathematics Goal", "Instructional Focus", and "Other". We collected 244 forms for the observed teachers. The observation forms were parsed into units of analysis that were feedback units distinguished from the next by turns in content. In sum, there were 3,595 feedback units in the teachers' observations forms (μ=14.79).

Based on Schwartz et al. (2018), pre-determined codes for the observation forms were used to code the feedback units. The first level codes determined whether a feedback unit was mathematical (M) or instructional (I) in focus. Feedback related to mathematical thinking, mathematics content, terminology, or notation, was coded as Mathematical (M). Comments related to instructional decisions that were not specific to mathematical content were coded as Instructional (I). The second level determined whether the comment was descriptive (D), suggestive (S), or complimentary (C). Descriptive refers to comments that summarize or describe a situation without any intended suggestion or judgement. Suggestive refers to comments that were meant to have the teacher consider alternatives or to question a move or explanation of content. A comment was coded as complimentary if it connoted a positive attribute of the lesson.

If a feedback unit was coded as descriptive (MD or ID), no additional sub-codes were assigned. Each of the *suggestive* second-level code was coded as either *consideration* (C) or *imperative* (I). A *suggestive, consideration* comment indicates that the mathematical objective or instructional focus was not hindered and that the observer was merely giving the observed teacher a question or alternative to consider. An imperative suggestion includes comments that stood in the way of the mathematical objective of instructional focus being met. A feedback unit with a complimentary code (MC or IC), was either general (G) or specific (S). General compliments consisted of phrasings such as "nice lesson", whereas specific compliments referred to a particular instance during the lesson. Examples of each set of codes are provided in Table 1.

Table 1: Coding Scheme with Examples

Level 1 Code	Level 2 Code	Level 3 Code	Examples
Mathematical (M)	Descriptive (M, D)		"Added all angles to make sure they were 180"
	Suggestive (M,S)	Consideration (M,S,C)	"A 'math talk' anchor chart may help guide discussions"
		Imperative (M,S,I)	"Never really answered the question (problem)"
	Complimentary (M,C)	General (M,C,G)	"Topic well covered"
		Specific (M,C,S)	"Loved the calculator analogy—have to know to use the tool properly"
Instructional (I)	Descriptive (I,D)		"Had the kids organized before the lesson started"
	Suggestive (I,S)	Consideration (I,S,C)	"Students may be more willing to share ideas if they can formulate them first on paper"
		Imperative (I,S,I)	"Wait time—need more"
	Complimentary (I,C)	General (I,C,G)	"Ms. K has a very approachable demeanor"
		Specific (I,C,S)	"These tips are a great foundation to encourage more group talk later on in the year"

Reliability

Each feedback unit was coded by two coders. After initial coding, any discrepancies were discussed until agreement was reached. Thus far, 338 feedback units from one team have been coded for this preliminary analysis. The initial agreement between the coders was 89% for Level I codes (M vs. I), 86% for Level II codes (D/C/S), 82% for Level III codes (C/I or G/S). By October, all ten groups and 3,595 feedback units will be coded and analyzed. Results from the first team, Teachers Being Outstanding (TBO) are presented below.

Results

The raw data for the 338 codes provided by the TBO team are provided in Table 2. The most frequent feedback was instructional descriptive (ID). Though the feedback units coded as mathematical were infrequent, when teachers provided feedback coded as mathematical, it was mostly descriptive in nature. Of the suggestive feedback, none were imperative and there were almost three times as many instructional suggestive consideration (ISC) feedback units than mathematical suggestive considerations (MSC).

Table 2: Counts for Each Code Across TBO Team

Code	MD	MSC	MSI	MCG	MCS	ID	ISC	ISI	ICG	ICS
Count	56	13	0	1	25	144	34	0	13	52

An analysis of Level I codes shows that there were 95 feedback units coded as mathematical and 243 codes as instructional. For the level two codes, there were 200 descriptive, 91 suggestive, and 47 complimentary. At level three there were 47 feedback units coded as consideration and 0 imperative feedback units. There were 14 general level 3 codes and 77 specific, however twice as many feedback units, coded as specific, were instructional in nature.

The percentage of each feedback unit code for the TBO group is provided in Table 3. The percentages represent the frequency of each code relative to the total number of feedback units given by the observer for all teachers observed. For each observer, the code that was most frequent is highlighted in gray. Across all six teachers, the most frequent codes were instructional in nature, with the instructional descriptive codes being the most common among the team of observers.

Table 3: Percentages of Each Code by Observer

	MD	MSC	MSI	MCG	MCS	ID	ISC	ISI	ICG	ICS
Ebony	18.87	1.89	0.00	0.00	0.00	58.49	7.55	0.00	5.66	7.55
Condi	27.27	0.00	0.00	0.00	4.55	31.82	4.55	0.00	9.09	22.73
Tammy	20.72	5.41	0.00	0.90	13.51	23.42	9.01	0.00	2.70	24.32
Karmen	7.84	0.00	0.00	0.00	13.73	39.22	11.76	0.00	3.92	23.53
Meegs	16.46	1.27	0.00	0.00	2.53	70.89	7.59	0.00	1.27	0.00
Jameka	0.00	19.05	0.00	0.00	0.00	19.05	33.33	0.00	9.52	19.05

Discussion and Conclusions

These preliminary results show that teachers tend to focus on instructional aspects of mathematics lessons and that it is more common for them to provide descriptive comments than suggestive or complimentary, despite the emphasis on constructive and meaningful feedback during the PD sessions preceding the IRs. It is rare for teachers to provide mathematical or instructional suggestions that they believe are imperative in nature. It will be important to conduct this analysis across the other nine IR teams and to disaggregate the results by observer and by team to determine if there are any differences in feedback based on the structure of the teams.

These results have implications for professional development and mathematics teacher leadership programs. Professional development and programs seeking to develop mathematics teacher leaders should consider developing activities to facilitate teachers' observation skills to include a critical eye for providing feedback to their peers. Whole group discussions, interspersed with IR observations, that provide opportunities to review the feedback and consider ways to make it more meaningful would allow for this type of intervention. A study that describes teachers' growth under this model would be illuminating.

Creating a network of teachers that can provide critical, non-evaluative feedback to one another has the potential to make small incremental and sustainable improvements to teachers' practice. The present study shows that teachers provide a range of different feedback types and also suggests that PD should focus on helping teachers provide more suggestive feedback.

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EXAMINING IN-SERVICE TEACHERS' DIAGNOSTIC COMPETENCE

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Teachers' diagnostic competence is essential for effective mathematics instruction. Prior studies have examined teachers' diagnostic competence using various approaches, such as asking teachers to assess students' erroneous work or anticipate potential learning difficulties. Few studies have examined how teachers interpret the significance of student errors, that is, to what extent the teachers think the flaws in students' work indicate a serious conceptual error or a trivial mistake that can be easily remediated. In this paper, we investigated the diagnostic competence of 2527 elementary in-service teachers by asking them to categorize errors in students' authentic place value errors in the context of decimal operations. Implications are discussed.

Keywords: Diagnostic Competence, Mathematical Error, Misconceptions, Decimal, Place Value

Perspectives

Teaching effectively and efficiently requires teachers to recognize 'what is' and 'how to' respond to the students' errors (Hill et al., 2008). Researchers (e.g., Artelt & Rausch, 2014; Schrader 2009; Südkamp et al., 2012) have defined diagnostic competence as the ability to anticipate or evaluate how well students perform on tasks. Diagnostic competence has been identified as a foundation of teaching expertise for decades (Weinert et al., 1990). Previous studies have worked on the conceptualization and measurement of teachers' diagnostic competence (Klug et al., 2013) as well as exploring how it affects students' learning (Guruzhapov et al., 2019; Helmke & Schrader, 1987). Researchers have assessed teachers' diagnostic competence by examining teachers' ability to analyze and identify errors in the students' work, anticipate common errors, and estimate the difficulty level of given tasks in order (e.g., Ostermann et al., 2018).

The present study is designed on the premise of defining teachers' diagnostic competence as how they infer the significance in the student errors. Such competence is important as it bridges teachers' diagnostic thinking of interpreting student work and making corresponding instructional decisions (Loibl et al., 2020). For example, while working on a multiplication problem such as 15 times 0.6, if a teacher considers a student response of 90 to be a *minor error* (e.g., a procedural error that misses the decimal point), he or she may respond by reminding students to add the decimal point after obtaining the solution of the mathematical operation. On the other hand, a teacher who regards this response as a *major error* (e.g., a conceptual error indicates a limited understanding of place value), might lead to a substantial intervention focused on the significance of the decimal point and place value. That is, perceiving an error as a *major error* more likely leads to conceptual instead of procedural remediation. This said we defined major and minor errors using the following text in the survey: "*Major errors* indicate a misunderstanding of key ideas that may persist even after sustained follow-up instruction; whereas *Minor errors* indicate a lack of awareness or inattention that can be addressed with brief follow-up instruction."

Markovits and Even (1999) reported a range of teachers' diverse interpretations and responses to instructional situations involving a decimal point. The data helped us in gaining an initial understanding of teachers' diagnostic competence on their knowledge related to decimal topics. To explore teachers' diagnoses of the significance of student errors more broadly, we focused on teachers' views concerning typical errors related to arithmetic operation with decimal notation. This research study was narrowly focused to better understand one case of teachers' diagnostic

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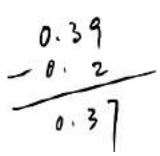
competency by answering the following research question: *How do in-service elementary teachers differ in their diagnosis of the significance of student errors with decimal place value?*

Method

The data for this study is drawn from a large-scale assessment focusing on evaluating teachers' knowledge for teaching rational numbers. For this paper, we included data from two items about teachers' diagnostic competence on decimal topics (see Figure 1). Both items are related to fourth-grade common core standards. Item 1 involves decimal subtraction (0.39 – 0.2 = 0.37) and Item 2 involves multiplication of a decimal and a whole number (15 × 0.6 = 90). The errors in both the items relate to the placement of decimal points. This error can be understood from two perspectives. If the error is understood to be procedural, it reflects students' missteps in completing an algorithm. In this case, the teacher may address the error by reminding the student about the correct steps of the algorithm. On the other hand, if the error is conceptual, it is reflective of students' inadequate understanding of place value. A teacher who considers the error as evidence of a misunderstanding, she or he may respond with more extensive instruction that is aimed at developing student understanding.

In this study, we tried to gain a better picture of how teachers interpret students' erroneous work. We provided definitions for the examinees of the two categories described above as major and minor errors, respectively. We then asked teachers to select the best option to complete the statement about each sample of student work, "In this student work sample, error or imprecision is (a) major and related to this topic, (b) minor and related to this topic, (c) related to a different topic, or (d) not evident." (Figure 1). Teachers who selected option (c) were asked to provide text to explain their reasoning.

Item 1



Considering the topic of place value, select the best option to complete the statement.

Item 2

Considering the topic of multiplying whole numbers and decimals, select the best option to complete the statement.

Figure 1. Two Item Samples

To answer the research question, we report the distribution of responses for all the options across 2527 elementary in-service teachers. To further understand teachers' reasoning on these questions, we analyzed all teachers' textual responses to option (c) (For Item 1, 3 were blank, thus n = 42; for Item 2, 3 were blank, thus n=36; see Table 2 and 3) using open coding (Cresswell & Poth, 2017). The teachers' exhibiting similar mathematical or pedagogical reasoning were grouped within one theme. For example, a teacher selected (c) related to a different topic and typed "Need to follow the same number value (tenth and hundredth)" for Item 1 which was categorized into the theme *place value* in Table 2. These themes and associated findings were discussed and reconciled during weekly group meetings.

Results

Teachers' Perspectives on the Significance of Student Errors

We found that teachers hold different perspectives on the importance of the same student errors. For the error in Item 1, 59% of the teachers interpreted it as a major error while 36% of them perceived it as a minor error. For the error in Item 2, 35% of participants indicated this error as a major error and 60% consider this is a minor and related error. A small percentage of teachers did not recognize the error in Item 1 (4%) or Item 2 (3%). For each item, around 2% of teachers thought the error was related to a different topic and reported their judgments on which topic was involved textually.

Table 1: The Distribution of Teachers' Perceptions of the Two Tasks

Option	Item 1		Item 2	
	N	%	N	%
Major and related to this topic	1482	59%	878	35%
Minor and related to this topic	904	36%	1521	60%
Related to a different topic	45	2%	39	2%
Not evident	96	4%	68	3%
Missing data	0	0%	21	1%

These data indicate that elementary teachers were more likely to classify the decimal notation error in the decimal subtraction problem as a conceptual error than they were the error in the decimal multiplication problem. From analyzing teachers' open-ended responses, we found more evidence to support this argument. We noticed that 42 teachers offered textual responses and 25 (60%) of them said the error was about place value (Table 2). We interpreted these responses as evidence that these teachers viewed the error as conceptual. Although these 25 teachers were able to identify this is a related error, they did not decide whether this was a major or a minor error, which may indicate the challenges of evaluating the importance of the error for these teachers. For example, one teacher responded "this is related to the topic, but the student could use interventions in decimal place value. This intervention could be beneficial to show him that .2 = .20 helping him to better line up his decimal number." The teacher has shown an understanding of the student's error and even offered intervention to remediate it, but she or he did not make a judgment about the importance of the error.

Table 2: Themes from Open-ended Responses on Item 1 (Option (c))

Theme of Responses (n = 42)	Number of Teachers (%)
Place value	25 (60%)
Computation/difference/algorithm/subtraction/operations	9 (21%)
Decimal addition vs subtraction	5 (12%)
Others	3 (7%)

Nine participants thought the error in Item 1 is related to broader topics such as computation, difference/subtraction, algorithm, and operations, which suggests these teachers noticed the error but did not necessarily interpret it to be related to place value error (Table 2). Five teachers thought the students should use addition instead of subtraction to solve the original word problem. In the "Others" category, one teacher thought students should use 0.02, one teacher suggested the student did not understand which number to subtract.

The teachers who offer textual responses to Item 2 (n = 36) held a more diverse interpretation of the student's decimal multiplication error. Contrary to Item 1, the teachers' responses were less dominated with the *place value* for Item 2 (n = 10, 28%). Four participants attributed the error to a lack of number sense. A teacher wrote that "[the student] doesn't understand the concept of a decimal - if you start with 15 groups of a number less than 1, your answer can't be larger than 15". Another teacher argued that the error related to "understanding the reasonableness of the answer due to values. For example, about 1/2 of 15 couldn't possibly be 60." These teachers seemed to identify a conceptual reason for the error but did not identify and describe the student's specific error, which may inhibit students' understanding of decimal (Markovits & Even, 1999). Nine teachers (25%) identified the error as procedural, relating to "carrying the decimal" or "placing the decimal point." Ten teachers (28%) thought the error was related to broad topics, such as "multiplying with decimals", "decimal", or "multiplication." Three teachers responded with something else such as "multiplying decimals is not a fourth-grade level math operation", "6", and "money".

Table 3: Themes from Open-ended Responses on Item 2 (Option C)

Theme open responses (n = 36)	*Number* (%)
Place value	10 (28%)
Number sense	4 (11%)
Decimal point placement	9 (25%)
Decimal multiplication or computation	10 (28%)
Others (e.g., not a 4th grade topic)	3 (8%)

In brief, from the teachers' written responses, we gained confidence in the larger finding that more teachers tended to identify the error in Item 1 as a conceptual error while more teachers interpreted the error in Item 2 error as a procedural error.

Discussion

Through this study, we found that two errors involving decimal notation were viewed by teachers in substantially different ways, with far more teachers classifying an error in decimal subtraction as major than a related error in decimal multiplication. These findings suggest the need to explore teachers' diagnostic competence concerning students' errors, and in particular to see how teachers perceive the importance of the error in students' mathematical learning. Such exploration goes beyond simply noticing students' errors because it requires teachers to identify students' errors, locate the error within certain mathematical topics, and justify its significance while providing conceptual remediation. This aspect of teacher knowledge may be more predictive of teachers' instruction because it requires teachers to apply their knowledge of student thinking to instructional decisions.

One limitation of this study is the possibility that some teachers held different interpretations of the terms major or minor error as these terms are not commonly used, which may affect the percentages of each option but may hardly affect teachers' general perception of each item. Although we provided the definitions before each item, some teachers may not have understood them as we expected. Thus, this study calls for future qualitative exploration of how teachers understand the major and minor error, the rationale of their option selections, and how they normally deal with such student errors through interviewing teachers and observing their teaching. How teachers interpret errors involving the decimal point in different mathematical contexts may deepen our understanding of teachers' knowledge for teaching decimals, an area in which little is presently known (Takker & Subramaniam, 2019).

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THE DESIGN, IMPLEMENTATION, AND IMPACT OF A COLLABORATIVE RESPONSIVE PROFESSIONAL DEVELOPMENT (CRPD) MODEL

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It is important to design professional development (PD) around teachers' professional thinking and needs. Researchers have explored how teachers center on and build upon students' thinking in mathematics teaching, but few studies have investigated how to identify and be responsive to teachers' ongoing needs while planning and enacting effective PD. As such, this study presents a Collaborative Responsive Professional Development (CRPD) model that arose from efforts to elicit and validate teachers' voices to design PD experiences that were relevant and meaningful to them. We share the rationale of the model design, its implementation during a two-year PD project, and its impact on teachers' instructional practice.

Keywords: Teacher Education – Inservice / Professional Development; Teacher Educators

Purpose

Avalos (2011) defined PD as "teachers learning, learning to learn, and transforming their knowledge into practice for the benefit of their students' growth" (p. 10). Putnam and Borko (1997) argued that teacher educators should engage mathematics teachers in PD in a way that is parallel to how we expect them to engage their students. Similarly, Spangler (2019) advocated that just as we are asking teachers to attend to their students' mathematical thinking, "we as teacher educators need to demonstrate the curiosity and intellectual humility that allows us to understand how and why something a teacher did or said came from a place that made sense to them" (p. 2). Additionally, researchers have noted the importance of supporting the affective side of teachers' professional growth throughout the PD process (Cross Francis, 2019, Cross Francis et al., 2019). The literature on effective PD indicates a necessary shift towards attending to teachers' needs holistically.

However, identifying and holistically responding to teachers' needs is a challenging task (Lee, 2005). Such efforts require a systematically designed PD process that situates teachers' thinking and voices as core components to all decision-making moves (Jez & Luneta, 2018; Lee, 2005). In this study, we share what we call a Collaborative Responsive Professional Development (CRPD) model (see Figure 1), aimed at systematically centering teachers' voices and needs in an iterative process of designing, implementing, and assessing the effectiveness of our PD curriculum. Our research question was: *To what extent was the implementation of the CRPD model effective for teachers' professional development?*

The Design of the CRPD Model

One core aim of the CRPD model is to ensure that teachers' voices, especially regarding their professional thinking and needs, were collected in various ways, analyzed through multiple perspectives, and utilized to inform instructional decisions. We share three big ideas that undergird the CRPD Model: the research on effective PD with mathematics teachers; creating a structure for collaboration on the design of PD activities; and creating structures for systematically collecting and being responsive to teachers' voices.

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The Research on Effective PD with Mathematics Teachers

We used three core features of effective PD as pillars upon which we built our CRPD model. The first feature is focusing on developing teachers' content and pedagogical knowledge for teaching mathematics (Darling-Hammond et al., 2017; Garet et al., 2001). This focus is responsive to strengthening elementary teachers' mathematical content knowledge and quality of mathematics instruction (Ma, 1999). The second feature is engaging teachers in active learning (Darling-Hammond et al., 2017; Garet et al., 2001; Loucks-Harsley, 1996) to strengthen their pedagogical skills. Guided by the Leading for Mathematical Proficiency Framework (Bay-Williams & McGatha, 2014), we aimed to support teachers' shifts in classroom practice. The third feature is enabling and promoting collaboration among teachers, so they feel they belong to a supportive community (Darling-Hammond et al., 2017; Galindo et al., 2014; Garet et al., 2001; Loucks-Harsley et al., 1996). Teachers built their professional learning communities in which they collaboratively reflected on and felt empowered and supported to make incremental, but powerful, shifts in their practice.

Creating a Structure for Collaboration on the Design of PD Activities.

University-school partnerships are a collaborative format for facilitating teachers' professional growth widely used in the field of mathematics education (Avalos, 2011; Bartholomew & Sandholtz, 2009). Our PD was designed as a two-year partnership between a higher education institution and elementary mathematics teachers from eight schools (Grades K-6) in two school corporations. There are challenges in building an efficient, effective partnership (Bartholomew & Sandholtz, 2009; Grossman, 1994; Winitzky et al., 1992). One way to strengthen the researcher-teacher (university-school) partnership is to work collaboratively with a range of stakeholders to develop a shared vision and identifiable goals (Association of Mathematics Teacher Educators, 2017) and enable stakeholders, especially teachers, to democratically engage in decision-making via a continuous and deliberate process.

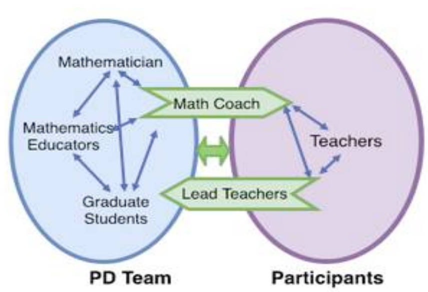

Figure 1: Forming a Structure for Centering on Teachers' Needs

The CRPD collaborative structure included two major groups: The University Team (PD Team) and In-service Mathematics Teachers Team (Participants). The Participants included classroom teachers and a Lead Teacher from each school. The PD Team included two mathematics teacher educators (MTEs), one mathematician, one mathematics coach (an experienced upper elementary and middle school teacher), and two international mathematics education graduate students (Figure 1). We should note the PD Team involved people with, at times, divergent perspectives on what they considered to be the most effective content and pedagogy curriculum or learning activities for the PD workshops.

One unique feature of this model is that the mathematics coach and lead teachers served as crucial conduits between the Participants and the PD Team. The lead teachers recognized the professional learning needs of the teachers at their schools from both practical and theoretical perspectives. Having lead teachers in the CRPD model helped create a sense of community among the participants as they felt safe sharing their concerns with someone they knew and trusted. One major role played by the coach was to visit teachers' classrooms daily to get an insight into the content and pedagogical needs of the teachers (Darling-Hammond et al., 2017; Gibbons & Cobb, 2016). The coach and lead teachers were instrumental in building trusting relations between the two groups, the PD Team, and the Participants, so we could interact effectively and symbiotically.

Creating Structures for Systematically Collecting Teachers' Voices

Teachers, as adult learners, do not enter any PD session as blank slates. Instead, they bring experiences, knowledge, skills, and dispositions which impact their PD engagement, contribution, and outcomes (Ball, 1996). For the PD sessions to be meaningful and relevant, we realized the need to create structures that would provide us with continuous feedback from teachers (Darling-Hammond et al., 2017; Yoon et al., 2007). Thus, we employed survey tools to identify teachers' PD goals, as well as their long-term and short-term needs.

First, to identify specific PD goals, we conducted an *Initial Survey* that went out to all teachers during the PD planning stage. Second, to ascertain long-term needs, each teacher completed an *Annual Reflection and Personal Growth Plan form*, at the beginning of 2016, 2017, and 2018, where they described what they wanted to improve in their teaching practice and shared important takeaways from the PD project. Third, to ascertain short-term needs, we designed a Workshop Feedback form to collect teachers' feedback, using both Likert-scale ratings and open-ended comments, on the quality and relevance of each learning activity in each PD session, as well as their needs and requests for future PD learning activities. In these three ways, the CRPD model design positioned teachers as active professional developers, having both a say about what professional growth they wanted and the means to attain their goals.

The Implementation of the CRPD Model

The project served 60 teachers from eight schools in two school corporations. Teachers' participation in the project included 80 hours per year, distributed among summer workshops (40 hours), two full-day workshops during the school year (one in the Fall and one in the Spring), and four after school sessions throughout the school year.

Adjusting Objectives for PD Workshops According to Teachers' Voice

Based on the *Initial Survey* results, we specified the main goals for this PD. After each PD session, the PD Team analyzed the *Workshop Feedback Form* via four stages. First, immediately after the workshop, the PD team quickly read through and sorted the surveys to identify those with fairly high Likert-scale ratings and those with lower ratings. Second, shortly after the workshop, we scanned the sorted survey responses and emailed that file to the PD team, who read through and analyzed the teachers' open-ended comments for themes. Third, email conversations took place about what themes we observed, and we brainstormed topics related to those themes that we might cover for the next PD. Finally, meetings were scheduled (in person and via Zoom) at which the PD Team worked together with the mathematics coach and the lead teachers to discuss our brainstorm ideas and to plan the agenda for the next PD based on the teachers' feedback, the coach's classroom observations, and the lead teachers' suggestions. At the end of each PD year, we used a similar process to systematically examine teachers' responses to the *Annual Reflection and Personal Growth Plan form*. This process systematically centered the teachers' feedback so that we were able to be responsive to it.

The Impact of the CRPD Model on Teachers' Professional Growth

Two foci of the PD were supporting teachers' understanding and implementation of the Standards for Mathematical Practice (CCSSO, 2010) and developing teachers' pedagogical content knowledge (Galindo et al., 2018). We examined the effectiveness of the CRPD model by analyzing teachers' statements about their major takeaways from the project and about shifts they had implemented in their instructional practice. Data sources included teachers' responses to the Reflection and Personal Growth Plan from 2017 and 2018.

From teachers' reflections about their main takeaways from the project, we found that the teachers' responses were largely centered on two process standards: 'Making sense of problems and persevere in solving them' (MPS1) and 'Constructing viable argument and critique the reasoning of others' (MPS3). Regarding MPS1, teachers acknowledged the significance of allowing students more time to explore problems to elicit multiple strategies and develop strong sense-making for the concepts. Teachers' responses signified the importance of using challenging mathematics tasks that elicit students' thinking and then assisting students in making sense of and persevering to solve the problem. For MPS3, the teachers initially expressed their inclination towards engaging students in discursive practices by promoting collaboration and communication. However, early on they were not reflecting on their role in establishing (or not) such a dialogic learning environment. Gradually, they realized that mathematical communication in their classrooms was too often teacher-dominated. One teacher wrote, "Teachers talk way too much and we all know it, we just can't stop…. I now let them share so much more and I just listen, whether it's right or wrong, I just listen. Then I ask a question, and I listen again. It is amazing the things our students can think of when we give them time to think and share." This teacher realized her role is to intentionally utilize questioning and listening to create ample opportunities and space for students to share their ideas and critique others' reasoning. These takeaways indicate the effectiveness of the PD project in promoting teachers' understanding and implementation of the mathematical practices.

In their reflections, teachers also wrote about their prior experiences as learners of mathematics and stated how experiences from this PD project changed their perceptions of mathematics and self as a mathematician. One teacher wrote, "I was not a confident math student during my school years and had a math phobia… I have learned so much from this experience. I even find myself using the strategies in my daily life. I am more confident in teaching math and helping my students. I actually love math now." Another teacher stated, "[As a learner] I memorized the algorithms we learned in school and didn't really ever question the why behind it. I feel that this training has helped deepen my understanding of math and math concepts." These statements point to the effectiveness of the PD project in positively impacting their relationship with mathematics as a subject.

Discussion and Implication

We sought to create a safe environment for the teachers to express their needs and provide honest feedback on PD learning activities, with an understanding that their voices would be heard, and the PD team would develop a responsive PD curriculum. We confirmed that cycles of emergent and responsive curriculum (Confrey & Lachance, 2000) development are a powerful tool for centering and addressing the needs of teachers. The CRPD model democratizes PD by sharing decision-making power among the Participants and the PD Team. Teachers are given an active voice in their learning, and the math coach and lead teachers serve as strong advocates for teacher participants. Being collaborative and responsive in curriculum design requires systematic and iterative cycles of planning, implementing, and reflecting during which teachers' voices are centered, valued, and utilized.

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POSITION AND SENSEMAKING IN REHEARSAL DEBRIEF DISCUSSIONS

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Following rehearsals of instructional practices, teacher educators often facilitate debrief discussions for participants to reflect on and make sense of their experience. This study explores the ways in which rehearsing and non-rehearsing teachers, who act as teachers and students respectively, make sense of that experience collectively and how their positions as teachers and students are reflected in their talk. Data from eight rehearsal debriefs conducted with in-service secondary mathematics teachers during practice-based professional development are examined.

Keywords: Teacher Education – Inservice / Professional Development, Learning Theory

Recent literature in mathematics teacher preparation focuses on how coached rehearsals can be used to support the learning of complex instructional practices (Ghousseini, 2017; Lampert et al., 2013). In these approximations of practice, one person typically takes on the role of teacher, while the rest of the group takes on the role of student. The rehearsal is facilitated by a teacher educator (TE) who takes on the role of coach (Kazemi et al., 2016). As these approximations of practice have gained prominence in both pre-service and in-service professional development, research has focused on the learning of participants who take on the role of teacher (Ghousseini, 2017; Lampert et al., 2013) and the structures used during the rehearsal to support the enactment of teaching practices (Kazemi et al., 2016). However, we know less about the ways in which debrief discussions following the rehearsal, in which both the rehearsing teachers (RTs) and non-rehearsing teachers (NRTs) participate, support the sensemaking and learning of both. In this paper, we contrast the experiences of rehearsing and non-rehearsing secondary mathematics teachers following rehearsals during a summer professional development institute. In particular, we describe how both RTs and NRTs make sense of their experiences during the reflective debrief discussions that follow each rehearsal, and ask how their position as student or teacher in the rehearsal is reflected in their talk.

Prior Literature

Often, after a rehearsal, all participants (RTs, NRTs, and TEs) engage in a debrief discussion. This kind of reflective post-rehearsal discussion, facilitated by the TE, represents an opportunity for public sensemaking, or "collaborat[ing] on sensemaking as a shared group goal" (Ruef, 2016). Through public sensemaking, RTs, NRTs, and TEs all have a chance to learn from one another and reflect on the rehearsal experience. Public sensemaking is also valuable from an analytic perspective, because it gives insight into some (but not all) of participants' sensemaking about a given experience. In this kind of reflective debrief structure, we wonder about what sensemaking RTs and NRTs share publicly and how they might collaborate to make sense.

What teachers can and do say during debrief discussions is influenced by the role they played during the rehearsal. While all participants are intended to learn as teachers, only RTs participate in the rehearsal as teachers; NRTs spend rehearsals acting as students. In other words, NRTs are asked to change their position, moving from student in the rehearsal to teacher during the debrief. How a person positions themselves or is positioned in an interaction can influence the obligations they feel (Aaron & Herbst, 2012; Herbst & Chazan, 2012). In particular, being positioned as a teacher or as a student can change how people react to the same contexts (Baldinger & Lai, 2019). Position is thus an important consideration that might help explain how public sensemaking is constructed in rehearsal debriefs.

We explore public sensemaking analytically through the lens of professional noticing. When engaging in the work of noticing, teachers *attend* to *interpret* specific details, imbuing them with meaning (Mason, 2002; Sherin et al., 2008). In the context of a debrief discussion, teachers might attend to and interpret an event that occurred during the rehearsal, elevating that event for public consideration. Because teachers' goals in participating in coached rehearsals are ultimately to inform and shape their teaching practice, teachers might also describe implications or connections that extend their thinking beyond the specific context of the rehearsal itself (Baldinger & Munson, 2020). This framework of noticing and implicating provides a lens through which we can describe the public contributions made during debrief discussions. Given this, we ask the following research questions: (1) How do RTs and NRTs use debrief discussions to publicly make sense of their experiences during rehearsals? (2) How is position reflected in RTs and NRTs sensemaking of the rehearsal during the debrief discussion?

Methods

Setting and Participants

This study took place in the context of professional development program for early-career (2nd-7th year) secondary mathematics teachers serving lower-income schools. This two-year fellowship included two-week summer institutes and ongoing online coaching during both school years. Our research considers the second summer institute, which focused on facilitating collaborative group work. Participants included 22 high school mathematics teachers from comprehensive public, magnet, and charter schools across the US.

Design

The summer institute culminated in a full day during which all participants had the opportunity to rehearse leading collaborative group work. Rehearsing teachers focused on the practice of conferring (Munson, 2018) to support productive engagement in cognitively demand task and equitable participation within collaborative groups. Teachers were randomly assigned to one of two rehearsal rooms. Four rehearsals were conducted in each room (eight total rehearsals), so that each teacher had the opportunity to rehearse the focal practice once and participated as an NRT three times. Additional math teachers were recruited to participate as NRTs to increase the class size in each room. After each rehearsal, the TE prompted the group to discuss the experience, beginning by asking RTs to share "some of the things in your head right now" and then pose any questions they had for the NRTs. Debriefs then pivoted to NRTs addressing any RT reflection questions and segued into general reflection. All debriefs ended with "final thoughts" from the RTs.

Data Sources and Analysis

This study draws on audio and video recordings of the debrief discussions following each of the eight rehearsals. Debrief discussions ranged in length from 14 to 23 minutes. Each was professionally transcribed for qualitative analysis. Each talk turn (n = 961) was coded by participant (RT, NRT, TE). All RT and NRT talk turns were then segmented and coded based on the type of sensemaking represented in the speech, (i.e., attending, interpreting, implicating), where *attending* and *interpreting* were defined by prior research (Sherin et al., 2011) and *implicating* was defined as making connections beyond the rehearsal (e.g., to the speaker's own classroom) or considering alternative pathways for the rehearsal. Any speech that did not fit these codes was not coded. Each talk turn segment was then coded for the position the speaker took (i.e., teacher, student). If an utterance took more than one position, it was further segmented such that each utterance could be given a single position and sensemaking code pair (n = 983). Code matrices were developed within and across the eight rehearsals to explore patterns of participation.

Findings

While the TEs structured the debrief by inviting RTs and NRTs to reflect during different parts of the discussion, the ways they participated in sensemaking were strikingly similar. Of the coded talk turn segments, 29% of RT talk and 34% of NRT talk attended to observable details about the rehearsal, while 47% of RT talk and 46% of NRT talk interpreted the details of the rehearsal, indicating the meaning the speaker made of events (see Figure 1). Together, these acts of public noticing made up approximately the same proportion of speech, 75% of RT and 80% of NRT coded talk turn segments. The vast majority of all contributions in the rehearsal debrief focused on noticing the rehearsal, making specific details of events public and offering ways of understanding the meaning of those events.

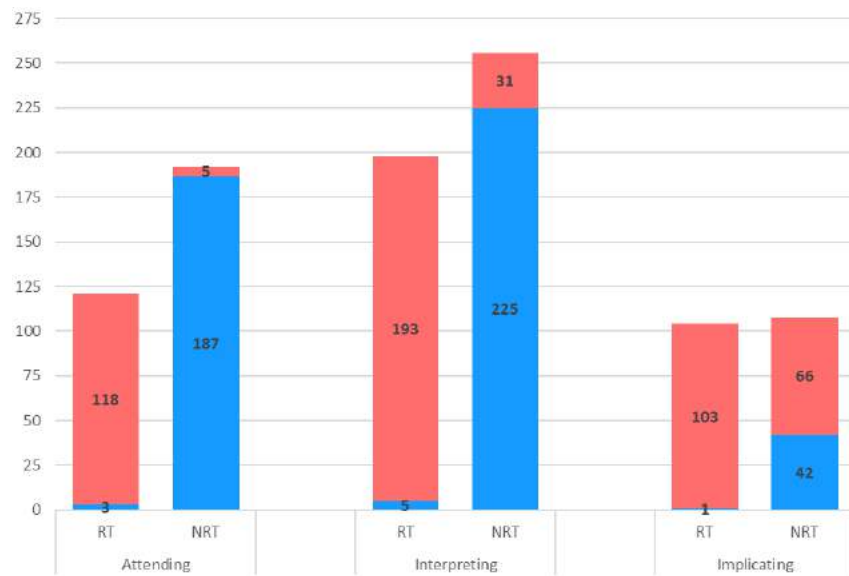

Figure 1: Attending, interpreting, and implicating talk turn segments for RTs and NRTs, by teacher (red) and student (blue) position of the speaker.

For the balance of coded talk turn segments, RTs (25%) and NRTs (20%) made implications, moving beyond the rehearsal as it occurred to consider the meaning the events might have for teaching or alternative scenarios for the rehearsal. All the data generated through noticing the rehearsal fueled reasoning about teaching and learning when implicating. These overall patterns of talk were not substantively different across the two rooms or any of the eight rehearsals.

While RTs and NRTs noticed and implicated in similar proportions, there were pronounced differences in the positions each assumed when speaking (see Figure 1). RTs overwhelmingly (98%) maintained their position as teacher when speaking, regardless of whether they were attending to or interpreting the rehearsal, or drawing implications beyond it. There was a coherence in their role throughout that is reflected in these data; as teacher-learners they were asked to act as teachers and learn from those acts as teachers. NRTs were asked to perform a more complex position move; as teacher-learners they were asked to act as students and learn from their experience as teachers. In contrast to RTs, NRTs overwhelmingly noticed the rehearsal from the position of student (92%). But when implicating beyond the rehearsal, NRTs took on a more complex stance, at times maintaining their rehearsal position of student (39%), but more often flipping to their learner position of teacher (61%).

Notably, while RTs were largely fixed in their position as teachers, they actively sought the perspectives the NRTs gained from being students, data to which they would otherwise not have access. These rhetorical moves from the RTs were striking in how they shaped the ongoing discussion in the debrief and contributed to the "expected" patterns described above. RTs asked pointed questions of their NRT colleagues to unearth how the teaching moves they used were experienced by the students in the rehearsal. They prompted NRTs with statements such as "I kinda want to know how it [the rehearsal] was from your [NRT] lens," "I'm curious to get feedback on… the groupworthiness of the task," and "Was there ever a moment when you [NRTs] wished we [RTs] had jumped in and we didn't?" to elicit the student position experience.

Conversely, when NRTs offered their own noticings, there is evidence in the data that these could fuel implications from RTs. For instance, after a string of 12 talk turns in when three NRTs attended to and interpreted a challenge they faced setting up graphical representations during the task, one RT made the following implication:

> What I'm hearing is, there's the potential for this to hang up a group? … So maybe if I had caught onto the idea that you were really hung up here, if I had observed that…I might have like, nudged you forward.

In this excerpt, the RT took up the NRTs' noticings to consider what they could have done differently, particularly if the RT had noticed, in the moment, what the NRTs did, that their struggle was impeding their mathematical progress. When NRTs offered their noticings as students, they were simultaneously supporting their own sensemaking of the experience and providing insight to RTs who wanted to understand how students experienced their teaching. These results foreground the interactive nature of sensemaking among RTs and NRTs.

Significance

The similarities of the proportions of talk across participates' roles, across both rehearsal rooms, and all eight rehearsals points toward the possibility that public sensemaking grounded in a shared experience like a rehearsal may more generally focus on noticing with a smaller portion of implicating beyond the experience. In previous work (Baldinger & Munson, 2020), we have suggested that debrief discussions may be venues for NRTs to develop adaptive expertise (Hatano & Inagaki, 1986) in the wake of rehearsal by promoting data-driven forward reasoning. This new analysis suggests that rehearsal debrief discussions may serve a similar function for RTs; future work could investigate whether the nature of what is noticed by RTs and NRTs and the types of implications they draw supports such a claim.

Position played a critical role in the ways that RTs and NRTs publicly made sense of their experience together. Prior research is premised on the safe environment that rehearsing among colleagues can provide to teacher-learners (Lampert et al., 2013), but the current research indicates that the roles NRTs play can provide an additional advantage. The rehearsal experience and the debrief discussion that followed offered participants a window into the experiences of students, supporting NRTs in learning from a rehearsal in which they did not teach and RTs in learning about the ways in which their pedagogical choices impacted students. NRTs were then not just safe colleagues with whom to approximate practice, but safe students from whom to elicit feedback on instruction. Future research could investigate how the structure of the debrief discussion can support RTs in gaining access to the student experience data in ways that can inform their learning from the rehearsal experience.

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AISSPOMMOOTSIIO'PA – A COLLABORATION TO IMPROVE TEACHING AND LEARNING MATHEMATICS AT THE KAINAI NATION

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The Kainai Board of Education, situated at the Kainai First Nations Reserve in Southern Alberta, Canada, initiated the Aisspommootsiio'pa project in 2017. The project intended to develop teacher leadership capacity aimed at improving mathematics teaching and learning at elementary and secondary levels. In this paper, we indicate the theoretical foundations for the project. We also report on its implementation during the first two years, which involved seventeen teachers, and offer suggestions for the extension of the project at a larger scale within the Kanai Nation.

Keywords: Elementary School Education, Mathematical Knowledge for Teaching, Teacher Education - Inservice / Professional Development, First Nations and Indigenous cultures

The Aisspommootsiio'pa Project

In 2017, the Kainai Board of Education (KBE) contacted the Werklund School of Education, University of Calgary, to initiate what was later called the *Aisspommootsiio'pa* project: Translated into English, *Aisspommootsiio'pa* means "supporting each other." Since then, a group of teachers engaged in a professional learning series that included reflection on classroom observations. One specific purpose of the project was to identify and develop mentorship capacity within KBE for future, sustained support for other teachers within the school district, which comprises two elementary schools, one secondary school, and one high school.

During the first year of implementation, a group of observers supported the project creating records of observed lessons for teachers to reflect on. Every observation included at least two observers who recorded images, notes, and events in a timeline, and who rated each lesson using an observation protocol based on a teaching model developed by the Math Minds Initiative (Preciado et al., 2019a). In the second year of the project, teachers also participated as observers and created the reports.

The first two years of the project are considered a pilot in preparation for a future intervention at a larger scale at KBE.

The RaPID Model

The work with teachers in the Aisspommootsiio'pa project followed, and ultimately informed, the Mind Minds Initiative. Research results from this initiative include a sustained improvement in student performance, as measured by the Canadian Test for Basic Skills (Nelson, 1997). This initiative has developed the Raveling, Prompting, Interpreting, and Deciding (RaPID) model for teaching based on research findings on classroom observation and student performance and engagement in mathematics for more than seven years (Preciado Babb et al. 2019a; 2019b). This empirically developed model is consistent with well-established theories of learning from the cognitive sciences, such as: embodied cognition (Varela et al., 1991); socio-cultural theory (Vygotsky, 1986); spatial reasoning (Davis, et al., 2015); conceptual metaphor theory (Lakoff & Johnson, 1999); and conceptual blending theory (Fauconnier & Turner, 2003). The model also draws from theories of influencing learning, such as: affordance theory (Gibson, 1979); variation theory (Marton, 2014); mastery learning (Bloom, 1968); meaningful learning (Novak, 2002); and expert–novice research (Ericsson, et al., 2006).

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Protocol for Classroom Observation

The Math Minds Initiative developed a classroom observation protocol, which was used in the Aisspommootsiio'pa project. Table 1 summarizes Levels 1 and 4 of the protocol used for the project in 2018 and 2019. A refined version with details of the RaPID model was later elaborated on by Preciado Babb and colleagues (2019a).

Table 1: Summary classroom observation protocol for Levels 4 and 1

	Level 4	Level 1
Connecting (a) (Raveling)	Contrasting ideas build on prior contrasts and toward a generalized learning target.	Contrasts do not build on prior contrasts. It is challenging to identify a learning target.
Connecting (b) (Raveling)	The known and unknown are systematically bridged; each new idea is anchored to prior understanding that has been effectively summarized to carry forward.	There is little attempt to bridge known and unknown; new ideas are not anchored to prior understanding
Prompting (a)	The teacher separates and effectively draws attention to the key idea; contrasts are clearly juxtaposed, highlighted, and appropriately sequenced.	Key critical features are overlooked.
Prompting (b)	Clear prompt [to each critical discernment] requires learners to make key distinctions.	Multiple ideas are presented before students are offered an opportunity to engage; those who attempt to engage on their own may fall behind.
Monitoring (a) (Interpreting)	Quick/perceptible means of response that provides meaningful information re: student understanding.	Students do not show evidence of understanding or are asked to indicate whether or not they understand: "What part don't you understand?" "Thumbs up if that makes sense."
Monitoring (b) (Interpreting)	All students checked.	Few checked; e.g., individual students called on to respond (out loud or on the board).
Adapting (a) (Deciding)	Group response attends to diverse needs.	Response limited to few learners; many waiting for help or extension; e.g., lengthy conversations with a single child.
Adapting (b) (Deciding)	The teacher clarifies contrasts and adjusts sequencing in ways that address all learners; e.g., clarifying a pattern of variation also allows extended consideration of the related variable(s).	The teacher repeats strategies that didn't work or are unnecessary; the repeated explanation is insufficient for those who didn't get it the first time and redundant for those who did
Engagement	All students engage together in continuously extending understanding	Many students do not participate.

The Aisspommootsiio'pa project included the adoption of resources developed by JUMP Math (https://jumpmath.org/) for teachers at the elementary level. JUMP Math is a member of Math Minds and the teaching resources have informed the development of the RaPID model.

Methods

In order to document the implementation of the Aisspommootsiio'pa project, a qualitative case study was conducted (Yin, 2018) to address the following questions: How do mathematics teaching practices and use of resources change when adopting the selected resource and participating in the learning series? What factors enabled or hampered the implementation of the RaPID model in the classroom?

A thematic analysis was conducted on surveys administrated at the end of each professional learning session (four session each year). These data were contrasted to the classroom observation reports (five rounds of classroom observation each year), which included images, notes, and timelines generated using the LessonNote app. Figure 1 shows the timelines generated for two lessons. The image on the left corresponds to what is called a *block lesson* in the Math Minds initiative, while the image on the right corresponds to a *ribbon lesson*. The latter assumes cycles of prompting, interpreting, and deciding, consistent with the RaPID model, whereas the former reflects minimal instruction and support to student during class.

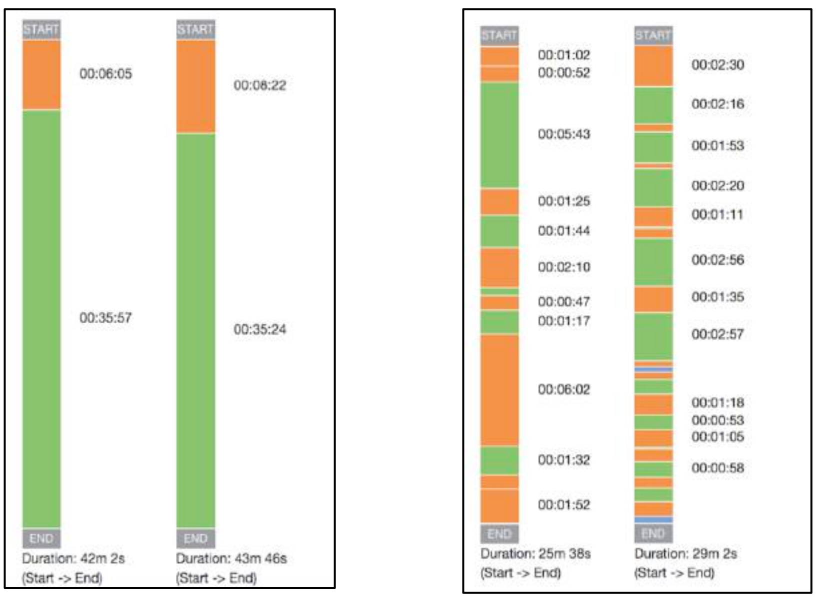

Figure 1: Timelines of classroom observation for two lessons: block lesson in the left and ribbon lesson in the right

Findings

Results of this study are presented as follows in three emergent categories.

Classroom Practices and Teachers' Learning

Teachers reported an appreciation of the RaPID model in the surveys administrated at the end of the professional learning sessions. Some teachers explicitly mentioned teaching practices related to the cycles of interpreting and deciding; this resulted in a more ribbon-like lesson, as shown in Figure 1. The survey entries also included learning about specific aspects of mathematics for teaching, such as different ways of understanding division (partitive vs. quotative). With respect to JUMP Math resource, some teachers indicated that the resource helped them to teach in "small chunks" that focused on critical mathematical features (Marton, 2014) required for students to understand the targeted concepts. Some teachers also indicated that the RaPID model started to inform their teaching of other subjects, not only mathematics.

Implementation Challenges

Despite evidence from surveys of teachers' learning and the impact of the implementation on their teaching practices, there was also indication of challenges for the implementation of the model. These challenges were confirmed by classroom observations and field notes.

One of the challenges in the project related to the adoption of the JUMP Math student teaching and practice booklets. The resources arrived late in the first year of the project and not all the teachers were following the program consistently. In fact, one teacher confessed that she only used the resource for the class that was observed as part of the project. Other teachers in the second year identified the need to try the resource for some years in order to become familiar with it and to be more confident using it.

Lack of time for debriefing and reflecting on observed lessons was also a challenge identified by some teachers. This challenge prompts to the need to allocate specific time for this purpose.

Some teachers also acknowledged the need for more targeted professional learning regarding the mathematics concepts; this was required to better "unravel" the concepts for students, prompt to critical features, interpret students' understanding, and make appropriate decisions.

Mentorship and Capacity Building

One of the observers in the project, also one of the authors of this report, provided feedback to teachers after classroom observation. These teachers emphasized in the surveys and with explicit comments in the professional learning sessions, the support they received from this other teacher, who holds a master's degree in mathematics education. This information was new to the KBE and prompted the need to identify mentorship capacity at KBE.

During the second year of the project, teachers reported being more comfortable having peers observing their class. They also indicated learning through classroom observation. This experience also helped to conceptualize mentorship as peer support in a horizontal fashion, as opposed to a vertical approach in which the mentor is regarded as an expert.

It was also noticed that observers required time to learn to use both the LessonNote application as a tool and the observation protocol used to guide classroom observations.

Discussion

This case study shows evidence of the viability of extending the Aisspommootsiio'pa project to other teachers in the school district and of developing mentorship capacity. The findings in the study can be translated into specific suggestions for this purpose, namely: considering time for teachers to become familiar with the teaching resources; allocating time for classroom debriefing and teacher reflection; training teachers to implement classroom observations; building a trustful relationship between classroom observers and teachers; and embedding teacher professional learning for all teachers involved in the project. These suggestions could be followed with other means of data that could be used to assess the impact of the project on mathematics student learning at the Kainai Nation.

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"THIS IS YOU. THIS IS YOUR FAMILY": CASE STUDY ON ATTENDING TO MATHEMATICAL LANGUAGE DEVELOPMENT

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Language is a vital component in mathematics classrooms and researchers have thoroughly examined how language functions in instruction. However, less is known about how teachers think about language enacted in their own classrooms. In this report, we describe how a teacher, Olivia, explicitly attended to language, particularly with emergent bilinguals. We describe affordances and tensions as she thought through language in the context of a professional development and in video-stimulated recall interviews.

Keywords: Classroom Discourse, Elementary School Education, Teacher Education - Inservice / Professional Development

There is a large amount of research on the role of language in mathematics classrooms. Scholars characterized language as a high need area for research given the growing diversity in the United States, diversity of language in the classroom and its impact on mathematics education (e.g., Barwell et al., 2017). Language supports constructing mathematical concepts, positioning of individuals and groups, developing mathematical argumentation, and shaping mathematical communities (Herbel-Eisenmann et al., 2017). Even with such a rich connection between language and mathematics, there is little research on teachers' thinking about language in their own instruction (Hajer & Norén, 2017). This dearth reflects how researchers' voices and interpretations, not teachers, are elevated in the research on language in mathematics classrooms. In this report, we add to this work by conducting a case study on Olivia, a third-grade teacher, and how she made sense of language in her instruction. We ask: *What are affordances and tensions Olivia perceived as she attended to mathematical language development?* We describe how Olivia's attention to both heritage (i.e., Spanish) and mathematical language was tied to mathematical development and participation and access, particularly for Latinx students.

Theoretical Framework & Professional Development Model

In order to explore how Olivia made sense of the language in her classroom through tensions and affordances, we frame our understanding of language and teacher knowledge. We view language similarly to how Gee (2005) described small-d and big-D discourse (which we will write as discourse and Discourse, respectively). First, he described discourse as "language-in-use" or the material of communication such as words and gestures in order to "design or build things." In this report, we use "language" at times to describe mathematical language but also as heritage languages (e.g., Spanish, English). Second, Gee described Discourse as other "language stuff" that enact specific identities and activities such as ways of acting and believing. We see Discourse bearing on how Olivia thinks about her language use as a marker of not only doing mathematics but also enacting a particular identity.

We view the nature of teacher knowledge as embedded in practice as opposed to an outside body of knowledge (e.g., contained within academia) that needs to be "learned" by teachers. Cochran-Smith

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and Lytle (1999) described teaching as “an uncertain and spontaneous craft situated and constructed in response to the particularities of everyday life in schools and classrooms” (p. 262) and thus, teacher knowledge draws from “their own reasoning and decisions, and their own inventions of new knowledge to fit unique and shifting classroom situations” (p. 267). This knowledge is highly sensitive to time and space. Based on this view of teacher knowledge, we co-designed and facilitated a professional development environment for elementary teachers called Learning Labs (LLs) (see Kazemi et al., 2018 for an elaboration of the model) focused on developing practices of mathematical argumentation. Additionally, this model aligns with Cochran-Smith and Lytle’s (1999) description of teacher knowledge as generated by teachers and based on their own actions.

Context, Data, and Analysis

Lockwood Elementary is situated in an urban area in the United States Midwest. In 2018-2019, the school served 443 students including 40% Latinx students. We began our LLs in January of 2019, having completed 8 LLs to date. Olivia, a 14-year bilingual Latina teacher participating in the LLs, taught at Lockwood Elementary for eight years by the time of the LLs. She taught in a third-grade Dual Language Immersion program (DLI) where native Spanish and native English speakers were placed in the same class with instruction in both Spanish and English. We conducted a case study (Merriam, 1998) in order to “develop an intensive, holistic description and analysis of a single, bounded unit” (p. 232-233). Olivia’s central belief is supporting language and empowering students, making her participation an important case to study. We reviewed materials from 2 interviews, written artifacts and verbal contributions in the 8 LLs, and 2 video-stimulated recall interviews (VSRs) from filmed classroom lessons with Olivia. Because Olivia frequently made statements about mathematical language development across data sources, through analysis we used the sensitizing question, “When Olivia talks about mathematical language development, what does this allow her to know about her students and do as a teacher?” and, “When Olivia talks about language development, does she perceive an opposing concept?” to identify affordances and tensions, respectively.

Results

In this section, we describe the affordances and tensions Olivia experienced as she considered supporting her students’ language development. We provide examples from written work during the LLs and one VSR lesson, where Olivia facilitated a choral count by fourths.

Affordances

Mathematical connections. Olivia’s focus on precise language (i.e. discourse) afforded her to challenge students’ thinking and help foster connections. For instance, during the VSR, she wanted to know how students would describe the count after three-fourths. She wanted students to “verbalize specific fractions where we’ve specifically been talking about how to simplify and making connections to the wholes.” As she expected, students had varied responses after three-fourths including “four-fourths” and “one.” Olivia facilitated a conversation asking students to prove the connection between them with a drawing. Additionally, Olivia said the task “opened up another conversation and then they were making connections to money and then we ended it where we made a connection to the clock and looking at a quarter and how we say the time.”

Engagement. Olivia predominantly focused on engaging with mathematics and participating in a classroom community (i.e. supporting Discourse) through language support. In LL4, Olivia recalled how conversation can have power and meaning, claiming it is about

> listening and trying to understand what someone else is saying. What someone is trying to restate… this goes back to the power piece, where we’d finished turn and talk and there was still [chatting] and [a student] said ‘no no no wait, I want to share something.’ He was trying to make a point.

In interviews, Olivia explained that through a language focus, she wanted more participation for emergent bilingual students (EBs). By trying a new instructional activity called choral counting, she noticed how it could engage EBs, such as her student, Esteban, and how he participated in the conversation about selecting "one" or "four-fourths." She recounted asking Esteban "who was very, very emergent with his English" what the class should say, "and he raised his hand and very clearly he said four-fourths. So it was such a small moment, but I feel like it was so important for him because he was able to participate."

Tensions

Content vs. language. In many instances, Olivia viewed developing language as separate from developing mathematical ideas. As Olivia worked to encourage students to have meaningful discussions, she struggled to find a balance between language socialization and mathematical content because she described these two in opposition. In LL6, Olivia reflected on students' shared mathematical ideas,

> It's been challenging rephrasing or paraphrasing [students'] ideas not because of the math but because of their language development. At the beginning I noticed many students were eager and talking around. But with their vocab and grammar it was really hard to understand the ideas they were trying to get across.

In the VSR, Olivia talked about the same tension while watching a lesson she taught in Spanish. The VSR was also conducted in Spanish. She apologetically reflected,

Yo admito que en la elección me alejé un poco del contenido de matemáticas porque quería enfocar más en aprovechando que los niños estaban tratando de explicar todo lo que podían… Es un proceso. Quizás alejé demasiado a las matemáticas, pero me hizo sentir y pensar que era obvio o es obvio de qué tanto, qué tanto tengo que regresar o repasar acerca de cómo compartir y cómo usar las palabras en matemáticas cuando estamos compartiendo.	I admit my decision to move away from the math content because I wanted to focus more on taking advantage of the fact that the children were trying to explain everything they could… It is a process. Maybe I took math away too much, but it made me feel and think if it was obvious or how obvious how much I have to go back or review how to share and how to use words in math when we are sharing.

Olivia recognized students need time to express their ideas but focusing on Spanish language development can take away from mathematical ideas. Although she saw the importance of language and mathematical content, she struggled with balancing the two.

Speaking Spanish vs. Spanish speakers. Prominent in the interviews, Olivia described a disconnect between the language of instruction, Spanish, and students who spoke Spanish. Olivia felt speaking Spanish as something students do but also something empowering. She saw language as part of one's identity. In the initial interview, Olivia reflected on the lives of Latinx students and if this has any bearing on their identity and how they participate in school,

> I feel is there more privilege, more access to resources outside of school that [White students] are able to partake in that my Latino students aren't able to… do my Latino students—do they feel empowered being native Spanish speakers and being in this program where there's this part where the Spanish, that's their language, that's them, their identity? And I do wonder how they feel as Latino students in [the DLI] program… the students don't want to use their Spanish and I see a lot of my Latino students that they start to because they

> are hesitant in Spanish. And that, for me, part of that identity, it's like, ‘But this is you. This is your family.’

After the first set of LLs, Olivia noted an improvement in the quality of student contributions in discussions. However, her Latinx students would still hesitate, “I see so many of my Latino students, even during Spanish when we're having conversations… this is their first language, and they're hiding in the conversations or they're not raising their hands.” Olivia wants “them to be proud of being able to participate in the conversation or even dominate the conversation the way native English speakers may be able to dominate more in English.” In the VSR, Olivia described,

Había un grupo de niños acá, algunas niñas que siempre están compartiendo en español y son muy buenos modelos para el español y el lenguaje, que tenían mucha pena y tienen tanta confianza cuando están hablando en español, pero tienen menos confianza cuando se trata de matemáticas.	There was a group of children here, some girls who are always sharing in Spanish and are very good models for Spanish and language, who get embarrassed but are confident when they are speaking in Spanish, but have less confidence when it comes to math.

Such moments demonstrate a tension Olivia experienced between improving discourse (students speaking) and hesitant Discourse (confident participation). It functioned as an overarching goal of empowering students, particularly Latinx students, their language, and their culture.

Discussion and Implications

We highlighted tensions and affordances an elementary school teacher, Olivia, perceived as she attended to developing language in her mathematics class. Underlying all the affordances and tensions we described, a central belief guided Olivia’s sentiment, “This is you. This is your family.” For Olivia, the lines between discourse and Discourse are blurred. Developing ways of speaking and heritage languages is not just about engaging in mathematical discussions or using appropriate words. Her attention to language is rooted in empowering Latinx students.

Olivia demonstrated attending to language is complex. Researchers have prescribed how teachers should think through and about language diversity in classrooms and confronting English dominance (e.g. Palmer, 2009). Working in a DLI setting affords Olivia to work on this confrontation but still handle tensions of balancing student empowerment, language development, and mathematical content. Due to space constraints, we could not further attend to how affordances and tensions of language impacted Olivia’s instructional considerations around attending to mathematical vocabulary, supporting students to engage in each other’s ideas, and building a math talk community. More work is needed to understand how teachers think through these ideas and how they learn to address this balance. Additionally, research on language should not just pay attention to language as it relates to status, power, or language resources (Barwell et al., 2017), but also how teachers make sense of these ideas inside instruction.

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TEACHERS' COGNITIVE PROCESSES DURING THE TEACHING OF FRACTIONS AND MULTIPLICATIVE PROBLEMS

LOS PROCESOS COGNITIVOS DE PROFESORES DURANTE LA ENSEÑANZA DE FRACCIONES Y PROBLEMAS MULTIPLICATIVOS

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This communication is about the case of a secondary school teacher and forms part of an investigation carried out with teachers. The aim is to identify the mathematical knowledge of the teacher, who has a vast educational experience when designing and teaching multiplicative problems with fractions, as well as to associate her reflections on her practice when she teaches those mathematical contents in the classroom. The methodological instruments applied in the tracing of this case were a questionnaire and three individual interviews of a didactic nature with feedback. This paper integrates only the tasks designed by the teacher in third interview that allow to show relevant data of the teaching that the teacher proposed and her corresponding purposes.

Keywords: Fractions, teacher's knowledge, multiplicative problems

Introduction

The relevance of the study of fractions and their operations is that it provides a basis for subsequent algebraic relationships. The understanding of rational numbers is fundamental for the development and management of mathematical ideas. In this sense, the teacher can propitiate suitable situations for the learning of Mathematics and guides the student towards the critical aspects of rational or fractional number knowledge (Kieren, 1988).

The knowledge that teacher owns for the teaching has been the subject of study for several decades. Researchers like Shulman (1986) suggest the importance of observing the transformation of the teacher's knowledge content in the knowledge content of instruction on a given topic. In this sense, Ball (1990) is focused on how teachers think about their pedagogical and mathematical knowledge and reasoning. In this case, we consider that teachers with experience in the class of Mathematics, show wealth in the various strategies and forms of long-term representation through the teaching of multiplicative problems associated with fractions. Derived from the above, we consider the data provided in this report to be relevant because they reveal the commonly inaccessible aspect of a teacher's thinking in usual practice, which allow us to interpret here the underlying thoughts that motivate their decisions; the reflections that arise from solving that tasks and if the awareness of this knowledge contributes to the educative improvement.

The main objective of this research is to identify the mathematic knowledge of in-service teachers, with a vast experience in decision-making situations and tasks design for the teaching of contents related to multiplicative problems about fractions. The questions that guide this study are a) what is the teacher's knowledge for the design and teaching of multiplicative problems linked to fractions? b) What knowledge rise during the thoughtful self-analysis of their teaching practice?

Theoretical framework

Some researchers (Ball, 2000, Ball & Bass, 2000, Ball, Thames & Phelps, 2008, & Smith, Bill, & Raith, 2018) link the relevance of the firm knowledge of the teacher about the subject matter she (he) teaches to the learning opportunities that he (she) can provide to students starting from the understanding what the official curriculum mentions and to adapt the actions scheduled for teaching taking into account the needs of their students.

Hart (1981); Jensen & Hohense (2016); Sharp & Welder (2014) and Tirosh (2000), among other researchers, have contributed to the teaching of fractional numbers and their operations. Their results indicate possible deficiencies in the teaching and students' difficulties to solve this type of problems because operations such as multiplication and division with fractional numbers are based on the use of rules and algorithms, putting aside the ideas or multiplicative relationships to provide their explanations and also the students' tendency to attribute the properties to the operations with natural numbers to the operations with fractional numbers. This is particularly important because we identified the last aspect in association with the designs and conceptions on the own teaching realized by the teacher of present research.

The background of the teaching on fractional numbers leads to the first concepts that Kieren (1983) provided. Such researcher mentions that fractions are made up of subconstructs with four meanings: measure, quotient, reason, and multiplicative operator. These subconstructs are the basis for the knowledge of the rational numbers.

Researchers like Behr, Lesh, Post, & Silver (1983) suggest the importance of representing geometrical regions, sets of discrete objects y numerical line as the most widely used models to represent fractions in the elemental education.

Freudenthal (1983) identifies fraction as fracturer, comparator, and multiplicative operator. We can find fractions in an operator in three discernible modes: fracturing operator, ratio operator, and fraction operator. The fracturing operator is associated with situations where specific objects are acted upon, separating them in equivalent parts. Through the ratio operator, we place two a magnitudes in a ratio, a magnitude correlated to another one. The fraction operator is recognized only in the number domain where it satisfies the need for the inverse of multipliers.

Vergnaud (1983) established conceptual fields and refers to them as problems and situations for the treatment of concepts, procedures, and representations of different types. Two of those conceptual fields that he established play an important role in the present research: additive and multiplication structures, in which the problems involve arithmetic operations. We will focus on the multiplicative structure that incorporates different semantic categories of verbal problems: isomorphism of measures and product of measures.

Methodological Design

This case study was carried out in a public secondary school under regulation of the Ministry of Education (Secretaría de Educación Pública, SEP) in Mexico City. The teacher who participated in our research has a Bachelor's Degree in Mathematics Education. We selected this participant because of the data she provided, her vast didactic and mathematical procedures and concepts, and the great communication she produces through her didactic design.

For data collection, we designed a questionnaire and an individual interview of didactic cut, with feedback supported by Valdemoros (2004) and Valdemoros, Ramírez, & Lamadrid (2015). The purpose of the questionnaire was to identify the knowledge and strategies of three Mathematics teachers when solving multiplicative problems linked to the use of fractional numbers. This instrument allowed us to choose the participant and some tasks of the interview.

The didactic interview with feedback included tasks already solved in the questionnaire to make explicit the thoughts that arise during teaching. The interview was oriented to think about the decisions and cognitive processes linked to the teacher's practice when she favors a strategy one another one; the representations she uses and how she link them to procedures to solve the related tasks with multiplication problems that involve fractional numbers. From this instrument, we took the data to prepare the present communication (whose research process has taken more than a year); we collected it through the application of several didactic interviews with feedback. The validation of interviews results is given through the triangulations of the instruments since different data collection methods will be used to study the posed problem in this research (in a next phase we will incorporate the observation in class, sessions carried out in a brief empirical course with others in-service teachers and new didactic interview with feedback).

Analysis of Results

Next is the relevant data of a previous teaching practice in which the teacher expressed the teaching processes she had utilized in her experience over 20 years. That is, we attempted to identify her teaching knowledge and what emerged from the reflection of her educational practice. We chose a multiplication problem and a division task with fractional numbers. Some fragments of the individual interview are exposed here, which include two tasks previously solved in the questionnaire to feedback on what the teacher does at the time she solves them. For the analysis of data, we considered the contributions Behr et al. (1983); Freudenthal (1983), Kieren (1983, 1988), Valdemoros et al. (2015) and Vergnaud (1983), among other researchers.

The multiplication problem

A rectangular building is on the corner of two streets. One of its fronts occupies a third of a street and the other occupies two-fifths of the other street. How much of the block does the building occupy? This task was taken from Jiménez (2015).

To solve the multiplication problem, the teacher used a pictorial representation (a rectangle); she partitioned it to represent the problem data and shaded it by saying "*Let's divide this front into thirds, we take the third part and from the other side of the fix we are going to divide it into fifth parts and we will take two of them.*" We asked her why she used this type of representation and she emphasized that the region with double shading allowed her to represent the graphic solution to the problem and show the multiplication fractions to students.

Subsequently, the teacher operated with the multiplication algorithm and associated the parts of the figure with the algorithmic answer, expressing: "*the students are going to realize that the result from multiplying the numerators corresponds to the part with double shading and the denominator corresponds the total of parts*. Perhaps the teacher used the pictorial representation to give sense to the operation; however, in this task she did not link the answer to the unit of reference. She focused on the main use of the algorithm, as reported researchers like Hart (1981); Kieren (1988), and Vergnaud (1983), among other researchers.

The inverse of the multiplication problem

A painter will paint a mural on a wall that has an area of $3\frac{1}{4}$ m^2; he knows the length is $2\frac{3}{5}$ m. What is the length of the area of the fence?

During third the interview, the teacher reaffirmed that the pictorial representation allowed her to make explicit to students the association of data of the figure with the formula: "*We know how to obtain the area: by multiplying the base by height over two; students know that from elementary school*". Immediately after, she wrote the formula of the area and substituted the data of the problem represented in the rectangle. She emphasized this strategy to make easier the teaching of this type of situation and said: "*we can find that value by substituting the formula of the area through an inverse operation; in this case is a division of fractions* $3\frac{1}{4} \div 2\frac{3}{5}$*, which will allow us to know the missing value*". The teacher referred to the inverse operation although she expressed it ambiguously (the observer seemed to refer to the notational displacements occurred in an equation), and so she justified the division of fraction.

The knowledge that emerged during the interview refers to the predominant use of rules and algorithms over the meaning of operations according (Hart, 1981), geometric representations as teaching models as stated by Behr et al. (1983), and to link the data to the problem. We also identified the use of fractions in the sense of measure, quotient, and multiplicative operator *indicated by* Kieren (1983).

The teacher suggested simplifying an equation to solve the problem of division; and so she vindicated the use of the rule to divide fractions, leaving aside the meaning of the inverse operation in multiplication as reported in the study by Valdemoros et al. (2015).

Discussion of results

From the data analysis, it is possible to infer that teacher's thoughts and decisions were defined by her teaching experiences, which also permeated her strategies based on her students' learning difficulties about the fractional numbers that she identified through her vast teaching experience. The teacher tried to solve this type of obstacle by embedding the teaching into a practice supported by the use of algorithms and rules that she considered students have strengthened since their passage through elementary school.

The teacher favored the use of a pictorial representation highlighting the area model (associated with calculating the area of the rectangle), which allowed her to justify operations with fractional numbers and link the algorithm to what she represented graphically. However, these representations did not provide enough elements to illustrate the case of division because she only used the geometrical representation to associate the data of the problem with the formula and replace them. She did not justify the meaning of the inverse operation of multiplication. She was based on an equation, as has been previously reported by Valdemoros et al. (2015). She also favored the use of the algorithm over the comprehension of intuitive relationships in which the operating situation was immersed.

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LOS PROCESOS COGNITIVOS DE PROFESORES DURANTE LA ENSEÑANZA DE FRACCIONES Y PROBLEMAS MULTIPLICATIVOS

TEACHERS' COGNITIVE PROCESSES DURING THE TEACHING OF FRACTIONS AND MULTIPLICATIVE PROBLEMS

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Esta comunicación sobre el caso de una maestra de secundaria se presenta en el marco de una investigación llevada a cabo con profesores. El propósito es identificar el conocimiento matemático de la maestra con amplia experiencia educativa, cuando realiza el diseño y la enseñanza de problemas multiplicativos ligados a las fracciones, así como vincular las reflexiones que realiza sobre su práctica cuando enseña esos contenidos matemáticos en el aula. Los instrumentos metodológicos aplicados en el seguimiento de este caso fueron un cuestionario y tres entrevistas didácticas individuales con retroalimentación. En el presente reporte se integran sólo tareas diseñadas por la maestra en la tercera entrevista que permiten mostrar datos relevantes de la enseñanza propuesta por la profesora y sus correspondientes propósitos.

Palabras clave: Fracciones, conocimientos del profesor, problemas multiplicativos.

Introducción

La importancia del estudio de las fracciones y sus operaciones reside en que hay un fundamento para las relaciones algebraicas posteriores. La comprensión de los números racionales es básica para el desarrollo y control de las ideas matemáticas, en este sentido, el maestro puede crear situaciones favorecedoras para el aprendizaje de la matemática y orientar al estudiante hacia los aspectos críticos del conocimiento de número racional o fraccionario (Kieren, 1988).

El conocimiento que posee un profesor para la enseñanza ha sido objeto de estudio desde hace varias décadas; investigadores como Shulman (1986) sugieren la importancia de observar la transformación del conocimiento del profesor en el conocimiento de la instrucción sobre un tema determinado, en este sentido Ball (1990) se enfoca en cómo piensan los maestros sobre su conocimiento y el razonamiento matemático y pedagógico. En este caso, consideramos que los docentes con experiencia en la clase de Matemáticas, muestran riqueza en las diversas estrategias y formas de representación prolongadas a través de la enseñanza de problemas multiplicativos ligados a las fracciones. Derivado de lo anterior, consideramos relevantes los datos aportados en este reporte, porque muestran los aspectos comúnmente inaccessibles del pensamiento de un profesor en la práctica habitual, lo cual nos permite presentar aquí los pensamientos subyacentes que motivan sus decisiones, las reflexiones que surgen ante la resolución de dichas tareas y si la toma de conciencia de esos saberes coadyuva a la mejora educativa.

El objetivo general de la investigación es identificar los conocimientos matemáticos de profesores en ejercicio con amplia experiencia ante situaciones de toma de decisiones y diseño de tareas para la enseñanza de contenidos relacionados a problemas multiplicativos acerca de las fracciones. Las preguntas que guían el presente estudio son: a*) ¿Cuáles son los conocimientos que posee el profesor para el diseño y la enseñanza de problemas multiplicativos ligados a las fracciones?, b) ¿Qué conocimientos emergen durante el autoanálisis reflexivo de su propia práctica docente?*

Marco Teórico

Algunos investigadores (Ball, 2000, Ball y Bass, 2000, Ball, Thames y Phelps, 2008 y Smith, Bill y Raith, 2018), vinculan la importancia de los conocimientos sólidos del profesor acerca de la materia que enseña y las oportunidades de aprendizaje que puede brindar a los estudiantes a partir de entender lo que menciona el curriculum oficial y hacer una adaptación de las acciones programadas para la enseñanza tomando como referente las necesidades de sus alumnos.

Investigadores como Hart (1981); Jensen y Hohense (2016); Sharp y Welder (2014), y Tirosh (2000), entre otros, han realizado aportaciones a la enseñanza de números fraccionarios y sus operaciones. Sus resultados apuntan a posibles carencias de la enseñanza y dificultades por parte de los estudiantes para resolver este tipo de problemas, debido a que operaciones como la multiplicación y división con números fraccionarios se sustentan con el uso de reglas y algoritmos, dejando de lado las ideas o relaciones multiplicativas para proporcionar sus explicaciones, tanto como la posible tendencia de los estudiantes a atribuir propiedades de operaciones de números naturales a las operaciones con números fraccionarios. Esto es particularmente importante porque los últimos aspectos los identificamos en asociación con los diseños y las concepciones sobre la propia enseñanza realizada por la profesora participante en esta investigación.

La enseñanza de números fraccionarios tiene antecedentes en los primeros conceptos aportados por Kieren (1983). Dicho investigador menciona que las fracciones están constituidas por subconstructos con cuatro significados: medida, cociente, razón y operador multiplicativo, estos subconstructos forman las bases del conocimiento de número racional.

Investigadores como Behr, Lesh, Post y Silver (1983) sugieren la importancia de la representación de regiones geométricas, conjuntos de objetos discretos y la recta numérica como los modelos más utilizados para representar fracciones en la educación elemental.

Freudenthal (1983) identifica la fracción como fracturador, comparador y operador. La fracción en un operador que se encuentra en tres modalidades discernibles: operador fracturante, operador razón y operador fracción. El operador fracturante se asocia con situaciones en las que se actúa sobre objetos concretos, rompiéndolos en partes equivalentes; mediante el operador razón colocamos las magnitudes en una razón: una con respecto a otra; el operador fracción actúa sobre el puro dominio del número, donde satisface la necesidad de inverso de los multiplicadores.

Por otra parte, Vergnaud (1983) establece campos conceptuales y se refiere a ellos como los problemas y situaciones para el tratamiento de conceptos, procedimientos y representaciones de diferentes tipos. Establece dos campos conceptuales como principales para esta investigación, estructuras aditivas y multiplicativas, en donde los problemas involucran operaciones aritméticas.

Para el trabajo interesan las estructuras multiplicativas, las que integran distintas categorías semánticas de problemas verbales: isomorfismo de medidas y producto de medidas.

Diseño Metodológico

Este estudio de caso se realizó en una escuela secundaria que pertenece a la Secretaría de Educación Pública en la Ciudad de México, participó una profesora con una Licenciatura en Educación Matemática. Elegimos a esta participante debido a la relevancia de los datos que ella aporta, la amplia experiencia didáctica y matemática que posee y lo mucho que comunica en sus diseños didácticos.

Para la recopilación de datos, diseñamos un cuestionario y una entrevista individual de corte didáctico con retroalimentación sustendada por Valdemoros (2004) y Valdemoros, Ramírez y Lamadrid (2015). El propósito del cuestionario fue identificar los conocimientos y estrategias de tres profesores de Matemáticas al resolver problemas multiplicativos vinvulados al uso del número fraccionario, este instrumento nos permitió elegir a la profesora y de él derivamos algunas tareas a la entrevista.

La entrevista didáctica con retroalimentación incluyó tareas ya resueltas en el cuestionario con la finalidad de hacer explícitos los pensamientos que surgen durante la enseñanza. La entrevista estuvo orientada a reflexionar sobre las decisiones y procesos cognitivos vinculados a la práctica de la maestra cuando privilegia una estrategia sobre otra; las representaciones que utiliza y cómo las vincula con los procedimientos para resolver tareas relacionadas con problemas multiplicativos donde involucra números fraccionarios. De este intrumento tomamos los datos para elaborar la presente comunicación (proceso de investigación que ha llevado más de un año); los recopilamos mediante la aplicación de varias entrevistas didácticas con retroalimentación. La validación de los resultados se da a través de la triangulación de los instrumentos ya que se van a utilizar diferentes métodos de recogida de datos para estudiar el problema planteado en esta investigación (en una etapa próxima agregaremos la observación de clase, sesiones en un breve curso empírico con profesores en servicio y una entrevista adicional con retroalimentación).

Análisis de Resultados

A continuación mostramos datos relevantes de la práctica docente precedente, donde la profesora manifiesta procesos de enseñanza ejercidos en su experiencia durante más de 20 años. Tratamos de identificar los conocimientos para la enseñanza que ella posee y lo que surge a partir de la reflexión de su práctica educativa. Hemos seleccionado para este reporte un problema de multiplicación y otro de división de números fraccionarios. Se presentan fragmentos de la entrevista individual, en donde se incluyen dos tareas ya resueltas con anterioridad en el cuestionario, con la finalidad de ir retroalimentando lo que la profesora hacía mientras las resolvía. Para el análisis de datos consideramos las aportaciones de Behr et al. (1983); Freudenthal (1983); Kieren (1983, 1988); Valdemoros et al. (2015) y Vergnaud (1983), entre otros investigadores.

El problema de la multiplicación

Un edificio de planta rectangular hace esquina con dos calles. Uno de sus frentes ocupa un tercio de una calle, y el otro frente ocupa dos quintos de la otra calle. ¿Qué parte de la manzana está ocupada por el edificio? Tarea tomada de Jiménez (2015).

Para resolver el problema de multiplicación la profesora utilizó una representación pictórica (el rectángulo), realizó particiones para representar los datos del problema y los sombreó diciendo:“ *vamos a dividir este frente en tercios, tomando una tercera parte y el otro lado del arreglo rectangular lo vamos a dividir en quintas partes de las cuales vamos a tomar dos*”; cuestionamos la razón de ese tipo de representación, ella hizo énfasis en que la región con doble sombreado le permitía representar la solución gráfica del problema y mostrar a los estudiantes la multiplicación de fracciones.

Posteriormente, la profesora realizó la operación con el algoritmo de multiplicación y relacionó las partes de la figura con la respuesta algorítmica expresando: *"los estudiantes van a darse cuenta de que el resultado de multiplicar los numeradores es la parte con doble sombreado y el denominador corresponde al total de las partes"*. Posiblemente, la profesora utilizó la representación pictórica para dar sentido a la operación, sin embargo, no relacionó la respuesta con la unidad de referencia, se enfocó en el uso preponderante del algoritmo, situación reportada por Hart (1981); Kieren (1988) y Vergnaud (1983), entre otros investigadores.

Problemas del inverso de la multiplicación

Un pintor va a realizar un mural en una pared que tiene un área de $3\frac{1}{4}$ m^2, sabe que de largo mide $2\frac{3}{5}$ m. ¿Cuánto mide la longitud que corresponde al ancho de la barda?

Durante la tercera entrevista, la profesora reiteró que la representación pictórica le permite explicitar a sus estudiantes la asociación de los datos de la figura con la fórmula: *"sabemos que el área se obtiene multiplicando la base por la altura, eso lo saben desde la primaria"*. Inmediatamente ella escribió la fórmula del área, sustituyendo los datos del problema representados en la figura geométrica (rectángulo), hizo énfasis en esta estrategia para facilitar la enseñanza de este tipo de situaciones y agregó: *"podemos conocer ese valor en la sustitución de la fórmula del área a través de una operación inversa que en este caso es una división de fracciones* $3\frac{1}{4} \div 2\frac{3}{5}$*, nos va a permitir conocer el valor que hace falta*", la maestra hizo referencia a tal operación inversa, aunque lo expresó de un modo ambiguo, ya que a los ojos de la observadora parecía referirse a los desplazamientos notacionales generados en una ecuación, con esta acción justificó la división de fracciones.

Los conocimientos presentes en la entrevista se refieren al uso preponderante de reglas y algoritmos sobre el sentido de las operaciones según Hart (1981), el uso de representaciones geométricas como modelos de enseñanza conforme a lo planteado por Behr et al. (1983), y para relacionar los datos del problema. Identificamos el uso de la fracción con el significado de medida, cociente y operador de acuerdo a Kieren (1983).

Un pasaje importante de la entrevista es el momento en que la profesora sugiere el despeje de una ecuación para resolver el problema de división, con lo anterior la maestra justifica el uso de la regla para dividir fracciones, dejando de lado el significado de la operación inversa de la multiplicación como se ha reportado con anterioridad (Valdemoros et al., 2015)

Discusión de resultados

Con el análisis de datos, es posible suponer que los pensamientos y decisiones de la profesora estaban definidas por su experiencia de enseñanza que también impregnaron sus estrategias basadas en las dificultades de aprendizaje de sus estudiantes sobre los números fraccionarios identificados a través de su vasta experiencia de enseñanza. La maestra trató de resolver este tipo de obstáculo integrando la enseñanza en una práctica respaldada por el uso de los algoritmos y reglas que a su consideración los estudiantes tenían afianzados desde su paso por la escuela primaria.

La profesora privilegió el uso de la representación pictórica destacando el modelo del área (asociado al cálculo del área del rectángulo), el cual le permitió justificar las operaciones con números fraccionarios y relacionar el algoritmo con lo que realizó gráficamente. Sin embargo, estas representaciones no aportaron los elementos suficientes para ilustrar el caso de la división, porque sólo utilizó la representación geométrica para asociar los datos del problema con la fórmula y sustituirlos, no justificó el significado de la operación inversa de la multiplicación, se apoyó en despejar una ecuación, como se ha reportado con anterioridad Valdemoros et al. (2015). Asimismo, favoreció el uso del algoritmo sobre la comprensión de las relaciones intuitivas en la que estaba inmersa la situación operatoria.

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CHALLENGES INFLUENCING SECONDARY MATHEMATICS TEACHER'S TRANSITION TOWARDS TEACHING WITH VIRTUAL MANIPULATIVES

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To increase teachers' use of virtual manipulatives and tasks within secondary mathematics classrooms and support changes to teachers' instructional practice, this study investigated situational challenges influencing teachers' implementation efforts during the course of a professional development (PD) opportunity. Identified situational challenges included: using Chromebooks, teachers' curriculum resource package, student needs, instructional time/planning, and teachers' collaborators. To promote the success of future PD opportunities, recommendations for acknowledging and embracing the situational challenges are provided.

Keywords: Technology, Instructional Vision, Teacher Education – Inservice / Professional Development

Despite expectations for teachers to use technology to support students' sense making and mathematical reasoning (AMTE, 2017), teachers claim that they are not prepared to use technology effectively in their instruction (Albion, Tondeur, Forkosh-Baruch, & Peeraer, 2015). Effectively teaching *with* technology describes teachers using technology to promote students' development of understanding through communicating and reflecting on mathematics, as well as through using and connecting mathematical representations (Reiten, 2018b). Using an interactive whiteboard and a virtual manipulative (VM) to explore how changing the slope and *y*-intercept of a graph changes its equation is an example of teaching *with* technology. Students reflect and build on possible relationships shared by themselves and peers. Through the use of the VM, they dynamically see the resulting graphs when the equations are changed. A VM is "an interactive, technology-enabled visual representation of a dynamic mathematical object…that presents opportunities for constructing mathematical knowledge" (Moyer-Packenham & Bolyard, 2016, p. 13). Teaching *near* technology describes using technology in a manner that does not promote opportunities for students to communicate, reflect, or connect mathematical representations. Using technology merely as an attention grabber (an example of teaching *near* technology) is a misuse of technology (Suh, 2016).

Though teachers have been encouraged to implement VMs for decades (e.g., NCTM, 2000), their use of VMs decrease as students get older (Moyer-Packenham & Westenskow, 2013). To increase teachers' use of VMs and tasks within middle and high school mathematics classrooms and support changes to teachers' instructional practice, this study investigated a targeted professional development (PD) opportunity aimed at supporting teacher learning (Driskell et al., 2016). To promote the success of future PD opportunities, this study investigated situational challenges teachers experienced (Yamagata-Lynch & Haudenschild, 2009) during the course of the PD opportunity that influenced their use of VMs and tasks. Which leads to the question at the core of this study, *what challenges faced the teachers, participating in a targeted PD opportunity, as they transitioned from teaching* near *towards teaching* with *VMs?*

Methods

Fourteen teachers participated in a year-long professional development (PD) opportunity aimed at supporting their use of VMs and tasks aligned with their curricular units (Reiten, 2018a, 2020). Grounded in activity theory (Engeström, 1987), this study investigated the teachers' participation during a PD and their reported practices related to implementing VMs and tasks. Rather than

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studying teachers' practices in isolation, teachers' reported practices were considered mediated by several factors (e.g., tools and mediating artifacts, community members, rules). Specifically, this study investigated the situational challenges the middle and high school (i.e., secondary) mathematics teachers faced as they transitioned from teaching *near* technology towards teaching *with* technology.

Figure 1 is an example of an activity system in this study. Teachers' conversations and reflections throughout the PD were used to investigate the situational challenges teachers experienced (Yamagata-Lynch & Haudenschild, 2008) as they began shifting their practices towards teaching *with* technology. Situational challenges or tensions (i.e., internal contradictions between and within components of an activity system) are opportunities for growth and learning (Engeström & Sannino, 2010). Interviews with four volunteer teachers (two 6th grade, one 6th/7th intervention teacher/former 8th grade teacher, and one high school teacher) provided insight into teachers' reactions to tools and VM tasks introduced during the PD, teachers' practices (during the PD and their classroom practices related to implementing VM tasks), and teachers' thoughts regarding what supported their implementation efforts.

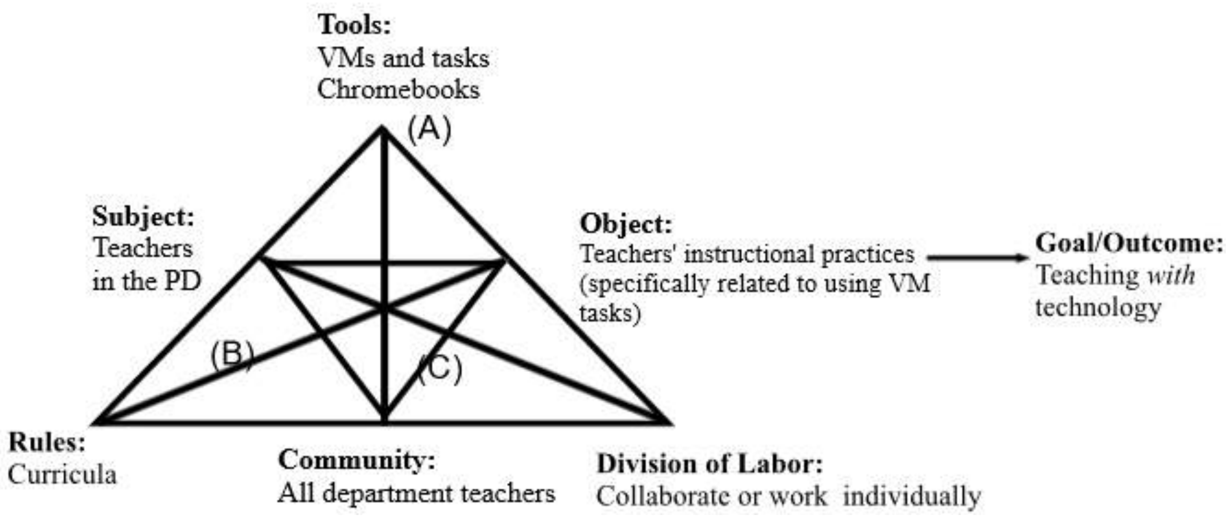

Figure 1: Example of an Activity System and Identified Challenges

The constant comparative method (Glaser & Strauss, 1967) was used as it drew our attention to the situational challenges (tensions) teachers experienced through simultaneous coding and analysis. Transcripts were coded to identify challenges based on what teachers reported (e.g., time, limitations of tools) and contradictions the researcher identified in the data (e.g., use of time during PD). Challenges within the same category were compared and category definitions were refined based on the commonalities and themes between coded data excerpts within the tension category. Data excerpts contained at least a complete sentence, and often consisted of multiple sentences (or small paragraphs) focused on the same topic. The number of data excerpts is provided to give readers a descriptive understanding of the relevance of various tensions.

Findings

Drawing from the ways teachers described using technology in their classroom, findings indicate that at least 11 out of 14 teachers in the PD transitioned towards teaching *with* technology. Initially, teachers reported using technology due to district, parent, and student expectations or because they thought the students may enjoy a particular technology-based task or game. By the end of the PD, teachers were selecting VMs and tasks based on their potential for supporting student understanding (Reiten, 2020). However, as teachers made changes to their practice, they experienced challenges within and between components of an activity system for the PD (e.g., see *A*, *B*, and *C* in Figure 1). Numbers in parentheses indicate the number of data excerpts for the identified challenge. The most common challenges related to: using Chromebooks (35), their curriculum package (26), student needs (24), use of worktime during the PD (25), and collaborators (22). It is posited that these challenges are relevant to other PD opportunities aimed at supporting teachers to teach with technology as the identified challenges extended beyond the particular technology tool to consider aspects of the teachers' community (e.g., teachers with whom they taught as well as the students in their classroom) and structure of the PD opportunity.

Figure 1 highlights three challenges confronting the eighth grade teachers (i.e., Erin, Mari, Pam, and Stan) in the PD that influenced their implementation of VMs and tasks. Occurring within and across components of an activity system for the PD, challenges teachers faced as they strove to teach *with* VMs and tasks included *(A)* limitations of the tools, *(B)* their curriculum resource package, and *(C)* their collaborators.

During the November PD session, as the eighth grade teachers were critiquing a VM task, Stan, Mari, and Erin became frustrated. They wanted to either enter specific side lengths for right triangles or have the side lengths always be integers. Neither option was capable with this VM (see *A* in Figure 1*)*. Specifically, Stan said, "I wish that this would be whole numbers. I wish it would stick to whole numbers. 'Cause the decimals, that doesn't even register with ME, [Erin: Right] if those are equal." Due to rounding errors, Stan and Erin thought that some of their students might struggle to identify the pattern in the table, thus feeling uncertain whether this VM would be worthwhile. Due to this challenge (located within the *tools* component), the teachers chose not to implement this particular VM with their students.

The eighth grade team of teachers also found it challenging to integrate VMs and tasks within their instruction due to their curriculum investigations building on each other *(*see *B* in Figure 1*)*. Mari wanted to use a VM to replace an investigation or as a pre-teaching tool with her "lowest students." She was not able to do so because all 8^{th} grade math teachers needed to implement the same thing. Replacing an investigation in one part of the unit with a VM could lead to investigations in the following lessons needing to be modified due to students not having the background information from previous investigations. Specifically, Stan stated, "[w]ith our curriculum, if you were going to use this (the VM), it would have to be in addition to or a summary. Or a reflection. Because otherwise, why do you do investigation one and two? … There's no reason to use Section 3.1, if you're not going to continue on." This situational challenge existed between the *rules* and *object* components of the activity system depicted in Figure 1. Additionally, drawing from his experience with a previous K-12 math leader, Stan was adamant that VM tasks could not replace investigations. Rather they needed to "trust and stick to the curriculum" even when students struggled to understand the investigation (see *C* in Figure 1). This situational challenge existed between the *community* and *object* components of the activity system depicted in Figure 1. Meaning Mari's instructional practices related to implementing a VM task were influenced by teachers in her community (e.g., Stan) beyond her control. Stan's belief in the role of his curriculum resource package as well the curriculum itself influenced if and how he chose to implement a particular VM task.

Discussion and Conclusion

Despite the expectation for secondary mathematics teachers to use technology tools in an effective and innovative way, many teachers report that they are not prepared to do so (Albion et al., 2015). How teachers are supported to teach *with* technology as opposed to *near* technology is an important issue facing the field. Teachers do not work in isolation, rather a variety of components influence their practices related to teaching *with* technology. Ultimately, the situational challenges described earlier influenced teachers' transition towards teaching *with* VMs and tasks. When acknowledged and embraced, these challenges provide opportunities for teachers to grow in their understanding of how to teach *with* technology tools. When designing opportunities to support teachers to teach *with* technology, the challenges highlighted in this study are important to consider and address.

The following recommendations highlight embracing challenges as opportunities to support teachers' growth rather than as something to be ignored. Because teachers often do not teach in isolation, it is important to address the ways teachers' peers support their integration efforts. When designing PD, intentionally integrating opportunities for collaborators to discuss and recognize personal beliefs related to the role of technology, curriculum, and so forth is important.

To address challenges related to teachers' curricula packages, consider supporting teachers in aligning specific VMs or tasks to their curricular units. Providing examples of VMs or tasks aligned to instructional units may initially support teachers in understanding the different ways tasks may be used in relation to their current curriculum (e.g., supplementing, replacing, or introducing their curricular investigations). Another way to start this process might be to have teachers observe other teachers teaching with technology while implementing the same curriculum. Debriefs after the observations provide additional opportunities to address challenges related to the role of the curriculum in deciding when, how, and why to add in specific technology based tasks. In the case of this PD, teachers were initially provided specific VM tasks aligned to their unit. As the PD progressed, teachers took on the responsibility for selecting VMs and tasks to explore that aligned to their learning goals.

Rather than focusing only on positive aspects or benefits when using a specific technology tool, acknowledging the limitations of a particular tool and potential ways to address the limitations may support teachers' comfortability with the tool. It is especially important to acknowledge how tool limitations may influence students' engagement in the mathematics and strategies for addressing the limitations. In the case of the 8^{th} grade teachers, they chose to not use a VM task due to concerns over whether students would recognize the intended pattern due to rounding errors. Potential strategies for addressing this concern include giving students dimensions of triangles to explore, reviewing with students the influence of rounding errors when squaring decimals, and looking for a new VM. Due to the cumbersome nature for creating triangles with given side lengths, the teachers chose not to use the initial VM and instead explored a different one that allowed students to enter measures of specific leg lengths.

The aforementioned suggestions are in response to the challenges revealed in the examples with the eighth grade teachers. It is important to keep in mind, that teachers may experience the various situational challenges differently compared to their peers. Therefore, we recommend providing opportunities for teachers to see successful integration efforts and reflect on their practice. Furthermore, though the recommendations stem from working with teachers during a PD, we posit these recommendations are important to consider when preparing and supporting pre-service teachers to teach *with* technology.

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HOW FACILITATORS DEFINE, DESIGN, AND IMPLEMENT EFFECTIVE EARLY CHILDHOOD MATHEMATICS PROFESSIONAL DEVELOPMENT

CÓMO FACILITADORES DEFINEN, DISEÑAN E IMPLEMENTAN TALLERES DE DESARROLLO PROFESIONAL EFECTIVO EN EDUCACIÓN MATEMÁTICA PARA LA INFANCIA

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The call to improve mathematics outcomes for children ages zero to eight requires the development of effective professional development approaches for early childhood mathematics educators. In this study, we looked at how six facilitators created workshops on spatial reasoning, mathematical play, number sense, and theories of learning for early childhood educators. Drawing on Desimone's components of effective professional development, we interviewed these facilitators to understand how they defined a successful professional development and how these definitions aligned with the workshops they created. Interviews showed that all the facilitators in this study designed their workshops to be engaging and interactive for their participants while drawing on the components of coherence, collective participation, and duration.

Keywords: Teacher Education - Inservice / Professional Development, Early Childhood Mathematics, Spatial Thinking, Teacher Knowledge

Introduction and Purpose

Scholars and educators have called for initiatives to improve mathematics outcomes for children ages zero to eight. This calls for effective professional development (PD) to be implemented on ways of thinking about early learning as multimodal, playful, and responsive to the varied sociocultural and linguistic contexts in diverse communities. As PD is crucial in supporting teachers' knowledge and skills that lead to changes in classroom practice (Garet et al., 2001), it is necessary to understand how PD facilitators approach such a vital call. In this study, a nonprofit math and science education center aimed to engender ways of thinking about early math learning through an extensive initiative that partnered with roughly 100 early childhood educational leaders across a Western U.S. state. Central to this initiative were PD workshops led by mathematics coaches who focused on the areas of spatial reasoning, number sense, mathematical play, and theories of learning for children ages 0-8. As the National Council of Teacher of Mathematics (NCTM, 2014) acknowledges the critical role of mathematics coaches in enhancing teacher capacity and positively influencing teacher beliefs, this study aims to understand the goals of the PD workshops established by the mathematics coaches themselves. We asked the following research question:

1. How do experienced facilitators use current research on successful PD to inform their own workshops?

Conceptual Framework

We drew on Desimone's (2009) five dimensions for effective PD as a lens to understand how the PD facilitators structured their workshops. See Table 1 for descriptions of these dimensions.

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Table 1: Components of Effective Professional Development

Framework Component	Definition
Coherence	Incorporating participants' individual goals with that of the larger group (Gordon, 2004).
Duration	Follow-up activities and ongoing support in the form of coaching and interacting with colleagues (Ball, 1996).
Content Focus	Textbooks, kits, curriculum units, and other forms of content that focus not only on what the content is, but how it is learned by students (Garet et al., 2001).
Collective Participation	PD that is developed and administered for groups of teachers that come from the same school or department that allows teachers to work together to discuss content, skills, and problems that they experience in their teaching (Garet et al., 2001).
Active Learning	Providing first-hand experiences with the content where teachers can actively participate instead of passively learn through lecture-based sessions (Penuel et al., 2007).

Methods

Study Context and Participants

This study is part of an ongoing project focused on providing and evaluating an early childhood mathematics professional development offered to approximately 100 participants representing 30 educational agencies across a Western U.S. state. For this study, we focused on the workshops designed and facilitated by the PD mathematics coaches during the week-long PD held in July 2019. Participants attended 90-minute sessions about learning theories, culturally relevant pedagogy, spatial reasoning, number-sense tasks, and mathematical play, all centered around enhancing early childhood mathematics education.

Data Collection and Analysis

We conducted one-on-one interviews with each PD workshop facilitator (n=6) after their sessions. Facilitators were asked questions about the professional development workshops they designed for this session, including how they defined a successful PD, how they imagined their participants would implement the content and theory they presented, and what research they based their work on. Using emergent coding (Strauss & Corbin, 1994), we looked to see what themes arose when facilitators defined effective PD and described how they designed their workshops. In particular, we examined ways in which the facilitators' descriptions of their workshops compared or comported with what they described as the components required for a successful PD. This analysis provided insight about the many different evidence-based approaches the facilitators took to create and implement their workshops.

Results

Below we address our findings for the workshops on Mathematical Play, Spatial Reasoning, Theories of Learning, and Number Sense.

Mathematical Play

Peter's session engaged participants in a discussion about the various ways that they play in their own daily lives and asked participants to consider whether "learning" might also be a way to describe these activities. Peter challenged his participants to not think of play and learning as separate ideas, but as potentially one and the same: "Children develop meaning by interacting with objects and posing problems through play." After some time for group discussion about this idea, Peter presented information on the history of early childhood play. Next, participants were given time to play with non-traditional pattern blocks and reflect on what problems they posed during their play, what mathematics they drew on in their play, and how this playful experience impacted their identity as a mathematics learner.

When asked about what defines a successful PD in his post-interview, Peter specifically mentioned all five components of Desimone's (2009) framework. Peter explained the importance of creating PD that engages participants in the kinds of experiences they would do with their own learners. Drawing on Desimone's framework for content-focused active learning, Peter anchored the workshop by drawing on participants' experiences with play, which resonates with creating a coherent PD experience. Peter described a practice-focused approach to support participants in implementing strategies in their own contexts and highlighted the importance of treating participants as professionals, drawing on play as a way to enact a more equitable approach to PD. Lastly, Peter described learning math through play as a way to promote attitudes and dispositions toward mathematics that are playful and fun rather than intimidating or unapproachable.

Spatial Reasoning

At the beginning of their session on spatial reasoning, Shane and Ana encouraged participants to think about how they got from the parking lot to the room where the session was held. Then, participants created written instructions or a visual sketch for traveling the distance. Shane and Ana defined spatial reasoning as the concepts, tools, and processes involving the location and movement of objects and persons, either mentally or physically, in space; they also introduced Piaget's three mountain task and the importance of spatial reasoning for mathematics learning. Next, participants engaged in nine different spatial reasoning activities, including Piaget's three mountain task and water level task, mental rotation visualization tests, and mental folding. Participants engaged in a group discussion about the challenges they encountered when engaging in these activities and whether their view of the importance of spatial reasoning had changed.

In their post-PD interview, Shane and Ana explained that they wanted participants to leave with an expanded understanding of spatial reasoning. Furthermore, Shane and Ana wanted to connect what they presented in the spatial reasoning session to other sessions that were offered at the week-long institute. In this way, Shane and Ana's session aligned with Desimone's components of content focus and active learning. Finally, Shane and Ana hoped that participants would provide similar learning experiences for the teachers they worked with. In order to encourage this post-PD implementation, Shane and Ana discussed the need to support their participants in applying what they learned in their own contexts, whether they worked with infants/toddlers or preschool-aged children.

Theories of Learning

Sam and Evelyn began their session by engaging participants in a discussion about what it feels like to be a learner, "to have the participants experience describing an object as a child would. We wanted the participants to put on the hat of a learner." In order to mirror what it is like to develop a concept as a child, they introduced an unfamiliar word: "Tutusa". Tutusa was a made-up concept the PD facilitators developed that represented objects that weighed the same but looked different. Sam and Evelyn gave a few visual examples and non-examples of Tutusa, then provided cubes of different sizes, color, and weight, as well as measurement scales so that participants could work in small groups to determine the meaning of Tutusa. At the end of the session, each group was asked to *nonverbally* share what they believed Tutusa meant. Nonverbal communication was an added challenge to engage participants in communicating meaning without words, through gesture or movement.

In their post-PD interview, Sam and Evelyn stressed the importance of creating an engaging and interactive activity by drawing on the backgrounds, knowledge, and needs of their participants. In this way, they placed an importance on coherence throughout their session. Sam and Evelyn also highlighted how a successful PD needs to include regular follow-up and ongoing support for all participants. As Evelyn explained, the follow-up is "just as significant as the institute itself, if not more." These responses align directly with Desimone's components that attend to duration, collective participation, and active learning.

Number Sense

Becky's session on number sense engaged participants in recognizing developmental progressions for various number concepts, such as number word sequence, one-to-one correspondence, and strategic reasoning. Becky showed six different videos of preschool-aged children reciting number word sequences to twenty, for example, and asked participants to look for cues that could provide insight in the child's counting processes. She then provided participants with concrete "what to do" strategies to support children in various stages of development. For example, for a child in an early stage, she highlighted how teachers could support children in developing one-to-one correspondence. Becky's goals for the session included wanting to engage participants in thinking about how young children come to think about numbers and number concepts. Becky acknowledged that "counting is a complex learning experience," so she wanted to support participants in thinking about how young children "make connections to place value" or other such concepts.

In her post-PD interview, Becky positioned herself as a learner, taking a reflexive stance towards her own facilitation practices. She stated that she not only hoped participants learned from the session–ideally, she intended to learn from participants as well. She wanted to draw on the expertise in her audience to help everyone in the room "understand more deeply and connect to other learnings they have had." In preparing her session, Becky explained, she developed an agenda but would likely end up changing her plan depending on the identities and experiences of her audience. In reference to Desimone's framework, Becky's responses align with both coherence and active learning; she activated participants' prior experiences and leveraged these for learning in her session.

Discussion and Conclusion

Overall, participants stressed that a successful PD includes follow-up coaching and ongoing support, both of which are consistent with the importance of duration in effective PD. In addition, the PD facilitators created workshops that were engaging for their participants, drawing on the active learning component. We argue that the four sessions presented here are exemplary cases of rigorous and ambitious PD aligned with current research and grounded in the needs of the practitioners in the room. This report reveals important insights about how experienced PD facilitators approach their practice to provide a professional learning event that seeks to go beyond the week-long institute itself.

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THE DESIGN AND IMPLEMENTATION OF AN INTERVENTION TO SUPPORT AND RETAIN EARLY CAREER MATHEMATICS TEACHERS

EL DISEÑO E IMPLEMENTACIÓN DE UNA INTERVENCIÓN PARA APOYAR Y RETENER PROFESORES DE MATEMÁTICAS RECIÉN EGRESADOS

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This study reports on efforts over several years to design and implement a yearlong intervention intended to support secondary mathematics teachers in their early years of teaching. The intervention is designed to support these teachers' development of meaningful professional relationships with a school-based mentor and to create an online community of practice for support with other professionals. The intervention itself consists of early career teachers and their mentors participating in monthly professional development sessions such as online meetings, Zoom panels with experts, and collaboratively reading and discussing timely, purposeful, and relevant content. The intervention is designed to not over burden the participants and to be feasible for national implementation with little funding. The goal of the intervention is to try to retain secondary mathematics teachers in the profession by providing them with meaningful and targeted support.

Keywords: Teacher Education - Inservice / Professional Development, Teaching Tools and Resources

Purpose of Study

Half of all teachers leave the profession within the first five years, and this rate is highest for mathematics positions in high poverty schools (Fantilli & McDougall, 2009; Goldring et al., 2014). Furthermore, half of all current teachers in the U.S. retiring in the next five years (Foster, 2010), enrollment in teacher preparation programs declining, and teacher turnover is costing America $7.3 billion annually (National Math + Science Initiative, 2013), which represents a crisis for public education in the U.S. These conditions lead to classrooms staffed with underprepared/unqualified teachers, which profoundly affects the mathematical preparation of students in high school, college, and beyond. Experts agree that addressing the mathematics-teaching crisis meaningfully will require building a more cohesive system of teacher preparation, support, and development (Mehta, Theisen-Homer, Braslow, & Lopatin 2015). The purpose of this study is to report on the design and implementation of a cost effective, easily replicable intervention for early career secondary mathematics teachers with the goal of positively impacting teacher retention. We also present lessons learned over two years of implementing the intervention and provide suggestions for future research.

Background

Transforming the preparation of secondary mathematics teachers across the U.S. is at the core of the Mathematics Teacher Education Partnership (MTE-P). Since its inception, this initiative has continued to improve mathematics teacher education across the nation (for more information about the partnership see (Martin et al., 2020). MTE-P has established guiding principles and five Research Action Clusters (RACs) to carry out these principles. The authors of this paper are members of the RAC guided by a focus on teacher retention and induction in line with standard P.5-Recruitment and Retention of Teacher candidates, which is included in the Standards for Preparing Teachers of Mathematics (AMTE, 2017).

Novice teachers often feel isolated and those feelings of isolation are often associated with teachers leaving the field (Carroll & Fulton, 2004; Schlichte, Yssel, & Merbler, 2005). This RAC is grounded in the perspective that teacher retention would improve with the development of communities of practice to provide a support network to draw upon, including online communities (Wenger, 2011). Communities of practice are "groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly" (Wenger, 2011, p.1). Wenger further shared three features that characterize communities of practice: a domain of interest, a community (members who participate in joint activities and discussions), and shared practice. For our work, our domain of interest is teaching high school mathematics during the early years of a teacher's career. The community consists of early career teachers, mid-career mentoring teachers, curriculum specialists, and university program coordinators and mathematics teacher education faculty. The practice of focus is teaching mathematics. We recognize that the work of retaining teachers requires, in part, a focus on developing relationships within the educational community and promoting connectedness within the larger community (Minarik, Thornton, & Perreault, 2003).

Past Work: Driving the Design

To respond to the teacher retention crisis, the RAC created a survey as an initial step to study the current support systems of early career secondary mathematics teachers. One research question guiding this work was: *What is the perceived scope, nature, and impact of professional support for early career mathematics teachers?* This survey was created through an iterative design and vetting process that extended from the fall of 2014 to early 2016. The main goal of the survey was to better understand the degree to which early career mathematics teachers perceived various learning opportunities as influential to their interest in teaching mathematics. By better understanding current support systems, the RAC could develop interventions that would strengthen and replicate systems that were working and attempt to improve broken ones. The survey consisted of 25 questions asking respondents to report on their current support systems, job satisfaction, projected longevity in the field, and other related topics. The survey was given in November of 2016 and gleaned 141 responses from teachers across the nation. Results from this study are presented in Amick et al. (2020).

The vast majority of novice teachers had received mentoring or coaching from someone at their school site, and almost (89%) found that experience to be moderately or very influential to their enthusiasm for teaching mathematics. This finding is consistent with other research on induction programs (Ingersoll & Strong, 2011; Youngs et al., 2019). In their review, Ingersoll and Strong (2011) found that induction programs and *especially teacher mentoring programs* positively influenced novice teachers' satisfaction, commitment, and/or retention. Further, Ingersoll (2012) found that retention was significantly impacted when a mentor and novice teacher taught in the same subject area and had a common planning time, as well. He also found that having multiple induction supports had a strong positive effect on retention. Thus, the work of this RAC, to develop systems that can effectively support early career teachers with the overarching goal to increase their job satisfaction and longevity in the field, was built upon past research, as well as the survey results.

Methods

Our overall methodology for this work has been a design experiment approach (Cobb et al., 2003), focusing on a problem in practice and pragmatically designing an intervention to impact that problem with multiple iterations of implementation and (re)design. We used constant comparative analysis to modify the intervention and methods as the investigation evolves based on new findings from analyzing the data collected.

Design

Due to both current research in the field and the RAC's survey results pointing towards mentoring as an extremely impactful induction experience, our group focused on the mentoring relationship as the basis for the first year of implementation of our intervention. The first year intervention was implemented throughout the 2018-2019 academic year. The intervention was designed to provide targeted support to first-year teachers by: (1) strengthening the mentor/mentee relationship through monthly communications; (2) suggesting targeted discussion topics between the mentor/mentee teachers; (3) and providing synchronous online meetings to build a professional community. In keeping with a design experiment approach, the intervention was modified over the course of the year, based on continuous analyses, in an effort to improve the intervention. After the first year of implementation the team went through a (re)design process during the summer of 2019 to prepare for the second year of implementation.

In order to avoid overburdening early career teachers, the intervention was designed to include only one hour of active participation each month. Furthermore, the intervention was designed to engage the early career teachers with their mentors to allow them to take part in an online community such that each pair had opportunities to engage in learning about research-based teaching practices together. Including mentor teachers in the study, was meant to provide a supported space for mentor teachers and first-year teachers to build positive relationships.

The monthly engagement activities were selected/designed by the research team to be timely, and several of the selected topics are also aligned to the Common Core Standards of Mathematical Practice and NCTM's Effective Mathematics Teaching Practices. For example, in September an email was sent with several self-care resources and asked the participants to peruse and discuss with their mentor teachers. In October, a Zoom panel was put together where participants have a sounding board to vent frustrations, ask questions, and seek advice. The panelists included teachers who are past their first few years of teaching, but who are still in their early years of their careers as to be relatable to the participants. We ended each year with an anonymous feedback form for the early career teachers to complete. One addition we made during the second intervention year was the creation of a Facebook group for participants to engage. We posted resources frequently (at least once a week) on the group feed to attempt to create an online space for dialogue and further support online community development.

Participants

To recruit early career teachers, the researchers on this project extended email invitations to recent graduates of their teacher preparation programs. For our first year of implementation, we only invited teachers in their first year of teaching. Participants were asked to commit one hour a month to the study, and to recruit a mentor teacher if they did not have one assigned to them. We strongly suggested that mentor teachers also teach math. The participants for both years of implementation included a diverse group that taught a variety of courses and grade levels (6-12), and have settings that range from large urban districts to small rural schools. During year 1, we had seven teachers volunteer to participate and during year 2 we improved our recruitment of early career teachers and had 15 teachers register.

Data Sources

The main data sources of this study consist of feedback from the participants over the course of the year via email, and a mid-year and end-of-year survey for each intervention year. The end-of-year survey is used to collect information on how useful the new teachers found each of the monthly interventions to be in supporting them and what supports they still wish they had. There are also questions for the new teachers asking specifically if the support received had an impact on whether or not they intend to continue in the profession in the future.

Results

We recently completed our second year of implementation and thus far, the results have been mixed as to the usefulness of the intervention. For those participants who remain engaged with the group we have received overwhelmingly good feedback as to the usefulness of the Zoom teacher panels. The participants report it being helpful to connect and talk to others that are and have been in similar situations. They also greatly appreciated the very practical advice. Unfortunately, the attendance for the Zoom teacher panels has been low with only 2-5 early career teachers participating. In addition, though we have encouraged the engagement of mentor teachers we have so far not had any join our conversations. Many of our participating teachers also reported that they had very little interaction with their mentor teachers. The teachers we have received feedback from often report feeling overwhelmed, primarily with issues of student engagement and planning. We have had little success with engaging early career teachers with Facebook in spite of many attempts to try to draw them into the conversation. Overall, we have struggled to keep teachers involved and engaged in the intervention beyond the first few months, which has led us to begin to rethink our approach.

Discussion/Summary

Similar to the results that we have seen, Youngs et al. (2019) in their synthesis of research on teacher induction programs that lead to retention of STEM teachers found that interventions with first year teachers seem to have little effect, which they attribute to teachers likely being overwhelmed. We are now considering focusing on teachers in their second and third years of teaching and would recommend that focus for future research. Another consideration is to focus on mentors or teacher leaders for the intervention and how to help them support groups of mentees. This would be a significant shift in focus for our interventions and recruitment of participants but might help to develop strong mentorship teams focused on a mentor. We have also considered taking a school team approach and involving an administrator as past research has shown the value of perceived administrator support (Youngs et al., 2019). We propose one approach could be to have school-based teams including all early career teachers, 1-2 mentor teachers, and an administrator and focusing on how to develop relationships and build community in the teams. We continue to look for impactful ways of supporting and retaining early career teachers and we believe the lessons we have learned will be useful to others in mathematics teacher education.

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COLLABORATIVE LEARNING WITHIN AN INFORMAL COMMUNITY: HOW ONLINE SPACES CAN CATALYZE CHANGE

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Online learning communities are an increasingly prevalent informal learning site for teachers. These sites offer an emotionally and philosophically supportive space for teachers who advocate for change. In this study, our analysis of the interactions transpiring within one mathematics education Facebook group illuminates a critical conversation taking place, instruction based on students' perceived abilities. Teachers discuss systems level tracking and classroom level ability grouping to catalyze change and subvert the structures that produce tracking in schools. This collaborative environment both informed and empowered teachers to make educative decisions about inclusive strategies that built heterogeneous groups of learners.

Keywords: affect, emotion, beliefs, and attitudes; equity and diversity; informal education; technology

The negative effects of tracking are clearly articulated within the National Council of Teachers of Mathematics' publications *Principles to Action* (2014) and *Catalyst for Change* (2018), yet this practice continues to dominate K-12 classrooms. Choices made based on ill-informed assumptions about students' academic ability (Ladson-Billings, 1997) have led school systems to track students into mathematics classrooms that often do not prepare students for futures in STEM fields (Oakes, 1990). Course sequencing and perceived ability grouping continue to have negative effects on students such as: the continuation of social reproduction (Reichelt, Collischon, & Eberl, 2019), lack of student motivation (Lessard, Larose, & Duchesne, 2018), and lower beliefs about one's mathematical ability (Mijs, 2016).

Teachers are becoming more aware of the effects of tracking and some are attempting to change the oppressive system from within their schools and classrooms. The purpose of this study is to highlight the voice of teachers engaged in a social media network seeking support on how to make a change in their schools. In what follows, we begin by situating our work within social learning perspectives and previous research on the negative effects of tracking. We then outline our findings and discuss how a social media space can emotionally and philosophically support teachers as they advocate for change.

Framing

Teacher Learning

Grounding our work within the context of social learning, "communities of practice" offer a lens to examine the dynamic interactions that occur within an informal learning environment. Wenger (2006) defines communities of practice as "groups of people who share a concern or passion for something they do and learn how to do it better as they interact regularly" (p. 1). Communities of practice foster a sense of belonging where members share a common goal. Teachers' participation and interaction within these spaces allow them to develop shared knowledge, learn together, and support one another in their practice (Wenger, 1998).

Teachers' beliefs about teaching and learning play a significant role in shaping their instructional practice (Schoen & LaVenia, 2019). Kyndt and colleagues (2016) utilized the results of their meta-analysis on informal teacher learning to identity three key learning outcomes that resulted from participation in these learning communities: (1) improved content knowledge, (2) stronger pedagogical knowledge and skills, and (3) a change in attitudes and identities. Macia and Garcia (2016) further explored the nuances of informal learning spaces and found that teachers often entered into these spaces to support the context-specific needs of their classroom. For example, one contextual demand teachers frequently sought advice on was how to best group students for mathematics instruction. These findings suggest that an informal learning community can serve as a source of instructional support for mathematics teachers.

Ability Grouping

In mathematics instruction, ability grouping is a teaching practice implemented by many teachers (Anthony & Hunter, 2017). Ability grouping refers to students that have been grouped based on their perceived academic ability, which is often determined by their performance on an assessment. While teachers identify the use of ability grouping as a way to cater to their students' diverse learning needs and raise student performance (Hunter, Hunter, & Anthony, 2019), these practices continue to fuel the inequities in our education system. In the era of No Child Left Behind, categorizing students based on their performance with particular labels has contributed to the specific language that teachers use when talking about students (Datnow et al., 2018). For instance, using words such as "high" or "low" communicates the belief that students have fixed mathematical abilities. In order to help teachers shift their beliefs about students' abilities and vision of equitable mathematics instruction, opportunities for professional learning and growth within a supportive environment are needed.

These opportunities expand beyond traditional face-to-face learning environments as the demands placed on teachers continue to grow. For this reason, informal learning communities have become particularly appealing. These communities offer a flexible space for teachers to collaborate, advocate, learn from one another, share ideas or resources, seek information or support, and reflect on one's own knowledge or practice with other teachers from around the world (Macia & Garcia, 2016). In this study, we examine one Facebook group where teachers network together to build a learning community focused on mathematics education.

Methodology

In this qualitative study, a grounded theory approach (Charmaz, 1983) was employed to conceptualize the nature of the interactions taking place in a mathematics education Facebook group. The group was created by a university-based mathematics education research group and had 14,943 members at the time of data collection. In this space, members can pose questions, celebrate successes, share struggles, elicit support, and share resources. On average, the group generates seven original posts per day, and 95 comments on existing posts. In this study, data gathered from this Facebook group includes 2,600 original posts with comments.

The constant comparative elements of grounded theory (Charmaz, 1983) warrant the use of an inductive content analysis (Roller & Lavrakas, 2015) to identify themes and patterns within the data. The coding process took place across three phases. In phase one, the research team open-coded a subset of the data to form emergent themes (Creswell & Poth, 2018) and generated an initial codebook. The research team met to discuss the 49 initial codes and collapsed them into 15 overarching themes. In phase two, the research team drew on these 15 themes to analyze a different subset of data. Additional codes emerged and two themes were added, yielding a total of 17 overarching themes. To establish intercoder agreement (Creswell & Poth, 2018), the research team analyzed a final subset of data and achieved an 84% agreement score. In phase three, the entire data set was hand-coded using the 17 themes.

For the purposes of identifying conversations around tracking, only posts coded for the following three themes were included: (1) beliefs about teaching and learning mathematics, (2) challenges experienced by teachers and students when teaching and learning mathematics, and (3) any mention of students' experiences learning mathematics. A total of 147 posts were included and analyzed.

Results

In what follows, we report on one prevailing topic, tracking, that emerged as teachers discussed their experiences in this informal learning community. Our analysis suggests that members often come to this community to share their frustration and solicit help to become change agents - teachers actively trying to address inequitable practices within their schools. Teachers appear to recognize the negative impact tracking can have on students at a classroom and systems level; however, there is a disconnect between what they believe and the pressure they receive from colleagues, administration, parents, and district leaders to enact ineffective grouping practices. We draw upon two illustrative posts that represent our findings around how this online learning community discusses systems level tracking and classroom level ability grouping to catalyze change and undo, or subvert, the structures that produce tracking within today's schools.

System Level

In one interaction, a teacher described the negative impact tracking has had on her high school students. She wrote, "I have been teaching the 'lower' track now for 4 years and most of my students tell me they feel stupid for being in my track. Kids make fun of them and feel like they are better than them because of what track they got placed in." At a staff meeting, this teacher advocated against the use of tracking in mathematics but was met with opposition. "I guess, I was surprised of [sic] the resistance."

In her post, she continued by asking, "Was wondering if anyone has had any success in convincing change at their schools?" The post generated 54 replies, which led to a critical conversation around this type of practice. Many of these replies created a feeling of connectedness among the group members that engaged in the conversation. That is, they seemed to share similar beliefs about tracking. Some replies built a bond through words of encouragement, such as, "You have planted the seed! You are right - all kids DO deserve better." Other members of the group shared similar experiences where they too were met with resistance and unsuccessful in their own attempts to lead change. In another reply, a teacher wrote, "I have had absolutely no luck at all. My school is very set...and I can't see anything changing not even in the medium-term future."

This post demonstrates a desire to make a change at a systemic level. These teachers seem to think beyond their own instruction and consider the changes needed to transform education on a larger scale. While teachers may recognize the need for a systemic change in education, they also seem aware of the impeding barriers and obstacles. To mitigate the impact of system level choices, many group members that replied to the post suggested that the original poster take on the issues within her own classroom and mitigate the negative effects through practices she could control immediately.

Classroom Level

One reply encouraged the original poster to address systemic issues of tracking within her own classroom, "You should just go all 'Stand and Deliver' on them and teach the 'lower' track so well that they surpass the other track!" Similarly, another teacher wrote, "Keep your mouth shut and prove them wrong! Do solid teaching with your 'low' kids and let them prove that your methods work on the lowest kids. I love proving people wrong with data." Both of these replies called for the teacher to subvert the systemic level practice of tracking by using good teaching practices, which are less likely used in lower tracks (Mijs, 2016), to create noticeable change through student performance.

Members of the Facebook group discussed their experiences and struggles with ability grouping within their individual classrooms and the impact this has had on their students. One teacher shared how his colleague is "stuck on grouping." As this teacher described his colleague's students, he used language such as, "fast kids" and "slower kids," though he indicated he did not care for those terms. This teacher described the tension between promoting heterogeneous grouping practices and the pressure of time constraints and curriculum expectations. To help his colleague shift away from these harmful teaching practices, he posed the following question to the group, "How can I help her [colleague] with this transition without homogenous grouping and/or what recommendations do you have to help get her students to being able to work together and help each other?" Interestingly, this post did not evoke the same level of engagement. While the previous post generated rich dialogue among members, replies to this post were limited. These replies guided the original poster, and the readers of the interaction, towards inclusive strategies that built heterogeneous groups of learners.

Discussion

Educators have leveraged the use of social media, such as Facebook, as a platform for informal learning. Given its flexibility and appeal, teachers gravitate towards these spaces to improve their knowledge and practice (Anderson, 2019). In our analysis of one Facebook group, we found teachers engaged in critical conversations to catalyze change and undo the structures that produce tracking within today's schools. Research has consistently found that ability grouping fails to benefit the student and further exacerbates the inequities in our school systems; however, conversations within this online community suggest that these practices continue to prevail. Our findings suggest that membership in this type of learning community provides individuals affective resources (Brodie, 2020) through emotional support, which is often overlooked in these spaces. This group also provides emotional support through confirming other members' feelings of frustration, while encouraging them to continue to fight for change at their schools.

While some teachers in this Facebook group recognize the impact of ability grouping and hope to dismantle these practices on a systems level, others navigate these waters within their own classroom. As members from the group share their subversive practices and call for others to "prove them wrong", they are advocating for Creative Insubordination practices such as, "using the master's tools" and "flying under the radar" (Gutiérrez, 2016, p. 54). These practices encourage teachers to work within the system and use required tools, such as student assessment and imposed ability groups, to produce outcomes contrary to those often expected. Using Creative Insubordination practices allow teachers to be instigators of change while still working within the confines of a deeply flawed system that continues to impose tracking.

Online learning communities, such as this Facebook group, provide technologically-enhanced ways for teachers to build their understanding of how mathematics is taught and learned. Further research within social media facilitated learning communities should investigate how these spaces are changing the field's perception of collaborative learning within a community of practice. These online spaces are critically important to some educators and will continue to serve as a professional learning site for teachers.

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TEACHER EDUCATION (IN-SERVICE) / PROFESSIONAL DEVELOPMENT:

POSTERS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

SELF-DIRECTED LEARNING FOR RURAL MATHEMATICS TEACHERS

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Twenty-seven percent of American public-school teachers are located in rural areas of the country (National Center for Education Statistics, 2012), yet their professional learning needs and experiences are under-studied. Limited research has highlighted some of the particular challenges of being a rural teacher of mathematics: geographic isolation, professional isolation, and insufficient opportunities for high-quality professional learning (Royster as cited in Cady & Rearden, 2009). One strategy used to address these issues has been to form inter-district networks of faculty for purposes of professional learning, but this has been met with uneven success (Howley & Howley, 2005).

This study considers a model of teacher professional learning within a rural inter-district learning cooperative situated within economically disadvantaged Appalachia. The model utilized invites teachers to direct their own learning, by first identifying their learning needs and then requesting financial grants of up to $1,000 from the cooperative in order to realize them. Similar to Slavit and McDuffie (2013), we draw on adult learning theory to explore the individualized and self-directed nature of the participating rural teachers' professional learning experiences and examine the following research question: *How do rural mathematics teachers describe their motivation, needs, and learning within self-directed professional learning experiences?*

The authors analyzed videos of the final summative presentation from eight teachers receiving the cooperative grant. For these analyses, three broad codes from Knowles (1975) were utilized: (1) identify areas of growth (why participate), (2) finding people and resources to learn from (what is needed), and 3) evaluate the learning that they have experienced (lessons learned). Analytic memos were created for each of the three broad themes.

The reasons for participation teachers identified often reflected the situation of their practice in a rural, socio-economically disadvantaged region and/or the needs of their students as a whole. Two teachers considered the economic outlook of the region, choosing to supplement their current practice with skills that might support their students in creating or capitalizing upon emerging economic opportunities. These teachers worked to incorporate computer programming and making skills into their mathematics classrooms. The resources teachers utilized to realize their goals were largely unsurprising. Grant monies were used to purchase tools for makerspaces, laboratory materials for cross-curricular STEM units, computer science applications, and pieces of technological equipment. All teachers reported successful learning experiences. While some teachers shared unexpected learnings, such as re-imaging the use and application of educational technology, others reported constraints on learning often associated with technology adoption.

Findings from this study suggest that when rural teachers are supported in designing and self-directing their own learning, their motivation for doing so is often grounded in the needs of their students and the needs of the local community. Most of the resources purchased with the mini-grants were typical educational supplies, but some also reflected the cultural heritage of the region. While material resources were procured with relative ease, expertise was not. This was evidenced both within the constraints of technology implementation and teachers' first-time attempts to design cross-curricular content. Future research should consider the ways in which rural teachers can access and draw upon others' expertise to inform their self-directed professional learning experiences.

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EXPLORING VIDEO COACHING PRACTICES OF ONLINE MATHEMATICS COACHES

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Coaching has become a common practice to support teachers (Coburn & Russell, 2008; Foltos, 2014; Knight, 2007). While much of the coaching research has focused on the roles and responsibilities of coaches (Gibbons & Cobb, 2017; Mudzimiri et al., 2014) there is a lack of empirical research on what occurs in the interactions between a coach and teacher and the mechanisms by which these interactions support teacher learning. This study aims to understand both *how* and *why* coaches engaged in specific coaching practices.

We analyzed interviews with four mathematics coaches in which they reflected on the purposes, goals, and practices they perceived as critical for supporting teachers in a fully online coaching model. For three years, these coaches had used an online content-focused coaching model with rural middle school mathematics teachers. The online coaching model was an adaptation of West and Staub's (2003) content-focused coaching, which prioritizes focusing on mathematical content knowledge and student thinking. In this study, the coach and teacher co-planned a lesson via Zoom, the teacher enacted the lesson and recorded with a Swivl robot and iPad, and the coach and teacher debriefed the lesson via Zoom. The lesson video was uploaded automatically to a shared library, through which the coach and teacher viewed and annotated the lesson video prior to the post-lesson conference.

Analysis and Findings

We used Barlow et al.'s (2014) dimensions of coaching purposes: interacting with teachers about mathematics content, promoting teacher reflection, and negotiating professional relationships between coach and teacher. This framework helped us to identify specific coaching practices and connect the coaches' rationale to how these practices supported teacher learning. For example, coaches described doing the mathematics of the lesson with the teacher in the planning meeting because it afforded richer discussions of how students learn the mathematical content and how instructional decisions would influence whether they met the goals of the lesson. Coaches also reported that doing the mathematics together provided opportunities for the coach to deepen the teacher's mathematical content knowledge.

Additionally, several coaching practices were indicated as being critical to prompting teacher reflection and negotiating professional relationships: annotating the lesson video, making suggestions, and discussing evidence from the video related to the learning goals of the lesson and instructional practices goals of the teacher. The coaches indicated these practices led to collaborative and more reflective relationship with a teacher.

The results of these analyses provide examples of how and why specific coaching practices can support the development of teachers' instructional practices. We believe this study will support the coaching research community.

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ELEMENTARY TEACHERS' BELIEFS ABOUT TEACHING MATHEMATICS AND SCIENCE: IMPLICATIONS FOR ARGUMENTATION

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Teachers in the elementary grades often teach all subjects and are expected to have appropriate content knowledge of a wide range of disciplines. Current recommendations suggest teachers should integrate multiple disciplines into the same lesson, for instance, when teaching integrated STEM lessons. Although there are many similarities between STEM fields, there are also epistemological differences to be understood by students and teachers (see, e.g., Conner & Kittleson, 2009). How to teach STEM lessons without ignoring the unique characteristics, depth, and rigor of each discipline is an open question (Kertil & Gurel, 2016). This study investigated teachers' beliefs about teaching mathematics and science using argumentation and the epistemological and contextual factors that may have influenced these beliefs.

This qualitative study was conducted in a professional development course designed for elementary teachers. Participants in this study included 14 teachers from a rural school district in the southeastern United States. Data sources include video recordings and transcripts from one in-class meeting of the course and two semi-structured interviews with each participant.

The majority of elementary teachers in this study believed argumentation was an important part of teaching all subjects. Their beliefs about argumentation in science suggest that they see argumentation as inherent in the learning of science: "Scientific inquiry is very similar to argumentation" (Gloria, Int. 2). Teachers' statements about argumentation in mathematics, on the other hand, suggest that students need to know the mathematical content prior to engaging in argumentation. For instance, "I didn't [engage them in argumentation] because of understanding …they were not intellectually ready for that concept" (Bill, Int. 2).

Teachers' beliefs about different epistemological underpinnings of mathematics and science, along with contextual constraints, led to different beliefs and intentions for practice with respect to argumentation in these disciplines. The contextual constraint of testing and the amount of curriculum the teachers perceived as essential focused more attention on the teaching of mathematics, which could be seen as benefiting student learning of mathematics. On the other hand, the perception of science as involving wonder, curiosity, and inherently positive and interesting ideas may lead to the creation of a more positive learning environment for the teaching of science. These questions remain open and need to be studied further: What are the consequences of perceiving argumentation in mathematics as limited to concepts already well-understood? Can integrating the teaching of mathematics and science lead to more exploratory and inquiry-based teaching of mathematical ideas alongside scientific ones?

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THE SEMIOSPHERE: A LENS TO LOOK AT LESSON STUDY PRACTICES IN THEIR CULTURAL CONTEXT

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Gallimore (1996) claims that changes in teaching and learning practices are challenging. He ascribes this resistance to change to the fact that "we are dealing with cultural matters", and not just psychological and pedagogical issues (*ibid*., p. 230). Cultural aspects have therefore become one of the focus of research attention in Mathematics Education in the last twenty years (Bartolini Bussi & Martignone, 2013). How cultural and social aspects affect teacher critical reflection during professional development experiences of in-service and prospective mathematics teachers? I address the issue of how to deepen culturally sensitive understandings of such processes. I am inspired by Lotman's concept of Semiosphere (Lotman, 1990) that I identify and use to read the processes of teachers' professional development experiences. Strengthened by the tradition of the Italian school in Research in Mathematics Education and rooted in it, which grants considerable importance to semiotic studies (Arzarello, 2006; Bartolini Bussi, 1996), I propose the Semiosphere as a theoretical lens that attempts to react to Skott and Møller's call (2020) to look at the issues of policies and culture in the teachers' local professional development setting, and to react to the need underlined by Yves Chevallard (1981) to take into account the codetermination of the various knowledge signs into the Noosphere.

In Italy, as a foreign cultural element, Lesson Study (LS) has been implemented in order to allow mathematics teachers and researchers to reflect on and thus to question their own didactic practices and intentionality (Bartolini Bussi & Ramploud, 2018; Mellone, Ramploud, Martignone & Di Paola, 2019). Designing, implementing and observing, and afterwards reviewing a one-hour lesson have been uncommon spaces for collaborative reflection of Italian mathematics teachers, because of their cultural tradition. Even critical reflection therefore becomes a cultural activity and, as such, pervasive and not easy to study. We need a culturally sensitive lens that can help us to identify and study reflection practices. Through the qualitative analysis of a LS experience, looking at the dialogues between teachers and their practices of shared critical reflection, I can state that the Semiosphere highlights the asymmetries between the systems of signs that exist in a culture, in a practice, in a methodology, in a professional development path, or in a lesson planning. It is in this space that the process of cultural transposition takes place. In fact, as pointed out by Vygotsky (1999) signs do not appear as *mediators* of activity, as is the case in other sociocultural approaches, but as an *integral part* of human thinking and human activity. The Semiosphere allows to keep identifying the constituent elements of a reality even from the identification of elements external to it. In fact, precisely because of its asymmetric and non-homogeneous character, based on dialogue, the Semiosphere creates not only its own internal organization, but also its own type of external disorganization. It defines what is not itself. The LS teachers' meetings can be pictured as a multidimensional dialogue in the Semiosphere during which each choice of teaching/learning, in contact with another, can become "more aware" (Jullien, 2005). Here the critical dialogue and reflection of the teachers, if read from the point of view of the Semiosphere, do not lose contact with the reality in which they are born. So, the problem of possible integration between Lotman and Chevallard lenses according to the Networking of Theories approach (Radford, 2008) arises spontaneously. The analysis of the institutional aspects and the levels of co-determination seems enriched by a dynamic interchange perspective, and vice versa this can be integrated with the aspects

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of power and the institutional constraints typical of a school system governed by laws. Future studies could tell us about the connection of the two theories as lenses for professional development practices.

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FROM TERRIFIED TO COMFORTABLE: A FOURTH-GRADE TEACHER'S JOURNEY IN TEACHING CODING

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Keywords: Teacher Beliefs; STEM / STEAM; Teacher Education - Inservice / Professional Development; Elementary School Education

Collective Argumentation Learning and Coding (CALC) is a project focused on providing teachers with strategies to engage students in collective argumentation in mathematics, science, and coding. Collective argumentation can be characterized by any instance where multiple people (teachers and students) work together to establish a claim and provide evidence to support it (Conner et al., 2014b). Collective argumentation is an effective approach for promoting critical and higher order thinking and supporting students' ability to articulate and justify claims (Nussbaum, 2008). The goal of the CALC project is to help elementary school teachers extend the use of collective argumentation from teaching mathematics and science to teaching coding. Doing so increases the probability that teachers will integrate coding in regular classroom instruction, making it accessible to all students.

A total of 32 elementary school teachers participated in the CALC project, which included a semester long course focused on coding content and strategies to implement collective argumentation in the classroom. Teachers were interviewed before and after the course. Ten teachers were selected to participate in the enactment phase, in which classroom observations and stimulated recall interviews were conducted. All class meetings, observations, and interviews were audio- and video-recorded. Data were analyzed using previously established analysis methods for beliefs and argumentation (Conner et al., 2014a; Conner et al., 2011; Kim et al., 2013). We highlight Gloria (pseudonym), a fourth-grade teacher from Cohort 1 because of the extent to which she went from fear of coding to fluent implementation.

Initially Gloria was comfortable engaging her students in argumentation, explaining they already used it in mathematics with Cognitively Guided Instruction (CGI). However, she was "terrified" (Int 2) about learning to code because she didn't view herself as proficient with technology. She was willing to overcome her fear of coding because she saw the value in providing her students with coding experiences that would help them develop necessary skills for our increasingly technological society. Gloria saw coding fitting naturally into mathematics, but she asked for ideas on how to integrate coding into other subjects: "Yeah, I can do shapes. What else can I do that's beyond that? What's something I can push the limit on?" (Int 2).

In our first observation, Gloria requested not to wear a microphone and asked members of the CALC project to co-teach with her. In this lesson, her students engaged in argumentation while coding prebuilt robots to travel around a rectangle with a specified perimeter. In our final observation, Gloria successfully implemented a lesson that included students programming more complex robots to travel a map of the Oregon Trail. Reflecting on this lesson, Gloria was "proud" that her support moves had become less leading, allowing for more student discovery and argumentation (Post-obs Int 2). In the course of three months, Gloria's instruction progressed from using simple coding activities to more sophisticated coding platforms. This progression in her coding instruction paralleled the change in her personal feelings about coding as she moved from "terrified" to "comfortable with it" (Post-obs Int 2).

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GETTING A FOOT IN THE DOOR: EXAMINING MATHEMATICS COACHES' STRATEGIES FOR GAINING ACCESS TO CLASSROOMS

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Mathematics coaches are called upon to provide productive, job-embedded professional development for teachers (Gibbons & Cobb, 2017), but their capacity to do so hinges on gaining access to classrooms. Even in schools or districts where coaching is pervasive, teachers are under no obligation to participating in coaching and often have autonomy about whether, when, and for what they invite coaches into their classrooms to do joint work (Saclarides & Lubienski, 2020). Much of what we know about access comes from skilled practitioners (Killion, 2008, Knight, 2017) and a modest body of empirical research (Hartmann, 2013; Mangin, 2005), both of which point to the challenges and importance of gaining access. This study asks: What strategies to elementary mathematics coaches draw upon and enact when negotiating access to teachers' classrooms? What relationships exist between these strategies?

Eleven full-time, school-based elementary mathematics coaches in a public school district located in a southeastern, metropolitan area of the United States were interviewed using a semi-structured protocol. All interviews were audio recorded and transcribed. All statements that described strategies, or action the coach took to gain access to classrooms, were identified and inductively coded to describe the nature of the strategy. Strategies were counted for each coach as either present or absent and then clustered into larger categories based on related functions through an iterative process. Last, statements in which multiple strategies were discussed in tandem were used to create a model for how coaches coordinated strategies to gain access.

Mathematics coaches reported 33 distinct strategies for gaining access to classrooms, with each coach reporting 10 – 20 strategies (median=13). These strategies spanned two related tiers. In the first tier, all coaches engaged in *relational* and *structural strategies* to position themselves and their coaching work as embedded in school routines, creating conditions to move into classrooms. In the second tier, coaches drew from four strategy types (*pitching in, cloaked coaching strategies, indirect strategies, direct offers*) to gain physical access to classrooms. The strategies in this second tier varied in their directness. While direct offers were open invitations to coaching, other types offered non-coaching assistance in the classroom, engaged teachers in coaching while avoiding directly describing the coach's intent, or created opportunities for teachers to approach coaches. These less direct strategies enabled access when coaches perceived that direct offers would be rejected. Coaches deliberately selected the types of strategies they used with different teachers, based on their perceptions of teachers' dispositions and experience.

Gaining access was found to be complex work for all coaches interviewed, requiring a suite of well-coordinated strategies. Future research might investigate whether and how gaining access for mathematics coaching, particularly in the elementary grades studied here, might require different types of strategies from coaching in other disciplines.

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COTEACHING AS PROFESSIONAL DEVELOPMENT: A STUDY OF SECONDARY MATHEMATICS TEACHERS PARTNERING TO TRANSITION PRACTICE

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Keywords: Teacher Education – Inservice / Professional Development; Standards (broadly defined); Mathematical Knowledge for Teaching

There have been many efforts to move mathematics teaching towards student-centered instruction based in problem solving and reasoning. Despite these efforts, Cohen and Mehta (2017) found that U.S. mathematics classrooms remain mostly teacher-centered and instruction remains procedural. Kennedy (2005) explains the ideals are "barely visible in the complex landscape of competing intentions and the multiple areas of concerns that are important to teachers" (p. 61). Currently the field recognizes that professional change occurs through intensive, prolonged, and focused models. One promising model is *coteaching*, where two teachers work collaboratively, which provides "immediacy of the relationship between thought and action" (Roth, 2002, p. 59) and allows for risk-taking and moving away from 'conservative pedagogies' (Gallo-Fox, 2010). Coteaching, without a hyphen, is defined as "a commitment to coplanning, copractice, and coreflection" (Murphy & Martin, 2015, p. 277). These three stages can provide mutual understanding and subsequent learning through zones of proximal development (ZPD). ZPD can then be used as a lens to capture professional growth where interpersonal interactions are transformed to the intrapersonal plane and integrated into each teacher's practice. Murphy, Scantlebury, and Milne (2015) connected six elements in their ZPD framework for coteaching, which will be discussed in the findings below to address the research question: *How can secondary mathematics teachers in a coteaching partnership serve as resources for each other's professional growth towards reform-oriented standards?*

This study analyzes a single, holistic case study (Yin, 2018) of two secondary mathematics teachers striving to accomplish progressive standards in their coteaching of four Algebra 1 sections. The design of the study emerged as part of the researcher's teaching practice, resulting in a naturalistic inquiry of an authentic situation. Sources of data include audio recordings of researcher-participant coplanning and coreflection meetings, some of which developed into responsive interviews, coenactment of lessons, and teaching material artifacts. Qualitative thematic analysis was done in relation to the six elements of coteaching. As an example, in one of the first lessons that altered the coteacher's typical lesson structure, she was challenged to consider how non-direct instruction opens up unpredictable situations. She states: "I just have to think these things out in my head like you know what if it's not working out the way I want it to work out? This is, this is different." This 'bud of development' (Vygotsky, 1978) represents the coteacher's hesitant but promising transition towards enacting student-centered lessons.

Through partnering with another, issues in the classroom can be explicitly recognized and alternatives can be reflected upon and enacted. Furthermore, joint reflection of enacted lessons "provides the opportunity for the deconstruction of those experiences and the reconstruction of a shared meaning in a way that transforms understandings and changes practice" (Crow & Smith, 2005, p. 491). Implicit beliefs about teaching and learning can be critically analyzed, conflicts may be resolved, and sensitivity can increase. The complex landscape of teacher beliefs, dispositions, experiences, and knowledge can be influenced through the coteaching model.

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ONE TEACHER'S LEARNING TO FACILITATE ARGUMENTATION: FOCUS ON THE USE OF REPEATING

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Engaging students in collective mathematical argumentation is a practice leading to authentic learning in mathematics classrooms (see, e.g., Krummheuer, 1995). Researchers have reported multiple cases of expert teachers supporting collective argumentation and described student learning in these cases (e.g., Krummheuer, 2007). However, little is known about how novice teachers learn to support their students in making mathematical arguments. Our goals of professional development (PD) with beginning teachers included stimulating learning through reflection and documenting the learning as evidenced by both practice and conversations about practice. The following research question guided our study: *How did the ways in which a beginning mathematics teacher used repeating to support collective argumentation change and mature over time while participating in professional development activities?* For this study, we focus on the PD activities in which a secondary mathematics teacher, Jill (a pseudonym), participated during her first 3 years of teaching. During the PD activities, we engaged Jill in identifying and diagramming arguments from her teaching using Toulmin's (2003) diagram, and in analyzing and reflecting upon her own practices with respect to supporting argumentation using Conner et al.'s (2014) Teacher Support for Collective Argumentation framework. To encourage and facilitate Jill's learning to support argumentation, a mathematics teacher educator-researcher (MTE-R) asked questions and provided feedback and assistance to her during 14 one-on-one PD meetings, structured similarly to stimulated recall interviews. We video-recorded all the meetings. To answer our research question, we focused on instances in which Jill identified, analyzed, and critiqued her repeating actions, such as restating and displaying. We wrote memos describing changes in Jill's use of repeating over time. The analysis is in the initial stage and is ongoing. In year 1 PD, we see Jill enacting one of two repeating actions rather naturally in her practice as she restated student contributions to the collective. Through focused reflections on her practice with the guidance of the MTE-R, Jill learned more about how she could use both restating and displaying actions to support collective argumentation. As we followed Jill's learning into year 3 PD, we found that Jill used both repeating actions in purposeful ways. She also used these actions in strategic ways to support her students in taking a more active role in the classroom discourse and argumentation. Jill serves as a case to demonstrate how a novice teacher learned to support argumentation through stimulated recall interviews of practice.

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DEVELOPING ARGUMENTATION PRACTICES FOR TEACHERS

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Mathematical argumentation is an important feature in the development of conceptual understanding for students (Osborne et al., 2019; Staples & Newton, 2016). Research has generally focused on how argumentation plays out in the classroom, with little focus on how teachers learn this complex work. This study explores teacher's understanding in facilitating argumentation and its implementation over time in the context of a professional development initiative. We address the following questions: How do teachers understand argumentation as a practice? How do teachers implement argumentation in their classrooms as they participate in the professional development? Our work is framed around an understanding that teacher learning is contexual, with a focus on interactions in the content (Greeno & Engeström 2014) and built off of teachers' practices (Kazemi and Hubbard, 2008). We consider argumentation as reasoning about a claim to build agreement across a community, as established by Knudsen et al. (2018).

Eight elementary teachers participated in Learning Labs (Gibbons et al., 2017), a series of monthly professional development sessions with interim support by coaches on implementing the practices. Each Learning Lab consisted of a cycle of new learning, planning a lesson using mathematical argumentation, enacting the lesson, and a debrief of the experience. Data included field notes from each Learning Lab, teachers' written reflections, and pre- and post-interviews for each teacher. We conducted cross-data analysis, with the sensitizing question of *how do teachers understand argumentation and how do they make plans to facilitate argumentation in their classrooms?* We analyzed perceptions and actions involving argumentation over the series of Learning Labs to understand moments of teacher insight and change regarding argumentation.

Findings show a change over time in teachers' understanding and facilitation of argumentation in practice. Early understandings focused on argumentation as explaining one's thinking. Teachers grappled with the distinction between explaining and justifying and with how to implement argumentation in the classroom (see Ghousseini et al., 2019). Over time, teachers developed more nuanced ideas of what counts as argumentation (making claims, providing evidence). Their new understandings helped them generate supports for students to participation in argumentation, ranging from claim comparisons to creating rough drafts of ideas. For example, one teacher worked to provide a set of claims for students to promote discussion that focused on justifying support or disagreement with each. Another teacher worked on language supports for argumentation and forms of modeling justification to move beyond students simply explaining a strategy. While the growth shown in teachers shows a more complex and practice-oriented understanding of argumentation, the differences across individual teachers represent the unique ways they connected to the professional development experiences. These findings show the effectiveness and significance of explicit and practice-oriented professional development for teachers' understanding of mathematical argumentation.

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REMOTE ENGAGEMENT IN EARLY MATHEMATICS PROFESSIONAL DEVELOPMENT: USING TANGIBLE ARTIFACTS TO MEDIATE PARTICIPATION

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Keywords: Teacher Education – Inservice / Professional Development, Early Childhood Education, Distance Education, Embodiment and Gesture

Although the field has recognized the importance of early mathematics education for young children, many early childhood educators do not have access to high-quality, engaging professional development in math or science (McClure et al., 2017). In particular, educators in rural communities lack opportunities to grow professionally due to geographical isolation or under-resourced programs. In addition, many professional development opportunities offer prescribed programs that do not encourage active participation and are not connected to teachers' existing practices or approaches (Kennedy, 2016).

Building on frameworks for effective professional development (Fishman, Davis, & Chan, 2014) and embodied design (Abrahamson & Lindgren, 2014), we used video conferencing technology paired with tangible materials to engage a cohort of remote online early childhood educators in four sessions of professional development in early mathematics education. Each session was one hour in length and included opportunities for online teacher learners to engage with tangible materials (e.g., Froebel gifts, triangle construction materials) to explore mathematical concepts central to early childhood development. Our approach was designed to 1) engage teachers as learners with carefully designed materials to develop their own understanding, and 2) open pathways for mediated participation through the sharing of physical constructions via video conferencing. In this poster, we focus on the following research question: How do participants' material constructions and interactions act as mediating resources in their participation in remote online professional learning?

We video recorded two of the professional learning sessions, surveyed participants, and interviewed a sample of participants about their experiences in the professional learning. Our findings suggest that tangible materials allowed for common sense-making and active participation throughout the sessions. Furthermore, tangible materials served as resources for mathematical engagement and dialogue in spite of educators' remote participation. This poster will highlight the professional development approach and suggest several implications for the use of tangible materials to enhance online professional development engagement.

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PROMOTING COACHES' LEARNING THROUGH DOING THE MATH TOGETHER

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Mathematics coaching is deeply complex work that requires coaches to develop and draw upon multiple forms of expertise related to mathematics, mathematics teaching, and mathematics coaching (Polly, Mraz, & Algozzine, 2013). Yet, few studies have explored how professional learning experiences for mathematics coaches might support their development of these forms of expertise (Jackson et al., 2015). Through an analysis of a representative instance in which elementary mathematics coaches participated in a professional development activity called *doing the math* (Loucks-Horsley et al., 2010), we aim to contribute to the growing body of research focused on how coaches' own professional learning opportunities might be structured.

Our study took place in Hamilton School District, which is a public school district located in a metropolitan area in the southeastern United States. We partnered with one district administrator and 12 elementary mathematics coaches. Eight of the coaches were entering their fifth year as coaches, while three were in their first or second year. Data sources included video data of these professional development sessions (n=6) as well as interview data (n=15) in which participants described their own learning. All data were professionally transcribed.

To understand what opportunities for professional learning were opened up or closed down, we first coded for representations of practice and epistemic claims in the *doing the math* transcript segments. Next, we addressed whether a representation of practice or an epistemic claim was centered on students, mathematics, mathematics teaching, and/or mathematics coaching. To understand our participants' perspectives on the benefits and drawbacks of participating in *doing the math*, we engaged in an open coding process using interview transcripts (Creswell, 2013).

Our analysis showed that as the coaches engaged in *doing the math* together, opportunities were opened up for them to discuss students' and their own mathematical thinking, the mathematical concepts and disciplinary practices included in the task, how those concepts and practices were related to grade level expectations, how tasks could be put to practical use by teachers, and the ways in which teachers can enhance mathematical access for all students. Yet, our analysis also showed that through *doing the math*, explicit conversations about mathematics coaching were not typically available for discussion. Our participants discussed four benefits of engaging in *doing the math*: being placed in the seat of a learner, deepening their own understanding of the mathematics standards, deepening their understanding of how to support students' access to mathematical tasks, and sharing the task as a resource with teachers. They also cited two drawbacks to consider when implementing *doing the math* as a professional development activity, including a lack of time and teacher resistance.

Our study adds much needed research describing how coaches' professional learning might be structured. Future research might consider exploring how to interweave explicit conversations about coaching into coaches' professional learning opportunities.

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CHALLENGES IN IMPROVING AND MEASURING MATHEMATICS DISCUSSION LEADING PRACTICE

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This study is an effort to address the challenge of supporting the enhancement of teaching practice. Our model situates professional development (PD) in mathematics instruction occurring in a summer program for fifth grade students. This PD model has two parts. First, participants engage in "legitimate peripheral participation" (Lave & Wenger, 1991) in teaching in this fifth grade classroom through structured conversations about the lesson plans, close observation of teaching, and analysis of student tasks. Second, participants engage in focused learning on leading mathematics discussions through simulations and rehearsals. Two groups of teachers participated, one onsite with a facilitator, and the second at a remote site with an in-person facilitator who delivered the leading mathematics discussion professional development. We study the impact of our PD model. Specifically, we ask: Does teachers' participation impact their own teaching practice, and if so, in what ways?

Twenty-one teachers participated across the two groups. We collected and analyzed a set of pre- and post-videos of classroom discussions. Participants were asked to record three mathematics discussions two months before the PD occurred and three such lessons two months after participation. A tool that captured techniques named in our decomposition of discussion (Selling et al., 2015), including advanced techniques utilized by experienced teachers, was applied to all videos by two research team members.

Prior to the intervention, the means of technique usage of the remote participants were higher than those of the onsite group on almost every dimension ($p < .05$). Thus, we share the findings for the two groups separately. The onsite group (lower pre-intervention mean) did not appear to be leading discussions before the intervention. They showed slight increases in both orienting students to the thinking of others and concluding discussions. Since the intervention was focused on orienting students, likely an unfamiliar area of work, we hypothesize that this was the focus of their practice post-intervention. Conversely, the remote group (higher pre-intervention mean), who appeared to be leading discussions before the intervention, decreased on several categories and showed near significant growth on connecting and extending student thinking. One possible explanation for these decreases is the timing of the post-data collection at the beginning of the year when they may have been explicitly teaching their students how to engage in discussion, leading to fewer instances of particular discussion-leading moves. The increase in connecting and extending may have been due to readiness to take on this difficult work.

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A MATHEMATICS TEACHER'S CURRICULAR DECISIONS

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Although research uncovered some factors that inform mathematics teachers' curricular decisions, it is less clear how they make such decisions. I used the practical knowledge framework to explore the construct of image with a veteran eighth-grade mathematics teacher. Images are teachers' perceptions of teaching that involve emotionality, morality, and are intimately connected to personal and professional narratives. Data consisted of transcripts of conversations, weekly journals, and student-written work. The analysis uncovered two images: 1) Bringing the outside in, and 2) Reading students and moments. Both images express how the teacher made curricular decisions: Using lessons that brought the outside in and following her students. This study expands research on curriculum by illuminating how teachers make curricular decisions and therefore revealing teachers' practical knowledge in practice.

Keywords: curriculum development, narrative inquiry, mathematics teachers

Research has identified diversity in teacher curricular decisions (Heaton, 2000; Lampert, 1985; Remillard & Bryans, 2004; Remillard et al., 2009; Sztajn, 2003). *Curricular decisions* are those involving selection of tasks as well as pedagogical approaches taken to teach such tasks. Although Remillard et al. (2009) identified factors interacting in teachers' curricular decisions, they called for exploring the nature and decision-making process.

Practical Knowledge Framework and the Image Construct

Teachers' practical knowledge ([PK], Elbaz, 1983) framed this study. Teachers face various situations and "draw on a variety of sources of knowledge to help them to deal with these" (Elbaz, 1983, p. 47). Rather than describing knowledge in the form of cognitive structures, Elbaz (1981, 1983) described teachers' knowledge as situated in experience. *Image* is a construct derived from PK and refers to teachers' perceptions of their teaching that involve emotionality, morality, and connections to personal and professional narratives (Clandinin, 1985; Elbaz, 1983). I explored: How does an eighth-grade mathematics teacher make curricular decisions?

Methodology and Methods

Building from a collaborative relationship (Suazo-Flores, 2016), Elizabeth, the author, narratively inquired (Clandinin & Connelly, 2000) into Lisa's curricular decisions while both planned and taught a lesson over three months. Lisa was an eighth-grade mathematics teacher. Elizabeth conducted narrative analysis (Polkinghorne, 1995) to transcripts of conversations (Clandinin & Connelly, 1994), students' written work, and Elizabeth's personal weekly journal.

Findings and Conclusion

Two images convey how the teacher made curricular decisions: *Bringing the outside in* and *Reading students and moments*. Lisa enjoyed teaching lessons that would take students on field trips and embedded in real-world contexts. Lisa also paid attention to what students were doing and saying to decide her next teaching step. This study contributes to existing studies in curriculum-development by illuminating how teachers make curricular decisions and therefore revealing teachers' PK.

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FROM HIGHLY RECEPTIVE TO HIGHLY SKEPTICAL: ENGAGING ALL TEACHERS THROUGH RESPONSIVE PD FACILITATION

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We suggest five PD participant portraits to initiate a discussion on how best to support and facilitate PD with a wide range of participants. Our preliminary findings include five teacher portraits. We discuss how facilitators were responsive to all teachers' needs.

Keywords: Teacher Education – Inservice / Professional Development

Professional developments (PD) play a central role in efforts to improve teachers' mathematical content knowledge, pedagogical content knowledge, and beliefs about what it means to 'do mathematics' (Ball, 1990; Hill, 2007). Creating Algebra Teaching Communities for Hoosiers was the result of a Math-Science Partnership grant from Indiana's Department of Education in 2015. The study involved 15 middle and high school urban teachers, with a focus on enriching teachers' knowledge and skills for teaching algebra. This study examines: How do teacher portraits help facilitate the activities in the PD experience?

This study focuses on exploring how PD facilitators used facilitation techniques to support participants based on character portraits (Sztajn, Borko, & Smith (2017), and contributes to an area of research needed on skillful facilitation techniques (e.g., Bobis, 2011; van es, 2014) to prepare and support PD facilitators. Findings culminated in five teacher portraits.

Highly Skeptical Teacher (HST) is an experienced teacher but is uncomfortable being observed by colleagues. HST doubts students can be successful with the PD tasks. Others followed the skepticism because of HST's experience in the classroom. Facilitators probed questions to interrupt preconceived perceptions of students. **Cautiously Receptive Teacher (CRT)** is eager to apply the theories into practice but struggles to bring ideas into reality in the classroom. CRT is hesitant to try new things, but gradually over time buys into the vision of the PD. Trying out activities with students was the best technique to convince CRT of novel teaching practices. **Highly Receptive Teacher (HRT)** is highly reflective and collaborative. HRT sees the potential of all students to be mathematical learners and makes connections between teaching, the PD, and everyday life experiences. PD facilitators would ask HRT to point out students' mathematical thinking. **Box-Checker Teacher (BCT)** is extremely organized, thrives on explicit directions and timeline, and most comfortable with direct instruction. BCT's intense focus on clear tasks and schedules, and high anxiety made the group dynamics tense. PD facilitators solicited input from BCT on the clarity of expectations. **Lopsided Engager Teacher** (LET) has great relationships with all students, even the most disruptive, and is deeply troubled when other teachers do not believe that all students can learn mathematics. LET displays turns of both low engagement and intense engagement. PD facilitators stoked this passion to engage in rich discussions, and showed empathy to situations where relationships take priority over learning.

This study begins a conversation about mathematics teaching facilitation and how best to support and facilitate with a wide range of participants.

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TEACHING AND CLASSROOM PRACTICE

RESEARCH REPORTS

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A CASE OF SHARED AUTHORITY DURING A STUDENT DEMONSTRATION IN A MATHEMATICS CLASSROOM

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By applying a conversation analytic approach to analyze the fine-grained manifestation of authority in classroom interaction, we examine the interaction between a teacher and a student while the student presented a solution at the whiteboard. This case highlights the complexity of shared authority and the ways that authority is construed by discourse practices and negotiations over the ownership of knowledge. We offer a nuanced interpretation of how the teacher and student shared epistemic authority through their joint activity. Further, we argue for the importance of distinguishing teachers' epistemic authority and deontic authority to further our understanding of how a teacher can share authority with students during instruction.

Keywords: Classroom Discourse, Authority, Student Demonstration, Conversation Analysis

The flow and concentration of authority in mathematics classrooms can impact students' identities as knowers and doers of mathematics (Boaler & Greeno, 2000; Esmonde & Langer-Osuna, 2013, Langer-Osuna, 2017). In our view, authority is not static but rather is relational (e.g., a teacher is presumed to have more authority than a student) and interactional (e.g., a student sharing a solution may imbue the student with authority). Moreover, our analysis focuses on the relative nature of authority, meaning participants may have greater or less authority, but this may shift over the course of interaction. In this paper, we explore a case of one mathematics teacher's attempts to share authority when a student (i.e., a *demonstrator*) shared a solution at the front of the room. This case study of a teacher negotiating authority with students during a common classroom activity (i.e., presenting a solution at the board) offers an opportunity to examine the complexity of shared authority in mathematics classrooms. Our analysis highlights the ways that authority is construed by discourse practices and negotiations over the ownership of knowledge.

This paper uses data from a partnership with mathematics teachers designed to support teachers to do action research on their discourse practices. Here, we build on that research using a conversation analytic (CA) approach to foreground the role of language and knowledge in the sharing of authority in mathematics classrooms. We approach our analysis under the assumption that knowledge is public and interactionally managed and occurs within and is constituted as a situated discursive practice (Barwell, 2013; Edwards, 1993). By attending to how teachers and students *perform* knowledge rather than what knowledge they *have* (Byun, 2019), our analysis centers the knowledge displays and negotiation that are an integral part of the process of sharing authority in classroom interactions. In this paper, we argue that the physical arrangement and position of a student alone—such as the physical location of a student at the front of the room—may not be enough to account for how authority is shared in the classroom. Instead, authority can be shared through discursive moves that orient classroom members to the knowledge domains in which one or more students are the primary authorities.

Authority in Mathematics Classrooms

Teachers exercise tremendous authority in their classrooms (Amit & Fried, 2005; Oyler, 1996; Wagner & Herbel-Eisenmann, 2014b). In mathematics classrooms, teacher authority is also influenced by the institution of schools and common discourses of mathematics. As a student in an

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interview stated, “[mathematics] is not like literature where someone can say this way, and someone can say this way" (Amit & Fried, 2005, p. 158). This commonly held belief reflects the common practice of mathematics teachers determining what is correct or incorrect as arbiters of truth. As mathematics educators try to move from teacher-centered to student-centered ways of teaching, sharing authority has been a central theme in the discussion (e.g., Ball, 1993; Cobb, Wood, Yackel, & McNeal, 1992; Cohen, 1990; Hamm & Perry, 2002; Lampert, 1990).

Scholars have examined authority and authority relations in mathematics classrooms from multiple perspectives. Amit and Fried (2005) found that students portray parents, teachers, and peers as authority figures while placing teachers at the center of authority. Wagner and Herbel-Eisenmann (2014b) found that teachers attributed authority to not only people but also institutions (e.g., school board) and tools and artifacts (e.g., textbooks, manipulatives). Authority in mathematics classrooms thus has complex origins and relations that manifest in overt and covert ways (Wagner & Herbel-Eisenmann, 2014a).

Authority can also have different discursive functions in unfolding classroom interactions. Oyler (1996) distinguished two dimensions of teacher authority: teacher being *an* authority and being *in* authority. These two kinds of authority, respectively, represent a content dimension of mathematical knowledge and a process dimension of organizing and orchestrating learning activities. Similarly, Langer-Osuna (2016) examined both intellectual authority and directive authority in student-to-student interactions. Although these two kinds of authority are interdependent (Langer-Osuna, 2016), it is important not to conflate these two. As we argue in this paper, process authority can be a resource for teachers to share content authority in the unfolding classroom interactions. To further examine teacher authority in the context of classroom interaction, we draw on two major discussions on authority in social interaction: epistemic authority and deontic authority.

Epistemic and Deontic Authority: A Conversation Analytic View

CA scholars found that participants orient to relative authority in two broad ways during interactions. First, epistemic authority concerns the relative difference in participants' depth of knowledge at hand. To be clear, the concern is not with the depth of knowledge in participants' minds, but rather how the participants treat themselves and others as more or less knowledgeable in the interaction. As we discussed earlier, institutional roles such as being a teacher or a student, in part, shape the epistemic authority. CA approaches can uncover how such authority manifests in a local, interactional context. For instance, Heritage and Raymond (2005) illustrated how, when a person assesses and describes something before others share, they are often ascribed with more epistemic authority than doing so after someone else. They observed when a participant with less authority makes a claim first, the participant mitigates the associated epistemic authority by downgrading their assertion (e.g., pre-facing with "I think," adding a tag question, seeking confirmation). Thus, epistemic authority can originate from not only being *a teacher* but also from a local, interactional role such as being a *teller of news* or *teller of trouble*. In our case, a student being someone who shares a solution at the board afforded her with epistemic authority, though limited.

Second, deontic authority concerns someone’s "right to determine others’ future actions" (Stevanovic & Peräkylä, 2012, p. 298). Although epistemic authority is about "description," deontic authority is about "prescription" (p. 298). In a classroom setting, process-authority aligns with deontic authority, with which a teacher directs the future actions of the class, such as selecting who will speak next and choosing a topic for the class to discuss (Mehan, 1979). Directives and proposals are often associated with deontic authority, but the recipients of these actions may resist the speaker’s deontic authority by refusing to comply or framing their compliance as their autonomous action (Kent, 2012). As Steveanoci and Peräkylä (2012) stated, "[d]eontic authority is an interactional achievement, claimed, displayed, and negotiated at the level of the turn-by-turn sequential unfolding of the interaction" (p. 315).

A CA approach fits our investigation on teacher authority in classroom interactions for at least two reasons. First, teacher authority is "created and maintained through interactions" (Oyler, 1996, p. 23). A CA approach brings our attention to teacher authority situated in interactions, thereby allowing us to see how authority is instantiated and shifts as the interaction unfolds. Second, a CA approach offers a systematic way to formulate empirically grounded interpretations of authority in social interaction. Based on its root in ethnomethodology, a CA approach attends to how participants orient to authority based on the subtle details of moment-to-moment interactions. Based on this theoretical grounding, we investigated the following research question: *How do epistemic and deontic authority instantiate and interact with each other when a student is a demonstrator?*

Data and Methods

This study was part of a larger partnership focused on supporting secondary mathematics teachers to use action research to examine their classroom discourse by focusing on, for example, issues of status, and positioning in mathematics classrooms. Data came from classroom videos of participating teachers that they collected to examine changes in their discourse patterns. Among multiple cases of student demonstrations, this case of Ms. Reed was selected for this report because on the surface, her case illustrated evidence of overt teacher authority. She regularly used directives (e.g., “stop,” “continue,” “ask”), and the length of the student demonstration was approximately 13 minutes, which was longer than many of the other observed student demonstrations. This was intriguing for us because we anticipated that the greater the authority that a teacher exercises, the less knowledge can be shared from demonstrating students, thus resulting in a shorter demonstration length. This led to a more fine-grained analysis of this selected case to understand how teacher authority manifested during this relatively longer student demonstration.

Following the tradition of conversation analysis, we adopted the Jefferson Transcription System (Jefferson, 2004) to capture a range of speech features (e.g., delays in response, elongated pronunciation, intonation changes) that may be significant to examine authority in interaction. Here, we only report the following features: silence in 1/10 sec, (.x); silence shorter than 0.3 sec, (.); overlapping talk, [ ]; vowel elongation, : ; emphasis, _ ; unrecoverable speech, ( ), as we referred to them in our findings section. By examining both what the teacher does and how students respond, we examined how the teacher and students are orienting to both epistemic and deontic authority. Based on the participants' orientation, we made an empirically grounded interpretation of authority.

Findings

In this section, we analyze transcript extracts from a demonstration by Anika, an 8th grade student. Before the demonstration, students were engaged in two warm-up problems, each of which asked them to create an equation of a line passing through two given points. After the students solved the problems individually, Ms. Reed called on students to share their answers. Anika volunteered to share her solution. Ms. Reed asked Anika to share her method at the front of the room on a digital interactive whiteboard. Our analysis highlights the nuanced ways that authority was shared through the joint discursive activity of members in the class. This analysis also points to the importance of distinguishing teachers’ deontic and epistemic authority.

Exclusive Deontic Authority

Extract 1: Use of Directives

```
083  ANIKA:    .hh so then I would (.) set up my equation lik:e
084            (          ) over (2.5)
085            so: this would be x two: and then this would x one
086            so then it would be negative two: (1.0)
087  Ms.R:     okay sto:p (.) question.
088  JOHN:     u[:m  ] why is the uh seven uh y two and the fi- and
089  Ms.R:      [John]
090  JOHN:     the uh negative two x two.
```

Throughout the demonstration, Ms. Reed acts with deontic authority in at least two ways. First, she stops the speaker to direct the class to another activity (line 87). Ms. Reed's falling intonation when she says "question." indicates that this is a directive rather than seeking a question. Second, Ms. Reed selects the next speaker (line 89). Note that John starts his speech with "um" (line 88) occupying the conversational floor with his speech and vying for his right to speak. As soon as Ms. Reed names John as the next speaker, John starts his question. In both accounts, students orient to Ms. Reed's deontic authority with immediate compliance. That is, Anika stops her demonstration and John initiates and begins his question with full compliance with Ms. Reed's directives.

Sharing Epistemic Authority

Contrary to Ms. Reed's deontic authority, we found that Ms. Reed orients to Anika's epistemic authority, thereby sharing epistemic authority with Anika. Ms. Reed does so with particular kinds of actions (e.g., highlighting Anika's epistemic access, seeking confirmation). Some of these actions occur even before Anika walks to the front of the room. We illustrate this pre-work with Extract 2 below.

Extract 2: Constructing Anika as an Expert

```
023  ANIKA:    so I found the slope and then I (substituted it)
024            into it. (                        )
025  Ms.R:     okay so (1.0) we are gonna look at (.) your method.
026            (0.5)
027            Tanner? as you listen to Anika is that- does that
028            sound similar to (.) what approach you took?
029            (.)
030            she said y two minus y one::, (1.0) she did some
031            subtracting over x two minus x one,
032            (1.0)
033  TANNER:   (             )
034  Ms.R:     so can I- can I ask you:?
035            (0.5)
036            what is tha:t formula= where did you get that.
037            (1.0)
038  ANIKA:    u:m I learned it (        )=
039  Ms.R:     =okay
040  ANIKA:    during the summer.
```

After Anika described her approach to the problem, Ms. Reed says "we are gonna look at your method" (line 25). With her lexical choice of the possessive pronoun, "your", Ms. Reed orients to Anika's ownership of the method. This lexical pattern continues. As Ms. Reed asks Tanner to compare his ideas with Anika's (lines 27-28), she indicates that the method originated from Anika by pre-facing her clauses with "she said" and "she did" (line 30).

Another salient point in this extract is the way Ms. Reed makes Anika's exclusive epistemic access public by asking Anika about the source of her method (line 36). Although what Anika names as the source of knowledge is unrecoverable for the analysis (line 38), we can see, in contrast to deontic authority, Anika takes up her epistemic authority. Anika expands her turn by adding "during the summer" (line 40) despite Ms. Reed's earlier acknowledgment token, "okay" (line 39). Anika's expansion of her turn shows Anika's orientation to her exclusive epistemic access to the method since "the summer" refers to a time beyond the school year, which lies outside of the class's shared experience.

In these interactions Anika is constructed as a person with knowledge that the rest of class may not have. In the following extracts, we discuss instantiations of Ms. Reed's orientation to Anika's epistemic authority. Notably, some of the orientation is displayed with Ms. Reed's deontic authority.

Extract 3: Treating Anika with Epistemic Authority

```
JOHN:     u[:m  ] why is the uh seven uh y two and the fi- and
Ms.R:      [John]
JOHN:     the uh negative two x two.
          (1.5)
ANIKA:    u:m I guess it's just the way it's placed? so if this
          was the: first two coordinates and this was second
          two coordinates that-
JOHN:     (            ) closer to the x two uh (0.5) closer to
          y intercept is:  (1.0) for (0.5)
Ms.R:     I think she mea:ns that's the first point?
          (0.5)
JOHN:     a[::h      ]
Ms.R:      [and that]'s the second point=
JOHN:     =okay
Ms.R:     that's the x from the first one, and the y from the
          first one so that's how she said x one y one and
          that's the x: and y from the second one.
          (.)
          is that what you mean?
ANIKA:    yeah
```

When Ms. Reed has students ask questions, John poses a question (line 88). Anika, thus not Ms. Reed, answers John's question. The source of knowledge is Anika in this question and answer sequence. This contrasts with the deontic authority that Ms. Reed exercises by selecting John as the next speaker, as discussed earlier. Ms. Reed exercises her deontic authority to coordinate the activity of others and get the work of teaching done (Oyler, 1996). However, Ms. Reed tacitly acknowledges that Anika is the person from whom the flow of knowledge originates. For instance, Ms. Reed downgrades her epistemic stance as she explains Anika's method. In line 97, she pre-faces her statement with "I think" and finishes her turn with a rising intonation (noted as "?") despite her statement's declarative syntax.

Most notably, after revoicing Anika's method (lines 100-104), Ms. Reed seeks Anika's confirmation (line 106), to which Anika responds with a positive confirmation, "yeah" (line 107). This interaction marks a significant shift in how Ms. Reed overtly shares authority with Anika. By seeking confirmation, Ms. Reed once again downgrades her epistemic stance and signals to the class that she is not the expert over the information being clarified. With both Ms. Reed's confirmation-seeking and Anika's response, both of them treat Anika with epistemic authority.

Extract 4: Centering on What Anika Does

```
ANIKA:    so then (.) y is equal to negative two fifths
          ((Anika writes on the board))
          times (1.8) three (1.0) plus (       )
Ms.R:     k (.) stop (.) questions so far?
          (.)
          what is she doing?
          (2.0)
          we talked about how she found her slo:pe and that was
          similar to just the change in y over change in x that
          we done (0.7) Azad (.) what is she doing?
AZAD:     exploiting the x y coordinates: to (.) find b.
          (1.7)
Ms.R:     Molly?
MOLLY:    what's the three in parentheses?
Ms.R:     ask (0.7) Anika
MOLLY:    what's [the  th-]
ANIKA:           [it's  x.]
```

Ms. Reed continues to exercise deontic authority throughout the student demonstration. Once again, Ms. Reed uses a directive, “ask” (line 165). Prior to this, Molly poses a question (line 164). From the extract, it is not clear who the recipient of the question is. Nonetheless, Ms. Reed treats Molly's question as not directed to Anika, and she uses a directive to direct her question toward Anika (line 165). Molly, in turn, starts to repeat the same question (line 166), but Anika answers Molly's question even before Molly finishes her repetition (line 167). In other words, Anika projects Molly’s question to be identical to the question Molly asked earlier, and she acts as if the recipient of the original question was herself. This is another illustration of how Ms. Reed’s deontic authority manifests overtly. Ms. Reeds suspends the ongoing questioning and answering activity and makes the recipient of the question relevant to the degree that the identical question has to be repeated.

Although Ms. Reed’s use of deontic authority may seem pedantic, it plays an important role in the lens of epistemic authority. By having Molly direct her question to Anika, Ms. Reed orients to Anika with epistemic authority over the information being presented (i.e., Anika’s method). In a similar vein, Ms. Reed stops Anika’s demonstration and prompts the class to ask questions (line 154). Because no student responds (line 155), Ms. Reed asks, “what is she doing?” (line 156). Note Ms. Reed’s use of the pronoun “she”, which indicates Anika as the agent of the activity. This utterance implicitly orients to Anika’s epistemic authority since Anika has the primary right to describe what she is doing.

Ms. Reed then selects Azad as the next speaker. The interesting feature of this question and answer sequence is the absence of evaluation or feedback after Azad’s response. This contrasts with the common Initiate-Response-Evaluate (IRE) pattern (Mehan, 1969) and reinforces how Ms. Reed is sharing epistemic authority with Anika because Ms. Reed is tacitly deferring her epistemic authority that otherwise would have been used to confirm the correctness of Azad’s response.

We initially hypothesized Ms. Reed’s exclusive deontic authority and her deferring epistemic authority throughout Anika’s demonstration. Our fine-grained examination, however, revealed deviant cases of such a claim (i.e., moments of Ms. Reed’s asserting epistemic authority). In ethnomethodological studies, considering deviant cases is crucial to develop a more nuanced interpretation of the phenomena (Heritage, 1984). In the following, we present one of these deviant cases and further explore how Ms. Reed does, and does not, share epistemic authority.

Deviant Case: Not Sharing Epistemic Authority

Extract 5: Choral Chant

Ms.R: two divided by five (.) is what
(1.2)
S?: (point) (.) four
Ms.R: point four. (0.5) negative point four. (0.5)
<u>neg</u>ative two divided by five is,
(0.9)
STs: (negative point four). ((Choral Chant))
Ms.R: negative two point four because (.) negative divided
by a positive (.) is negative (0.8) two divided by a
negative five is,
STs: negative - [WOO]
Ms.R: negative [point four] because a positive
divided by a negative is
(1.1)
STs: a negative
Ms.R: so: (.) as long as you have one negative (0.8) you
can place it (.) <u>wherever</u> you like in your fraction
(.) because it has the very same value. (.) okay?

Ms. Reed facilitates choral chants through which she rhetorically constructs common knowledge within the classroom (Edwards & Mercer, 1987). The choral chant occurs after a student asks a question relating to the equivalence of the fractions (-2)/5, -(2/5), and 2/(-5). In contrast to Extract 4, in which Ms. Reed did not evaluate Azad's response, Ms. Reed confirms each response during the choral chant with her repetition (lines 239, 243, and 247 through 249). This indicates that Ms. Reed, not Anika, had epistemic authority during this interaction despite the fact that Anika is still at the front of the room. Further, in these IRE sequences, Ms. Reed justifies each response as indicated by her use of "because" (lines 239, 243, and 249). Her consistent justifications can be explained as her efforts to orient to the authority of mathematics as a discipline; yet, Ms. Reed remains to be the person who confirms the correctness.

Although we do not include the corresponding transcripts here, we also note other moments when Ms. Reed asserted her epistemic authority during Anika's demonstration. For instance, when one student asked if Anika's method is similar to what they have learned before, Ms. Reed offered confirmation, "exactly what it is," again with her account of how Anika's method relates to their prior learning. These deviant cases lead to a more nuanced understanding of how epistemic authority was shared during Anika's demonstration. Within the knowledge domain of Anika's method, Ms. Reed orients to Anika's epistemic authority. However, when the topic of discussion deviates from Anika's method (e.g., how her method connects to the class's prior learning, equivalence of fractions), Ms. Reed retains her epistemic authority to confirm the necessary knowledge for students to meaningfully engage with Anika's method.

Discussions and Implications

The push to develop student-centered classrooms should not imply the abdication of teacher authority (Oyler, 1996). This case demonstrates that teachers' deontic authority can aid in designating students' epistemic authority. Discussions about student-centered classrooms that misconstrue the role of teacher authority can paralyze teachers' efforts to facilitate productive mathematical discussions. Rather, there is a need to further examine how teachers can utilize deontic authority to share epistemic authority with students so that students can share and co-construct knowledge productively against a backdrop of institutional constraints such as curricular requirements and time limits. In professional development settings, this distinction can be helpful in designing a range of teacher moves that can be used to share epistemic authority (Herbel-Eisenmann et al., 2017).

This case also highlights the importance of how mathematics knowledge is introduced and framed by the teacher. Anika's method is quite conventional and is based on procedures that are often introduced in textbooks. However, Ms. Reed and the class treated her method as novel and, more interestingly, as a knowledge domain over which Anika had primary authority. This shows the situated nature of mathematical knowledge within a community in terms of authority. Although Ms. Reed might have known the conventional nature of Anika's method, Ms. Reed performed knowledge in ways that imbued Anika with primary epistemic authority. This had the effect of reconfiguring Anika's knowledge, converting it from a conventional method learned during the summer into an owned resource lying within Anika's epistemic domain. This was made possible through Ms. Reed's deontic authority, the exercise of which enabled her to demarcate lines along a terrain of mathematical knowledge. By setting boundaries around who owns what knowledge, Ms. Reed was able to portray Anika as someone who was not merely repeating a textbook method.

We also see potential for this study to contribute to discussions of equity in mathematics education research. As Byun (2019) discussed, teachers need authority to control the topic of classroom discussion so that students are steered toward different knowledge domains that can position minoritized students in more powerful positions. Without this deliberate reshaping of the epistemic terrain, participation patterns would likely continue to marginalize groups of students with particular social markers (e.g., race or gender). Although we did not attend to the racialized and gendered aspects of authority in classroom interactions in this study, we suggest further investigating the question of who teachers share epistemic authority with and its consequences for equity in mathematics classrooms. For instance, this case could be seen as an example of a female student of color who re-authors a conventional mathematical method through joint activity with her teacher, thereby challenging dominant conceptions about the authorship of mathematical ideas. Further work along these lines may offer valuable insights into equity issues in mathematics classrooms.

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HIGH SCHOOL MATHEMATICS TEACHERS' ORIENTATIONS TOWARD ENGAGEMENT

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The purpose of this research study is to characterize secondary teachers' orientations toward mathematics engagement. Results indicated that these 16 high school mathematics teachers tended to emphasize a cognitive dimension for engagement in their orientations, usually intertwined with an additional dimension (affective, social, or behavioral). Understanding teachers' thinking about engagement is a critical step toward helping teachers improve their practice to support their students' engagement in mathematics learning.

Keywords: High School Education; Teacher Beliefs; Instructional Vision; Affect, Emotion, Beliefs, and Attitudes.

In this study, we describe how high school mathematics teachers think about mathematics engagement. Teachers' efforts to engage students during their instruction are likely to be informed by their thinking about mathematics engagement. Understanding mathematics teachers' orientations toward engagement at the secondary level is particularly important because students' motivation and engagement has been found to decline over time as students move through levels of education. For instance, Chouinard and Roy (2008) found that students' self-efficacy, enjoyment, and sense of the utility of mathematics decreased as they move through middle school and into high school, and students became more disengaged over time in high school.

Students' motivation and engagement is malleable, socially situated, and influenced by teachers' instructional practices in the moment and by the classroom climate (Anderson, Hamilton, & Hattie, 2004). Teachers' instructional practices can impact students' motivation and engagement, and engagement is an important step on students' path toward learning mathematics. We conjecture that understanding high school teachers' orientations toward engagement is essential for supporting teachers to create secondary mathematics classrooms that disrupt declines in students' motivation and engagement.

Mathematics Engagement

Engagement in school manifests as students' expression of affect, beliefs about themselves, sense of belonging, and observable behaviors in the school setting (Jimerson, Campos, & Greif, 2003). Engagement is thus a complex meta-construct that simultaneously accounts for cognitive, affective, and behavioral dimensions (Fredricks, Blumenfeld, & Paris, 2004). Middleton, Jansen, & Goldin (2017) extended these dimensions to add a fourth with respect to mathematics learning: social engagement. Mathematics engagement is an interactive relationship between students and the subject matter, and it is manifested in the moment through expressions of behavior and experiences of emotion and cognitive activity; engagement is constructed through opportunities to do mathematics, as situated in both current and past experiences (c.f., Middleton, Jansen, & Goldin, 2017).

For students to learn mathematics, they must be engaged with experiences that support learning. In a study of almost 4,000 middle school and high school students in Western Pennsylvania, researchers

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found that higher levels of cognitive, behavioral, emotional, and social engagement predicted students' course grades in mathematics (Wang, Fredricks, Yea, Hofkens, & Linn, 2016). According to Greene (2015), it is well-established in prior research that motivation constructs such as students' self-efficacy support students' engagement in ways that lead to learning. However, it is possible that some teachers might speak about engagement in ways that are not always connected to learning, instead more connected to students' behaviors.

Teachers' thinking about students' mathematics engagement

Very few prior research studies have been conducted on secondary teachers' thinking about mathematics engagement, but some relatively recent research from Australia provides insights. Skilling, Bobis, Martin, Anderson, and Way (2016) conducted interviews with 31 secondary mathematics teachers from ten schools. Their results indicated that teachers in their study tended to describe students' engagement in terms of students' behavioral, affective, or emotional engagement; they spoke less often and less extensively about students' cognitive engagement. About one-third of these teachers reported an instrumental orientation, such that they strove to provide students with examples of how mathematics was a part of their lives outside of school. Some of these teachers also emphasized a relatedness dimension of engagement as they reported making efforts to build relationships with students to promote engagement. Their stance, which we also adopt, was that multi-dimensional orientations toward engagement would be more productive for teachers to hold.

Bobis, Way, Anderson, and Martin (2016) investigated changes in teachers' thinking about engagement, particularly among teachers who initially thought about engagement in terms of students' behavior primarily. After professional development, these teachers began to view engagement as more multi-faceted, beyond behavior management, and more than whether students were on-task. For the purposes of this study, we view behavioral engagement as the least productive dimension of engagement, because students could be on-task but not intellectually connecting with mathematics. We view cognitive engagement a potentially productive dimension, as it focuses is on students' mathematical thinking and learning.

Teachers' orientations

The term "orientation" is usually not defined explicitly in research literature on teaching and teacher education. Researchers' use of the term seems to imply that an orientation is a constellation of beliefs (e.g., Ambrose, 2004) or a set of perspectives and dispositions (Remillard & Bryans, 2004). It is particularly compelling to consider the root idea of "orienting," as these ways of thinking about teaching and learning can provide a direction for teachers' decision making. In this study, we define teachers' orientations toward mathematics engagement to be the set of teachers' beliefs about what it means for students to interact with mathematical tasks and each other productively during mathematics class, and together this set of beliefs provides direction for how teachers would enact instruction to engage their students.

By "beliefs," we mean what a teacher holds to be true. Beliefs are different from knowledge in that they are personal truths (Rokeach, 1968), and they have stronger, more affective components than knowledge (Nespor, 1987). Beliefs must be inferred by what a person says or does; they cannot be directly observed (Pajares, 1992). According to Rokeach (1968), ''All beliefs are predispositions to action'' (p. 113). Similarly, Aguirre and Speer (1999) explain that beliefs are "conceptions, personal ideologies, world views and values that shape practice and orient knowledge" (p. 328). Following Leatham (2006), we assume that teachers' beliefs are sensible to them, so we do not attempt to investigate whether teachers' actions appear consistent with their beliefs from a researchers' perspective.

Our stance toward teachers' orientations reflects that beliefs can cluster together within a system of beliefs. Green (1971) writes of beliefs having varying levels of psychological strength, some beliefs in a cluster are more central and others are more peripheral. Our investigation of teachers' orientations toward engagement targets the cluster of beliefs about the meaning of mathematics engagement, identifying which are more central in their belief clusters.

This study was guided by the following research question: *What are secondary teachers' orientations toward mathematics engagement?* We investigated dimensions of engagement that teachers reported when talking about engagement in interviews about their teaching practice.

Methods

This exploratory study was conducted during the second year of a three-year NSF-funded project designed to investigate engagement in high school mathematics classrooms. In Fall 2018 and Spring 2019, project team members interviewed 16 teachers in two states (one in the Southwestern region of the United States and one in the Mid-Atlantic region). Schools in these areas of the country use different curricular approaches: integrated mathematics in the Mid-Atlantic and topics-based courses in the Southwest. The three Mid-Atlantic schools implemented a block schedule with approximately 90-minute class periods. In the Southwest, the class periods were approximately 50 minutes long.

We gathered data from six schools (three from each state). In the Mid-Atlantic, the schools' demographics ranged from 9-30% low income, 24-57% white, 27-46% Black, 7-24% Latinx, and 5% or less Asian-American, Native American, or mixed-race students. In the Southwest, the schools' demographics ranged from 85-94% low income, 2-5% white, 1-15% Black, 74-96% Latinx, and 5% or less Asian-American, Native American, or mixed-race students.

Teachers were recruited for this study by soliciting nominations from district curriculum supervisors and mathematics coaches. The 16 participating teachers averaged 10.8 years of teaching experience, ranging from 1 to 27 years. Twelve teachers had earned a Master's degree. They self-identified their races as follows: 14 white, one Asian-American, one Hispanic/Latinx. They self-identified their genders as eleven female and five male.

Data Collection

Each teacher completed a baseline survey online at the start of each course. Survey items were open-ended, such as: *In your own words, what does "engaging students with mathematics" mean?* Interviews took place at the end of the semester in the Mid-Atlantic region, where schools had block scheduling, and it was at the end of the academic year in the Southwestern region. Interviews lasted from 35 minutes to about an hour and 15 minutes. Prior to the interviews, we video recorded classroom observations between two and four times per class period; observations targeted a lesson activity that the teacher conjectured would be likely to engage students. Interview questions included: *What are some of your favorite strategies you use to engage students? Why do you use these?* and *Can you tell me about a time when you have successfully engaged students with mathematics?* During the interview, we also asked teachers to elaborate on their definition for engagement on their baseline survey. Additionally, the interviews incorporated a video viewing session protocol; we showed teachers a video clip of their lesson, and we asked them to reflect on their students' engagement and their approaches to engaging them.

Data Analysis

The goal of our analysis was to describe the orientations each teacher used to conceptualize students' mathematics engagement. We compared descriptions of orientations to map a framework of ways in which mathematics teachers think about engagement along six dimensions identified in the engagement literature. Any of these dimensions could be either central or peripheral in a teacher's orientation, or in the constellation of beliefs the teacher held about mathematics engagement.

Each interview transcript had three sections, and we applied four levels of analysis to the transcripts. The interview's three sections were: (a) teachers' definitions for mathematics engagement, (b) teachers' strategies for engaging students, and (c) teachers' reflections on students' engagement during a classroom video episode. Interviews were transcribed prior to analysis. Each interview was coded independently by two researchers. All disagreements were resolved.

We operationally defined a teacher's orientation to be composed of dimensions of engagement that they reported in the three sections of the interview. Each transcript was analyzed at four levels: (1) we applied descriptive coding techniques (Saldaña, 2013) to identify the dimension(s) of mathematics engagement to which each teacher was oriented [see Table 1].

Table 1: Indicators for Dimensions of Mathematics Engagement

Dimension of Engagement	Indicators of Engagement that Teachers Described
Affective	Students' emotional responses, interest, attitudes, and expressions of values.
Cognitive	The process of students coming to understand, learn, and make sense of mathematics.
Social	Students interact with one another for the purpose of learning mathematics.
Behavioral	Observable actions of students, including whether or not they were on task.
Relatedness	Enactments of interpersonal care or personal connections between the teacher and students and among students.
Instrumentality	Students see mathematics as useful and relevant to their lives.

(2) We analytically identified the centrality of the dimensions in each interview section based on the teacher's use of repetition, level of detail, and emphasis terms. Central dimensions had two of these three features (repetition, detail, or emphasis). Peripheral dimensions did not. (3) We analytically determined the degree to which a dimension was central to a teacher's orientation by identifying whether the dimension was central to the teacher across more than one section of the interview. (4) Finally, we applied axial coding across each teacher's interview (Saldaña, 2013) and compared teachers' central dimensions to identify categories of orientations.

Results

In Table 2, we summarize the central and peripheral dimensions of mathematics engagement reported by these teachers at the case-level, or across the interview for each teacher. Central dimensions are labeled with a shaded 1 and peripheral with an unshaded 2. Zero indicates no evidence of this dimension in teachers' responses.

Table 2: Teachers' Orientations toward Mathematics Engagement

Teacher	Cognitive	Affect	Social	Behavioral	Instrumentality	Relatedness
Elise	1	1	1	1	0	0
Ken	1	1	1	2	2	1
Chloe	1	1	2	1	2	0
Addie	1	1	2	2	2	0
Jessica	1	1	2	2	0	0

Nancy	1	1	0	2	1	1
Jimena	1	2	1	2	2	2
Julie	1	2	1	2	0	2
Nicole	1	2	2	1	2	2
Peter	1	2	2	1	0	0
Colton	1	2	2	2	1	2
Tori	1	2	2	2	1	0
James	1	2	2	2	0	2
Rachel	1	2	2	2	0	0
Anne	2	1	2	2	1	2
Craig	2	2	0	1	2	0

All of these secondary mathematics teachers expressed a multi-faceted orientation toward mathematics engagement--they all spoke about engagement with at least four dimensions, either at the central or peripheral level. Table 3 (below) shows four prevalent orientations that teachers reported.

Table 3: Four Prevalent Orientations Toward Mathematics Engagement among Secondary Teachers

Orientations toward mathematics engagement	Summary of each orientation
Cognitive-Affective	Support students' learning of mathematics while cultivating enjoyment, interest, and a desire to learn mathematics.
Cognitive-Social	Learning mathematics is a process of coming to know mathematics through discourse.
Cognitive-Behavioral	For students to be actively involved in mathematics learning, teachers must manage students' behavior.
Cognitive-Instrumental	Opportunities to learn are enhanced by connections between school mathematics and students' lives.

All but two of these teachers (87.5%) reported a central cognitive dimension in their orientation such that engagement involves students thinking mathematically or making sense of mathematics. This is important because previous research indicates that mathematics teachers tend to think about engagement in terms of its behavioral dimension primarily, unless they had had support from professional development (e.g., Bobis, Way, Anderson, & Martin, 2016; Skilling, Bobis, Martin, Anderson, & Way, 2016). Our findings illustrate counter evidence, indicating that secondary teachers can strongly consider learning processes when they talk about mathematics engagement.

Teachers consistently paired statements about cognitive engagement with statements about one or more other dimension. Affective, social, and behavioral engagement, and instrumentality were viewed as critical contributors to students' efforts to grapple with and understand important mathematical concepts and procedures. Rather than speaking about goals for cognitive engagement and then speaking about other facets separately, teachers' responses concurrently emphasized the role of affect, social or behavioral dimensions in supporting cognitive engagement. In particular, when speaking about strategies and practices that contribute to students' math engagement, these facets were reasoned as important for helping students engage in learning the mathematics productively.

Relatedness and instrumentality were the dimensions reported least by these teachers, but these dimensions were still a part of most of these teachers' orientations. At least half of the studied teachers mentioned relatedness and instrumentality in at least one of the sections of the interview at some level (either central or peripheral). It is interesting to note that these dimensions displayed the most variability in teachers' responses. Half of the teachers in the sample did not mention relatedness as somewhat central to their conceptions of engagement, and six did not mention instrumentality.

Cognitive-Affective Orientation toward Mathematics Engagement

Six secondary teachers reported a cognitive-affective orientation toward mathematics engagement (Jessica, Addie, Chloe, Elise, Ken, Nancy). They said that students are engaged when they invest their thinking in order to learn (cognitive) and their investment will increase if they enjoy the experience and feel a desire to learn (affective). Addie wrote that engagement is "Where students are excited to learn the beautiful world of mathematics." (baseline survey) "I think it's taking where students are at, because that's where they're comfortable, and then expanding their knowledge in different ways that aren't necessarily lecture based in order to get students really interested in mathematics." (22-25, interview). When Addie spoke about engagement in terms of learning (cognitive), she also wrote about affective experiences of excitement, beauty, comfort, and interest. Nancy said, "I feel like engagement is having them be actually, like, cognitively thinking about the mathematics that are happening and not just copying down the notes… it's about having them actually think about it… having, like, some sort of level of fascination or even just curiosity, or seeing a goal with it. Just, kind of, finding a purpose in it." (39-46, interview). Nancy valued fostering curiosity (affective) so that students engage in deep mathematical thinking (cognitive). The teachers spoke about productive opportunities for students to understand mathematics in ways that also provided powerful opportunities for students to develop strong relationships with the discipline of mathematics.

Cognitive-Social Orientation toward Student Engagement

Four teachers (Julie, Jimena, Elise, and Ken) reported a cognitive-social orientation toward engagement. They described engagement as investing thinking in order to learn (cognitive) through a process of coming to understand mathematics through discourse (social). (We note that some teachers, such as Elise and Ken, were examples of more than one of these four orientations, because their orientations contained more than two dimensions.)

Julie's baseline assessment provided a concise example of a cognitive-social orientation. She wrote, "Engaging students with mathematics means finding ways for students to think about and discuss mathematics in a way that deepens their understanding." Thinking (cognitive) and discussing (social) were integral to Julie's view of engagement.

These teachers tended to characterize cognitive engagement as a process of grappling with mathematics. For instance, Elise said, "I want them to say, 'What am I doing that's not making sense?' Or, 'What pieces could I be missing that are not connecting?' I want them to ... if they find an answer, interpret that answer. Is it a useful answer? Does it answer the problem you're trying to figure out? Does the answer make sense?" Similarly, Julie said, "So I like to ... from time to time, after we've done a concept, to kind of pose a question that forces them to really, first of all, think on their own. Can they generate their own thought? But then to have those discussions with their peers to see, 'Well, what do you think about that? I didn't think about it. How can we maybe expand on each other's ideas to see different ways of viewing the same kind of problem?' " (27 – 39, interview) These teachers talked about learning and understanding in ways that involve making sense, interpreting their work, and wrestling with concepts as they talk them through with peers.

Cognitive-Behavioral Orientation to Students' Engagement

A cognitive-behavioral orientation toward mathematics manifested in four teachers' talk (Nicole, Peter, Chloe, and Elise). They reported that they wanted students to be actively involved in the learning process (cognitive), but they also reported having to manage behavior so this would happen (behavioral). On his baseline survey, Peter wrote, "Engaging students means that all students are working on what are supposed to. All students are actively participating in their own learning, with no exceptions. Engaged means 'doing.'" Elise spoke about engagement challenges as being about the range of ways students engage behaviorally and cognitively.

> You have different levels of engagement. You have the kid that hasn't even attempted to pick up a pencil… the kid that looks like he's listening or she's listening, but hasn't even read the question or hasn't tried to understand the directions and the task. And then you have the kids like, 'Oh, I got an answer. I'm done.' … It's such a big spectrum of engagement and lack of engagement that you try to address every day. (50-59, interview)

Engaging their students cognitively and behaviorally was reported by these teachers as something they constantly worked to accomplish. For instance, Nicole talked about cold calling (behavior management) as a way to engage students to think about mathematics (cognitive engagement), as she said,

> I kind of force them to be a little bit more engaged for the Popsicle sticks. And then also, if they didn't know the answer, they had to listen to somebody else, and then they had to repeat it back. Like, [student] didn't know what to do, so somebody else gave the answer. And then, I made [student] repeat it so that he was listening, at least. I don't know if that's considered engagement, because to me, he's just listening, and he's just repeating. But, at least it's trying to get them to think. If I could see the rest of the class, I believe most of them ... No, probably 50% of them were actually engaged, because I'm hearing talking in the background. I don't know where that came from. (370-380, interview)

The teachers who intertwined behavioral engagement with cognitive engagement spoke about using classroom management practices to bring about productive behavior in hopes that it would lead to stronger intellectual investment among students.

These teachers spoke about the cognitive dimension of learning in ways that appeared to be more closely aligned with procedural fluency than conceptual understanding. When teachers articulated a cognitive-behavioral orientation, there was a focus on getting answers over sense making, modeling procedures through lecture, and guided practice of steps to solve a task. This perspective on cognitive engagement contrasted with teachers who reported a cognitive-affective, cognitive-social, or cognitive-instrumental orientation, which illustrated a focus on interpreting and understanding mathematics.

Cognitive-Instrumental

A cognitive-instrumental orientation toward engagement was illustrated by three teachers (Tori, Colton, and Nancy) who talked about opportunities to understand (cognitive) being enhanced by connections between school mathematics and students' lives (instrumental). Tori reported the following on her baseline assessment: "For me, engaging students with mathematics means using the real-world information to understand the concepts in mathematics, and hopefully apply what the students have learned to their personal lives and become lifelong learners of mathematics." Colton described engagement on his baseline survey as "Giving them something more to connect with." Nancy reported the power of connecting mathematics and students' lives. When discussing functional relationships, such as whether the relationship between time and location is a function, she said, "I think that part of the reason why it was so engaging is because some of those things allow them to challenge math. … I think it's cool when they can make that, like, real-world connection."

These teachers could leverage opportunities to connect with mathematics (instrumental) in ways that enhanced students' opportunities to understand mathematics (cognitive).

Discussion

We conjecture that quality mathematics instruction can be best supported when teachers go beyond a focus on behavioral engagement in their orientations. The secondary teachers in our study emphasized cognitive engagement primarily, with other dimensions serving as supports and influential catalysts for helping students engage cognitively. If teachers tend to prioritize cognitive engagement in their orientations, and if they hold multiple dimensions of engagement in their orientations, they are likely to have productive resources and strategies they can call upon to support their students.

Future research could investigate whether teachers' instructional practice varies depending on their orientations toward mathematics engagement. It is possible that a teacher who holds multiple dimensions toward mathematics engagement in their orientations is more flexible in their approach to engaging students. Alternatively, it may also be possible that teachers who hold only two dimensions – such as cognitive-affective, cognitive-social, or cognitive-instrumental – could effectively engage their students in mathematics learning. The necessary and sufficient conditions for improving student cognitive engagement through integration of two or more dimensions is an open question.

To support teachers in developing their orientations, we propose two goals for teachers' learning about mathematics engagement: (1) Teachers can strive to more fully enact their orientations toward mathematics engagement in their teaching practice; and (2) Teachers can work to enhance additional dimensions toward mathematics engagement in their orientations. Teachers' orientations reveal some insight about their instructional vision for engaging students with mathematics (Munter, 2014). With appropriate coaching support or collaborative inquiry with teachers who hold similar orientations, teachers may be able to approach enacting teaching in ways that more closely align with their orientations. Alternatively, teachers could learn to take up instructional strategies aligned with additional dimensions of engagement to further develop their practice.

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SUPPORTING ENGLISH LEARNER STUDENTS' DEVELOPMENT OF THE MATHEMATICS REGISTER THROUGH AN INSTRUCTIONAL INTERVENTION

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We describe an instructional intervention designed to help teachers engage English learner (EL) students in mathematical problem solving and learn the mathematics register. The "Discursive Assessment Protocol" (DAP) integrates Pólya's classic problem solving framework with research-based instructional strategies that benefit EL students. The research-based instructional strategies are grounded in theories of academic language development. A sample problem-solving episode is provided that demonstrates how an EL student wrote a "multiplication story" involving fractions and what we learned from using the DAP to support him and other EL students develop the mathematics register in English. Among the implications of this study is the value of selecting tasks that are not only worthwhile mathematically, but worthwhile in that they have potential to develop students' mathematics register.

Keywords: Equity and Diversity, Marginalized communities, Problem Solving.

Schools are struggling to meet the needs of English Learners (ELs) in the United States (Borjian, 2008; Valenzuela, 2005). ELs largely enter U.S. schools performing below English Proficient (EP) students in core academic subjects (Abedi & Gándara, 2006) and dropout rates for ELs are considerably higher than EP students (Borjian, 2008; Kanno & Cromley, 2013). Schools experiencing an influx of EL students must adjust to meet these students' educational needs (Barrio, 2017; Irizarry, 2011). In this paper, we describe an instructional intervention designed to help teachers support the mathematical learning of their EL students and how it informed instruction during a problem-solving episode. The "Discursive Assessment Protocol" (DAP) integrates Pólya's (1945/1986) classic problem solving framework with research-based instructional strategies that benefit ELs. The research-based instructional strategies are grounded in theories of academic language development that afford EL students repeated and consistent opportunities to express their mathematical ideas and negotiate meaning with others (Moschkovich, 2013; 2015). The integrated design of the DAP is intended to guide teachers to provide their students with needed supports to learn and use the mathematics register during problem solving episodes.

Since the fall of 2019, we have been working with "Ms. Ware," a 5th grade teacher in an urban school district. A goal of this work has been to examine the DAP as an instructional intervention in Ms. Ware's mathematics classes to understand how effectively it guides her to elicit students' mathematical reasoning and develop their use of the mathematics register in English during problem-solving episodes. In all of her classes, Ms. Ware teaches a high percentage of ELs. She is fluent in Spanish and is devoted to providing her EL students with a high quality education in mathematics. The research question we explore here is: How does the DAP used during problem-solving episodes inform how to support EL students to develop the mathematics register in English?

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Background on the Discursive Assessment Protocol

Early iterations of the DAP built upon and were an extension of a clinical interview protocol (see Kitchen & Wilson, 2004; Kulm, Wilson, & Kitchen, 2005). The DAP was designed for use with individual ELs or groups of EL students during mathematics problem-solving episodes, but can also be used with the general student population. During piloting of the DAP with middle school students over the course of two years (2007-09), the DAP guided teachers to provide students with opportunities to ask questions, to be creative, to test and revise their solution strategies, and to explore mathematical ideas deeply (see Kitchen, Burr, & Castellón, 2010; Castellón, Burr, & Kitchen, 2011). Such instruction is a clear departure from instruction historically found in schools that serve high percentages of low-income EL students in which the memorization of math facts, algorithms, vocabulary, and procedures are the focal point of instruction (Kitchen, DePree, Celedón-Pattichis, & Brinkerhoff, 2007; Moschkovich, 2013). Moreover, the DAP is intended to help teachers provide students with opportunities to make sense of the language demands of mathematical problems as well as to provide scaffolded supports for EL students to engage in mathematical discourse to explain their ideas and to listen to and make sense of the ideas of others. As students engage in mathematical discourse, they build on their prior experiences and knowledge to achieve more advanced understandings of mathematical concepts (Ryve, 2011).

Incorporating Pólya's (1945/1986) four-stage problem solving framework, the DAP is designed to be administered during problem-solving episodes involving worthwhile mathematics tasks, ensuring that students have something to talk about (Silver & Smith, 1996). In the example that we provide here, we used a performance assessment task (referred to simply as "task" throughout) that is publicly accessible for free through the Illustrative Mathematics (IM) project. Rich tasks such as performance assessment tasks "engage students in thinking and reasoning about important mathematical ideas" (Franke, Kazemi, & Battey, 2007, p. 234). Though the use of such tasks does not guarantee high-level student responses, cognitively demanding tasks provide the means for teachers to engage students in mathematical discourse in which students are actively sharing their thinking, comparing their solution strategies, making conjectures, and generalizing (Silver & Smith, 1996).

An important goal of instruction for EL students should be amplifying rather than complexifying English language speech or text (Zwiers et al., 2017). This entails providing students with multiple opportunities to understand mathematical ideas and terms by providing support for learning with concrete materials such as manipulatives and mathematical models, engaging students in think-alouds, and using culturally relevant and authentic contexts. ELs need repeated opportunities to understand the problem at hand, not only because English is a second language, but because the learning of mathematics is embedded within the linguistic patterns of academic language development. Academic language has been defined as the linguistic expectations of students to learn, speak, read and write about academic subjects such as mathematics (Schleppegrell, 2004). Discipline-specific registers can further refine academic language. Described as words, expressions, and meanings specific to mathematics (Secada, 1992), the mathematics register is the disciplinary-specific reading, writing, listening and speaking norms of content teaching and learning that is more complex than everyday English. It is helpful to think of the academic register as a series of resources that promote meaning making, or a set of linguistic features, such as words, symbols, and forms (Schleppegrell, 2004). A unique feature of the DAP is that instructional strategies designed to support the development of EL students' mathematics register, also referred to as English as a Second Language (ESL) instructional strategies, are incorporated throughout its four stages, such as acknowledging and using gestures, integrating cognates, revoicing, and incorporating graphic organizers (see Figure 1 below).

1. Understand the problem	
Check for understanding	- Students underline important information in problem - Teachers ask: "What is the problem asking you to do? What do you know that can help you figure this out?" - Students define new words and begin using them in sentences.
Teacher deliberately incorporates ESL strategies	- Students use a picture, diagram, or some type of mathematical representation to concretely model problem.
Teacher maintains high expectations and recognizes students' intellectual assets	- Teachers look for opportunities to highlight students' mathematical ideas with other students.
2. Create a plan to solve the problem	
Students create plan to solve problem	- Teachers ask: "What strategy, representation or tool will work best to solve the problem?" - Teachers assess student understanding of their plan.
Teacher deliberately incorporates ESL strategies	- Teachers integrate graphic organizers and mathematical models during small group instruction and discourse.
3. Carry out the plan to solve the problem	
Teacher engages students in mathematical discourse and meaning making	- Teachers engage whole class in mathematical discourse and asks questions while highlighting student work. - Teachers integrate the mathematics register in discourse and instruction.
Teacher continues to use deliberate ESL strategies	- Teachers use gestures, cognates, revoicing, graphic organizers and mathematical models.
Students refine and revise their solutions	- Teachers do not need to be overly concerned in this stage about students' production of "correct" English.
4. Looking back	
Students reflect on their solutions	- Teachers ask: "Does your solution make sense? How do you know? What questions do you still have at this point?"
Teacher works to help students use the formalized mathematics register	- Students write up their final solution to the problem using the mathematics register.

Figure 1: The Discursive Assessment Protocol (DAP)

The first stage of the DAP involves understanding the task/problem (Pólya, 1945/1986). In this stage, Pólya advocates that students consider a picture, diagram, or some type of mathematical model that could be helpful for solving the problem. Modeling helps EL students learn the mathematics register by touching the objects that represent mathematical ideas and repeatedly hearing and then repeating the words represented by these objects (Garrison & Mora, 2005). After developing a mathematical model to make sense of the problem, students share their ideas with peers to solicit feedback and modify their models. To ensure ELs have repeated opportunities to understand the problem at hand, teachers ask questions during this stage such as "What is the problem asking you to do?" and "How are you going to figure this out?" In this stage, teachers need to define words used in the problem under consideration. In the second stage, students devise a plan to solve the problem (Pólya, 1945/1986). Working in small groups, students share their ideas with peers and their teacher to get feedback on their solution strategies. This process ensures that students have opportunities to reflect upon their problem-solving strategy to determine whether the strategy makes sense. Students need support in this stage to develop self-regulation strategies such as devoting significant time to analyze and plan how to attack the problem similar to accomplished problem solvers (Schoenfeld, 1985).

In the third stage, students carry out their plan to solve the problem (Pólya, 1945/1986). Students have opportunities to share their mathematical thinking with peers and their teacher through mathematical discourse. When possible, the teacher seeks to integrate the mathematics register in discourse and instruction, though does not need to be overly concerned with students using "correct" English. A key in this stage is that the teacher asks meaningful questions and actively works to highlight and build on students' ideas to support students reflecting on their mathematical thinking and errors (Schoenfeld 1985). Instruction should leverage ELs' knowledge in their first language as a means to help them comprehend a second language (Cummins, 2000). To support ELs in particular, the teacher "re-voices" students' explanations, references students' mathematical ideas, and asks clarifying questions. In this manner, the DAP functions as a formative assessment tool, supporting teachers to examine, understand, and leverage students' mathematical ideas and thinking as a means to inform their instruction (Kitchen, 2014). In the fourth stage, students look back at their solutions and check their results (Pólya, 1945/1986). In this stage, the teacher asks: "Does your solution make sense? How do you know? What questions do you still have at this point?" In addition to reviewing and checking their answers, ELs need opportunities to explain their ideas using the mathematics register. In this stage, students explain their problem solutions in writing with the expectation that they will include the mathematics register in their write ups. Having had time to think about, solve and revise their solutions also means students' anxiety level, the affective filter (Krashen, 2009), has been lowered and ELs may have more confidence explaining their ideas in writing.

Research Methodology

Starting in the fall of 2019, our research team (Richard, Libni and Karla) has been collaborating with Ms. Ware to implement the DAP during problem-solving episodes with her two 5th grade mathematics classes. Both of these classes have a high percentage of ELs (20% or more). To date, we have co-taught with Ms. Ware during problem-solving episodes on four occasions. We employed a team teaching approach to co-teach in which instruction was divided up among the four of us (Cook & Friend, 1995; Sileo & van Garderen, 2010). Each problem-solving episode typically lasted between 40 and 60 minutes and involved students solving a performance assessment task. During each episode, we worked to follow the four stages of the DAP as students solved a given task. Primarily in the second and third stages of DAP implementation, all three members of the research team circulated throughout the classroom with Ms. Ware, asking individual and groups of students questions engaging in discourse. Prior to each problem-solving episode, we planned how we hoped to co-teach during the episode, identified questions to ask, and discussed English words and phrases that we hoped to develop during instruction to support EL students' emerging mathematics register. During the problem-solving episodes, we videotaped Ms. Ware and students who had provided consent. At the conclusion of these episodes, we collected all the work students had created.

To illuminate how the DAP informs instruction, a sample student solution to a performance assessment task is provided. Specifically, we highlight how one EL student, "Fernando," solved a given task and how his solution informed us vis-à-vis how to support Fernando and other EL students to more fluently construct English sentences in the "multiplication stories" that they devised. The data used in the example provided came from copies of collected student work samples and from videotapes made during the problem-solving episode. Student work samples and videotapes were interpreted using interpretative methods (Creswell, 2014). The student work samples were initially read or viewed as a whole, followed by a period of open coding to reflect upon and clarify how students were solving a given task and how they used the mathematics register to express their solutions. An iterative process of coding, reflecting upon, and then clarifying what we learned from reviewing student work samples then took place (Miles, Huberman & Saldana, 2013). This process went through multiple revisions as the data were repeatedly read and reviewed to check the consistency of findings. This process continued until either no new categories were developed or consistency was achieved. After we established how to characterize students' solutions to tasks posed, we searched for commonalities and differences across these solutions to further examine how the DAP could be used to inform instruction intended to support the development of EL students' mathematics register. We went through a similar process when reviewing videotapes of problem-solving episodes.

Research Findings

We now offer an example of how through the use of the DAP, we gained insight into how to support the development of an EL student's mathematics register in English during a problem-solving episode. Immediately prior to the problem-solving episode, Karla and Libni led a brief lesson on fractions to Ms. Ware's students. They had students identify unit fractions in both columns and rows of a rectangular whole similar to the rectangular whole displayed in Figure 2. In several exercises, students identified equivalent fractions in diagrams given to them such as 1/5 and 2/10. Karla and Libni also had students identify the fraction created when two of these fractions overlapped, something Ms. Ware had been doing with her students for at least a week prior to this lesson. In addition, students began identifying an equation that could be derived through fraction multiplication. The purpose was to emphasize the meaning of fractions as operator (e.g., 1/4 of 1/3). Following this brief lesson, the four stages of the DAP were administered during implementation of the following IM task shown in Figure 2.

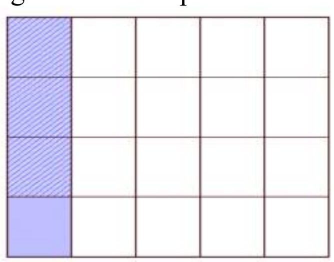
The diagram below represents one whole. 	Problem: Write a multiplication story that could be solved using this diagram with its two types of shading. Explain how your story context relates to the diagram provided (http://tasks.illustrativemathematics.org/content-standards/5/NF/B/4/tasks/2075).

Figure 2: Task Implemented with the DAP

Ms. Ware introduced the task by asking three different students to read the problem out loud to the whole class and give brief explanations about what the problem was asking as a means to check for understanding (first stage of the DAP: Understand the problem). In addition, as planned prior to the lesson, Ms. Ware began asking questions we had collaboratively identified such as *What is the problem asking you to do? What is the whole? What is a multiplication story?* She also checked for understanding of the terms "two types of shading" and "relates." Once Ms. Ware was satisfied that her students, for the most part, understood the task because a number of them could express what the task was asking of them, she moved on to the second stage of the DAP.

Initially working alone, students started devising a plan to solve the task (second stage of the DAP: Create a plan to solve the problem). It was in this stage that Fernando devised a multiplication story involving videogames. In his written solution, he started by making sense of the diagram given in Figure 2; he identified the whole, circling the entire diagram and writing "The Whole." He also identified both of the fractions represented in the diagram (3/4 and 1/5). Lastly, Fernando wrote the following expression that he believed was represented in the diagram: 3/4 x 1/5.

In his write-up (stage 3 of DAP, Carry out plan), Fernando created a story that was mathematically sophisticated involving videogames:

"Sam got to play videogames 1/5 of an hour. After he did his homework he got to play 3/4 of the 1/5 that he played. How much does he play after homework?"

During the third stage of the DAP, teachers engage the whole class in mathematical discourse and ask questions while highlighting student work. Fernando shared his story in his small group. At one point, he also responded to another student's story during a whole class discussion that took place. The language that he used in the context of his story is specific to the operator concept of fraction (e.g., *"he got to play 3/4 of the 1/5")*. The operator subconstruct has two different interpretations, as stretcher/shrinker and as a duplicator/partition-reducer. The difference between the two is that with stretcher/shrinker, the transformation of the fraction results in the same number of units of different size (e.g. 3/4 should be interpreted as 3 x [1/4 of a unit]), and with the duplicator/partition-reducer the fraction result elicits a different number of units of the same size (e.g. 1/4 x [3 units]) (Charalambous & PittaPantazi, 2007). The operator subconstruct can also be considered a function, a set of operations that need to be done to get a result (Lamon, 2007). In this case, Fernando used the duplicator/partition-reducer interpretation.

While Fernando's story is mathematically sophisticated, the clarity of the story could be improved in at least two ways. First, he could modify the second sentence to read, "*After he did his homework, he got to play 3/4 of the 1/5* of an hour *that he had already played*." The inclusion of the phrase "of an hour" in Fernando's sentence clarifies the amount of time that he originally played videogames. Another option is to modify the sentence to read, "*After he did his homework, he got to play 3/4 of the* amount of time he had already played." Secondly, in *"How much does he play after homework?,"* it is unclear whether Fernando is asking for a unit of time (e.g., hours, minutes) or possibly some number of videogames. To clarify, the question posed could be modified to reference a unit of time. For example, the question could be *"How much* time *does he play after homework*?" or *"How* many hours *does he play after homework*?" These potential modifications are examples of sentence frames (Wisconsin Center for Education Research (WCER) (2014). Fernando's story informed us about how sentence frames such as "How __ of __ of an hour" or "How many hours" could have helped him and other students to tell their stories using fluent sentences that included details (Coleman & Goldenberg, 2009). As we progressed through the four stages of the DAP, we came to recognize the complex language needed to develop a multiplication story. Rather than simply devising questions and addressing keywords in this problem-solving episode, Fernando and other EL students would have benefitted from being given sentence frames that they could have applied directly in their stories.

Discussion and Implications

In this paper, we described what we learned from using the DAP during a problem-solving episode to support Fernando and other EL students to develop their use of the mathematics register in English. Fernando created a multiplication story in response to a task that demonstrates his mathematical understanding of the part-whole notion of fractions as well as the concept of fraction as operator (Charalambous & Pitta-Pantazi, 2007; Lamon, 2007). While Fernando's story is mathematically sophisticated, it was also the case that the story could be improved with the addition of a few key phrases. We observed this during the third stage of the DAP when students were presenting their stories to peers and the entire class.

This example demonstrates how the DAP can be a helpful tool to inform instruction about how to support EL students with the linguistic expectations associated with writing mathematics related stories. Specifically, in this case, how the introduction of words and expressions through the use of sentence frames could support the development of students' English language fluency in the domain of mathematics. Undoubtedly, the demands of writing a multiplication story are linguistically complex (Martiniello, 2008). To address this complexity, an implication for instruction is how through the explicit use of sentence frames (WCER, 2014), we could have helped EL students such as Fernando address this complexity by providing them at the initiation of the problem-solving episode with expressions such as "how much time" and "how many hours" that they could have used in their stories.

After observing Fernando's response and other students' responses to the task, we noted the importance of not only identifying potential questions and key phrases and words needed to support EL students during task implementation, but the value as well of identifying potential language supports for students such as sentence frames that students could have used in their stories. The use of the DAP helped us gain this insight. In addition to providing guidance on how to integrate ESL strategies as students worked through Pólya's (1945/1986) four stages of problem solving, the use of the DAP informed us about how to support students' burgeoning mathematics register to construct fluent and detailed sentences involving time.

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DISENTANGLING THE ROLE OF CONTEXT AND COMMUNITY IN TEACHER PROFESSIONAL DEVELOPMENT UPTAKE

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The study presented here utilized a cross case comparison of three different professional development programs to examine the contextual factors associated with uptake related to what teachers learned related to content, pedagogy and the resources used in their professional development (PD) workshops. From a theoretical perspective this study draws on a situative perspective to guide our analyses on how uptake across different PD projects impacted teacher learning. Findings indicate that teachers' perceptions of learning may be associated with explicit goals and intentions of the PD program, their perceptions of community and the relevance of the content, pedagogy and resources provided to their everyday work in mathematics classrooms. Differences were found to be related to where they fell on the adaptive-specified continuum.

Keywords: Research Methods, Professional Development, Teacher Knowledge, Teaching Tools And Resources

One central challenge for the field of teacher professional development (PD) is how to design interventions that target teacher knowledge, while also maintaining a focus on instructional practice and student learning (Jacobs, Koellner, Seago, Garnier & Wang, 2020). A number of researchers have worked to address this challenge and there is now a strong research base delineating critical design features of effective PD (e.g., Borko, Jacobs & Koellner, 2010). The consensus in the current PD discourse about features of effective PD include a focus on mathematics content, student learning of content, active learning opportunities for teachers, coherence, duration, and collective participation (Sztajn, Borko, & Smith, 2017). Although some PD programs that adhere to design recommendations by the literature have produced encouraging results (e.g. Franke, Carpenter, Levi & Fennema, 2001), others have proven much less successful (e.g. Jacob, Hill & Corey, 2017). We believe that context, as well as the nature of the PD on the adaptive-specified continuum, might be key to helping us understand and uncover impact aspects related to how teachers perceive their learning of content and pedagogy as well as their use of resources.

At present, very little is known about the degree to which context impacts teachers' learning from PD. The one area that researchers have focused on and have found some evidence of how context plays a role in teacher learning is social and political contexts of schools and their impact on the implementation and effectiveness of mathematics of PD.

The study reported here goes beyond social and political aspects that impact PD and includes multiple contextual factors related to what teachers take up and implement after participating in a particular PD. The study uses comparative case study analysis to examine three different and distinct professional development programs that are geographically situated across the US, focused on different mathematical content, and different PD structures. We aim to disentangle the role that context plays in uptake of PD content, pedagogy and resources of these three ambitious PD projects by analyzing teachers' perceived uptake in these areas.

Theoretical Frameworks

Situative theorists define learning as changes in participation in socially organized activity (Greeno, Collins, & Resnick, 1996). They consider the acquisition and use of knowledge as aspects of an individual's participation in social practices. With respect to professional learning, situative theorists

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focus on the importance of creating opportunities for teachers to work together on improving their practice and locating these learning opportunities in the everyday practice of teaching (Ball & Cohen, 1999). All three PDs were designed around this premise. A situative perspective suggests that groups of teachers who take part in different PD workshops using different materials, with different facilitators, and are situated within different educational contexts (e.g., different geographical locations within the United States) might have very different learning opportunities and experiences impacted by the role of context.

PD Model Continuum: Adaptive Through Specified

PD models fall on a continuum from adaptive to specified (Borko, Koellner, Jacobs & Seago, 2011). On one end of the continuum are *adaptive* models, in which the learning goals and resources are derived from the local context and shared artefacts are generally from the classrooms of the participating teachers. In these models, the artefact is selected and sequenced by the facilitator and/or the participating teachers, and the related activities are based on general guidelines that take into account the perceived needs and interests of the group. On the other end of the continuum, specified models of PD typically incorporate published materials that specify in advance teacher learning goals. In video-based specified PD, the video clips are typically pre-selected and come from other teachers' classrooms.

The nature of what teachers take up and use across the continuum has the potential to shed light on factors that are associated with the teacher learning related to content and pedagogy. This study examines three professional developments that fall on different parts of the continuum. The goal is not to determine which types of PD are "best" because each has its affordances and challenges, but rather to better understand the variance of teacher uptake and use within and across these PD experiences. Understanding and deeply analyzing and unpacking variance among and between types of PD offers the potential to identify the factors that impact uptake and use from PD. This paper examines how teachers' self-reported uptake differs across PDs located at different points on the adaptive-specified continuum. Specifically, one is highly adaptive, one is highly specified, and one lands in the middle. We believe conducting a cross case comparison will aid in helping us understand the factors associated with uptake related to content, pedagogy and resources.

Oerview of TaDD Project

This three-year impact study, Taking a Deep Dive (TaDD) is collecting qualitative data from three large U.S. National Science Foundation PD projects in order to use case studies and cross case analysis to further inform what teachers take up and use in different PDs in different contexts and why some teachers appear to take up and use more than others and why some PDs have better results than others. This paper uses a comparative case analysis and focuses on the portion of the TaDD study that investigates self-reported learning related to pedagogy, content and resources taken up and used from the following three NSF PD projects one to two years after the project and funding ended. In the next section, we briefly describe the three different PD projects.

Learning and Teaching Geometry (LTG)

The first NSF project, LTG, an efficacy study of the learning and teaching geometry professional development materials: Examining impact and context-based adaptations, sought to improve teacher's own knowledge and instructional strategies in transformations-based geometry. This PD consists of 54 hours of highly specified video-based PD that is grounded in modules of dynamic transformations-based geometry which is aligned with the Common Core State Standards in mathematics (CCSSM). Through video analysis, teachers work together to solve problems and further their knowledge in mathematics teaching in the domain of geometry. The PD allows teachers to better support students in their attempt to gain a deeper understanding of transformations-based geometry through activities like rate of change on a graph, scaling activities, and similarity tools. The material strongly connects to other critical domains including similarity, proportional reasoning, slope, and linear functions. LTG is a specified PD as the content and pedagogical goals of the PD are clearly articulated for each workshop in the packaged materials.

Lesson Study (LS)

The second NSF project, Collaborative research: TRUmath and Lesson Study: Supporting fundamental and sustainable improvement in high school mathematics teaching (LS), aimed to engage in design research to develop and implement a replicable model for a coherent, department-wide approach to professional learning focused on creating classroom environments that produce students that can be powerful mathematical thinkers. In the PD, teachers work to create lesson plans that are focused and coherent and allow for a deeper and richer understanding of mathematics and the ability to make connections and implement curriculum more effectively. In this project, teachers were taught the *TruMath* framework. This is an observation instrument that can be used to analyze classroom interaction across different dimensions. Teacher teams engaged in LS as a way to work on specific shifts in teaching practice that aligned with the TRU dimensions. LS is an adaptive form of PD that utilized the TRU framework but allowed for teachers' ideas to guide the workshops.

Visual Access to Mathematics (VAM)

The third NSF project, Visual access to mathematics: Professional development for teachers of English learners (VAM), aimed to build skills in mathematical problem solving and communication through the use of visual representations. This PD consisted of face-to-face PD as well as online workshops where teachers implemented problems from the PD and shared their student work to discuss access for English Learner's (EL's) and all students. The project investigated the instructional strategies and supports that teachers of EL's need to provide access to mathematical learning while advancing academic language development. The approach was grounded in the use of visual representations, such as diagrams and geometric drawings, for mathematical problem-solving with integrated language support strategies. The intended goals of VAM were to help teachers to properly select appropriate visual representations for the use of different rational number task types and communication tools to show and explain mathematical thinking. VAM fell in the middle of the adaptive-specified framework as the face-to-face workshops had specified and intentional goals and the online professional learning meetings were guided by the teachers and used artefacts of practice, mainly lesson plans, to guide their discussions.

Methodology

Sixty-six participants from the three NSF projects took a 32-question survey LTG had 28 participants, VAM had 25 and Lesson Study had 13). This survey included questions that asked participants to reflect back on their PD experience and characterize their past and/or current use of the PD content, pedagogy and materials. The survey included seven Likert scale questions, where participants responded to statements on a scale of 1-10, as well as eighteen follow up questions that allowed the participants to explain and provide more details about their numeric response. To analyze the data, we used descriptive statistics and ANOVAs to understand the differences and similarities between uptake by project (LTG, VAM, Lesson Study) with ANOVAs followed by pairwise comparisons. Given the small sample sizes in this study, we report significance levels at the $p<.10$ level as well as the typical $p<.05$ level.

Qualitative responses were coded to move deeper into the data and unpack the quantitative results. Three project researchers coded the qualitative responses to better understand teachers' perceptions of uptake after participating in professional development. The seven Likert scale questions were used as the baseline and the coded eighteen qualitative questions were used to analyze participants perception and vision of uptake from their learning experiences in PD. Finally, we compared the differences among and between programs and present case studies of each project.

Results

We examined one-way differences by project by finding averages of the seven Likert scale questions on the survey. Likert survey questions ranged from 1 (not at all) to 10 (a lot). In comparing the three projects, VAM participants had consistently higher average ratings than LTG and Lesson Study. We found 6 areas that were significant at $p<.05$ and one at $p<.10$. While all three projects reported a high degree of established community within their respective PD experiences, VAM participants reported a stronger ($p<.10$) sense of community than Lesson Study participants. Furthermore, VAM participants reported greater ($p<.10$) use of materials and resources than Lesson Study. Other significant differences include VAM participants reporting higher levels of district support than both Lesson Study and LTG ($p<.05$). Reports of content and pedagogy use, as well as how well the facilitator met the goals of the participants, were significantly higher ($p<.05$) for VAM than LTG.

Table 1: Average teacher survey descriptions of PD uptake and use, by group: means and standard deviations (N=66)

Likert Scale Survey Question: Codes for Likert Scale Survey Questions ranged from 1 (not at all) to 10 (a lot)	LTG (n=28)	VAM (n=25)	LS (n=13)	*p*-value
To what degree, was the community established in your PD?[1]	8.46 (2.12)	9.24 (1.01)	8.00 (1.41)	p<.10
How much did your district support your attendance and implementation of the PD in your classroom?[2,3]	5.21 (3.34)	7.44 (2.62)	4.77 (3.32)	p<.05
How much of the content from your project do you use in your classroom?[3]	5.29 (3.10)	7.24 (2.28)	5.54 (2.67)	p<.05
How much of the pedagogy from your project do you use in your classroom?[3]	5.54 (3.12)	7.56 (1.92)	6.62 (2.90)	p<.05
How much of the resources/materials from your project do you use in your classroom?[1]	4.57 (3.34)	6.16 (2.76)	3.69 (2.72)	p<.05
To what degree was the facilitator focused on the intended goals of the PD from your own perspective?[3]	7.57 (2.67)	9.16 (1.38)	7.69 (2.53)	p<.05
To what degree was the facilitator meeting the needs of the participating teachers?[4]	6.54 (2.72)	8.84 (1.34)	6.46 (2.33)	p<.001

Of the three studies, teachers in the VAM project had higher self-report data on several dimensions related to the uptake of content, pedagogy and resources. In order to better understand specific

1 VAM > Lesson Study, p<.10.
2VAM > Lesson Study, p<.05.
3VAM > LTG, p<.05.
4VAM > Lesson Study and LTG, p<.01.

uptake, we analyzed the qualitative data using a coding manual consisting of 33 codes that span content, pedagogy and resources in general, as well as specific areas such as representations, technology, principal and coach support, facilitator impact and theoretical alignment. Three project staff initially coded three surveys from each project and compared their results, discussed discrepancies and resolved differences by refining and agreeing upon codes.

VAM

Almost all of the VAM participants were able to identify representations from the PD that they used to teach relevant content including ratio, proportion, percent, dilation, and scaling. Additionally, approximately 50% of participants mentioned specific pedagogical strategies such as the Three Read Strategy that they learned in the PD. Participants were also able to describe how they used resources such as specific tasks, applets, and computer-based activities from the PD in their classroom practice. Only two participants didn't respond or identify any specific uptake from the PD. The majority (92%) of the participants responded with an abundance of uptake. One participant explained, "I use the number line as often as I can. I try to help students see that it can be used with multiple patterns as an underlying skill for the double number line." Some participants took up general strategies that could be used across mathematics lessons and others in other content areas. For instance, one person explained,

> "Generally, I've found that using visuals to access mathematics and the conceptual understanding in math has greatly benefited my students. The teaching strategies around using visuals to support understanding is something I use regularly."

In regard to principal and district support, the majority of VAM participants reported high levels of support. Others reported support related to release time. In general, no one reported anything negative related to support. Participants also reported that principals were generally supportive. In terms of community, VAM participants reflected on the collaborative nature of both the on-line and in-person sessions. One participant noted,

> "I felt part of the community at the PD, my voice was heard and mattered. The small zoom sessions were also helpful and supportive. Getting to know and ask questions to a smaller cohort was less intimidating and educational. The moderators were so supportive and helpful and always followed through with any issues that needed follow-up or extra clarification."

Participants also felt they benefited from working on problems together as learners, "As we explored activities and experienced them as learners ourselves, we really opened up to one another and got to know one another."

LTG

Participants reported lower levels of uptake for LTG than VAM. About 50% of participants responded "none" or "nothing" in terms of content they currently use in their classroom. Several noted that this was because they were not currently teaching geometry. The participants who did report content uptake mentioned specific transformation-based content from the PD. For example, one participant noted, "Rotations, translations and dilations are helpful for students to see and visualize different possibilities for real world problems they are interested in solving." In terms of pedagogy, LTG teachers talked about using dynamic strategies that were closely related to the transformation-based geometry content, including the use of manipulatives and representations. Other teachers mentioned more general pedagogical strategies that were modelled by the facilitator of the PD, such as strategies for facilitating discussions, incorporating vocabulary and helping students develop explanations. As one participant noted, "I learned how to let students have a discussion to sort out their own ideas and practice defending their answers. In terms of resources, 75% of participants described how they currently used specific resources, such as patty paper or tasks and activities from the binder they were given at the PD. However, 25% of participants reported that they did not use any resources from the PD. As described by this participant, this may be due to the content or grade level the teacher is currently teaching: "I do not use any of the paper materials that we were given in the binder because it does not apply to 6th grade or is very introductory for 7th grade." In terms of district and principal support, teachers generally responded neutrally, and many stated that the district was not involved and sometimes not aware of the PD. Principal support varied greatly between school sites. Several reported that the principal was very supportive whereas other teachers reported lack of support. In terms of community, most teachers noted that they felt part of the community and enjoyed working on problems and discussing strategies together. For example, one participant noted that there were, "lots of discussion and opportunities to share. All opinions and strategies and thoughts were valued." Another added, "We worked during PD hours together and discussed our shared teaching experiences at other times. The PD leaders made sure everyone felt engaged and included."

LS

The LS project also had much lower responses than VAM when looking at the quantitative findings for each category. LS participants did not perceive that they took up any content. Not one of the respondents referred to specific mathematics content in their responses. Most responded that they didn't have anything to report or that it was not applicable. On the other hand, when responding to pedagogical uptake, many responded positively and focused on different aspects of pedagogy that they took up and new instructional strategies that they were continuing to try to use. Three participants mentioned the TRU framework that was used to analyze lessons related to effective instruction throughout their PD. Other than two respondents that said the pedagogy was not applicable, the positive respondents shared different strategies they took away from the PD. One participant commented,

> "I engage in much more formative assessment with students and I constantly try to elicit more student thinking to determine how students are thinking through problems and then tailoring my instruction to meet students' needs for understanding in real-time. I also think that I am much more focused on the central mathematics and big ideas of a unit or a lesson. This has allowed me to tweak my lessons, so I can make better decisions about which content is ancillary, extension or extraneous."

The other three responses were focused on assessment and questioning. Other LS teachers focused on questioning. For instance, one teacher reported that she has changed, "questioning strategies during a lesson to cultivate student's critical thinking." It is not clear why four out of 13 teachers in the LS program had very targeted pedagogical uptake related to assessment, questioning, and meeting students' needs whereas the rest of the participants found little to respond to related to content, pedagogy and resources. One participant did note, "Developing a focus for the work that applied to all teachers concerns proved to be difficult." In terms of resources, none of the LS teachers reported acquiring or using any resources from the PD. Three reported using some strategies with other teachers in planning lessons. In regard to principal and district support teachers reported a range of feelings from negative to positive but as a whole appeared to fall somewhat in the middle. One teacher might sum this up the best when explaining, "Somewhat? The district didn't play an active role in either supporting us or hindering us." In terms of community, most reported they felt part of the community and several highlighted the importance of the community builders that were done at the beginning and end of each session. However, one reported mixed feelings about their colleagues and community, "There were moments of brilliance during these sessions. Unfortunately, the department, in general, always leaves a bad taste in my mouth because we have a lot of dead weight in our department. It makes everything more difficult"

Discussion and Implications

Teachers perceptions of uptake differed across the three sites. It appeared that where a PD falls on the continuum may have impacted self-report data. VAM, which falls in the middle of the continuum, had the highest self-report data among teachers. The more specified the goals, the clearer teachers were able to indicate whether the PD was useful to the types of mathematics classes they were currently teaching. On the other hand, if the PD was more adaptive and the nature of the goals and intentions were evolving, teachers appeared to indicate quite different aspects of the PD that were relevant to their planning and teaching. If a PD was both adaptive and specified at times, more teachers had positive and similar experiences associated with uptake.

For instance, the VAM PD was specified enough to allow participants to recall and identify specific resources or pedagogical tools (such as double number lines or the Three Reads Strategy) that they could use in their classrooms. It was also adaptive enough that participants had time to think about how to modify the tools and resources that they learned about during the PD. On the other hand, the adaptive nature of LS might have made it difficult for participants to report how they were using the skills they learned because some of the goals and intentions of the program were not articulated during the PD. The goals were evolving simultaneously during the PD as teachers were engaging in developing one lesson plan per cycle for one specific classroom. In addition, teachers may have had a harder time generalizing the relevance from a particular lesson study cycle to their specific classrooms and contexts. The adaptive nature of the LS PD has many more complexities than the other two. For example, the goals and intentions are continually evolving and therefore teachers may take up very different aspects of the PD that are relevant to their teaching. This is unique to adaptive PD because it has the potential to meet teachers where they are at. On the other hand, the specified nature of LTG might have impacted self-report in that if teachers were not teaching transformations-based geometry, they may not have been able to explicitly identify content, pedagogy or resources when they were teaching other content, even if there were underlying connections that could have been made. Many of the LTG teachers were able to identify tasks and tools when asked but at the same time may or may not have found the narrowly focused and specified content relevant to their current teaching. However, the teachers who were currently teaching geometry reported positive levels of uptake because they highly motivated to use and then teach the LTG content with their students.

The nature of the PDs and the ways in which participants felt as if they were members of the community may also influence uptake. The LS PD was the only one of the three where teachers visited each other's classrooms. The impact of visiting classrooms on community needs to be explored as we hypothesize that this may make teachers more vulnerable and may influence their perceptions of their learning. On the other hand, VAM was the only PD that had both an online and in-person community as part of the professional development. This role of this on teacher learning also needs to be explored further. We argue that this self-report data does not indicate whether classroom practice or student learning was impacted. More research is needed on the impact of context on classroom uptake. Case study classroom videotape data and interviews will provide us more information on the classroom uptake of participants in these three studies and will help us investigate the ways in which the three different types of PD have different affordances and constraints on teacher uptake.

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SUPPORTING INCLUSION OF STUDENTS THAT TYPICALLY STRUGGLE WITH MATHEMATICS IN COGNITIVELY DEMANDING SMALL-GROUP DISCOURSE

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This study examined how two small-group discourse types (Reflective and Exploratory) supported the inclusion and enacted levels of cognitive demand of students who typically struggle with mathematics in real-world, task-based assessment activities. The study focused on 11 fifth-grade students within a larger study involving 97 fifth-grade students engaging with 24 mathematics task-based assessment activities. Results showed that students that typically struggle with mathematics were more likely to participate in group discourse during reflective discourse. Additionally, discourse contributions by these students were more likely to be high cognitive demand during reflective discourse. Reflective discourse provided students with time to think through and write down their own strategies which may have increased student confidence and willingness to engage in explaining and justifying their thinking.

Keywords: Cognition; Classroom Discourse; Elementary School Education; Equity and Diversity

Task-based assessments provide educators with an understanding of students' ability to apply their mathematics understanding to real-world situations. Mathematics discourse can also provide a formative assessment of students' cognitive understandings and misconceptions. Research on mathematics discourse is divided on timing - after (Reflective) or during (Exploratory) student engagement with mathematical tasks –to best support the inclusion of all students' ideas. Thus, the purpose of this study was to examine how these two discourse types supported the inclusion and enacted levels of cognitive demand (CD) for students in four Grade 5 classrooms who typically struggle with mathematics, when engaging with 24 task-based assessment activities.

Theoretical Perspective

Cognitive demand is the number and strength of the connections within and between mental networks, or schema, to solve a specific task (Webb, 1997). High student-enacted levels of cognitive demand (HCD) are defined as students' mental actions that require two or more schema connections to make inferences or connections between mathematical ideas or contexts. HCD can promote deeper student understanding of mathematics properties and procedures, increase students' ability to solve related mathematics problems, and reinforce mathematics connections (Stein, Grover, & Henningsen, 1996). This is especially important for students that typically struggle with mathematics.

Multiple factors can influence student-enacted CD when engaging with mathematics tasks. The Discourse and Interpretation Influences on Cognitive Demand Framework in Figure 1 illustrates a few of these factors. The intended CD, the number of connections anticipated for a student to complete a task (e.g., the designed tasks at specific DOK levels in this study), as well as the implemented CD, teacher actions that encourage different student physical and mental actions (e.g., using recall or reasoning), can influence what and how a task is presented to the students (Webb, 1999; Boston & Smith, 2009). However, research shows that students' interpretation of a task primarily influences their enacted CD (e.g., Otten, 2012). Students may enact multiple elements of HCD or low cognitive demand (LCD) in response to a single task (e.g., using recall of facts to support counter-argument).

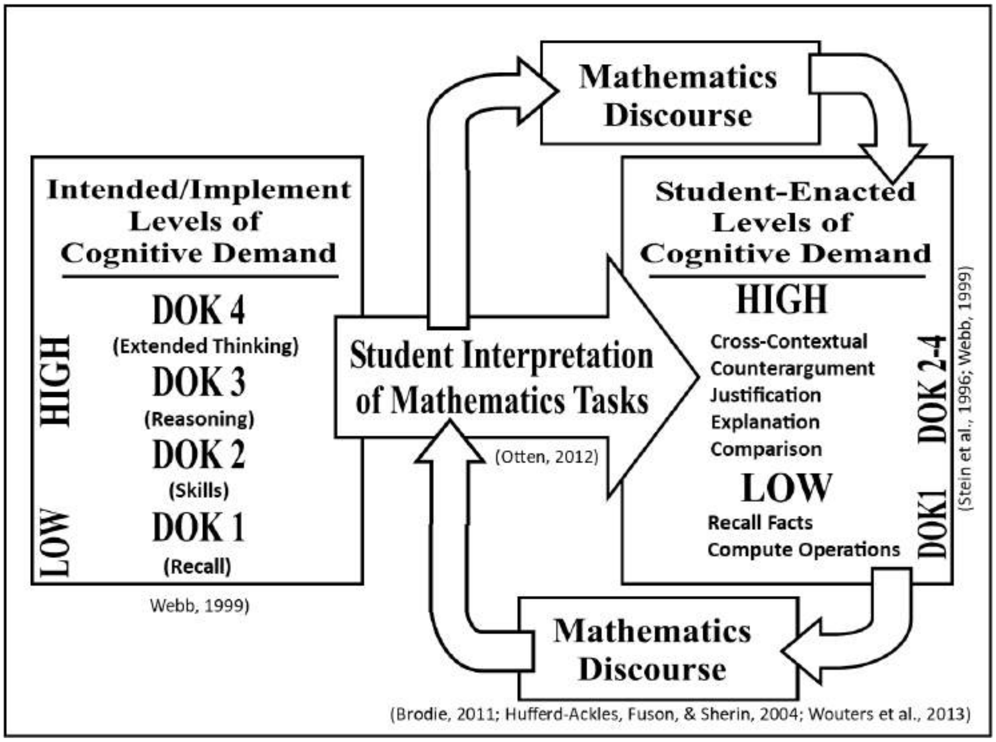

Figure 1. Discourse and Interpretation Influences on Cognitive Demand

As seen in Figure 1, mathematics discourse can influence students' interpretation of mathematics tasks. Mathematics discourse has the potential to increase student-enacted CD by eliciting student participation in HCD actions such as evaluating and reasoning about mathematical properties or procedures (Charalambous & Litke, 2018; NCTM, 2014).

Placement of discourse may also influence verbal and written student-enacted CD. Reflective Discourse takes place after students have engaged with the tasks independently, while Exploratory Discourse takes place while students are engaging with the task. Students' internal discourse and written CD after engaging with Reflective Discourse can influence their verbal contributions towards the social mathematics discourse. Group reflection on differences between verbal contributions, may prompt a change in students' interpretation of the task to develop a more advanced conception of the mathematics task (e.g., Silverman & Thompson, 2008) and may even prompt a change in student-enacted CD. Students' interpretations of mathematics tasks will also influence their verbal contributions towards exploratory discourse as students' personal understandings are negotiated to form a group understanding of the problem (Bruner, 1986; Clements & Battista, 2009; Forman, 2003) and the written CD for the task.

Researchers are conflicted on which placement of discourse is the best. Walter (2018) explains that delaying discourse until students have sufficient time to process the mathematics or write down their own ideas can promote the inclusion of students who might otherwise be ignored, such as typically struggling students. Rojas-Drummond and Mercer (2003) contest that waiting to engage in discourse until after the task is complete results in LCD cumulative talk where students, such as those that struggle with mathematics, simply agree with one dominant idea. Instead, they recommend Exploratory Discourse, which allows students to inclusively discuss relevant information in a timely manner.

Methods

This study utilized a mixed methods research design (Tashakkori & Teddlie, 2010) to answer the research question: How do two types of small group discourse (exploratory and reflective) support the inclusion and enacted levels of cognitive demand (CD) of students that typically struggle with mathematics during real-world tasks?

Participants

This paper focused on 11 of 97 fifth-grade students from a larger study (Litster, 2019). Focus students were identified as typically struggling with mathematics based on self-identification (N=7) during group discourse (e.g., I'm usually wrong/I just don't really get how to do math) or identified with a mathematics IEP (N=4). Six students were identified as English Learners (ELs).

Procedures

Students in four fifth-grade classrooms completed two real-world, task-based assessment activities (12 tasks per set), in groups of two or three. Typically struggling students were in different groups during the study. Students worked collaboratively with a small group on one set, engaging in Exploratory Discourse of solutions and strategies; and individually on a second set, followed by Reflective Discourse of solutions and strategies. Using a crossover design (Shadish, Cook, & Campell, 2002), two classes completed Set-1 with Exploratory Discourse and Set-2 with Reflective Discourse, while the other two classes completed Set-1 with Reflective Discourse and Set-2 with Exploratory Discourse.

Data Sources and Analysis

There were two main data sources in this study: students' written work relating to the mathematics tasks and video/audio recordings of students' interactions and discourse while engaging with the tasks.

The researcher used qualitative magnitude coding to identify and "quantitize" (Saldaña, 2015, p. 86) the enacted levels of CD in students' verbal responses to the mathematics tasks as high cognitive demand (HCD) or low cognitive demand (LCD). Twelve percent of the data were double coded with a relatively high inter-coder reliability (α=0.9212) (Hayes & Krippendorff, 2007). The researcher created frequency tables to compare quantitative results relating to enacted CD. Timestamps from the video data were used to identify time engaged in active mathematics discourse.

The researcher qualitatively coded students' verbal responses using pattern and structural coding to identify patterns in students' actions that seemed to increase or decrease typically struggling students' participation in the small group mathematics discourse

Results

Results found that overall, Reflective Discourse was more likely to support higher levels of cognitive demand in discourse contributions by typically struggling students than Exploratory Discourse. Table 1 shows a comparison of the cognitive demand in student's discourse contributions, based on ability and discourse type.

Table 1: CD of Mathematics Discourse Contributions by Ability and Discourse Type

Mathematics Ability	Reflective Discourse		Exploratory Discourse	
	LCD	HCD	LCD	HCD
Typically Struggling	37.71%	62.29%	50.72%	49.28%
Everyone Else	32.46%	67.54%	29.92%	70.08%

As seen in Table 1, students that typically struggled with mathematics were engaged in HCD discourse for a larger percentage of time during reflective discourse than exploratory discourse. Additionally, there was a smaller difference between ability groups in the percentage of time students engaged in HCD discourse during reflective discourse (5.25%) than exploratory discourse (20.8%).

Reflective Discourse was also more likely to support the inclusion of typically struggling students than Exploratory Discourse. Most groups of students engaged in active mathematics discourse for about 12 minutes per task-set, with an average difference of less than one across the discourse types. However, not every student participated in the discussion for all 12 tasks in each task-set. Table 2 shows the count and average number of tasks where a student remained silent and did not participate in the discussion for an entire task, based on ability and discourse type.

Table 2: Count of Tasks Where a Student Remained Silent

Discourse	Struggling Students (n=11)	Everyone Else (n=86)
Reflective	11 (1.00 per student)	45 (0.52 per student)
Exploratory	57 (5.18 per student)	21 (0.24 per student)

As seen in Table 2, students that typically struggle with mathematics were more likely to remain silent during the discussion of tasks during Exploratory Discourse than during Reflective Discourse. Additionally, there was a larger discrepancy between ability groups in the average number of tasks where a student remained silent for exploratory discourse (5 tasks) than for reflective discourse (<1 task).

Qualitative analyses supported the quantitative results. During Exploratory Discourse, the typically struggling students often said little to nothing during the entire set and were often ignored by their group. For example, the only phrase one student with an IEP, Penny, said while completing a set using Exploratory Discourse was, "You are going too fast." After this comment, her group told her the answers and what to write on her paper after they solved each task. Other struggling students often went without speaking for three or four tasks in a row. Most students who were silent during exploratory discourse appeared to be engaged in active listening (e.g., looking at other students in group and writing on paper in response to other students' comments). However, students who struggled with the mathematics were more likely to disengaged with the tasks completely (e.g., playing with paperclips in their desk).

In contrast, during Reflective Discourse, the typically struggling students were more likely to ask for help. For example, Penny, who talked once during Exploratory Discourse, contributed 19 different times during Reflective Discourse. Most of Penny's contributions confirmed shared answers (e.g., "Yeah, I got that too"); however, occasionally she would ask her group for help or contribute new ideas. An example of this is found in the excerpt below where the group is engaging in reflective discourse after trying tasks 1-3 from Task-Set 1 on their own. In these tasks, students are calculating revenue from movie sales [T1], book sales [T2], and comparisons between the two [T3] for Harry Potter and the Half-Blood Prince.

Wendy: What did you get down here [pointing to T2] cause I didn't finish.
Penny: I am pretty slow writer and thinker so I did not make it down there. But show me what you did.
Wendy: I did .5 million times 12.99. [points to work for 500,000 times 12.99 that is half finished]
Penny: It says round. $12.99 is like $13 so can you just do 13 times 5?
Xander: So I did that and I got 65.

Wendy: So on this one [points to T3], I am pretty sure this is the movie [points to T1] and this is the books [points to T2]

Penny: Paperbacks means books

Wendy: So yea, so up here [T1] is 247.24 million and the highest one here [T2] was just 65 million so I think down here people saw the movie more.

Penny: Yea, I think they saw the movie more. That's [pointing to 247.24 on her page] way bigger than that [pointing to 65 on Xander's paper].

Xander: The price [on the books] was bigger than $7 [price of movie ticket] and the price was bigger but it is a smaller number [total book revenue].

Penny: Yea and I think people don't really like reading any more.

[Video 210H1, 0:31-2:35]

In this excerpt, we can see that, not only is Penny asking for help, her group is willing to explain their answers rather than just asking her to copy them. Additionally, Penny was able to add ideas and strategies to the discussion, such as rounding 12.99 to 13 for an easier computation problem, a comparison to justify an agreement, and a possible context for the lower book revenue. Although no one in the group caught the mathematical error for multiplying 13 x 5 instead of 13 x .5, there was evidence of high verbal CD by all three students: Penny's counterexample for rounding $12.99 to $13, Wendy's justification of why people saw the movie more, and Xander's comparison of the inverse relationship between T1 and T2.

One factor that may help explain the increased discourse and HCD contributions by typically struggling students is that groups engaged in reflective discourse often had one student (not typically struggling) who started the discussion by asking about a task they struggled with. For example, in this excerpt Wendy moved the discussion directly to Task 2 because she struggled to complete her very large (and unnecessary, though mathematically accurate) calculation. Seeing other students struggle may have helped students who typically struggle feel more comfortable asking for help or offering ideas. It also may have helped the other students in the group feel more patient with their explanations because they could point to their work during their explanation or compare their work to the struggling student's work to find where their calculations diverged.

Qualitative analyses also noted a pattern in validation of student ideas. During exploratory discourse, struggling students' ideas were often devalued by other members of their group. However, this was not the case during Reflective Discourse. The following excerpts from the group working with Miguel, a Latino EL student with a Math IEP, provide examples of this behavioral pattern. In this first excerpt, Miguel, Tara, and Ryan are engaging in exploratory discourse to solve task 5 in Task-Set 1. In this task, students are comparing different sources of online movie ratings for Harry Potter and the Half-Blood Prince.

Ryan: So Amazon is 3 3/4, minus iTunes, which is 4.

[Ryan and Tara start to subtract the fractions]

Miguel: It's 1/4

Tara: You have to show your work

Miguel: I just know it

Tara: Then how did you do it in your head? Cause you can't minus that. That would equal 1 ¾ so how did you do that in your head?

Miguel: It's just like . . . [pointing to line plot] .

Tara: [interrupting] If you can't say it then you need to do it.

Miguel: Maybe if you do 4 minus 3 and you get 1 . . . and then minus ¾

Tara: You can't do that or you would get negative. You have to change this to improper fraction and that makes 4/1 and this [points to 3 ¾] makes 9/4 and you have to make a common denominator so this is 16/4 and then you subtract so it's 7/4 and then you have to change it to a mixed number so the answer is 1 ¾.

Miguel: Yours is probably correct.

[Video 460H2, 11:30-13:20]

In this excerpt, we can see that even though Miguel had the correct answer and what appeared to be a valid strategy to get to the answer, Tara's interruption did not give him the time to fully work out his reasoning to support his answer. In the end, Miguel conceded that Tara was probably correct, even though her calculations were incorrect. This pattern of concession was similar among other typically struggling students whose divergent answers or strategies were not valued or explored (e.g., "Okay, you're the smart one, so what's the answer;" or "You are probably right cause you are usually right.").

In this second excerpt, Miguel, Tara, and Ryan each completed Tasks 6-10 in Task-Set 2 on their own and came together during Reflective Discourse to discuss their answers and strategies. In these tasks, students are calculating and comparing production and retail costs for the Diary of a Wimpy Kid book series. In the excerpt from Task 7 below, they are discussing their calculations for possible fractional discounts [e.g. ½] off the rounded retail price [$60].

Tara: So on this one I multiplied and ½ times 60 is 30.

Miguel: I divided.

Tara: No, so you have to multiply. You change this to an improper fraction so 60 is on top of 1 and then 60 times 1 is 60 and . . .

Miguel: [interrupts] But it's the same answer [points to paper].

Ryan: Me too. So 1/3 of 60 is like 20.

Tara: Yep and 1/4 times 60 is 40.

Miguel: It's 15. Cause 40 + 40 + 40 + 40 is more than 60.

Ryan: So 1/5 of 60 is 12.

Miguel: And 60 divided by 10 is 6.

Tara: So 1/7 times 60 is . . . 49?

Miguel: [Shows Tara his division] so it rounds to 8.

Tara: Ok [revises her answer]

[Video 460D2, 2:51-3:54]

In this excerpt, Miguel had a chance to try out his strategies before comparing answers with Ryan and Tara, which may explain why he appears more confident in his answers and in his strategies than he was in the exploratory excerpt. During Reflective Discourse, Miguel was able to show Tara that his strategy produced the same answer as hers on the first few calculations. This may have helped Tara to accept his strategies and answers on the later calculations that did not match her answers. One reason Reflective Discourse was more likely to support the inclusion of typically struggling students may be that it provided students time to think through their own strategies. Miguel was able to conceptualize a unit-fraction discount as a whole number division problem, a strategy that was only used by one other person in the entire study.

One exception to these patterns was Summer, a Native American EL student with an IEP. Summer's group was very patient with her and never took over her work. For example, if Summer said, "I need help," her group would ask, "Which part," or "Would you like to try . . . [specific strategy from class]." Additionally, Summer's group would use questioning techniques such as "Do you remember how [teacher's name] taught us last week?" or "Okay, so what do you need next to be able to do that?" at least three times before offering a specific suggestion for the next step. During both exploratory discourse and reflective discourse, Summer's group waited to continue until Summer was confident in her work. Group support and validation may have provided Summer with the support to ask questions, request more time to complete a task, and offer the occasional strategy to complete a task, regardless of discourse type.

Conclusions

In conclusion, both discourse types elicited HCD discussions such as evaluating and reasoning about mathematical properties or procedures, similar to other research results (e.g., Charalambous & Litke, 2018; NCTM, 2014). However, Reflective Discourse practices were more likely to support the inclusion and HCD discourse contributions of students that typically struggle with mathematics, during the real-world tasks in this study. These results are similar to other research on reflective discourse practices such as Kalamar (2018) who found that by the end of her three-week intervention, 100% of minority and typically struggling students in an intervention class were participating during Reflective Discourse.

Similar to Walter's (2018) findings, wait time may play an important role in increased participation of typically struggling students. Reflective discourse practices allow students time to think through and try out their strategies prior to the discussion. Having a clear train of thought, as well as a tangible artifact to refer to, may have increased student confidence and willingness to participate in the discussion. In this study, students who do not typically struggle with mathematics were more likely to admit when they were struggling with a problem or answer during reflective discourse than during exploratory discourse. This may have also contributed to an environment where the typically struggling students were more willing to ask questions or propose their own ideas.

Finally, the results relating to Summer and her group bring forward the need for future studies relating to group norms, such as those used by Summer's small group, that may support the inclusion of students who are traditionally excluded from small group discourse, regardless of discourse type.

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PROFILING PRODUCTIVE MATHEMATICAL TEACHING MOVES IN 4TH-8TH MATHEMATICS CLASSROOMS

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As the field draws to greater consensus around components of productive mathematics classrooms, the need increases to answer questions about how we operationalize and measure these components and how we can actualize such practice in classrooms. This research report shares an analysis from a larger project aimed at describing and quantifying student and teacher components of productive classrooms at a fine-grain level. We share results from this analysis of 39 mathematics lessons with a focus on teacher moves and catalytic teaching habits that characterize these lessons. A cluster analysis identified four profiles of lessons differentiated by the existence of these catalytic teaching habits and subsequent work with student ideas. Further, these clusters appeared to account for differences in student contributions in lessons.

Keywords: Instructional activities and practices

There is general consensus that productive mathematics classrooms are ones where student thinking is integral and student discussion permeates (Jacobs & Spangler, 2017). A mathematics classroom can be conceptualized in terms of the interrelations between student(s), teachers, and content, often deemed the instructional triangle (e.g., Hawkins, 2002). The relationships between these areas can serve to position students towards engagement in meaningful mathematics. In this work, we share an analysis of residue from the instructional triangle measured via the Math Habits Tool (MHT; Melhuish, et al., 2020). The MHT was developed to capture *mathematically productive components* of classrooms in terms of both what teachers and students do in-the-moment. We conjectured that we could categorize different types of mathematics classrooms based on the existence of, and patterns within, habits of mind/interaction (student engagement in mathematics and with each other), catalytic teaching habits (teaching moves to engender student engagement with the content), and teaching routines (teaching structures that position and encourage students as contributors to mathematics).

In order to begin this discussion, we share results from an analysis of 39 lessons spanning grades 4-8. These results illustrate a partition of lessons into four clusters differentiated based on how student ideas were prompted/treated: student ideas are not prevalent and teacher-prompts are limited as well as unvaried (cluster 1); student ideas are in discourse, prompts are more prominent yet unvaried (cluster 2); student ideas are in discourse with varied prompts for student ideas with a focus on expanding ideas (cluster 3); student ideas are in discourse with varied prompts for student ideas with a focus on students engaging with each others' ideas (cluster 4).

Literature Background

Researchers and policy documents alike have identified attention to student mathematical thinking as a key component of mathematically productive classrooms (Jacobs & Spangler, 2017; National Council of Teachers of Mathematics, 2014). In Jacobs and Spangler's (2017) overview of the literature on teaching, they have unpacked this attention into two core teaching practices: noticing students thinking and orchestrating classroom discussion. Classrooms that incorporate these practices

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provide settings where student ideas serve as the grounds for moving the mathematics forward. Teachers research student ideas, provide space for student ideas to become part of the classroom discourse, and develop a shared community in which students take ownership of mathematics.

These core teaching practices have one commonality: teaching centered on students' mathematical ideas. We have identified four essential purposes for teacher moves related to students' mathematical ideas: engaging students in idea generation, researching student ideas, engaging students in expanding their ideas, and orienting students to each other's ideas. Productive idea generation can be supported through mechanisms such as allowing for private reasoning time (e.g., Kelemanik, et al., 2016) and providing and maintaining high cognitive demand tasks (e.g., Stein, et al., 1996). Researching student thinking occurs when teachers press for sharing reasoning and meaning behind student ideas to allow teachers to attend to, interpret, and decide how to respond to student thinking (e.g., Jacobs, et al., 2010). Expanding ideas then includes prompts for students to justify their responses (e.g., Boaler & Staples, 2008) or reflect on their thinking (e.g., Schneider & Artelt, 2010). The last essential component to a student idea-driven classroom is orienting students' to each other's ideas. This involves bringing student ideas into the public discourse to establish common ground (e.g., Staples, 2007), and asking students to interpret and compare each other's ideas (e.g., Stein, et al., 2008).

Theoretical Orientation and Analytic Framework

The theory underlying our work is that of the instructional triangle (Hawkins, 2002). Hawkins posited that instruction can be viewed through the relationships between teachers, students, and content. Lampert (2001) expanded on this work in analyzing her own practice, noting how instruction occurs through the arrows and introduced that the teacher plays a mediating role on the arrow reflecting the relationship between student and content. Further, Cohen et al. (2003) brought attention to not just "student" but the additional layers of interactions between students within a classroom.

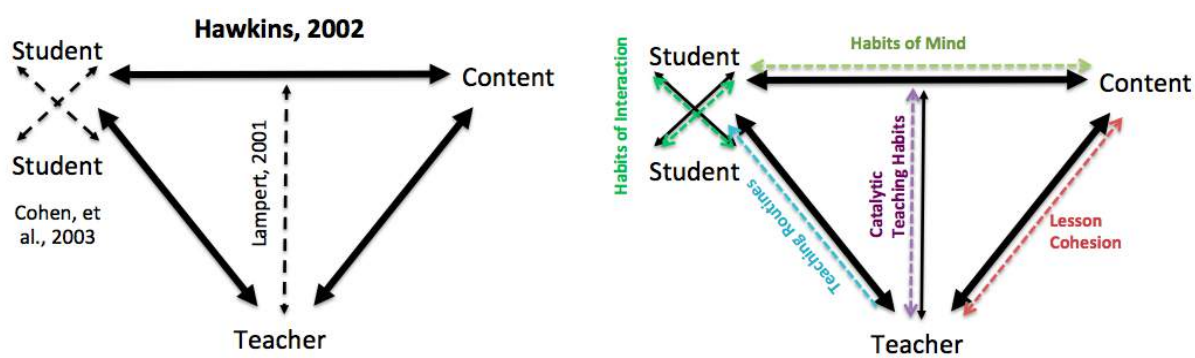

Figure 1: Instructional Triangle and the MHT Framework

In our operationalization of this triangle, we focus on key relationships within those triangles: productive ways students engage in mathematics (habits of mind), with each other around mathematics (habits of interaction), productive teaching structures (teaching routines), and individual teaching moves that can serve to catalyze student productive engagement with mathematics (catalytic teaching habits.) In this report, we focus on the category of catalytic teaching habits (CTHs) as they are both highly observable and play a key role in mediating the relationship between students and content. Recall, CTHs are prompts by the teacher to bring about students' engagement within the lesson and are represented in the instructional triangle by the line, added by Lampert, originating from the Teacher to the line connecting Students to Content. The CTHs can be found in Figure 2 in the Results section. We will also briefly analyze students' habits of interation (HoIs) which relate to communicating mathematical ideas primarly mapped to the Student-to-Student interactions within the instructional triangle. Examples of HoIs include explaining mathematical ideas, critiquing and debating, comparing ideas, exploring multiple pathways, and asking genuine questions.

Methods

For the scope of this project, we are analyzing lessons from two school districts: elementary schools (grades 4-5) from a large urban district and middle schools (grades 6-8) from a mid-size urban district, both in the United States. We are analyzing a lesson at the end of the year from all of the teachers from the middle school group, and a stratified random sample, according to Mathematical Quality of Instruction (Hill, 2014) scores, from the larger set of elementary teachers. Currently, we have coded 39 lessons (19 from 4-5 and 20 from 6-8).

Two trained coders independently watched each video-recorded lesson and qualitatively coded the classroom interaction by interpreting teacher moves and student contributions using the MHT codebook as a guide. The unit of analysis was at the contribution level so a student (or group of students) explaining one idea would be a single unit. Each substantive mathematical contribution from a student(s) would have a single HoI code. Similarly, a teacher press would be a unit. During this process, coders took detailed notes to keep a record of their rationale for assigning particular codes. Then, the two coders met to discuss and reconcile their individual interpretations until agreement was reached. In addition to student and teacher interaction, coders also noted portions of class time spent on whole class discussion (or teacher lecture), small group work, and individual work as well as rated each lesson holistically across a number of categories including *overall teacher* and *overall student*. Each lesson was rated a 1, 2, 3 or 4 within each holistic coding category where 1 represents the lowest rating and 4 represents the highest rating. The *overall teaching* captures the degree to which teacher moves reflect catalytic teaching habits, teaching structures including productive routines (such as selecting and sequencing), and ultimately if teachers prompted students towards justifying or generalizing (a requirement for a score of 4). Similarly, the *overall student* code captured whether students were engaged in math habits of mind and interaction with a scores ranging from no engagement (1), some engagement (2), engagement in many habits (3) and engagement with many habits including justifying or generalizing (4). We calculated Krippendorff (2004)'s alpha for *overall student* at 0.679 and *overall teacher* 0.764, both meeting the acceptable cutoff for reliability.

At the current stage, we focus our detailed analysis on the CTHs. To simplify this analysis, we considered a binary variable for each CTH on each lesson (occurred or did not occur.) We then conducted a two-step cluster analysis using a log-likelihood distance in order to cluster together lessons that were similar in terms of CTH profiles. Four clusters provided a fair classification with each cluster containing at least eight lessons. We further situate these clusters in relation to the summary variables: *overall student* and *overall teacher* scores.

Results

An Overview of CTHs in Lessons

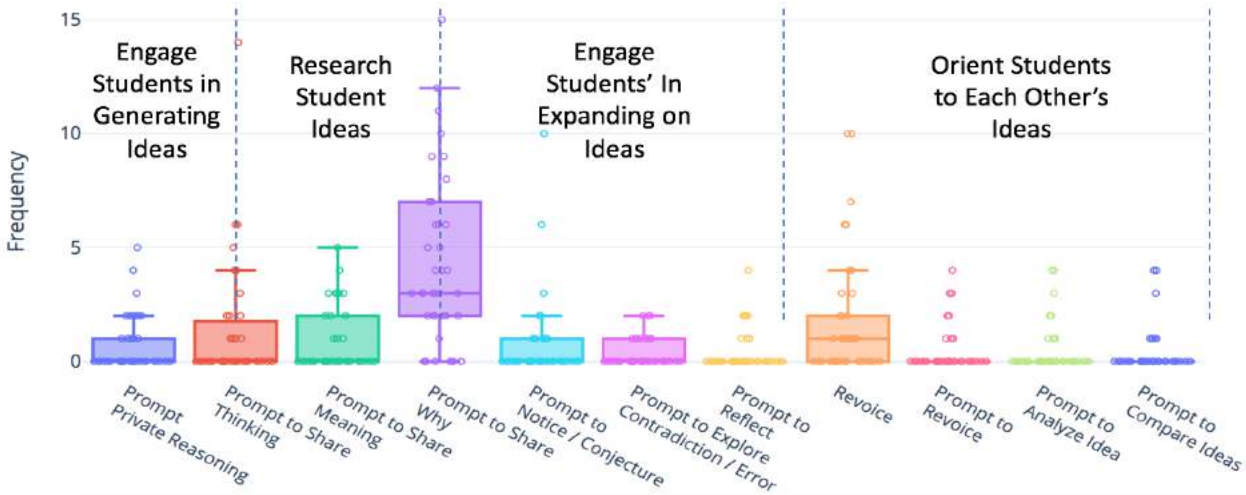

Figure 2: Boxplots Reflecting Frequency of CTHs Per Lesson

An overview of the frequency of various CTHs can be found in Figure 2. This figure contains box plots along with data points (each dot is one lesson) representing the range in frequency for each class. Notice that prompts to *share why* are most common, with well over half of the lessons including multiple occurrences. In contrast, the majority of lessons had zero prompts for students to analyze ideas (see the density of dots at 0 frequency).

Profiling Lessons Based on Types of CTHs Present

Catalytic Teaching Habit	Cluster 1 (14 lessons)	Cluster 2 (8 lessons)	Cluster 3 (8 lessons)	Cluster 4 (9 lessons)
Prompt for Private Reasoning	0%	0%	100%	77.8%
Prompt to Share Thinking	14.3%	25%	87.5%	44.4%
Prompt to Share Meaning	35.7%	0%	75%	44.4%
Prompt to Share Why	57.1%	100%	100%	100%
Prompt to Notice/Conjecture	28.6%	12.5%	25%	44.4%
Prompt to Explore Contradiction/Error	7.1%	62.5%	12.5%	44.4%
Prompt to Reflect	0%	12.5%	37.5%	55.6%
Revoices Student Idea	14.3%	87.5%	87.5%	55.6%
Prompt to Revoice Student Idea	0%	12.5%	0%	88.9%
Prompt to Analyze a Student Idea	0%	12.5%	25%	66.7%
Prompt to Compare Student Ideas	0%	0%	0%	88.9%

Table 1: Percentage of Lessons in Each Cluster with Relevant CTH Occurrence

Table 1 contains the cluster analysis results where the percentages in each cell indicate what percentage of lessons in that cluster contain the given CTH (e.g. 14.3% of lessons in cluster 1 contain a prompt to share thinking). The coloring reflects the density of lessons in that cluster containing that CTH (red: less than or equal to 25% of lessons, yellow: between 25% and 75%, green: 75% or more of lessons). Lessons in cluster 1 tended to have few CTHs with no or few variations in prompts related to students contributing ideas. Lessons in cluster 2 all included at least a basic prompt for a student to share their idea, and most included a teacher revoicing a student idea, pushing student ideas into the classroom public space. Cluster 3 reflected a greater variation in CTHs with the majority containing not just *explain why* prompts, but also general prompts to share thinking and share understanding of a mathematical idea. However, like cluster 2, student ideas were revoiced by the teacher. Cluster 4 reflected less variation in initial idea sharing mechanisms, but substantial prompts for students to engage in each other's ideas. Notice the extremely high proportion of lessons in cluster 4 that contained *prompt to revoice student idea*, *prompt to analyze a student idea*, and *prompt to compare student ideas*. These moves shift the intellectual responsibility back to the students to make sense and engage with one another's ideas.

Examples of CTHs and Corresponding Student Contributions By Cluster

In order to further situate these clusters, we share a representative exchange from a classroom in each cluster.

Cluster 1, Minimal Use of Student Ideas. The following exchange comes from a 7th grade classroom where students were guided by the teacher during a lesson about demonstrating the Pythagorean theorem using pictorial models. While displaying a visual of a right triangle, and pointing to the right angle, the teacher asked:

Teacher: What are we claiming that we have here? [pointing to right angle]
Students: Right Angle

The teacher endorsed the response, by repeating it, and continued with their explanation which included prompts for students to give short answers to teacher questions. This exchange was of a unidirectional nature (Brendefur & Frykholm, 2000) in which the teacher directed the instruction, and students were asked closed questions. Such exchanges typified lessons in cluster 1 in which few, if any, CTHs occurred. Note, an exchange such as this, with short "fill in the blank" style answers, were not categorized as any of the *Share* codes (2nd, 3rd, 4th in Table 1). The MHT codes require a request for more substantive student contributions before such thresholds are reached.

Cluster 2, Some Use of Student Ideas. In cluster 2, the lessons were characterized by *share why* CTHs and teacher revoicing of student ideas. For example, consider the following exchange from an 8th grade classroom that focused on converting between scientific and standard notation:

Teacher: It's in the tens place, but why I'm [sic] adding a zero? (CTH: Share why)
Student: In the tens, the power says two numbers. (HoI: Explain)
Teacher: Oh, very good, it says that there are two numbers. (CTH: Teacher revoice)

In this exchange, the teacher prompted the students to explain why a zero was added into the tens place based on the power. The student provided a short explanation, and then the teacher revoiced the explanation. This interaction was typical within the entirety of the lesson in which the teacher guided the exploration, but requested contributions beyond short answers from students that were then acknowledged/evaluated and revoiced.

Cluster 3, Use and Press for Expansion of Student Ideas. This next exchange comes from a 4th grade classroom during a lesson centered on understanding the definition of a pyramid. A student volunteered the definition as a solid with a polygon base. The teacher began working with this idea:

Teacher: Hmmm, what is a polygon. A square is a polygon. Triangle is a polygon. What makes those polygons? (CTH: Prompt to Share Meaning)
Student: They are closed shapes that have a straight line. (HoI: Explain)

At this point, the teacher prompted the student to repeat the idea, then the teacher revoiced the idea, “straight lines, closed shapes.” The teacher then continued to prompt students to add onto this definition. In contrast to the prior example, the teacher prompt was more than just a prompt to share reasoning, but to connect a student idea to mathematical meaning.

Cluster 4, Presses for Engagement with (and Uses) Student Ideas. The subsequent exchange comes from a lesson about volumes of rectangular prisms in a 5th grade classroom. A student was presenting their solution to a problem that asked them to find volume, and the teacher asked this student to compare their contribution to a previously discussed student strategy:

Teacher: [Student A], how could you tell [Student B] that he actually did something similar, what did he actually use? (CTH: Prompt to Compare Student Ideas)

Student A: Well, he used addition... and it’s basically the same thing because with this like the 3 is kind of like right here and then it would be like 7 times the width like that. (HoI: Compare)

The teacher followed up by restating Student A’s multiplication strategy to explicitly connect to the addition strategy offered by Student B. Having students explain and compare each other’s ideas distinguished cluster 4 from the other clusters.

Situating the CTH Profiles in Terms of Student Contributions

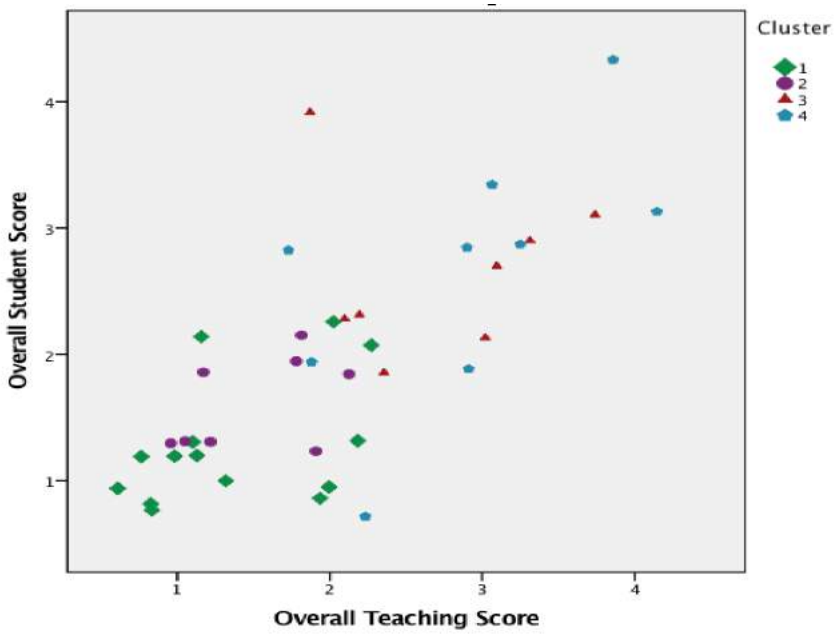

Figure 3: Scatterplot Representing Lessons on Overall Student and Overall Teaching Scores by Cluster. Note: the dots are jittered to be visible.

In order to further situate these results within the whole lesson overall, we explored how these clusters related to *overall student* and *overall teacher* scores (see Figure 3). First, we note that there *is* a substantial positive relationship between *overall teaching* and *overall student* scores as measured via the MHT. An increase in *overall teaching* score was associated with an increase in the odds of higher *overall student* score, with an odds ratio of 2.494 (95% CI, 1.410 to 3.579), Wald $\chi^2(1) = 20.314$, $p < .001$. If we then look at our lesson clusters we can discern that cluster 1 tends to have low ratings for both teacher and student. Cluster 2 includes both low and middling scores on both, and lessons in clusters 3 and 4 tend to have higher *overall teaching* and *overall student* scores.

Figure 4: Boxplots of Frequencies of Student Contributions by Cluster (as reflected by Habit of Interaction occurrences)

Further analysis examines whether these CTH prompts for more student-to-student engagement (as evidenced in cluster 4) actually relates to increased amounts of such interactions by students. A glance at student contributions paints a similar picture (see Figure 4). When comparing the number of occurrences of habits of interaction (HoI; a proxy for student math idea contribution), a one-way ANOVA identified significant differences between clusters ($F(3)=11.1215$, $p<.001$). The average number of student HoI in a cluster 1 lesson was 3.00 (sd=2.386), cluster 2—8.25 (sd=8.892), cluster 3 – 19.875 (sd=10.789), and cluster 4 – 28.889 (sd=19.915). A Tukey HSD post hoc test identified significant differences between cluster 1 and 3, 1 and 4, and 2 and 4. This reflects that variation in catalytic teaching habits seems to correspond to the number of math contributions from students.

Discussion and Future Research Plans

This analysis serves as an initial view into profiles of different mathematics classrooms. In nearly all of our analyzed lessons, teachers pressed for students to contribute to the lesson. However, the nature of these presses and how teachers worked with student ideas varied. We identified two types of lessons that were not characterized by rich use of student ideas: lessons with minimal prompting for mathematical reasoning and lessons in which student ideas were asked for and revoiced by the teacher. We also identified two distinct, productive types of lessons: those focused on generating and expanding mathematical ideas and those focused on engaging students with each other's mathematical ideas. We conjecture that both types of lessons are essential for students to engage richly with mathematics.

The aim of the MHT is to complement existing analyses of classrooms including: qualitatively robust analyses of classrooms (e.g., Stein, et al., 2008; Staples, 2007), and quantitative analysis based on a set of overall scores (e.g., Lynch, Chin, & Blazar, 2017). This existing research base has established the productivity of particular teaching moves in case studies, and that measures of overall quality of instruction can be linked to student achievement. Our work acts as a bridge between the detail of qualitatively analyzing lessons and the power of quantitative analysis of many lessons. By identifying (literature-based productive) teacher moves and student contributions in-the-moment across many lessons, we are able to profile various types of lessons. In this phase, we are creating profiles via the types of CTHs occurring. This is paired with an initial analysis of student contributions that reflect a high degree of relationships between CTH lesson profile and student contributions.

Through the course of this project, we plan to enrich and expand these initial profiles by including more than twice the number of lessons presented here and incorporating further information about student discourse. For example, this initial analysis reveals that the presence of CTHs can statistically cluster lesson types; that these lesson types are related to correlations between higher *overall teacher* scores and higher *overall student* scores. For example, lessons in cluster 4 are associated with high *overall student* scores as well as included more CTHs prompting student-to-student interaction, and nearly 10 times as many student HoI codes when compared to cluster 1 lessons. Furthermore, the student codes for habits of mind (HoMs), of particular interest to many educators, could be further linked to specific CTHs as well as to larger teaching routines. Focusing primarily on HoIs, we have not discussed HoMs in detail in this report. Where HoIs focus on the existence of student contributions and interaction, HoMs focus on the mathematical practices embedded within those interactions. For example, a student can share their thinking (an HoI) and within that exchange can make reference to a graphical *representation* as well as generate a *conjecture* (HoMs). The skeleton of such a relationship is building—CTHs are denser and more varied within clusters 3 and 4 and lessons in those clusters are largely associated with high *student overall* codes. Furthermore, clusters 3 and 4 also reveal far greater frequencies of student HoI codes.

In our next phase of analysis, we plan to expand out the components in the cluster analysis, move towards a distance metric that is non-binary to account for frequency, and begin cluster analysis leveraging the timing of the teacher and student contributions. We look forward to further exploration of the mechanisms that may promote more student interactions and deeper student engagement with mathematics.

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TEACHING AND CLASSROOM PRACTICE:

BRIEF RESEARCH REPORTS

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PRACTICES OF FACILITATORS WHEN PLANNING MATHEMATICAL MODELING ACTIVITIES IN AN INFORMAL SETTING

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This qualitative study used Yinger's (1980) process model to investigate mathematical modeling facilitators' planning practices. Four university-based facilitators planned and implemented mathematical modeling tasks during a summer camp for middle and high school youth. Analysis of interviews and planning documents revealed that in the problem-finding phase, facilitators engaged in joint work (brainstorming) and used necessary resources (social, material, and conceptual) and prepared lesson plans. In the second phase, facilitators elaborate on the instructional ideas they had generated at the problem-finding stage. Finally, in the third phase, they categorize their solutions as successful or not and could either add to their repertoire of knowledge (routinization) or go through the process all over to come up with a refined solution. Strategies employed by participants in this study extend our knowledge regarding the facilitators' practices at the planning phase of modeling activities

Keywords: Modeling; Instructional Leadership; Teaching Tools and Resources; Instructional Activities and Practices; Informal Education

Introduction and Background

Mathematical modeling, an emerging branch of K-12 mathematics learning, is a process by which mathematicians develop and use mathematical tools to represent, understand, and solve real-world problems (Lesh and Doerr 2003; Ang, 2004). Students do not always see connections between life and mathematics and miss opportunities to apply what they learn in mathematics to the situations around them (Tran & Dougherty, 2014; Verschaffel, De Corte, & Vierstraete, 1999). Mathematics teachers play a significant role in making students see the relationship between mathematics and the real-world through effective teaching.

Effective teaching is a product of a well-planned instructional process. Efficient planning is essential in teaching mathematical modeling because of its demanding nature for teachers during its implementation. This demanding nature occurs because teaching is more open and less predictable in modeling situations (Carlson, in press; Cai, et al., 2014). During planning, teachers anticipate and prepare to respond to students' mathematical ideas, questions, and challenges. They consider strategies students are likely to use, develop responses to them, as well as strategies that might be productive to highlight during the lesson (Smith & Stein, 2012).

Purpose of the Study

This study describes the planning practices of four facilitators (faculty members) of modeling activities during a summer camp for middle and high school youth. This study will provide insights for researchers who are working to understand how facilitators plan and implement mathematical modeling activities, especially in an informal setting. In this study, an informal setting refers to the modeling environment where activities are independent of the class curriculum, age, or grade. Research questions investigated are: What problem-finding strategies do facilitators exhibit when planning modeling activities? How do facilitators elaborate, investigate, and adapt these strategies in their planning of modeling activities? How do facilitators evaluate these strategies?

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Conceptual Framework

Yinger (1980) identified three stages of planning: the problem-finding, problem formulation/solution design, and implementation, evaluation, & routinization. He theorized that the problem-finding step is the discovery of a potential instructional idea that requires further elaboration at the problem formulation stage. At the formulation stage, the dilemmas from the problem-finding stage are continually expounded and verified mentally until a desirable solution is obtained through elaboration, investigation, and adaptation. The planner relies on their repertoire of methods for solving problems during elaboration and investigation phases. Yinger viewed the implementation, evaluation, and routinization phase as part of teacher planning, and this is where the formulated solutions are tried out and assessed. Whatever the outcome is at this stage, successful or not, it forms a basis for knowledge and experience for future planning.

Methodology

Setting and Participants

This study is a part of a larger project focused on rural youth learning to use mathematical modeling to investigate issues in their home communities. Twenty-nine youth and nine adult mentors from six rural communities participated in a summer camp where they worked on modeling activities. The facilitators were the participants for this study, and they are all university mathematics teacher educators and researchers on the larger project.

Data Collection

Data sources included planning documents of modeling activities during camp and semi-structured interviews with the four participants. See modeling activities in Table 1.

Table 1: Description of modeling activities at the summer camp

Task	Description	Purpose
Cookie Task	This activity allows youths to decide what chocolate chip cookie is "best." The problem springs from a real-world situation, and its solution allows approaches from many perspectives.	• To use qualitative attributes in a mathematical model - How do we deal with a primarily qualitative question like "best" in mathematical modeling.
Agent-Based Modeling	This activity models some complex systems in the world. For instance, modeling a population involving healthy and sick people and how the system evolves with time.	• To learn how a technology-based approach could be used to model a real-life scenario—for instance, using NetLogo to model sick and healthy people.
Mapping Activity	Activity that enables modelers to create representations of their lived experiences in map format.	• To know the importance of maps and realize how values effect info stored on maps.

The planning documents for these activities were studied and used to develop an interview protocol that served as the primary data source. I conducted an average of 60 minutes of semi-structured interviews with each participant roughly 18weeks after the camp. I coded the transcripts, organized the codes into themes, and then organized the themes around phases of the process model.

Data Analysis

I commenced the analysis by looking at lesson plans facilitators prepared for camp activities. Essential keywords that captured the initial analysis of facilitators' intentions formed the basis for the interview protocol developed for the interview. These keywords include activity goals - modeling goals, mathematics goals, & statistical goals; resources for facilitation; modeling practices; anticipated challenges and strategies to respond; and modelers' exploration. I organized raw data from participants' interviews to make sense of the contents and then conducted open and axial coding (Corbin and Strauss, 2007). Then I looked for connections that linked categories of codes to the three phases of process model: problem-finding, problem formulation; Implementation, evaluation, and routinization.

Results

I organized the findings of this study by the three stages of the process model to provide a holistic understanding of the data collected.

Problem-finding

When planning modeling tasks, facilitators engaged in joint work (brainstorming). They used social, material, and conceptual resources, prepared lesson plans and relied on teaching experiences. They also embraced multiple perspectives, constructed questions to launch the activity, guessed modelers' ability, and considered the interest of modelers. Sally, in her words, said: "Theresa and I met and kind of just worked through it. Furthermore, during the prep week, actually doing it and getting feedback from there helped us figure out more details of how we are going to facilitate it." Facilitators also conducted ethnographic research about modelers' environment and their interest in understanding youth characteristics. About the Agent-Based Model, Edward said:

> One of the groups talked a lot, during our ethnographic research in their community, about physically closing off some lanes of traffic and one of the ways that you can understand or make predictions about that is through an agent-based model.

Knowledge and past experiences of facilitators interrelate with teaching goals and materials (resources) when finding instructional ideas for a new modeling task. Edward described how his previous experience with Agent-based Modeling directed him to locate the right human resource who helped out in his planning.

> I have some familiarity with the Agent-Based Modeling from doing it in some of my classes as a graduate student. Nevertheless, it was not really on my radar. Here is what happened, Kelvin, who was sort of founding participant in this project, is vast into technology. One of the technologies that he was thinking about was agent-based modeling, which made sense to me because I had experience with it.

When giving an account on how her background knowledge with modeling interrelates with the teaching goals when preparing for the cookie task, Isabella, said the following:

> Thus, the cookie task – we thought of a few reasons for choosing it. One is that we thought it was something that would be fun and low stakes for the first night of camp. We thought that it would be something that all students know something about it. Moreover, we also thought that it was an activity that would facilitate work through a lot of the modeling cycle.

These scenarios show how facilitators discover potential instructional ideas for efficient planning. These ideas are then elaborated at the problem-formulation stage of their plan.

Problem formulation

At the problem formulation stage, knowledge and experiences of facilitators played essential roles. Their past experiences enabled them to anticipate what modelers might do on the tasks. To do this efficiently, they needed to elaborate on the instructional ideas they had generated at the problem-finding stage. At this stage, elaboration means creating a suitable response to the ideas discovered in the problem-finding phase. Isabella described doing this via deep thought: "As a team, we thought deeply about kind of different knowledge bases they might bring to bear on problems, especially their local problem-solving practices."

When elaborating, facilitators perceived their roles like that of a "coach" who gives support Theresa recounted that her role is to set modelers up to interpret the real-world to their mathematical world with little or no help from the facilitator. To achieve this, facilitators needed to anticipate modelers' behavior during Implementation. They narrated that modelers bring different initiatives to modeling space, a situation that facilitators must speculate when planning.

Implementation, evaluation, and routinization

Implementation, evaluation, and routinization is the last phase in the instructional planning process. In this phase, facilitators try out solutions developed at the problem-solving stage. Facilitators in this study did this through the preparation week preceding the camp week for evaluation. Sally gave the following response to a prompt: “And during the prep week, actually doing it and getting feedback from there helped us to figure out more details of how we are gonna facilitate it.” At this stage, facilitators considered solutions to be successful or not. If successful, they would add the solutions to their repertoire of knowledge (routinization). Otherwise, they would iterate the process for a refined solution.

Discussion and Conclusion

This study sought to document facilitators' planning practices of modeling activities. From the results and analysis, I found some practices to be general to facilitators in this study. These results suggest that facilitators build on their previous facilitating knowledge to anticipate what modelers would do. The facilitators paid attention to modelers' interests and what they care about when engaging in problem-solving. Facilitators also considered how modelers would engage with the task and think about multiple perspectives modelers would bring into the modeling space. Facilitators anticipate what modelers will do and will not do on a task and identify where and when modelers might deviate from a better approach to solving a task.

For this reason, facilitators thought about modelers' possible approaches to solving modeling tasks. Facilitators considered necessary resources to accomplish tasks and considered multiple perspectives in their approaches by brainstorming with colleagues to anticipate what modelers would do in various dimensions. Finally, they thought about questions modelers could ask and then prepare some canned responses. This study agrees with Munthe & Conway (2017) on how planning involves shared knowledge construction and professional learning. This study also finds that planning is a process of preparing a framework guiding teachers' actions. As Young (1998) found what teachers do at planning to include identifying content, developing a timeline, identifying goals, skills, and objectives, deciding on instructional materials and so on, this study finds that facilitators of modeling identified settings, resources, and teaching goals as necessary when planning modeling tasks. As this study took place in an informal setting, findings herein may not necessarily be generalizable to a curriculum-based classroom but can be helpful. For future work, researchers may consider investigating facilitators' planning practices in a formal setting. The findings from this study are productive strategies that could be reproduced by researchers, especially in an informal setting.

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EXPLORING TEACHERS' CONSTRUCTIONS OF EQUITY IN MATHEMATICS EDUCATION: AN ECOLOGICAL APPROACH

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Research in mathematics education over the recent decades has resulted in a large number of definitions, conceptual framings, and operationalizations of what it means to do equitable teaching. An exploration of the activity system of equitable mathematics teaching is necessary to synthesize current literature and to work from teachers' current understandings of equity in education. A systems-approach to exploring equitable mathematics teaching is necessary to capture how individuals navigate structures of culture, power, and privilege to engage in equity work. We must re-center the voices of teachers to understand how they construct notions of equity in mathematics education. The object-constructions held by teachers inform their goals for instruction, which then influence the types of instruction enacted in classrooms.

Keywords: Equity and Diversity; Instructional Activities and Practices; Instructional Vision; Systemic Change

Equity research in mathematics education has only grown in the last few decades, resulting in a variety of perspectives on and definitions of equitable mathematics teaching and learning. The PME-NA Equity Statement captures a broad swath of framings: "include ideas ranging from access to educational resources, to positioning students as capable and humans as valid sources of knowledge, to questioning the curriculum and high stakes assessment practices, to promoting critical social justice perspectives of mathematics as sociopolitical." These variations in definitions and resultant frameworks guiding practice exist for researchers as well as practitioners. It is critical to clarify how teachers define equity, for this "directly relates to how we seek to both measure and achieve it in our schools" (Gutiérrez, 2002, p. 152). Teachers work within classrooms, school departments and districts, and broadly as part of the professional community, to achieve equitable mathematics teaching. Explicating how teachers construct their understandings of equity and social justice provides context for unpacking the goals they hold for instruction.

Researchers have explored teacher identities in the classroom and how that impacts attention towards equity and social justice in schools, broadly, or pedagogy, locally (Wager, & Foote, 2012). Others attend to what teachers are disposed to notice in classroom interactions (Edwards, 2011; Hand, 2012) or the orientations that drive decision-making in instructional moments (Schoenfeld, 2010). One's personal experiences, values, and beliefs influence the ways teachers engage in the profession, especially around equity work (Gutierrez, 2002; Wager & Foote, 2012). Commentaries on the roles power structures, in the form of culture (Louie, 2017, 2018), race (Martin, 2009), class, and whiteness, among other social identifiers the perpetuate hierarchies of inequity, explore how the teaching and learning of mathematics is inherently situated within systems that privilege certain perspectives of the discipline (Rubel, 2017). Although there is a rich body of research on equity in mathematics education that focuses on teacher conceptions and identity, and a separate but equally fertile body of literature on equity in mathematics education that addresses issues of power and oppression, literature that attends to both is still emerging. Reed & Oppong (2005) worked with teachers on their definitions of equity, noting how race and class influence how equity is carried out in practice. Bartell (2013) explored how teachers' goals for instruction may align with goals for social justice. While these arguments are critical, we must consider how systems interrelate with activity on an empirical scale to understand how these relationships play out within instructional

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settings, departments and districts, as well as interactions among members of the profession. A common implication for future scholarship across these studies explicates this need: a call for systems-focused research on *equitable mathematics teaching* that investigates how structures of power and culture interact with teachers' goals for instruction and their resultant practice (Louie, 2018).

Systems-focused research would target the integration of micro-, meso-, and macro-environments that influence how teachers make sense of and work towards more equitable forms mathematics teaching and learning. Bronfenbrenner states, "studies of learning should take into account the social ecology that forms the context for human activity. An ecological approach considers the development of an individual in relation to the "immediate environment, and the way in which this relation is mediated by forces emanating from more remote regions in the larger physical and social milieu" (1979, p. 13). Teachers' commitments to equity may draw upon their personal experiences in and outside of the classroom environment, as well policies or practices held as normative within their school organization, understandings of the field of mathematics educators interested in social justice work, and broader understandings of the ways societal hierarchies of power and privilege shape teaching and learning for individuals. An ecological approach frames ones' experiences within the cultural and historical milieu that make meaning through personal and professional commitments to equity and how one acts in service of those commitments.

Theoretical Framework

I leverage Engeström's (1987) Cultural-Historical Activity Theory (CHAT) as a way to capture how teachers navigate interlocking systems to engage in the activity of equitable math teaching. Activity "involves people operating jointly in a persistent system of relations with other people and institutions," asking us to conceptualize equitable mathematics teaching as something which is constantly developing through the joint work of teachers with others in the community (Foot, 2014, p.9). Communities are not defined by proximity, but span place and space in pursuit of some shared values or goal (Wenger, 1998). Members of an activity system identify a specific need, or an object, that drives collective action. In this instance, we consider the collective of teachers across the profession actively oriented towards the object of equity in mathematics education.

Objects and Object Construction

The motivating need, or object, of an activity system is a complex idea that cannot be explicitly identified or captured, but rather, shifts and expands as actors within the system work to achieve it. An object is worked-towards on an individual level, by subjects setting and achieving goals through actions. Individuals within the activity systems may hold varying constructions of the object under focus that shape the goals they set (Engeström, 1987). For example, some teachers may consider equity in mathematics education as the equitable distribution of opportunities to learn, while others may prioritize curriculum that are relevant to students' lives (Bartell, et al., 2008). Engeström (1990) notes that the historical development of object-constructions - in this case, what equity in mathematics education has looked like and meant throughout time - affords and constrains how teachers perceive of and engage in it, including the resources and conceptual tools they take up to guide their work. It is also important to note the teacher's personal experience can include their learning journey, professional experience, their positions within power structures, and environmental characteristics of their classroom, school, and surrounding contexts (Foot, 2014).

A teacher's object-construction of equity in mathematics education informs their goals for instruction. Goals may directly or indirectly relate to the object-conception held by the subject; for example, a teacher whose object-conception of equitable mathematics teaching is that of Teaching Mathematics for Social Justice (TMfSJ) (Gutstein, 2006), which includes the use of socio-political mathematical tasks in the classroom as a key component, their goal might be the implementation of a particular curriculum across the year. Another goal for that teacher may be incorporating reflective questions and discourse into their mathematics tasks so that students can engage actively in reading the world using mathematics. Both of these goals are tangible, actionable steps the teacher can work to achieve that serve the object-conception of engaging in TMSJ. The object as each individual has constructed it will lead to different actions within the activity system. Unpacking the ways teachers construct equity and social justice in relation to mathematics teaching and learning provides opportunities to clarify how they move towards instructional goals that align or contradict those intentions (Bartell, 2013).

The research question explored in this presentation is part of a larger study that attempts to explicate the activity system of equitable mathematics teaching. Foot (2014) comments that "understanding an activity system requires understanding its object" (p. 10); thus, to understand the object of equity in mathematics teaching, we must first explore how teachers involved in the activity system construct their object-conception and related goals for teaching. Thus, this session explores how teachers committed to equity and social justice construct the object of equity in mathematics education. Further, how do teachers draw upon micro-, meso-, and macro-levels of educational systems in their constructions and resultant goals for instruction?

Methodology

This study collaborates with secondary math teachers committed to equitable mathematics teaching to understand how they construct the object of equity in math education. Participants are mathematics educators at a non-profit educational organization for rising middle school students in the Bay Area. This program's mission is explicitly oriented towards creating equitable educational spaces for students, and this mission is a key factor in hiring. Educators in this organization have made an explicit commitment to equitable teaching through their employment status and program-offered professional development opportunities to reflect on their teaching and inequities in education. All participants are licensed educators, yet their experiences teaching in a non-traditional learning environment offer considerations for disrupting existing educational systems and transforming spaces for learning towards more equitable ends (Freire, 2000; hooks, 1994; Martin, 2009).

The participants engaged with questionnaires and follow up interviews to explore their commitments to equity in teaching mathematics. The questionnaires provided a baseline operationalization for how each teacher constructs equity in mathematics education and how they see it play out in an ideal classroom setting. A series of three interviews following the questionnaire allowed opportunities to probe for more detail and to have participants explain their experiences and perspectives that inform their object-construction. Each interview, and subsequent analysis, attended to a different layer of micro-, meso-, and macro-level ecological systems. Analysis of the data included iterations of structural coding and inductive thematic coding (Auerbach & Silverstein, 2003). First, data from both sources was linked for participants and segmented by topic, which provided context for codable instances and captured detail on the ways teachers saw equity issues in their practice. Next, I applied structural codes, noting when teachers drew upon micro- (such as classroom tools or norms), meso- (like site or program policies for mathematics teaching), and macro-systems (for example, the resources available in the broader professional community for TMfSJ or ideological systems like racism or whiteness) as they construct and work towards goals for equitable mathematics teaching and learning. I coded all teacher responses, allowing their language to drive the creation of themes for how teachers in the activity system of equitable mathematics teaching construct the object of equity. Across these codes, trends emerged that outline the landscape for how teachers make sense of equity in their practice. Throughout this process, I continuously engaged in member-checking with participants to accurately amplify their voices and regularly constructed memos to process my positionality and understandings of participant experiences (Auerbach & Silverstein, 2003).

Findings and Discussion

The study is ongoing, and thus, there are no clear themes to report as of yet. However, the expectation is that teachers generally conceptualize equity in ways that have been previously discussed in the field, though not consistently explicitly linking to the frameworks with which their constructions of equity are aligned. Teachers describe aspects of equity in mathematics to contextualize how these constructions are worked upon in practice, connecting to their goals for mathematics teaching and learning. These goals will provide nuance to aid in explicating how equitable mathematics teaching is understood and taken up by committed practitioners, including understanding the tangible goals for instruction each is oriented towards. Finally, these responses illuminate how teachers recognize, draw upon, and negotiate concentric systems of education. For example, how might one teacher's construction of equity in mathematics education as a status concern between students, drawing on the work of Complex Instruction (Cohen & Lotan, 1995) (macro-) align or contradict with departmental expectations for tracking students into courses (meso-) or their instructional strategies for inviting classroom discourse (micro-level).

This session contributes to the field of research on equity in mathematics education by centering teachers' constructions of equity and attending to how these constructions shape and are shaped by their goals for instruction. The lens of Engeström's (1987) CHAT provides opportunity to highlight ecological systems teachers work within as they negotiate their practice. I draw explicitly on notions of objects and object-constructions to understand how teachers committed to equity in mathematics education makes sense of this driving object and how their constructions are both similar and different. Further, I consider the link between one's construction of equity in math education and the goals they hold for instruction to understand how teachers are acting towards their object-conceptions. These results provide a more nuanced understanding of how teachers take up the work of equitable mathematics teaching within their educational contexts.

This research is part of a larger study that aims to articulate the activity system of equitable mathematics teaching. As objects are one of the centering tenets of an activity system, it is paramount we begin describing the activity system with the collective themes for how teachers construct equity in mathematics education. Future goals of this research include understanding how teachers committed to equity work towards their goals for instruction, employing equitable mathematics teaching practices and navigating systems in their disruptive action. This study will support grounding research on equitable mathematics teaching in the lives and work of teachers committed to equity. I also claim that the systems-level approach will bring light to the contradictions and tensions across everyday professional practice, which in turn opens space for professional development, restructuring of school policies, and future research on equitable mathematics teaching to explicate and reduce these challenges so that the field can more successfully move towards our object of equity in mathematics education.

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PRESERVICE TEACHERS' PERCEPTIONS OF DEPICTIONS OF MATHEMATICS TEACHING PRACTICE WHEN ENDURING INDIVIDUAL CHARACTERISTICS ARE INTRODUCED

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This study investigates the extent to which the introduction of enduring individual characteristics of students and teachers in depictions of teaching practice produces systematically different responses from preservice teachers. Enduring individual characteristics include characters' skin tones, names, and descriptions of the school and school community. Fifty-six preservice teachers were randomly assigned to one of two survey depiction formats: one including enduring individual characteristics of students and teachers. Teacher practices and student problem solving were held constant across both formats. Results indicate that, for several survey items, participants responded differently depending on the survey depiction format they were assigned. Interpretations of results suggest that enduring individual differences may be of critical importance to include in rich media resources utilized in mathematics teacher education.

Keywords: Teacher Beliefs, Teaching Tools and Resources, Teacher Education - Preservice

Purpose of the Study & Guiding Framework

Online, rich media platforms are transforming the ways individuals across a range of professions are prepared and practice. One such platform, *LessonSketch*[1], allows mathematics teacher educators and preservice teachers to develop and engage with materials where users can create, share, and discuss scenarios that represent classroom interaction (Herbst & Chieu, 2011). Initial uses of LessonSketch deliberately provided depictions of teaching practices absent of individual characteristics. As Herbst et al. (2017) describe in prior work, *LessonSketch* characters were nondescript characters whose role was to depict practice rather than individuals. However, later updates to the platform began to incorporate contextual markers in teaching classrooms, such as skin tone, hairstyles, and body size. Furthermore, Herbst et al. (2017) describe the differences between the original, generalized depictions in earlier versions of LessonSketch as *enacted individual differences* (e.g. facial expressions, body orientation), and the updated contextual markers as *enduring individual differences* (e.g. body size, race, gender, or class).

The introduction of the option of incorporating enduring individual differences in depictions of instructional practice allows for the opportunity to explore the complex nature of the role of enduring individual differences in preservice teachers' perceptions of classroom interactions. While teachers may outwardly and consciously hold beliefs that all children can learn mathematics, a life immersed in the social discourse of gender, racial, and wealth hierarchies may lead them to rely on enduring individual differences in their interactions with students in ways that teachers may not be aware (Clark, Whitney, & Chazan, 2009). This study aims to explore the instability of the relationship between teacher resources, instructional practice, and student learning due to a host of normative, instrumental, and situational factors that influence a teacher's affective and cognitive resources in

[1] *LessonSketch* is designed and developed by Pat Herbst, Dan Chazan, and Vu-Minh Chieu with the GRIP lab, School of Education, University of Michigan. The development of this environment has been supported with funds from National Science Foundation grants ESI-0353285, DRL-0918425, DRL-1316241, and DRL-1420102. The graphics used in the creation of these storyboards are © 2015 The Regents of the University of Michigan, all rights reserved. Used with permission.

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varied and specific racial contexts and at particular moments in time (Chazan, Herbst, & Clark, 2016). Our research question for this exploratory study is: *Do preservice teachers systematically respond differently to rich media depictions of mathematics classrooms when enduring individual differences are introduced?*

Methods

Our research question has important sub-questions. As we seek to investigate whether the introduction of enduring individual differences produces systematically different responses to depictions of teaching practice we also want to know, if so, where? And, which teacher practices produce different results? Further, we contend that enduring individual differences such as skin tone may contribute to systematic differences due to implicit bias (Greenwald & Krieger, 2006). It should be noted that this study is an exploratory one; the broader research questions we provide cannot be answered substantially through this study alone. Further work and refinement are necessary.

Participants were presented with scenarios of mathematics classroom interactions. The design of scenarios and survey questions was guided by several frameworks utilized in teacher education, mathematics teacher education, and mathematics education research (Hiebert, 1986; Martin, 2000; McKown & Weinstein, 2008; National Governors Association, 2010; TeachingWorks, 2020). Participants then answered 120 questions related to the scenarios. Participants were randomly assigned to view and respond to a format of the scenarios with one of two different degrees of individuality: *enacted only individual (henceforth enacted) difference and enacted and enduring (henceforth enduring) individual difference.* Enacted individual difference depictions (Figure 1) do not contain any visual or descriptive markers such as skin tone of students and teachers; enduring individual difference depictions (Figure 2) contain such markers. The depicted students' mathematical thinking and students' mathematical practices are held constant across both formats. The depicted teacher's instructional practices are also held constant across both formats. Twenty-eight preservice teachers responded to the enacted individual difference survey format and 28 preservice teachers responded to the enduring individual difference survey format.

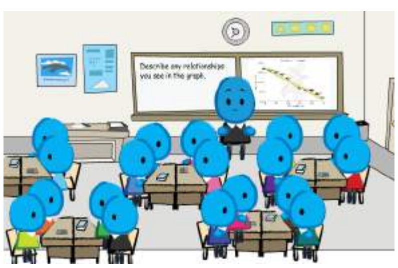

Figure 1. Enacted Individual Difference

Figure 2. Enduring Individual Difference

The survey consisted of three sections: a division scenario, a multiplication scenario, and questions related to school and classroom context. The majority of survey questions were measured on a 6-point Likert scale (from strongly disagree to strongly agree).

Results

Results are indicated in the tables below. The items contained in the tables refer to items where preservice teachers assigned to the enacted difference form responded significantly differently to the preservice teachers assigned to the enduring difference. In Tables 1 and 2, items are grouped by the extent to which they focus on teacher practice or student thinking.

Table 1. Division Scenario

Scenario 1: Students are placed in pairs and assigned division problems. Student A uses a non-traditional algorithm and Student B is confused. The teacher interacts with the pair of students and encourages them to practice the traditional algorithm. Student B raises questions suggesting that he has some conceptual understanding of the nontraditional division algorithm.

Survey Format		Enacted difference (Blue skin tones)		Enduring difference (Brown skin tones)		Mann-Whitney U
Perceptions of Teacher Practice		Mdn.	Mean	Mdn.	Mean	Sig.
Q20	The teacher is effectively setting up and managing small group work.	3	2.75	2	2.18	0.038
Q30	The teacher is effectively checking for student A's procedural understanding of division.	3	3.04	2	2.25	0.015
Q31	The teacher is effectively checking for student B's conceptual understanding of division.	2	2.46	2	1.71	0.007
Q32	The teacher is effectively checking for student B's procedural understanding of division.	2	2.39	1	1.57	0.001
Perceptions of Student Thinking/Cognition		Mdn.	Mean	Mdn.	Mean	Sig.
Q26	Student B is likely to do well on the division problems on the chapter test.	2	2	3	2.82	0.003
Q41	For each student, indicate if they are most likely above level, on level, or below level-- Student B*	0	0.18	0.5	0.54	0.011

* Q41 was measured on a 3-point scale, from below-level (0) to above-level (2)

Table 2. Multiplication Scenario

Scenario 2: Students are placed in pairs and assigned multiplication problems. Student D uses the traditional algorithm. Student C computes answers through use of the partial product method. The teacher acknowledges that partial product method but encourages both students to use the traditional algorithm for efficiency and accuracy on the test.

Survey Format		Enacted difference (Blue skin tones)		Enduring difference (Brown skin tones)		Mann-Whitney U
Perceptions of Teacher Practice		Mdn.	Mean	Mdn.	Mean	Sig.
Q53	The teacher is effectively managing small group work	3	3.29	2	2.68	0.030
Q70	The teacher should review the traditional algorithm to multiplication with all students.	5	4.71	4	4.21	0.036
Perceptions of Student Thinking/Cognition		Mdn.	Mean	Mdn.	Mean	Sig.
Q76	Student C was likely assigned as the helper in the group.	2	2.21	2.5	3	0.036

Table 3. School and Classroom Context

School and classroom context: Participants are asked to respond to questions related to how they perceive aspects of the school and classroom.

Survey Format		Enacted Difference (Blue skin tones)		Enduring Difference (Brown skin tones)		Mann-Whitney U
		Mdn.	Mean	Mdn.	Mean	Sig.
Q86	Do you think this is most likely a 3rd, 4th, 5th, or 6th grade class?*	4	3.75	4	4.11	0.014
Q88	I would feel comfortable teaching this class.	5	4.71	4	4	0.003
Q89	I would feel comfortable teaching at this school.	5	4.54	4	3.64	0.001
Q90	I would want access to instructional support if I were teaching this class.	5	4.64	4	3.86	0.006
Q91	I would want access to instructional support if I were teaching at this school	5	4.75	4	3.79	0.000

*Q86 was measured on a 4-point scale, from 3rd grade (3) to 6th grade (6).

Discussion

The results of this study suggest that, for several survey items, preservice teachers' perceptions of depictions differ when the depiction formats vary by the inclusion or exclusion of enduring individual differences of depiction characters. In particular, preservice teachers perceived that teachers' practices associated with the management of small group work and checking for student understanding was less effective when brown skin color tones of characters were introduced to the depiction. Furthermore, preservice teachers were more likely to assign a higher grade level to characters with brown skin tones. Lastly, preservice teachers reported that they would be less comfortable teaching the class or in the school when brown skin tones were introduced. Overall, findings suggest that further exploration is needed to better understand if preservice teachers' perceptions are influenced by the introduction of enduring individual characteristics, and, further, if influenced by the introduction of specific racialized enduring individual characteristics such as brown skin tones.

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INSTRUCTIONAL TENSIONS FACED WHILE ENGAGING HIGH SCHOOL GEOMETRY STUDENTS IN SMP3 TASKS

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This study engaged HS geometry students in the reasoning-and-proving process through the use of novel tasks aligned with Standard for Mathematical Practice (SMP3) (construct viable arguments and critique the reasoning of others). The tasks facilitated opportunities for students to engage in SMP3 by (a) proposing a conjecture; (b) drafting an argument for their conjecture; (c) critiquing each other's arguments; and (d) revising their arguments based on peer feedback. In this study, we describe the instructional tensions that surfaced during the implementation of the tasks and the way the teacher addressed those tensions in her class (Berry, 2007). The two most common tensions were between action and intent when launching the tasks and between telling and growth during the draft and critique phases. Findings raised important questions of how to support students in learning what counts as a mathematical conjecture or critique.

Keywords: Instructional activities and practices, Reasoning and Proof, Instructional Vision

Introduction and Purpose

The Standards for Mathematical Practice (SMP) articulate eight domains of mathematical thinking students should gain expertise in across K-12 grades (National Governors Association [NGA] Center for Best Practices & Council of Chief State School Officers [CCSSO], 2010). Specifically, SMP 3 states that students should "construct viable arguments and critique the reasoning of others." Historically, constructing formal deductive arguments (proofs) has been restricted to high school geometry courses (Herbst, 2002). Proof tasks in commonly used U.S. Geometry textbooks provide opportunities for students to engage in some aspects of SMP3, such as posing a conjecture, constructing a proof, investigating a statement, and developing a rationale (justification) for mathematical claims with varying degrees of frequency across categories (Otten, Gilbertson, Males, & Clark, 2014). In contrast, the textbooks analyzed provided relatively few opportunities for students to find a counterexample and did not explicitly ask students to respond to the reasoning of others or construct arguments with the goal of communicating to their peers. Although Otten and colleagues (2014) did not report the percentage of textbook exercises where students were asked to engage in multiple forms of reasoning-and-proving activity within the same task, the differences between categories suggests that students are not consistently engaging in the multifaceted process described in SMP 3.

The purpose of this study was to implement a series of novel tasks designed to engage high school geometry students in the reasoning-and-proving process (Stylianides, 2007) in alignment with the multifaceted approach described in SMP 3. Specifically, the tasks were novel in that students were asked to (a) propose and investigate their own conjecture instead of one provided for them; (b) critique each other's arguments; and (c) revise their argument based on peer feedback instead of teacher feedback. In this preliminary study, we analyze the instructional tensions (Berry, 2007) that arose when implementing the tasks. In doing so, we contribute greater insights into challenges that classroom teachers might face when navigating across classroom cultures, towards one that is centered around students' mathematical ideas instead of one based on ideas presented by the teacher or textbook.

Theoretical Framework

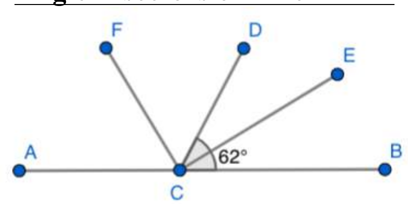
Teacher's instructional decisions are shaped by their personal knowledge and beliefs as well as their obligations to a variety of stakeholders, including the mathematics discipline, individual students, interpersonal dynamics in the classroom, and their broader institutional context (Herbst & Balacheff, 2009; Herbst & Chazan, 2003). Tensional dilemmas, or tensions, surface when there is a contradiction between their beliefs, knowledge, and obligations such that there is no clear decision that adequately addresses all of their concerns (Lampert, 1985). Some teachers choose to prioritize one obligation over another, while other teachers, such as Lampert (1985), instead try to manage the tensions through instructional decisions that reduce the dilemma without completely resolving it. For example, Berry (2007) described six instructional tensions she experienced in her role as a teacher-educator: telling and growth; confidence and uncertainty; action and intent; safety and challenge; valuing and reconstructing experience; and planning and being responsive. Although instructional tensions have been described across multiple contexts (e.g., Berry, 2007; Herbst, 2003; Rouleau & Liljedahl, 2017, Webel & Platt, 2015), more research is needed with respect to teachers' experiences when enacting the Standards for Mathematical Practice (SMPs).

Methods

Instructional Sequence

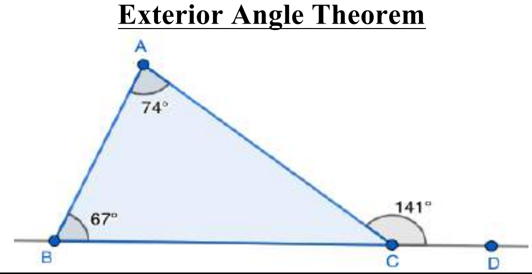
The tasks used in this study were developed with the goal of engaging students in multiple facets of SMP3. Two of the tasks had been previously implemented with secondary students (Conner, 2018); the remaining three tasks were constructed using similar design principles. Each task allowed students to develop a conjecture about a geometric relationship involving an infinite class of objects. The diagonals of a parallelogram and classes of similar polygon tasks also allowed for students to pose and investigate multiple correct conjectures (see Figure 1).

Angle Bisectors of Linear Pair



Given: $\underline{CE}$ bisects $\measuredangle BCD$; $\underline{CF}$ bisects $\measuredangle ACD$

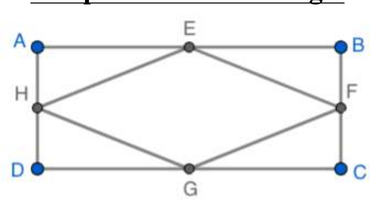
Exterior Angle Theorem



Diagonals of Parallelograms

Draw a few parallelograms on your paper. Draw in the diagonals. Make a conjecture about the diagonals of all parallelograms.

Midpoints of a Rectangle



What quadrilateral is formed when you connect the midpoints of a rectangle?

Classes of Similar Polygons

After describing what it meant for all quadrilaterals to be similar, students were asked to conjecture which classes of polygons (e.g., squares) were all similar to one another.

Figure 1: SMP3 Tasks

The teacher launched each task by posing a scenario for students to consider through the use of a verbal description or a computer-generated representation. Students then formed an initial conjecture about the generalization of the relationship. Next, the teacher either discussed the individual/group conjectures with the class and had all students prove the same conjecture or students proved their own conjecture without a whole-class discussion. For example, students wrote a proof for their own conjectures about the diagonals of a parallelogram. Their conjectures included: diagonals are congruent, diagonals are perpendicular, diagonals create two pairs of congruent triangles across from one another, and diagonal intersect at each other's midpoint. Students worked individually or in small groups to develop a draft argument proving or disproving their conjecture. Once draft arguments were completed, students exchanged papers and provided feedback to one another. Students then drew on their initial arguments and peer feedback to complete a final draft of their argument. The task concluded with a whole-class discussion around how to prove one of the conjectures, which drew on ideas from students' written work. In instances where there were multiple student-conjectures, the remaining ones were discussed in class but not proven.

Context

The study took place in three geometry classes, all taught by the second author, located in a rural high school in the Midwest region of the United States. Proof-writing was a regular part of instruction, with proofs written weekly in class and, less frequently, assigned as homework. Each task was completed in 1 – 1.5 class periods (roughly 60 - 90 minutes).

Data and Analysis

Data for this study consists of a HS geometry teacher's oral reflections after each of the five tasks (see Figure 1). During the reflection process, the first author asked open-ended questions, such as "How do you think the task went?" and "What issues arose during the lesson?" Since the classes were not video recorded, the teacher consulted students' written work and was read portions of the researcher's field notes to help recall what happened.

Using Berry's (2007) framework, the researcher coded the teacher's reflections after each task for the instructional tensions that surfaced and then looked for themes across tasks. Next, the researcher qualitatively coded the reflections for instances where the teacher described how she navigated the identified tension during the lesson. In order to establish trustworthiness and reliability, the researcher and teacher conducted a member check on the themes and how she addressed the identified tensions in her teaching (Lincoln & Guba, 1985).

Findings

Action versus Intent

The teacher's goals (intent) was to provide students with opportunities to engage in different facets of SMP3 and improve their proof writing skills. Her goal for students to pose and investigate their own conjectures resulted in tensions regarding how to introduce the tasks in a way that did not undermine this goal. For instance, after noticing that students had relied on examples during a previous task, she described questioning how to introduce the diagonals of the parallelogram task in a way that would encourage students to generalize past specific examples.

> I was so hesitant. I didn't want to label the angles. And I didn't do one [diagram] as a class collectively. Trying to get them *again* to generalize past the examples, cause now that I had that experience with, 'oh, they just draw in examples'… how to word my language to try to get them to move that way initially, and not waiting until the revisement [discussion] period.

In this task, the teacher's actions at the beginning of the task did not undermine her goal to have students form conjectures. Instead, having students construct multiple examples and discuss their conjectures in small groups resulted in them realizing on their own when a conjecture was false.

During the exterior angle theorem, the teacher ultimately launched the task in a way that guided students towards the specific angle relationship, despite her goal of having students pose their own conjecture. The teacher initially told students to "make a conjecture about the exterior angle of a triangle and its interior angles". This resulted in the student conjecture that $m\measuredangle ACB$ and $m\measuredangle ACD$ added to 180°. Recognizing that their conjecture would not result in meaningful reasoning-and-proving activity, the teacher and researcher decided to guide students towards the anticipated angle relationship using a series of questions about what the students noticed in the diagram. "We had a purpose, so at that moment it was less about individual student and more about whole class so we could move forward" (Teacher).This tension between focusing on the intended mathematical content and allowing students to engage in the SMP3 process using their own conjectures was present throughout the lessons.

Telling versus Growth

Throughout the draft and critique phases of the lessons, the teacher experienced tensions between giving students direct feedback or guidance and allowing them to discover and improve their arguments on their own. For example, when a group's draft argument did not match their conjecture, the teacher struggled to make sure she was not saying "too much to them," hoping other groups would notice and provide that feedback. When a student asked if they could create a drawing to prove their conjecture, the teacher struggled to respond while also being "very conscientious of not saying that they were right or wrong." During the critique phase, the teacher felt like she had to encourage students to write down their questions and comments and "give them permission to be critical" when providing feedback to their peers.

The teacher used the whole class summary as a way of resolving prior tensions to directly address issues in students' work related to their justifications, precision in language, and generality of their arguments. She drew on students' ideas throughout the proof construction process to show she valued the thinking they did in the previous lesson phases.

> I remember trying to think of how to tie in what they were doing to what I was saying. So you guys used examples and this is how we go further. […] I remember trying to draw on what they did, so that it didn't seem like a waste of time.

Across the lessons, the teacher prioritized students' growth and voice during the beginning parts of the lesson. During the summary, she built on students' comments while also making sure the argument encompassed all cases and used mathematically precise language.

Implications

Teachers, often implicitly, navigate tensions throughout their lessons as a result of competing obligations that surface (e.g., Cohen, 1990; Herbst, 2003). In this study, the specific tensions surfaced in part due to a desire for students to have ownership in all stages of the tasks. When preparing teachers to incorporate SMP3 into their practice, it can be helpful to acknowledge these potential tensions and support teachers in reflecting on how they might navigate them in their class. Although the tensions experienced were not specific to the novel task used (see e.g., Rouleau & Liljedahl, 2017), the focus on SMP3 surfaced additional questions around how to support students in developing understanding of what counts as a mathematical conjecture or useful critique. Specifically, to what extent should teachers intervene when students pose conjectures that will limit their reasoning-and-proving opportunities (e.g., a conjecture that is a direct application of a definition)? What are ways teachers can support students in providing meaningful critiques? How can teachers balance the tension between developing students' understanding of proof and providing opportunities to engage in the different facets of SMP3?

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MATHEMATICS INSTRUCTORS' ATTENTION TO INSTRUCTIONAL INTERACTIONS IN DISCUSSIONS OF TEACHING REHEARSALS

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In our project, we develop curricular materials to support prospective secondary teachers' development of MKT and provide professional development (PD) opportunities for instructors who will teach with these materials. In this paper, we examine the ways in which mathematics faculty engage in the teaching rehearsal debriefs included in the PD to answer the question: To what instructional interactions do instructors of mathematics content courses attend during rehearsal debriefs enacted in PD? Findings show that mathematics instructors attend to all types of interactions but attention is influenced by instructors' mathematical knowledge.

Keywords: Mathematical Knowledge for Teaching, Teacher Education – Preservice

Purpose of the Study

The preparation of secondary mathematics teachers spans content and pedagogy, and includes development of mathematical knowledge for teaching (MKT; Ball et al., 2008; CBMS, 2012; AMTE, 2017). However, teachers perceive a disconnect between tertiary mathematics experiences and secondary teaching practice (Goulding et al., 2003; Ticknor, 2012; Wasserman et al., 2018; Zazkis & Leikin, 2010). To address this disconnect, the MODULE(S^2) Project (Lischka et al., 2020) has designed educative curricular materials (Davis & Krajcik, 2005) to be implemented in undergraduate mathematics content courses, including those often taught by mathematicians (Murray & Star, 2013), that situate mathematical content in pedagogical settings and utilize high-leverage teaching practices (e.g., Ball et al., 2009). To provide support for instructors implementing the materials, the project organizes professional development (PD) opportunities in which instructors receive support in enacting elements of the materials with which they may be unfamiliar. One tool used in the MODULE(S^2) Project PD is teaching rehearsals with group debriefs (Ghousseini, 2017).

Although there is much literature regarding PD with K-12 mathematics teachers (e.g., Farmer et al., 2003; Loucks-Horsley et al., 2010), there is less known about how tertiary instructors interact with and take up PD. The purpose of this paper is to draw on the experiences of the MODULE(S^2) Project PD in an exploratory case study (Yin, 2014) to develop more understanding of PD with tertiary instructors. We address the following research question: To what instructional interactions do instructors of mathematics content courses attend during rehearsal debriefs enacted in PD?

Theoretical Perspective and Framework

We define *instruction* to be the "interactions among teachers and students around content, in environments" over time (Cohen, et al., 2003, p.122). Calling upon Lampert (2001) and Cohen and colleagues (2003), these interactions can be modeled by an instructional triangle (see Figure 1), which demonstrates the interactions between teachers, students, and content.

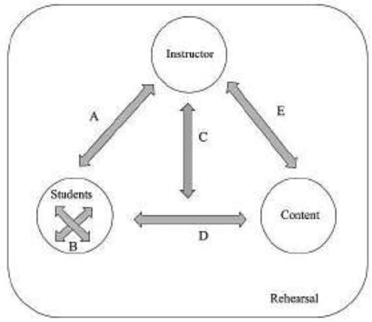

Figure 1: The instructional triangle

Approximations of practice (Grossman, et al., 2009) are useful in supporting teachers to gain experience with and knowledge of the various interactions demonstrated in the instructional triangle. One such approximation is a teaching *rehearsal* in which novice instructors are engaged in the "deliberate practice of well-specified instructional activities" with support from knowledgeable others (Ghousseini, 2017, p. 191). Teaching rehearsals provide instructors the opportunity to engage in approximations of future instruction supplemented by knowledgeable feedback. Ghousseni concluded that the structure of these rehearsals provided prospective secondary teachers the opportunity to "improve their performance in response to feedback that drew on mathematics and student learning of mathematics" (p. 198), bringing together the development of teaching practice and understanding of content knowledge. We build on Ghousseini's work with rehearsals and guided debriefs in the context of PD with tertiary instructors and assert that teaching rehearsals and debriefs may similarly provide opportunities for mathematics instructors to develop teaching practice and content knowledge needed for the preparation of prospective secondary teachers.

Methods and Modes of Inquiry

Contexts, Participants, and Data Sources

The participants in this study were three mathematics faculty from different undergraduate institutions who were engaged in PD for the implementation of MODULE(S^2) Project materials for an algebra content course. Henceforth these mathematics faculty will be referred to as *participants*. Data collected for this study includes video recording of the three teaching rehearsals, video recording of the respective three debriefs, artifacts from the rehearsal lessons, and reflections from the participants.

During a teaching rehearsal, one participant takes on the role of *acting instructor* and the remaining participants take on the role of *acting students* (i.e. the prospective teachers in the undergraduate courses in which materials will be implemented). During a teaching rehearsal, the acting instructor prepares a lesson from the MODULE(S^2) materials and teaches for approximately 10 minutes. These ten minutes of rehearsal are video recorded, then immediately played back to all participants. Following the viewing of the recorded rehearsal, a facilitator from the MODULE(S^2) project conducts a debrief in which participants discuss what occurred in the recorded lesson. For this report, we focus on the videos of these debriefs.

Three teaching rehearsals occurred during the PD, giving each participant the opportunity to serve as an acting instructor once and as an acting student in the other two rehearsals. The lessons chosen for the three rehearsals were respectively on the concepts of inverse functions, the covariational view of functions, and relations. The general goal of the rehearsals and debriefs was to develop instructors' skills in enacting the high-leverage practices embedded within the curriculum materials. Thus, our goal was for instructors to focus on interaction C in Figure 1.

Analysis

Two researchers separately coded each debrief video using a priori codes based on the instructional triangle framework and then came together to reconcile their coding. We categorized every statement made by participants during the debrief as referencing interaction A, B, C, D, or E as labeled in Figure 1. To clarify the coding, consider the following vignette from the debrief of the rehearsal on the topic of covariational and correspondence view of a function:

> During a discussion regarding the clarity of a hypothetical secondary student's quote in the lesson materials, the acting instructor comments that they did not realize that the quote could be misleading until one of the acting students pointed out its obscurity. In reaction to this comment, an acting student stated a way in which they had misinterpreted the quote during the rehearsal lesson. A second acting student followed this with an insight into this misleading quote that they discussed with a fellow acting student during the rehearsal.

Using the categories of interactions indicated in Figure 1, the acting instructor's statement would be coded as referencing a category C interaction, the first acting student's statement as referencing a category D interaction, and the second acting student's statement as referencing a category B interaction. We describe the trends that emerged in codes across the three debriefs to reveal evidence of participants' attention to aspects of instruction during the debrief discussions.

Results

During Debrief 1, participants discussed a rehearsal in which the acting instructor taught a lesson on the topic of relations and their inverses. The debrief began with an acting student commenting on how the acting teacher's use of precise mathematical language when discussing the definitions of range and codomain could, "promote students' mathematical precision." A second acting student shifted the conversation toward how the acting instructor used both table and ordered pair representations of relations during their lesson. The acting instructor replied, "our goal [in the rehearsals] is to have multiple representations." The same acting student continued by asking if the discussion of defining the domain and codomain sets "had emerged as a consequence of [their] discussion" during the rehearsal. "Yes...this comes up all the time when I talk about functions and relations...I think that it is really important" replied the acting instructor. Participants continued the discussion of leveraging student reasoning by pointing out that the acting instructor had written a suggested incorrect answer on the board. "Was the teacher giving enough space?" the acting instructor asked the group, wondering if students had enough individual thinking time during the rehearsal, as the debrief concluded.

In Debrief 2, participants discussed a rehearsal in which the acting instructor taught a lesson on how a secondary student may think about the topic of correspondence and covariational views of functions. "I really liked how the instructor asked 'why, as a teacher, would it be important to figure out their reasoning?'" an acting student commented to begin the debrief. "The teacher kept a complete poker face and let us go with that," a second acting student pointed out when the acting students incorrectly categorized the hypothetical student's view of a function. Some confusion as to whether this categorization was actually incorrect arose from this statement. For the remainder of the debrief, the participants discussed the acting instructor's and acting students' conceptualizations of the difference between a correspondence and covariational view of a function. "I guess the 'co' in covariation and correspondence means that you have to look at both variables," the acting instructor responds to an acting student claiming that correspondence only requires reasoning with one variable.

In the final debrief (3), participants discussed a rehearsal in which the acting instructor taught a lesson on the topic of graphs of relations. "I liked...having the chance to have individual thought...before getting into groups," an acting student began the debrief. They then pointed out a moment in which the acting instructor admitted to the acting students that they were unsure themselves of the answer to these questions. The acting instructor responded that they wanted to explore the questions with a "level of authenticity." During the rehearsal, the acting instructor posed a question that encouraged the acting students to think about the differences between the definitions of a graphs of a relation and of an equation, which acting students said "seemed open ended." "You said 'let's look at an example'," an acting student pointed out an instructional decision to move the lesson forward. The acting instructor then stated that they aimed to collect "helpful student comments or quotes that [the instructor] can then revisit." From this, the participants discussed how they know when it is appropriate to use different teaching strategies. Table 1 displays how our coding reflected the discussions in these three debriefs.

Table 1: Instructional Triangle Coding Counts for the Debriefs

	A	B	C	D	E	Content
Debrief 1	4	0	11	3	6	0
Debrief 2	1	2	21	7	2	13
Debrief 3	4	0	12	0	5	1
Total	9	2	43	10	13	14

Discussion

This study aimed to build knowledge concerning the aspects of instruction to which tertiary instructors attend during debriefs of teaching rehearsals enacted in PD. Our data shows that instructors participating in debrief discussions attended to each component of the instructional triangle, with the majority of the discussions attending to interaction C (how the instructor interacts with students and content together), which was the goal of the PD. However, in one debrief (Debrief 2), the instructors lack of comfort with the mathematical content superseded the ability to focus on student thinking. Instead, attention focused on the content itself. These results show that if the participating instructors are developing necessary content knowledge, this may influence the focus of attention during the debrief.

This work demonstrates that teaching rehearsals are a useful tool to engage tertiary instructors in discussions of student thinking. Further, these results point to the need to structure PD in a way that first supports participants content knowledge development prior to requesting participants to focus on student thinking. Similar research in other content areas is needed to identify concepts for which a focus on student thinking will be best supported by first supporting instructors' content knowledge.

Acknowledgement

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ORCHESTRATING BOTH STUDENT AUTHORITY AND ACCOUNTABILITY TO THE DISCIPLINE WHEN GUIDING STUDENTS PRESENTING A PROOF

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Teachers often find it hard to balance between justice to the students' input and leading the class towards the decided goal. We focus on how the teacher orchestrates the balance between whole class student authority and accountability to the discipline. In the case a student presents the result from group work but at some point needs help. The teacher de-personalises the discussion and directs the class' attention to the subject and not to individual students. Thereby, the class is treated as a community with a shared authority. By the end, collective learning has taken place.

Keywords: Classroom discourse, Communication, High school education, Reasoning and proof

Introduction

The teacher's responsibility is to lead classroom discussions that build on student thinking and guides the class to "strike an appropriate balance between giving students authority over their mathematical work and ensuring that the work is held accountable to the discipline" (Stein et al., 2008, p. 332). Teacher-class discussions were analysed as acquisition of mathematical knowledge (Prediger et al., 2015). Less attention has been paid to how teachers manage situations where students' presentations fail to present the group's end result or provide understandable explanations. How can the teacher respond without simply taking over the explanation? This paper focuses on a case where the teacher intervenes during a student's presentation and manages to give clear responses without "outshining" the students.

Theoretical framework

Participation

The *participationist perspective* denotes all approaches where learning is conceptualized as participation in classroom discourses and collectively implemented activities (Sfard, 2008). Learning mathematics is a process of enculturation into mathematical practices including discursive practices and how they are interactively established in classroom micro-cultures. Mathematical practices capture collective mathematical development and describe interactively established ways of joint action in mathematics classrooms. The participation perspective intertwines *discursive participation*, taking part in discourse practices according to discursive norms and *epistemic participation*, taking part in the joint epistemic processes of knowledge constitution (Erath et al., 2018).

Discursive approach and collective learning

Cobb et al. (2011) write about the collective learning of the classroom community as the evolution of classroom mathematical practices. In line with this, Lerman (2002) outlines the principles of a cultural, discursive psychology, where learning is an initiation into the practices of school mathematics including learning to speak mathematically. The teacher has a vital role in showing what is approved within the discourse, i.e. the accountability to the discipline. Furthermore: "interactions should not be seen as windows on the mind but as discursive contributions that may pull others forward into their increasing participation in mathematical speaking/thinking" (Lerman, 2002, p. 89), which is in line with Sfard's (2008) view of learning as a combination of acquisition and participation.

Gravemeijer (2004, p. 126) points to "the proactive role of the teacher in establishing an appropriate classroom culture, in choosing and introducing instructional tasks, organising group work, framing topics for discussion, and orchestrating discussion". In line with this, Stein et al. (2008, p. 320) emphasise the importance of "whole-class discussions in which the teacher actively shapes the ideas that students produce to lead them to more powerful, efficient, and accurate mathematical thinking." In the Discursive Approach by Sierpinska (2005), the teachers' role in classroom conversations is similarly characterised by an obligation to lead the discussion in the direction of relevant mathematical ideas and themes. In line with this, we introduced the term *captivating dialogue* (Andresen & Dahl, 2018) to situations where students are progressively initiated into the practices of school mathematics through a whole class discussion.

Research question

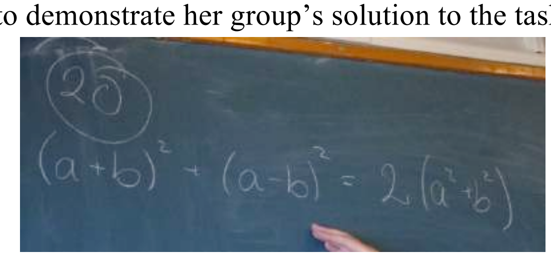
How can a teacher's orchestrate a balance between student authority and accountability to the discipline while guiding the presentation of students' group work?

Methodology

The data consisted of video recordings (30 hours) of teaching during the autumn of 2013 in eight Norwegian upper secondary classrooms as part of the EU research project *KeyCoMath* about students' strategies for creative problem-solving (Andresen, 2015, 2018). The aim of the project was to develop and study teaching that encourages students' inquiry, and intellectual autonomy. The teachers were experienced teacher who volunteered to develop exploratory mathematics tasks to their own classes with the purpose of stimulating student inquiry. This paper focuses on one sequence (6 minutes, 28 seconds, translated to English) and discusses the interactions between the student at the blackboard, the rest of the students, and the teacher. The utterances are not analysed as isolated events but as they occur in a context of sequential utterances and the analysis does not evaluate each utterances in terms of whether or not they are evidence of learning, as we perceive learning as a result of a combination of a series of events.

Data and analysis

Tina teaches a mathematics class of 24 students from a larger town. The excerpt is from the final lesson of a 'Mathematics Day' where the students worked in groups with tasks from 'Proofs without words' (Nelsen, 1993). Each task asks for an explanation of the connection between its figure and its formula. This type of tasks was novel to the students. The excerpt shows a student (Ingvild) who volunteered to demonstrate her group's solution to the task in Figure 1.



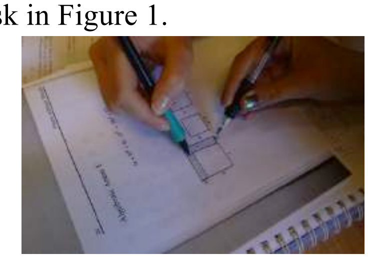


Figure 1: Task as shown on the blackboard (left), student work from book (right)

Ingvild appears calm and relaxed. The task and a drawing of the square is seen on both the blackboard and the book (see Figure 1, left & right & Figure 2, left side of the left equation).

Ingvild: We are supposed to deduct something, right?
Ingvild: But okay. I was thinking ... it was ... [looks at Tina]
Tina: They [the class] are the ones you are supposed to explain it to

Ingvild: Okay [smiling]. It is $(a + b)2$ because ehh we have a times a here [points to the smaller square inside the bigger square and writes a^2 on this square] and ehh no [looks at Tina and appears doubtful]

Tina: Help [aimed at the class]

Tina: There is a lot of help [several hands have been raised]

After 41 seconds Ingvild hesitates and looks appealing at Tina who stands next to the blackboard. Two points of interest: i) Tina gives the *authority to Ingvild* to explain something to the whole class and not only to Tina. Here, Tina emphasizes the *class' understanding* rather than checking the correctness of the result or *Ingvild's understanding*. ii) Ingvild gets stuck almost immediately, but Tina does not take over the explanation but directs Ingvild's attention to the class and requires Ingvild to get help from the class ("There is a lot of help"). By that, Tina assigns the *authority to the class* and encourages interaction between Ingvild and the class.

Next, different students in the class contribute to the task's solution, and one student says: "On the one side it is $a + b$ and the same down. Therefore, in a way it becomes $a + b$ times $a + b$ and we can write this as $(a + b)^2$ …". To which Ingvild responds: "But where does a^2 come from [points at a^2 in the square]". Several students then start to explain at the same time. After 1 minute and 18 seconds Tina interrupts and says: "Someone needs to come up [to the blackboard] and explain it". No one volunteers but several students provide explanations from their seat. Ingvild frequently replies "Hmm", nods and points at the mentioned places on the figure. In our interpretation, the class accepts the authority given by Tina and willingly participates in explaining. Ingvild can follow the suggestions and although she failed to explain the task on her own, she does not appear timid by the situation and her peers do not appear to ridicule her.

After 2 minutes and 20 seconds, Ingvild takes over again and draws the second square with side length $a - b$ (see Figure 2, left) and explains:

Ingvild: And it becomes $a - b$ and $a - b$ [points to each side and then looks at Tina]

Tina: Hm-hm [accepting sound]

This time, Tina does not ask the class for help, but indicates that Ingvild is on track, which we interpret as Tina showing accountability to the discipline by focusing on mathematical content. Until 4 minutes and 48 seconds into the recording, Ingvild draws the rest of the figures seen in Figure 2 while she and some students are discussing what to do. Next, Ingvild hands out the chalk with a happy smile, as if she thinks she has finished. But Tina intervenes:

Tina: I do not quite understand it all. [Tina points to a^2 and b^2]. But these ones? [pointing at the two strips on the left side of the right equality while turning towards the class]

Until 6 minutes and 17 seconds into the recording, Tina exchanges with different students, including Ingvild. The intriguing part of the reasoning, illustrated in Figure 2 (right side), appears to be that the dark rectangles (a times b) 'overlap' in the second square on the left side, and, therefore, must be added (as b times b) to the last square on the right side. Tina asks questions like: "How big is this piece?" to make sure that all the areas in the squares on the left side of the equality sign are represented at the right side of the equality sign. Then Tina asks: "Is it correct?" and concludes by saying: "So based on this where we have drawn – thank you [to Ingvild] – two squares". At this point the class apparently impulsively applauses while Ingvild returns to her seat. Visible signs of agreement are students' nodding and confirming answers, and nobody asks more questions although the atmosphere is forthcoming.

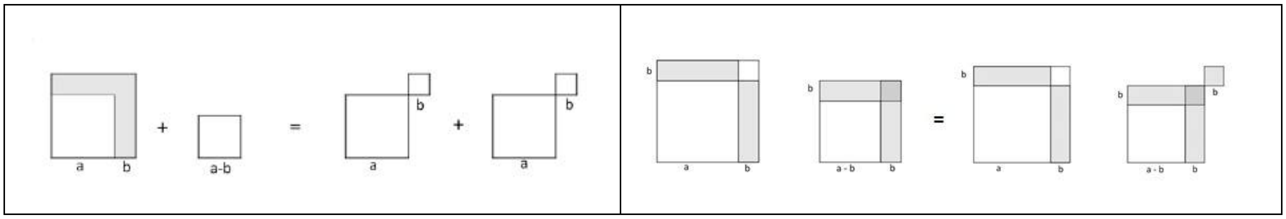


Figure 2: Figures from the task (left), figures from the blackboard (right)

Discussion

Accountability and authority

In our interpretation, Tina shows accountability to the discipline by spending almost one fourth of the sequence's time (towards the end) to make sure that all figures to the right have been covered by the figures to the left. The accountability is balanced with student authority as Tina directs focus of attention to the interaction between the whole class and the subject on the blackboard rather than focusing on the interaction between Ingvild and herself. Further, Tina gives the authority to the community of students when she requests help. We also see that teacher authority is not the same as teacher monologue, Tina orchestrates is in complete control even though she is at the background most of the time.

Classroom culture, participation and collective learning

Ingvild had volunteered and does not exhibit discomfort when she gets stuck. Tina thanks Ingvild during the conclusion of the sequence, and the class applauds even though it was not a brilliant presentation. This shows a classroom culture with ample space for student authority and for discussion. We also see that Tina does not only focus on student authority. Stein et al. (2008) describes that sometimes a focus on student thinking is perceived to imply that the teacher "must avoid providing any substantive guidance at all" (p. 316). In Tina's case, providing substantive guidance is not in itself a contradiction to a student-centred classroom culture. Tina manages the balance and establishes a classroom culture in which the students through discursive participation create the basis for collective learning. By the end of the sequence, collective learning (Cobb et al., 2011) has taken place and *the class* knows the solution. Tina ensures that the class is on the path which is in line with Sierpinska's (2005) views of the role of the teacher as someone who has the responsibility of leading a class in a relevant direction.

Conclusions

In this paper we address how a teacher orchestrates the balance between accountability to the discipline and authority to the students. We focus on students in the classroom as a group and analyse a sequence from a lesson where a student presents the result of group work. The teacher avoids the face-to-face communication with the presenting student, which could otherwise have been the teacher's choice of action when a student gets stuck in the explanation. Thereby the teacher manages to insist on the inclusion of the whole class into the discussion. Furthermore, the teacher avoids taking over from the student and giving the explanations using her authority. Rather, the teacher encourages and supports the rest of the class to develop the appropriate explanations. In our interpretation, the students' authority therefore remains acknowledged together with relevant mathematics that is held accountable to the discipline.

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REVIEWING THE LITERATURE ON FLIPPED MATHEMATICS INSTRUCTION: A QUALITATIVE META-ANALYSIS

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Flipped instruction is often viewed in relation to what is done outside of class (e.g., watching instructional videos) but it is also important to attend to what happens in class. Flipped instruction also has similarities to inverted or blended learning, but "flipping" terminology has garnered enough traction in practice and research as a contemporary phenomenon that it is worthwhile examining it on its own terms. In this research brief, we presented an overview of some initial findings from an ongoing meta-analysis of literature on flipped mathematics instruction. Understanding the research methods previously used to study flipped instruction and the contexts in which those methods were used, will provide future researchers and practitioners with a greater understanding of the impact of flipped instruction on the teaching and learning of mathematics at all levels.

Keywords: Technology; Instructional Activities and Practices; Curriculum Enactment; Research Methods.

Introduction

With the rise of YouTube and other video platforms, flipped instruction—also called "flipped learning" or "flipped classrooms," defined by videos or other multimedia assigned as homework rather than skill practice or problem set homework—has become more prevalent over the past decade (Smith, 2014; Talbert,2018). It has been implemented most often in mathematics and science, especially at the post-secondary levels (Uzunboylu & Karagozlu, 2015). But even in K12 schools, more than 10% of teachers report flipping mathematics lessons at least once a week (Banilower et al., 2018).

As an innovation, flipped mathematics instruction has been predominantly teacher driven, with individual teachers deciding to try it as a way to, for example, accommodate students who miss class or have difficulty following a live lecture and to free up more time in class for active student work (de Araujo, Otten, & Birichi, 2017). Practical implementations, therefore, were outpacing research until recently when a surge of empirical studies on flipped instruction began (Talbert, 2018). The emerging literature, however, encompasses studies with different foci in terms of the outcomes of interest, from student attendance (Asarta & Schmidt, 2015) to their attitude and engagement (Clark, 2015) to measures of content learning (Ichinose & Clinkenbeard, 2017). Even studies that focus on similar outcomes have produced potentially conflicting results. For example, Clark (2015) had positive findings in favor of flipped mathematics instruction but De Santis and colleagues (2015) had neutral-to-negative findings.

Because of the wide range of contexts and foci for research on flipped instruction, and because of the contradictions in preliminary findings, it is important to systematically review the literature. The specific question guiding this review was, In what ways and to what extent has prior research examined flipped mathematics instruction? This literature review study complements existing reviews such as that of DeLozier and Rhodes (2017) that examined instructional activities that are included in studies of flipped instruction and Zainuddin and colleagues (2019) who, like us, examined methodological approaches and overarching results but which looked across multiple subject areas over a short period of time (2017-2018). Our study will focus on flipped mathematics instruction, specifically, and will include a broader timespan.

Framing Flipped Instruction

Flipped instruction is often viewed in relation to what is done outside of class (e.g., watching instructional videos) but it is also important to attend to what happens in class. Bergman and Sams (2012), for example, wrote about flipped instruction but focused largely on ways of using newly-available in-class time. de Araujo et al., (2017) have also pointed out the importance of planning for in-class activities, which is what separates flipped instruction from fully-online instruction. Flipped instruction, because of the possibility of content delivery occurring at home, also has similarities to inverted (e.g., Strayer, 2012) or blended (e.g., Graham, Woodfield, & Harrison, 2013) learning, but "flipping" terminology has garnered enough traction in practice and research as a contemporary phenomenon that it is worthwhile examining it on its own terms.

Method

For this qualitative meta-analysis of the literature on flipped instruction in mathematics classroom, the authors identified the publications through searches on multiple databases and individual journals, excluded the relevant publications using criteria (e.g., empirical, mathematics focused), and screened and recorded each article's details (e.g., definition of flipped instruction, methodology, findings). The details of each phase will be unpacked in the following sections.

Article Identification

To identify the publications relevant to this qualitative meta-analysis, we conducted our initial search in the ERIC database. Using the search terms "flip*" and "flipp*," our search focused on titles, abstracts, and keywords. Our search was restricted to peer-reviewed empirical articles (means have some forms of research questions, methods, and findings in them) published and available as of August 2018. During our initial search, we realized that some scholarly journals in mathematics education (e.g., ESM, SSM) or computer journals (e.g., EJMSTE) were not listed in the index on the ERIC database, we expanded our searches to the individual journals listed in top 7 mathematics education journals, as defined by William and Leatham (2017), or appeared within top 10 mathematics education either Scopus or Google Scholar Metrics. We did the same to the computer journals ranked within top 10 on either Google Scholar Metrics, Scopus, or Web of Science. For these individual journals, we searched peer-reviewed empirical articles using the search terms focusing on titles, abstracts, and keywords on their website or through ProQuest. If the individual journals did not allow us to search using either of titles, abstracts, and keywords, then we expanded our searches to full text if the option was available. If the individual journals did not have a searchable engine on their website, then we used Google Scholar and searched the full text using the same search terms.

Article Inclusion and Exclusion

Overall, as of August 2018, after further removing duplicates, we retrieved 1148 entries (822 from the ERIC database and 326 from the individual journals). For the 1148 publications, we read through their abstracts to check whether each article focuses on flipped instruction for teaching and learning mathematics (e.g., geometry, college algebra, statistics). Thus, we used the following criteria of inclusion: peer-reviewed empirical article, flipped instruction, and content area. Two raters individually read through the abstract and individually examined each criterion as "Yes," "Maybe," or "No." If the examination of each article was not matched, the raters discussed until they agreed with one or another. Also, if the abstract is not available, the raters skimmed through the full article and examined the criteria. After the first round of coding, there were 105 Yes's, 851 No's, and 192 Maybe's. For the Maybe's, both raters skimmed through the full article and examined the criteria, and it turned out that there were 12 Yes's and 180 No's. Consequently, we identified 117 Yes's and 1031 No's after the initial round of coding.

Coding the Literature

Both the screening and coding of the studies were conducted by the authors. To ensure the quality of these as key steps in our qualitative meta-analysis, we utilized a spreadsheet to organize and record the details of each study. The authors developed an initial coding scheme and recorded each article's definition of flipped instruction, research questions, overall methodology, details of methodology and data sources, participant information, mathematics content of focus, measured outcomes, and findings. Using the initial coding scheme, the authors coded two articles together in order to get familiar with the coding scheme and to test how the coding scheme works. Then, the research team deviated the articles and coded independently, and then met to discuss any issues or concerns that emerged while coding the articles. After discussing the issues that arose during the coding process, the team decided to add two more dimensions—length of study and details of the flipped classroom and comparison classroom (if applicable)—to the coding scheme.

Findings

We conducted a synthesis of literature within each category using an inductive and iterative process. As of February 2020, 97 of the 117 articles selected for inclusion in this qualitative meta-analysis were coded. The findings presented in this paper represent a brief overview of the methodology, geographic location, mathematics course, participant grade band, and findings in favor, against, or mixed of flipped instruction from the 97 coded articles. Coding of remaining articles is ongoing. Findings resulting from ongoing synthesis of literature involving the theoretical frameworks and definitions of flipped instruction guiding each study, instruments and measures, specific mathematics content, and activities used within each study, if provided, will be presented in future manuscripts.

Methodology and Context of Included Studies

The methodology used by the researchers and contexts of the included studies provided a picture of how and where flipped instruction was being studied. Of the 97 coded articles, we found that nearly 49% of the studies used quantitative research methods, 12% used qualitative research methods, and 39% used both qualitative and quantitative methods. Study participants included elementary, secondary, and post-secondary students in the United States, Canada, Europe, Asia, Africa, and Australia. Our synthesis of the literature revealed that an overwhelming majority of the studies were conducted within post-secondary institutions in the United States (n = 71); and very few studies were conducted in elementary classrooms (n = 3) (see Table 1).

Table 1: Geographic Location and Grade Band of Participants

Grade Band	Total	*Geographic Location				
		United States	Canada	Europe	Asia	Australia
Elementary K - 5	3	3	0	0	0	0
Secondary 6 - 12	22	15	1	1	2	0
Post Secondary	71	52	2	2	3	2
TOTAL	97	70	4	3	5	2

*if specified

The mathematics content studied ranged from fourth-grade mathematics content through graduate level mathematics courses. Post-secondary mathematics course content represented nearly 70% of mathematics content in classrooms using flipped instruction, with statistics (21.5%) and calculus (15.2%) courses being the majority (see Figure 1). Few studies included elementary mathematics content. Of the 7 studies that included elementary mathematics content, participants in 4 of those studies were undergraduate students majoring in elementary education.

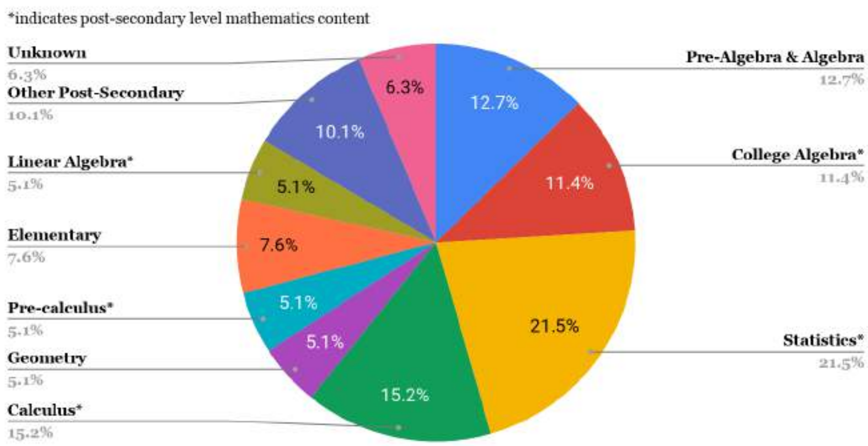

Figure 1: Mathematics Content

Findings in Favor, Against, or Neutral of Flipped Instruction

Findings from included studies revealed numerous positive findings in favor of flipped instruction in mathematics classrooms. Of the 57 included studies (from the 97 coded studies) that measured mathematics achievement of students in classrooms with flipped and non-flipped instruction, 53 of those studies reported at least one statistically significant result in favor of flipped instruction. Fifteen studies reported at least one result that did not show a statistically significant difference in student achievement; and, 3 studies reported at least one statistically significant result in favor of non-flipped instruction. Additional reported findings included both positive and negative reports of participants' perceived impact of flipped instruction on mathematics achievement, level of anxiety, class attendance, motivation, and study habits.

Conclusion

In this research brief, we presented an overview of some initial findings from an ongoing meta-analysis of literature on flipped mathematics instruction. Understanding the research methods previously used to study flipped instruction and the contexts in which those methods were used, will provide future researchers and practitioners with a greater understanding of the impact of flipped instruction on the teaching and learning of mathematics at all levels. As coding and synthesis of the studies included in our meta-analysis continues, the findings presented in this research brief are expected to change.

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CONNECTING IDEAS AND GESTURING DURING WHOLE-CLASS DISCUSSIONS

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This study extends our understanding of teachers' use of gestures during mathematics instruction. In particular, I examined the relation between teachers' gesture and the kind of mathematical connection verbally identified during whole-class discussions. Analysis of video-recordings of two teachers implementing a common unit of instruction revealed, in general, the teachers were more likely to use pointing and writing gestures rather than depictive gestures to make mathematical connections or support connection-making. However, the teachers used gestures differently during discussions based on the kind of mathematical connections discussed. These differences included the use of more than one type of gesture for an entity in a connection and whether both entities of a connection co-occurred in speech and gesture.

Keywords: Communication; embodiment and gesture; instructional activities and practices

In their review of research on learning and teaching with understanding, Hiebert and Carpenter (1992) found that explicit attention to mathematical connections during instruction was generative for students' learning, promoted recall, and supported students to develop a positive disposition toward mathematics. Unfortunately, The Third International Mathematics and Science Study (TIMSS) 1999 Video Study revealed there were few opportunities for and practically no discussions of mathematical connections in US mathematics classrooms (Hiebert et al., 2003). Interestingly, teachers in higher, achieving countries, such as Japan, were more likely to not only discuss connections but also to use gesture while doing so (Richland, 2015). While there is a growing body of evidence that gestures are beneficial for student comprehension (c.f., Hostetter, 2011) and support students' contributions during a discussion (Alibali et al., 2019), it is unclear if there is any relationship between teachers' gestures and the specific kind of mathematical connections made during instruction beyond connecting representations. This paper describes how two teachers' gestures varied in relation to the kind of mathematical connection being discussed during whole-class instruction.

Theoretical Foundation and Constructs

Embodiment and Situative Perspectives on Gestures

Broadly defined, *gestures* are movements of the body, usually of the hands and arms, for the purpose of communicating, and they sometimes accompany speech (McNeill, 1992). To understand how and why teachers gesture during instruction, I draw on the theoretical perspectives from embodied and situated cognition. From an embodied perspective, gestures emerge from simulated actions or perceptual states (Hostetter & Alibali, 2019). For example, asking an individual to think about a cup is also likely to activate the mental actions needed to hold a cup and so the individual may produce a "cupping gesture" with one hand. However, individuals may produce gestures that do not have roots in simulated actions or perceptual states. For example, a teacher may point at a mathematical object so that students may follow the referent of her speech (e.g., Is this [points at an expression] the same as this [points to a second expression]?). From a situated perspective, gestures are a semiotic resource that support interaction by developing, refining, or clarifying ideas (Goodwin, 2000). For example, Keene, Rasmussen, and Stephan (2012) argued that a sequence of gestures between an instructor and students over a series of lesson supported students' understanding of equilibrium solutions.

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

In this study, I followed Alibali et al. (2014) in distinguishing between depictive, pointing, and writing gestures. *Depictive gestures* are "gestures that portray aspects of semantic content directly, via hand shape or motion trajectory, either literally or metaphorically" (Alibali et al., 2014, p. 76). Depictive gestures align with an embodied perspective of gestures. *Pointing gestures* are "gestures that indicate objects, locations, or inscriptions, usually with an extended finger or hand" (Alibali et al., 2014, p. 76). *Writing gestures* are "writing or drawing actions that were integrated with speech in the way that hand gestures are typically integrated with speech but that were produced while holding a writing instrument (usually chalk or marker) and that involved writing to indicate or illustrate the content of the accompanying speech" (Alibali et al., 2014, p. 76). Pointing and writing gestures align with a situative perspective of gestures.

Mathematical Connections

M*athematical connections* are the discursive ways in which an individual or community makes or describes a relationship between two or more mathematical entities. Entity is meant to encompass ideas, concepts, objects, representations, procedures, or methods. An individual or community may make a mathematical connection in variety of ways such as *connecting through comparison* (e.g., $\sqrt{a^2 + b^2}$ is the same as $\sqrt{b^2 + a^2}$), *connecting through logical implication* (e.g., If two distinct lines have the same slope, then the lines are parallel), *connecting methods* (e.g., Using the Pythagorean theorem or the distance formula can be used to find the distance between two points), or *connecting specifics to generalities* (e.g. A 6-8-10 triangle is an example of a Pythagorean triple; Singletary, 2012).

Methods

Participants and Data Collection

Melissa and Robin (pseudonyms) were selected to be part of this study from a larger research project that followed a cohort of secondary mathematics teachers in their teacher preparation program. Melissa and Robin were white females in their early twenties. They co-planned and co-taught an advanced 9th grade coordinate algebra course together during their student-teaching. Course goals included leveraging algebra to deepen and extend students' understanding of geometry. The data included lesson materials from one unit of instruction and video-recordings of the enactment of those lessons in two different class periods. This included 8 instructional days with Melissa as the focus teacher and 6 instructional days with Robin as the focus teacher. Each lesson recording was approximately 70 minutes in duration.

Data Analysis

First, I transcribed all video recordings of the lessons and included screen captures of the teachers' gestures with a short description. Then, I reduced the data to episodes of whole-class discussions about content-related activity (e.g., discussing the solution to a mathematical task) and not the day-to-day operation of school (e.g., checking attendance). From the reduced data, I then coded for connecting-periods (i.e., moments in whole-class discussions when a student or teacher made a mathematical connection). I will call connecting-periods just periods for simplicity. I excluded any periods if the mathematical connection in the period had already been discussed previously. This exclusion was done because Alibali et al. (2014) found that teachers were more likely to use gestures when the connection was novel to students. Next, using the Mathematical Connections Framework (Singletary, 2012), I coded the kind of mathematical connection expressed in the period. Finally, I coded and described the modalities (speech and/or gesture) used by the teacher for each entity in the connection.

Results

Across the lessons, there was a total of 60 periods. The teachers generally used at least one gesture (e.g., depicting, pointing, or writing) during a period (45 of 60). In about one-third of the periods, the teachers used two or more gestures to accompany speech about a mathematical connection (21 of 60). There were 13 periods when the teachers did not use any gestures and 2 periods where I was unable to determine if a teacher used a gesture due to the position of the camera (e.g., a teacher walked off camera).

The teachers' gestures differed depending on the kind of connection. For instance, they were more likely to use two or more types of gestures with speech to support discussions when connecting through comparison (13 of 23) or connecting methods (8 of 11). In contrast, there were few instances of a teacher gesturing with two or more types when connecting specifics to generalities (3 of 14). There were no instances of a teacher supporting discussions of connecting through logical implications using two or more types of gestures (0 of 11). In fact, it was somewhat common for connections through logical implication and connections of specifics to generalities to be unaccompanied by teachers' gestures (7 of 12 and 5 of 14, respectively).

Table 1. Modalities across kinds of connections

	Kind of mathematical connection			
	Comparison	Logical implication	Methods	Specifics to generalities
Two or more gestures with speech	13	0	8	3
At least one gesture with speech	21	4	11	9
Unable to determine	1	1	0	0
No gesture	1	7	0	5
Total	23	12	11	14

Furthermore, the teachers generally expressed both entities of a mathematical connection with gestures when connecting through comparison (14 of 23) and connecting methods (9 of 11) during instruction. In contrast, the teachers seldom expressed both entities when connecting through logical implication (2 of 12) and connecting specifics to generalities (3 of 14).

Table 2: Gesture use for entities within each kind of connection

	Kind of mathematical connection			
	Comparison	Logical implication	Methods	Specifics to generalities
One entity	7	2	2	6
Both entities	14	2	9	3
Neither entity	1	7	0	5
Unable to determine	1	1	0	0
Total	23	12	11	14

Lastly, for all kinds of connections, teachers often used pointing and writing gestures with speech over depictive gestures in relation to a single entity of a mathematical connection. This finding is in agreement with what Alibali et al. (2014) found. Therefore, there was no relation between the kind of mathematical connection and the type of gesture. Table 3 outlines the type of gesture for at least one entity of a mathematical connection in relation to the kind of mathematical connection. Note that the sum for each kind of mathematical connection is different in Table 3 than the other tables because one entity of a mathematical connection could have been expressed multimodally (e.g., with a pointing and writing gesture).

Table 3. Type of gesture and kind of mathematical connection

Type of gesture (for at least one entity)	Kind of mathematical connection			
	Comparison	Logical implication	Methods	Specifics to generalities
Depictive	4	1	5	1
Pointing	16	1	8	4
Writing	14	3	7	8

Discussion

Novice teachers do gesture when discussing mathematical connections or supporting students' connection-making during instruction. This outcome is a distinctive shift from the TIMSS 1999 Video Study results and most likely reflects the recent emphasis on facilitating student-centered mathematical discussions in mathematics teacher education in the US. Novice teachers' use of gestures during discussions is also important because teachers' use of gestures has been found to lead to greater student comprehension (c.f., Hostetter, 2011) and promote students' contributions during a discussion (Alibali et al., 2019). Further, these novice teachers seldom used gestures when connecting through logical implication and connecting specifics to generalities. This is noteworthy because gestures are a semiotic resource for students' meaning making and a teacher's gestures may be a resource for moving students to a more productive meanings of logical implications (Hoyles & Küchemann, 2002) or to more sophisticated generalizations (Ellis, 2007). However, I do not argue that all the connections were productive for students or that gestures alone always lead students to develop productive meanings of connections. For example, Lobato et al. (2003) described how a teacher's use of ambiguous language of "goes up by" when describing the slope of a line along with her use of a sweeping gestures along one column in a table of values may have contributed to students' overgeneralization of slope as a difference rather than a ratio. One productive direction for future research is to determine if teachers are able to notice whether and how their own gestures are (not) productive for students' mathematical connection-making.

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ONE TEACHER'S ANALYSIS OF HER QUESTIONING IN SUPPORT OF COLLECTIVE ARGUMENTATION

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The use of questioning is an effective strategy for orchestrating collective argumentation. However, teachers with minimal experience facilitating argumentation may conceive of effective support as providing little to no verbal input in the argumentation. In this study, we analyzed one teacher's analysis and critique of her support for collective argumentation during her first three years of teaching. We argue that learning to analyze her support for collective argumentation enriched the teacher's understanding of questioning. More specifically, by explicitly identifying how her questions elicited components of arguments from students, the teacher re-evaluated her questions, focusing on purpose rather than form. Implications from this study draw connections between learning to facilitate argumentation and the dilemma of telling that teachers encounter when trying to teach mathematics in ways that honor students' thinking and sense-making.

Keywords: Classroom discourse, High school education, Instructional activities and practices

Teachers have a pivotal role in orchestrating argumentation. Teacher moves such as revoicing and establishing social and sociomathematical norms are supportive of mathematical argumentation (Forman et al., 1998; Yackel, 2002). Further, researchers (e.g., Hunter, 2007; Martino & Maher, 1999) have recognized that teacher questioning is a key factor in supporting mathematical argumentation. For instance, Martino and Maher suggested that a sequence of questions that offer an opportunity for generalization help students to build mathematical arguments. However, teachers have difficulties incorporating questioning strategies in their classroom teaching, even when supported by curricular materials (e.g., Sahin & Kulm, 2008). Furthermore, some researchers argued that teachers may not have a clear understanding of what effective questioning strategies are or how to implement them in order to support argumentative discourse (e.g., Kosko et al., 2014; Zhuang & Conner, 2018). In particular, Kosko et al. (2014) found that some teachers envision mathematical argumentation being left to the responsibility of students with relatively limited input from the teacher. The purpose of this paper is to demonstrate how a teacher learned to analyze her support for argumentation while also co-developing an understanding of her role in supporting argumentation with a special focus on questioning as a strategy for supporting argumentation.

Theoretical Perspective and Conceptual Framework

Drawing from a situative perspective, we conceptualize learning as socially constructed; it takes place through interaction with other human beings, within a specific context, and through active engagement and participation in meaningful practices (Lave & Wenger, 1991). In this study, the situative perspective led us to attend to a teacher's participation and use of analytic tools when discussing selected video representations of her teaching. As the teacher examined, commented on, and critiqued her support of argumentation with another more experienced other (i.e., the mathematics teacher educator-researcher, MTE-R), they built and negotiated the meaning of the practice of supporting argumentation in school mathematics.

Following Toulmin's (1958/2003) model of argumentation, an argument consists of at minimum a *claim* (statement whose validity is being established), *data* (support provided for the claim), and *warrant* (statement that connect data with claims). In this paper, we focus on collective argumentation (i.e., individuals working together to determine the validity of a claim). According to the Teacher Support for Collective Argumentation (TSCA) framework (Conner et al., 2014), teachers can support collective argumentation in three ways: directly contributing to the argument (e.g., providing a claim), asking a question (e.g., requesting an action or information from students), or using other supportive actions (e.g., repeating a student's claim to the class). For the purpose of this paper, we focus on the teacher's questions and her critique of those questions.

Methods

Participant and Data

Jill (a pseudonym) was a participant in a 6-year longitudinal study focused on understanding how beginning teachers learn to facilitate collective argumentation. Jill agreed to participate in the final phase of the study, which was to follow her into her first three years of high school teaching. For this paper, we analyzed data from the first and third years of Jill's teaching because we noticed a significant shift in Jill's participation in analyzing her support for collective argumentation between those two time points and that this contrast provided insights into her understanding of questioning to support argumentation. Data includes 6 classroom observations in her first year and 9 classroom observations during her third year. The research team video-recorded each lesson observation, collected lesson artifacts (e.g., worksheets), and made field notes. After each lesson observation, the team identified episodes of argumentation in the video-recordings and referred to lesson artifacts and field notes as needed to make sense of what happened in the video-recordings. In post-lesson interviews (Interview 6 through Interview 19), the third author interviewed Jill to discuss her supportive actions with respect to collective argumentation by having her analyze selected argumentation episodes from the lesson's video-recordings. The focus of these interviews was to assist Jill in analyzing her support for argumentation, understand Jill's goals for the lesson, and gain insights into Jill's perspective of the school context in which she worked. All post-lesson interviews and video clips were transcribed as data sources.

Data Analysis

At the first stage of analysis, the research team diagrammed episodes of argumentation identified in Jill's lessons using a revised Toulmin's (1958/2003) model (as described in Conner, 2008). The team classified all of Jill's supportive actions for argumentation using the TSCA framework, including Jill's direct contributions to arguments, questions, and other supportive actions. In the second stage, the team developed a codebook to identify moments when Jill analyzed her support for argumentation. The subset of the codes included *identifies argument* (i.e., teacher identifies an argument or episode of argumentation), *identifies component* (i.e., teacher identifies data, claim, or warrant of an argument), *identifies support* (i.e., teacher identifies a question or other supportive action), *teacher critique of support* (i.e., teacher's evaluation of her own support or observation about the presence of support or lack thereof), and *teacher analysis of support* (i.e., teacher categories or otherwise gives ideas about what kinds of support she provided). After coding all the post-lesson interviews from Jill's first and third year, the team generated reports of all the instances of these codes in the data. The team used these reports to compare Jill's analysis of her questioning over time. This analysis is ongoing; initial results are presented in this paper.

Results

Year one: "Very leading on my part, I think"

During her first year of teaching, Jill did not perceive a teacher's questioning as essential support for students to make arguments. For example, Jill asked the MTE-R at the end of the first post-lesson interview, "How [do I] get them (students) to actually form arguments themselves without me having to do it for them? Like without me having to say 'Well, why do you think that?' You know, dictating every little step of it." (Interview 6). Typically, when the MTE-R asked Jill to describe what she noticed after watching video clips of her first year of teaching, Jill described her questioning as leading. For example, "I said, 'Well, what are the slopes of the two lines?' and then she [the student] said, 'Well, they're opposite reciprocals'...So it was very, very leading on my part, I think." (Interview 8). We interpreted Jill's description of leading similarly to how she described her questioning in the first interview as "dictating every little step." In other words, a question was leading to Jill if it resulted in the claim or warrant that she was expecting students to make in the argument.

In an attempt for Jill to see her questioning as supportive of argumentation, the MTE-R asked Jill in the last interview during her first year of teaching to provide examples of leading questions that she used during an episode of argumentation. As Jill went through the transcript, she began to re-evaluate some of her questioning. For example, Jill stated, "I think, where I say, 'Wait, what else do we know?' that was not a leading question. That was very open" (Interview 8). This question had the potential to elicit an unexpected claim from students, and she evaluated it as not leading, which supports our interpretation of her meaning for "leading." The MTE-R next assisted Jill in identifying how her questioning supported students in contributing claims or warrants (Interview 8):

> MTE-R: You say, "How do we know these are right angles?" So, you're emphasizing, okay, the claim here that we're looking at. It is these are right angles, right?
>
> Jill: These are right angles (nodding).
>
> MTE-R: And so, then a student says, "Because of the slope." So, you are then saying, 'Okay, let's go with that. Because of the slope, what do we need to know about the slope essentially?' Right?

Jill often described her questioning in her first year of teaching as leading, but she seldom considered how her questions supported students to make contributions to the argument, such as providing a claim or warrant. We argue that having Jill examine her questioning in relation to supporting students to contribute claims or reasoning for the claims assisted her to reconsider the purpose of her questioning and how it was a useful strategy to support collective argumentation. The MTE-R provided these opportunities to Jill over the course of her second and third years of teaching. Jill identified argument components (e.g., claims or warrants) and her supportive actions, such as questioning, in relation to students' contribution of those components.

Year three: "But that's different than leading"

At the end of her third year of teaching, Jill was provided with an episode of argumentation from her class and asked to identify what she did to support students to contribute components of the argument. Jill pointed out several questions she used to support students' contribution of claims or warrants. For example, Jill stated, "Okay so… 'Why does 6 not work?' would be [what] got her to say that [warrant] but it goes to [give reason for] that [claim]. So that [question] was my support for that part of this little [warrant]" (Interview 17). Jill even identified claims or warrants that were unprompted by her: "And then...so her friend said that [claim]. I don't think I said anything really" (Interview 17). After identifying all of her supportive actions (questioning and other supportive actions), Jill reflected on her questioning without any prompting.

> Jill: So, really, I think I didn't, I didn't say too many leading things here.

MTE-R: Huh-uh (affirmative).
Jill: Which is probably what made this argument good. Because I didn't say anything.
MTE-R: No, you said things.
Jill: Well I just, I gave her…
MTE-R: You said appropriate things.
Jill: Slight direction. Nothing leading.
MTE-R: True, true.
Jill: But slight direction I think sometimes is necessary because they're still new with things.
MTE-R: Oh yeah. Mm-hmm (affirmative).
Jill: But that's different than leading them. (Interview 17)

This was a shift in Jill's analysis of her questioning in relation to her observations from her first-year interview. Recall, Jill initially asked the MTE-R how to get students to make arguments without her having to "dictate every little step." By her third year of analyzing her questioning, Jill described her questioning as supportive of getting students to contribute to the argument and reflected that asking those questions "sometimes is necessary."

Discussion

Kosko et al. (2014) hypothesized two reasons for why teachers envisioned providing minimal scaffolds, such as questions, during argumentation: lack of teaching experience with argumentative discourse or falling victim to the conception of "not telling" (Lobato et al., 2005). This study provides support for the latter hypothesis. Early in Jill's analysis of her support, she critiqued her questions as "too leading" based on their *form* (i.e., a question that does not allow for multiple contributions from students) rather than their *function* (i.e., getting students to make claims or provide explicit warrants). Reformulation of telling in terms of function rather than form was an important consideration to make explicit for Jill when first learning to analyze her support of mathematical arguments. Jill's analysis of her questioning with assistance from the MTE-R and the TSCA framework (Conner et al., 2014) supported her to reformulate the purpose of her questioning. Lobato et al. (2005) also argued for the reformulation of telling in terms of conceptual rather than procedural content of the new information and the relationship of the "telling" action to other teacher actions. While Jill initially focused on the form, rather than the function, of her questions to support argumentation, it is reasonable that these other reformulations may need to be explicitly addressed with teachers as they learn to facilitate argumentation. Nonetheless, this study provides evidence for the interaction between a teacher's learning to facilitate argumentation and the dilemma of telling regarding the form and function of her questions when trying to honor students' mathematical thinking.

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STEM INSTRUCTORS' NOTICING AND RESPONDING TO STUDENT THINKING

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Effective instruction is enabled by an instructor's attention to student thinking and their ability to respond. Further, instruction that is student-thinking-centered can lead to increased conceptual understandings and improved learning experiences for students. Despite this evidence, there is limited research about college instructors' noticing of student thinking. The purpose of this study is to better understand what college STEM faculty notice, and how this enables or constrains their ability to respond to student thinking. STEM faculty, who taught introductory courses, were filmed and interviewed using a semi-structured stimulated recall protocol. Transcripts were analyzed using open coding and thematic analysis. Initial results highlight differences in how faculty elicit student thinking, what is noticed about student understanding, and how this impacts the degree to which faculty can be responsive to student thinking.

Keywords: Instructional activities and practices, Post-secondary education, STEM

There have been calls for increased attention to the teaching of introductory undergraduate science, technology, engineering, and mathematics (STEM) courses and professional development (PD) for those who teach these courses, in an effort to improve enrollment and retention rates in STEM disciplines (Bok, 2013; Holdren & Lander, 2012). Effective instruction is enabled by an instructor's attention to student thinking (e.g., Erickson, 2011). Further, instruction that leverages student thinking can lead to increased conceptual understanding and more positive learning experiences for students (Carpenter et al., 1989; Thornton, 2006). The purpose of this study is to investigate what college STEM instructors notice about student thinking, and the ways in which instructors respond as they make instructional decisions.

Research on Professional Noticing

The noticing required for effective teaching is specialized and goes beyond simply being observant (Ball, 2011). Most scholars agree that it consists of attending to and making sense of particular events during instruction (Sherin, Jacobs, & Philipp, 2011). Jacobs, Lamb, and Philipp (2010) narrow this scope and describe professional noticing as three interrelated skills: attending, interpreting, and deciding how to respond to students' mathematical strategies.

Despite the growing amount of research investigating teacher noticing at the K-12 level, there is little known about what this construct looks like at the college level. Amador's (2014) work investigating future mathematics teacher educators' noticing, one of the few studies at the post-secondary level, found no significant changes in participants' noticing over the short term of the study, but suggested that teachers continued engagement with noticing and reflecting could promote professional growth in this area. This is also supported by evidence that instructors can develop their ability to notice through PD (e.g., van Es & Sherin, 2002).

In order to support college STEM instructors' ability to notice and respond to student thinking, it is first essential to gain a better understanding of what instructors notice, and how they respond through instructional decisions that leverage student thinking. In this paper, I investigate the following research questions: (1) What do college STEM instructors notice about their students' understanding? (2) In what ways are college STEM instructors responsive to their students' understanding (i.e. use student thinking to inform instructional decisions)?

I focus this study broadly on college STEM instructors since little is known about what instructors notice at the post-secondary level, and how this interacts with how they respond to student thinking. It is worth noting that when investigating teacher noticing at the K-12 level, it is common for both science and mathematics education researchers to draw on literature from both disciplines, which points to the similarity and translatability of this work across disciplines. For example, there is evidence at the K-12 level that both science and mathematics instructors attend to students process skills or errors as novices and can develop skills for noticing the disciplinary substance of student thinking (e.g., Stockero, 2014; Barnhart & van Es, 2014). Focusing on STEM more broadly will also afford the opportunity to consider disciplinary differences that may arise and would be beneficial to consider in designing and implementing PD to support instructor noticing and responding at the college level.

Methodology

Faculty from various STEM departments, including Math, Biology, Physics, and Chemistry, who were recognized by their colleagues as individuals who made thoughtful decisions about their teaching were invited and agreed to participate (N=8). Participants were experienced instructors who regularly taught introductory STEM courses. For this proposal, I focus on two participants, Dr. Bio (full professor in Biology) and Dr. Chem (career-line instructor in Chemistry) who taught large enrollment (70 and 270, respectively) introductory STEM courses.

Participants were interviewed before and after a class period which they selected to have filmed (the target class). The pre-observation interview was a semi-structured interview designed to elicit the instructors' goals for class, knowledge of student understanding regarding the topic to be covered, and how this knowledge impacted their planning for class. Clips from the target class (2-5 clips) were selected using selection criteria, and included moments where student thinking was illuminated. These clips were used in the semi-structure post-observation interview to prompt discussion about student thinking and instructional decisions. Interviews were transcribed and analyzed using open coding and thematic analysis to identify emergent themes related to the ways in which instructors notice and respond to student thinking.

STEM Instructors Eliciting, Noticing, and Responding to Student Thinking

The thematic analysis illustrated similarities and differences between how participants gained insight into student understanding by eliciting student thinking, what was noticed about students' understanding, and then how this enabled or constrained their ability to respond.

Eliciting Student Thinking

Dr. Bio and Dr. Chem both created opportunities in class for students to share their thinking, though they differed in what their eliciting allowed them to learn about their students' thinking. Dr. Chem strategically designed free response clicker questions to draw out connections and common student errors. She anticipated where students would struggle and leveraged that as an opportunity to engage students. Dr. Chem stated, "I know where they're going to get hung up. So, I purposely designed questions to get them hung up, because I think that if you do it wrong it helps you remember how to do it right." Additionally, before class, Dr. Chem worked through the clicker questions, anticipating common student mistakes, so that she could address these errors in class. She said,

> I calculated the wrong answer beforehand … [because] when the results come in for the question, I look at how many people answered it wrong, and there's more than one wrong way to do it, and I look at the most common ways and address it.

When using clicker questions, Dr. Bio created multiple-choice questions that would set students up to answer incorrectly. He said, "I'm able to set up what I think is a logical straw man for them. They almost always go for [it]." This type of question does not create an opportunity to elicit student thinking and to gauge how students are thinking about the content.

Noticing Student Thinking

Dr. Bio and Dr. Chem noticed different things about student thinking. Dr. Bio noticed that students did not always take away the main points that he was trying to communicate, saying:

> Because obviously, I'm focused on them getting this one point. And then somebody is telling me, 'Oh, what about this?' And I was like, that's sort of in there, but that's not where I was going. And sometimes I realize … that where I'm going is not where the class is going.

Dr. Bio did not regularly gauge where students were at with their understanding. Thus, he was limited in what he could notice about student thinking, and did not have an opportunity to understand what constrained students' ability to make the connections he desired. Dr. Bio relied on end of semester surveys for feedback on areas that students felt were challenging; he said,

> [I think about the] student comments from the end of the semester, about their perception of the course being disorganized. So, I have, over the years, taken that to heart as constructive criticism and try to make the connections more apparent and meaningful to them.

This comment highlights that he responds to student confusion by working to improve his course from semester to semester, but he is not equipped with the knowledge or tools to assess how impactful these changes are on student learning.

Dr. Chem notices when students are stuck by their facial expressions, understands what students are likely struggling with, and anticipates how she might respond. She said,

> I put the question up, and then I anticipate they're going to read the question, they're going to start working, and then they're going to look at me really perplexed. I wait until I get the look, and then I ask them if they're stuck and they are. So, I will say, 'do you remember this … from earlier this semester?' And then they go, 'Oh', and then they start working again.

Additionally, Dr. Chem used a variety of approaches (including clicker questions, whole class discussions, eavesdropping, facial expressions, interactions in office hours) to gauge where students were at with their understanding, and thus could adjust her plans for class accordingly.

Responding to Student Thinking

Dr. Bio and Dr. Chem responded differently based on what they noticed about student thinking. Dr. Bio made changes each semester, but rarely made changes during class or between classes, saying, "I don't always incorporate [it] right on the fly. And sometimes it's a year delay between when I got that question [and] when I can actually address it in class". Since Dr. Bio did not create opportunities to elicit and notice the substance of student thinking, he was constrained in his ability to make changes that were rooted in students' understanding. He discussed putting himself "in the mindset of the student" when bridging parts of class that were disjoint, saying,

> [I am] going through those notes sections [of the slides] and making sure that they are written in such a way that a student looking at that slide, if she's confused by it, should be able to read those notes and come away saying, 'Oh, now I see what this slide is about.'

Although Dr. Bio provided opportunities for students to ask and answer questions during class, if the students' contribution was not what he had thought about prior to the lesson he tended to redirect back to his prescribed plan for the class. For example, Dr. Bio said,

> I figured that kind of response was kind of off base. I mean ... it just kind of threw me. And so that's probably why I immediately, before I lost the thought of - what do I really want to accomplish with this - I just went ahead and told them.

Dr. Bio's vision of what he wanted to accomplish in this class period, and his struggle to connect the students' thinking with his goal, constrained his ability to notice how this student was understanding the content and the connections that the student was trying to make. Dr. Bio cared about student learning and wanted to improve semester to semester as an instructor, but he did not have the knowledge or the tools to effectively engage with and respond to student thinking.

Dr. Chem regularly made changes from one class to the next, and made changes each semester. Specifically, Dr. Chem planned for instruction based on students' current understanding of the material, stating, "I won't move forward if they're not getting it. There's no reason to. I actually wanted to get farther on Tuesday then I did, but the class wasn't ready to go farther." Dr. Chem also discussed adding clicker questions for the next class to provide students more opportunities to engage with key concepts. Dr. Chem made the following comment after discussing a topic her students were currently struggling with: "I have a new question that will address it. … I've posted the annotated slides so they can see how to do it, but … we're going to do another question that's similar to that [in class]." This comment highlights that Dr. Chem provided her students with the resources to revisit what they were struggling with in addition to creating an additional opportunity to revisit the material. Dr. Chem also discussed making changes from semester to semester, saying,

> I'll take notes of like - 'I'll need to spend more time on this', or '[students] really struggled with this'. Then when I get ready for class the next semester, I go back to that and that's what prompts me to make new slides.

Discussion

Both Dr. Bio and Dr. Chem were experienced instructors, but they noticed different things about their students' thinking, which then enabled or constrained their ability to leverage and respond to student understanding. Specifically, Dr. Bio noticed that students were struggling to make connections, and did not understand what was underlying these difficulties. Consequently, he was limited in what he could respond to. Dr. Chem noticed specific student struggles and built on her existing knowledge of student thinking. Her awareness of students' understanding enabled her to respond both during in-the-moment instruction and in her planning.

Dr. Bio's goal for instruction was to help students appreciate the connections between the seemingly disconnected content through the sharing of his knowledge. Dr. Chem, on the other hand, viewed her course as an opportunity to challenge common student errors and to support students in developing a more complete and correct way of thinking about the content. Consequently, it seems as if Dr. Bio and Dr. Chem had differing dispositions towards student thinking, which impacted their approach to instruction and the opportunities that they created to notice and leverage student thinking. This highlights that it could be important to foster a responsive disposition that values student thinking when developing PD to support faculty in their ability to elicit, notice, and respond to student thinking.

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HYBRID ENVIRONMENTS OF LEARNING: POTENTIAL FOR STUDENT COLLABORATION AND DISTRIBUTED KNOWLEDGE

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An exploratory study of the impact on transforming mathematics teaching and learning practices into the classroom is presented by means of introducing a hybrid learning environment, in this case, designed to address the topic of functions in the first year of finance at college. This topic is normally covered in two weeks in the classroom. In this exploration, the students worked independently on the topic using materials or resources available in a digital teaching platform throughout the first week. In addition, the topic was addressed in the classroom under the teacher's guidance during the second week. The results show collaboration between students to refine or validate their conceptions, which also could support connectivist hypothesis of distributed knowledge.

Key words: Teaching tools and resources, distance education, post-secondary education, communication

Introduction

According to Heffernan et al. (2012, p.101), if school practices must change in order to keep pace with the development of new technologies and to meet students' expectations regarding their use, then the efforts on teacher’s education and in-service teacher development must be altered, there must be a greater number of interactive educational technologies developed in the cloud and implemented in the classroom.

In the exploratory study we are presenting here, we worked on the design and set up of a hybrid scenario of learning. Participant students worked autonomously during one week of the first semester of finance at college. In the following week, teaching and learning were continued now into the classroom under the teacher's guidance. The usage of this hybrid scenario of teaching and learning, in this case on functions, allowed us to investigate possible productive collaborations between students as a consequence of their autonomous work within the activities in the digital platform. Here we report what was done by the students, it suggests a significative transformation of usual teaching and learning of mathematics in the classroom, and also allow us to advance connectivist learning hypotheses.

Theoretical Frame and Methodology

It is noteworthy that the work of Sutherland and Balacheff (1999) early on announced the possibility, now already materialized, of online courses or digital devices for teaching freed from tutoring by the teacher, accessible outside of school and operated via digital media, such as the Internet. By means of online materials or devices, in this case, like videos or forums as digital tools, students are left with the responsibility of unchain their own forms of appropriation of knowledge, and it is mainly through the exchange of opinions between peers that are attained possible advances in one subject's learning. (Downes, 2009).

One of important theories underlying the design and implementation of online and hybrid learning environments is connectivism, mainly developed by Siemens (2006) and Downes (2009). Connectivism is a theory of learning that emerged linked mainly to the use of the Internet, as well as virtual education. Many researchers still question what this theory explains, provides or suggests, for

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example, regarding the incorporation of technology in the classroom (see, for example, Kop & Hill, 2008). Whether it could do this regardless of previous theories or as an extension of some of them, or of theoretical models that have so far been applied to study the integration of technology in school (to see some of these models, see Zbieck & Hollebrands, 2008; Olive et al., 2010; Ruthven, 2014). However, according to Downes (2009), what connectivism has to exhibit is to what extent is an emerging theory, and empirically proving in what sense is a new paradigm that would specifically explain the case of network learning and collective distributed knowledge.

Finally, it is also important to highlight that student productions become registered data when working within a digital teaching platform, and availability of all these records in order to classify and analyze them is one of the advantages of using and designing digital teaching platforms (Dedé & Richards, 2012), since in this way teachers in charge of conducting courses in the classroom can then have in advance these type of records and use them as a diagnosis of difficulties or opportunities for points to be addressed in their classroom.

Thus, for the concentration and interrelation of the students' productions, in this exploratory study an Excel sheet was used and the SOLO taxonomy of Biggs and Collis (1982) was applied. SOLO taxonomy is an analysis tool for a structured classification into four levels of development or evolution of student knowledge around a concept. In general, according to these authors, the four possible levels of classification, starting from the simplest to the most complex, are the following: pre-structural, uni-structural, multi-structural and relational. This taxonomy allowed to identify the refinement and validation of the students' conceptions, formulated through their communication exchanges in the forums on the subject.

Having at hand all these data allowed us the identification of student communication exchanges for productive collaboration. It should be noticed that here the term productive collaboration between students refers precisely to the refinement or validation of conceptions between the students carrying out productive collaboration or critical communication exchange.

Next, in Figure 1, a small part of the concentrate and classification of the students' productions is presented.

Analysis and Results

As previously mentioned, this exploratory study sought to identify cases of collaboration or productive interrelation between students. Below it is shown an image of the classification we accomplished of the different levels of development of the students' conceptions on the subject, extracted from our analysis or classification of their participation in the forums. Likewise, a paradigmatic example of student productive collaboration is also presented.

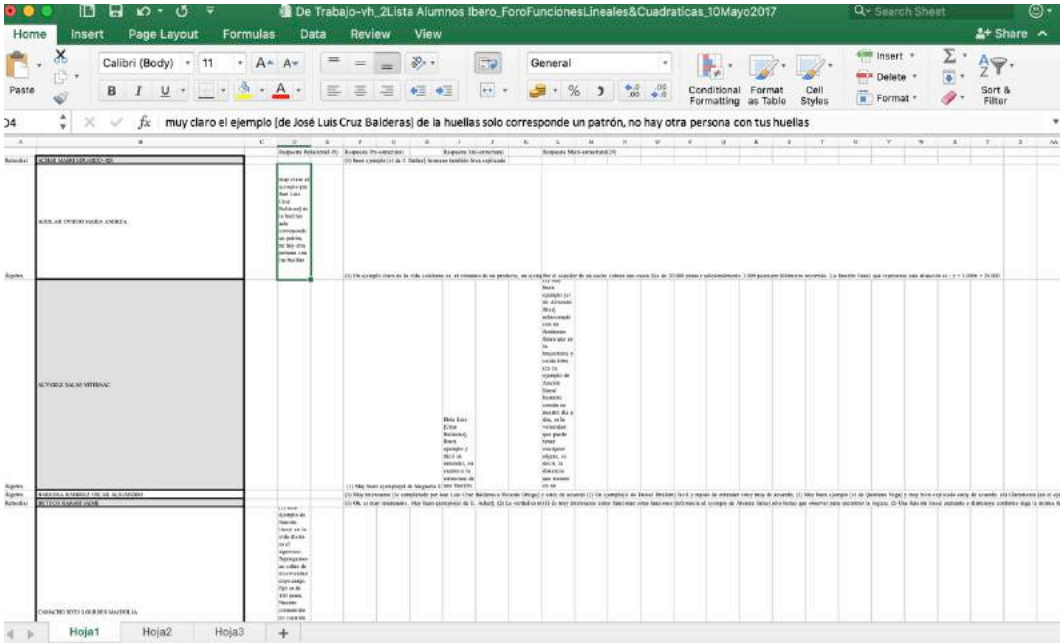

Figure 1. Synthesis of the students' productions, according to the SOLO taxonomy, in search of identifying critical communication exchanges or productive collaboration.

Examples of student production at different levels of development according to the SOLO Taxonomy

Pre-structural level

Student MAA: "A clear example of everyday life is the consumption of a product, an example is the purchase of phones, there are different phones: price levels, with x = the phone and y = the price depending on which phone you would like, the price increases, but all phones have the same function: communicating. [Another] excellent example [in the one given by CS], it was very clear to me how we apply linear functions in daily life. [It is also] an excellent example [the one of AS] because it helps you understand what a linear function is, very simple, with an example from daily life."

Uni-structural level

Student ALA: "When throwing a ball, it first goes up and forward, then falls while continuing to advance, thus forming a path shaped like an inverted parabola."

Multi-structural level

Student YAS: "Very good example [the one given by BA] related to a physical phenomenon that is the trajectory and free fall. [Another] example of a fairly common linear function in our day to day is the speed that any object can have, that is, the distance it travels in a given time. Speaking a little more specifically, assuming that a car on a flat road tends to travel 20 km in 5 minutes, with a linear function, the distance it will travel in 25 minutes could be determined. The algebraic expression, in this example, could be f (x) = 4x. Where x represents the minutes' time you want to calculate to see the distance traveled."

Relational level

Student MAA: "The example [by student JL] of the footprints is very clear, only one pattern corresponds, [because] there is no other person with your footprints"

Student LMC: “Another example of a linear function in daily life is as follows. Let us suppose an electricity charge whose fixed amount is for 100 pesos. Our consumer in question has had a consumption of x watts amount and each watt price is 2 pesos. The function would be expressed in the following way: f (x) = 2x + 100. Thus 2 is our value in a, and 100 is the constant that we add. [Also] a very good example [the one of AS] about something that has an application in daily life, although it should be mentioned that this is only valid for uniform movements (where speed is constant). [In addition, I also] understand the example [the one given by CF] and it seems valid to me, but I consider that because having two antecedents for the same image (this does not meet the definition of function) we would need just one person wanting two things at the same time and not that two people are the same since x1 could be equal to x2 without this affecting the function as long as f (x1) is equal to f(x2). [Finally, also] I agree with another example [the one from MAA], two phones can have the same price, therefore, the same image can have two antecedents, but a singular phone would not have two prices (obviously if we only talk from a provider) so an antecedent could not have two images.”

From the examples and classification presented here, it is clear, according to Heffernan et al. (2012, p. 92), that the materials and activities developed on a digital teaching platform can be used in a multiplicity of ways, among others, so that students receive feedback from their classmates on their actions, which can later be capitalized on reviewing the topic in class or solving questions associated with feedback on the exam. Furthermore, according to data issued from our exploration, these devices also could serve to unchain refinement or validation of students' conceptions of the subject to be learned, as it will be shown by the following paradigmatic example of this type of interrelation or student productive collaboration.

Productive collaboration between students: A paradigmatic example

An example of feedback, or productive collaboration between students, which from our point of view shows the refinement or validation of the concepts at stake on the subject, is shown below.

JL: "Hello ..... my example is fingerprints. There is only one pattern for each person. ”

...

RO: "An example of a function in soccer could be a free kick to the goal because it starts from a zero point, rises and falls again."

JL: "Hello RO, I agree with your example, as long as it is specified that the function is the position of the ball in a certain time when making the free kick."

It is clear, in the case of the communication exchange between JL and RO (given by means of a forum), that the feedback that JL provides to RO is crucial to validate his function’s example, which was formulated in a so schematic way. Practically, it is JL contribution that rescues the visualization of the phenomenon provided by RO’s example, in fact, completes and reformulates it. It is to say that JL filters, refines, and produces a formulation of a function that underlies in the visualization of the phenomenon initially provided by RO.

In summary, the knowledge or formulation of the function at stake did not reside in a single location but rather through a reformulation of a confluence of information originated, in this case, by the exchange of critical information or productive collaboration between two individuals who sought to investigate mathematical functions, a common subject of interest, that finally produced feedback to each other, what is consistent with connectivist learning or distributed knowledge as pointed out by Downes (2009).

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EMBRACING PROVING INTO EVERYDAY LESSON BY PROBLEM POSING

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Proof plays significant roles in the context of school mathematics and is a tool for enhancing student's understanding of mathematics. Lack of opportunities for proving in textbook has been documented. This study was conducted to consider an instructional way to make proving as everyday lesson by formulating more opportunities than did textbooks. The guiding assumption of this study is that conjectures which students come up with can be initiatives for learning how to prove. This preliminary study will show that problem posing is a strategic tool with potential to bridge everyday instruction and the practice of proving so as to teach how to prove more meaningfully and authentically.

Keywords: Reasoning and Proof, Instructional Activities and Practices, Classroom Discourse

Proof and proving have been considered as central in the context of school mathematics with its roles (Knuth, 2002a) which are "inseparable in doing, communicating, and recording mathematics" (Schoenfeld, 1994). In *Principles and Standards for School Mathematics* (NCTM, 2000), authors argue that "Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena." (p. 59). However, it yields difficulty for students to understand it and for teachers to teach it (Stylianides, Stylianides, & Weber, 2010). The disagreement of its centrality for *all* in secondary school mathematics exists among in-service teachers (Knuth, 2002a). Even worse, there are not many opportunities available for students to engage in reasoning and proving in textbooks (Bieda et al., 2014; Thompson, Senk, & Johnson, 2005). Thus, to cultivate a context where students are introduced to proving, engage in the practice, and, ultimately, recognize proving as fundamental in learning of mathematics, need to authentically formulate opportunities beyond those available in textbook should be met. That way, with opportunities to engage in proving in a mathematically meaningful way rather than to take part in a mere ritual as spectators—such as reading and understanding proofs given in textbook without formulating or exploring conjectures, both students and teachers can be more fluent in proving. In this report, a-year-long study of problem-posing activity with particular interest in proving and teacher's instructional interventions which foster student's reasoning and developing proof will be analyzed.

Literature Review

Proof and Reasoning in School Mathematics

Proof and reasoning are neither mere content to be learned with chosen topics nor reserved for certain grade levels. NCTM (2000) states "Reasoning and proof should be a *consistent* part of students' mathematical experience in prekindergarten through grade 12. Reasoning mathematically is a habit of mind, and like all habits, it must be developed through consistent use in *many contexts*." (p. 56, italics added). In the context of secondary school mathematics, the only place in which proof is substantially treated is geometry (Knuth, 2002b). The proofs in the subject and do not show the variety of ways of proving (e.g., proof by contrapositive, reductio ad absurdum). As Thompson, Senk, & Johnson (2005) argued "Because many research studies have shown that writing proofs is difficult for students at all levels, it seems to us that students need more opportunities to engage in varied aspects of proof-related reasoning in order to become more fluent in reasoning and proving."(p. 286) Furthermore, there are not many opportunities for students to engage in reasoning and proving tasks across textbooks which are considered to be primary sources of teaching and

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learning mathematics (Bieda et al., 2014; Thompson, Senk, & Johnson, 2012). Mathematics teachers need to take on an active role in teaching reasoning and proving beyond what is available in textbook and have strategic knowledge of instructional practice and its relation to student's learning of proof (Stylianides & Ball, 2008) and how it can be more impactful.

In *Proofs and Refutations*, Lakatos (1976) exemplified use of examples when exploring, formulating, qualifying a conjecture, developing a proof and making revisions when encountered with counter examples—either *global* or *local*. Although there exists heterogeneity in appearance among them, mathematically similar objects enable observers to notice regularity between them and the regularity becomes a mathematical conjecture—possibly to be proven true thus to be a theorem. For teacher's specific interest and intent to teach certain theorems, some may argue that designed examples can be given to students as resources to experience transition from empirical arguments to formal proofs. However, the main focus of this study is not on teacher's designing or displaying examples as intended for teaching specific content but on teaching how to strategically generate examples with same constraints in order for students to look for examples (or counter examples) not restricted to those within their reach.

What is problem posing? Silver (1994) defines the term as "both the generation of new problems and the re-formulation, of given problems." (p. 19). According to the author, problem posing can also offer insight into solution of a problem: when developing a proof, posing problems can be a pathway to gain insight into proof. As a way of posing problems, Brown & Walter (1983) suggested "What-If-not?" strategy which new problems can be generated by varying some of the given conditions of a problem. For example, after solving a problem that a sum of two even numbers is even or odd, one can pose a new problem with a question "what would it be if I add two odd numbers?". Lockwood et al. (2013) studied how a mathematician uses examples when proving and disproving. By referring back and forth to examples of relevance to a conjecture, the mathematician gained insight of proof by leveraging idea of one insightful example. In the same line with what Balacheff (1988) called a *generic example*, a representative example of the domain of a conjecture suffices to be developed to a proof by *syntactic proof production* (Weber & Alcock, 2004) or *transformation of images* (Harel & Sowder, 1998). However, unlike teachers and mathematicians, this may be improbable for students to do as such.

Methods

Participants

Geographically located at the vertical center of the Korean peninsula, the school where was the locus of this study is a high school with male students only and located in an urban area. Nearly all students intended to enroll the school to prepare for their admission to college.

As a high school mathematics teacher and the researcher in this project, I had taught junior high students for 3 years and started to teach high school students for the first time by the time this study began. The guiding assumption was that every student is *a theory builder* (Carey, 1985) who can come up with a conjecture or a plausible argument which makes the most sense to them based on their observations and that most of students are able to develop and write proofs by themselves or with a little help offered by *a teacher* or *a more capable peer* (Vygotsky, 1978).

Data Collection

The data collected for the study include student's written assignments, teacher's verbal and written communication with individual students, and two video-taped lessons of which duration is roughly 50 minutes. Based on "What-If-Not?" (Walter & Brown, 1983), for consistency in structure and organization of the assignment, it was structured in a worksheet. Before administering the work sheet weekly, the instructor demonstrated how to use it and explained what is expected as the end result in each step. Until everyone reached understanding of the activity, there had been discussions and negotiations of what it means to be *true*, *valid*, and *appropriate* (Stylianides, 2007; Stylianides et al., 2016) when evaluating validity of a proof.

Data Analysis

Since it may be premature to present a framework which will be used in the later analysis, I shall present the working framework in the process of conducting an initial analysis through the general inductive approach (Thomas, 2006) to highlight themes of relevance to the purpose of this study.

Preliminary Results and Discussions

"What-If-Not" strategy has a potential to offer a strategic way for students to better identify and understand what assumptions are given and conclusions they should prove. For example, one of students in the class was attempting to solve a problem: find the maximum area of a rectangle inscribed in a given isosceles triangle. The student reached at a solution which was not the solution of the problem since he solved a problem without taking the condition "isosceles" into account. Then, after some conversation with him, it came into his attention that he left the condition out. As described in this instance, the student was able to take all the conditions into account after discourse with the teacher. Then, the teacher posed a question as an extension of the problem: "what if the triangle is a right triangle? Or what if the triangle is an acute triangle?" Even though it took a few days for the student to figure out how to solve it, the student reported the teacher that this extended discourse with him led the student to use the strategy in evaluating his understanding of problems by manipulating the given.

Problem posing can offer a strategic way for students to generate examples beyond the individual potential example space (Watson & Mason, 2005) and gain insight into how proof looks like. As it enhances student's understanding of what constraints are given and should be verified by making negation or eliminating and reinstating some of the given and the to-be-verified, it can scaffold student's generation of examples under the conditions met by examples or counter examples. There was a student had issues with qualifying examples and non-examples based on the given constraints. He often said that it is difficult to come up with a counterexample when attempting to disprove a conjecture. The teacher asked the student to think of examples meeting the *least* (i.e. the maximum number of the given conditions as many as he can consider simultaneously into account) subset of the given. Then the teacher demonstrated how to add the rest of the given one by one and accordingly prompted the student to list a number of examples each time. The teacher demonstrated how to qualify examples by adding an additional constraint and left the student doing the rest to reach at being able to generate examples or counterexamples thereafter. The student soon became capable with manipulating the constraints by leaving out and reinstating some of them. This student seemed to show the potential of "what-if-not" strategy as a way of generating examples (or counterexamples) beyond his reach in that he did not try to recall the examples from his experience but qualify examples by adding the given condition into his consideration and narrowing them down to the domain of the argument of his interest. This is not meant to argue that the strategy itself suffices to extend the example space but that it has potential to do as such only with teacher's careful consideration and helpful prompts rather than simply offering the caveat. The kinds of prompts which are crucial in teaching and learning of proof will be identified and discussed in what follows next.

Teacher's role is crucial and critical in the success of teaching and learning of proof. As documented in Stylianou & Blanton (2011), teacher's role becomes of more importance in the teaching proof. In this study, it was the teacher who extended the discourse with individual student to offer an opportunity to engage in exploring and revising conjectures, developing a proof, and prompting student to revise the proof for the greater proximity to the degree of formal proofs. This study will identify three types of prompts by what the teacher intended to elicit from students at a given time: those for *justification, elaboration,* and *generalization*. The instructional intent of prompts was to extend and structure discussions and to attend to what can improve student's proof in terms of precision (in use of mathematical terms, expressions, or representations), clarity (in use of language), and generality (of the proof). The working definitions of the prompts are as follows:

1. *prompts for justification* are meant to point out unexplained parts and request to fill logical gaps or challenge truth of conjectures assumed to be true or referred to in student's argument;
2. *prompts for elaboration* are meant to call attention to what requires clarification or discrepancy between what is intended by student and understood by others; and
3. *prompts for generalization* are meant to pose questions which possibly lead to generalization of part of student's reasoning or examples.

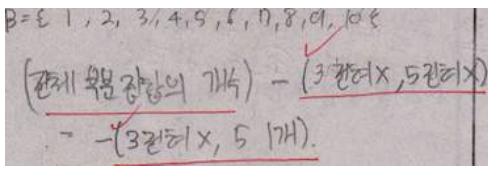

Figure 1. Examples of *Elaboration*

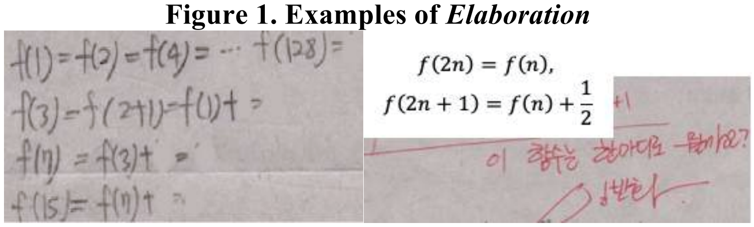

Figure 2. Examples of *Generalization*

There are a few limitations in this study that should be examined through research involving different individual participants, classroom culture, and society. As Cobb & Yackel (1996) pointed out, the results of a well-designed (or well-controlled) research can hardly argue that the study is conducted independently of any aspect of the context of the socio-cultural or individual (or psychological) peculiarity of the participants, the classroom, and the society involved. I acknowledge that this study is not the case that the results are drawn independently of the individuals, the classroom culture, and the society. Future research in the similar perspective toward problem posing and instruction of proof taken in this study will let light be on the unpaved paths I have not taken in this study and nourish the literature.

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PROFILING THE USE OF PUBLIC RECORDS OF STUDENTS' MATHEMATICAL THINKING IN 4TH-8TH MATHEMATICS CLASSROOMS

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Centering class discussions around student mathematical thinking has been identified as one of the critical components of teaching that engages students in justifying and generalizing. This report shares analysis from a larger project aimed at describing and quantifying student and teacher components of productive classrooms at a fine-grain level. We share results from this analysis of 39 mathematics lessons with a focus working with public records of students' mathematical thinking.

Keywords: Instructional activities and practices, Teacher Education-Inservice/Professional Development

The goal of *Working with Public Records of Students' Mathematical Thinking* is to make student thinking available to all students (Ghousseini, 2009), and to maintain common ground (Staples, 2007). This may look like recording student ideas (Cengiz et al., 2011; Staples, 2007) and engaging the class to work with it. Publicizing student work has the potential to position students as contributors to mathematics (Cohen, 1994). We illustrate an analysis of how public records of students' thinking were used in 39 lessons of grades 4-8 classrooms to productively generate meaningful student discourse. Students can learn mathematics when engaging with each other around mathematics (Schwartz, Black, & Strange, 1991). Teacher prompts that elicit reflection, communication, and meaningful explanations regarding a student's work and their thinking have been identified as essential and beneficial for mathematical learning and understanding (Hiebert, et al., 1997; Henningsen & Stein, 1997; Hiebert & Wearne, 1993; Kazemi & Stipek, 2001).We hypothesized that lessons in which teachers engaged students in examining public records of students' mathematical thinking would generate more and higher-levels of student discourse. Our research questions were: (1) How prevalent are public records of students' mathematical thinking within the lessons? (2) Do lessons that contain public records include more student-level engagement, specifically higher-level cognitive engagement? Do those lessons that also contain selected and sequenced public records include even more than those with either a) no selected and sequence public records and b) more than those with no public records at all?

Theoretical Orientation and Analytic Framework

There is a general consensus in the mathematics education community that high-quality mathematics classrooms are those in which student voices are heard, and student thinking is leveraged as the means to move instruction forward (e.g., Ball, 1993; Jacobs & Spangler, 2017; Nasir, & Cobb, 2006; Schoenfeld, 2011; Turner, Dominguez, Maldonado, & Empson, 2013). Enacting practices that foreground student thinking is complex, requires intentional and strategic moves, and persistence in enacting these moves over time (Staples, 2007; Boaler & Staples 2008; Franke, Kazemi & Battey, 2007). Mathematically productive teaching routines are a set of teaching routines designed for accessing and working with student mathematical thinking. Research

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has emphasized attending to students' mathematical thinking as one of the most essential aspects of impactful teaching (Jacobs & Spangler, 2018; Lampert et al., 2013).

One such teaching routine is *Working With Public Records of Students' Mathematical Thinking* (described above). This routine can be situated within the teaching routine *Working With Selected and Sequenced Student Math Ideas*. The goal of this routine is to advance student understanding by fostering connections related to the core mathematical ideas on which the lesson/task focuses. Once teachers have learned about how their students are thinking, they need to choose how to build ideas with the whole class by selecting and sequencing how student ideas are shared (Stein, Engle, Smith, & Hughes, 2008, Stein & Smith, 2011).

Methods

The 39 coded lessons for this project stem from two urban school districts in the United States: grades 4-5 were from a large urban district, whilst grades 6-8 came from a mid-sized urban district. The 20 lessons from the middle school were taken from each teacher at the end of the school year, and the 19 lessons from the elementary school teachers were a stratified random sample, according to Mathematical Quality of Instruction (Hill, 2014) scores. Because this paper focuses on two teaching routines (*Working With Public Records of Students' Mathematical Thinking* and *Working With Selected and Sequenced Student Math Ideas*), all lessons were coded for those two teaching routines. Each lesson was also coded for Students' Habits of Mind (HoM) and Habits of Interaction (HoI). HoI focus on students' verbal interaction with the teacher as well as with one another. HoI include *Explaining* their thinking, asking *Genuine Questions*, *Revoicing* other students' contributions, *Private Reasoning Time*, *Compare* logic and ideas for similarities or differences, exploring multiple *Pathways* to solving a problem, and *Critique* one another's ideas. HoM can happen within an HoI and focus on the cognitive activity embedded within their verbal interaction. HoM are noted here as *Representations (Reps), Connections* within and across two mathematical concepts, strategies, or structures, *Regularity and Structure* using patterns, properties, or mathematical structures, *Metacognition (Meta)* or reflection on their own thinking, recognizing, examining, or using their own or each other's *Mistakes,* engage in *Meaning of tasks and terms, Justify* their thinking, and *Generalize* ideas. To summarize, HoI are the ways a student can interact with others whereas HoM are the mathematical activities embedded within such an interaction. These codes were developed for a larger study involving the Math Habits Tool, which was developed to capture *mathematically productive components* of classrooms in terms of both student and teacher in-the-moment actions.

All coding was completed by graduate students who took part in a three-day coding training camp that focused on the various student and teacher-level codes used in this project. Each lesson was then assigned to two graduate students to code independently. After each coder had completed their initial coding of the lesson, the pairs of coders meet to compare their independent coding and reconcile any differences and disagreements. Disagreements that could not be reconciled between the two coders were sent to a third person for final decision.

In considering our research questions, we grouped the 39 lessons into three themes: (1) lessons containing public records where at least two public records were selected and sequenced; (2) lessons containing at least one public record, but none that were selected and sequenced; and (3) lessons containing no public records. We then compared those groups in terms of quantity and type of HoI, and HoM within the lesson.

Results

Of the 39 lessons, 26 lessons (67%) did not contain public records of students' mathematical thinking (Group 3), thus, student work was not displayed and worked with at all. Of the 13 lessons (33%) that did include public records of students' mathematical thinking, six were further situated in a *selecting and sequencing* routine (Group 1), while seven were not (Group 2). Thus, only 33% of the lessons contained student work that was actively displayed and worked with, and about half of those were situated in a selected and sequenced routine.

Across all three groups, the Habits of Interaction *Explain* and *Questions* were frequently used. Similarly, the Habit of Interaction *Private Reasoning* occurred sporadically throughout some of the lessons. Thus, explaining mathematics, asking genuine questions, and prompting students to use private reasoning about mathematics are habits of interaction that are seemingly not dependent on reasoning with students' work within a public record, so we removed those three HoI from our next level of data analysis. *Generalize* was not present in any of the lessons, so it too was removed from the next level of data analysis.

We found that while there was generally infrequent use of higher-level Habits of Mind and Interaction across all 39 lessons, lessons that did use public records engaged students in higher-level HoM and HoI more frequently than lessons that did not. Furthermore, lessons that selected and sequenced the public records were found to include student engagement in these codes more often than lessons that did not select and sequence their public records. (See table 1.)

Table 1: Percentage of Lessons in Each Category with Relevant Student Habit Occurrence

	Compare	Pathways	Revoice	Critique	Reps	Connect	Structure	Mistakes	Meta	Meaning	Justify
Group 1	83%	67%	83%	50%	83%	67%	33%	50%	50%	17%	50%
Group 2	14%	0%	29%	14%	86%	29%	43%	14%	29%	0%	0%
Group 3	8%	4%	0%	0%	23%	12%	8%	4%	0%	12%	8%

Moreover, we found this to not only be true of the lessons, but within the public records themselves. Next, we consider how the student engagement within a lesson compares to the engagement specifically during a public record portion of class. Table 2 highlights the average percentage frequency of a student habit for a whole lesson in the group's top row, and the average percentage frequency of a student habit for the public records portion of a lesson in the group's bottom row. For example, of all the higher-level student habits used in Group 1 lessons, 15% were *Compare* and 8% of those habits took place within a public record. Because Group 3 lessons contained no public record, there are no student habits within a public record to display (i.e. the second row is empty). Notice that 50% or more of the student habits in Group 1 lessons happen within a public record, and with the exception of *Compare*, *Critique*, and *Mistakes*.

Table 2: Frequency Percentage of Habits Per Lessons & Public Record in Each Group

	Average Time	Compare	Pathways	Revoice	Critique	Reps	Connect	Structure	Mistakes	Meta	Meaning	Justify
Group 1	56:50	15%	7%	14%	4%	28%	12%	2%	3%	8%	2%	5%
	17:46	8%	4%	10%	2%	22%	8%	1%	2%	7%	2%	5%
Group 2	58:30	2%	0%	14%	5%	43%	17%	10%	2%	7%	0%	0%
	09:02	0%	0%	12%	2%	26%	14%	7%	0%	5%	0%	0%
Group 3	45:00	11%	9%	0%	0%	32%	7%	20%	2%	0%	7%	11%

Discussion and Future Research Plans

Only 13 lessons (33%) engaged students using a public record of students' mathematical thinking. Only 6 (15%) lessons selected and sequenced the public records. Lessons with public records showed a higher percentage usage of higher-level cognitive engagement. Lessons which selected and sequenced the public records engage student in higher-level mathematical habits consistently more than lessons that did not. In fact, on average, lessons that selected and sequenced the public records of students' mathematical thinking showed a 46% increase in higher-level cognitive engagement compared to lessons that did not.

One explanation for this drastic difference is that in selecting and sequencing public records, students are exploring multiple pathways, comparing strategies, and inevitably critiquing and debating any contradictory or different ideas. Thus, by selecting and sequencing students' ideas, teachers make these habits of interaction more accessible for the students and can more advantageously create a dialog around multiple ideas.

Close to 50% or more of the student codes in Groups 1 and 2 lessons occurred within public records. Thus, public records are creating a time for students to engage in mathematical discourse more frequently and at a higher-level than time outside of the public record.

Although important to make student thinking available to all students and work with it, it is not enough. Providing access to students' ways of thinking offers ways of engaging; however, without selecting and sequencing the engagement is shallow and less frequent. Thus, by selecting and sequencing the public records of students' mathematical thinking, an exploration and dialog using the habits of interaction can be sparked to ignite the higher-level conversation that leads to deeper, more frequent usage of habits of mind such as making meaning and justification. Therefore, as evident from literature (Stein, Engle, Smith, & Hughes, 2008, Stein & Smith, 2011), having students' present their ideas to the class is not as effective in creating productive student discourse as carefully monitoring, selecting, and sequencing student ideas. Moreover, the results we have stated here illustrate the effects on students' engagement when a teacher effectively selects and sequences students' mathematical ideas. Further work will involve continued analysis of 61 more lessons to see if this pattern still holds. Additionally, this work focused on only student engagement in the lesson, but future work will also include analysis on teacher prompts for student engagement.

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AN INSTRUCTOR'S ACTIONS FOR MAINTAINING THE COGNITIVE DEMANDS OF TASKS IN TEACHING MATHEMATICAL INDUCTION

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Mathematical tasks are central to students' learning since they can influence and structure the ways in which students think about mathematics. Carefully selected tasks have potential to broaden students' views of a subject matter and facilitate their mathematical growth. However, research identifies that cognitive demands of tasks may change as the tasks are enacted during instruction. For this reason, it is important to understand what instructors can do to maintain the intended cognitive demands. In this paper, we investigate a teacher's actions for maintaining high-level cognitive demands of tasks in teaching proof by mathematical induction. Our findings suggest that the method of quasi-induction (Harel, 2002) may be considered as an example of a productive scaffolding strategy for assisting students in mastering proof by induction.

Keywords: Classroom Discourse, Reasoning and Proof, University Mathematics

Proof by mathematical induction is a technique for proving statements about natural numbers. To prove that proposition $P(n)$ holds for any natural number n, one needs to check that (1) $P(1)$ is true (the base case) and (2) if $P(k)$ is true for some fixed but arbitrary natural number k, then $P(k+1)$ is also true (inductive implication). The principle of mathematical induction poses conceptual difficulties to college students (Dubinsky, 1991; Harel, 2002; Movshovitz-Hadar, 1993; Stylianides, Stylianides, & Philippou, 2007).

Carefully selected tasks can help students overcome cognitive obstacles associated with proof by mathematical induction. Mathematical tasks play a crucial role in students' learning. They can shape students' conceptions about the subject. Furthermore, they offer an opportunity for teachers to lower their authority in the classroom, in turn allowing students to create mathematics for themselves. However, one must be able to strike a balance when determining the appropriate difficulty of task for a student. When tasks do not significantly challenge the student, they may become routine or discourage creativity. In contrast, if a task is too difficult, students may make insufficient progress toward the intended mathematical goal.

The cognitive demand of a task represents its level of difficulty (Stein, Grover, & Henningsen, 1996). The cognitive demands of tasks for K-12 students have been well documented (Spears and Chávez, 2014; Bieda, 2010; Henningsen and Stein, 1997). However, to our knowledge, cognitive demands of have not been extensively explored at the undergraduate level. This study aims to contribute to the research on cognitive demands of tasks by considering problems an instructor used in teaching proof by mathematical induction. Specifically, the purpose of this case study is to investigate a teacher's actions during the enactment of high-level tasks. Results address the following research question: *what are the teachers' actions for maintaining high-level cognitive demands of tasks in teaching proof by mathematical induction?*

Theoretical Framework

The present study is guided by the Mathematical Tasks Framework (Stein et al., 1996). This model describes the evolution of a task through three phases of classroom: as written in instructional materials, as set up by a teacher in the classroom, and as implemented by the students. Ultimately, a mathematical task should lead to student learning.

For the purposes of this study, we distinguish between *planned* and *enacted* mathematical instruction. Planned instruction refers to how teachers plan a mathematical task and how they pose it in the classroom. Enacted instruction is the actual teaching that occurs, including the active roles of both teachers and students (Remillard, 2005). The Mathematical Tasks Framework represents planned instruction by the first two phases and enacted by the third one.

The framework further specifies two dimensions of tasks, task features and cognitive demands, that may affect the transition between phases. This study is centered around cognitive demand. Cognitive demand refers to the variation in the kind of thinking processes required of students while engaging with the tasks. According to Stein et al., (2009), there are four levels of cognitive demands: a) memorization, b) procedures without connections, c) procedures with connections and d) doing mathematics. The first two levels are traditionally considered low-level demands, while the latter refer to high-level demands.

The enactment of mathematical tasks of high cognitive demands allows students to develop sense-making and reasoning skills, critique their peers' solutions, formulate examples and counterexamples, create viable justifications, and properly communicate their reasoning. Therefore, implementation of high-level tasks may be beneficial for students' mathematical growth. Planning the implementation of a task is crucial to students' learning that occurs around this task. However, research identifies that teachers have difficulty enacting high-level tasks even if they were planned as such (Boston & Smith, 2009). Teachers may either maintain the high cognitive level or they may lower it to make tasks more accessible for students. Stein and Smith (1998) suggest a list of factors associated with the maintenance of high-level cognitive demands. These factors include teachers giving sufficient time, making conceptual connections, pressing for justifications, and scaffolding student thinking and reasoning.

The term scaffolding has been used in various contexts. Anderson (1989) highlighted the importance of the Vygotskian notion of scaffolding in supporting students' high-level thinking processes. Henderson and Stein (1997) define scaffolding as a teacher's assistance in response to a student's struggle with a task. This assistance enables the student to complete the task alone, but does not reduce the cognitive demands of the task. William and Baxter (1996) separate the constructs of analytic and social scaffolding. Social scaffolding refers to the scaffolding of social norms; analytic scaffolding is the "scaffolding of mathematical ideas for students" (p. 24). Speer and Wagner (2009) consider analytic scaffolding as guiding students "further toward the desired mathematical goal(s) by using selected student contributions" (p. 536). For the purposes of our data analysis, we put these ideas together and define the construct of scaffolding as a *teacher's pedagogical strategies or actions toward the desired mathematical goals in response to or in anticipation of students' struggles.* Furthermore, we introduce the term *productive scaffolding* to refer to *scaffolding that maintains the cognitive demands of the task*.

Data and Methods

This study used one white male instructor's materials and three episodes of teaching proof by mathematical induction at a large public research university in the southeastern United States. The course is a junior-level course designed to teach mathematics majors typical mathematical proof techniques. The data used are part of a larger project studying cognitive components of proof by mathematical induction. For this project, research-based instruction was developed and implemented. Teaching episodes were video and audio recorded and transcribed by the authors.

We analyzed the instructor's lecture notes as written prior to instruction. All mathematical tasks were categorized using the Tasks Analysis Guide (TAG) (Stein et al., 2000) with respect to Stein et al.'s (1996) levels of cognitive demands. Once a consensus was reached between us, we went through two phases of video analysis and coding. During the first round of video analysis, we classified each mathematical task's level of cognitive demand as it was presented in the classroom. We then coded the teacher's actions for instances of Stein et al.'s (2009) factors for maintaining cognitive demands, using Vygotsky's notion of scaffolding. After identifying the factors, we returned to our data to closely study the instances of scaffolding using our definition.

Results and Discussion

During the observed teaching episodes, the instructor used PowerPoint slides with three tasks displayed on three big screens around the room (Figure 1).

1. For each of the following parts, decide whether the given information is enough to conclude that the following claim is true.

   Claim: $P(n)$ is true for all $n \in Z^{+}$.

   If the given information is not enough, offer a brief explanation on why (perhaps listing a value of n for which $P(n)$, is not known to be true).
   a) $P(1)$ is true and there is an integer $k \geq 1$, such that $P(k) \rightarrow P(k+1)$.
   b) $P(1)$ is true and for all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
   c) For all integers $k \geq 1$, $P(k) \rightarrow P(k+1)$.
   d) $P(1)$ is true and for all integers $k \geq 2$, $P(k) \rightarrow P(k+1)$.
2. a) Prove that for all natural numbers n, 3 divides $8^n - 5^n$.
   b) Prove that if 3 divides $8^6 - 5^6$, then 3 divides $8^7 - 5^7$.
3. Prove that for all natural numbers n, $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

Figure 1: Tasks

When presenting the tasks to the class, the instructor provided students with little preliminary explanation. He typically displayed the tasks on the screens and encouraged students to work in small groups. For this reason, we can claim that these tasks were set up by the teacher in the same way that they were presented in the instructional materials. In the following discussion, we will refer to the tasks of first or second phase (Stein et al., 1996) as planned.

We further used TAG as an instrument to analyze the cognitive demands of tasks. All the tasks exhibited key attributes of "Doing Mathematics" tasks (see Stein et al., 2000). The problems required complex and non-algorithmic thinking and considerable cognitive effort. They also encouraged students to explore and understand the nature of mathematical concepts and to access relevant knowledge and experience.

The first round of analysis revealed the presence of most of the factors associated with maintaining high-level cognitive demands of tasks (Henningsen & Stein, 1997, Boston & Smith, 2009). First, the instructor seemed to allot an appropriate amount of time for the students to engage with the tasks through small-group discussion. Second, students received formal instruction on logical implication before they were introduced to the principle of mathematical induction. Given that logical implication is an important part of each task, we can argue that the tasks build on students' prior knowledge. Third, the instructor constantly questioned students by asking them to rephrase and justify their reasoning. He also frequently made conceptual connections between the tasks and students' solutions and modeled high-level performance through presenting counterexamples to students' erroneous claims.

Scaffolding

Avital and Libeskind (1978) introduced the method of "naïve induction" to assist students in overcoming their bewilderment of the transition from the base case to the inductive step. Naïve induction has been elaborated by Harel (2002) and labeled as "quasi-induction." Prior research has identified quasi-induction as a fruitful instructional approach (Harel, 2002; Cusi & Malara, 2008). This method engages students in repeated application of the inductive implication $P(k) \rightarrow P(k+1)$

for beginning values of k, reinforcing the logical reasoning that is essential in proof by mathematical induction. More specifically, students first establish that $P(1)$ is true. Then, they create a chain of logical implications $P(1) \rightarrow P(2), P(2) \rightarrow P(3), P(3) \rightarrow P(4),$ and so on. After considering these first few implications, students can then infer that the process must continue until eventually $P(n)$ is shown to be true.

The idea of quasi-induction was built into the overall observed instruction. In anticipation of students' struggle with formal proof, Tasks 1 and 2 were designed to engage students in quasi-inductive reasoning. Quasi-induction was first explicitly introduced by one of the students in the discussion of Task 1c who said, "1 works, so $P(1+1)$ works, so 2 works. And you can plug 2 back in for k and the logic repeats itself." In response to the student's reasoning, the instructor discussed mathematical rigor of quasi-induction, but accepted the suggested solution. Furthermore, during the group work on Task 3, students in one of the groups were not engaged with the task. The instructor suggested they use quasi-induction: "Sometimes it helps to do – just try a bunch of cases, just to get a feel of what's going on." This prompt allowed the task to still have the key attributes of Doing Mathematics while making it more accessible for the students.

Task 2a was introduced at the very beginning of the first class. One of the students suggested using the binomial expansion to prove the statement. The teacher acknowledged this idea but encouraged the students "to practice something inductive." The instructor anticipated students' struggle with formal solution using proof by mathematical induction. For this reason, after the students went through Tasks 1a-1d, he presented an "easier" Task 2b. The use of Task 2b did not reduce the cognitive demand of Task 2a. However, the students were given means to generalize the proof of $P(k) \rightarrow P(k+1)$ from a particular case of logical implication $P(5) \rightarrow P(6)$. Therefore, we consider the idea of generalization from a particular case as another example of what we call productive scaffolding.

Conclusion

This study provides an example of instructor's strategies for maintaining high-level cognitive demands of tasks in teaching proof by mathematical induction. The actions discussed above may inform instructors in preparation for and implementation of teaching mathematical induction. Our findings are consistent with the factors suggested by the extant literature (Henningsen & Stein, 1997, Boston & Smith, 2009 Stein & Smith, 1998). Namely, the teacher actively built upon students' prior knowledge, constantly asked students to explain their reasoning, and purposefully facilitated conceptual connections. We also report that scaffolding plays a central role in teaching proofs. In the context of proof by mathematical induction, the method of quasi-induction is suggested to be an example of what we call a productive scaffolding.

Although formal mathematical induction may be considered as a generalization of quasi-induction, there is still a cognitive gap between the two, which the students are not always able to bridge. Harel (2002) described this gap as a difference in perception of the inference $P(n) \rightarrow P(n+1)$. Future research must elucidate scaffolds as students attempt to bridge the gap.

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RELATIONSHIPS WITH MATHEMATICS: THE IMPORTANCE OF AGENCY AND AUTHORITY

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Mathematics classrooms are spaces where teachers provide students with opportunities that will inevitably shape their conceptions of the subject and their own abilities to learn it. Therefore, it is important to understand how a classroom community defines mathematical knowledge, mathematical practice, and what it takes to be a person who is successful in mathematics. The study uses interviews with teachers and students in two classrooms plus a district wide survey to understand the relationships with mathematics they construct. The paper ends with a discussion of specific areas of pedagogy that could support the development of productive relationships with mathematics by more authentically centering student thinking in the classroom.

Keywords: Teacher Knowledge, Equity and Diversity

Introduction

For many, math is seen as requiring rote memorization and the regurgitation of procedures with little to no room for free thinking. Research shows that such approaches to math learning are related to low achievement (Boaler & Zoido, 2016, PISA, 2012, Gray & Tall, 1994). Mathematics classrooms are spaces where teachers provide students with opportunities that will inevitably shape their conceptions of the subject and their own abilities to learn it. Therefore, it is important to understand how a classroom community defines mathematical knowledge, mathematical practice, and what it takes to be a person who is successful in mathematics. This paper builds upon Boaler's 2002 framework for a relationship with mathematics in terms of knowledge, practice, and identity. The study uses interviews with teachers and students in two classrooms plus a district wide survey to understand the relationships with mathematics they construct. The paper ends with a discussion of specific areas of pedagogy that could support the development of productive relationships with mathematics by more authentically centering student thinking in the classroom. These include a shift towards open and project-based curriculum and an increase in value placed on student mistakes and struggle.

Literature Review

Aguirre, Mayfield-Ingram, and Martin (2013) define mathematics identity as "the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the contexts of their lives" (p. 14). A student's mathematics identity will be formed in part by the ways they have been positioned in their particular learning context (Holland et. al, 1998). However, these conceptions of math identity are missing a key component to students' experience in the mathematics classroom-- the behaviors and practices they are expected to engage with while doing mathematics.

Teacher expectations of student mathematical behavior can be thought of through the lens of agency and authority. Agency refers to the extent to which students are able to express and use their own ideas in mathematical problem solving and authoring (Boaler, 2002). Student agency depends deeply upon the beliefs held by both teachers and students about what is expected of students in their role as problem solvers. This ranges from one end of a spectrum where the students are able to approach problem solving creatively, using their own ideas and methods to the opposite end where students are expected to use one specific procedure that's been told to them by another authority (i.e. the teacher or textbook). Part of this type of agency involves the extent to which students see this aspect of problem solving as being within their control. Gutstein (2007) found that when students experienced a strong sense of agency in their problem solving, they were empowered to interrogate knowledge sources and critically analyze material rather than simply receive it as truth. This sense of agency resulted in students deconstructing representations using mathematics to deepen their understandings of new material rather than searching for a pre-determined solution strategy.

Similar to the concept of student agency in the classroom is the concept of student authority. According to Cobb, Gresalfi, and Hodge (2009), authority in the mathematics classroom pertains to who decides what constitutes mathematical legitimacy. In some classrooms, this authority could lie solely with the teacher or textbook whereas in other classrooms it may be shared between the students, teacher, and textbook. Amit and Fried (2005) found that when the teacher is the main authority figure, students oftentimes use mathematical concepts introduced by the teacher unreflectively. In other words, students blindly reproduce what the teacher has shown without further thinking. These authors also found that this particular authority dynamic can hinder the productivity of collaborative learning efforts. For example, when students are working in groups, but the teacher is seen as the authority, little impetus exists for authentic collaborative problem solving.

Relationships with Mathematics

Boaler (2002) introduced the concept of a disciplinary relationship (see Figure 1). As knowledge, practice, and identity develop for a mathematics student, they each contribute to an overall relationship with mathematics. First, we consider the identity aspect of the relationship which is broken down into two parts: beliefs about one's role, and one's mindset. Within a mathematics learning setting, students will develop their own beliefs about what it means to learn math. A student may expect to be a passive receiver of knowledge, an active participant exercising agency and mathematical authority, or somewhere in between (Belenky, Clinchy, Goldberger, & Tarule, 1986). Additionally, they may believe that their mathematical abilities are static and fixed, or that they can be cultivated and grown (Dweck, 2005). Second, we consider the knowledge and practice aspects of the relationship with mathematics which together constitute the student's beliefs about the nature of mathematics and doing mathematics. The student may believe that knowledge in mathematics is made up of facts and procedures to be memorized, or they may see math as a web of ideas connected by logic and reasoning (Boaler & Zoido, 2016, PISA, 2012, Gray & Tall, 1994). Finally, the student may see the practice of doing mathematics as effortlessly and quickly understanding material or requiring struggle, learning from mistakes, and creative thinking (Boaler, 2015).

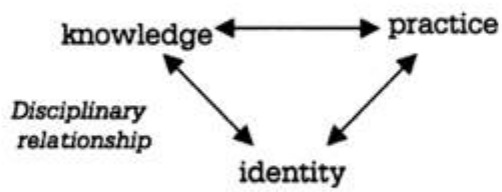

Figure 1. Framework for Disciplinary Relationships adapted from Boaler, 2002

Oftentimes, when a student has developed an unproductive relationship with mathematics, they respond to creative approaches in problem solving by stating something like, "am I allowed to do that?" As part of Boaler's 2002 study, she observed many classrooms referred to as "traditional" and "reform"- oriented. One of the most observable differences between these two types of classes were the role that agency played in each of them. Within the traditionally oriented classrooms, students were expected to follow standard procedures of the discipline. In these cases, the students' relationships with mathematics revolved around the agency and authority of the discipline. The students expected to follow the procedures and practices defined by the discipline.

In the reform-oriented classrooms students were "required to propose 'theories', critique each other's ideas, suggest the direction of mathematical problem solving, ask questions, and 'author' some of the mathematical methods and directions in the classroom" (Boaler, 2002, p.45). While it appeared that these classes offered more agency to students, Boaler clarifies that these students engaged with what Pickering (1995) calls the "dance of agency" between the established methods of the discipline and their own knowledge and practices (Boaler, 2002). (Note: Pickering (1995) found that professional mathematicians also engage with the "dance of agency" when developing and discovering new mathematics.) Through this "dance of agency" students would employ standard procedures coupled with their own ideas to adapt and extend methods in new and unknown contexts. The students developed mathematics relationships that gave them a sense of agency and allowed them some authority over their mathematical knowledge construction.

It is important to note that these students' opportunities to engage with the "dance of agency" were dependent on a number of things including the presence of an engaging project based curriculum but also the practices that teachers expected students to exhibit (Boaler, 2002). When it comes to mathematical practices (or ways of engaging with mathematics), the field provides what seems like a never-ending list of possible practices that teachers might focus their classroom towards (CCSS Mathematical Practices, 2010; NCTM Process Standards; Stipek et al., 1998). These practices include: communication, reasoning, proof, representations, justification, argument, sense making, and many more. While each of these practices are undoubtedly important, teachers cannot be expected to prioritize each one to the same extent so they must make choices based on their pedagogical beliefs or the needs of their students.

The decision of what to prioritize is complex in today's mathematics classrooms where students enter with an increasingly diverse range of strengths and needs. That is not to say that heterogenous classrooms are an issue. In fact, many research studies have shown that all students benefit from learning in de-tracked and heterogenous classrooms (Boaler, 2006 & 2011; Burris, Heubert, & Levin, 2006; Horn, 2008; Porter et al, 1994). However, knowing that the expectations for practice in the mathematics classroom will form the normative identity and ultimately influence the students' relationships with mathematics, it is imperative to know what teachers emphasize and how these priorities are taken up by students. This leads to the research question guiding this study: How are teachers and students constructing relationships with mathematics together?

Research Methods

Study Context and Participants

Although no two students will have the same relationship with mathematics, theoretically, the widest range of differences in these relationships would be apparent in a heterogeneous classroom where students represent a great range of prior experience and achievement levels. For this reason, this study focuses on heterogenous Algebra 1 classrooms.

The data for this study comes from a research project on a large urban school district in the Bay Area that had recently implemented a district-wide change of their mathematics course taking sequence by de-tracking mathematics through sophomore year and enrolling all 9th grade students in Algebra 1. Previous to this policy, students took algebra in eighth grade and 40% of students were failing and re-taking Algebra 1 (Hull Barnes & Torres, 2018). Since the implementation of the de-tracking policy, this failure rate has dropped to just 8% (Hull Barnes & Torres, 2018).

The student body of the district is culturally diverse with about 35% of students identifying as Asian, 27% Latinx, 15% White, 7% African American, 5% Filipino ,1% pacific islander, and <1%

American Indian (Facts, 2018). Furthermore, approximately 55% of students are considered Socioeconomically Disadvantaged, 29% of students are designated as Language Learners, and 11% are students diagnosed with Special Educational Needs (Facts, 2018). The students at Park High School represent an even more diverse community than that of the district with a higher percentage of socioeconomically disadvantaged students and greater percentage of students of color.

The participants in this study include two Algebra 1 teachers at Park High School and 6 of their students, for a total of 12 students. See Table 1 for more details about the participants.

Table 1: Participant Details

Teachers	Teacher Details	Student Pairs and designation[1]
Ms. Anderson	Early career teacher	Jackie and Kim (high achieving)
		Silvia and Arthur (turn around)
		Leta and Jose (low achieving)
Mr. Lang	Veteran teacher	Teresa and Mario (high achieving)
		Chantel and Steven (turn around)
		Marina and Lucy (low achieving)

Data

The data includes interviews with teachers and students in the two focal classrooms and results from a survey on mathematical mindset administered to 555 9th grade algebra students across the district. The interview questions for both teachers and students were developed using Rubin & Rubin (2008) as a guide. Teachers were interviewed during their prep periods and students were interviewed in pairs during class time.

The Mathematical Mindset Survey was developed by the youcubed research team and validated through previous research studies. The survey contains 27 questions with Likert scale answer options: Strongly Disagree, Disagree, Somewhat Disagree, Somewhat Agree, Agree, and Strongly Agree. The survey was conducted using the Qualtrics online software and offered in English, Spanish and Chinese. There are approximately 4,750 ninth grade students in the district, and the email requested teachers to give the survey to at least one of their 9th grade Algebra 1 classes. The distribution of the survey resulted in a total of 555 student responses to the survey representing a sample from 8 different high schools and 23 different teachers within the district.

Methods

Interview Data

To analyze this data, the researcher followed the analysis guidelines for inductive coding found in Miles, Huberman, & Saldana (2013). First, she open-coded each teacher interview to generate an initial set codes that were subsequently collapsed the codes into a broader set of codes and shared these with an expert in mathematics education for feedback. This resulted in a final teacher codebook that was applied to both teacher interviews.

From here, the researcher conducted a theme analysis which included exporting all excerpts coded with the same code into an excel spreadsheet and re-reading the excerpts, making note of the general theme(s) coming up in that code. From this process the teacher themes were generated and written up as one summary paragraph per theme.

[1] Teachers nominated a pair of students for each designation, "turn around" refers to students who started the school year low achieving but had improved throughout the school year.

Following the analysis of the teacher interviews, the researcher completed the same process with the student interview data which resulted in summary paragraphs for the student themes.

Since the research question asks about how the teachers and students construct relationships with mathematics together, the next step in the analysis required making connections across the two data sets. This process started with a comparison of the theme paragraphs from both sets of data and then a grouping strategy to create general themes. In some cases, this process was straight forward because both teacher and student themes already matched. For example, both the student and the teacher analyses resulted in a theme around mathematical authority, so those were grouped together as a general theme across both data sets. However, for themes that were less straight forward, the researcher would group similar ones together and then generate a heading for that theme. As she made her way through the list of teacher and student themes, she would first try to place the theme into one of the already existing headings, adjusting the title of the heading to better suit the themes included. In the cases where she was unable to reasonably connect the theme to a heading, she would generate a new heading. This was an iterative process that resulted in four general themes that cover all of the teacher and student themes where the most noteworthy findings centered upon agency and authority in the classroom.

Survey Data

The survey response data was first downloaded from the Qualtrics site and uploaded into the STATA quantitative data analysis software by another member of the research team. Then, the researcher created a table for each survey question that displays the spread of student responses by both frequency and percentage. To draw connections between the interview data and survey data, the researcher combined all agree answers into one metric and all disagree answers into another.

Findings

Both the teacher and student interviews surfaced a theme around agency and authority. The teachers want their students to work with one another to make mathematical decisions, choose methods, share ideas, and come to their own understandings around the content and take control of their learning. In an effort to encourage these practices, both teachers report trying to take a step back so that students can have genuine experiences of doing math with one another – and making mathematical decisions - without the teacher as the main authority.

> Ms. Anderson: I think that if I was constantly stepping in, it doesn't... I think that that just puts me back at this position of: 'I hold all the knowledge and I'm in charge of all of this.' And there are already enough times when I am that, and I am playing that role. And I've had some experiences this year where I blatantly did something wrong, and the kids didn't say anything to me. And I was like, 'Guys, what? You let me go through that whole thing.' And they were like, 'We figured you must have been right.' And I was like, 'Well, I'm flattered that you think I'm great.' Like, 'No, if you... You guys need to trust, trust yourself.' So I think that there are so many times where I already have all that power and control that if I'm gonna let a student go to the board, I don't wanna make that be a pseudo experience. I would rather have them actually be in control of it.

Mr. Lang shares his thinking around the value of students exercising mathematical authority and sharing their work with the class, especially when the class is struggling with a particular topic.

> Mr Lang: I think the more that when we work on something and there's some sort of place where we get stuck, to have a student or a student group go and present their solutions or how they're thinking about it. We're not ready for the more— I think I read this is largely a common thread in Chinese mathematics classes where they actually look for people to do good mistakes, if you will, to present it on the board and have that be learning— I don't think we're there yet, we're probably more showing the thinking and the steps towards what's gonna be a productive solution, but to have more students talk both so that they can get their ideas heard and be seen as students that can

> bring good mathematical knowledge to the class, as well as their articulation of what they understand about math, I believe, really helps them sort of solidify whatever understanding they develop.

Although Mr. Lang would like for his students to have agency and authority in approaching math through their own different ways of thinking, he sees his class as ready to discuss correct or “productive” thinking only. Herein lies a tension, there is both a sense of freedom and confinement of student thinking embedded in his statements: he wants them to feel the freedom to express their ideas but confines which ideas he values. This focus on correct thinking and answers is reflected in the student responses in terms of their beliefs in their own mathematical agency and authority in the classroom.

The students’ feelings of agency in mathematical problem solving are limited to choosing and utilizing different resources (such as calculators or peers) rather than choosing or creating their own mathematical ideas. For example, Kim, a high achieving student, explains that she is aware that she can utilize a variety of resources to succeed in math.

> Kim: I think what we need to do to be successful in our class is to ask more questions and ask for help, and really use the kind of resources we have around us, like our teachers and our peers, to help us.

This response is similarly reflected in the survey results where 89.1% of students agreed with the statement: “I am in charge of my own learning journey in math.” However, only 66% agreed that “In math class I feel creativity is valued”. While many students feel that they are in charge of their “learning journey”, fewer see creativity valued in the classroom. If we consider creativity as original thinking, then we can understand this result to communicate that students do feel agency over their math learning more generally, but not necessarily mathematical agency, meaning, students are not expressing and utilizing their own mathematical ideas. Students feel free to utilize resources (including the teacher and peers) but when it comes to sharing their unique ideas about math (creativity) they feel less freedom.

This apparent lack of expressions of mathematical thinking is closely intertwined with the students’ perception of mathematical authority. For these students, the act of deciding mathematical legitimacy was whittled down to merely deciding which answers are right and which ones are wrong rather than an interrogation of another’s mathematical reasoning or justification. For example, Silvia, a “turn around” student, was asked how she when the mathematical work she is doing is right and gave the following explanation:

> Silvia: We don't know. [chuckle] I mean, I don’t know when I’m doing my work, I don't know if it’s right or wrong. I would probably just ask a teammate or Ms. Anderson but yeah.

Of the twelve students interviewed, eleven students believed that their peers can help them decide what is right and wrong but that their teacher is the ultimate authority on the material. Perhaps the greatest detailed expression of the teacher’s mathematical authority came from Lucy’s explanation of what she does to decide if her work is right.

> Lucy: Oh, I just ask the teacher time and time again like, "Is this right?" And then when the teacher sees the problem on the work we try to show, then the teacher just sits down with us and then explains it deeply and deeply like, "What you need, what's this thing called and what's that thing called?" And then you answer it and then you... And then once the teacher is like, "That's correct," then you write it down with it. You write down step-by-step, how do you do that and how you do that, which is really helpful.

The belief that the teacher is the main source of mathematical authority appears to also be shared by a portion of the students surveyed. The results showed that 42% of students agree with the statement: “The teacher is the only one that knows if I understand or not”.

The focus on correct or incorrect answers rather than mathematical thinking seemed to manifest in a fear of mistakes and struggle. The students interviewed talked about struggle as something that is negative and should be avoided. For students, struggling is a sign that you don’t understand rather than a key part of the learning process.

> Kim: Whenever I get stuck, it makes me feel frustrated, and it's really uncomfortable because I feel like I could do it, but it just stalls, and my brain is like blank.

For Kim, and other students, getting stuck brings forth feelings of frustration and an inability to keep moving forward. Silvia expressed how continued instances of struggle begin to discourage her from mathematics learning.

> Silvia: I wouldn’t say I hate math, but it is frustrating. I was raised to be always a good kid, so I always like to be really good at what I’m doing. But when it comes to math, when I don’t get something, it just feels so frustrating. I'm just like, ‘you know what? Nevermind. Forget it. I’m not doing this,’ and I kinda just get stuck with that mindset…Yeah, it is really frustrating ‘cause you’re trying to actually be present in the group and trying to help other people. But then you’re just like, ‘Welp, I’m stuck.’ And it’s kind of (pause) ugly to have to be asking other people constantly about what's going on.

This communicates a low level of student agency as the students become debilitated by their own signs of struggle. These sentiments are reflected in the survey responses, where just over half (52%) of the students surveyed responded that they agree with the statement: “It is important not to make mistakes in math”, and 77% agreed with the statement: “I feel discouraged when I get a low grade in math”. The majority of students see mistakes and struggle as negative indications of one’s math learning and math ability. Furthermore, feelings of agency are thwarted as students feel discouraged by their struggles.

Overall, both the teachers and students varied in their reports about agency and authority in the classroom. The teachers want students to have agency and authority in their classrooms and they attempt to cultivate these dispositions by asking students to present their work to the class or share their thinking with peers. However, Mr. Lang notes that he feels his class is not ready to share incorrect thinking with one another. The students explain that they do share their work with one another, but with a focus on obtaining the right answers rather than sharing their thinking. Both teachers emphasize a desire for students to see themselves and one another as mathematical authorities rather than relying on the teacher without question. However, for eleven of the twelve students, the ultimate authority lies with the teacher.

Discussion and Conclusion

Together, the teachers and students in this study have constructed particular relationships with mathematics that can be thought of in terms of identity, knowledge and practice. In terms of identity, students seem to expect to take on a role that is not entirely passive but not quite active either. They report interacting with their peers but mostly for the purposes of verifying they are getting right answers instead of for the purpose of sharing their mathematical thinking. Mathematical knowledge has taken the form of the ability to decide what answers are right or wrong. In terms of practice, teachers and students value producing right answers above exploring the value of mistakes and struggle. Teachers report focusing student attention on what’s deemed “productive”. For the students, even though they see effort as important for success, there is a tension here where students actually wish to avoid struggling and spending too much time on any one topic as this feeling of struggle becomes debilitating for them, as noted by Kim who finds that when she gets stuck, “my brain is like blank”. The lack of value placed on struggle and mistakes works against the feelings of agency for students.

The teachers feel that students are not ready to discuss mistakes and instead focus on student solutions that are “productive”. This begs the question of who are the students that are producing what Mr. Lang describes as “productive” solutions and “good mathematical knowledge”. The avoidance of engaging with struggle and mistakes keeps the focus on being right and therefore devalues asking too many questions or asking for too much support. Furthermore, it increases the risk of sharing your thinking if you are not sure it is correct. Although the teachers express a desire for students to experience agency and authority in the classroom, it appears that the development of these dispositions are thwarted by a fear of mistakes and struggle on behalf of both the teachers and the students. Students are focused on asking questions to obtain the right answers or making sure they are doing the right steps rather than engaging in collaborative mathematical sensemaking.

An expansion of what is mathematically valuable would allow a wider range of students to see themselves (and be seen) as productive and important contributors to the mathematics community and expand beliefs about who is good at math.

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ARTICULATING THE STUDENT MATHEMATICS IN STUDENT CONTRIBUTIONS

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We draw on our experiences researching teachers' use of student thinking to theoretically unpack the work of attending to student contributions in order to articulate the student mathematics (SM) of those contribution. We propose four articulation-related categories of student contributions that occur in mathematics classrooms and require different teacher actions:(a) Stand Alone, which requires no inference to determine the SM; (b) Inference-Needed, which requires inferring from the context to determine the SM; (c) Clarification-Needed, which requires student clarification to determine the SM; and (d) Non-Mathematical, which has no SM. Experience articulating the SM of student contributions has the potential to increase teachers' abilities to notice and productively use student mathematical thinking during instruction.

Keywords: Classroom Discourse, Communication, Instructional Activities and Practices

Productive use of student mathematical thinking during instruction is a critical aspect of effective teaching (National Council of Teachers of Mathematics, 2014). Along with other researchers (e.g., Sherin et al., 2011), we see noticing, in particular *attending* to student mathematical thinking, *interpreting* it, and *deciding* what to do with it (Jacobs et al., 2010), as critical skills that support this productive use. Although teachers who are adept at productively using student mathematical thinking might have developed intuition and skills that allow them to notice important aspects of student contributions, such practice needs to be unpacked to support more novice teachers' learning (Boerst et al., 2011). Toward that end, in this paper we draw on our experiences researching teachers' use of student thinking to theoretically unpack the work of attending to student contributions to articulate the student mathematics in those contributions. Our goal is to contribute to the knowledge base for developing teachers' abilities to notice student mathematical thinking during instruction, abilities that lay the groundwork for productive use of that thinking.

We conceptualized a set of high-leverage instances of student mathematical thinking—what we called **M**athematically significant pedagogical **O**pportunities to build on **S**tudent **T**hinking, or MOSTs (Leatham et al., 2015). We proposed that *building* on MOSTs—turning the MOST over to the class for them to collectively make sense of it—was a productive way to use student mathematical thinking (see Van Zoest et al., 2016 for an elaboration of *building*). The first (of six) criteria for determining a MOST is "Student Mathematics" and requires evaluating a given student contribution to determine whether students' words and actions provide "sufficient evidence to make reasonable inferences" (p. 92) about what the student is saying mathematically. This way of conceptualizing *student mathematics* (SM) is how we view what it means to attend to the mathematics in student contributions. Attending to students' mathematics in this way requires attending to what students are and are not saying and being careful about the inferences we make in that regard. Such attention positions a teacher to "confidently articulate" the SM of the student contribution so that they can interpret that thinking and decide what to do with it based on the mathematics the contribution makes available for the class to engage with.

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There is evidence that experience articulating SM can positively impact teachers' noticing. Teuscher et al. (2017) studied two pairs of student teachers, of which one pair had had experience articulating the SM for student contributions in a data set of secondary mathematics lessons. The differences between the two pairs' written reflections on student mathematical thinking during lesson observations were striking. All four consistently attended to student mathematics, but not to the same level of detail. The two who had previous experience articulating SM demonstrated skill in doing so in their reflections. The other two were only moderately able to provide a detailed articulation. This study suggests that developing teachers' skills in articulating SM positions them to attend to the mathematics of student contributions.

The Work of Articulating: Four Categories of Student Contributions

According to our view, articulating SM requires that one provide a reasoned argument for any inferences. Based on our experience justifying such inferences, we propose four distinct articulation-related categories of student contributions that require different teacher actions:

1. Stand Alone Contributions require no inference to determine the SM.
2. Inference-Needed Contributions require inferring from the context to determine the SM.
3. Clarification-Needed Contributions require student clarification to determine the SM.
4. Non-Mathematical Contributions have no SM because they are not mathematical.

Stand-Alone Contributions

Stand-alone student contributions are the easiest instances to identify the SM for, because the student contribution itself is the SM—no inference is required. This category of student contribution is straightforward and makes clear what mathematics the contribution makes available for the class to engage with. For example, consider a student contribution during an introductory lesson on adding fractions, where a student asks: "Is $\frac{2}{5}+\frac{1}{5}=\frac{3}{10}$?" The SM of this contribution is simply: *Is* $\frac{2}{5}+\frac{1}{5}=\frac{3}{10}$*?* The statement is clear (though not mathematically correct) and complete. This SM demonstrates that SM need not be true, and also that SM can come in a variety of forms, including questions.

Inference-Needed Contributions

Conversational norms dictate that we do not always use complete sentences or make explicit references. Instead, we use pronouns and take other communication shortcuts. Students do the same. Because it is impossible to know exactly what students are thinking, teachers make inferences about their students' contributions. These inferences are based on observations of what students say, gesture, and write. Thus, these shortcuts often need to be filled in to make sense of what mathematics the student contribution makes available. In these situations, although the work of inferring the SM can be done, it requires making inferences from the context. Teachers must take care, however, to not infer beyond the evidence provided by the student contribution. In particular, there is a tendency to fill in the gaps with what one wants to hear. When a student makes a contribution, it is their mathematics that needs to be attended to.

Suppose, for example, a class is asked a general question such as, "Do you understand?" and a student responds, "No." We know that the student does not understand something, but their contribution does not provide evidence of what they do not understand. In contrast, if the class was asked, "Is Ax + By = C a linear equation?," it could be reasonably inferred that if a student says "No," they actually mean *No, Ax + By = C is not a linear equation*. Here, it is reasonable to infer that the student is answering the teacher's question and the italicized statement articulates the mathematics that the contribution makes available and thus is the SM of the contribution.

In drawing such inferences, one must stay as close to the context as possible. For example, if a student says, "Can it ever have two y-intercepts?" in the context of an introductory discussion about the slope-intercept form of linear equations, a reasonably inferred SM is: *Can a graph of a linear* *equation ever have two y-intercepts?* Although it is possible that this student is wondering about the multiplicity of y-intercepts for graphs of all types of equations, the contextual evidence suggests it is more likely that they are thinking only about linear equations.

This section illustrates how articulating the SM makes explicit things that were implicit because of communication norms and the context, but does so without altering the mathematical content of the student contribution. The resulting SM is a clear articulation of a reasoned inference of what the student is expressing mathematically in the contribution.

Clarification-Needed Contributions

Clarification-needed contributions require additional information from the student to determine the SM. These contributions do not contain enough information to reasonably infer the SM; thus, we cannot reasonably articulate their SM. Sometimes clarification-needed contributions are students' attempts to articulate ideas that are particularly insightful and relevant. This is why it is critical for teachers to learn to recognize when clarification is needed and how to productively seek that clarification.

There are several ways in which a student contribution that appears mathematical may not contain enough information to reasonably infer the contribution's mathematics. For example, students often express general confusion by saying things such as, "I don't get it." Without further information we cannot reasonably infer the mathematics underlying their confusion. Sometimes students' contributions are too convoluted to make sense of what they are saying without clarification. For example, during a discussion about why ¼ times 3 is ¾, a student may state, "The 3 is like 3 and then you have a ¼." The student recognizes that there is a 3 and a ¼ involved, but how they see the relationship between these numbers is unclear. Thus, there seems to be mathematical thinking going on, but we cannot infer what it is.

Another subset of clarification-needed contributions are clarifiably ambiguous (Peterson et al., 2019). These contributions have two or more viable interpretations, and we cannot make a reasoned argument for which one best articulates the SM of the contribution. Consider the interchange when a teacher says, "Could we use unit rate to solve the proportion $\frac{6}{4} = \frac{x}{10}$?" and a student responds, "Yes, by dividing." We can infer that the student is saying, "Yes, we can use unit rate to solve the proportion $\frac{6}{4} = \frac{x}{10}$ by dividing." The latter part of the sentence, however, is ambiguous; there is no indication of which quantity would be divided by which other quantity. There are several legitimate possibilities for these quantities, resulting in multiple interpretations for this student's statement. The student might be saying "divide 6 by 4" to get 1.5 or they might be saying "divide the numerator by 2 and the denominator by 2" to simplify 6/4 to 3/2. Both of these are viable interpretations for what the student might mean by dividing. Of course, there are other possible interpretations that might reveal misconceptions about the "unit rate" strategy or about proportions in general. Thus we cannot with any level of confidence infer the SM. In order to articulate the SM of this contribution, we would need to ask the student to clarify what is being divided by what. Regardless of the reason that clarification is needed, moving forward without clarifying such contributions could lead to misunderstandings. Students could think that different ideas are being considered, leading to cross-talk and general confusion. Also, without knowing the SM of a contribution, teachers would not be able to determine whether that thinking is worth pursuing.

Non-Mathematical Contributions

Sometimes students say things like, "I need a pencil" that clearly have no mathematical content. Other times we have evidence THAT students are thinking, but there is not enough evidence to infer whether WHAT they are thinking is mathematical. For example, instances of general agreement (e.g., "Okay" or "Yeah") in response to a vague teacher question (e.g., "Does this make sense?" or

"Was this problem the same as the ones last week?"). Even when a student is engaged in mathematics, they can make contributions that have no mathematical content. For example, a student describing their graph might say, "I made my line pink because pink is my favorite color." Non-mathematical contributions have no SM to infer.

Summary and Conclusion

We identified four categories of student contributions based on the inferability of their SM. The Stand Alone category requires no inference by a teacher because the student contribution and its SM are the same. The Inference-Needed category requires drawing on the context to infer the SM of the contribution. For both of these categories, we are able to articulate the SM of the contribution. For the Clarification-Needed and Non-Mathematical categories, we are not able to articulate an SM for the contribution; the Non-Mathematical because there is no mathematics involved and the Clarification-Needed because it needs clarification to articulate the SM. To make a Clarification-Needed contribution the focus of a whole-class discussion in its current state would likely be unproductive—at best wasting valuable instructional time and at worst introducing misconceptions. In our own work articulating the SM of student contributions from a variety of classrooms where students are given the opportunity to share their thinking, we have found many Inference-Needed and Clarification-Needed contributions in every classroom—the types of contributions that require drawing (or deciding not to draw) inferences. Thus, reflection on or observation of almost any mathematics lessons provides ample opportunities to practice this critical work of attending "within" (Stockero et al., 2017) student contributions. Attending to student contributions with the necessary precision to articulate the SM is one important aspect of the "close listening" (Confrey, 1993, p. 311) teachers need to facilitate meaningful classroom mathematics discourse. Such listening is "not mastered instantaneously" but is truly a "habit of listening" (p. 312). Experience articulating the SM of student contributions has the potential develop this habit and increase teachers' abilities to notice student thinking during instruction.

A classroom where a wide range of student contributions are available creates a complicated environment in which to carry out the work of teaching. Teachers must continually decide which students' ideas to make the object of a class discussion and which to respond to in other ways. In this paper we unpacked the process of figuring out what students are expressing mathematically, a foundational skill for the productive use of student thinking. Being deliberate about making reasoned inferences of the SM of a student contribution sets teachers up to make informed decisions about whether and how they use the student thinking that is available to them.

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MAKING ADDITION VISUAL: SUBITIZING AND SCAFFOLDING

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In this qualitative empirical study, we discuss new perspectives on how the teaching of addition can be made visual for young learners. Our research is framed by scaffolding (specifically the development of conceptual discourse through representational tools) and subitizing. Subitizing is generally described as seeing how many suddenly without counting each individual item and representation tools are resources that support students and teachers to discuss and reflect on mathematical ideas. We describe one of many short Grade 2 classroom episodes that occurred weekly during an entire school year in Canada. Episodes were centered around the use of small round manipulatives that were arranged based on subitizing literature. We make initial claims about making the teaching of addition visual through student subitizing abilities, correctly solving an arithmetic problem and student explanations and responses to what they saw and did.

Keywords: Number Concepts and Operations, Elementary School Education.

Objective

By examining a short classroom episode on addition through the lenses of scaffolding and subitizing, this study aligns with the PME-NA goal of deepening and understanding the psychological aspects of the teaching and learning of mathematics. Using subitizing, a construct that is firmly rooted in psychology (Kaufman et al., 1949; Revkin, et al., 2008), leads to new perspectives and pathways to discuss the issues of teaching number concepts and operations. This qualitative study pairs mathematics education literature (e.g., scaffolding) with psychological concepts (e.g., subitizing) with the aim of examining the question: How can the teaching of addition be grounded in visualization through capitalizing on subitizing?

The difficulties and issues young students face when learning about how to solve basic arithmetic problems are well documented in mathematics education research (e.g., Boaler, 2015; Baroody et al., 2009; Jordan & Montani, 1997) and is even apparent in media pieces with public calls for 'back to the basics' (Rushowy, 2019). Boaler (2015) states, "[w]hen students focus on memorizing…they often memorize facts without number sense, which means they are very limited in what they can do and are prone to making errors" (p. 2). Similarily, Kamii and Domenick (1998) suggest that when young students are pushed to memorizing algorithms too soon (such as the traditional addition algorithm where students stack the numbers and "carry"), the algorithms "unteach" place value which in turn hinders the acquisition of number sense.

'How to' manuals and daily teaching activities have appeared and have great potential to address issues of basic arithmetic. The two we identify as having the most potential and influence are 'number talks' and 'making thinking visible'. These activities and manuals could benefit from conceptual framing and infusing/weaving of psychological research with explicit connections to teaching. Number talks are brief daily talks where students talk about their strategies to mentally solve computational questions (Humphreys & Parker, 2015). Number talks (Parrish, 2010) align with research findings that suggest that students learn basic arithmetic gradually over time (Bruce & Chang, 2013). 'Making thinking visible' for the purpose of enhancing teaching can be found in the form of books for teachers (e.g., Hull, Balka, D. S., & Miles, 2011). Ritchart, Church and Morrison (2011) explain how active use of knowledge (including retention and understanding) is achieved through learning experiences that require learners to think about and with what they are learning.

Understanding and explicating student thinking is a difficult task (Leatham et al., 2015) because thinking is largely invisible and often conceived as an internal process (Ritchhart, Church, Morrison, 2011).

Here, ideas from 'number talks' and 'visible thinking' inspired us to develop short classroom episodes on addition and subtraction where small round manipulative were spatially arranged based on recommendations in subitizing research. In essence, these lessons aimed to make individual student's addition and subtraction solution strategies visible so that they could be discussed.

Conceptual framing

Subitizing is generally described as "instantly seeing how many" (Clements, 1999, p. 400). Clements (1999) categorized subitizing into two types: perceptual and conceptual. Perceptual subitizing is "[r]ecognizing a number without consciously using other mental or mathematical processes and then naming it" (Clements, 1999, p. 401). Whereas, conceptual subitizing applies the perceptual process repeatedly and quickly uniting those numbers. For example, a child can perceptual subitize "4" by simply recognizing it and naming it and conceptually subitize 4 by recognizing 4 consists of 2 groups of 2. It is important to note that the way objects are spatially arranged can impact the ease at which a student subitizes (Clements, 1999). Indeed, studies (e.g., Mandler & Shebo, 1982) about subitizing have concluded that students make less mistakes (i.e., find it easier) when dots appear as they do on dominoes—e.g. 10 as two sets of five as you would see a five on a dice face.

Subitizing is deeply linked to visualization and images, and it has clearly been suggested that subitizing should be capitalized on for the learning and teaching of addition (Clements, et al., 2019). Clements (1999), while making these recommendations, draws on the work of Markovits and Hershkowitz (1997) to go as far as saying that "[c]onceptual subitizing is a component of visualization in all its forms…and [c]hildren refer to mental images when they discuss their strategies" (p. 403). Noteworthy is that a literature search on subitizing and number operations reveals many pieces from psychology and points to a scarcity of classroom-based research in how subitizing can be/has been used in the service of teaching addition.

The wide use of scaffolding in math education is apparent in Bakker, Smit, and Wegerif's (2015) literature review. In 2006, Anghileri related scaffolding specifically to math contexts based on previous research on scaffolding outside of math (e.g., Wood et al., 1976). She put forward three levels of scaffolding with the aim to provide language that can be used to describe actual acts of teaching in mathematics classrooms. Scholars (e.g., Bakker, Smit, and Wegerif, 2015), have used Anghileri's three leveled framework to analyze data from math classrooms.

Level 3: *developing conceptual thinking* comprises of two subcategories: making connections and developing representational tools. This level is "less commonly found but identified as the most effective interactions" (p. 47). Level 3 scaffolds are most relevant to this study because of representational tools. Representational tools support students and teachers in building conceptual discourse, as they "constitute as a resource that students can use to express, communicate, and reflect on their mathematical activity" (p. 48). Anghileri (2006) explains that representational tools can provide "powerful visual imagery" (p. 47) and are important because "[m]uch of mathematical learning relates to the interpretation and use of systems of images, words, and symbols that are integral to mathematical reasoning" (p.47). The most common form of representational tools take the form of symbolic records of students' ideas that occur through teachers noting students' interpretations and solution strategies. Representational tools can take other forms such as graphs, and in Dove and Hollenbrands' (2015) study which examined scaffolds provided by high school geometry teachers, Geometer's Sketchpad was considered a representational tool.

In terms of most effective teaching and Level 3 scaffolds, Anghleri (2006) gives a sense that mathematics classrooms should go beyond the individual and there should be evidence of "shared", “togetherness”, “communal” and “cooperation”. This is apparent when she describes “real mathematics learning in the classroom” in terms of “struggle for shared meaning… a process of cooperatively figuring things out determines what can be said and understood by both teacher and students” (p. 46). She explains that with Level 3 scaffolding “understanding comes to be shared as the individuals engage in the communal act of making mathematical meanings” (p.49). Meaning that effective teaching (or *developing conceptual thinking)* can be evidenced through the development and the use of representational tools that are communal.

Methods

This study took place in a Grade 2 class, in which, researchers and the classroom teacher co-planned short (typically 7-8 minute) weekly lessons/episodes, throughout the 2018-2019 school year. There were approximately 20 students involved in each episode which took place in a predominantly English-speaking public school in a highly populated Canadian city. One of the researchers (not the assigned teacher) acted as the lead teacher during the classroom episodes.

During the lessons, the study participants were seated on a carpet together in front of the screen. Students were given an arithmetic problem (18+12) and asked to think about how they would solve it. Small circular objects (flat circular candies that are ¾ cm in diameter) were laid out and projected on a screen to represent numbers in questions. Numbers were often colour coded (i.e., for 18+12; 18 was represented by blue objects and 12 by green objects, Figure 1).

Eight of the 42 collected episodes were analyzed for this report. One episode was chosen because our conceptual framework points to the development of representational tools being communal acts that involves ‘shared’ and ‘together’. Hence, we looked for what we call ‘chorus’ in the data. These are incidents in the videos where students joined in ‘chorus’ speaking with the teacher or in response to teacher questions about what they saw. It is difficult to identify, from the audio of the video recorded data, how many students form chorus but it is clear that it is much more than 10 students. Data was analyzed using Powell et al.’s (2003) model for studying the development of learners’ mathematical ideas and reasoning using videotape data.

Results (Episode of 18+12)

The episode begins with the teacher directing students' attention to a pile of 18 blue candies and a pile of 12 green candies (with no structured arrangement) and asking students how many blue candies there are. A student comes to the projector and begins to move the candies one by one to create two strings of 3 candies each. The teacher interrupts the student and says: “Can I offer you an idea? I would really like 5s” and the teacher arranges five of the blue candies as you would see on the face of a dice. Before the student begins to re-arrange the objects, the teacher addresses the entire class with: “Does everyone agree this is five?” as she circles the five candies she re-arranged with her finger. A chorus of students calls out “yes”. Subitizing is confirmed with immediate student chorus confirmation to the question “can you see it right away?”.

The student rearranges the rest of the blue candies and declares 18. The teacher verbally repeats the number 18 and asks the student “How do you know that?”. The student explains he (re)composed the larger number 18 by saying “five, ten, fifteen, eighteen” while simultaneously pointing at the manipulatives. The teacher says: “OK. Five, ten, fifteen, 16, 17, 18”. There is a chorus of students that join for “five, ten, fifteen” only the teacher says “16”, more students join for 17, and then there is a chorus for 18. Another student moves up to the projector and rearranged the green candies in a similar way. The student states: “five, ten, eleven, twelve” providing evidence of conceptual subitizing. The students are then told the goal of the teaching episode is to figure out how many candies are on the projector (i.e., to solve 18 + 12) and figure out as many strategies as they can.

The teacher invites another student to the projector to use counting on as a strategy to solve 18 +12. The student starts by recognizing the 18 blue candies. Then the teacher and student count one-by-one together using their hands and fingers "19, 20, 21, …" until 30 (note that the manipulatives are not used). The teacher then prompts for connection making by pointing to the candies individually as she says "19, 20, 21, …" until 30. There is no audible chorus with her as she spoke and she asks how many students used the strategy of counting on. Two other students lift their hands to indicate they used a solution strategy of counting on by 1s.

The teacher asked, "who used a different strategy?" A student offers a strategy that counts by 5s and 10s. The student starts by pointing at two blue groups of 5s that are already formed and saying "five, ten, fifteen, twenty, twenty-five [pause] then I moved these two and put them together" as she moved the two green candies to be with the three blue candies to form a five as you would see on the face of a dice (Figure 1).

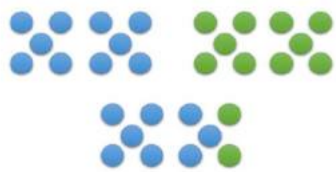

Figure 1: 18 + 12

Discussion and/or Conclusions

We conclude that addition was made visual, as teaching was grounded in visualization. Given the issues that students experience with addition, this study should be of great interest, as we offer a different way to teach addition that follows underlying principles of teacher resources ('Number Talks' and 'making thinking visible') but extends practical suggestions by infusing math education research and psychology research. In essence, we have shown how the teaching of addition can be made visual through using small objects that are spatial arranged in specific ways that are inline with research on subitizing.

Our results evidence teaching grounded in visualization through conceptual subitizing ("a component of visualization in all its forms") and the development and use of a representational tool that is communal. Students used conceptual subitizing to identify 12 and 18 items. There are indications of a representational tool that is communal when students respond in chorus and when solving 18+12. Spatially arranging small round manipulatives in a very specific way provoked one student to describe how she solved 18+12 by composing the last group of 5 (conceptually subitizing 5) by composing 3 blue and 2 green candies.

Although it has been recommended that subitizing can and should be used to support students in arithmetic, there is a scarcity of studies that respond to these recommendations and calls by enacting and researching them in actual mathematics classrooms. Significantly, through conducting classroom based research and analyzing our data through scaffolding and subitizing, we highlight how number talks that focus on making thinking visible by capitalizing on subitizing can be used to make the teaching of addition grounded in visualization.

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LINKING A MATHEMATICIAN'S BELIEFS AND INSTRUCTION: A CASE STUDY

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Teachers' beliefs impact their instructional choices, but characterizations of that relationship are limited in college settings. Based on interviews and classroom video from three units of instruction, this paper examines a full-time instructor's stated beliefs about teaching and ways these beliefs manifested in their teaching. The instructor made curricular choices clearly aligned with their stated beliefs about math, learning, and teaching. Day-to-day instructional choices reflected these beliefs as well, but tensions between beliefs also manifested. Characterizations of the interactivity of classes are provided through descriptive and quantitative measures. These characterizations of instruction highlight changes in instruction throughout the semester.

Keywords: Classroom Discourse; Teacher Beliefs; University Mathematics

Beliefs impact the ways people perceive, interpret, and respond to situations (Pajares, 1992). Thus, numerous studies have examined teachers' beliefs, including three handbook chapters on teachers' beliefs in math education (Thompson, 1992; Richardson, 1996; Philipp, 2007). However, less is known about mathematicians' beliefs and their impact on instruction. Similarly, limited research has been conducted on semester-long college instructional practice. In response to these gaps, this study addresses the following research questions: (1) How did an instructor describe their beliefs about math, learning, and teaching? (2) How can their instructional practice be characterized? (3) What relationship exists between their beliefs and instructional practice?

Literature Review and Conceptual Framework

Extensive research describes the coordination of beliefs into a belief system. Philipp (2007) synthesized previous belief system characterizations as: "A metaphor for describing the manner in which one's beliefs are organized in a cluster, generally around a particular idea or object" (p. 259). Prior work on beliefs highlighted how they are influenced by a teacher's view of the nature of math (Ernest, 1991), prior school experiences, and immediate classroom situations (Raymond, 1997) as well as their effect on instructional practice (Wilkins, 2008). One of the distinctions between studies is how researchers address perceived inconsistencies in teachers' statements and actions. Early studies examined differences between what a teacher claimed and what they did (e.g. Cohen, 1990). Later studies examined both teachers' beliefs and practices before drawing conclusions (e.g. Schoenfeld, 2003; Speer, 2005; Speer 2008) and emphasized the importance of observing teachers for a long period to see how beliefs impact instruction (Skott, 2001).

While extensive research has been conducted on K-12 teachers' beliefs (e.g., Beswick, 2012), fewer studies have examined teachers' beliefs or instruction at the university level. Weber (2004) examined a real analysis professor's lecture-based teaching but observed the teaching style varied based on the material. Johnson, Caughman, Fredericks, and Gibson (2013) examined teachers' priorities for instruction while using Inquiry-Oriented (IO) materials, especially noting content coverage concerns, goals for student learning, and student opportunities to discover mathematics. Surveys of abstract algebra instructors have examined influences on lecturers' teaching (Johnson, Keller, & Fukawa-Connelly, 2018; Johnson, Keller, Peterson, & Fukawa-Connelly, 2019). Those most influential (in order of frequency) were their experience as a teacher, experience as a student, and talking to colleagues. These instructors self-reported their time spent on types of instruction, leaving questions about how to characterize college teaching.

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The theoretical framework in this study is Leatham's (2006) construct of sensible systems. This framework posits that belief systems can be organized such that beliefs that seem contradictory to an outsider are not examined together by the teacher holding the belief, allowing "inconsistent" beliefs to coexist. Alternatively, certain beliefs could be held as ideal while others are given priority in specific situations. Generally, he suggested that if a researcher concluded a teacher's beliefs were inconsistent, the researcher did not have all of the information.

Methods

In this case study, the instructor participant, Dr. Bailey (a pseudonym), was a full-time instructor teaching an introductory abstract algebra course. Bailey's class met three times per week in 50-minute periods that were a mixture of lecture and "lab" days. They engaged in two semi-structured interviews (Fylan, 2005) lasting one hour each. Interviews were audio and video recorded and coded using thematic analysis (Braun & Clarke, 2006). Classroom data were collected in the middle of a unit on groups and through the whole units on group isomorphism and quotient groups. Classroom data were analyzed with the Toolkit for Assessing Mathematics Instruction–Observation Protocol (TAMI-OP) (Hayward, Laursen, & Westin, 2017) and the Inquiry-Oriented Instructional Measure (IOIM) (Kuster, Johnson, Rupnow, & Wilhelm, 2019).

The TAMI-OP is an observation protocol that aids recording what the instructor and students do in a classroom, broken into 2-minute segments of instruction. The IOIM was a rubric that provided a way to characterize how IO a class was. The IOIM uses a five-point scale and scores seven practices (below) that reflect the principles of IO instruction.

1. Teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point.
2. Teachers elicit student reasoning and contributions.
3. Teachers actively inquire into student thinking.
4. Teachers are responsive to student contributions, using student contributions to inform the lesson.
5. Teachers engage students in one another's reasoning.
6. Teachers guide and manage the mathematical agenda.
7. Teachers support formalizing of student ideas and contributions and introduce formal language and notation when appropriate. (Kuster et al., 2019)

Results

Instructor Beliefs

Bailey highlighted mathematicians' search for theorems as a purpose of math: "So I think mathematics is the search for theorems which…I would take to mean things that both can be proven…and then also the actual pursuit of proof…" Bailey emphasized actively doing math to learn it: "I'm a firm believer in learning by doing is best, so…every class I try to give the students something to do even if it's…here I'm gonna put this…example on the board for two minutes, let you guys work on it…." They based these ideas on how they learned: "I have to be …coming up with my own examples or coming up with my own proofs and just really synthesizing for it to stick." They were aware that how they learned could differ from how others learn, just as people have different ways of thinking in other contexts: "Different people have different frames for interpreting politics…so I think the same applies to learning."

Bailey discussed the role of different types of instruction within a class period when addressing the nature of teaching math. On lecture days, they would focus more on exploring the definitions and proofs in the class with a few smaller examples worked in. On lab days, they would expand the interaction that students were engaged in, especially for addressing examples.

They valued lecture as a way to make sure they taught all of the intended material and were satisfied with the interaction/coverage balance struck with two lectures and one lab per week.

Bailey identified two main ways that their beliefs about the nature of math, learning math, or teaching math were reflected in their instruction: the use of different types of instruction to reach different types of learners and an emphasis on students doing mathematics.

> It reflects my belief that people learn in different ways, and so, try not to use the same style throughout and also do different things….All my undergraduate mathematics classes were what I've been referring to as lecture.…I wasn't great at following what was going on in the lectures at that point in time. The group work is the kind of thing that would have helped me, so…putting in that different element for maybe people who do learn in a different way.

Their beliefs about creating a variety of learning opportunities for their students sprang from their experiences as a learner. In this case, the lack of alignment between their experiences and what would have helped them appeared to be formative. This relates to Johnson et al. (2018), in which the second most reported influence on instruction was experiences as a student.

Characterizing Instruction

Instruction is characterized based on data and analysis from the IOIM and TAMI-OP. IOIM practice scores are listed by practice (e.g. column P1 shows Practice 1 scores) with lecture scores on the left and lab scores on the right. TAMI-OP data rates are presented to the nearest whole percent. Counts of time blocks refer to numbers of 2-minute blocks (e.g. 9/31 segments lecturing means 9 of the 31 2-minute segments had some time spent on lecturing).

Table 1: IOIM Practice Scores in Lectures/Labs

Unit	P1	P2	P3	P4	P5	P6	P7
Groups	2/4	2/4	½	2/3	2/3	3/2	1/2
Isomorphism	2/3	2/2	2/2	2/3	1/2	3/2	2/2
Quotient Groups	2/2	2/2	2/2	1/2	1/1	3/3	1/1

In the Group unit, the lectures received low (1) to medium (3) IOIM scores, and the lab received medium-low (2) to medium-high (4) scores, as shown in line one of Table 1. These scores indicate the lecture days were not well aligned with IO instruction whereas the lab days were somewhat aligned with IO instruction. Similar scores were given in the Isomorphism unit. In the Quotient Group unit, the lab days received scores similar to lecture days; the only difference was on Practice 4, where the lab score was higher. Students engaged in less discussion with each other on lab days at all but one table, which depressed the IOIM scores. Across the three units, lecture scores held steady or decreased, except for Practice 7 in the Isomorphism unit. There, the lab started the unit, allowing some informal notation and ideas to come from the students before isomorphism was fully explained. The lab scores decreased or held steady except for Practice 6, where students were given more closure in a whole class setting in the last unit.

The results from the IOIM are also reflected in the TAMI-OP. In Table 2, we see lecture days in the Group and Isomorphism units were dominated by the instructor lecturing and included less time for students to work individually or in groups, whereas the allocation of time was flipped on the lab days. In the Quotient Group unit, more time was spent lecturing and students spent less time working than in previous units. Furthermore, unlike the previous units, where labs received a full day each time, this unit's labs received only partial days or spread over two days.

Table 2: Time Averages Across All Three Units

Day	Segments Lecturing	Segments Students Working	Segments Student Presenting	Segments Whole Class Discussion
Group Lecture	49/51	9/51	0/51	0/51
Group Lab	3/25	23/25	0/25	0/25
Group Ave.	68%	42%	0%	0%
Isomorphism Lecture	47/51	5/51	0/51	0/51
Isomorphism Lab	6/26	26/26	0/26	0/26
Isomorphism Ave.	69%	40%	0%	0%
Quotient Gp. Lecture	168/177	12/177	0/177	0/177
Quotient Gp. Lab	4/36	36/36	0/36	0/36
Quotient Group Ave.	81%	18%	0%	0%

Combining the information from the IOIM and the TAMI-OPs paints a picture of a class strongly guided by the instructor's mathematical knowledge but with some opportunities for student exploration. The mathematical authority rested with Bailey, who was in charge of moving the class forward. As the semester progressed, students were given less time to work and the amount of time the instructor spent lecturing increased, especially in the final unit.

Discussion and Conclusion

Dr. Bailey's stated beliefs about the nature of math focused on the structure of mathematics and the search for theorems. Their instruction reflected a belief in math as the search for theorems through their emphasis on proof in lecture, which they addressed by lecturing twice as much as they provided labs. Most of the time on lecture days was devoted to presenting proofs of theorems and thinking through implications of the work the instructor did at the board. However, the existence of two types of instructional days, opportunities to work on problems for extended periods, and opportunities to interact aligned with Bailey's stated desire to use many types of instruction to reach many types of learners. Although most groups experienced largely lecture and individual work time in class instead of varied amounts of discussion, this was still more instructional variety than might be expected in a "typical" lecture class. Bailey noted that their previous semester's section had been more interactive, so it is possible this was more due to the students' preferences than Bailey's intention. Here we have a tension between Bailey's belief that students should be interactive and that students should be free to make choices about how they want to learn. In keeping with Leatham (2006), it seems Bailey acted more on the latter belief, indicating they considered aligning to students' learning preferences more important than the incorporation of discussion while learning math.

Bailey seemed to intend to enact the interactive classroom described in the interviews. However, as the semester wore on, other factors seem to have gotten in the way. When behind their schedule, they pressed to finish by reducing the student work time to half days for labs. The instructor did not state a desire to reduce student work time, so it is possible they did not notice they were shifting how much time they spent on different activities. Nevertheless, this raises questions for further research on the influence of instructional pressures across a semester.

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OPERATIONALIZING ACCESS FOR STUDENTS: MAKING MEANING OF TASKS, CONTEXTS, AND LANGUAGE

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Untangling the relationships between teaching, learning, and content is complex. This study focuses on one aspect of these relationships, i.e., the at times challenging role that language can play in mathematical tasks, discussions, and student access. The authors analyze two video banks to identify and operationalize combinations of teacher and student actions that support student access to mathematical tasks and language.

Keywords: Classroom Discourse, Instructional Activities and Practices, Equity and Diversity

Mathematics education reforms and standards movements highlight the vital role that language and discussion plays in teaching and learning (National Council of Teachers of Mathematics (NCTM), 2010, 2014; National Governor's Association Center for Best Practices & Council of Chief School Officers (NGACBP & CCSO), 2010). Yet, the complex nature of mathematical tasks and discussions can become an obstacle for students' participation in mathematics classrooms (Aguirre & Bunch, 2012; Chval & Chavez, 2012). In order to support student access, teachers must develop practices that facilitate student access to mathematical tasks and language (Boaler & Staples, 2008; Chval, Pinnow, & Thomas, 2014; Staples, 2007).

This study comes from a decades long collaboration between K-12 schools, a nonprofit education organization focused on professional development and coaching for teachers of mathematics, and mathematics education faculty. In recent years, the partners collaborated to develop an app-based observation tool (Melhuish & Thanheiser, 2017) designed to provide teachers with formative assessment data about their implementation of observable mathematical teaching and learning practices. As part of this work, the authors are refining the tool to add or amplify student and teacher practices that support access to mathematical tasks and language. In alignment with this goal, this study was guided by the following research questions: (1) what observable teaching practices support students in making meaning of mathematical tasks and language, and (2) how might students engage in these making meaning practices?

Theoretical Orientation

Hawkins (2002) represents effective instruction by the relationships that exist between and among the vertices of the instructional triangle (see Figure 1a). In this triangle, the teacher builds a relationship with the student for purposes of understanding the student's relationship to the content, and then the teacher responds in ways that engage the student in thinking about and interacting with others and ideas that are intended to lead to a deeper understanding of the content. Lampert's (2003) expands on Hawkin's triangle by explaining, through examples from her own practice, how the problem space of teaching occurs along each of the arrows connecting the vertices of the triangle. Lampert adds a fourth arrow to the diagram to represent the relationship between the teacher and the arrow between students and content (see Figure 1b). Both Lampert (2003) and Cohen, Raudenbush, and Ball (2003) write explicitly about the need to consider how these triangular relationships function in the context of a teacher's work with individual students as well as in a classroom full of students. Cohen et al. (2003) make this explicit by adding a representation of multiple students interacting at the "student" vertex.

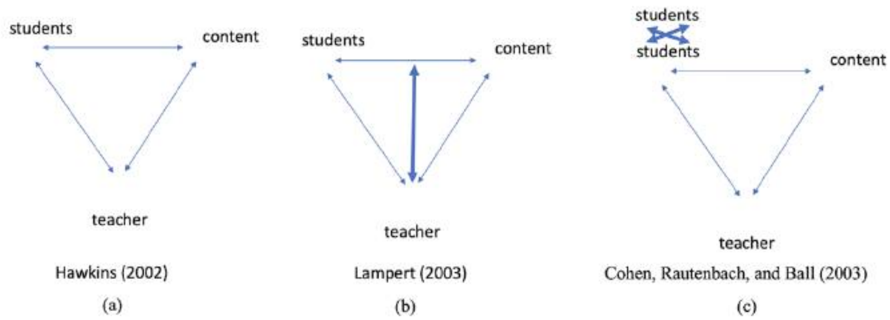

Figure 1. Three different representations of the student, teacher, content triangle.

The study team is working to delineate the complexities of these relationships in ways that make the actions both observable and learnable. By overlaying the triangle on the tool, one can see that the relationships are embedded (see Figure 2). Teachers initiate and enact catalytic teaching habits (CTH) and teaching routines (TR) to elicit student ideas and/or in response to what they understand students to be saying, doing, or understanding. These teacher actions are designed to prompt students to engage in habits of mind and interaction as a means of deepening their understanding of mathematical content. This study focuses on the project team's efforts to operationalize specific components of the tool (see highlighted text in Figure 2). What results is a smaller set of actions by and among teachers, students, and content that we hypothesize will support students in making meaning of tasks, contexts, and language.

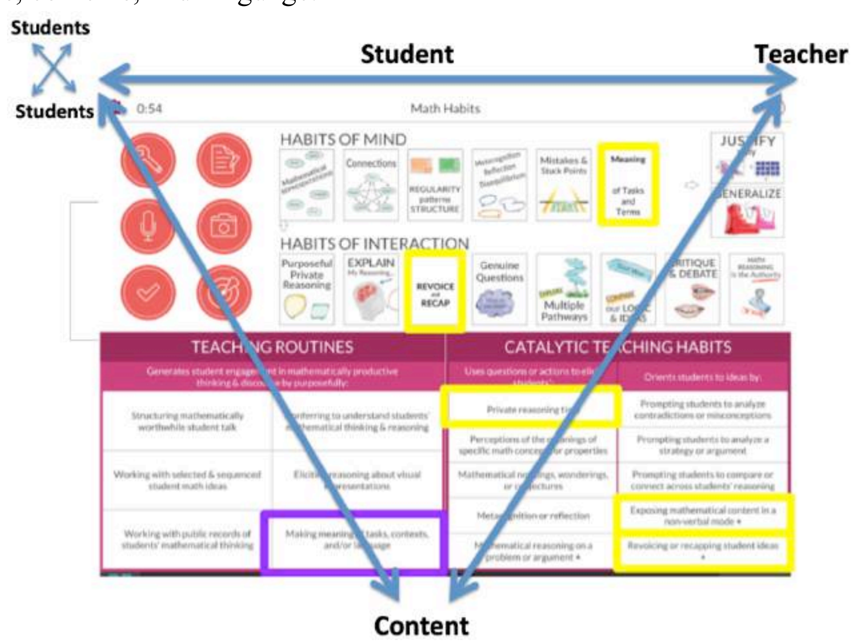

Figure 2. Teacher and student moves that support access to learning opportunities.

Methods

We drew upon two video banks of mathematics lessons spanning K-8 classrooms to identify teacher and student actions that supported students in making meaning of tasks and language. We initially developed a codebook that operationalized research-based practices (e.g., Ball, 1993; Jacobs &

Spangler, 2017; Nasir, & Cobb, 2006; Schoenfeld, 2011; Staples, 2007). Initial data analysis began with the development of a codebook with decision rules for the coding process and descriptions for each code. For example, we created distinct rules for coding TR stanzas and CTH stanzas with each stanza representing a discrete coded section of transcript data (Saldaña, 2013). TRs were defined as a collection of teacher-initiated moves that engaged students in prolonged mathematical discourse and/or productive thinking, while CTHs were defined as single teacher moves to elicit or focus student thinking. Additionally, student contributions were classified as a habit of mind (ways in which students engage with the mathematics) or habit of interaction (ways in which students engage with each other around the mathematics). These codes were then further refined through testing in classrooms and with video. Two researchers independently coded video transcripts, and then met to compare coding and resolve inconsistencies to reach interpretive convergence (Saldaña, 2013).

Findings

We present two excerpts that explicate the ways in which teachers might support students in making meaning of tasks and language. In the first transcript, the teacher implements a teaching routine to support students in making sense of a mathematical task before they start working on the task. The task states, “In a school gymnasium, 375 students have gathered for an assembly. The students are seated in 15 equal rows. How many students are seated in each row?”

Transcript	**Code(s)**
Teacher (reading the task aloud): In a school gymnasium, 375 students have gathered for an assembly. Okay, any questions there? Everybody knows what an assembly is? Although we haven't had many this year. We are going have one tomorrow. *Student*: We are? *Student*: Two *Teacher*: Two tomorrow. So we're going to be in an assembly tomorrow. Everybody gathers in the gym and we'll watch a show or something like that, presentation. …	**TR** (turns on): Making meaning of tasks, contexts, and/or language
Teacher: Okay, the students are seated in 15 equal rows. Any questions on rows? Rows are side to side. Columns are up and down. Anybody has been to a sporting event? *Student*: I have. *Teacher*: Usually your ticket says row so and so. So rows are like all the seats going across. Although we don't have rows in our assembly, some places do. How many students are seated in a row, in each row? Alright	**CTH**: Perceptions of the meanings of specific math concepts or properties **TR** (turns off): Making meaning of tasks, contexts, and/or language

Figure 3. Teacher practice analyzed with the meaning-making codes.

Here, the teacher uses the students’ shared experiences of going to assemblies in their school and possibly attending sporting events to make meaning of the context of the task. Within this longer TR, the teacher then uses a CTH to define the specific mathematical concepts of rows and columns. Noticeably, the students were not observed actively contributing to making meaning of the task or language.

Conversely, in this second transcript the teacher and students both engage in making meaning of the task and language. This excerpt occurs after the students had been working through several story problems. The teacher implements the TR of making meaning of tasks and language after he notices that the wording of a particular task was confusing to some students. This task states, “How many periods of time, each ⅓ of an hour long, does a 8-hour period of time represent?”

Transcript	Code(s)
Teacher: So the question says how many period of times right? How many periods of time. Does it ask you how much time? Okay, it's asking how many periods of time. So what is that? It's kind of confusing right there. Periods of time. What is that mean periods of time?	**TR** (turns on): Making meaning of tasks, contexts, and/or language
Student: Periods of time means how many sections of time there is in certain amount of time. So, there is two periods of time in an hour that's half an hour long.	**HOM:** Meaning of Tasks & Terms
Teacher: So, who heard Gabby said about period of time?	**CTH:** Revoice and recap student ideas
Student: Basically, she said that each…okay…in this equation periods of time would be 8.	**HOM:** Meaning of Tasks & Terms **HOI:** Revoice & recap
Teacher: Eight, eight 1/3. That's you said. You haven't started middle school yet so you don't have period right? So period of time in that would be like you got a math class the first period. Second period you go to English, third period, fourth period okay. So that's periods of time. So we don't want to know that math is 60 minutes. We just call that first period or second period, third period. So periods of time. They are not looking for a time here. They are looking for how many periods of time. Does that make sense? So we are not going to have time here. We are going to have a number period. Like, how many periods do you have in middle school? I have 6 periods. How long are those periods? 50 minutes each or 60 minutes each, okay? That's different.	
Student: That's what I thought. It says how many, not how long.	**HOI:** Compare our logic and ideas
Teacher: Great! So, how long would be the time, right? ...Is it helpful for you guys in this conversation to know what [student] just said. He is helping clarify what we are talking about. It says how long are they so we want time. How many is how many periods are there. Different thinking on that.	**CTH:** Revoice and recap student ideas **TR** (turns off): Making meaning of tasks, contexts, and/or language

Figure 4. Student teacher interaction analyzed with the meaning-making codes.

During this TR, the teacher first elicits students' understandings of the concept of periods of time rather than merely defining a period of time. This leads to a student engaging in the meaning making HOM. The teacher then extends this with a CTH by asking other students to revoice the original student's ideas. The teacher adds to this definition by introducing the real world context of class periods in middle school. Finally, the clarification is made that a period of time refers to how many not how long, which leads to a student spontaneously engaging in a HOI to compare their thinking with the thinking being discussed. Taken together, these excerpts show how a teacher might implement a TR to support students in making meaning of task and language, and how the engagement of students during this TR may vary based on teacher responses.

Discussion

This work comes from a focus on how to support student access to mathematical content and discussions. We build upon Cohen and colleagues' (2003), Hawkin's (2002), and Lampert's (2003) conceptualization of the instructional triangle in order to support this goal. Through multiple rounds of theoretical and empirical exploration, we have identified teacher and student actions that appear to support students in making meaning of tasks and language. By explicitly naming these teacher and student actions, we hope to bridge the theory to practice divide by supporting teachers in learning about and implementing these practices in their own classrooms.

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MATHEMATICS TEACHERS' EPISTEMIC DISPOSITIONS AND THEIR RELATIONSHIP WITH TEACHER INSTRUCTION AND STUDENT LEARNING: A SYSTEMATIC RESEARCH SYNTHESIS

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Some research suggests that teachers' beliefs and thoughts about the nature of mathematical knowledge and knowing (broadly termed epistemic dispositions) comprise an important factor that influences their practice. However, to date, there is no systematic review of the empirical literature on mathematics teachers' epistemic dispositions. The purpose of this systematic research synthesis was to assess the existing empirical literature in order to (a) describe mathematics teachers' epistemic dispositions, (b) to identify whether such dispositions correlate with teacher's use of constructivist teaching practices, and (c) correlate with student learning outcomes. A systematic assessment of 30 relevant studies suggest that teachers, on average, hold constructivist epistemic dispositions regarding mathematics. Few studies reported correlations between epistemic cognition indices and teacher practice or student outcomes.

Keywords: epistemic cognition, epistemic beliefs, epistemological beliefs, teacher beliefs

Cognitive processes involved in constructing and evaluating arguments—called *epistemic cognition*—has been well studied in the educational psychology literature. Epistemic cognition concerns itself with the thinking that people do about what they know and how they know it (Chinn, Rinehart, & Buckland, 2014; Sandoval, Greene, Bråten, 2016). For example, a learner engages in epistemic cognition when they explain "how they know" that a mathematical assertion is true or justified. A common object of investigation in epistemic cognition research is people's beliefs about the nature of mathematics, mathematical knowledge, and processes of knowing—sometimes termed *epistemic beliefs* (e.g., Cooney, 1985; Ernest, 1989; Muis, 2004; Thompson, 1984). Existing research syntheses suggest that students' epistemic beliefs support their motivation, selection of productive problem-solving strategies, and achievement outcomes in mathematics (e.g., Muis, 2004) and are involved in teachers' lesson planning, evaluation of student work, and instructional techniques (e.g., Maggioni & Parkinson, 2008). Yet, despite several decades of research consistently confirming that epistemic cognition plays a crucial role in facilitating teaching and learning in many disciplines, little to no research focuses on synthesizing findings regarding teachers' epistemic cognition in the domain of mathematics.

The purpose of this systematic review was therefore to synthesize the existing work on epistemic cognition in mathematics teaching in order to specify teachers' epistemic dispositions and identify whether epistemic dispositions are associated with instructional practice and student achievement. Specifically, we sought to answer three central questions: (a) *What are teachers' epistemic dispositions towards mathematics?* (b) *To what extent are epistemic dispositions associated with teacher instruction? (c) To what extent are epistemic dispositions associated with student learning?*

Theoretical Framework

Epistemic cognition can be defined as the thinking that people do about knowledge and knowing (Greene et al., 2016). A common focus in epistemic cognition research is on the beliefs that people hold about knowledge and knowing—or epistemic beliefs—which are studied both as both a domain-general and domain-specific construct. Three decades of research from various disciplines have yielded multiple domain-general models of epistemic cognition that broadly fall into three categories:

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developmental, multidimensional, and philosophically informed models (e.g., Sandoval et al., 2016). *Developmental models* of epistemic cognition investigate how people's views of knowledge progress through a series of levels over time (e.g., Kuhn, 1991; Moshman, 2015; Perry, 1970). *Multidimensional models* explore epistemic cognition as a set of multiple, relatively independent dimensions of beliefs (e.g., Hofer & Pintrich, 1997; Schommer, 1990). *Philosophically informed models* more broadly conceive of epistemic cognition as encompassing not only beliefs, but cognitive processes that take into account motivation, emotion, and practices that dynamically interact with beliefs in context (e.g., Chinn et al., 2014).

Theoretical Models of Epistemic Cognition Specific to Mathematics

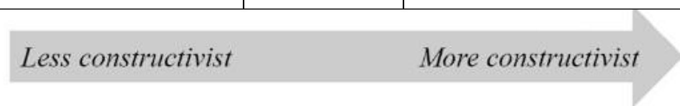
Much of the literature on mathematical epistemic cognition focuses on individuals' beliefs about mathematics and the nature and acquisition of mathematical knowledge (e.g., epistemic beliefs; Ernest, 1989; Thompson, 1984). The most commonly cited model of teachers' beliefs about mathematics is that of Ernest (1989). Ernests' model posits that teachers' beliefs about what mathematics is impacts their beliefs about how students learn, how teachers should teach, and subsequently impact their enacted model of how students learn (e.g., their teaching practices and how they utilize classroom resources like textbooks). Ernest proposes three categories of epistemological beliefs that increase in their level of sophistication: instrumentalist, Platonist, and problem-solving. Individuals that hold an *instrumentalist* perspective believe that mathematics is a set of unrelated rules and facts. Instrumentalists view mathematical statements as mere consequences of a set of arbitrary mathematical rules. Math teachers that adopt an instrumentalist perspective might view math statements as "just a collection of disconnected formulas" to be memorized and reproduced that are ultimately disconnected from our experience in the world. *Platonists* hold the view that mathematics is a unified body of objective mathematical knowledge and that mathematics is discovered. This can be illustrated by the teacher who believes that that mathematical knowledge is highly interconnected, builds upon itself, and exists in an unchanging almost transcendent world of objective mathematical knowledge. A Platonist teacher might believe that the best way to communicate mathematical knowledge to their students is to expose students to math knowledge in a logically consistent way. The *problem-solving* perspective holds that mathematics is dynamic, expanding, and is a human invention. This perspective stems from the view that mathematics is essentially a human invention constructed from subjective experience in the world. Teachers that hold a problem-solving perspective might believe that mathematical knowledge is a construction used to describe individual experience of the world, that math is a language to describe the world around us, and that the best way for students to learn mathematics is to co-construct knowledge through discussion and interaction in the classroom.

Additional mathematics-specific theoretical models of epistemic cognition are similar to Ernest's (see Table 1). Felbrich and colleagues (2012) and Daepepe and colleagues (2016) also posit categorizations of teachers' epistemic dispositions that lie on a continuum of less to more constructivist (scheme-related, formalism, and process-related). Two of Blömeke's three categories are similar, with the third category, the application perspective, being somewhat unique in that it represents a teacher with the perspective that math is a tool that can be applied to accomplish various tasks.

Teachers' mathematical beliefs are also predicted to shape their perceived role in the classroom, intended outcomes, and enacted instructional practices. For example, Ernest's (1989) model predicts that teachers' epistemic beliefs inform their espoused and enacted models of teaching and learning mathematics as well as their use of classroom materials. Briefly, this model posits that teachers' constructivist epistemic beliefs are expected to correspond with teaching practices that subsequently support student learning.

Table 1: Four Developmental Models of Teachers' Beliefs about Mathematics.

Ernest (1989)	Instrumentalist	Platonist	Problem Solving	
Felbrich (2012)	Math is Static Science		Math is a Dynamic Process	Application
Blömeke (2008)	Scheme-Related	Formalist	Process-Related	
Daepepe (2016)	Absolutist		Fallibilist	



As it stands, the epistemic cognition frameworks reviewed here posit that teachers generally progress from less to more constructivist mathematical beliefs and that these views on the nature of mathematics shape teachers' espoused models for teaching and learning and their enacted practices. However, it should be noted that such developmental models of epistemic cognition concentrate on epistemic beliefs and are limited in that they do not consider the multi-dimensionality or context-sensitivity of epistemic cognition as proposed in the educational psychology literature (e.g., Hofer & Pintrich, 1997; Chinn et al., 2014). As such, we operationalized epistemic cognition to include multidimensional and philosophically informed models and cast a wide net for retrieving relevant information about the topic, despite there being no math-specific theoretical models that are widely used that take these perspectives.

Method

Inclusion criteria. This review investigates empirical research on epistemic cognition of instructors within the domains of educational psychology and mathematics education. Studies were selected if they examined *teachers'* thinking about mathematical knowledge and knowing that could be identified as satisfying one or more of the components of the operational definition outlined above. These components included beliefs about the nature of knowledge in mathematics, justifications of knowledge in mathematics, sources of knowledge in mathematics including teachers' perspectives on the acquisition of mathematical certainties (i.e., proof). We included articles, dissertations, reports, and book chapters published in English.

Search procedures. Relevant empirical literature was identified by searching online databases, PsychINFO and ERIC, with the following search command: "(teach* OR instruct* OR profess* OR faculty) AND (epistem* OR proof* OR prove OR proving OR (math* NEAR/6 belief*)) AND (math*)," no additional restrictions were placed on the search. This search resulted in a total of 810 items, of which a total of 30 texts met the inclusion criteria and were selected for this review after multiple rounds of screening (screening procedures available upon request).

The 30 papers were then coded to capture characteristics of the theoretical framing, study setting, participants, internal validity, and external validity (Cooper, 2016; codebooks available upon request). Papers were broadly categorized by whether they addressed one or more of the three main research objectives to (a) describe teachers' epistemic cognition about mathematics, (b) identify whether there is a relationship between epistemic cognition and teaching practices, and/or (c) identify whether there is a relationship between epistemic cognition and student learning outcomes. Some texts were applicable to more than one category.

Preliminary analysis. For this preliminary analysis, we recorded the direction of effects—we noted whether each study found that teachers held constructivist dispositions or not, and whether these dispositions were positively or negatively correlated with reform-based instructional practices or with student learning. We then tallied up the direction of effects across these studies. The secondary reference section presents a list of the articles cited in the review.

Preliminary Results

The empirical literature identified in this synthesis tended to centralize epistemic beliefs as the object of investigation. Of the 30 items reviewed, all 35 of them appeared to be explicitly focused on assessing static epistemic beliefs using developmental or multidimensional conceptions of epistemic cognition (rather than philosophically informed models that consider the context-sensitive nature of epistemic cognition). Study samples ranged from pre-service K-12 teachers, and in-service teachers of preschool up through undergraduate and graduate instructors. Of 30 texts, 12 were qualitative, 16 were quantitative, and 2 were mixed methods.

RQ1: What are teachers' epistemic dispositions towards mathematics?

We assessed sample means of teachers' beliefs about mathematics from quantitative studies to judge whether their epistemic dispositions towards mathematics were constructivist or not. Of the 17 studies presenting relevant means, 13 of them (76%) revealed that teachers on average held constructivist beliefs about mathematics knowledge and knowing. Qualitative findings were consistent, but suggest that these dispositions were context dependent.

RQ 2: To what extent are epistemic dispositions are associated with teacher instruction?

To answer the second research question, we tallied the direction of effects of correlations between constructivist epistemic dispositions and teachers' reform-based teaching practices. Of the thirty papers, only four of them reported such correlations, all of which (100%) were positive and significant.

RQ3: To what extent are epistemic dispositions associated with student learning?

To answer the third research question, we tallied the direction of effects of correlations between constructivist epistemic dispositions and student learning outcomes. Of the thirty papers, only two studies presented correlations between epistemic dispositions and student learning. Both correlations were positive, but only one was significant.

Significance

We sought to assess the empirical literature on mathematics teachers' epistemic cognition to describe their epistemic dispositions and identify potential relationships with their practice and student learning outcomes. A systematic review of 30 journal articles, book chapters, reports, and dissertations begin to suggest that teachers lean towards constructivist perspectives regarding mathematical knowledge and knowing. A few studies show that these constructivist dispositions are correlated with reform-based teaching practices. However, due to the very small number of studies linking such beliefs with specific teaching or student learning outcomes, we recommend that more research is needed to establish such links.

We also found that all of the literature identified in this search conceived of and measured epistemic cognition as a unidimensional, static construct. Future work should also build from epistemic cognition models that centralize the role of context and frame epistemic cognition as a situated process rather than capturing only static beliefs.

We also note that issues of race, gender, and class were all but absent from this body of literature. Existing research suggests that teachers' seemingly innocuous beliefs about the nature of mathematical ability are not gender-neutral (Copur-Gencturk, Thacker, & Quinn, 2019). Such evidence suggests that implicit racial and gender biases may belie the seemingly harmless beliefs about the nature of mathematics and mathematical knowing. Future research should explore potential relationships.

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DISTRICT CERTIFIED CULTURALLY RESPONSIVE TEACHERS AND THEIR ELEMENTARY MATHEMATICS TEACHING PRACTICE: A MULTI-CASE STUDY

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This multi-case study examines how three elementary teachers, all certified by their school district in culturally responsive teaching (CRT) through professional development opportunities, implement mathematics teaching practices that support CRT. Furthermore, this study examines the CRT certification process in the focal district and the structures that support teachers in their enactment of CRT. Data were collected via interviews, questionnaires, observations, teacher journals, and other reportable data. The teachers' CRT practices in mathematics fell into four large quadrants aligning with the work of Hammond's (2015) Ready for Rigor framework. The findings expand upon the literature and provide us with a more informed understanding of what CRT looks like in elementary mathematics classrooms with teachers who have been certified in CRT from a district developed and applied certification model.

Culturally Relevant Education, Elementary School Education, Equity and Diversity, and Instructional Activities and Practices

Purpose & Theoretical Framework

The achievement of historically marginalized students has been an ongoing concern for stakeholders. Gay (2010) stated, "The achievement of students of color continues to be disproportionately low at all levels of education, and the need to change these dismal conditions is even more pressing" (p. xxvii). While addressing student achievement in mathematics education, Bonner (2014) emphasized how data from the National Center for Education Statistics (NCES, 2009), indicated that from 1990 to 2007 there was, "little progress in closing the persistent mathematics achievement gaps between certain groups" (p. 377). More recently, data from the National Assessment of Educational Progress (NAEP; 2017) reported that though there were not significant changes in racial and ethnic disparities from the previous years, the scores of White students remain higher on average than those of their Black and Latinx peers, indicating that while the achievement gap is smaller than it was in 1990, disparities are still prevalent. Although there are numerous reasons why such achievement gaps persist between students of color and their White counterparts in mathematics, including but not limited to, tracking/leveling, access to resources, institutional racism, and stereotype threat, research has shown that the achievement of historically marginalized youth is likely to increase when learners have positive mathematical identities (e.g., Borman & Overman, 2004) and cultural identities (e.g., Moll, Amanti, Neff, & González, 1992). Teachers play a significant role in forming student perceptions and fostering the development of such identities (Schoen, Cebulla, Finn, & Fi, 2003).

Thus, Gay (2010) stated, "Culturally responsive teaching is a means for unleashing the higher learning potentials of ethnically diverse students by simultaneously cultivating their academic and psychosocial abilities" (p. 21). Though the theoretical framework for CRT has informed the educational community for quite some time, scholars (e.g., Hammond, 2015) continue to discuss the challenges of operationalizing CRT in practice. Mathematics education in particular has produced limited research examining the teaching practices of culturally responsive teachers in pre-kindergarten through 12th grade (preK-12) (Thomas & Berry, 2019). Bonner (2014) offers three reasons for why this might be the case, including: 1) the majority of the works are specific to one population such as African American learners (e.g., Ladson-Billings, 1994); 2) there is a broad focus

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

on content and practice, making it non-mathematics-specific (e.g., Gay, 2010); and, 3) the works remain largely theoretical (e.g., Greer et al., 2009).

The purpose of this multi-case study is to examine CRT practices in elementary mathematics with three teachers who have been locally recognized and certified in CRT by their school district. Furthermore, the intent is to examine the CRT certification process in the focal district and the structures that supported the teachers in their enactment of CRT with historically underserved students, in their efforts to address the achievement gap. Though there is variability in how to define achievement gap (e.g., test scores, course enrollment patterns, cognitively demanding learning opportunities, etc.), this study is utilizing the terminology to emphasize the gap in standardized test scores, based upon how the focal district is operationalizing the construct and their desired outcome. This study is not about "gap gazing" rather language surrounding the achievement gap has been made explicit to describe the context within the district (Gutiérrez & Dixon-Román, 2011). This study is grounded in CRT both in theory and practice.

Methods

Research Questions

1. How do teachers become fully certified in CRT, and what structures support teachers in their enactment in the focal district?
2. How do three elementary teachers, who have been certified in CRT, implement mathematics teaching practices? How does the mathematics instruction support CRT?

Site and Sample

William County (pseudonym) is located in a southeastern state and it is known for its diverse student population inclusive of over 90 spoken languages. There are approximately 14,000 students enrolled in elementary schools in the district. In an effort to close the achievement gap in William County, district leaders created a CRT certification program for preK-12 teachers, administrators, and counselors. Since the program's enactment in 2016, 40 individuals have been certified across the district. The majority of the certified teachers are elementary, and to date, no secondary mathematics teachers have received certification.

I secured the consent of Skylar, Elizabeth, and Clay (pseudonyms). Skylar and Elizabeth are both Black women and Clay is a White man. Skylar and Elizabeth both teach at River Elementary (pseudonyms for school names) and Clay is at Ivy Elementary. The participants teach mathematics in different grades such that Skylar is pre-kindergarten, Elizabeth is third-grade, and Clay is fourth-grade. Their years of teaching experience range between five and 11 years. Additionally, their ages range from late-20s to late-40s. All of the teachers were part of the most recent cohort to receive certification. To incorporate multiple perspectives, I also draw upon the voices and the actions of district leaders.

Data Gathering Procedures

Mapping cultural reference points questionnaire. The first module of the district's CRT certification focuses on teachers recognizing their own cultural lenses. To inform my understanding of how the teachers are pushing themselves outside of their own cultural boundaries, I had them complete a questionnaire (Hammond, 2015) following the first interview.

Teacher journals. A key component of CRT examines how teachers react in the moment and how they use those experiences to inform their practices in the future, as seen in Bonner (2014). Following each observation, I asked the teachers to briefly reflect upon their own instruction using a journal protocol to gauge awareness.

Interviews. The first, semi-structured interview focused on the teachers' perceptions of their enactment of CRT in mathematics education and their experiences with the certification process. The second teacher interview focused on emerging themes surrounding the teachers' CRT practices and their perceptions of district structures.

Classroom observations. I observed the actions of the teachers inclusive of both how they gained knowledge of their students and how they used such knowledge to inform their teaching practices. I observed (and video recorded) the teachers (using protocol) for at least 10 full mathematics lessons (1.5 hours each; time of a unit). Data were collected using fieldnotes.

Other reportable events. Other reportable events in this study is multifaceted, including: informal, unstructured interviews and conversations that arose; various forms of artifacts such as assignments, student work, photographs, and the teachers' certification portfolios; and data points from involvement in community partnerships, division meetings, and school-wide meetings in public spaces.

Data Analysis

The information gathered from the questionnaire served as preliminary data that led toward the development of other methods; particularly, by helping to inform the observations. The journal reflections were re-read, and compared to the data for the corresponding classroom observations to analyze teacher awareness. Both interviews with the teachers were recorded and transcribed to allow for member checking. All fieldnotes were transferred into write-ups, and analytic memos were written intermittently to document emerging themes and inferences from data collection. Data sources were triangulated and re-read and re-coded to document emerging patterns and themes of CRT practices. I compared confirming and disconfirming evidence and continued to adjust my findings until all of the evidence was accounted for. Additionally, I engaged in peer debriefs and consulted with experts in the field about emerging themes and patterns, aligning with the theoretical frameworks to ensure trustworthiness.

Findings

District Professional Development & Structures

The focal district has enacted a CRT certification program inclusive of three professional development modules and three characteristics of focus for monthly cohort meetings. The three modules include: 1) recognizing your cultural lens, 2) engaging diverse learners, and 3) ensuring equitable parent participation. The characteristics of the CRT certification state that culturally responsive teachers: 1) acknowledge and incorporate the importance of cultural heritage of all students, while reflecting on their own personal cultural influences; 2) provide multi-cultural instruction and differentiation for relevance and rigor; and, 3) build positive learning partnerships with students and families. To receive certification, teachers have to demonstrate within their portfolios that they are working to enact CRT and present evidence of student achievement. The compilation of their work is also presented at a district-wide Equity Conference. Furthermore, district structures are in place with the purpose of continuing to influencing the teachers' learning and implementation of CRT. These structures are evidenced in the county's Equity Model that acts as a structural hierarchy of support (See Thomas (2020) for further discussion of the Equity Model and district-level support structures.).

CRT in Mathematics Classrooms

During my time working in the classroom with the teachers, it became evident that their conceptions of CRT were highly influenced by the work of Hammond (2015) and the *Ready for Rigor* framework. However, similarly to other works surround CRT, some of the components of each quadrant exemplified particular tenets that are more thoroughly captured in other literature (e.g., Gay, 2010; Ladson-Billings, 1994). The findings have been outlined in Figure 1. The quadrants are oriented to mathematically model the coordinate plane and the ways in which the teachers went about building CRT at the beginning of the school year. However, it is important to acknowledge that after the initial phase (of consecutive order), this is very much viewed as a continuous cycle without particular attention to order and the quadrants are not mutually exclusive. Furthermore, gaining knowledge has been placed at the center of this model.

Although mathematics teaching and learning are embedded throughout Quadrants 1-3, Quadrant 4 on information processing is most specific to mathematics education. All three teachers emphasized the importance of helping their students to first develop growth mindsets or mathematical mindsets (Boaler, 2016) when tackling challenging tasks that require cognitive demand and problem solving. Furthermore, in the domains for relevance, mathematical representations (Berry et al., 2017), and discourse, I examined how such standards-based practices (NCTM, 2000) were accompanied with CRT strategies to help students process information. The teachers viewed these as strategies for "stimulating brain growth to increase intellective capacity" (Hammond, 2015, p. 17). For sample excerpts and a thorough examination of the findings on CRT in practice as demonstrated in Figure 1 refer to Thomas (2020).

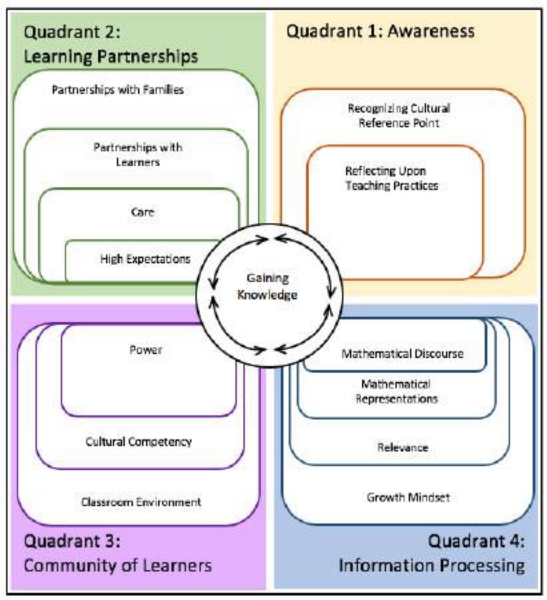

Figure 1: Findings of CRT in Elementary Mathematics Classrooms

Discussion & Significance

The study is significant because its findings expand upon the literature (e.g., Bonner, 2014; Thomas & Berry, 2019) and provide us with a more informed understanding of what CRT looks like in elementary mathematics classrooms. Furthermore, as indicated by Cai et al. (2017) there is a need in the field to link research and practice, and this study attempts to bridge that gap with CRT. This work is unique because the teachers are certified in CRT and supported by district structures in enacting CRT practices. This study continues to inform our understanding of how to operationalize CRT in mathematics, and it gives us insight into the professional development and support structures that may influence the implementation of such pedagogy.

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DOCUMENTING MATHEMATICAL LANGUAGE: DISTINCTION-MAKING AND REGISTER-FITTING

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This paper advances theory for language use in mathematics learning contexts. The theory arises from a cross-sectional longitudinal study of student language use in Grades 3 to 11, both in English first-language contexts and French Immersion contexts. We point to translanguaging and the language-as-resource metaphor to consider the goals educators have for documenting students' mathematical language. We problematize deficit-oriented assessment of mathematical language and differentiate between using language for distinction-making and for register-fitting. Both are important. We introduce a tool for documenting language repertoires to recognize students' language strategies, including distinction-making and register-fitting.

Keywords: Classroom Discourse, Communication, Cross-cultural Studies, Probability

The understanding of any mathematics is mediated by language. There is a reciprocal relationship between language and conceptualization: language repertoires are necessary to convey an idea, and the language used shapes the way people conceptualize. This reciprocity led us to to identify children's language repertoires in a range of contexts. Here we present a tool for documenting language repertoire and we explain how it helps us think about theory.

The research data from which we draw examples comprised a cross-sectional longitudinal study in English-medium and French Immersion instructional contexts in an Anglophone region in Canada. We worked with students in Grades 3, 6, and 9 in the first year, Grades 4, 7, and 10 in the second year, and Grades 5, 8, and 11 in the third year.

Second language acquisition literature has shown that people are generally good at picking up and using the language strategies employed by others in interaction (Ellis, 1997; Long, 1985, 1996, 2007; Swain, 2000, 2008)—in other words, people are naturals at learning language. We claim that first language acquisition works similarly—people pick up and use the language strategies used by others. To listen for students' language strategies (as opposed to students' ability to understand and then use the strategies we exhibit), we deemed it necessary to design mathematical tasks that do not have us saying or writing language strategies we foresaw students using. We found it possible to avoid the specialist language of prediction in our tasks by constructing a narrative context for our questions.

We introduced the game of Skunk with a narrative like this in each classroom: "I was picking strawberries in the forest. When my basket was quite full, a skunk wandered into the berry patch. I ran away so the skunk would not spray me. And I lost the berries in my basket." Participants had a pile of beans (representing the berries), a cup (the basket), and a bowl (home). When the researcher rolled the die and called out the number, participants put that number of berries in their basket. A 6 represented the skunk. When it was rolled, everyone would lose the berries in their baskets. If they had "gone home" (dumping their beans into their bowl) before the appearance of the skunk, their berries were safe. We played seven rounds.

The day after playing the game in the classroom, we interviewed groups of students and played again but, instead of the die, we used six cards bearing the numbers one to six (the skunk). The interviewer would not replace the cards into the deck until the deck was completely played out, at which time it would be reshuffled. Thus the participants experienced the difference between independent and mutually exclusive events in probabilistic situations. During the card game, the

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interviewer would ask the participants to say why they made their choices about when to "go home." After the game, the interviewer would ask participants about specific things they had said earlier, asking for clarification on meaning. We had students work in pairs to encourage them to dialogue about their choices (e.g., "should we go home or stay in the berry patch?").

After transcribing the interviews, we collated students' language strategies for identifying certainty or uncertainty, organizing them into charts—one chart for each interview. The chart identifies each strategy and when it was used by referring to the turn number. A turn begins and ends with a change in who is speaking. The structure of the chart emphasizes who is the first person to use each language strategy and who uses it after that. We show one such chart here (Table 1), from an interview with four English-medium Grade 6 students. The strategies are presented in the order that they are used. For example, the first language strategy for uncertainty was Bal's use of 'probably' in turn 21. Bal used 'probably' again in turns 40, and 149. (Names are pseudonyms. "Int." refers to the Interviewer.)

Table 1: Grade 6 English group – expressions of certainty and uncertainty

Certainty	Certainty (continued)	Uncertainty
Simple Assertion		'probably'
1st user: Col (23, 44, 60)	…	1st user: Bal (21, 40, 149)
2nd user: Int. (49)		'it depends'
3rd user: Bal (62, 122,	'got to' / 'gotta'	1st user: Daz (22)
129)	1st user: Adi (260, 265)	2nd user: Adi (33)
'have to'	2nd user: Int. (262, 289)	'usually'
1st user: Int. (49, 81, 255,	3rd user: Daz (283)	1st user: Bal (25)
262, 279, 295, 297, 299)	4th user: Adi (284)	'you never know'
'sure'	'you know it's got to'	1st user: Daz (35)
1st user: Researcher (121)	1st user: Col (261)	2nd user: Bal (238)
'you know'	'has to'	3rd user: Int. (239)
1st user: Daz (197)	1st user: Adi (265)	'could'
2nd user: Int. (361)	'a rule'	1st user: Daz (35)
'probably'	1st user: Col (290)	2nd user: Int. (303, 305, 311,
1st user: Daz (197)	2nd user: Daz (291)	313, 317)
'I know'	3rd user: Int. (293, 295, 357)	'I/you think'
1st user: Col (226)	'can't'	1st user: Int. (30, 66, 69, 126,
'need to'	1st user: Int. (327, 329, 336,	128, 141)
1st user: Col (228, 294)	352, 361, 363)	2nd user: Col (41, 106)
2nd user: Daz (282)	'not allowed'	3rd user: Bal (105, 125)
'you know you're going to'	1st user: Bal (337)	'not sure'
1st user: Bal (238)	2nd user: Int. (338, 354, 361)	1st user: Int. (45, 47)
'you never know'	'impossible'	'I don't know'
1st user: Bal (238)	1st user: Col (340)	1st user: Daz (227)
…	2nd user: Int. (341, 344, 354,	'a chance'
	361)	1st user: Bal (304)
		2nd user: Int. (305)

Problematizing deficit assessments of mathematical language

We acknowledge that we found it hard to avoid deficit assessment even though it was our expressed intention to avoid it. For example, we expected students to use modal verbs to make distinctions in degrees of certainty—as we had found in earlier work (e.g., Wagner, Dicks & Kristmanson, 2015)—

ranging from negative root modality (e.g., 'it is not six') to positive root modality (e.g., 'it is six') and different levels of modulation in between.

However, we noticed that Adi, Bal, Col and Daz here did not use some common modal verbs. For example, after giving the students a chance to use the modal verb 'can't' on their own, the interviewer used it multiple times (starting in turn 327), even explicitly asking the students what it meant. But the students did not use it. It was clear that they understood it, because when asked about it they made distinctions between things that are 'not allowed' (Bal in turn 337) and things that are 'impossible' (Col in turn 340). If we were to rate their language use on a checklist, how would we assess their use of the word 'can't'? We can say they did not use the word. But it would be inappropriate to say they do not have the word in their repertoires. The fact that they did not use it does not mean they cannot use it. To illustrate further, we consider the word 'impossible'. If we had ended the interview a little sooner, before turn 340, we would not have known that it was in Col's repertoire. We can say that at least Bal and Col understand 'can't' because they responded well to questions using the word. They even demonstrated sophisticated understanding by making distinctions between impossibility due to logic and due to authority. Rowland (2000) has documented the language of this distinction.

Further, we see that Bal and Col responded with understanding to the word 'can't'. What can be said about the others in the group? We argue that it would be inappropriate to say that Adi and Daz did not understand 'can't'. While they did not use the word nor respond directly to the word being used, there was no reason for them to speak about it because Bal and Col already did so. We entered the research project with a principled decision to avoid deficit assessment. We found it difficult to avoid deficit approaches in our read of the data. Ultimately, our data gave us evidence to reject deficit assessments of language.

Translanguaging for distinction-making and register-fitting

Distinction is a word that appears multiple times in our theorizing above. Our stance of seeing language as resource (Martínez, 2017; Moschkovich, 2007, 2013; Planas & Setati-Phakeng, 2014; Ruiz, 1984) led us to appreciate the language work done by the students in our data across the ages. This led us to ask what goals we would promote for mathematics educators in relation to mathematical language. We settled on these three: (1) understanding mathematical concepts, (2) ability to use language to make mathematical distinctions, and (3) ability to sound knowledgeable (fitting the genre, the grammar, the lexicon)

We assume that all mathematics educators are interested in supporting students to develop understanding of mathematical concepts. With an interest in mathematical language, it is common to say that students should also be able to communicate their mathematics. This goal compels us to ask what it means to communicate mathematics. We differentiate between successful communication of an idea, which we call *distinction-making*, and using 'correct' language, which we call *register-fitting*. As shown by the students Adi, Bal, Col and Daz, and by the students in every other interview in our research from Grades 3 to 11, it is possible to communicate conceptual distinctions without using conventional language.

We claim that communicating mathematics successfully means being able to make mathematical distinctions in a way that others understand. We use the new theory of *translanguaging* here, introduced by García and Wei (2014), to challenge the neat boundaries people often imagine around languages. We aim to appreciate the range of language strategies people use, no matter how they cross lines of recognized languages (e.g., English or French), and variations within languages (mathematics registers, dialects, etc.).

For example, in relation to prediction, it is important to have language strategies that distinguish between certainty and uncertainty. This can be done with adverbs like 'certainly' and 'possibly.' Bal used adverbs—'probably' thrice, and 'usually' once. The distinction can also be made with adjectives, such as 'impossible' which was used by Col first in this interview. The distinction can be made with modal verbs as noted above and with a distinction between knowing and thinking, also noted above.

Further sophistication is possible with distinctions between levels of certainty (modulated certainty)—for example, Bal's adverbs 'probably' versus 'usually.' Other than that distinction we did not find modulated certainty in this group. For further examples we can point to data from a group of three Grade 9 French Immersion students. (We do not show the table due to space restrictions.) To identify a higher probability, Enk said 'plus de chance' (more chance) early in the interview (turn 14). Much later, Gyl said 'une bonne chance' (a good chance) (turn 127). To identify lower probability, Gyl said 'un sur trois chance' (one in three chance) early in the interview (turn 20) and 'une petite chance' (a small chance) later (turn 90).

In addition to making distinctions clearly, we have seen that mathematics teachers value students using 'proper' words 'properly'. We use the word *proper* in quotes because it can only refer to loosely defined expectations for standard lexicon ('proper' words) and standard grammar (words used 'properly'). This means that educators want to induct students into a community of mathematicians, who, presumably, use the words and grammar of the mathematics register—the specialized methods of communication used amongst the mathematically literate (Barwell, 2007; Halliday, 1974; Pimm, 2007). We see this *register-fitting* as different from distinction-making. However, there is a connection: as people move to using more conventional language it becomes easier for others to understand their meaning. For example, if we use language that you know and use, you will more likely understand us. This is the unitary force of language in Bakhtin's (1981) metaphor—the centripetal force (Barwell, 2014).

French Immersion contexts are especially interesting to us in terms of register-fitting because there are two explicit goals: learning mathematics and learning French. We suggest similar goals in first-language medium classrooms too: learning mathematics and learning to communicate mathematics, which would include both distinction-making and register-fitting.

Two of the four strategies used by students to modulate certainty in the French group used improper French: 'un sur trois chance' (turn 20), and 'une petite chance' (turn 90). We could criticize the students for using improper French (a deficit assessment). Alternatively, we could honor them for using the language strategies available to them to make the distinctions they intended to make. Strong language speakers are inventive with language to communicate their ideas. There are plenty of other examples of 'improper' language being used powerfully across the corpus of data in both French and English interviews.

But deficit views of 'proper' language are even more complicated. We consider the Francophone interviewer, who said 'c'est peut-être pas' ('it might not be') in turn 98. This too was successful communication in non-standard French (the speaker dropped the 'ne' in front of the verb 'est'). However, in oral French it would be weird to say the standard French 'ce n'est peut-être pas.' The interviewer was using standard oral French. We believe that anyone who has transcribed natural language data will have realized that people do not often speak in proper sentences.

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TEACHING AND CLASSROOM PRACTICE:

POSTER PRESENTATIONS

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USES OF THE FREE MATH TEXTBOOK IN ELEMENTARY SCHOOLS IN MEXICO

USOS DEL LIBRO DE TEXTO GRATUITO DE MATEMÁTICAS EN LAS ESCUELAS PRIMARIAS DE MÉXICO

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Keywords: Curriculum analysis, Elementary School Education, Learning tools

Our research is a case study, qualitative, exploratory and descriptive, in which we analyze the free textbook for mathematics and consider, among other variables: the structure, the didactic situations, the expected learning, etc. The participants at the beginning of the research were seven teachers from a Primary School located in the municipality of Toluca: Two from first grade and one from second to sixth, with their students respectively. In the profile of the teachers we consider the following features: Gender, age, degree attended, professional training, institution that accredits their studies and professional experience

The information was obtained by applying a questionnaire to the teachers and another to their students. In the following we show an excerpt of them.

Questionnaire for the students	**Questionnaires for teachers**
ON THE POSSESSION OF THE BOOK When did you enter school this year, did you get a math book? ABOUT KNOWING THE BOOK AND ITS PARTS Do you like your math book? Why? What is the most interesting thing about your book or about math? ON THE USE OF THE MATH BOOK Do you use the math book in your math classes? (never, sometimes, always) In your math classes do you use the book as a whole class, at the beginning, at the end or homework? Does someone in your house help you solve the problems in the math book? ON THE WORK SEQUENCE WITH THE BOOK When you are presented with a challenge from the book: What do you do first? Next? In the end? Do you discuss with your classmates different strategies to solve it? Do you agree to solve it? Do you participate in group discussions? Do you communicate what you do? Do you always understand what others are saying?	The following questions are related to the free math textbook. How important is it to you in conducting math class? In relation to the didactic situations raised are: In relation to the didactic situations raised consider: When in class do you use it In relation to the mathematical concepts to build the didactic situations are: Do you use other books in your math class? What difficulties do you and the student face when using the new textbook: Didactic situations of the free textbook difficult to deal with. Do you consider that the textbook has defects? Which ones? Do you believe that parents can support students to carry out the activities in the textbook? Why? Are your students capable of producing their own ideas and procedures? YES NO Why? What is more important, the exercise or construction of the concepts? Why? Do students' ideas bring any new knowledge to you? Why? What do you consider before engaging your students in a new challenge? Have you solved all the challenges? Why? Are the challenges adjusted to the context and conditions of your children? But because? If they don't fit what do you do? Provide an example

USOS DEL LIBRO DE TEXTO GRATUITO DE MATEMÁTICAS EN LAS ESCUELAS PRIMARIAS DE MÉXICO

USES OF THE FREE MATH TEXTBOOK IN ELEMENTARY SCHOOLS IN MEXICO

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Palabras clave: Análisis Curricular, Educación Primaria, Herramientas de enseñanza

Nuestra investigación, es un estudio de caso, de corte cualitativo, exploratoria y descriptive, dónde analizamos el libro de texto gratuito de matemáticas y consideramos, entre otras variables: la estructura, las situaciones didácticas, los aprendizajes esperados, etc. Los participantes al inicio de la investigación fueron siete profesores de una Escuela Primaria ubicada en el municipio de Toluca: Dos de primer grado y uno de segundo a sexto, con sus estudiantes respectivamente. En el perfil de los profesores consideramos los siguientes rasgos: Género, edad, grado que atiende, formación profesional, institución que acredita sus estudios y la experiencia profesional

La información la obtuvimos mediante la aplicación de un cuestionario a los profesores y otro a sus estudiantes. En lo siguiente mostramos un extracto de los mismos.

Cuestionario para los alumnos	Cuestionarios para los profesores
SOBRE LA POSESIÓN DEL LIBRO ¿Cuándo entraste a la escuela este año te entregaron un libro de matemáticas? **SOBRE EL CONOCIMIENTO DEL LIBRO Y SUS PARTES** ¿Te gusta tu libro de matemáticas? ¿Por qué? ¿Qué es lo más interesante de tu libro o de matemáticas? **SOBRE EL USO DEL LIBRO DE MATEMÁTICAS** ¿En tus clases de matemáticas utilizas el libro de matemáticas? (nunca, a veces, siempre) ¿En tus clases de matemáticas utilizas el libro toda la clase, al principio, al final o tarea? ¿Te ayuda alguien en tu casa a resolver los problemas del libro de matemáticas? **SOBRE LA SECUENCIA DE TRABAJO CON EL LIBRO** Cuando te presenta un desafío del libro ¿Qué haces primero?, ¿después? ¿Al final? ¿Comentas con los compañeros distintas estrategias para resolverlo? ¿Se ponen de acuerdo para resolverlo? ¿Participas en las discusiones grupales? ¿Comunicas lo que haces? ¿Comprendes siempre lo que los demás exponen?	Las siguientes preguntas están relacionadas con el **libro de texto gratuito de matemáticas**. ¿Qué tan importante es para Usted en la conducción de la clase de matemáticas? En relación a las situaciones didácticas planteadas son: En relación a las situaciones didácticas planteadas considera: En qué momentos de la clase lo utiliza: 5. En relación a los conceptos matemáticos a construir las situaciones didácticas son: ¿Utiliza otros libros en su clase de matemáticas?¿Qué dificultades enfrenta Usted y el alumno al utilizar el nuevo libro de texto: Situaciones didácticas del libro de texto gratuito difíciles de abordar. • ¿Consideras que el libro de texto tiene defectos? • ¿Cree que los padres de familia pueden apoyar a los alumnos a realizar las actividades del libro de texto? ¿Por qué? ¿Tus alumnos son capaces de producir ideas y procedimientos propios? ¿Por qué? ¿Qué es más importante, la ejercitación o construcción de los conceptos? ¿Por qué? ¿Las ideas de los estudiantes aportan algún conocimiento nuevo para ti?¿Por qué? ¿Qué consideras antes de involucrar a tus estudiantes en un nuevo desafío? ¿Haz resuelto todos los desafíos?¿Por qué? ¿Los desafíos se ajustan al contexto y condiciones de tus niños?¿Por qué? Si no se ajustan ¿Qué haces? Proporciona un ejemplo

Agradecimiento

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WHAT MATTERS TO MIDDLE SCHOOL MATHEMATICS TEACHERS: RESULTS FROM A THREE-YEAR PROFESSIONAL DEVELOPMENT PROGRAM

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This poster session describes the results of a ranking activity from a three-year professional development (PD) program for middle level mathematics teachers. Not surprisingly, teachers in their first year of the PD program valued observing other teachers use questioning techniques with students and classroom observations more than any other of the seminar style activities. Some subtle shifts such as planning with other teachers and activities involving sorting student work were valued higher for year three teachers rather than year one and year two teachers. The results are consistent with the types of activities teachers desire in PD programs (Matherson & Windle, 2017).

The conceptual basis for the three-year professional development program involved two core components. The first component involves knowledge of content and students (Ball, Thames, & Phelps, 2008). The focus of the PD is exploring diverse student approaches to solving middle grades mathematics problems and serves the dual purpose of increasing middle school teachers' content knowledge and their understanding of students' thinking within specific mathematics content areas and topics. The second component of the professional development is examination of potential learning trajectories with the goal of planning and predicting student responses and questions that will promote productive mathematical discourse (Sztajn, Confrey, Wilson, & Edgington, 2012).

This study answers the following research questions:

1. What activities in professional development specifically focused on middle school mathematics do teachers value the most?
2. Are there shifts in what teachers value from year one participation to year three participation?

Data sources were part of a survey that was given to teachers at the end of each year of the three-year professional development program. The structure of the program involved three days of seminar style sessions in which teachers viewed videos of students solving problems, sorting student work, readings, discussions, planning problems to pose, and videos of teachers posing problems. Three of the four classroom embedded sessions involved sorting student work and then watching the host teacher orchestrate the sharing and questioning students. The fourth classroom embedded day involved planning problems and interviewing individual students.

Fifty-two teachers were asked to rank 10 professional development activities from 1 (most valuable) to 10 (least valuable). The 10 activities are: watching videos of students solving problems, predicting student solution strategies, observing in classrooms, interviewing individual students, sorting student work, planning with other teachers, learning about problem types, videos of classroom instruction, questioning techniques, and readings. The highest ranked activities of year one and year two teachers were questioning techniques and classroom observations. However, year 3 teachers ranked predicting student solution strategies and planning with other teachers highest, suggesting shifts in priorities over the three years of professional development.

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TRANSLANGUAGING MOVES IN ELEMENTARY MATHEMATICS CLASSROOM NUMBER TALKS: UNDERSTANDING LINGUISTIC REPERTOIRES

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Keywords: Computational Thinking; Instructional Activities and Practices; Classroom Discourse

Mathematics education research advocates for practices that celebrate all students' mathematical reasoning and ways of knowing (Turner & Drake, 2016). For multilingual teachers, translanguaging theory opens up how mathematical knowledge is shared and understood in multilingual mathematics classrooms. Translanguaging, a dynamic view of language acquisition, posits that instead of viewing multilinguals as having separate language registers, we instead view their access to multiple linguistic repertoires in specific contexts (García & Kleifgen, 2010). Number Talks are a structured series of computation problems selected and sequenced and presented to elementary students in order to build mental computation and relational thinking (Bray & Maldonado, 2018; Parrish, 2010; Humphreys & Parker, 2015). This study investigated the translanguaging moves that revealed the mathematical thinking of multilingual elementary students while engaging in number talks.

We conceptualize a translanguaging stance in the mathematics classroom as the deliberate choice by teachers to create a space in which children's mathematical thinking and language practices are positioned as powerful resources during mathematics instruction (García, Ibarra Johnson & Seltzer, 2017). Further, we posit that Number Talks are a beneficial activity for mathematics instruction in multilingual contexts due to the open-ended nature of the activity. Students may share their thinking, unencumbered by language separation requests, as they share their mental computation, all while the teacher facilitates a discussion of ongoing analysis of the mathematical relationships that are revealed in students' strategies (Bray & Maldonado, 2018).

Six number talks that occurred in a 3rd grade two-way dual language classroom at a Southwest school were analyzed for this study. Of 23 students, 22 identified as Latinx (with families from Mexico, El Salvador, Honduras and Puerto Rico), and one student identified as both Black and White. Number Talks were video recorded and transcribed, and focused on multidigit subtraction, multiplication and division, and unit fraction multiplication. We used multimodal analysis (Jewitt, 2009) because it is particularly helpful to identify how bilingual learners use semiotic resources other than spoken language to participate (Domínguez, 2005).

Two themes emerged from analysis of the Number Talks: translanguaging teacher moves to facilitate the mathematical flow of ideas, and ongoing community mathematical knowledge building moves. Translanguaging teacher moves emerged at various points in the Number Talk, both to invite students into conversation, to scribe students' oral strategies, and to ask for further reflection and discussion. Community mathematical knowledge building occurred between students, and often began with a mathematical question or puzzlement. Our study highlights the need for opening up the ways in which multilingual students draw upon their linguistic repertoires to build community and individual mathematical knowledge.

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USING TRANSLANGUAGING TO RE-EXAMINE AND DECONSTRUCT EARLIER FINDINGS ABOUT BILINGUAL MATHEMATICS EDUCATION.

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In my 2005 doctoral thesis, I addressed the question of what is meant by individuals' growing mathematical understanding within a particular bilingual situation wherein those individuals use words with no direct or precise translation between English, a dominant Western language, and Tongan, an indigenous Pacific vernacular. As a result of that study, a defined number of structural categories of "language switching" were then identified which in turn provided a useful way of describing the pattern in which the studied bilingual individuals alternated between the two aforesaid languages. Since then, one remaining challenge is how exactly this type of *languaging* can be formalized and practiced.

This study employs the socio-linguistic theory of "translanguaging" as an alternative framework for analysing bilingual teachers' language acts and as a new lens that allows me as researcher to re-examine and deconstruct my earlier categorisations and findings about the role of the two aforesaid languages in bilingual individuals' mathematical discussions and teaching. This alternative view of "bilingualism" recognizes that bilingual individuals may have only one *language system*, not two, and that effective instruction would involve finding ways to help these individuals draw on all their linguistic resources, their full repertoire, when learning academic content in a new language.

This study also employs and thus continues to demonstrate the power of Pirie and Kieren's Theory (Pirie & Kieren, 1994), which, along with its associated diagrammatical model, were presented and discussed previously at a number of PME meetings. Of particular interest and a focus in this new study is how translanguaging as a process, which is said to be accessible through a bilingual's prior knowledge, may directly be related to Pirie-Kieren's innermost *Primitive Knowing* layer – the starting point or "base knowledge" for the growth of mathematical understanding. This is an important link in using translanguaging as an analytical lens in this study as well as in re-examining the results of my earlier research work.

Video recordings of bilingual mathematics teachers in two high schools in Tonga, which is a small island country in the South Pacific, were made in 2019. Several episodes of these bilingual teachers' classroom language use are included in this poster presentation to illustrate the results and findings of the study. The preliminary findings not only corroborate many aspects of my earlier research work but also offer a different perspective. For while the new study supports the view that translanguaging is a normal yet personal practice in this type of bilingual classrooms, it also recognizes that effective instruction involves identifying clues that can help students draw not only on their entire linguistic resources and repertoire, but also on their *primitive knowing*.

However, there appears to be no one-size-fits-all remedy to translanguaging. It varies among bilingual individuals in how it is deployed to facilitate learning or mathematical understanding. It also puts a damper on my desire to continue searching for an effective systematic pedagogical approach toward using two languages in a bilingual classroom environment. This new realization comes from viewing the language system that underlies what bilingual individuals actually speak as personal and unique, even when there is a commonly shared cultural identity (Otheguy, Garcia, and Reid, 2015), as is the case in Tonga. If bilingual individuals, like Tongans, are allowed the flexibility of translanguaging and thus access to mathematical terms and images in either language, such a dynamical practice would allow bilingual teachers and students alike to creatively interact and co-construct mathematical meanings (Manu, 2005).

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Using translanguaging to re-examine and deconstruct earlier findings about bilingual mathematics education.

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ALTERNATIVES TO MATHEMATICS CLASS DISCUSSIONS: INCLUSIVE WAYS TO CONNECT AND EXPAND CHILDREN'S IDEAS

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Keywords: Classroom Discourse, Inclusive Education, Equity and Diversity

Teachers' efforts to orchestrate mathematics class discussions frequently result in the monopolization of the discussion by a few students (Webb et al., 2019). The purpose of this paper is to explore how a teacher learned to transform mathematics whole class discussions toward being more inclusive of multiple students. I report on a study where students' perspectives informed a teacher-researcher collaboration to develop these inclusive alternatives in a third-grade Spanish immersion classroom. I ask: How did a teacher make sense of students' perspectives on mathematics class discussions to develop alternative ways to socialize ideas? I argue that developing inclusive ways to connect and expand children's ideas involves learning about how students navigate class discussions and challenging unquestioned teaching practices.

Methodology

I followed a participatory research approach (Fals-Borda, 1987), where I collaborated with the teacher to make sense of students' perspectives on mathematics class discussions, and to develop inclusive ways to socialize students' mathematical ideas. Data sources included audio-recordings of focus groups with students, video-recordings of mathematics lessons, and audio-recordings of interviews and collaborative data analysis sessions with the teacher. I used a social semiotics analytical framework, which acknowledges that students' multiple ways of developing and contributing ideas is part of the multimodal nature of mathematical activity (O'Halloran, 2015). The teacher and I collaborated analyzing data to interpret relationships between the students' and the teacher's perspectives. These interpretations informed how we co-developed teaching strategies to transform class discussions. Interpretations also informed how the teacher flexibly and responsively approached students' interactions in her class.

Summary of Findings

Initially, the teacher considered her role in mathematics class discussions to involve distributing uniformly student talking time. In contrast, students valued opportunities to influence others' thinking, and they experienced mathematics class discussions as overcompetitive. For students, influencing others' thinking involved more than contributing ideas through spoken utterances.

The teacher flexibly and responsively drew on two teaching strategies to promote inclusive ways to socialize mathematical ideas. The group ambassador's strategy offered opportunities to develop initial ideas in a small group and then influence the work of a different group. Ideas circulated both within and across groups, thus serving one purpose typically reserved to class discussions. In the spotlighting gestures strategy, whenever the teacher noticed gestures that communicated relevant aspects of a mathematical idea, she directed the class attention to such gestures. Students observed the gestures, and adapted and incorporated them in their own mathematical activity. Spotlighting gestures helped children communicate ideas and make sense of others' ideas without the linguistic demands that class discussions frequently impose.

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LEARNING THROUGH ACTIVITY (LTA) IN SUPERIOR EDUCATION: THE CASE OF THE HEINE-BOREL THEOREM

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Keywords: Classroom Discourse, University Mathematics, Learning Trajectories, Heine-Borel Theorem.

The research program called Learning Through Activity (LTA) had its origins in Simon (1995). This research program has the objective of creating an integrated theory of conceptual learning, and the design of instructional mathematical activities through the use of Hypothetical Learning Trajectories (HLTs). Thus, this HLTs play a crucial role in the LTA program. This theoretical framework implements the constructivist theory of education, developed by Piaget (1970), as well as its applications in mathematics pedagogy by von Glasersfeld (1995).

In the last decades, the community of mathematics educators has become very interested in continuing to expand this theoretical framework, in an effort to incorporate social, cultural and psychological theories in the teaching-learning processes involved in a classroom context.

During this years, according to Stylianides & Stylianides (2009, 2018), research about teaching through this kind of activities has shown very promising results in the basic levels of education, and this being the case, HLTs as part of a mathematical teaching cycle have become one of the main referents about how to design activities to guide students learning according to the constructivist theory of education, for example (Leikin & Dinur, 2003; Simon, Kara, Placa & Avitzur, 2018; Stylianides & Stylianides, 2018). However, although HLTs seem to be a very promising way to approach mathematics pedagogy, there has been little research about their implementation in the higher levels of mathematical education (Simon et al, 2018).

In this Poster, we present a synthesis of a didactical proposal, based on the LTA program's approach, that includes a main HLT with the goal of guiding the student towards a proof of the Heine-Borel theorem, and other auxiliary HLTs, that will provide the student with the necessary tools to prove the theorem. We do this with the intent to investigate the efficiency of the LTA framework in the higher levels of mathematical education.

The proposed trajectories begin with the definition of open covers, and continue through some supporting theorems, such as the theorem that guarantees the compactness of closed subsets of compact sets, and the theorem that guarantees the compactness of any K-cell, before culminating in the proof of the Heine-Borel theorem, which states that any subset A, of the Euclidean space Rn, is compact if and only if A is closed and bounded.

The study is addressed to university students beginning their studies in the topology of Rn in the Faculty of Sciences of the "Universidad Nacional Autónoma de México" (UNAM), the most important public University of México. We based the mathematical part in Rudin (1986), and other classical texts such as Bartle (2011).

Lastly, this poster will also include a synthesis of a metric based on Toulmin's argumentative model that will be used to evaluate the knowledge acquired by the students through the learning trajectories. In the last years, this model has been used by several researchers in mathematical education to analyze non formal arguments, for example (Pedemonte, 2007; Simpson, 2015; Zazkis, Weber & Mejia, 2016; Herrera, Rivera & Aguirre, 2019).

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MATHEMATICS TEACHER EDUCATORS AS CULTURALLY RESPONSIVE PEDAGOGUES: INTRODUCING A FRAMEWORK FOR "GROWING CRP"

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Keywords: Culturally Relevant Pedagogy; Mathematics Teacher Educators; CRP framework

Overview and Research Question

Research on culturally responsive pedagogy (CRP) in mathematics education and teacher education has focused on either preparing new teachers for diverse classrooms (e.g., Aguirre et al., 2012; Ukpokodu, 2011) or on taking a critical stance toward disrupting dominant/oppressive paradigms in mathematics classrooms (e.g., Gutiérrez, 2017; Willey & Drake, 2013). The tendency has been to focus on developing the CRP of prospective/practicing teachers (PTs) and/or the mathematics curriculum, rather than that of mathematics teacher educators (MTEs).

Our research responds to Averill et al.'s (2009) challenge to "critically reflect on [our] own culturally responsive practices, ideally in discussion with other practitioners, teacher educators, and students" (p. 181). We present a collaborative self-study of two mathematics teacher educators (MTEs) developing our own CRP. We address the question, "What do MTEs learn from attempts to grow and reflect on their own CRP?" In this presentation, we share our newly developed *MTE Framework for Growing CRP*. To date, no specific tool has been proposed for supporting and guiding the professional growth of culturally responsive teacher educators.

Conceptual and Methodological Framework

Our research takes a layered approach to collaborative self-study (e.g., Hamilton & Pinnegar, 2013; Hug & Möller, 2005) such that our individual self-studies in our respective institutions converge to a second layer of collaboration through conversations on how we enact our developing CRP. Our framework includes four key reflective questions, which we constructed from careful review of a few sources (see, for example, Aguirre & Zavala, 2013; Lingard & Keddie, 2013): (1) How do my pedagogical practices draw on my students' mathematical discourse and funds of knowledge? (2) How am I addressing concerns for balance between deep and rigorous mathematical knowledge with issues of culture, equity, social justice and language diversity? (3) What struggles and resistances do I experience as I attempt to disrupt dominant forms of pedagogy that my students express more comfort with? (4) How is my own identity and experiences of being a teacher interacting/integrating with key principles of what it means to be culturally responsive? In the framework, each key question is further clarified by sub-questions.

Data Collection and Significance of Study

Our collaborative self-study model includes individual journaling on the reflective questions of our self-study framework, and monthly meetings with each other for the collaborative layer. Data collection is ongoing with an aim of continually refining our framework for growing MTE CRP. In fact, the processes of applying, refining, and revising the framework itself help work toward the goal of growing as culturally responsive pedagogues. This poster brings to light ways that MTE collaborative self-reflection can support efforts to grow one's own CRP practice— in collaboration with other MTEs and also with one's students (PTs), making the framework available for use by other teacher educators who wish to reflect on their own CRP.

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BEGINNING TEACHERS' EQUITABLE AND AMBITIOUS NUMBER TALKS

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Number Talks (NTs) are instructional routines that offer students a participation structure for students to use mental mathematics to solve computational problems. Responding to a call for understanding ways teaching practices could be ambitious and equitable (Jackson & Cobb, 2010; Kazemi, Franke, & Lampert, 2009; Lampert et al., 2013), we investigate the structure of the routines and components of NTs as enacted by beginning teachers (BTs).

We analyzed the 21 videos of NTs by creating transcripts of the lessons. We then drew on Parrish's (2014) and Parker and Humphrey's (2018) descriptions of NTs to identify phases in NTs. We identified a structure of NTs: *introducing*, *collecting*, *idea sharing*, and *closing* phases. This structure was consistent across the data set. We then drew on Cazden's (2001) concept regarding the *initiate-respond-evaluate/feedback* (IRE/F) pattern that is prevalent in mathematics classrooms (Lawrence & Crespo, 2003) to parse transcripts initially coded as *idea sharing* into manageable units for analysis, called segments. We then characterized each of the segments by their function (e.g., sharing strategies, comparing ideas). We looked for patterns across these characterizations. We analyzed segments by attending to patterns of who was talking within each segment type as well as overall patterns of individual's talk throughout the duration of the NT. We then contrasted segments that contain a mathematical error with those that do not.

Our analyses suggest two findings. First, BTs followed a routinized structure across this set of NTs. Within that structure, we identified important variation within the *idea sharing* phase that have implications for ambitious and equitable teaching. Second, segments coded as *strategy plus, teacher strategy,* and *comparing* created more opportunities (as compared to segments coded as strategy) for multiple students to engage with mathematical ideas. Further, in these three segment types, students engaged in a variety of ways. In this poster, we focus on *strategy plus* segments to illustrate the potential for NTs to be both ambitious and equitable.

The recognizable structure of NTs across our dataset is notable. We see NTs as a type of transportable container through which BTs can develop an ambitious and equitable practice. NTs are transportable in the sense that their recognizable and reproduceable structure offer supports for BTs to engage in complex ambitious and equitable instruction. Though much current literature focuses on structures and routines to implement NTs, our analyses indicates that the structure itself does not inherently create ambitious and/or equitable NTs. We identified the *idea sharing* phase as a critical point for further development and investigation. It is here where we found distinctions between ambitious and/or equitable NTs related to the types of segments in which BTs engaged students.

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THE INTERPLAY BETWEEN A VISUAL TASK AND ELEMENTARY STUDENTS' MATHEMATICAL DISCOURSE

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This research was conducted by a fourth-grade teacher and doctoral student in mathematics education in conjunction with their advisor, a professor of mathematics education.

A growing body of research in mathematics education has highlighted the importance of recognizing mathematics learning as a socially mediated activity. Indeed, mathematics education researchers have increasingly focused on how classroom dialogue can facilitate students' creation of shared understandings. Aligned with this theoretical heritage, we recognize that human life and learning are inherently social and rooted in communication. We also recognize that student discourse is connected to student cognition and thus learning. Accordingly, this study relied on socio-cultural discourse analysis (Hennesy, et al., 2016, Mercer, 2010) both as a theoretical and a methodological tool to examine the nature of dialogue in one classroom in the context of students' collaborative work on one visual task. We ask, given the centrality of task selection to fostering discourse, how the use of a visual task, as an instructional tool, might affect students' peer-to-peer discourse practices?

Methods

The goal of this study was to identify specific discourse practices students utilized while collaborating on a visual mathematics task. A focus group of 4, fourth-grade students' interactions on one task was used as a data source for analysis.

The participants worked on a task (Boaler, 2017, p. 32) that asked them to work together to find patterns. Students were each nine to ten-years-old and represented a range of academic abilities. The group discussion was videotaped and transcribed.

Transcriptions were coded using Hennessy et al.'s Scheme for Educational Dialogue Analysis (SEDA) (Hennessy, et al., 2016). SEDA offers a scheme for analyzing discourse practices, specifically outlining different practices.

Findings

A total of 111 utterances were coded. Students most frequently conjectured and made their reasoning explicit by utilizing visual models presented in the task. Additionally, "Explicit reasoning" and "build on ideas of others" accounted for the majority of students' communicative acts during the discussions.

Group members exhibited different patterns of practice and adopted different roles. Despite differences in discourse moves, the majority of communicative acts consisted of students' conjectures and ensuing explanations using the task's visual models. Learners' comments frequently relied on the visuals with statements such as, "Look, (points to paper), there are triangles all over the place." Group members relied on the visual task as they considered proposed ideas, built on them, and utilized them in their own work.

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A UNIVERSITY INSTRUCTOR'S ORCHESTRATION FOR SUPPORTING HIS STUDENTS' PROGRAMMING FOR MATHEMATICS

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In our on-going five-year naturalistic research, we analyze how math majors and future math teachers learn to use programming for mathematics investigation. Using the instrumental approach (Trouche, 2004) as a framework, we present some exploratory results on how the instrumental orchestration of an instructor (Bill) supports the activity of his students.

Rabardel (1995) describes how people, through their instrumental geneses, appropriate an artifact and turn into an instrument. Trouche (2004), proposed the concept of instrumental orchestration to refer to the teacher's organization: the arrangement and didactic use of artifacts in the class to steer the student's instrumental geneses. As an extension to this concept, Drijvers et al. (2010) consider three instrumental orchestration's components: (i) the *didactical configuration* – "an arrangement of artifacts in the environment; (ii) the *exploitation mode* – "the way the teacher decides to exploit a didactical configuration for the benefit of his or her didactical intentions"; and (iii) the *didactical performance* – which "involves the ad hoc decisions taken while teaching" (Drijvers et al., 2010; p. 215). Bill's data includes the course syllabus and the assignment guidelines; assessment grading rubric and 6 instructor interviews.

Bill's **didactical configuration** involved mathematical and social considerations and a web of ideas and actions that provide a creative structure for drawing connections between programming and mathematics (Buteau et al., 2020). The choice of programming technology and the general guidelines of their use was established in 2000 by the mathematics department at his university. Bill follows this didactical configuration.

The **exploitation mode** relates to his aims and the didactical design of the assignments, which involve modeling, problem-solving, simulations, and explorations; and where "mathematics is for programming", and "programming is used to do and understand mathematics". Bill provides his students with guidelines for each assignment including, in written form: worksheets guiding them through several steps or parts. For Bill, the mathematical content should be interesting for his students, related to computing and real-life phenomena. Bill's **didactical performance** aims at supporting and empowering his students while also taking into account and promoting the affective aspects (e.g. motivation, as well as creativity). He supports students by guiding their activity through individual interactions (in the lab mainly) or/and through collective interventions and discussions in lectures (Sacristán et al., 2020). In our on-going work, we analyze how the instrumental orchestration proposed by Bill, in particular the artifacts used in class, the design of the assignments and how he interacted with his students, support the development of his students' instrumental geneses of programming for mathematics. In the poster, we will illustrate the above in the case of a particular student in Bill's class.

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TECHNOLOGY

RESEARCH REPORTS

THE EFFECTS OF A TECHNOLOGY COURSE WITH COLLABORATIVE DESIGN ON PROSPECTIVE TEACHERS' KNOWLEDGE AND BELIEFS

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This study investigates the effects of a technology methods course containing a unique collaborative design experience on prospective elementary and secondary mathematics teachers' technological beliefs, computer algebra system (CAS) beliefs, and technological pedagogical content knowledge (TPACK). Overall gain scores on all three instruments were statistically significant. Moreover, gender and level (elementary vs. secondary) were statistically significant predictors of TPACK gain scores. However, the influence of level on TPACK gain score was different for female prospective teachers (PTs) than male PTs. Even in the case of low gain scores PTs displayed beliefs that were aligned with productive uses of technology in the classroom. PTs showed greater gains on knowledge subdomains associated with technological knowledge than on technology free subdomains (e.g., pedagogical content knowledge).

Keywords: technology, teacher beliefs, teacher knowledge

Technology plays an increasingly pervasive role in our everyday lives and that influence extends into school classrooms. Yet research suggests that technology is often used to support current educational practices instead of as a catalyst for change (e.g., Cuban, Kirkpatrick, & Peck, 2001). The mathematics education community has created an extensive body of research that has recognized the important role that both knowledge (e.g., Meagher, Özgün-Koca, & Edwards, 2011) and beliefs (e.g., Kim et al., 2013) play in shaping teachers' decisions around the use of technology in school classrooms. One particularly popular conceptual framework for thinking about the knowledge that teachers need to possess in these classrooms is technological pedagogical content knowledge (TPACK) (Mishra & Koehler, 2006). A variety of approaches have been used to promote TPACK among practicing and prospective teachers. One of the more popular approaches involves collaboration in the design of technology-infused lessons (e.g., Koehler & Mishra, 2005). This study examines the effect of a technology methods course containing a unique collaborative design environment on prospective elementary teachers' (PSETs') and prospective secondary mathematics teachers' (PSTs') beliefs about technology in general, beliefs about computer algebra systems (CAS), and their TPACK knowledge.

Background

TPACK is one of the most frequently used frameworks to conceptualize and research the knowledge that teachers who teach successfully with technology need to possess. A variety of interventions have been found to positively influence the TPACK of prospective teachers such as technology rich field experiences (Meagher, Özgün-Koca, & Edwards, 2011), collaborative design experiences (e.g., Agyei & Voogt, 2012), and engaging students in solving mathematics problems with technology (Meagher et al.). Wang, Schmidt-Crawford, and Yin (2018) reviewed 88 empirical studies and found that modeling of the integration of technological, pedagogical, and content knowledge in university courses and by practicing teachers was an effective way of increasing the TPACK of prospective teachers (PTs). Their synthesis also suggests that gaining experience teaching with technology, engaging in peer mentoring, and learning technological knowledge are important in developing PTs' integrated knowledge domains such as technological content knowledge (TCK).

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An extensive collection of research has highlighted the connections between beliefs and teaching practices (e.g, Kim et al., 2013) and numerous studies have investigated teachers' beliefs with regard to technology. One way to conceptualize teacher beliefs regarding technology is what I refer to as the *role of technology in mathematics classrooms* which consists of a continuum with *doing mathematics* on one end and *learning mathematics* on the other end. Beliefs aligned with doing mathematics include the mastery principle (Fleener, 1995), "old school" (Erens & Eichler, 2015), and the restriction of CAS black box techniques (Doerr & Zangor, 2000). Individuals professing a learning mathematics position do not believe that students must learn fundamental ideas before technology; technology can be used as a tool to learn mathematical ideas (Lagrange, 1999). Proponents of a doing mathematics position argue that students should not use technology until they have learned the concepts or procedures that the technology can perform. An assumption hidden within this position, which is in contrast to the learning mathematics position is that students learn mathematics solely through paper-and-pencil work, not with technology. A belief that is aligned with the doing mathematics position is that even if technology is only allowed until students have acquired the paper-and-pencil skills they can still lose proficiency with these skills if technology is used too frequently, often described as technology becoming a "crutch" (e.g., Schmidt, 1999).

Beliefs are often connected to other personal characteristics. Tharp, Fitzsimmons, and Ayers (1997) found that practicing secondary teachers used technology more extensively in the classroom if they possessed less rule-based perspectives of mathematics. Teo and colleagues (2008) found that constructivist teaching beliefs of PTs were positively correlated with both constructivist and traditional use of technology while traditional teaching beliefs were negatively correlated with a constructivist use of technology. There is also an extensive body of research highlighting connections between gender and technology (e.g., Sanders, 2006).

Previous research has uncovered connections between TPACK and gender. For example, Bulut and Işiksal-Bostan (2019) found that male PSETs had significantly higher scores than female PSETs in TPK, TK, and TPACK. The relationship between TPACK and beliefs is mixed. For instance, Niess (2013) found that teachers' TPACK levels were occasionally connected to their beliefs. Similarly, Smith, Kim, and Mcintyre (2016) investigated the TPACK and beliefs held by four prospective middle grades teachers. Two of the teachers appeared to show relationships between beliefs and TPACK, where more student-centered views of mathematics teaching and learning were aligned with higher levels of TPACK for one teacher. More teacher-centered views of mathematics teaching and learning were aligned with lower levels of TPACK for another teacher. The results for the other two teachers were less clear.

This study builds on this extensive collection of research to investigate the effects of a technology methods course involving both PSETs and PSTs on their beliefs and TPACK. The technology methods course at the center of this study contains components that have been found to have significant impacts on the TPACK of PTs (Wang et al., 2018) as well as a previously uninvestigated collaborative design environment. This study was designed to answer three research questions.

1. In what ways does a technology methods course involving a collaborative design experience influence PTs' technological beliefs and CAS beliefs?
2. In what ways does a technology methods course involving a collaborative design experience influence PTs' TPACK and related knowledge subdomains?
3. Does a technology methods course involving a collaborative design experience differentially impact PTs' beliefs or TPACK knowledge depending on the gender or level (secondary vs. elementary) of participants?

Methodology

Frameworks

The TPACK framework (Mishra & Koehler, 2006) was used to understand the knowledge gained by PTs as a result of the activities comprising the technology methods course. This framework highlights the separate and interconnected nature of three different knowledge areas. By separate I mean that knowledge exists that is solely, technological, pedagogical, and content in nature that teachers must possess in using technology successfully in the classroom. For instance, purely technological knowledge comes into play when students "break" a pre-constructed technological document and the teacher must deploy his/her/their technological knowledge to diagnose and repair the problem. By integrated I mean that in addition to TPACK which involves the complex interplay of technological, pedagogical, and content knowledge there exist three other integrated knowledge types: pedagogical content knowledge (PCK); technological content knowledge (TCK); and technological pedagogical knowledge (TPK).

The learning by design framework (Koehler & Mishra, 2005) guided the construction of collaborative design experiences that PTs experienced in the technology methods course as the center of this study. The framework involves learning-by-doing and extended design work on authentic problems. Specifically, learning-by-doing involves two components: construction of lessons involving technology and the teaching of those lessons in middle school and high school classrooms. Authentic problems are those that teachers working in schools frequently encounter such as how to incorporate technology into a textbook lesson that does not currently contain technology or how to develop technology-rich activities that help students to develop conceptual understanding of important mathematical ideas. The course instructor often acts as a facilitator or problem-solving expert.

Context

The study took place in a large university in the midwestern U.S. that is known for its teacher preparation program. In the past, the technology methods course taught in the mathematics department, only enrolled PSTs, but the development of an Elementary Education Mathematics Major with a certification across grades K-8 necessitated the creation of another course focusing on technology use at the middle school level (grades 6-8) for these individuals. Since the development of the middle school mathematics technology course both courses have been taught at the same time and place and by the same instructor. The class met for two 100-minute sessions a week for 12 weeks. The course where the data for this study were collected was taught during the Spring 2019 semester. A total of five prospective elementary teachers (PSETs) and four PSTs were enrolled in the jointly-held course and chose to participate in the study.

PTs enrolled in both classes developed lesson plans and student activity sheets. The lesson plan involved components such as lesson objectives, places where students might struggle, how student struggles would be addressed, answers to lesson questions, and estimated time required for students to complete various lesson components. The student activity sheet involved a warm-up (if the PT chose to include one), activities and questions students were to complete, and oftentimes an exit ticket. The focus of the lesson was on conceptual understanding, the use of technology to help students learn the objectives of the lesson, the use of one or more high cognitive demand tasks (Stein & Smith, 1998), and the inclusion of at least one class discussion. All lessons taught in area classrooms involved middle school mathematics. The use of teaching experience with lessons involving technology has been found to positively affect prospective teachers' TPACK knowledge (Wang et al., 2018).

The class involved two different types of group lesson planning structures: brainstorming and refinement. Brainstorming involved the development of general ideas about a lesson without the creation of specific lesson elements. Refinement involved the presentation and critique of a student activity sheet. Brainstorming occurred if the PT was struggling to develop a lesson plan and student activity sheet while refinement was used if the PT had already completed a lesson plan and student activity sheet. PTs engaged in the development of lesson plans and student activity sheets individually, as part of a large group consisting of the entire class, and working with the instructor of the course. All of the PTs wrote a paper detailing the planning, enactment, and reflection regarding their lesson.

In addition to the presentations and brainstorming sessions, the PTs engaged in the following activities in the technology methods course: solving mathematics problems using technology; completing journal entries designed to make their beliefs regarding technology transparent to them; reading mathematics education articles involving technology and reacting to them; exploring the symbolic manipulation capabilities of CAS, the completion of a project involving the solution of an infinite class of optimization problems using graphical, tabular, and CAS capabilities; and considering how technology can be implemented in mathematics textbook lessons that do not currently use technology. Each PT created a lesson plan and student activity sheet which were either presented to the classroom for critique and refinement or began as brainstorming sessions for a total of nine lessons involving technology.

Instruments

A technology beliefs survey was administered to PTs on the first day of class and again on the last day of class. The beliefs survey was adapted from Schmidt (1999) in the following ways. First, the words calculator or calculators were replaced with technology. Second, items involving practicing teachers that referenced components of their work that were not applicable to prospective teachers (e.g., perspective of parents of their students) were removed. A frequently used TPACK questionnaire (Schmidt et al., 2009) consisting of 58 items measuring seven different knowledge domains was administered during the first day of class and again during the last day of class.

This questionnaire contains items in four different content areas: mathematics; literacy; science; and social studies. In addition to measuring TPACK (five items), the questionnaire also measures technological knowledge (TK) (seven items), content knowledge (CK) (three items), pedagogical knowledge (PK) (seven items), pedagogical content knowledge (PCK) (one item), technological content knowledge (TCK) (one item), and technological pedagogical knowledge (TPK) (four items). This TPACK questionnaire was used with the group of PSETs as this was the population for which the instrument was developed.

The questionnaire was adapted for PSTs (resulting in 44 items) by removing the CK, TCK, and PCK items related to literacy, science, and social studies and replacing them with similar items related to the students' minor degrees (e.g., history). Given the work that the PTs completed with CAS described earlier, I also administered a CAS beliefs survey (Lavicza, 2010) on the first day and last day of class to determine whether their beliefs regarding this powerful technology had changed as a result of the activities in the technology methods course. This survey, consisting of 20 items, was adapted as the original was intended for faculty teaching mathematics at the university level. For instance, the word *mathematicians* in the item, CAS enables mathematicians to work on problems more efficiently, was replaced with *students*. The technology beliefs survey, CAS beliefs survey, and TPACK questionnaire consist of Likert scale items that range from strongly disagree to strongly agree. None of the PTs were enrolled in another course that involved the use of technology during the Spring 2019 semester, but all were taking either foundational education courses or courses involving pedagogical components. Thus, there is a potential that the gains seen on the instruments with regard to pedagogy could be a result of these other courses. Three out of four of the PSTs were enrolled in mathematics content courses during Spring 2019, but this was a modern algebra course that did not highlight the connections between its content and school mathematics.

Analysis

I assigned a numerical score for each of the Likert scale items (strongly disagree – 1, disagree – 2, neutral – 3, agree – 4, strongly agree – 5). For each item of the technology belief survey and CAS beliefs survey, the differences between the first and last administration were calculated and the sum was found for each PT. Items that were negatively worded were reverse scored. The mean of the totals across the group of PSETs and the group of PSTs were found. All of the collected data were examined for trends. As there were different numbers of questions for the knowledge subdomains in the TPACK questionnaire, each PT's difference was divided by the number of questions that contributed to that difference for reporting purposes. The mean of these values was reported for each PT as an average gain value that enables comparisons to be made across different knowledge subdomains. The assumptions for the statistical tests (e.g., normality) were met and an alpha level of .05 was used for all statistical tests. The paired samples *t*-test was used to test for statistical significance for gain scores on each of the three instruments. Effect sizes were found by converting a *t*-value into an *r*-value (Rosnow & Rosenthal, 2005). A factorial ANOVA test was run on the gain scores for technological beliefs, CAS beliefs, and the TPACK questionnaire with gender and level (elementary and secondary) as independent factors.

Results

Overall, PTs scored higher on the second administration (M = 130.11, SD = 12.424) than the first administration (M = 110.22, SD = 4.324) on the technological beliefs survey and this result was statistically significant, $t(8) = -4.850$, $p = .001$, $r = .86$. In the factorial analysis, the independent factors of gender and level as well as the interaction were statistically non-significant. The changes in technological beliefs for PSETs and PSTs are shown in Table 1. The beliefs score changes for PSTs were greater than the score changes for PSETs. Three questions were common across both groups in terms of the greatest change between administrations of the technological beliefs survey. The first of these involved the belief that technology can damage students' paper-and-pencil skills and become a "crutch." A total of four out of nine of the PTs started out agreeing or strongly agreeing with this statement and shifted to disagreeing with this statement. The remaining five teachers either disagreed with the statement across both administrations of the survey or moved from disagree to strongly disagree. The second of these questions stated that students who use technology in high school mathematics classes learn mathematics better than those who do not use technology. Six out of nine of the PTs moved from disagree/neutral with regard to this statement to agreeing with it. The last statement involved prospective teachers' lack of confidence to teach mathematics involving technology. Six out of nine PTs agreed with this statement at the start of the class, but by the end of class they all disagreed with the statement.

Overall, PTs scored higher on the second administration (M = 69.89, SD = 8.42) than the first administration (M = 57.78, SD = 6.14) on the CAS beliefs survey and this result was statistically significant, $t(8) = -3.210$, $p = .012$, $r = .75$. In the factorial analysis, the independent factors of gender and level as well as the interaction were statistically non-significant. The changes in CAS beliefs for PSETs and PSTs are shown in Table 1. There were several low change scores in the table. For instance, Logan had no change on the technological belief survey, Liam had a CAS belief change of only two, and Madison had a change of -1 on the CAS belief survey. All of these PSETs possessed a number of initial beliefs that were aligned with an environment where technology is seen as a valuable tool to assist in the teaching of mathematics. For instance, Logan professed initial beliefs on 19 out of the 39 technological beliefs questions that were aligned with practices presented in the class. Across all PTs, two questions had highest gains from first to last administration: CAS promotes students' conceptual understanding; and CAS can be used to develop more engaging lessons.

Table 1: Changes in Beliefs for PSETs and PSTs

Prospective Teacher[a]	Technological Belief Change	CAS Belief Change
	PSETs	
Liam	26	2
Sophia	20	20
Amelia	13	6
Logan	0	15
Madison	14	-1
Mean	**14.60**	**8.4**
	PSTs	
Noah	12	6
Emma	40	23
Olivia	35	33
Mason	19	5
Mean	**26.50**	**16.8**

[a]All names are pseudonyms.

Overall, PTs scored higher on the second administration (M = 118.78, SD = 7.19) than the first administration (M = 102.44, SD = 12.78) of the TPACK questionnaire and this result was statistically significant, $t(8) = –4.599$, $p = .002$, $r = .85$. There was a significant main effect of gender on the TPACK gain score F (1, 5) = 14.12, p = .013, *,ω-2.=.15*. The main effect of level on the TPACK gain score was statistically significant F (1, 5) = 7.73, p = .039, *,ω-2.=.08*. Additionally, there was a statistically significant interaction between gender and group on TPACK gain score F (1, 5) = 59.904, p = .001, *,ω-2.=.67*. In other words, the influence of level on TPACK gain scores is different for female PTs than male PTs. Specifically, male PSETs had higher TPACK gain scores (M = 18.5, SD = 3.54) than females PSETs (M = 9.67, SD = 2.08). Female PSTs had higher TPACK gain scores (M = 33.00, SD = 2.83) than male PSTs (M = 7.50, SD = 4.95).

The knowledge gains by content subdomain for PSET and PST are shown in Table 2. For both groups the technology methods course appeared to have only a moderate influence on their content knowledge, technological knowledge, pedagogical knowledge, and pedagogical content knowledge. Both groups experienced the greatest knowledge gains in the TCK and TPACK areas. PSTs also experienced larger gains in the area of TPK.

Amelia experienced a loss of four in the area of TPK on the questionnaire. This occurred because on three of the five statements she moved from strongly agree to agree resulting in a drop of negative three. On the fourth statement in this area she had no change from agree while on the last statement she moved downward from agree to unsure. Mason also had a sum for a content subdomain (TK) that was negative. On four of the seven questions comprising this area, he had no change from agree or strongly agree on the initial and final questionnaire. On the three other questions, he moved from agree or strongly agree to unsure. Noah also had a sum of negative one in the knowledge subdomain of TCK that included one question. On this question, he moved from strongly agree to agree. In sum, despite decreases in change scores for some of the PTs that resulted in overall decreases the final rating on the majority of these statements was still in the agree category.

Table 2: Changes in Knowledge for PSETs and PSTs

Prospective Teacher	TK	CK	PK	PCK	TCK	TPK	TPACK	Total
				PSETs				
Liam	1	0	4	1	2	2	6	16
Sophia	0	1	3	0	2	3	3	12
Amelia	4	0	2	0	2	-4	4	8
Logan	1	1	6	0	0	5	8	21
Madison	1	1	2	1	0	3	1	9
Average Gain	0.2	0.2	0.49	0.2	**1.20**	0.36	**0.88**	13.20
				PSTs				
Noah	2	0	1	0	-1	2	0	4
Emma	7	1	12	2	1	5	7	35
Olivia	9	3	3	1	2	6	7	31
Mason	-4	0	1	0	2	6	6	11
Average Gain	0.61	0.33	0.61	0.75	**1.00**	**1.19**	**1.00**	20.25

Discussion

As a group, across all three instruments, PTs performed statistically significantly better on the second administration than the first administration. Thus, the collection of activities in the technology methods course appeared to positively influence PTs' beliefs and their TPACK. In general, where PTs displayed smaller changes in beliefs, their initial beliefs were already aligned with environments where technology was seen as an important tool in learning mathematics thus they had less room to change. This suggests that simply experiencing activities involving the use of technology to learn mathematics as all of the PTs did in previous courses can promote positive beliefs involving technology.

The PTs demonstrated less growth in CK, PK, and PCK as a result of the technology methods course than they did in the areas of TCK, TPK, and TPACK. This suggests that while topics regarding general pedagogical knowledge, content knowledge, and pedagogical content knowledge emerged during design work, teachers may not have perceived the work as occurring in these domains as the lessons were centered around the use of technology. That is, the PTs might have primarily seen the design work as involving technology. Indeed, on the technological belief survey one of the items of greatest change was their confidence in developing technologically based lessons. This result aligns with the work of Koehler and Mishra (2005) in which their collaborative learning environment resulted in greater connections among technology, pedagogy, and content. The differential gains on TK between PSETs and PSTs might have been a result of the particular lessons the groups developed. PSETs tended to create lessons involving technological applications that were already constructed while PSTs' lessons required them to learn and deploy more technological knowledge.

The PTs had limited work with CAS during the methods course as they used it to learn mathematics at the beginning of the course and solve optimization problems at the end of the course; none of the PTs created a lesson involving CAS. Nonetheless, they made statistically significant gains on this survey. This may have been due to a spillover effect involving the extensive design work in technology. Importantly, the technology methods course and its limited use of CAS resulted in a shift to envisioning the CAS as a tool to develop students' conceptual understandings much like the teachers Lagrange (1999) investigated. This is an interesting finding as PTs did not specifically use CAS in activities focused on conceptual understanding. This finding suggests that the collaborative design experiences and their focus on conceptual understanding affected PTs' beliefs in a different type of technology than where this work occurred.

As mentioned earlier, there were a few negative gain scores sprinkled throughout the results even though many of the PTs with these values were still agreeing with beliefs that were aligned with the use of technology to promote mathematics understandings and greater TPACK knowledge. These losses might have reflected a correcting of overly optimistic beliefs or knowledge as a result of deep engagements with technology during the collaborative design process.

Despite previous findings with regard to gender and technology (Sanders, 2006) this study found no relationship between gender and technological beliefs or gender and CAS beliefs. However, gender, level, and the interaction between gender and level were significant predictors of TPACK gain scores. Female PSETs had lower initial scores than male PSETs on the TPACK questionnaire overall. Thus, male PSET gain scores might have been lower than females because there was less room to grow. Female PSTs had the lowest initial TPACK scores among all both groups, about 20 points lower than female PSETs giving them more space to grow. PSETs had experienced more mathematics courses that incorporated technology and one more mathematics methods course than PSTs. These factors might have translated into higher initial TPACK scores. The higher TPACK scores among male students overall is similar to the findings of Bulut and Işiksal-Bostan (2019).

In sum, these findings illustrate the effectiveness of a technology methods course on PTs' beliefs and TPACK knowledge. The study is limited by its reliance on self-report data and the small sample size. In the future, I intend to examine other data (e.g., PT classroom enacted lessons involving technology) to move beyond self-report data in understanding the effect of the course on PTs' TPACK. Moreover, the study focused on the effect of the technology methods course as a whole on PTs' beliefs and knowledge and while the collaborative context was described in detail, the effects of this unique factor on PTs were not isolated. My future research intends to more carefully investigate the effects of this unique activity on PTs' beliefs and knowledge. The omnipresence of technology in today's classrooms necessitates that teachers be prepared to use it in ways that draw on its unique affordances and its potential to change mathematics instead of in ways that support traditional instruction (Cuban et al., 2001).

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INVESTIGATING UNDERGRADUATE STUDENTS' GENERALIZING ACTIVITY IN A COMPUTATIONAL SETTING

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Computational activity is increasingly relevant in education and society, and researchers have investigated its role in students' mathematical thinking and activity. More work is needed within mathematics education to explore ways in which computational activity might afford development of mathematical practices. In this paper, we specifically examine the generalizing activity of undergraduate students who solved combinatorial problems in the context of Python programming. We demonstrate instances of generalizing in terms of Ellis et al.'s (2017) framework, and we argue that some opportunities were facilitated and supported by the computational setting in which the students worked.

Keywords: Cognition, Computational Thinking, Programming and Coding

Introduction and Motivation

Computation is an increasingly essential aspect of science and mathematics, and researchers and policy makers within STEM broadly (Blikstein, 2018; NGSS Lead States, 2013; Weintrop, Beheshti, Horn, Orton, Jona, Trouille, & Wilensky, 2016), and mathematics education especially (e.g., Benton, Saunders, Kalas, Hoyles, & Noss, 2018; Buteau & Muller, 2017; Cetin & Dubinsky, 2018; Hoyles & Noss, 2015; Feurzeig, Papert, & Lawler, 2011), are making the case for more attention to be paid to computing in the field. Central to current interest in computing are questions related to whether and how computing might strengthen students' mathematical thinking and activity, including their engagement in mathematical practices. The question of whether such computing allows for transfer of content knowledge or practices is a source of debate, with some researchers making the case for and some against arguments that evidence of transfer exists (see Tedre & Denning (2016) for discussion). Acknowledging this debate, we investigate such questions with a qualitative exploration in which we demonstrate how computing may support students' engagement with a mathematical practice – generalization.

Our research question is: *In what ways did a computational setting support undergraduate students' generalizing activity on combinatorial tasks?* We hope that by providing qualitative interview data, we can gain some insight into how students engage in generalization in a computational setting. We see the paper as serving both a specific and a broader purpose. First, the paper is meant to highlight specifically how students engage in the particular practice of generalization with the use of programming (within the domain of combinatorics). Our results thus shed light on generalization as a practice, and they also illuminate implications for generalization within combinatorics. More broadly, this paper exemplifies of how students can make connections to and engage in mathematical practices within a computational setting.

Relevant Literature and Theoretical Perspectives

Literature and Theoretical Perspectives on Computation

In this paper we focus on *machine-based computing*, which we take to be the practice of developing and precisely articulating algorithms that may be run on a machine. We make two distinctions in this characterization. First, while computing can encompass many kinds of activity, we focus on activity that involves developing, articulating, and implementing algorithms. Second, while such algorithm

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development could occur strictly by hand, we focus on activity that uses a machine. Further, we distinguish machine-based computing from engagement with technology more generally, which might include using computer algebra systems or dynamic geometry software – for us, machine-based computing involves not just using software, but engaging in algorithm design and implementation in some capacity. In our study, the specific machine-based computing in which our students engaged was programming in Python.

Mathematics education research has a history of using computers to enhance students' mathematical reasoning, beginning with Papert's (1980) introduction of Logo to help young children. Recently there seems to be an increasing amount of attention being paid to computation in education research, perhaps due in part to Wing's (2006, 2008) re-popularization of the term *computational thinking*. We have seen considerable attention paid recently toward examining the role of computing in mathematics education (e.g., Benton, et al. 2018; Buteau, Gueudet, Muller, Mgombelo, & Sacristan, 2019; Buteau & Muller, 2017; Cetin & Dubinsky, 2018; DeJarnette, 2019; Lockwood, DeJarnette, & Thomas, 2018; Lockwood & De Chenne, 2019).

We acknowledge that there is some discussion about whether or not computing can be effective in helping students engage in other practices and skills (e.g., Tedre & Denning, 2016). However, in spite of such debates, we think it is still worth investigating the degree to which computational activity might in fact support students' thinking and engagement in mathematical practices. Part of our reason for this is that we feel there are useful frameworks within mathematics education (such as Lobato's view of actor-oriented transfer) that may bring fresh perspectives toward questions of the role of computing in mathematics education. We draw on recent work by Lockwood et al. (2019), who interviewed research mathematicians in academia about their use of computing in their work. While these mathematicians suggested many benefits of computing, they also "framed computing as allowing for some other important habits of mind or practices related to their work" (p. 9). While these mathematicians did not name the practice of generalizing specifically, we view our work as building on these findings by Lockwood et al. The mathematicians reported in Lockwood et al.'s study shared their own beliefs and experiences, and we wanted to explore and demonstrate some of their claims with student data, offering insights into how students' engagement with computation can actually give opportunities for students to connect computing to the practice of generalizing.

Literature and Theoretical Perspectives on Generalization

Generalization is an essential aspect of mathematical thinking and learning, and there has been much work that has established the importance of generalization in mathematics education. Such work has included investigations into students' generalizing within algebraic contexts (e.g., Amit & Neria, 2008; Ellis, 2007a, 2007b, Radford, 2008; Rivera & Becker, 2007, 2008), and there has also been exploration into generalizing activity among undergraduates in areas like calculus, linear algebra, and combinatorics (e.g., Dubinsky, 1991; Jones & Dorko, 2015; Kabael, 2011; Lockwood, 2011; Lockwood & Reed, 2018). Researchers have also proposed theories about the nature of generalization, providing some categories and distinctions of generalizing activity (e.g., Ellis, 2007a; Harel & Tall, 1989; Harel, 2008). Together these studies provide rich insight into the nature of generalization in many settings. We contribute to this body of work by examining generalizing within the context of machine-based computing, and we hope to identify and understand specific ways that a computational setting might support students' generalizing.

Broadly, Ellis (2007a) followed Kaput (1999) and defined generalization as "engaging in at least one of three activities: a) identifying commonality across cases, b) extending one's reasoning beyond the range in which it originated, or c) deriving broader results about new relationships from particular cases (p. 444), and we similarly adopt that broad characterization. We also draw upon Ellis, Lockwood, Tillema, & Moore's (2017) Relating-Forming-Extending (R-F-E) framework of generalizing activity in characterizing generalization. Ellis et al. (2017) emphasize three different generalizing activities, which build upon a previous taxonomy that Ellis (2007a) had developed. In *relating,* students establish "relationships of similarity across problems or contexts" (p. 680), and so students make connections among situations they have encountered. In *forming*, students engage in "searching for and identifying similar elements, patterns, and relationships" within a single task (p. 680). Here, students may be attending to regularity and articulate some general pattern or relationship that they observe. In *extending, "*students extend established patterns and regularities to new cases" (p. 680). This might typically involve some increased abstraction (such as moving from numerical cases to arguments involving variables). Ellis et al. also discuss ways in which these generalizing activities are interrelated – for instance, relating and forming may help students start to identify some regularity, which can then facilitate their extending to more general cases. This categorization offers language by which to characterize generalizing activity that we observed in our students. We focus on instances of relating and forming in this paper.

Mathematical Discussion and Motivation for Focusing on Combinatorics

There are a couple of reasons that we focus on combinatorics in this paper. First, the computational setting is particularly well-suited for combinatorial problems, in the sense that some of the features of the programs (loop structures, conditional statements) serve to highlight important combinatorial concepts. We have articulated this phenomenon elsewhere, including demonstrating students' uses of conditional statements to reason about types of counting problems (Lockwood & De Chenne, 2019) and highlighting the computer's effectiveness in helping students verify solutions to counting problems (De Chenne & Lockwood, in press). We believe that combinatorial problems provide rich contexts in which students can solve mathematical problems in computational settings. In addition, combinatorial tasks are well suited to generalization, and researchers have previously explored students' generalizing activity on combinatorial problems (e.g., Lockwood, 2011; Tillema & Gatza, 2018). Our work builds on such studies by illuminating ways in which the computational setting supports generalizing within combinatorics. We thus aim to contribute both to work on generalization and work on combinatorics, building on our knowledge base of students' generalizing activity within the field of combinatorics especially. Finally, on the whole, combinatorial problems can be difficult for students to solve (e.g., Batanero, Navarro-Pelayo, & Godino, 1997; Lockwood & Gibson, 2016), and we see value in investigating ways to improve students' combinatorial experiences. In this case, by focusing on generalization within a computation setting, we gain insight into how students might understand and generalize ideas within the particular domain of combinatorics.

Methods

Data Collection

We draw on two data sources for this paper, both of which were part of a broader study investigating the role of computing in teaching combinatorial ideas. The broader study is ongoing and includes multiple paired and small group teaching experiments and one round of classroom implementation. We narrowed our focus to these two data sources for the sake of space and because they provide illustrative examples of relating and forming. First, we conducted a paired teaching experiment (in the sense of Steffe & Thompson, 2000) with two undergraduate students, Charlotte and Diana (all names are pseudonyms). They were chemistry majors recruited from a vector calculus class, and they participated in selection interviews, which indicated that they had not taken courses in discrete math, they were not familiar with combinatorial formulas, and they had no prior programming experience. Second, we share results from an individual interview with a computer science (CS) student, Allen. He was a CS major recruited from an introductory class in computer science. He indicated on a recruitment survey that he had not taken a class in discrete mathematics and that he had programming experience. In both cases, the students sat at a computer and worked on combinatorial tasks, writing in a Python coding environment while the interviewer asked clarifying questions. Charlotte and Diana participated in 11 total interview sessions during which they solved a variety of counting problems. Allen participated in 3 total interview sessions during which he wrote programs to list all outcomes of counting problems. In the final interview (from which the data in this paper is taken), we asked him only to solve a counting problem, and we then prompted him to explain how he would verify his solution.

Data Analysis

The data are part of a project in which we explored students' combinatorial thinking and activity within a computational setting, and we were not explicitly targeting generalization in this project. However, as we reviewed data it became clear that students were engaging in generalizing activity, and we wanted to examine that activity more systematically. For analysis, we surveyed both sets of data for instances that illustrated each form of generalization in the R-F-E framework. We particularly sought examples that would highlight the role of the computer and ways in which it supported students' generalizing activity. We identified a number of episodes of relating and forming in our data sets, and we chose the two episodes discussed in this paper as representative examples. Together we discussed additional generalizing in our data, and we articulated ways in which the computer in particular facilitated generalizing activity, which we elaborate in the Results and the Discussion and Implications sections.

Results

We provide two examples of students engaging in generalizing activity, and we focus especially on *relating* and *forming* within the R-F-E framework. We do not provide data related to extending in part due to space, and also because we hypothesize that the computer is particularly useful in supporting relating and forming, and students can then extend ideas and relationships by hand. We elaborate this point in the Discussion Implications section.

Relating

We demonstrate one particular sub-category of relating that Ellis et al. (2017) described: *relating objects*, which involves forming a relationship of similarity between two or more present mathematical objects. We demonstrate an instance of relating objects in which Charlotte and Diana connected back to work on a prior problem they had done (both sets of students engaged in more relating, but we do not have space to offer additional examples). We highlight the tendency of students to copy and paste, then edit, code from prior problems (we call this repurposing previous code), which we feel is a feature of the computer that particularly supported relating. On the one hand, repurposing code could seem just like practical, time-saving technique, but we argue that this is actually important for generalization for a couple of reasons. First, the code itself gives students a new aspect of the problem to which and from which they can relate. Because the code can be seen as encapsulating and representing a counting process, students can identify similarities between the representation of code on various problems. Our students drew on similar structures and features of code as they copied and pasted work from prior problems. For instance, at one point, Diana asked Charlotte, "Do you want me to copy this code from over here [a previous problem], since it's really similar?" This suggests that she perceived similarity between code they had written in the past and a current situation. Notably, the computer specifically facilitates that similarity by allowing such repurposing easily to occur. With very little effort, students get to duplicate and then adjust something they did previously. Such adjustments could be done by hand, but there is something about editing and adjusting real time that allows for efficiency. We also hypothesize that because copying and pasting reduces students' work load, it may incentivize their looking for similarity between solutions, thus encouraging generalizing activity.

As an example of relating, we offer an instance of Charlotte and Diana repurposing code. They had previously worked on a problem about enumerating people, *How many ways are there to rearrange 5 people: John, Craig, Brian, Angel, and Dan?*, reasoning about code in Figure 1a. This code counts arrangements of five people – the nested loops cycle iteratively through each element in the set People, and != (not equal) prevents elements from being repeated (more details about such problems are in Lockwood & De Chenne, 2019). We later asked: *Write some code to list and count the number of ways to arrange the letters in the word PHONE. How many outcomes are there? What do you think the output will look like?* As they started this problem, they had the following exchange.

```
arrangements = 0
People = ['John', 'Craig', 'Brian', 'Angel', 'Dan']

for p1 in People:
    for p2 in People:
        if p2 != p1:
            for p3 in People:
                if p3 != p1 and p3 != p2:
                    for p4 in People:
                        if p4 != p3 and p4 != p2 and p4 != p1:
                            for p5 in People:
                                if p5!=p4 and p5!=p3 and p5!=p2 and p5!=p1:
                                    arrangements = arrangements+1
                                    print(p1, p2, p3, p4, p5)
print(arrangements)
```

```
arrangements = 0
People = [P, H, O, N, E]

for p1 in People:
    for p2 in People:
        if p2 != p1:
            for p3 in People:
                if p3 != p1 and p3 != p2:
                    for p4 in People:
                        if p4 != p3 and p4 != p2 and p4 != p1:
                            for p5 in People:
                                if p5!=p4 and p5!=p3 and p5!=p2 and p5!=p1:
                                    arrangements = arrangements+1
                                    print(p1, p2, p3, p4, p5)
print(arrangements)
```

Figure 1a, 1b: Code for the People and PHONE problems

Charlotte: Okay. So, I feel like, yeah, basically just gonna be the same as this one because, I mean PHONE has five letters and this had five people. So, I feel like we can maybe just copy the same code.

Diana: Yeah, and then like edit it to be like PHONE.

The interviewer then asked her why their idea would work, and Diana said the following:

Diana: It works because there's like the same number of things that you're arranging. So, like you're arranging people here, there's five of them and then you're arranging letters here and there's five of them. And it's still the same as like you're not repeating any letter, so you keep the not equal to expressions. And, yeah.

Diana's comments highlight what she perceived as similar about the situation – they were still arranging five objects, and they still did not want to be able to repeat any object. So, she noted that they still wanted to maintain the same fundamental features of the code, which was the "not equal to expressions." They edited the code from Figure 1a just to have the letters P, H, O, N, and E instead of the people (Figure 1b). They ran the code, which correctly printed all 120 arrangements of letters in the word PHONE. The fact that they left much of their code the same as the People problem is noteworthy, as it suggests that they were attuned to the fact that some features of the code were or were not essential to solving the new problem. Notably, they did not change what they perceived as features that would not change their process or output (for instance, they did not re-name the set People, and they kept and p1 through p5 as their variables). That is, they recognized that the underlying structure was the same between the problems, but some other features, like the names of the set and variables, did not matter. This gives insight into what they deemed as relevant similarities or differences among the two problems.

To summarize, the computational setting afforded students with opportunities to repurpose code, and by doing so they related current situations to prior work. The computational representation of the code signified a particular counting process, and the computer's capacity to allow such code to be repurposed facilitated the students' generalizing activity of making connections among problems. We demonstrated one instance of this, but there were multiple examples throughout the teaching experiments of such activity. In this way, this example of relating within the computational setting lets us see one way in which computational settings could afford some unique opportunities for engagement in the practice of generalization.

Forming

Next, we offer an instance of forming. Ellis et al. (2017) distinguish between types of forming, and we focus on *searching for similarity or regularity* (searching to find a stable pattern, regularity, or element of similarity across cases, numbers, or figures), and *identifying a regularity* (identification of a regularity or pattern across cases, numbers, or figures). We share Allen's work on the Books problem, which states *Suppose you have 8 books and you want to take three of them with you on vacation. How many ways are there to do this?* Allen originally solved this problem incorrectly by finding the number of ways to arrange three of the eight books, 8*7*6 = P(8,3), rather than selecting three of the eight books, 8*7*6/6 = C(8,3). (We will refer to this correct answer as C(8,3), even though Allen did not yet know a closed form for it, and he could only find the value on the computer using his program.) After writing code that listed the outcomes of arranging the books, he noticed that outcome 132 appeared in the output after the outcome 123, and he realized that he had not been correct, stating "that would not be a good combo because it already appeared up here; it's just in a different order." He thus saw he needed to correct his solution, and to do so Allen wrote the code in Figure 2. When run, this code prints all three number combinations in ascending order, thus ensuring that each combination is printed exactly once, and it correctly outputs 56 as the total number of combinations.

```
count = 0

for i in range(1,9):
    for j in range(i+1,9):
        for k in range(j+1,9):
            count = count + 1
            print(str(i)+" "+str(j)+" "+str(k))
print count
```

Figure 2: Allen's code for the Books problem

While Allen's code correctly counted the number of outcomes of the Books problem, he did not initially offer any justification for a counting process or mathematical expression. However, he remarked that it was interesting that 56 = 8*7, which was his original answer of 8*7*6 divided by 6. So, he realized that a ratio of his answer over the correct answer was 6 (that is, P(8,3)/C(8,3) = 6). He wondered what would happen if the problem were selecting from only 7 books instead of 8. After extending his original (and incorrect) solution to seven books, P(7,3) = 7*6*5, he predicted that the ratio of P(7,3)/C(7,3) would be 5. (One potential rationale for this prediction is that he reduced the total number of books by 1 (from 8 to 7), and so he reduced his prediction by 1). Allen then decided to use the computer and his computational experience to explore these relationships more systematically. First, he adjusted the code from Figure 2 to count the number of outcomes for 7 books (rather than 8), yielding 35, and he computed the ratio of P(7,3)/C(7,3), yielding 6 (which contradicted his prediction of 5). Allen then adjusted his code to allow for him to explore more examples efficiently. In particular, he created a function that would let him explore multiple numbers of books within a range, from which he always selected 3 books. For each number of books, the program computed the ratio of his estimated guess (P(n,3) with the actual correct value that he found computationally (which is C(n,3)). Allen's code is displayed in Figure 3, and the output shows verified that the ratio P(n,3)/C(n,3) was 6 in each case (again, we write P(n,3)/C(n,3) for clarity, as Allen thought about this ratio as his original solution divided by the actual solution). We view Allen's initial exploration and creation of this function activity as an instance of forming, namely *searching for similarity or regularity*, as he was making predictions and looking for and expecting to observe patterns. Then, when he ultimately determined that the ratio was 6, we take this as an instance of *identifying a regularity*.

```
In [23]: def fun(book):
             count = 0
             for i in range(1,book+1):
                 for j in range(i+1,book+1):
                     for k in range(j+1,book+1):
                         count = count + 1
                         #print(str(i)+" "+str(
             return (book*(book-1)*(book-2))/cou

In [24]: for i in range(4,21):
             print fun(i)

         6
         6
         6
         6
         6
         6
```

Figure 3: Allen's identification of regularity via programming a function

Allen then decided to extend his work by changing the number of books being selected in his code; that is, rather than selecting three books, he wrote code to select four, five, and then six books. It is noteworthy that Allen had not yet provided justification for the constant 6, and so the decision to extend his work seems to stem from his desire to observe a pattern (thus we view this as an instance of forming). Further, the computer facilitated this forming activity by allowing him to adjust his previous code so other numbers of books were selected. For each of these new instances, he divided the answer from his original solution method by the actual answer, and he observed a constancy in each case. Essentially, Allen fixed an m and used his code to calculate $P(n,m)/C(n,m) = m!$ as n ranged for values of from $m + 1$ to 21, (again, he expressed this ratio as his original solution divided by the actual solution). The constant he found in each case represented the number of ways to arrange the books after the books have been selected. After finding values in cases three through six, he remarked "Okay, I think I found a really high-level relationship that is several layers." We asked him to elaborate, and Allen stated the following.

> Allen: So, this is the number of books. This is three books, four books, five books, six books. So, if that's the case, then with two books it should be 3. Three would be 6, four would be 24, five would be 120, and six would be 720… what I noticed is each time you go up, you multiply by the next number. So, 6 times 4 equals 24, which multiplied by 5 equals 120, which multiplied by 6 equals 720.

Allen went on to observe that "these are all factorials." Using this information, he constructed and justified a closed form for $C(n,m)$ (we do not include analysis of this data due to space).

To summarize this episode, Allen used his code to find a constant ratio between his (incorrect) original solution and the correct solution he computed. He then identified a pattern between these constants as the number of books being selected was increased. We observed *searching for similarity or regularity* when Allen identified the constant 6 in his work on selecting three books. Then, we observed *identifying a regularity* when Allen found a pattern among constants as the number of books selected increased. We argue that the computer was fundamental in this process, as Allen generated these constants by writing and implementing code. The computer, and the outcomes generated, seemed to afford Allen the opportunity to search for patterns and identify relationships. In Allen's case, he used the computer in two important but different ways. First, he used the computer to generate answers to problems he could not yet solve by hand (computing the correct number of combinations before he knew the closed form of $C(m,n)$). Second, he wrote a function to generate multiple cases, which allowed him to search for regularity in multiple cases efficiently. This episode thus sheds light on how the computational setting supported Allen in the specific generalizing activity of forming.

Discussion and Implications

In this paper, we have offered instances of students engaging in generalizing activities of relating and forming (in terms of Ellis et al.'s (2017) R-F-E framework) within the context of programming in Python. We have tried to make the case that in these cases the computer offered specific affordances for generalizing activity. These included copying and pasting to support relating, and generating correct solutions to multiple problems in order to support forming. While some of these activities would technically be possible by hand, the manipulation that the computer allows seemed to expediate this practice of generalization for the students. As an additional note, we mostly observed the computer being used to support relating and forming, and extending that we observed among students tended to be done by hand (students often extended formulas or expressions they had written by hand). We thus hypothesize that the computer may be most effective for supporting relating and forming, which then contribute to extending. More work is needed to explore whether and how extending arises explicitly via computational activity.

There is more to study specifically about the role of the computer (and specifically machine-based computing) in facilitating students' generalizing activity. We have focused on combinatorics, but such computing may elicit generalizing in other ways in other domains. Researchers could explore ways that the computer might facilitate other kinds of generalizing activity in other domains or in other kinds of problems. Further, we have focused on one perspective of generalization, drawing explicitly on Ellis et al.'s (2017) framework, but researchers could consider other possible framings of generalization to consider the computer's role in supporting students' generalization. In addition, our results demonstrate instances in which students engage in the practice of generalization. However, there are many other practices, and Lockwood et al. (2019) have suggested that other practices like proving or problem solving might closely be related to the kind of machine-based computing described in this paper. Thus, future research could be conducted on ways in which computing might support other mathematical practices.

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MIDDLE SCHOOL STUDENTS' DEVELOPMENT OF AN UNDERSTANDING OF THE CONCEPT OF FUNCTION

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Middle school students (n=144) worked with an applet specially designed to introduce the concept of function without using algebraic representations. The purpose of the study was to examine whether the applet would help students to understand function as a relationship between a set of inputs and a set of outputs and to begin to develop a definition of function based on that relationship. Results indicate that, by focusing on consistency of the outputs the students, at a rate of approximately 80%, are able to distinguish functions from non-functions. Also, students showed some promise in recognising constant functions as functions, a known area of common misconceptions.

Keywords: Middle School Education, Technology, Representations and Visualization

Introduction

The concept of function is considered to be one of the most important underlying and unifying concepts of mathematics (e.g., Leinhardt, Zaslavsky, & Stein, 1990; Thompson & Carlson, 2017). Students have experiences with functions, or function behaviour, from the very earliest grades usually through pattern exploration. Study of functions continues up to and through high school with a formal treatment of functions as arbitrary mappings between sets. Indeed, in the Common Core State Standards for Mathematics function is given its own domain in grades 9-12 (Common Core State Standards Initiative, 2010).

Much of the lack of depth of knowledge of the concept can be attributed to the privileging of algebraic representations (function as algebraic rule) or graphical representations (function as graph that passes the vertical line test) and a consequent lack of focus on the general relationship (see e.g. Best & Bikner-Ahasbahs, 2012; Breidenbach et al., 1992; Carlson, 1998; Thompson, 1994). What might a group of students who have never encountered the concept of function learn by encountering it in a novel representation? Can they learn to think of a function as a relationship between inputs and outputs with some rules about the outputs rather than something that is defined by an algebraic rule? These are the questions that guided the current study.

Related Literature

Prior to secondary school, opportunities for study of functions are limited in scope (Best & Bikner-Ahasbahs, 2012; Carlson & Oehrtman, 2005; Vinner & Dreyfus, 1989) and focus mainly on pattern recognition and study of covarying quantities, most often related to an underlying linear structure (Blanton et al., 2015; Stephens et al. 2017, Ellis, 2011). For example, in Blanton et al. (2015) 6th grade students are given the tasks "People and Ears: The relationship between the number of people and the total number of ears on the people (assuming each person has two ears)" (p.520) to study the function type $y = x + x$ and "Age Difference: If Janice is 2 years younger than Keisha, the relationship between Keisha's age and Janice's age (Carraher et al., 2006)." (p. 521) to study the function type $y = x + 2$. In other words, the functional relationships typically encountered in elementary and middle school years are designed to prepare the (mathematical) ground for studying linear relationships (y = mx, y = x + b, y = mx + b) i.e. the privileging of algebraic representations begins early in the study of functions. Leinhardt et al. (1990), in a meta-study of research on function, and Mesa (2004), in a study of 24 middle grades textbooks from 15 countries, note the difficulty for students in apprehending the modern, abstract definition of function depending, as it does, on the mapping of one set of elements to another emphasising the difference between function and relation (many-to-one acceptable, one-to-many not acceptable); whereas, the work on function in early grades builds on the intuitive notion of a 1-1 correspondence and the historical development of function rested on covarying quantities.

Even in secondary school functions are typically introduced as very limited classes such as linear and quadratic, with attendant graphs and tables, with the result that students regularly consider functions to be mathematics objects solely defined by an algebraic formula (e.g., Best & Bikner-Ahasbahs, 2012; Breidenbach et al., 1992; Carlson, 1998) and have difficulty identifying constant functions as functions (Bakar & Tall, 1991; Carlson, 1998; Rasmussen, 2000). Instruction and curricular materials often emphasize procedures and algebraic manipulations when studying functions and research shows that students then have difficulty in understanding different representations and different contexts for functions (Carlson & Oehrtman, 2005; Cooney et al., 2010). At the heart of many student difficulties is a shallow understanding of the definition (Ayalon et al., 2017; Panaoura, et al., 2017). Students who have an algebraic view of function and who use procedural techniques to identify functions and non-functions struggle to comprehend a general mapping between sets (Carlson, 1998; Thompson, 1994).

Exposure to, and facility with, various representations of functions, i.e "flexible use of functions . . . within and between all kinds of representations and also between different functions" (Best & Bikner-Ahasbahs, 2012, p. 877), has been shown to be a critical component of a rich understanding of function (Best & Bikner-Ahasbahs, 2012; Dubinsky & Wilson, 2013; Martinez-Plandi & Tigueros Galsman, 2012). Furthermore, researchers have found promising results when using novel contexts and non-standard representations of functions such as dynagraphs, arrow diagrams, and directed graphs (Dubinsky & Wilson, 2013; Sinclair, Healy & Sales, 2009). The purpose of this study is to examine the effect of a specially designed applet on middle school students' ability to develop an understanding of the concept of function.

Methods

Context

Previous research (Meagher et al., 2019) has shown the promise of a vending machine representation as a "cognitive root" (Tall, McGowen, & DeMarois, 2000) for the study of functions. Thus, we designed an applet, *Introduction to Function,* (https://tinyurl.com/y2dramsb) as a mechanism for learners who have never encountered the concept of mathematical function and, therefore, do not associate the concept with any particular representation, to learn the basic elements of function. The goal was for the students to learn that a function is a relationship between of a set inputs that are matched with a set of outputs in a consistent and, therefore, predictable manner.

The *Introduction to Function* task is a GeoGebra book that consists of seven pages and has an accompanying worksheet. On the first two pages are two vending machines each of which consists of four buttons (Red Cola, Diet Blue, Silver Mist, and Green Dew). When a button is clicked it produces none, one, or more than one of the four different colored cans (red, blue, silver, and green), which may or may not correspond to the color of the button pressed (see Figure 1). The students are told that the first machine on each page is an example of something called a function, and the other is not a function, with their task being to identify what is the difference between the behaviour of the machines that makes one a function and the other not.

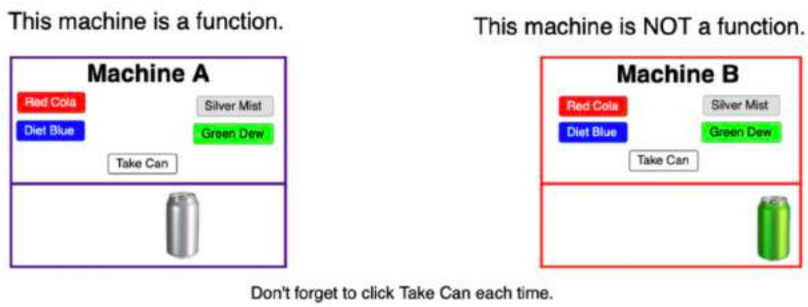

Figure 1: Screenshot of Introduction to Function

The machines on the first two pages work as follows:

This One is a Function		This One is Not a Function	
A	Red – Red Blue – Blue Silver – Silver Green – Green	B	Red – Red Blue – Blue Silver – Random Green – Green
This One is a Function		**This One is Not a Function**	
C	Red – Blue Blue – Red Silver – Silver Green – Green	D	Red – Red Blue – Random Silver – Silver Green – Green

Figure 2: Machines A – D

Note that Machines B and D are not functions because one of the buttons, when clicked, will produce a random can (i.e. not always the same result). Note also that in Machine C the colour of the output can does not correspond to the input button pressed, but that the non-matching can is consistently produced. After the first two pages there was a whole group discussion in which students discussed the first two pages, with the goal of consolidating their ideas.

The next four pages of the GeoGebra book consist of pairs of machines with the students being told that one of each pair is a function In each case there is a random element in the non-function. The machines work as follows:

	Which One is a Function?			
Page 3	E	Red Cola – red Diet Blue – blue Silver Mist – silver Green Dew – random color	F	Red Cola – silver Diet Blue – green Silver Mist – red Green Dew – blue
	Which One is a Function?			
Page 4	G	Red Cola – random color Diet Blue – random color Silver Mist – random color Green Dew – random color	H	Red Cola– blue Diet Blue – silver Silver Mist – green Green Dew – red
	Which One is a Function?			
Page 5	I	Red Cola – 2 silver cans Diet Blue – green Silver Mist – red Green Dew – blue	J	Red Cola – red Diet Blue – blue & random color Silver Mist – silver Green Dew – green
	Which One is a Function?			
Page 6	K	Red Cola – pair of random color Diet Blue – blue Silver Mist – silver Green Dew – green	L	Red Cola – green Diet Blue – green Silver Mist – green Green Dew – green

Figure 3: Machines E - L

On the worksheet, students are asked to note whether each machine is a function or not a function and how they know. After they complete these pages students are given the prompt: "Using the terms 'input' and 'output' write a definition for function based on your exploration of the machines."

Participants

The *Introduction to Function* applet was used in fifteen seventh grade classrooms. These classrooms were across two different states (one Northeastern state and one Southeastern state) and five different teachers for a total of 144 students who engaged with the task. These students engaged with the applet towards the end of their seventh grade year and had not yet learned about the definition of function or function notation.

Data collection and analysis

Students worked in pairs (N = 72) to engage with the applet on a laptop that screen captured their work. Data collected were their worksheets, which include their definitions, screen recordings, and audio recordings. For this study our analysis focused on the students' worksheets. All data was coded by three researchers. Any disagreements were discussed until any discrepancies were resolved.

For the definitions we coded for use of the terms input/output, attention to output, and focus (Author et al., 2019). In terms of input/output, each definition was read for use of those terms in the definition for example, "M49_M62: No matter what input the output is the same" and "M117_M118: A function is when you get the same output." In terms of focus, each definition was coded regarding whether the definition indicated a function was a relationship (or mapping), an object, or neither. We referred to this set of codes as focus, as they indicated how the students "saw" function. If the definition indicated that the function relates to the input and output then the definition was coded as a relationship. For example, "VM_M91_M96 The word function may mean when you input something, even though you may not get what you asked for, you will only get one type of it." The code "object" was used when the definition referred to a function as something, such as the button, or the machine.

Finally, definitions were coded according to whether or not they attended to output. In order for a definition to be coded as attending to output, the definition needed to refer to an output having a pattern, or being the same or consistent. For example, "VM_M54_M59: Function is when you put in the input and the output will never change / will always be the same."

Analysis of the student worksheets proceeded along two dimensions: classification of whether the pairs of students correctly identified the machines E through L as functions and the students' justifications for their classifications. For the pairs of machines E&F, G&H, I&J, K&L, since students were told one was a function and one was not, it was possible to simply count the classification. Of course, the percentages should mirror each other i.e. the number of "corrects" for machine E should match the number of "incorrects" for machine F.

The students' written justifications for their machine classifications were open coded using a constant comparative method to look for themes (Creswell, 2014). The final codes for students' justifications are shown in Figure 4. Justification codes were not mutually exclusive, as a justification could have been coded based on inconsistency as well as using the context of the vending machines.

Code	Description
Justification based on inconsistency of output	Students' justifications use phrases indicating an attention to the inconsistency of the outputs. Examples include: "different colors", "random", "it changes".
Justification based on consistency of output	Students' justifications use phrases indicating an attention to the consistency of the outputs. Examples include: "consistent", "constant", "pattern".
Justification uses the context of the machine	Students' justifications describe the relationships between inputs and outputs using the vending machine context. Examples include: "because it always gives the wrong drink", "it gives random colored cans"
Justification unclear	There is not enough detail in the students' written response to classify their justification.

Figure 4: Justification codes

Results

Identification of the Machines.

The first element of analysis was to tally whether the participants were able to correctly identify which of the machines E-L are functions. Recall that participants worked through machine pairs A&B and C&D being told that A is a function and B is not a function and that C is a function and D is not a function, and that the concept established was that the machine should behave consistently even if the colour of the output can does not match the colour of the button pressed. Students classification of the machines is shown in Table 1.

Machines	Non-function reason	% Correct
E & F	Machine E: Green Dew has random output	81.3
G & H	Machine G: all outputs are random	95.8
I & J	Machine J: Diet Blue output is Blue & random	86.1
K & L	Machine K: Red Cola output is 2 random cans	80.7

Figure 5: Participants' correct identification of functions

At a first level of analysis this shows that, broadly speaking, the pairs of students were able to correctly identify which machines were functions. The percentage of correctly identified functions for the first four pairs of machines was at least 80% and ranged from 80.7% to 95.8%.

It is interesting to note that for the pairs E&F, I&J and K&L the correct percentage is very similar (between 80% and 86%). The exception is the machine G&H pairing which has a much higher percentage of students identifying it correctly. This can be explained as follows: the primary identifying factor for a machine not being a function was the random behaviour of one of the buttons. However, one has to press a button often enough to be able to identify the behaviour as random. In the case of Machine G, all four buttons give random output and, therefore, the threshold to identify random behaviour is lower. Furthermore, Machine G comes first and, therefore, students can very quickly identify Machine G as not a function and not concern themselves too much with Machine H.

Looking more closely at the incorrect answers for the first four pairs of machines we see that it is often the same pairs of students getting incorrect answers. 10 of the 14 (71.4%) pairs of students who made a misidentification of the E&F pair misidentified at least one other machine, with 5 pairs misidentifying all of the first four sets of machines except the G&H pairing. Furthermore, of the 22 pairs of students that misidentified at least one machine, only seven of the 22 (31.8%) had their first wrong answer after the first pair of machines E&F and six of those seven misidentified just one of the pairs E&F, G&H, I&J and K&L.

The result for Machine L with 80.0% of participants identifying it as a function is a potentially significant result since researchers have shown that students exhibit difficulties identifying constant functions as functions (e.g. Carlson, 1998; Rasmussen, 2000). However, it may be that many students identified Machine K (output from Red Soda is two random cans) as not a function and concluded that Machine L must be a function.

Characterizing Students' Justification of Functions and non-Functions.

To better understand the ways in which students were making sense of the machines, we analyzed their justification for whether or not each machine was a function or non-function (see Figure 5). Those that were determined to be functions were justified based on consistency of the input/output relationship and those determined to be non-functions were described as such based on the inconsistency of this relationship. One notable exception to this is the 11 students that used the language of inconsistency to justify their choices for Machine F (Red Cola → silver, Diet Blue → green, Silver Mist → red, Green Dew → blue). All 11 of the students that described this as inconsistent, also determined the Machine was not a function. We see that these students could not overcome the cognitive dissonance of a machine giving them a different colour output can from the input button pressed, even if it did so consistently. For example, one student (M90) described Machine E (R→r, B→b, S→s, G→ random) as "more consistent" than Machine F (R→s, B→g, S→r, G→b) which "randomizes things." The very next pair of Machines in the applet had a similar design (Machine H: R→b, B→s, S→g, G→ r), and only one student determined this to be a non-function using the reasoning of inconsistency. This suggests that the students refined their meaning for such a justification to be aligned with situations in which a single output results in different outputs. Examples of students' justifications based on inconsistency are shown in Figure 7 below.

Machine	Justification based on consistency	Justification based on inconsistency	Justification unclear
E	5	54	13
F	45	11	17
G	2	68	2
H	62	1	8
I	54	6	12
J	2	59	9
K	4	54	11
L	58	1	10
M	38	2	15
N	34	1	14

Figure 5: Characterizations of students' justifications for each machine

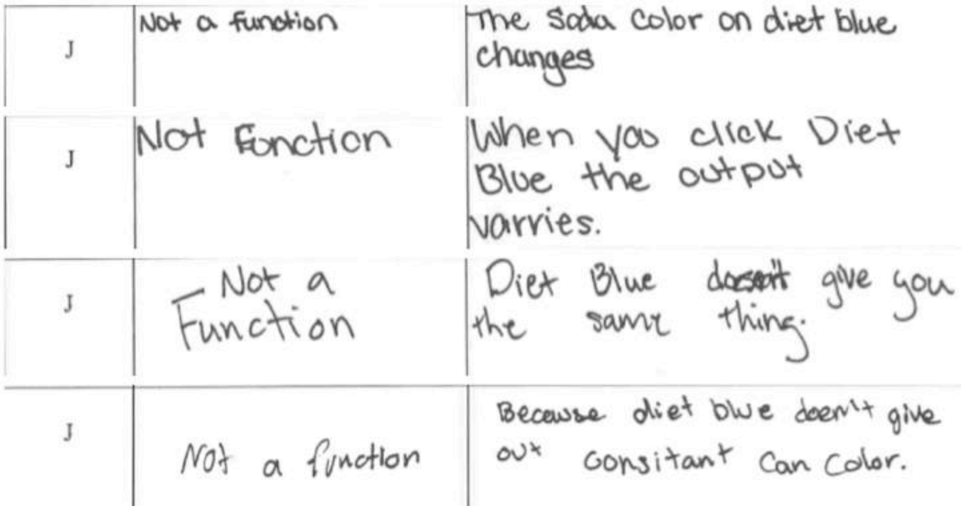

Figure 7: Examples of justifications based on attention to inconsistency of outputs

As is evident in the Machine F example above, the students' justifications provide insight to their misidentification of both functions and non-functions. For example, looking at the 13 pairs of students that misidentified Machine K (R→random pair) as a function it is evident that they either did not test the machine enough to see the random outputs that occurred when clicking Red Cola (e.g., "every color is functional, red produces 2 greens"), or they decided that since the rest of the buttons were consistent it was "close enough". For example, one pair wrote "mostly consistent" and another wrote "3 of the 4 function correctly." Furthermore, the inability to accept machines giving a different output from the button pressed, even if it does so consistently, persisted for a number of pairs. For example, Pair M17 & M20 said of machine J (R→r, B→b & random, S→s, G→g "The Blue one gives two but the others work."

It is notable that 80% of the student pairs used the language of the machine context in their justifications (see Figure 8 for examples). This suggests that having a realistic context in which to both think about and test their conjectures proved to be helpful in explaining their thinking.

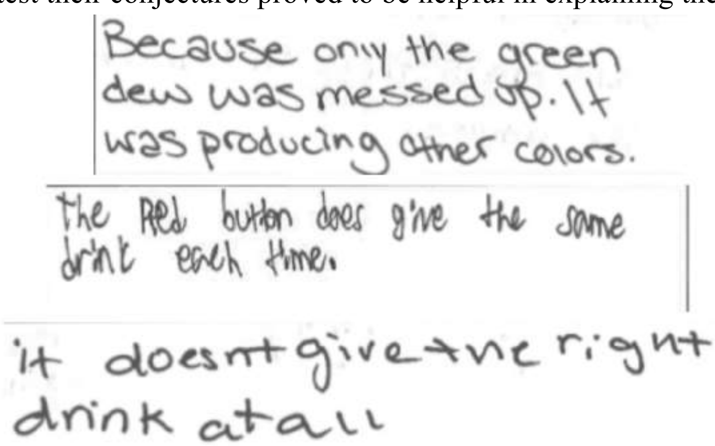

Figure 8: Examples of justifications that use the context of a vending machine

Definitions

One of the 72 pairs of students did not complete a definition on their worksheet. The remaining 71 definitions were coded using the codebook. In terms of the use of input/output 62 out of 71 (87.3%) definitions used the word input and 65 out of 73 (89.0%) definitions used the term output. Of course, the participants were asked to use these terms and, therefore, the result is not entirely surprising. Nevertheless, the result is promising in terms of establishing sets of inputs and outputs as a central aspect of the definition of function.

Perhaps the most interesting aspect of the activity was to examine the extent to which the participants would pay due attention to the outputs from the machines. Analysis of the definitions shows that 45/71 (61.6%) of the participants did pay attention to the output with definitions such as "When you input something, the output always will stay the same." However, 14/71 (19.7%) of participants, while paying attention to the output made an incorrect statement such as "Your input is your output and does not change."

In terms of focus, none of the participants described a relationship between inputs and outputs explicitly as a mapping between sets, and most definitions (43/71 (60.6%)) were coded as "neither object or relationship." A large number of participants' definitions (27/71 (38.0%)) were coded as "object" since they made explicit reference to the vending machine or the buttons of the machine. For example, "Whenever you input into the vending machine, you know the output which makes it reliable."

Conclusion

The purpose of this study was to explore whether seventh grade students who had not encountered the term function could use a specially designed applet to develop an understanding of a function as a relationship between inputs and outputs with some restrictions on the outputs. The non-standard representation of the *Introduction to Function* applet served to introduce the concept of function without algebraic representations. With the focus on the consistency, or otherwise, of the outputs the participants were able to correctly distinguish between functions and non-functions at least 80% of time. Some limitations of the study may be that the results were overdetermined by the discussion after the first two pairs of machines and that the participants might be seen to be simply playing a pattern recognition "game" with the rule "random bad, not random good." Therefore, more study would be needed to establish if the basic concept learned here transfers effectively to further study of function. However, even within this study, more than 60% used some appropriate language to describe the nature of the output in their definitions of function. In addition, contrary to a well-known misconception, participants may be able to recognise a constant function as a function.

Acknowledgments

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TECHNOLOGY:

BRIEF RESEARCH REPORTS

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DEVELOPING TPACK FOR MAKERSPACES TO SUPPORT MATHEMATICS TEACHING AND LEARNING

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In this project, we investigate how teachers develop the skills and knowledge to integrate makerspace technologies into mathematics lessons. Makerspaces are physical spaces that encourage creative design that often include emerging technologies such as 3D fabrication, coding, and robotics, and are being increasingly used to enhance mathematics instruction. Research suggests that for teachers to integrate any new technology into instruction, they must develop a specialized technological pedagogical content knowledge (TPACK), but little is known about how teachers develop TPACK for makerspace technology. We present emerging findings investigating how practicing mathematics teachers developed TPACK for makerspaces during a graduate technology course. Results suggest that despite similar experiences in the course, teachers varied significantly in their development of TPACK and integration of technology.

Keywords: STEM/STEAM; Teacher Education – Inservice/Professional Development; Teacher Knowledge; Technology

The issue of how teachers can integrate technology into their instruction in order to improve student learning of mathematics has been a focus of research for decades (Weglinsky, 1998), but continues to be a source of new questions (Cullen, Hertel, & Nickels, 2020). As new technologies are developed, researchers continue to wonder how these new technologies might impact students' mathematical thinking and learning. One emerging category of technologies which have the potential to transform student learning are those that are found in makerspaces.

Broadly considered, a makerspace is a physical space equipped with materials and technologies to encourage creative design (Cavalcanti, 2013). Some technologies currently found in makerspaces include 3D printers and other digital fabrication tools, robotics, microcontrollers (e.g., Arduino), as well as craft and circuitry tools. In this project, we investigate how teachers can develop technological and pedagogical content knowledge of makerspaces. This work looks "across cultures" as we investigate whether teachers can successfully integrate the "playful, growth- and asset-oriented, failure-positive, and collaborative" culture of makerspaces (Martin, 2015) into the context of their mathematics classrooms.

Introduction and Literature Review

Although makerspaces are relatively new, there is an emerging body of knowledge which suggests that they can be effective in improving student learning of mathematics. The use of makerspaces in mathematics instruction is informed by the cognitive theory of constructionism (Papert, 1980), which proposes that learning occurs by "actively constructing knowledge through the act of making something shareable" (Martinez & Stager, 2013, p. 21). Digital fabrication tools can expand constructionism to the creation of physical items. For example, students in a calculus class used 3D printers to create solids of revolution to create visual representations of integration (Propelka & Langlois, 2018). Some research has also suggested that makerspaces can be useful in developing

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teachers' pedagogical content knowledge (Corum & Garofalo, 2019; Greenstein, Fernandez, & Davidson, 2019). However, little is currently known about how teachers can build on these experiences in makerspace to inform their own instruction.

A long-standing body of research suggests that teacher professional development is critical to the successful integration of technology (e.g., Weglinsky, 1998). Building on Shulman's (1986) description of pedagogical content knowledge, researchers have described an integrated technological pedagogical content knowledge (TPACK) which combines expertise in technology with understanding of how it can be purposefully used to enhance student thinking of content ideas (Koehler & Mishra, 2009). Previous research suggests that professional development can be effective in developing teachers' TPACK (e.g., Bos, 2011), but that this specialized knowledge can develop in uneven or unexpected ways (Polly, 2011). In particular, Niess et al. (2009) proposed a set of developmental levels for TPACK to describe teachers' integration of a new technology into their mathematics instruction (Table 1).

Table 1. Developmental levels for mathematics teachers' TPACK (Niess et al., 2009)

Level	Teacher Knowledge and Technology Integration
Recognizing	Teachers can use a technology, but cannot yet integrate it into teaching
Accepting	Teachers see benefits of a technology and may use it for a teacher-led demonstration of a mathematical idea
Adapting	Teachers can include student use of technology in a surface or instrumental way to support previously-learned mathematics ideas
Exploring	Teachers can integrate a technology for effective learning of new mathematics
Advancing	Teachers can integrate technology to expand boundaries of students' mathematical practices

Niess et al. (2009) emphasize that teachers must go through these developmental stages separately for different technologies, and that particular features of each technology might impact teachers' learning. However, no research has specifically investigated the development of teachers' TPACK for makerspaces (which we refer to as MakerPACK). In order to address this gap in the literature, we investigated how professional development (in the form of a graduate-level, makerspace-augmented mathematics instructional technology course) can impact teachers' MakerPACK. In particular, we explored the following research question:

> How does practicing teachers' MakerPACK develop through their engagement with makerspaces, and to what extent are they able to use their MakerPACK to develop makerspace-augmented mathematics lessons?

Methodology

To understand how teachers' MakerPACK develops, we designed a makerspace-augmented mathematics instructional technology course. Our goal in course design and implementation was both to develop teachers' MakerPACK and to investigate that development, including how teachers demonstrated their MakerPACK through the creation of mathematics lessons.

Course Development and Structure

Informed by current trends in makerspace technologies, five modules were created to develop students' technological knowledge of emerging technologies. These technologies included paper circuits, 3D fabrication, coding, robotics, and microcontrollers. The primary instructional method was open-ended guided exploration to model best practices when integrating makerspace-augmented lessons into a classroom. Examples of guided explorations include determining the volume of an origami balloon using non-standard measurement tools, deriving Ohm's Law, using iterative programming to draw various polygons, and creating a binary counter. Each module included open make time for teachers to explore the technology on their own and to develop their own mathematics lesson to highlight how this work could be incorporated into their own classroom contexts. Two of the authors with extensive experience in makerspace technology and mathematics education were lead curriculum writers.

Participants and Implementation

The makerspace-augmented mathematics instructional technology course was offered at a large public university in the Mid-Atlantic region of the U.S. A total of eight graduate students, all of whom were experienced mathematics teachers, completed the course in the Fall 2019 semester, with seven students agreeing to participate in the study.

Data Collection and Analysis

In order to assess the extent to which the makerspace-augmented mathematics instructional technology course supported students' development of MakerPACK, we collected the "lesson concepts" students developed throughout the course. These lesson concepts consisted of an educational object using a specified technology, a description of how the technology could be used to teach a mathematics topic, and a reflection on the design process. We used a comparative case study approach to examine similarities and differences among teachers' development of MakerPACK. A sample of teachers' lesson concepts (each using coding to teach a mathematical idea) were analyzed using the components of the "Mathematics Teacher Development Model" as described by Niess et al. (2009). Two of the authors assessed the lesson concepts independently and then compared their assessments. When the authors' individually assessments were not aligned, they reviewed the lesson concept together in order to come to a consensus. Three lesson concepts were purposefully selected to illustrate the different levels of MakerPACK as observed during the makerspace-augmented mathematics instructional technology course.

Results and Discussion

Data analysis of the codes from the "Mathematics Teacher Development Model" (Niess et al., 2009) revealed that these three participants varied significantly in their MakerPACK development. None of these teachers had prior experience with makerspaces, and all three had similar experiences in the makerspace-augmented mathematics instructional technology course, yet their lesson concepts revealed quite different views and uses of technology. Looking across the components, we noticed three distinct profiles of MakerPACK: Accepting (Jenna), Exploring (Kyle), and Advancing (Lauren).

Jenna: Accepting

Jenna created a Scratch animation and activity to demonstrate geometric transformations for use in an eighth grade class. Her lesson included tightly teacher-directed instructions and little student autonomy. Jenna struggled with identifying an application for coding within her curriculum, and she expressed concern that the use of technology would divert students' attention from learning mathematics. When Jenna encountered technical difficulties, she changed the content of her lesson rather than persevering to find a solution, and stated in her reflection, "I struggled to justify the amount of time and effort required to not make a lot of mathematical progress." Across multiple components, analysis revealed that Jenna was at an accepting level of MakerPACK since she was willing to use the technology in a teacher-centered lesson, but similar to participants described by Niess (2013), her "concerns overshadowed [her] enthusiasm for the use of [technology] in instruction" (p. 181).

Kyle: Exploring

Kyle created a Scratch animation and student project to learn about piecewise function for use in a precalculus class. His reflection expressed enthusiasm about a strong fit with his curriculum, and his project gave students multiple options and significant mathematical autonomy, using a rubric rather than specific instructions to provide guidance. We also note that Kyle's view of the challenges of using new technologies was different than Jenna's. Kyle identified his own difficulties in creating a Scratch animation that required him to use mathematics beyond the specified topic (e.g., converting, scaling), and he planned for how he would attend to these challenges when implementing the lesson concept with students. Data suggests that Kyle was at an exploring level of MakerPACK since he intended to give students autonomy in the classroom to explore new mathematical content; he "displayed indications of transforming [his] knowledge by more clearly integrating mathematics, pedagogy, and [technological] knowledge" (Niess, 2013, p. 188).

Lauren: Advancing

Lauren created a Python program and a programming experience related to the Pythagorean Theorem for use in an eighth grade class. Lauren intended to use technology to expand students' mathematical practices, as the technology provided motivation for determining a generalized solution method for determining the unknown side length of a right triangle. Lauren recognized that the value of incorporating technology into this lesson extended beyond the identified instructional goals. She reflected that her initial errors and the trouble-shooting process gave her additional interest and ownership of her program, writing, "I hope coding brings out the problem solvers in my students." Lauren's lesson concept suggests she was at the advancing level of MakerPACK, in that she used her integrated technological pedagogical content knowledge to "willingly explore and extend the mathematics curriculum" (Niess, 2013, p. 189).

These specific findings indicate that these three teachers' development of MakerPACK varied in terms of the value they perceived in using technology, the level of student autonomy in their lessons, and their response to technical difficulties in using the technology. These findings reflect the integrated nature of TPACK, aligning with previous research suggesting that technical expertise, pedagogical practices, and beliefs about technology are closely linked.

Conclusion

Despite these three teachers having similar experiences in the makerspace-augmented mathematics instructional technology course, our analysis of their lesson concepts revealed wide variation in their development of MakerPACK. We hypothesize that their development of MakerPACK was mediated by their beliefs about mathematics teaching and learning. In weekly reflections, Jenna often revealed frustration with the technology and a desire for more explicit direction. Kyle and Lauren, however, revealed a willingness to engage in productive struggle and a desire for mastery of the technology. This hypothesis aligns with research suggesting that development of TPACK is often mediated by teachers' beliefs (e.g., Smith, Kim, & McIntyre, 2016). Future research is needed to more closely understand the relationship between teachers' beliefs and TPACK, as well as how TPACK for makerspaces can develop.

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INTEGRATING DIGITAL RESOURCES TO THE DOCUMENTATION SYSTEM OF A MATHEMATICS TEACHER IN A MEXICAN RURAL PRIMARY-SCHOOL

INTEGRACIÓN DE RECURSOS DIGITALES AL SISTEMA DOCUMENTAL DE UNA PROFESORA DE MATEMÁTICAS EN UNA PRIMARIA RURAL DE MÉXICO

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Resources play an important role in how subjects act and think. Teachers, throughout their professional career, develop ways of teaching mathematics —articulated by the organization and type of activities that guide their class, the resources they use and their forms of intervention; these are modified when a new resource is integrated. In this paper we present evidence of this phenomenon using the case study of a teacher who, as a result of her participation in a professional development course that we implemented, integrated digital resources into her documentation work, destabilizing her previous forms of teaching.

Keywords: Elementary School Education, Technology, Teaching Tools and Resources, Teacher Education – In service / Professional Development

Introduction

Although, in this digital age, the integration of digital technological (DT) resources into the teaching of mathematics is a social and pedagogical necessity (Sunkel, Trucco & Espejo, 2014), there are deficiencies in some sectors of the Mexican basic education system (Enríquez & Sacristán, 2017, 2019). In fact, DTs are given very little importance in primary-school teacher training (as can be seen in the syllabus of the official primary-school teacher training programs –SEP, 2018). In this report we present part of a study that looks at how primary school teachers in a rural area of Mexico, and who took part in a professional development (PD) course, integrate digital resources into their mathematics teaching. The study uses the Documentational Approach to Didactics or DAD (Gueudet & Trouche, 2009, 2012), to investigate the following question: *When teachers integrate digital resources into their teaching practice, what changes occur in the design and implementation of their mathematics classes, that is, in their documentation system*[1]*?*

The Documentational Approach to Didactics (DAD)

The DAD studies teachers' practice and their professional development by looking at their interaction with the resources they use (select, adapt, review, reorganize) for teaching mathematics (Gueudet & Trouche, 2009, 2012). The resources they use, are integrated into a resource system (SR).The documentation work is the set of interactions of the teacher with their SR, "within processes where design and enacting are intertwined" (Gueudet & Trouche, 2012, p. 24) . This documentation work produces documents, which are composed of recombined resources and the usage schemes associated with them; a teacher's set of documents is their documentation system (Gueudet & Vandebrouck, 2011). Throughout a teacher's professional career, their documentation system evolves as they work and experiment with old and new resources. Thus, the teacher's documentation work reveals their professional development, that is, the evolution of their practice, knowledge and beliefs (Gueudet & Trouche, 2012).

[1] "Documentation system" is defined further below.

Methodology

This report is part of a larger research that has included: a diagnostic study to investigate teachers' previous training in the use of DT and their access to these resources; an intervention phase, where we implemented a PD course for the integration of digital resources for teaching mathematics, to teachers who had participated in the diagnostic study; and an inquiry phase into the documentation work of the participating teachers when integrating DT into their practice. In the study, 67 teachers from 10 primary schools in a rural region of the Mexican state of Oaxaca, participated in the diagnostic study; these teachers in their majority, had little training in the use of DT for teaching mathematics; their use of DT in their practice was scarce; and their access to digital equipment was limited –because schools lack hardware and school policies limit the use of what little is available (Enríquez & Sacristán, 2017, 2019). This information was considered, in the intervention phase, for the design of the PD course aimed at promoting the integration of DT for mathematics teaching. The 67 teachers who participated in the diagnostic study were invited to participate in the course and associated study; 15 of them accepted. The PD course lasted 5 months (6 hours per week). It was based on theoretical models on teacher knowledge (Shulman, 1986; Ball, et al., 2008, Thomas & Palmer, 2014). It was also based on the experiences of the EMAT program for teaching mathematics in middle schools (Sacristán and Rojano, 2009) and used some of its materials (the Logo software and accompanying didactic guidelines –Sacristán, 2005; Sacristán & Esparza, 2005).

The PD course. It consisted of 3 modules, each focused on studying a certain DT resource – respectively, miscellaneous applets and interactive apps, Geogebra and Logo. The tasks of each module were presented in 4 stages: (i) The study of the resource; (ii) the design of mathematics lesson plans integrating the digital resources studied in the course; (iii) the implementation of the designed classes; and, (iv) group reflections on the implementation experiences.

Data collection. In order to study the changes in the participants' documentation systems, we used: (i) initial interviews of the teachers, and subsequent ones after each implementation of their lessons using DT; (ii) the lesson plans designed by the teachers during the course; (iii) in-class observations of the implementations of the designed lessons; (iv) the teachers' presentations of their class experiences, which they shared, during the PD course, with their colleagues.

We now present data from a teacher, Nohelia, who was a participant in the study.

The case of Nohelia

Nohelia's profile. This teacher has a bachelor's degree in primary education, with a master's degree in the development of teaching competencies, and she has also taken various DP courses, including one in mathematics and computing. At the beginning of the study, she had 9 years of teaching experience, and was in charge of the 5th grade in a rural primary school; her school had a computer lab with 18 units, a portable video projector, and the old hardware from Enciclomedia (SEP, 2012) –a computer, an electronic whiteboard and another projector.

Changes in the documentation system (DS) of Nohelia

Based on the data from the initial interview with Nohelia, we determined her initial documentation system (DS) –the resources, and how she used them, for her mathematics lessons, before the study. This DS changed after she participated in the DP course, when she integrated the studied DT resources to her practice. Nohelia's initial DS included several documents —e.g., activities to introduce topics, or to strengthen students' knowledge (or to help her students overcome weaknesses)– each with specific resources. Her initial DS (see Table 1) structured her lessons in four stages: (1) a reviewing previous knowledge, (2) sharing knowledge with the whole group, (3) developing a new topic, and (4) assessment.

Table 1. The documentation systems (DS) of Nohelia

Stage	Activity	Aim	Resources	Teacher's actions
		Initial DS *(without DT)*		
1	Solve problems in teams	Assess and review previous knowledge	Official activities and guidelines; Printed materials. Internet. The children's settings. Stories.	Following the resolution processes
2	Share solutions (whole group)	Deal with weaknesses and reinforce knowledge	Children's solutions Videos and concrete materials Exercises and problems	Guides Sets exercises Uses resources Decides problems for M3
3	Solve problems individually	Promote an expected learning	Official activities and guidelines; Printed materials. Internet. The children's settings. Stories.	Guides Deals with difficulties Sets problems
4	Solve textbook tasks	Assess the learning	Textbook	Guides. Assesses. Closes theme.
		DS1: *"Capacity measures [of liquid containers]"* ***interactive app***		
1	Pose questions on liquid volumes (liter, milliliter)	Assess and review previous knowledge	Questions	Pose questions & assess answers
2	Solve in teams problems on capacity of liquid containers	Promote an expected learning	Bottles, pails. Worksheet.	Guides
3	Solve individually problems on capacity of liquid containers	Promote an expected learning	*"Capacity measures"* ***App.*** Official activities. Worksheet.	Guides Deals with difficulties
4	Solve textbook tasks	Assess the learning	Textbook	Guides. Assesses. Closes theme.
		DS2: *Heights of triangles with* ***GeoGebra***		
1	Build triangles in teams (to describe characteristics)	Assess and review previous knowledge	Tangram Geometry set Colored pens and paper	Assess students' descriptions
2	Recognize the height of a triangle (individually)	Assess and review previous knowledge	Geometry set Pen and paper	Guides Reinforces knowledge
3	Solve individually the GeoGebra activity on heights of triangles	Promote an expected learning	**GeoGebra** Official activities. Worksheet	Guides Deals with difficulties
4	Solve textbook tasks	Assess the learning	Textbook	Guides. Assesses. Closes theme.
		DS3: *Constructing polygons with* ***Logo***		
1	Team game (characteristics of regular and irregular polygons)	Assess and review previous knowledge	Polygon bingo game Polygon comparison table	Designed and showed a comparison table of polygon characteristics
2	Draw regular polygons (individually)	Promote an expected learning	Geometry set Pen and paper	Guides
3	Construction of polygons with Logo (in pairs & individually)	Promote an expected learning	**Logo** Worksheet	Guides Deals with difficulties

In Nohelia's initial DS, according to the initial interview, the following aims and resources (shown in italics) were included at each stage: **Stage 1** (reviewing previous knowledge) and **Stage 3** (developing a new topic) both focused on having children solve tasks, with the difference that in Stage 1, the aim was to make a *diagnosis* of children's knowledge, through their problem-solving activity work in teams; while in Stage 3, the Stage 1 *results* (another resource in itself) are taken into account for posing tasks on which students work individually. Nohelia designed the tasks using the following resources: (i) the *curricular aims of the study program*, (ii) *materials* from the Ministry of Education's *teacher activity books* (SEP 1994); (ii) *didactic guidelines* from the teacher activity books, or purchased or found on the Internet; (iii) fragments of stories, adapted to the context of the students, to motivate and interest them. The problems to be solved were presented as *printed materials* or projected on *slides*. Between Stages 1 and 3, in **Stage 2** (sharing knowledge with the whole group), the student teams' shared their solutions to the initial tasks, and the teacher guided the discussions and strengthened skills (e.g., arithmetic ones), through *exercises*, *videos* and *concrete materials*, before moving on to the individual problem solving activity (Stage 3). Finally, in **Stage 4** (assessment), she used the official *textbook* and *program of study* as resources for assessing the expected learning. Additionally, other resources that influenced her entire lesson plan were the *program of study, colleagues' experiences, courses* taken, and *didactic literature.*

That initial DS was modified as a result of the PD course, where Nohelia designed three lessons integrating, respectively, the use of some of the proposed digital resources, giving rise to a new DS for each class (see Table 1): In DS1, situations were proposed to find the quantity of milk comparing containers in three different scenarios, "at home", "in the stable" and in a "processing plant", using an interactive app called "Capacity measurements". In DS2, Geogebra was used to design –based on a task from the teacher activity book– triangles where a spider has to descend from one vertex to the opposite base in order to trace their heights. And in DS3, Logo was used to construct polygons. The DS1 and DS2 lessons were carried out by Nohelia in her classroom with a computer and projector, and the DS3 lesson, in the media room.

For each new DS, Stage 1 still had as purpose to review the previous knowledge needed for the new topic; however, instead of posing a task, children were asked oral and written questions, and given games to play. In Stage 2, instead of being one in which knowledge was shared, the teacher posed other tasks: for DS1, she asked children to compare containers of different capacities (liters and milliliters), similar to what is included in the interactive app; in DS2, she used complementary tasks to those posed in Stage 1, in order to review student's previous knowledge about triangles, particularly in terms of the heights the triangles; and in DS3, the teacher and students drew regular polygons on the blackboard in order to analyze their sides and angles. In Stage 3, the aim of developing the new topic was kept, using worksheets to pose tasks, the solutions of which were discussed as a whole group, but adapted in the following ways: in DS1 and DS3, the DT resources replaced those that the teacher originally used, in order to select, design and implement the tasks. In DS2, the DT resource (Geogebra) was used in combination with one of the original resources (a task from the teacher activity book). In DS3, the use of Logo led to study the content in another way, in terms of the Turtle Geometry context, which required thinking of the angles as turns; it was thus necessary to become aware that these did not correspond to internal angles. Finally, **Stage 4** continued to focus on children solving tasks from the textbook, in order to assess their learning (except in DS3, where Stage 4 was no longer carried out, because it was not possible to coordinate the Logo tasks with the curricular content).

Final remarks

In the case presented, we observed that the integration of digital resources, as well as the training (the PD course), generated modifications to the teacher's document system, causing: resources to be substituted, or used in combination to previous ones (i.e., in DS2); for the topic's tasks (and worksheets) to be designed ad hoc by the teacher, instead of taken from other sources; and for some activities that the teacher used to carry out (such as assessing the solutions to the tasks), being done by the software. Each digital resource presented new possibilities (in terms of implementation, knowledge and even motivation), as well as some limitations: The interactive app was easy to use and was adapted to the curricular content, although the learning tasks were restricted to what was proposed in it; GeoGebra was difficult to use, but allowed the teacher to design her own digital material, according to her interests and the curricular content; and Logo brought about more drastic changes to the initial DS of Nohelia, because it modified how the curricular content was approached.

These experiences show how the integration of digital resources into the DS of teachers is a complex but necessary task: it demands, in order to have meaningful DT integration, the collaboration of teachers, trainers, researchers and authorities, as well as access to hardware, with a development of teachers' pedagogical technology knowledge (PTK) (Thomas & Palmer, 2014), as we attempted to achieve through the PD course.

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INTEGRACIÓN DE RECURSOS DIGITALES AL SISTEMA DOCUMENTAL DE UNA PROFESORA DE MATEMÁTICAS EN UNA PRIMARIA RURAL DE MÉXICO

INTEGRATING DIGITAL RESOURCES TO THE DOCUMENTATION SYSTEM OF A MATHEMATICS TEACHER IN A MEXICAN RURAL PRIMARY-SCHOOL

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Los recursos juegan un papel importante en la manera de actuar y pensar de los sujetos. Los profesores, a lo largo de su trayectoria profesional, han construido formas de enseñanza de las matemáticas —articulados por la organización y tipo de actividades que guían la clase, los recursos que utilizan y sus formas de intervención— que se ven modificados cuando un nuevo recurso es integrado. Aquí presentamos evidencias de este fenómeno a partir del caso de una profesora quien, al participar en un curso de desarrollo profesional que implementamos, integró recursos digitales en su trabajo documental, desestabilizando sus formas previas de enseñanza.

Palabras clave: Educación Primaria, Tecnología, Herramientas y recursos docentes, Capacitación docente / Desarrollo Profesional

Introducción

Aunque, en esta era digital, la integración de recursos digitales en la enseñanza de las matemáticas es una necesidad social y pedagógica (Sunkel, Trucco & Espejo, 2014) se observan carencias en sectores de la educación básica en México (Enríquez & Sacristán, 2017, 2019). De hecho, el uso de tecnologías tiende a no ser casi considerado en la formación docente (e.g., ver el perfil de egreso para profesor de educación primaria –SEP, 2018). En este reporte presentamos parte de una investigación que indaga cómo profesores de escuelas primarias de Oaxaca participantes en un curso de desarrollo profesional (DP), integran recursos digitales en su enseñanza de las matemáticas. El estudio utiliza la Aproximación Documental de lo Didáctico o ADD (Gueudet& Trouche, 2009, 2012), para responder la pregunta: *Cuando los profesores participantes integran recursos digitales a su práctica docente, ¿qué cambios se dan en el diseño e implementación de sus clases de matemáticas, o sea, en su sistema documental*[2]*?*

La aproximación documental de lo didáctico (ADD)

La ADD estudia la práctica del profesor y su desarrollo profesional a partir de su interacción con los recursos que utiliza (selecciona, adapta, revisa, reorganiza) para la enseñanza de las matemáticas (Gueudet & Trouche, 2009, 2012). Los recursos se integran en un sistema de recursos (SR). El conjunto de interacciones y procesos con el SR, donde se articulan diseño y puesta en práctica, conforman el *trabajo documental* (Gueudet & Trouche, 2012) del profesor. A través del trabajo documental se producen, *documentos* integrados por recursos recombinados y esquemas asociados de utilización; el conjunto de documentos de un profesor es su *sistema documental* (Gueudet & Vandebrouck, 2011). A lo largo de la trayectoria profesional del profesor, su sistema de documentos va evolucionando, al trabajar y experimentar con viejos y nuevos recursos. Así, el trabajo documental del profesor da cuenta de su desarrollo profesional, es decir, de la evolución de su práctica, conocimiento y creencias (Gueudet & Trouche, 2012).

2 "Sistema documental" se define más abajo.

Metodología

Este reporte se enmarca en una investigación que ha,consistido en: un estudio diagnóstico para indagar las condiciones de formación de los profesores y de acceso a recursos digitales; una fase de intervención, donde se implementó un curso de DP a profesores participantes en el estudio diagnóstico, para la integración de recursos digitales en la enseñanza de las matemáticas; y una fase de indagación del proceso de integración de recursos digitales de los participantes, a partir del seguimiento de su trabajo documental. En el estudio diagnóstico participaron 67 profesores de 10 escuelas primarias de la región Mixteca del estado de Oaxaca, México, quienes carecían de formación sobre el uso de tecnología digital (TD) para la enseñanza de las matemáticas hacían uso limitado de TD en su práctica, y su acceso a equipo digital era limitado –debido a un pobre equipamiento de las escuelas y a políticas escolares que limitaban su uso (Enríquez & Sacristán, 2017, 2019). Esta información se tomó en cuenta para el diseño del curso, en la fase de intervención, de DP para la integración de TD a la práctica docente de matemáticas. Se invitó a los 67 profesores participantes en el estudio diagnóstico a participar en el curso y estudio asociado; 15 de ellos aceptaron. El curso de DP duró 5 meses (6 horas semanales). Se fundamentó en modelos teóricos sobre el conocimiento base del profesor (Shulman, 1986; Ball, et al.,2008, Thomas & Palmer, 2014). También utilizó la experiencia del programa EMAT (Sacristán y Rojano, 2009) y algunos de sus materiales (el software Logo y las guías didácticas para uso – Sacristán, 2005; Sacristán & Esparza, 2005).

El curso. Consistió de tres módulos, cada uno dedicado al estudio de cierto tipo de recursos TD – respectivamente, applets e interactivos diversos, Geogebra y Logo. Las actividades de cada módulo se desarrollaron en un ciclo de 4 momentos: (i) El estudio de los recursos digitales; (ii) el diseño de planes de clases de matemáticas integrando el uso de los recursos estudiados; (iii) la implementación de las clases diseñadas; y (iv) la reflexión grupal de las experiencias.

La recolección de datos. Para indagar los cambios en los sistemas documentales de los participantes recurrimos a: (i) entrevistas iniciales a los profesores, así como posteriores a la implementación de cada una de sus clases con TD; (ii) los planes de clase diseñados por los profesores durante el curso; (iii) observaciones de las clases diseñadas; (iv) las presentaciones de sus experiencias de clase, compartidas con sus colegas del curso.

Ahora presentamos datos de una profesora, Nohelia, participante en el estudio.

El caso la profesora Nohelia

Perfil de la profesora. Nohelia es licenciada en educación primaria con una maestría en desarrollo de competencias docentes, y ha tomado distintos cursos de DP, entre ellos de matemáticas y de computación. Al inicio del estudio, tenía 9 años de experiencia docente, y estaba a cargo del grado 5º en una escuela primaria rural, donde se cuenta con un aula de medios con 18 computadoras, un proyector portatil, y un viejo equipo de Enciclomedia (SEP, 2012) –computadora, pizarrón electrónico y otro proyector.

Los cambios en el sistema documental (SD) de la profesora Nohelia

A partir de los datos de la entrevista inicial a Nohelia, determinamos su sistema documental (SD) inicial –recursos y formas de uso para su clase de matemáticas— al principio del estudio. Este SD cambió a partir del curso, cuando integró los recursos digitales estudiados a su práctica. El SD inicial de Nohelia incluía varios documentos —e.g., actividades de orientación, o de reforzamiento (para ayudar a sus alumnos a superar debilidades) —cada uno con recursos específicos. Su SD inicial (ver Tabla 1) orientaba la totalidad de su clase, en cuatro momentos: (1) repasar conocimientos previos, (2) socialización de conocimientos, (3) desarrollo del tema y (4) evaluación.

Tabla 1. Los sistemas documentales (SD) de Nohelia con sus momentos (M)

M	Actividad	Objetivo	Recursos	Acciones del profesor
		SD Inicial *(sin recursos TD)*		
1	Resolución de problemas en equipo	Evaluar y repasar conocimientos previos	Fichero. Guía didáctica. Internet. Impresiones. Contexto de los niños. Cuentos.	Seguimiento a los procesos de solución
2	Compartir soluciones en plenaria	Atender debilidades y reforzar conocimientos	Procedimientos de los niños Videos y material concreto Ejercicios y problemas	Orienta Usa recursos Pone ejercicio Decide problemas para M3
3	Resolución de problemas (individual)	Favorecer el aprendizaje esperado	Fichero. Guía didáctica. Internet. Impresiones. Contexto de los niños. Cuentos.	Orienta Atiende dificultades. Implementa problemas
4	Resolución de tareas curriculares	Evaluar el aprendizaje	Libro de texto	Orienta. Evalúa. Cierra el tema.
		SD1: *Interactivo* *"Medidas de capacidad"*		
1	Planteamiento de preguntas (litro y mililitro)	Evaluar y repasar conocimientos previos	Preguntas	Valora respuestas
2	Resolución de problemas de capacidad (equipo)	Favorecer el aprendizaje esperado	Botellas, cubetas, hoja de trabajo	Orienta
3	Resolución de problemas de capacidad (individual)	Favorecer el aprendizaje esperado	**Interactivo** "medidas de capacidad". Fichas, hoja de trabajo.	Orienta. Atiende dificultades.
4	Resolución de tareas curriculares	Evaluar el aprendizaje	Libro de texto	Orienta. Evalúa. Cierra el tema.
		SD2: *Alturas de triángulos con* ***Geogebra***		
1	Armar triángulos en equipo (describir sus características)	Evaluar y repasar conocimientos previos	Tangram Juego geométrico Marcadores y papel	Valora descripciones
2	Reconocimiento de alturas del triángulo (individual)	Evaluar y repasar conocimientos previos	Juego geométrico Papel y lápiz	Orienta Refuerza conocimientos
3	Resolución (individual) de actividad sobre alturas de triángulos	Favorecer el aprendizaje esperado	**Geogebra** Fichero; Hoja de trabajo	Orienta Atiende debilidades
4	Resolución de tareas curriculares	Evaluar el aprendizaje	Libro de texto	Orienta. Evalúa. Cierra el tema.
		SD3: *Construcción de polígonos con* ***Logo***		
1	Juego (características de polígonos regulares e irregulares) en equipo	Evaluar y repasar conocimientos previos	Lotería de polígonos Tabla comparativa de polígonos	Crea y muestra tabla comparativa de características de polígonos
2	Trazo de polígonos regulares (individual)	Favorecer el aprendizaje esperado	Juego geométrico Papel y lápiz	Orienta
3	Construcción en Logo de polígonos (en binas e individual)	Favorecer el aprendizaje esperado	**Logo** Hoja de trabajo	Orienta Atiende debilidades

En el SD inicial de Nohelia, de acuerdo a la entrevista inicial, se conformaron cada momento con los siguientes propósitos y recursos (mostrados en cursivas): los **momentos 1** (repasar conocimientos previos) y **3** (desarrollo del tema) se centraban ambos en poner a los niños a resolver problemas, con la diferencia que en el Momento 1, se buscaba hacer un *diagnóstico* de lo que sabían los niños, trabajando ellos en equipos; mientras que en el Momento 3, tomando en cuenta los *resultados* del 1 (otro recurso en sí), los alumnos trabajaban individualmente. Nohelia diseñó los problemas a partir de los siguientes recursos: (i) los *objetivos curriculares* del *programa de estudio*, (ii) *actividades* de los "ficheros" de la Secretaría de Educación (SEP 1994); (ii) orientaciones o *guías didácticas* de los ficheros, compradas o halladas en Internet: (iii) fragmentos de *cuentos*, adaptados al contexto de los estudiantes, para motivar e interesarlos. Los problemas se daban *impresos* o proyectados mediante *diapositivas*. Entre esos momentos, en el **momento 2** (socialización de conocimientos), se compartían las soluciones grupales de los niños a los primeros problemas, y la profesora reforzaba habilidades mediante *ejercicios* (e.g., aritméticos), *videos* y *materiales concretos*, antes de pasar a la actividad de resolución individual de otros problemas (Momento 3). Finalmente, en el **momento 4** (evaluación), el *libro de texto* y *programa de estudio* fueron los recursos utilizados para valorar los aprendizajes esperados. Adicionalmente, hay otros recursos que influyen en la totalidad de la clase –

tales como el *programa de estudio, experiencias de colegas, cursos* que ha tomado, y *bibliografía didáctica*.

Ese SD inicial se modificó a raíz del curso de DP, donde Nohelia diseñó tres clases integrando, respectivamente, el uso de algunos de los recursos digitales propuestos, dando lugar a un nuevo SD para cada clase (ver Tabla 1): En el SD1, se plantean situaciones para hallar la cantidad de leche comparando recipientes en tres escenarios distintos, "en la casa", "en el establo" y en la "planta procesadora", usando el interactivo "Medidas de capacidad". En el SD2, se usó Geogebra para diseñar, basándose en una actividad del fichero, triángulos por los que debía descender una araña desde un vértice hasta la base opuesta para trazar las alturas. Y en SD3 se usaría Logo para construir polígonos. Las clases de los SD1 y SD2 las implementó Nohelia en su salón con una computadora y un proyector, y la del SD3 en el aula de medios.

Para cada nuevo SD, en el **momento 1** se mantiene el propósito de introducir los temas repasando los conocimientos sobre el contenido a trabajar; sin embargo, en lugar de plantear un problema, se hicieron preguntas orales, escritas y juegos. El **momento 2,** en lugar de ser uno de socialización de conocimientos, fue uno para plantear otras situaciones: para el SD1, comparar recipientes de distintas capacidades (litros y mililitros), de manera similar a lo propuesto en el interactivo; en el SD2, actividades complementarias a las del momento 1 para evaluar los conocimientos previos sobre triángulos, en particular sobre sus alturas; y en el SD3, trazos de polígonos regulares para analizar sus lados y ángulos. En los **momentos 3,** se mantuvo el propósito de abordar el contenido mediante hojas de trabajo para plantear problemas, para luego discutir las soluciones en plenaria, con algunas adaptaciones: en los SD1 y SD3, los recursos TD suplieron los que originalmente utilizaba la profesora para seleccionar y diseñar problemas. En el SD2 se combinó el recurso TD (Geogebra) con uno de los recursos originales (una actividad del *fichero)*. En el SD3, el uso de Logo llevó a estudiar el contenido de otra manera, en términos del contexto de la Tortuga, pensando en los ángulos como giros, y requirió percatarse que éstos no correspondían a los ángulos internos. Finalmente, los **momentos 4** continuaron siendo la resolución de tareas del libro de texto para evaluar el aprendizaje (excepto en el SD3, donde no se llevó a cabo momento 4, ya que no pudo adaptar la actividad Logo, al contenido curricular).

Comentarios finales

En el caso analizado, observamos que la integración de recursos digitales, así como la capacitación (el curso DP) generaron modificaciones al sistema documental de la profesora, ocasionando: que se suplieran recursos, o se combinaran (i.e., en el SD2); que las actividades de aprendizaje (hojas de trabajo) del tema no fueran retomadas de otras fuentes, sino construidas por la profesora; que algunas actividades, que la profesora solía llevar a cabo (como el evaluar las respuestas a los problemas), fueran realizadas por el software. Cada recurso digital presentó nuevas posibilidades (en términos de implementación, conocimientos e incluso motivación), así como algunas limitantes: El interactivo fue de fácil manejo y adaptación al contenido curricular, aunque las tareas de aprendizaje se restringían a lo propuesto en éste; GeoGebra resultó de difícil manejo, pero permitió el diseño de material digital propio, de acuerdo al interés de la profesora y al contenido curricular; y Logo originó cambios más drásticos al SD inicial de Nohelia debido a que modificó la manera de estudiar el contenido curricular.

Estas experiencias muestran cómo la integración de recursos digitales, al SD de los profesores, es una tarea compleja pero necesaria: para lograr una integración significativa de las TD, demanda la colaboración de profesores, capacitadores, investigadores y autoridades, requiriendo acceso a equipo digital y de desarrollo del conocimiento pedagógico tecnológico (PTK – Thomas & Palmer, 2014), como se intentó mediante el curso de DP.

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ENHANCING STUDENTS' SPATIAL REASONING SKILLS WITH ROBOTICS INTERVENTION

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Spatial reasoning is a high-impact topic as it strongly predicts interest in, appreciation of, and success in STEM domains and careers. Yet, spatial reasoning is often under-used, underdeveloped, and ignored in current grade-school curriculum and teaching. Framed by the perspective of embodied cognition, our study explores changes in elementary students' spatial reasoning skills after participation in either a short-term or a long-term robotics intervention. We administered measures of spatial reasoning elements before and after two differently structured robotics interventions to students aged 9-10 years: a short-term (N=11) and two long-term (N=48). Statistical analysis revealed significant improvements to several different elements of spatial reasoning in both groups. Our findings suggest that programming robots in either the short- or long-term leads to improvements in spatial reasoning.

Keywords: Programming and coding; STEM / STEAM; Technology

The purpose of our study is to report changes in elementary students' spatial reasoning skills after participation in either a short-term or a long-term robotics intervention. Spatial reasoning is a high-impact topic as it strongly predicts interest in, appreciation of, and success in STEM domains and careers (Casey et al., 2011; Lubinski, 2010; Mix & Cheng, 2012; Mix et al., 2016; Wai et al., 2009). Spatial reasoning is highly malleable, and can be learned (Julià & Antolí, 2016; Sorby, 2009; Uttal et al., 2013).Yet, spatial reasoning is mostly under-used, underdeveloped, and ignored in current grade-school curriculum and teaching (Newcombe, 2010). A few studies have investigated the positive effects of spatial reasoning and robotics interventions in schools (Coxon, 2012; Francis et al., 2016; Julià & Anatolì, 2016, Julià & Anatolì, 2018; Khan et al., 2014; Verner, 2004). Our current study builds on previous studies to further investigate the malleability of spatial reasoning with short-term and long-term robotics interventions.

Background Literature

What is spatial reasoning?

While there is considerable debate about what spatial reasoning is (see Uttal et al., 2013), we draw upon Bruce et al.'s (2017) definition of spatial reasoning as the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects. Davis et al. (2015, p. 141) describe the emergent complexity of spatial reasoning skills as co-evolved and complementary nature of the mental and physical actions. The nature of these skills is entangled and emergent. Their description of spatial reasoning elements include ALTERING (*dilating/contracting, distorting/morphing, scaling, folding, shearing*), MOVING (*sliding, rotating, reflecting, balancing, sliding*), SITUATING (*dimension shifting, locating, orienting, pathfinding, intersecting*), SENSATING (*perspective-taking, visualizing, propriocepting, imagining, tactilizing*), INTERPRETING (*diagramming, modeling, symmetrizing, comparing, relating*), [DE]CONSTRUCTING (*de/re/composing, un/re/packing, re/arranging, sectioning, fitting*).

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Context

This study is based on a multi-year design-based research project investigating how robotics influences spatial reasoning. The data reported here is based on quasi experimental results investigating if robotics interventions impact spatial reasoning. The first iteration of the design reported here was based on a week-long robotics academy with elementary teachers and Grade 4-5 students held at a university (15 hours). The second and third iterations of the design reported here were year-long classroom interventions at a local school (40 hours) in Grades 4-5. It is beyond the confines of this brief research report to describe the structures of the interventions in detail. However, Table 1 summarizes the sequence and spatial reasoning elements engaged in each task. Every robotics task of the short-term and the long-term intervention involved multiple spatial elements.

Table 1. Spatial Skills Found in the Intervention Task

	Altering					Moving					Situating					Sensating					Interpreting					[De]constructing				
	dilating/contracting	distorting/morphing	scaling	folding	shearing	sliding	rotating	reflecting	balancing	directioning	dimension-shifting	locating	orienting	pathfinding	intersecting	perspective-taking	visualizing	proprioceptiing	imagining	tactilizing	diagramming	modeling	symmetrizing	comparing	relating	de/re/composing	un/re/packing	re/arraging	sectioning	fitting
Short-Term																														
Build the robot			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Robot Dance			x			x	x	x		x	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Polygon Task			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x		x	
Firebot Task			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Long-Term																														
Build the robot			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Robot Dance			x			x	x	x		x	x	x	x	x		x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
Polygon Task			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x	x	x	x
SnowBot			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x	x	x	x
Firebot			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x	x	x	x
Search and Rescue			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x	x	x	x
Grid Obstacle			x			x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x	x	x	x
Pooper Scooper			x	x		x	x	x	x	x	x	x	x	x	x	x	x	x	x	x		x	x	x	x	x	x	x	x	x

Methodology

Our study explores the following research question: Do elementary students' spatial reasoning skills improve after participation in either a short-term or a long-term robotics intervention? The data reported here is based on the quasi experimental pre- and post-test results collected before and after the interventions mentioned above. Participants in the short-term intervention included 11 Grade 4 students. Participants in the long-term intervention were from a different school and included 48 students total, whereby ten Grade 4 students completed the pre- and post-tests in 2017-2018. During the following school year 2018-2019, 19 Grade 4 students and 19 Grade 5 students respectively took the pre- and post-tests on spatial reasoning skills. The administration of the pre- and post-test instruments was similar for the short- and long-term interventions.

Description of pre- and post-test instrument. We developed an instrument that encompassed a broad range of spatial skills based on an amalgamation of established protocols. The spatial reasoning pre- post-test consisted of seven (different) task categories with 13 test items for the short-term intervention, and three additional test items for the long-term intervention. The short-term pre- and post-test included four Sorby drawing items, two paper folding items, two shape rotation items, two stereonet items, one building a 3D object from an image item, one moving a shape on a grid item, and one cross-section of a cube: isometric projection. The three additional items in the reasoning pre- and post-test of the long-term intervention included one more paper folding item, a block visualization/rotation item, and a pattern arranging item. Groups of three students were tested at a time. Each of the tests took approximately 45 minutes.

Analysis of results of short-term intervention. The data reported in this section details the Grade 4 results from the week-long intervention in 2016. The data consisted of one group of the same subjects at two different points in time (before and after the intervention). For such data, the Paired t-Test is an appropriate test to compare the means when the data is normally distributed (Huck, 2012; Mills & Gay, 2019). For data that is not normally distributed, the non-parametric Wilcoxon Signed-Ranks Test is more appropriate (Huck, 2012; Mills & Gay, 2019). To determine if that data is normally distributed, the Shapiro Wilks test is a suitable parametric test for a small sample size (Huck, 2012; Mills & Gay, 2019).

To compare spatial reasoning at the beginning and end of the week long robotics camp (15 hours) a paired t-test was conducted on the normally distributed Items 2, 3, 5, 10, 11, 12, and Total (pre- and post-test) and a Wilcoxon signed-rank test was used for the not normally distributed Items 1, 4, 6, 7, 8, 9, and 13 (pre- and post-test). All spatial reasoning test items saw improvement. There was a significant improvement for the Sorby drawing Item 2: $t(10) = -2.4$, $p < .05$; building a 3D object from a picture Item 11: $t(36) = -2.9$, $p = .05$, moving a shape on a grid Item 12: $t(11) = -2.4$, $p = 0.05$ and; the overall total: $t(11) = -3.05$, $p < .05$.

Analysis of Results of Year Long Intervention. To compare spatial reasoning at the beginning and end of the school year a paired t-test was conducted on normally distributed Items 1-16 (pre- and post-test) and a Wilcoxon signed-rank test was used on the non-normally distributed Items Total (pre- and post-test). The data combines the Grade 4 results from Year 2017-2018 and the Grade 4 and 5 results from Year 2018-2019. All spatial reasoning test items saw improvement. There was a significant improvement for the Sorby drawing Item 2: $t(47) = -2.4$, $p < .05$; paper folding Item 5: $t(47) = -3.4$, $p < .05$; rotation Item 9: $t(47) = -3.0$, $p < .05$; pattern arranging Item 16: $t(36) = -4.2$, $p < .05$, and; the overall total: $Z(-4.519)$, $p < .05$.

Discussion

In this study, we found how powerfully robotics interventions can improve spatial reasoning. We found similar results from both the short-term and long-term interventions. For the short-term significant results were observed in Sorby drawing, building a 3D object from a picture, moving a shape on a grid, and overall total score. For the long-term, significant results were observed in Sorby drawing, paper folding, shape rotation, pattern arranging and overall total score. The long-term intervention did not reveal a significant improvement to building 3D object from picture and moving along a grid. Perhaps this is because the students in the long-term study were already quite high in these two measures compared to the short-term group; there was not as much room to improve with these groups.

Previous studies also found improvements in spatial reasoning with robotics interventions. Verner (2004) reported improvements in three spatial reasoning skills: visualization, perception, rotation. Julià and Anatolì (2016; 2018) found improvements with spatial reasoning tasks. By drawing upon Davis et al.'s (2015) descriptions of spatial reasoning elements, we were able to provide a broader perspective of spatial reasoning than Verner (2004). Like Julià and Anatolì (2016; 2018), our instrument included paper folding, shape rotation, cube comparison and perspective taking. However, our test instrument also included Sorby drawing items, stereonets, building 3D objects from pictures, moving shapes on a grid and isometric projects. We not only looked at the spatial reasoning tasks as did Julià and Anatolì (2016; 2018), we also identified the spatial reasoning elements that were associated with each task. For example, visualization and imagining from SENSTATING were elements that appeared in every task and are integral to spatial reasoning as evident from most definitions (see Uttal et al., 2013).

Our identification of spatial elements within the tasks was important to show the compatibility of the test instruments with robotics. For instance, it may seem like the Sorby drawing item is completely different from programming a robot to move. However, the Sorby drawing item requires seven spatial reasoning elements: SITUATING (dimension shifting, intersecting), SENSATING (visualizing, imagining), INTERPRETING (diagramming), and [DE]CONSTRUCTING (de/re/composing, fitting). Francis et al. (2016) observed the engagement of these same spatial reasoning elements when students programmed robots to move.

Our findings complement our previous qualitative research which illustrated how spatial reasoning is engaged while building (Khan et al., 2014) and programming robots to move (Francis et al., 2016). These prior studies illustrated the complex and co-emergent nature of spatial reasoning which helped provide a basis for this current study that drew upon embodied cognition to investigate the results of *learning from action*. Our results provide some validation to Pouw et al.'s (2014) prediction that sensorimotor experiences are important for development of related concepts. In other words, this study along with our two previous studies (Khan et al., 2015; Francis et al., 2016) illustrate the power of sensorimotor experiences of robotics learning for improving spatial reasoning. Not only is spatial reasoning engaged, it also improves significantly.

As future work, we would like to explore how programming robots to move helps with learning mathematics. Spatial reasoning ability is correlated to mathematics achievement (Mix & Cheng, 2012) and computer programming is highly related to mathematics. Exploring connections could lead to unexpected and emerging insights for the teaching and learning of mathematics.

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TEACHERS' PROBLEM POSING IN PAPER-AND-PENCIL AND GEOGEBRA

PLANTEAMIENTO DE PROBLEMAS POR PROFESORES EN PAPEL-Y-LÁPIZ Y GEOGEBRA

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This paper shows the types of problems posed by of in service Mexican teachers in both paper-and-pencil and GeoGebra. The analysis and characterization of the posed problems were based on the model stated by Stoyanova (1998). According to the results, teachers can more easily pose problems in paper and pencil when dealing within a semi-structured situation. However, when using GeoGebra, they can more easily create problems within a free situation. These kinds of results indicate the necessity of professional development regarding the use of new technologies for mathematics teaching, where problem posing is fundamental.

Keywords: Problem posing, Technology, Teacher's knowledge.

Background

Problem posing is an important issue in mathematics teaching and learning. It is a useful mathematical activity because it helps evaluate content understanding, it encourages critical thinking, creativity and motivation, and it guides teacher's decision making. Furthermore, it is recognized that mathematics teaching and learning is modified in technological environments, therefore, problem posing is modified as well. In this regard, some researchers (e.g., Abramochivh & Cho, 2006; Fukuda & Kakihana, 2009, among others) have studied attitudes on the effect of problem posing in a paper-and-pencil environment versus the use of technology, in which they have identified, as a potential to pose appropriate problems, the possibility of direct manipulation of the mathematical object in technological environments.

In service teachers and future mathematics teachers usually are engaged in problem posing, but they are not aware of it. For instance, they pose problems when they adapt a problem from different sources and adequate it to the students' context. Crespo and Sinclair (2008) have stated that teachers should have similar experiences than those they want to set in their students. Therefore, problem posing should be present intheir teacher training and professional development. Thus, the research question that guided this study is: What kind of problems are posed by mathematics teachers in a paper-and-pencil environment, and which ones in a technological environment?

Theoretical Framework

According to Silver (1994, p. 19) posing mathematical problems, or just Problem Posing, is "the production of new problems and the reformulation of given problems. Therefore, posing a problem can occur before, during or after the solution of a problem". For González (2001), problem posing involves "to identify, create, describe and write a mathematical problem, individually or collectively, based on an initial situation–identified or created– by those involved in developing it" (cited in Rodriguez, García & Lozano, 2015, p. 103).

Regarding the classification of the type of situations in problem posing, Stoyanova (1998, cited in Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005) has proposed three categories: *Free* *situations, are those* that have no restrictions when creating a problem. *Semi-structured situations*, are those in which a drawing, a graph or a part of a story is given to pose the problem. And, *Structured situations*, are those which specifically refer to reformulating an existing problem.

Problem posing is important to promote mathematical thinking. In this regard, Ayllón and Gómez (2014) recognize that creating problems increases mathematical knowledge, because it encourages students to create connections between their already acquired knowledge. Among the different perspectives on problem posing, this paper is mainly based on Silver's (1994) ideas and on the proposed classification by Stoyanova (1998), for both, task design and data analysis.

Method

This paper is part of a current research project. Here, we are reporting the work of five teachers (T1, T2,…, T5), one male and four females (ages 27-37), for which their full work has been identified. Of the five teachers, one teaches at a junior high school, two teach at a high school, and one teaches at a college.

Task Design

Five problem posing tasks were designed, based on Stoyanova's (1998) classification. In this report, only the first two tasks were analyzed: *Task 1* uses a *free situation* to ask the teachers to pose a problem, related to any mathematical area.. *Task 2* uses a *semi-structured situation* toask them to create a problem regarding a given geometrical drawing Both tasks are divided into two parts; paper-and-pencil problem posing and problem posing in a GeoGebra environment.

Data Gathering

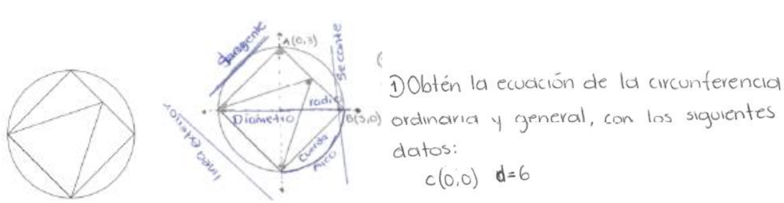
Data collection was carried out by means of a group interview in a workshop on problem posing, during three sessions. First, the five tasks were set to pose problems in paper-and-pencil (first part). Later, the same five tasks were used to pose problems using GeoGebra (second part. In the second part, each participant was given a GeoGebra file, to visualize and manipulate the given geometrical figure related to the semi structured situation. The sources for data analysis were the worksheets from each participant, the generated GeoGebra files and field notes.

Analysis and Results

Part 1: Problem posing in paper-and-pencil

Free situation. In Task 1 (creating a problem related to any mathematical area), the participants posed five problems. According to their answers, the following problems were posed: one geometry/calculus problem on optimization, one algebraic problem on second grade equations, two statistics problems on central tendency, and one financial mathematics on compound interest.

Semi-structured situation. For Task 2 (posing, based on a given figure, as many problems as they could), 38 problems were created, 37 Euclidean geometry problems and 1 problem on analytical geometry. Problems regarding Euclidean geometry are about areas (11), length (13), visualization of elements in the given figure, i.e., radius, diameter, chord, etc. (9), angles (2), and figure reconstruction using a ruler and a compass (2). As an example of the problems posed by the teachers, Figure 1 (right) shows the analytical geometry problem created by T4.



Find the circumference equation, ordinary and general, given that

Figure 1: Given figure (left). Problem posed by P4 (right)

Table 1 summarizes the total of posed problems in the first part, regarding the work on paper-and-pencil. Even though Task1 explicitly asks the teachers to pose one problem, the number of problems posed in Task 2 is higher. Which means that it was easier for participants to create problems when information is provided (semi structured situation) than when it is not (free situation).

Table 1: Summary, total created problems in paper-and-pencil

Task	Situation	Total problems
Task 1	Free	5
Task 2	Semi-structured	38

Part 2: Problem posing in GeoGebra

Free situation. In this type of situation participants created six problems (one of the participants created two). By their characteristics, these can be categorized as problems related to analytical geometry, Euclidean geometry, algebra and financial mathematics. Figure 2 shows the Euclidean geometry problem posed by T3, which involves the bisector.

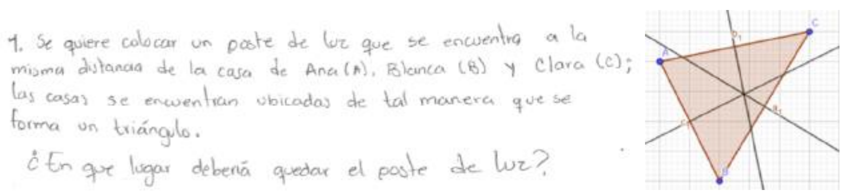


Figure 2: Problem Posed by P3 using GeoGebra

Semi-structured situation. This task involves a GeoGebra file. According to the results, 10 problems of this kind of situation were created when using GeoGebra referring to Euclidean geometry. The problems can be categorized as follows: 6 relating to areas, 2 relating to angles, and 2 to relating to length. Figure 3 (left) shows the given figure in this task (constructed in GeoGebra, and included in the given GeoGebra file used by participants) in which teachers based their problems. As an example, Figure 3 (right) shows the problems created by T5 using GeoGebra, the problem asks for measures of perimeter, angles, and the comparison of the areas of different figures involved, knowing that the observed square is 1 cm per side.

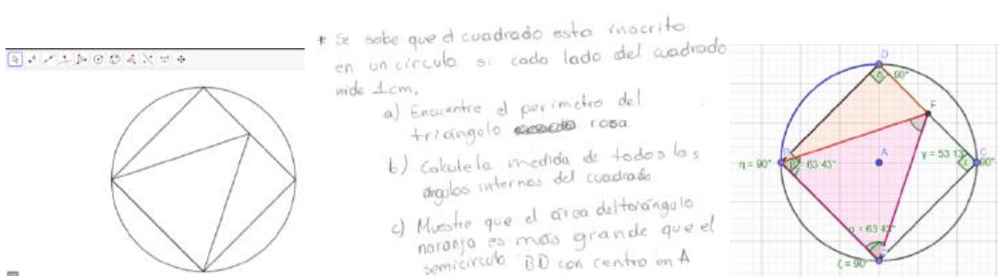


Figure 3: Given Figure (left). Problems posed by P5 (right)

Table 2 summarizes the problems posed by teachers during the second part. Based on the results, the number of problems created when using GeoGebra, in the semi-structured situation, decreases compared with the posed problems in paper-and-pencil (see Table 1). However, in both parts, the number of problems created in the semi-structured situation is higher than in the free situation.

Table 2: Summary, total created problems in GeoGebra

Task	Situation	Total problems
Task 1	Free	6
Task 2	Semi-structured	10

Conclusions

Regarding the free situation, in the first part, during the interview teachers expressed uncertainty about how to proceed to answer the task. This indicates the lack of experience of problem posing in their practice. Nevertheless, when they had the opportunity of using GeoGebra in the free situation, they interacted with the software in order to pose their problems by using it as a guide to explore the given figure, -see Figure 3 left,, or by including the use of GeoGebra as part of the problem.

In the semi-structured situation, when using only paper-and-pencil teachers assumed geometrical properties, which allowed them to pose a higher number of problems than in the second part. When the situation involved the use of GeoGebra, it was observed that teachers did not take advantage of the software's capacities (i.e., its dynamic feature), GeoGebra was used only as a means for static visualization. This may have happened because only participants T1 and T5 had previous experience with GeoGebra. Furthermore, the problems posed by the teacher did not mention the use of GeoGebra in solving the problems. Thus, even when previous reports (e.g., Abramovich & Cho, 2006; Fukuda & Kakihana, 2009) suggest the potential of the use of technology for problem posing, our results show the necessity for teachers to develop specific knowledge for mathematics teaching with technology in order to pose problems in technological environments. Therefore, models such as TPACK (*Technological Pedagogical Content Knowledge*, Mishra & Koehler, 2006, 2009), or KTMT (*Knowledge for Teaching Mathematics with Technology*, Rocha, 2013), among others, should be included in teacher training and development.

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PLANTEAMIENTO DE PROBLEMAS POR PROFESORES EN PAPEL-Y-LÁPIZ Y GEOGEBRA

TEACHERS' PROBLEM POSING IN PAPER-AND-PENCIL AND GEOGEBRA

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Este trabajo muestra los tipos de problemas planteados por profesores Mexicanos en servicio de diferentes niveles educativos tanto en un ambiente de papel-y-lápiz como mediante el uso de GeoGebra. El análisis y clasificación de los tipos de problemas fue realizado mediante la categorización propuesta por Stoyanova (1998). De acuerdo con los resultados, por un lado, los profesores plantean problemas con mayor facilidad, en papel-y-lápiz, cuando se trata de una situación semiestructurada. Por otro lado, al hacer uso de GeoGebra, se les facilita el planteamiento de problemas cuando la situación es libre. Lo anterior indica la necesidad de que los profesores cuenten con mayor apoyo respecto al uso de las nuevas tecnologías para la enseñanza de las matemáticas; donde el planteamiento de problemas es fundamental.

Palabras clave: Planteamiento de problemas, Tecnología, Conocimiento del profesor

Antecedentes

El planteamiento de problemas es importante en la enseñanza y aprendizaje de las matemáticas, puesto que puede ser tomada como una herramienta que ayuda a evaluar la comprensión de los contenidos, estimula el pensamiento crítico, la creatividad y motivación, y promueve la toma de decisiones de los profesores. Además, es reconocido que la enseñanza y aprendizaje de las matemáticas se modifica en ambientes tecnológicos; por la tanto, también el planteamiento de problemas. De esta manera, algunas investigaciones (e.g., Abramovich & Cho, 2006; Fukuda & Kakihana, 2009; entre otros) han estudiado sobre las actitudes y efectos del planteamiento de problemas en ambiente de papel-y-lápiz frente al uso de tecnología; al respecto, identifican la manipulación de objetos matemáticos en entornos tecnológicos como un potencial para plantear problemas apropiados.

Los profesores y futuros profesores, por su parte, en diversas ocasiones crean problemas sin ser conscientes de ello. Es decir, en el momento en que adaptan un problema extraído de otra fuente para que sea adecuado al contexto de sus estudiantes, llevan a cabo el proceso de planteamiento de un problema. Crespo y Sinclair (2008), afirman que los profesores deben tener experiencias similares a las que pretenden inculcar a sus alumnos, por lo que el planteamiento de problemas debería estar presente en los años de su formación. Por lo tanto, la pregunta de investigación que guió el estudio es la siguiente: ¿Qué tipos de problemas plantean profesores de matemáticas en ambiente de papel-y-lápiz, así como con el uso de GeoGebra?

Marco Teórico

De acuerdo con Silver (1994, p. 19) el planteamiento de problemas matemáticos, o *Problem Posing*, como se le conoce en la literatura es "la generación de nuevos problemas y la reformulación de problemas dados. Por lo que, plantear puede ocurrir antes, durante o después de la solución de un problema". Para González (2001), plantear problemas consiste en "identificar, crear, narrar y redactar un problema matemático, en forma colectiva o individual, a partir de una situación inicial identificada o creada por la(s) persona(s) que la realiza(n)" (citado en Rodríguez, García & Lozano, 2015, p.103).

Respecto a clasificaciones de tipos de situaciones en el planteamiento de problemas, Stoyanova (1998, citado en Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005), determina tres categorías: *Situaciones libres*, son aquellas que no tienen restricción alguna, sino que el creador decide el tema sobre el cual se planteará el problema. *Situaciones semiestructuradas*, se refiere a aquellas en la que se proporciona un dibujo, gráfico o parte de una historia sobre la cual se planteará el problema. Y *Situaciones estructuradas* las cuales tratan específicamente de reformular un problema ya existente.

De acuerdo con lo anterior, el planteamiento de problemas resulta importante para promover el razonamiento matemático. Al respecto Ayllón y Gómez (2014), reconocen que la invención de problemas provoca el aumento de conocimientos matemáticos, debido a que obliga a los estudiantes a crear conexiones entre los conocimientos ya adquiridos. Respecto a las diversas perspectivas reportadas en la literatura, con referencia al *problem posing*, el presente trabajo se basa principalmente en las ideas de Silver (1994) y en las clasificaciones propuestas por Stoyanova (1998), las cuales sustentan tanto el diseño de tareas, como el análisis de datos.

Método

Este trabajo es parte de un proyecto de investigación en curso. Del total de participantes, se da cuenta de sólo cinco profesores (P1, P2,…, P5), un hombre y cuatro mujeres (de entre 27 a 37 años) de quienes, hasta el momento, se tiene evidencia de su trabajo. De los participantes, dos imparten clases en nivel secundaria, dos en el nivel medio superior y uno en nivel superior.

Diseño de Tareas

Se diseñaron cinco tareas sobre el planteamiento de problemas, con base en la clasificación de Stoyanova (1998). Para efectos de este trabajo, se analizaron solamente las primeras dos tareas: La *Tarea 1*, dirigida al planteamiento de problemas por medio de una *situación libre*, solicita crear un problema del área de matemáticas de su preferencia. La *Tarea 2* se trata de una *situación semiestructurada*, en donde a partir de un dibujo geométrico se solicita plantear problemas relacionados con tal figura. Ambas tareas involucran dos momentos en el planteamiento de problemas: primero en papel y lápiz; segundo; en ambiente de GeoGebra.

Recopilación de Datos

La recopilación de datos se realizó mediante la implementación de un taller (entrevista grupal) dividido en tres sesiones. Primero se implementaron las cinco tareas referidas a plantear problemas en papel-y-lápiz (primer momento). Posteriormente, se utilizaron las mismas cinco tareas, para plantear problemas con ayuda del software GeoGebra (segundo momento). Para ello, cada participante contó con un archivo GeoGebra correspondiente, para visualizar y manipular la figura dada, para el caso de la situación semiestructurada. Para el análisis de datos, las fuentes de información fueron las hojas de trabajo de cada participante, los archivos GeoGebra y notas de campo.

Análisis y Resultados

Momento 1: Planteamiento de problemas en papel-y-lápiz

Situación libre. Para la Tarea 1 (inventar un problema del área de matemáticas de su preferencia), los participantes crearon un total de cinco problemas, uno por participante. De acuerdo con sus respuestas, fueron planteados: un problema sobre geometría/calculo (Optimización), uno de algebra (Ecuación cuadrática), dos problemas sobre estadística (Medidas de tendencia central) y uno de matemáticas financieras (interés compuesto).

Situación semiestructurada. En el caso de la Tarea 2 (inventar, a partir de una figura dada, ver Figura 1 izquierda, tantos problemas como pudieran) fueron creados 38 problemas. Los participantes crearon 37 problemas del área de geometría Euclidiana y 1 referente a geometría analítica sobre la ecuación de la circunferencia. Los problemas correspondientes a geometría euclidiana tratan sobre: longitudes (13), áreas (11), visualización de elementos de la figura dada e.g., radio, diámetro, cuerdas, etc. (9), ángulos (2) y recrear la imagen con ayuda del juego geométrico (2). Como ejemplo de los problemas creados, la Figura 1 (derecha) muestra el problema sobre geometría analítica creado por el participante P4.

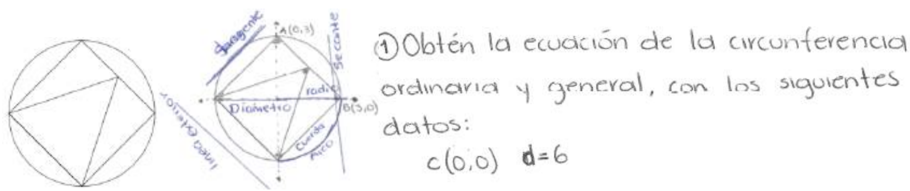

Figura 1: Izquierda, figura dada. Derecha, problema propuesto por P4

La Tabla 1 resume el total de problemas propuestos en el primer momento, referente al planteamiento de problemas en papel-y-lápiz. Si bien la Tarea 1 explicita crear un problema del área de su elección, el número de problemas creados en la Tarea 2 es considerablemente mayor, lo cual indica que para los participantes crear un problema resultó más fácil cuando se les provee de información (situación semiestructurada) a cuando no (situación libre).

Tabla 1: Resumen del total de problemas creados en papel-y-lápiz

Tarea	Situación	Total de problemas
Tarea 1	Libre	5
Tarea 2	Semiestructurada	38

Momento 2: Planteamiento de problemas en GeoGebra

Situación libre. En esta Tarea, los participantes crearon seis problemas (uno de los participantes creó dos). Por sus características, éstos pueden categorizarse como problemas de geometría analítica, geometría euclidiana, algebra y matemáticas financieras. La Figura 2 muestra el problema creado por P3; sobre geometría euclidiana, el cual involucra la mediatriz.

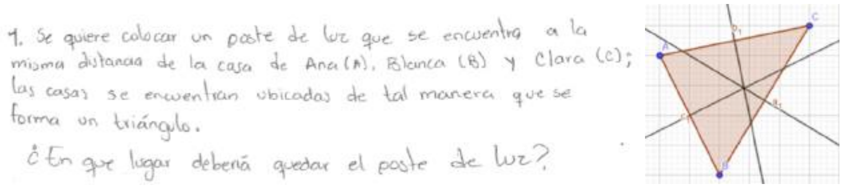

Figura 2: Problema propuesto por P3 con GeoGebra

Situación semiestructurada. En esta tarea se incluye un archivo GeoGebra. De acuerdo con los resultados, se crearon 10 problemas, todos referidos al área de geometría euclidiana. Por sus características, se pueden categorizar como problemas sobre: área (6), ángulos (2) y longitud (2). La Figura 3 (izquierda) muestra la figura dada (construida en GeoGebra) en la que cada participante se basó para crear sus problemas. Como ejemplo del tipo de problemas creados con ayuda de GeoGebra, la Figura 3 (derecha) muestra el creado por P5.

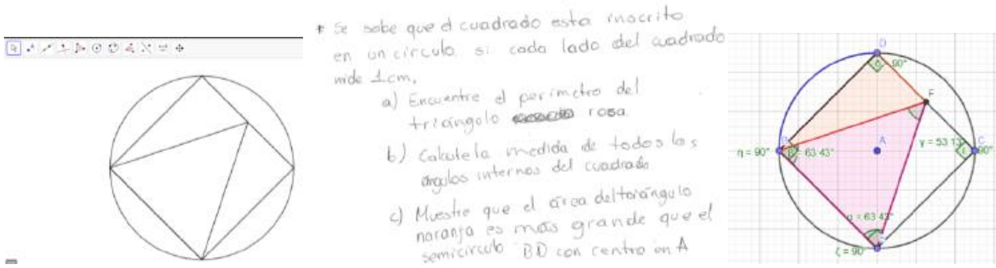

Figura 3: Izquierda, figura dada en GeoGebra. Derecha, problemas propuestos por el P5

La Tabla 2 resume el total de problemas propuestos en el Momento 2. De acuerdo con la Tabla 2, en GeoGebra, el número de problemas creados en la situación semiestructurada disminuye en comparación al ambiente de papel-y-lápiz (ver Tabla 1). Sin embargo, en ambos momentos, el número de problemas en la situación semiestructurada es mayor que en la libre.

Tabla 2: Resumen del total de problemas creados en GeoGebra

Tarea	Situación	Total de problemas
Tarea 1	Libre	6
Tarea 2	Semiestructurada	10

Conclusiones

Para la situación libre, en el primer momento, los profesores expresaron incertidumbre respecto a cómo proceder. Esto es indicativo de la poca experiencia sobre el planteamiento de problemas por parte de ellos en su práctica. Por otro lado, durante el uso de GeoGebra, la situación libre, les conduce a interactuar con el software, y de esta manera crearon los problemas, ya fuera que tomaran en cuenta GeoGebra como guía–al explorar–para la creación del problema, o bien sin necesidad de explorar, incluir su uso como parte del problema.

En la situación semiestructurada es notorio cómo los profesores, cuando trabajaron en papel-y-lápiz, asumen propiedades de la figura geométrica dada, esto les permitió plantear un número mayor de problemas. Cuando la situación involucra el uso de GeoGebra se observa que los participantes no explotaron la capacidad del software (e.g., su componente dinámica), sólo lo utilizaron como medio de visualización estática de la imagen, esto puede deberse a que únicamente los participantes P5 y P1 afirmaron haber utilizado GeoGebra en diversas ocasiones. Otra particularidad de los problemas propuestos es que no explicitan el uso de GeoGebra para resolverlo. En este sentido, aunque reportes previos (e.g., Abramovich & Cho, 2006; Fukuda & Kakihana, 2009) indican el potencial del uso de tecnología para el planteamiento de problemas, nuestros resultados muestran que, para ello, los profesores deben desarrollar conocimientos específicos sobre la enseñanza de las matemáticas con tecnología. Así, modelos como el TPACK (*Technological Pedagogical Content Knowledge*, Mishra

& Keohler, 2006, 2009), el KTMT (*Knowledge for Teaching Mathematics with Technology*, Rocha, 2013), entre otros, deben estar presentes tanto en el desarrollo profesional como en su formación inicial.

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PRESERVICE TEACHERS' PERSPECTIVES ON TECHNOLOGY INTEGRATION IN KINDERGARTEN THROUGH EIGHTH GRADE MATHEMATICS

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Studies have highlighted a multitude of beneficial student outcomes associated with the implementation of educational technology. However, there is a lack of understanding in both why and how preservice teachers intend to integrate technology into their future mathematics teaching. This small-scale study sought to examine preservice teachers' (N = 24) perspectives on technology integration within the context of elementary and middle school mathematics. The topics of primary interest in this study was preservice teachers' intended purposes of technology integration. Themes within responses to open-ended prompts were identified and interpreted through the lens of the SAMR model (Puentedura, 2006). Findings show that participants most frequently integrate technological resources in a way that augments a mathematical task. Implications for future research and teacher education are discussed.

Keywords: Technology, Preservice Teacher Education, Teacher Knowledge

The integration of technology into kindergarten through eighth grade (K-8) mathematics has been associated with a variety of benefits to students, teachers, and schools. Though various technological resources exist, particularly popular resources in K-8 mathematics are virtual manipulatives and mathematical games. Virtual manipulatives, defined as "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer et al., 2002, p. 373), have been shown to increase K-8 students' conceptual knowledge of several mathematics topics (Reimer & Moyer, 2005; Suh & Moyer, 2007), positive attitudes toward mathematics (Lee & Chen, 2015; Sen et al., 2017), confidence in mathematics (Yuan et al., 2010), and feelings of competency (McLeod et al., 2013). K-8 students with disabilities have benefitted from virtual manipulative use as well, demonstrating increased rates of learning (Root et al., 2017), greater accuracy (Bouck et al., 2014), and faster independence (Bouck et al., 2017; Bouck et al., 2018). Mathematical games, such as those offered by Math Playground (https://www.mathplayground.com/), have been shown to increase K-8 students' achievement regarding multiplication (Kiger et al., 2012), adaptive number knowledge, arithmetic fluency, and pre-algebra knowledge (Brezovszky et al., 2019). Technological resources also benefit teachers and schools, as many are free to access, available for use outside of the classroom, and decrease in-class time spent distributing and gathering materials during lessons (Moyer et al., 2002).

Due to these benefits, it is imperative that preservice teachers (PSTs) are competent in technology integration upon degree completion. However, sufficiently preparing PSTs to integrate technology in their future classrooms has proven to be a challenging task for teacher education programs. A common approach implemented by teacher education programs has been adding the requirement of a stand-alone educational technology course – an approach that 85% of institutions have adopted (Kleiner et al., 2007). However, these courses often lack content-specific context and classroom practice opportunities, as just 32% of institutions provide learning experiences where PSTs deliver technology experiences within elementary classrooms (Rose et al., 2017) and many PSTs feel unprepared to effectively integrate technology on their first day of in-service teaching (Tondeur et al., 2012). Research has uncovered several factors that explain PSTs' feeling of unpreparedness, including insufficient access to technology (Dawson, 2008), lack of technology skills (Teo, 2009), negative attitudes toward technology integration, lack of confidence in their ability to integrate

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technology, and the belief that their competence may be undermined due to students potentially having more knowledge about technology (Crompton, 2015). Gaining additional information regarding PSTs' perspectives on technology integration may prove beneficial to teacher education programs, current PSTs, and prospective PSTs.

This study sought to examine PSTs' perspectives on technology integration in K-8 mathematics. The aforementioned challenges associated with PSTs' integration of technology into K-8 mathematics inform the research question in this study: When prompted to select technological resources to enhance K-8 mathematics instruction after a two-day lesson about technology integration in K-8 mathematics, for what purpose do PSTs intend to use the selected resource?

Theoretical Framework

Puentedura's (2006) Substitution, Augmentation, Modification, and Redefinition (SAMR) model offered a theoretical perspective by which the intended purpose of a technological resource may be categorized. The SAMR model highlights four levels in respect to the impact that the integration of technology has on the design of a task within a lesson. Technology acts as a direct tool substitute at both the substitution and augmentation levels, but only provides functional improvement to the task at the augmentation level. The ability to significantly redesign tasks due to technology use occurs at the modification level, and technology use at the redefinition level allows for the creation of new tasks that would otherwise be inconceivable. Within the mathematics context of graphing functions, Dorman (2018) provided examples for each level of the SAMR model:

> At the substitution level, instead of printing off paper copies of the worksheet, an instructor could make the worksheet available online. At the augmentation level, students could complete the same questions on a Google Form, and the instructor could capture the answers for individual students to check for understanding. … At the modification level, … students could work in groups to analyze the different characteristic of functions as they graph them. Then, students could video record the characteristics and steps of how to graph functions. The video could be uploaded to a classroom website so that students can use it as a tutorial or study aid. At the redefinition level, students could create an online portfolio of all types of functions, and their graphs could include real-world applications that are modeled by the functions. (para. 3)

In this study, the SAMR model was utilized as a lens through which PSTs' intended purpose of mathematics technological resources were examined and through which PSTs' understanding of appropriate technology integration were interpreted.

Methodology

This study was conducted at a large university in the Northwest region of the United States. Participants (N = 24) were recruited from a K-8 Mathematics Methods course during the spring semester of 2020, which meets for two, 75-minute periods per week. All participants are PSTs majoring in elementary education which leads to licensure for teaching grades K-8. Participants were asked to respond to several prompts prior to, during, and following a two-day lesson about technology integration in K-8 mathematics. The design of the lesson was informed by Foulger et al.'s (2017) recommendations regarding teacher educator technology competencies and included: (a) an introduction to and exploration of mathematics technological resources; (b) modeling the alignment of K-8 mathematics content with both pedagogy and technology, and (c) collaborative activities in which PSTs designed mathematical tasks which utilized technological resources.

Open-ended prompts were posed to PSTs, including "Find one resource (include the URL) and answer the following questions: (1) For what grade level and CCSS [Common Core State Standards] would the resource be appropriate to use? (2) Explain how this resource might benefit a lesson." at the end of day two and (1) "Locate one resource (include the URL) and describe how you might use this resource to assess understanding in your future classroom." (2) "What do you think is the most practical application of technology in K-8 mathematics, and why?" after PSTs read Johnson et al. (2012) following the two-day lesson. The SAMR model was utilized to investigate the research question, with each response being coded as either substitution, augmentation, modification, or redefinition, according to PSTs' description of how the technological resource would be utilized.

Results

Of the PSTs recruited for this study, 19 consented that their responses may be analyzed for research purposes and 18 successfully completed both prompts. The recruited sample did not allow for an analysis based on demographic factors due to the fact that the vast majority of PSTs in the sample are White females in their third or fourth year of the elementary education program. Thus, demographic information was not gathered in this study.

Intended Purpose of Mathematics Technological Resources

The research question was examined with the following prompts: "Find one resource (include the URL) and answer the following questions: (1) For what grade level and CCSS would the resource be appropriate to use? (2) Explain how this resource might benefit a lesson." and "Locate one resource (include the URL) and describe how you might use this resource to assess understanding in your future classroom." Each response was coded according to the SAMR model based on the capabilities of the technological resource and PSTs' description of how the resource would be utilized in a lesson. In regard to the first prompt, the 18 PSTs who responded demonstrated a strong tendency to integrate technology into K-8 mathematics in a way that provides augmentation (n = 14). Four PSTs integrated technology in a way that modifies the task, while no PSTs described methods of integration where substitution or redefinition are utilized. Similar results were found in relation to the second prompt, in which PSTs favored augmentation (n = 14), while modification (n = 4) and both substitution and redefinition (n = 0) were less prevalent. It is worth noting that a total of 6 PSTs integrated technology in a way that modifies the task in response to at least one prompt.

The left side of Figure 1 displays the Pan Balance applet from the National Council of Teachers of Mathematics' (NCTM) Illuminations collection, which was selected by one PST as an opportunity to integrate technology to teach the commutative property, associative property, and distributive property. The PST supplied the equations located on the left side of Figure 1 and noted that this applet would benefit a lesson due to the visual representation of an equation being either equal to, greater than, or less than another equation. The affordances of the technological resource and rationale provided by the PST classify this instance of technology integration as an augmentation. Functional improvement is present, but a significant redesign of the task due to the integration of technology is not apparent.

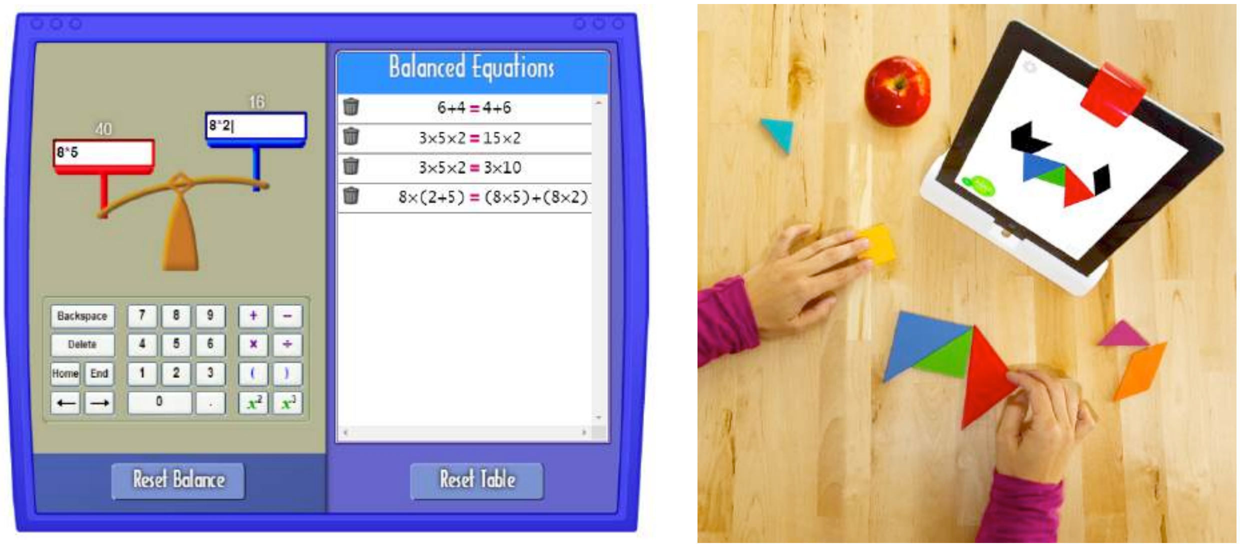

Figure 1: Augmentation - NCTM Illuminations' pan balance applet (NCTM, n.d.) and Modification - Osmo's tangram game (Osmo, n.d.)

The right side of Figure 1 presents the Tangram game to be used in conjunction with Osmo. Osmo, the red-colored device on the top of the tablet on the right side of Figure 1, utilizes the tablet's camera to scan the area directly in front of the tablet and then transfers that image to the tablet's screen. One PST selected Osmo as a technological resource to integrate into K-8 mathematics as a modeling task. The PST noted that Osmo allows for the concurrence of hands-on practice and technology integration in which students might explore the relationships between different shapes and construct/deconstruct various composite figures. The affordances of this game paired with the application described by the PST classify this method of integration as a modification task. The modeling task experiences a significant redesign via Osmo's Tangram game, though implementing this task is not entirely inconceivable without the utilization of the game via Osmo.

Discussion and Implications

Findings in regard to the research question are highlighted by PSTs' tendency to select and describe the integration of technological resources that augment a mathematical task. Similar results were found by Cherner & Curry (2017) when examining preservice English and social science teachers. While there is limited research of this topic within the context of mathematics education, this study uncovers a degree of understanding regarding PSTs' intended purposes of technology integration in a K-8 mathematics setting. These findings also have potential implications in regard to teacher education. The SAMR model was not presented to PSTs in the K-8 Mathematics Methods course, so it is possible that PSTs are simply unaware of the various degrees to which technology integration can impact the quality of a mathematical task. How might we encourage our PSTs to more frequently integrate technological resources in ways that modify and/or redefine the mathematical task? Future research is needed to examine the relationship of both the exposure to and discussion of the SAMR model to PSTs' design of mathematical tasks that utilize technological resources.

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PROSPECTIVE TEACHERS' APPROACHES TO PROBLEM-SOLVING ACTIVIES WITH THE USE OF A DYNAMIC GEOMETRY SYSTEM

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Using digital technologies when working on problem solving tasks allows students to engage in different ways to explore mathematical concepts as well as analyzing multiple approaches that emerge while solving the tasks. For this study, it was important to document the extent to which prospective teachers become aware of the potential of a Dynamic Geometry System (DGS) as a problem-solving tool. To this end, six prospective teachers participated in a series of problem-solving activities meant to be approached by using a DGS. Even simple tasks offer ample opportunities to explore mathematical concepts when representing them within a digital medium, and in turn, the DGS' affordances influence the way participants pose mathematical propositions and validate them while solving and extending problems.

Keywords: Technology, Problem-solving, Geometry, Teacher Knowledge.

Introduction

The introduction of a technological element in the mathematics classroom modifies the way concepts are addressed, thus creating a perturbation within the teaching system. Laborde (2002), mentions two aspects that need to be considered when a certain technology is to be introduced in a learning environment: The domain of knowledge (how are mathematical objects interrelated through such technology? Which aspects are preserved, and which ones are modified?) and the teacher-student interaction (What is the purpose being considered when using that technology? Is the technology used for learning or is it used to support the teacher's discourse?). Technological tools such as Dynamic Geometry System (DGS) have the potential to open up multi-interpretations of mathematical knowledge (Leung, 2017), objects within the DGS do not appear as exclusively virtual to students, but become materialized (Moreno-Armella & Hegedus, 2009) and, therefore, subject to experimentation. One of the most notable outcomes from the study of mathematical objects in a dynamic environment is the emergence of alternative ways to justify mathematical relations (i.e. using *point dragging* to verify if a figure holds a geometric property in certain conditions). Santos-Trigo (2019) mentions that the use of digital technologies in learning environments demands addressing what new pedagogies are needed to frame mathematical working in which learners participate in the construction of mathematical knowledge. To this matter, problem-solving activities can be exploited with the systematic use of digital technologies, allowing teachers and students to examine mathematical tasks from different perspectives that include a plethora of concepts, resources and representations (Santos-Trigo, Camacho-Machin & Olvera-Martinez, 2018). However, teachers' perspective on the nature of mathematical knowledge and the role of digital technologies will define the ways in which students interrelate conceptual knowledge when solving mathematical tasks. Thus, teachers need to rethink the nature of mathematical activities in classroom when students solve problems using a DGS (Moreno & Llinares, 2018). For this study, six pospective teachers participated in a series of problem-solving activities with the support of a DGS. To this effect, the research question that guided this work was: What are some of the ways prospective teachers explore mathematical ideas when solving problems using a DGS?

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Theoretical perspectives

Mediating tools are not epistemological neutral (Moreno-Armella & Sriraman, 2010). In a DGS, motion becomes a key element of mathematical representations. Therefore, a tool like a computer affects the cognition of the user, it reorganizes her ideas. In this way, the computer can no longer be considered as an agent that "does the task of the student” but provides students with a cognitive tool (Moreno-Armella & Sriraman, 2010). In a problem-solving environment, a DGS has the potential to enhance the use of heuristics like analyzing multiple particular cases, and fosters different problem-solving episodes like generate, explore and validate conjectures (Aguilar-Magallón & Poveda, 2017; Santos-Trigo & Moreno-Armella, 2016). These actions, however, are shaped by the subject’s expertise in using the tool. Throughout all problem-solving activities, it becomes important to pay attention to the transit in learners’ use of empirical approaches to the construction of geometric and analytic arguments to support results (Santos-Trigo, 2019). Santos-Trigo & Camacho-Machin (2013) proposed a framework to characterize ways of reasoning that emerge as result of using computational technology in problem-solving via four episodes: (a) comprehension episode, in which the solver needs to think of the task in terms of mathematical relations and how to use the DGS’ affordances to represent the problem (to generate a dynamic configuration); (b) problem exploration episode, where the tool is used to obtain empirically-generated conjectures; (c) search for multiple approaches episode, where students need to think of different ways to solve a problem in order to develop conceptual understanding of mathematical ideas; (d) Integration, a reflection of the different processes involved in the previous episodes.

Participants, methods, and procedure

The purpose of the study was to examine how high-school prospective teachers use a computational tool as a means of exploring multiple concepts derived from solving mathematical problems. Thus, this study is oriented to the analysis of cognitive processes exhibited by the participants and therefore, is of a qualitative nature.

Six prospective teachers participated in a problem-solving course as part of a master’s degree program at the CINVESTAV-IPN (Mexico City). They all had completed a university degree akin to mathematics and were attending the first semester of the program. The activities were conducted for 8 sessions of 3 hours each. The prospective teachers were encouraged to use a DGS (GeoGebra) as the main problem-solving resource. Firstly, they worked individually or in small groups and, subsequently, they presented their work to the group in plenary. Additionally, the participants were asked to prepare a report of their work in a text file and submit it to a google classroom platform. Data were collected through the information of the teacher’s reports and the video recordings of the sessions.

In this research report, I focus on the prospective teacher’s performance related to the following problem: Let ABC be a right triangle with perimeter 12. What are the lengths of the sides such that the triangle has maximum area?

It is important to note that, prior to the problem, participants worked on GeoGebra several construction problems collectively in plenary guided by the instructor. In this sense, they had knowledge about some of the DGS’ affordances.

Results and discussion

To present the results of this study, the participants’ work was structured around the problem-solving episodes described by the framework of Santos-Trigo & Camacho-Machin (2013). Also, it is also worth noticing the extent to which the use of a digital tool like GeoGebra modifies the way mathematical statements are stablished and validated (Santos-Trigo, Camacho-Machin & Moreno-Armella, 2016).

Comprehension episode. Most participants showed mainly algebraic procedures meant to find the function area $A,a.=,36a-6,a\text{-}2.\text{-}12-a..$ Subsequently, they didn't have any difficulties in finding that $,A\text{-}',a.=0$ if $a=12-6,\text{-}2.\approx3.515$, which is the measure of both legs of the right triangle with perimeter 12 such that its area is maximum. However, they were asked to represent the problem in a DGS. That is, they needed to construct a right triangle with perimeter 12 within GeoGebra. All of them showed the following procedure: define a slider a, and draw a circle with center A and radius a. Thus, the radius AB has side a. From B, trace a perpendicular line to AB. This line intersects a circumference with center B and radius $,12(6-a)\text{-}12-a.$ at point C. As a result, a right triangle ABC is obtained, such that its perimeter is always 12 (Figure 1).

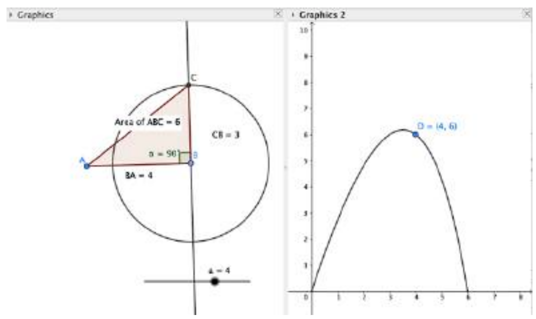

Figure 1: Dynamic configuration for Problem A

Exploration episode. How can the tool's affordances be used to find an empirical answer? Once the participants had a dynamic representation of the problem, they defined the point D: $(a, ,t\text{-}1.)$, where $,t\text{-}1.$ is the area of the triangle ABC. When moving the slider a, point D's trajectory can be visualized through the locus command. In consequence, the value of a can be adjusted so that (visually) point D is located at the vertex of the parabola.

Multiple approaches episode. The way this problem was been represented did not depend on the DGS' affordances, since the key aspect of the statement of the problem was approached in an algebraic way. Thus, participants were asked to re-interpret the given perimeter as a line segment of length 12 instead of a number. Using the coordinated axes, they traced AB with length 12 and placed a point C on the segment. In this way, AC is one of the sides of the triangle and by placing a point D on CB, segment AB will be partitioned into three segments. Are these segments always the three sides of a triangle with perimeter 12? It is important to see that this question does not appear when solving the problem by algebraic means, but in a digital medium, it is important to exploit the opportunities it offers to explore mathematical concepts. Figure 2 shows a triangle ACE such that $,AE.=,DB.$ and $,EC.=,DC..$ Whenever $,AC.>6$ (or more generally, $2,AC.>|AB|$) circumferences A and C will not intersect and, therefore, there will be no triangle. This is a crucial condition that must be stated when working with students because the motion of C must be limited in a way that ACE is always a triangle with perimeter 12 (or AB, in any case). Even though ACE is not necessarily a right triangle, it can be observed that when moving D, there is a position that results in angle AEC to be right. How to place point D such that the triangle is also a right triangle?

At this point, participants struggled to use an element of the dynamic configuration as a resource to obtain information about the mathematical relations involved. After working in small groups, two participants used the command *locus* to note that the motion of point E, because of moving D along BD, *seemed* to be an ellipse (Figure 3). What arguments could be used to support the validity of this conjecture? For different positions of *D*, it holds that $,AE.+,EC.=|CB|$, which means the sum of the distances from point *E* to *A* and *C* is always constant. Hence, point *E* moves on an ellipse with foci *A* and *C*.

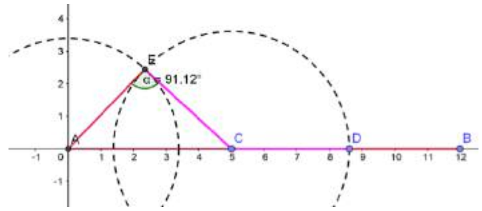

Figure 2. Drawing a triangle of a given perimeter

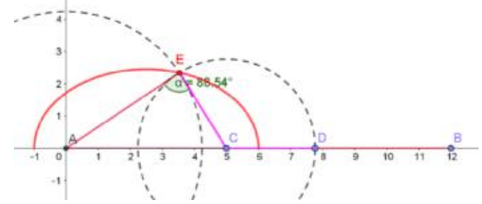

Figure 3. Locus of the point E as D moves along BC

Reflections on the problem. Two elementary forms of using loci within a DGS can be seen as crucial elements for concrete problem-solving strategies. On the one hand, the intersection of the ellipse and the circumference obtained in the analysis of the problem is a way of consolidating the heuristic of relaxing the conditions of a problem through an intersection point that unifies the solution of two subtasks (drawing a right triangle and tracing a triangle with a given perimeter). On the other hand, using a locus that represent a variation phenomenon can serve as a departure point for learners to make use of different geometric and algebraic resources to build and make sense of a robust dynamic configuration. The DGS also provides a scenario where mathematical discussions can be constantly extended. For instance, tracing the triangle considering AC as the hypothenuse rather than the side or restating the problem into considering an isosceles triangle instead of a right triangle with fixed perimeter.

Concluding remarks

In a digital medium, mathematical objects involved in the dynamic configuration of a problem become executable and react to the user's actions. This, in turn, allows the users to further extend their reflections or to find alternative approaches to the problem. In Problem A, the use of loci was underpinning in the formulation of conjectures and participants had to validate mathematical propositions stated in terms of the DGS affordances: what arguments can be stablished such that a certain property holds for the dynamic configuration when movable points are dragged? Prospective teachers were able to explore the concept of ellipse as a resource that can be useful in tasks related to the construction of triangles with a given perimeter. What is more, the concept of locus became a way to organize the use of problem-solving strategies such as solving similar simpler problems, simplify the conditions of the statement or solving many cases. As the activities developed, they became more prone to experiment with the tool and to find different solution paths that could open up for different kinds of mathematical discussions.

Teachers need to be exposed to environments where they can experience at firsthand how a digital technology affects the organization of mathematical ideas. When students use digital artifacts like a DGS, they rely on them to bridge the gap between mathematical ideas and their personal experiences through actions that can develop in the medium such as dragging or measuring attributes. If digital classrooms are to be successful, then teachers need to be fully aware of how these processes develop and relate to the generation of mathematical knowledge (Monaghan & Trouche, 2016).

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DESIGNING FOR AN INTEGRATED STEM+C EXPERIENCE

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In this paper we present an integrated design approach for bridging content between science, technology, engineering, math, and computational thinking (STEM+C). We present data from a design experiment to show examples of the kinds of integrated reasoning that students exhibited while engaging with our design. We argue that covariational reasoning can provide strong scaffolding in making integrated connections between the STEM+C content areas.

Keywords: Interdisciplinary Studies, STEM, Design Experiments, Computational Thinking.

To integrate math and science content, science materials often simply provide graphs while math materials often merely mention science terms for context. However, these efforts do not show the ways in which students' mathematical reasoning may influence their understanding of science, or how their scientific reasoning may influence their understanding of mathematical ideas. As English (2016) argued, in STEM integration there is a need for a more balanced focus on each of the disciplines, especially mathematics which is usually underrepresented. To illustrate this reciprocal relationship between mathematical and scientific reasoning, we looked for a design approach that could honor both math and science content. Specifically, our aim was to explore the research questions: (a) What kind of design integrates science and math for students? (b) What kind of reasoning do students display as they interact with this design?

Design Framework for Integrated Learning

First, we considered the power of covariational reasoning for bridging the two disciplines. Covariational reasoning is the mental coordination of simultaneous changes in two related quantities (Carlson et al., 2002). Mathematically, covariational reasoning has shown to be a strong building block towards the introduction of functions and graphing (Confrey & Smith, 1995). In terms of science, we considered that by engaging in covariational reasoning as they actively examine the interplay of variables in natural phenomena, students would develop deeper understandings of those phenomena than they might from exploring them only in terms of cause and effect relationships. To put it another way, there is a difference between reasoning about a cause and effect relationship, for instance, the depth of the rock affects its temperature, and reasoning covariationally about a relationship, for example, the temperature is changing as the depth of the rock is changing. This study of simultaneous change exhibited by covariational reasoning presented a promising route for supporting students' development of integrated forms of math and science reasoning.

We also considered the power of digital environments for designing simulations that dynamically model abstract mathematical and scientific concepts. We hoped that exploring a simulation would provide multiple trials and rapid feedback, supporting an inquiry environment (Meadows & Caniglia, 2019). Our goal was to encourage students to use the simulation to engage in inquiry practices such as questioning, developing hypotheses, collecting data, and revising theories (Rutten et al., 2012). This use of simulations to model and interact with data is also defined by Weintrop et al. (2016) as a form of computational thinking. They describe a taxonomy of computational thinking that includes practices such as Collecting, Analyzing, and Visualizing Data as well as Using Computational Models to Understand a Concept.

Finally, we gave careful attention to the design of tasks and questioning, aiming to shift students' attention to specific elements of the model and influence the nature of their interactions with those

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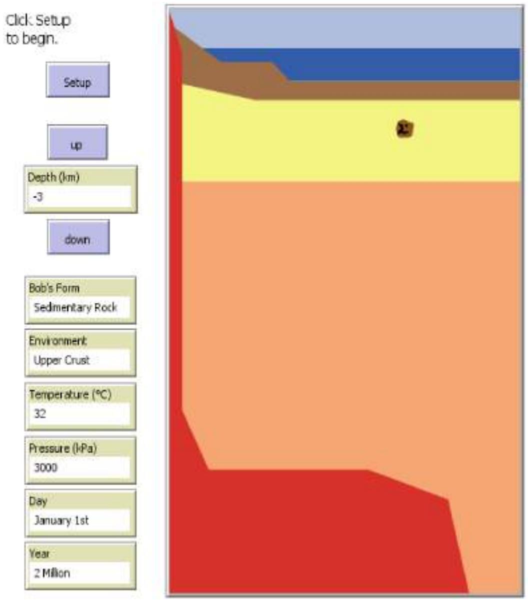
elements. The questioning was organized to encourage students to explore specific relationships, such as “What have you observed about how the temperature and pressure change as Bob moves deeper underground?” Our goal was to prompt the students to engage in some of the Carlson et al. (2002) mental actions of covariational reasoning. These include coordinating the change of one variable with changes in the other variable (MA1), coordinating the direction of change of one variable with changes in the other variable (MA2), coordinating the amount of change of one variable with changes in the other variable (MA3), coordinating the average rate-of-change of the function with uniform increments of change in the input variable (MA4), and finally coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function (MA5).

We also considered connecting the dynamic representations of relationships in the simulation with the graphing of those relationships. This connection was found to advance students’ conceptions of graphs of functions as a representation of coordinated change (e.g., Ellis et al., 2018). Students often fail to connect graphs with the covariational relationships they represent (Moore & Thompson, 2015), therefore, beyond simply describing and then having students graph these relationships, our goal was to ask them to use the simulation to collect data and graph these relationships. Our conjecture was that by engaging in these kinds of reasoning and practices as they interacted with our design, students would construct their own conceptual bridges in the context of an integrated STEM+C experience. In this paper, we present this design approach by providing an example using the phenomenon of the rock cycle, which is part of a larger collection studying science phenomena (e.g., Basu & Panorkou, 2019; Zhu et al., 2018).

An Example of Integrated STEM+C Design From the Rock Cycle

The earth science concept of the rock cycle describes the cyclical changes rocks experience due to the earth’s thermal energy. Like many natural phenomena, the rock cycle involves multiple variables. By identifying relationships such as the increase in temperature and pressure as depth below ground increases, we conjectured that covariation can be used as a link to integrate science and math. To encourage students to investigate for themselves how these quantities covary in the rock cycle, we developed the Bob’s Life simulation. The Bob's Life simulation (Figure 1) models the life of a rock named Bob near the sea on a volcanic island as he experiences different rock cycle processes and takes on different forms. The student controls Bob's depth in 1-km increments to investigate the behavior of the model. The simulation provides immediate visual feedback as the animation shows Bob moving and changing his form.

Though we used real geological data (e.g., Becerril et al., 2013) as a starting point to describe Bob’s path, these relationships are not always perfectly linear in the real world and can vary widely in real environments (e.g., de Wall et al., 2019). However, by sacrificing a certain amount of scientific realism, we were able to design the relationships between Bob's depth and the temperature and pressure to be piecewise linear functions with planned regions of friendlier numbers for students to graph. For example, in the Upper Crust environment we selected the numbers such that for each 1 km Bob moves down the temperature increases by 8 °C and the pressure increases by 1,000 kPa. This design choice maintains the scientific integrity of the model while creating an accessible data set for middle school students to investigate. The Bob’s Life simulation is thus a simplified model of real-world processes designed to have useful pedagogical features (Weintrop et al., 2016).



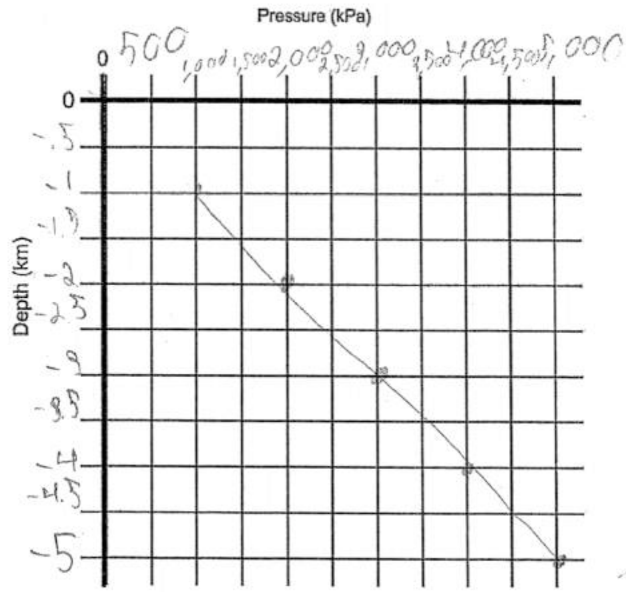


Figure 1 (left): The Bob's Life Simulation Interface
Figure 2 (right): Michael's Graph of Depth and Pressure

Students' Forms of Integrated STEM+C Reasoning

We focus on a whole-class design experiment (Cobb et al., 2003) in a middle school classroom and present episodes from a single pair of students, Laura and Michael, to illustrate examples of students' integrated reasoning as they interacted with our design. We adapted the Carlson et al. (2002) framework to identify episodes that illustrate students' covariational reasoning. We also remained open to students' use of other forms of reasoning, such as multivariational reasoning (Kuster & Jones, 2019) and computational thinking practices (Weintrop et al., 2016). At the same time, we examined how our design seemed to influence this reasoning by investigating the dialectic relationship between design and learning.

At the start of the design experiment, students were asked to freely explore the simulation. We included this free play to build students' interest and to encourage them to begin using inquiry practices (Rutten et al., 2012) as they interacted with the simulation and described what they noticed. During this time, we asked Laura about what she had noticed so far:

> Laura: I'd say, there's a rock named Bob, and you can control his depth. And if you go, the deeper you get, his form changes, his environment changes, his temperature, the pressure that's being put on the rock, the day, and the year change.

Laura's reasoning shows that the free exploration of the simulation offered a constructive space for her to both inquire into the behavior of the model and also to reason about how the rock's form, environment, temperature, and pressure change based on its depth. She coordinated changes in the variable she controlled, Bob's depth, with several other variables that were also changing simultaneously, illustrating MA1 reasoning. We would also argue that she exhibited multivariational reasoning since she was able to coordinate multiple variables at the same time.

After the free exploration, Laura was asked to use the simulation to investigate how the temperature and pressure change as Bob moves deeper underground. She responded that "the deeper you go, the more pressure's being put on it" and "the lower Bob goes in depth, the higher the temperature increases." Her responses show that she coordinated the direction of change of Bob's depth with the direction of change in pressure and temperature, engaging in both MA2 and multivariational reasoning. Immediately after giving the latter response, she used the simulation to illustrate by moving Bob several kilometers lower and pointing at the increasing number in the temperature readout. Laura's action shows that she used the simulation to understand what is changing and how it is changing in the model, illustrating a form of computational thinking.

Next, we asked students to collect data and graph the relationship between the variables. When asked to describe the numerical relationship in his graph (Figure 2), Michael said, "I got 1, as Bob moves 1 km deeper underground, the pressure increases by 1,000 kPa, because each km you go deeper, is 1,000 pressure compacted on Bob." Even though Michael's graph visually appears to be a falling line, his response shows that he viewed this as representing the increase in pressure as Bob moved deeper. This is evidence of MA3 reasoning as well as an indication that he might have been imagining his graph as an emerging record of this covariational relationship rather than a static shape (Moore & Thompson, 2015). This also shows that our design supported Michael in using computational thinking practices of collecting, analyzing, and visualizing data.

At the end of the experiment, students were asked to find intermediate values on their graphs. For example, we asked Michael to state the depth at which Bob would experience 4,500 kPa of pressure. Michael stated, "I think it's 4½ km underground." He explained that this is "Because 4 km underground is 4,000, and every .5 is 500, so if you do 4.5, that's going to be 4,500." Earlier, Michael had also observed, "1 km underground is 1,000, and if you add a .5 that's 500 more so that's 1,500." This shows that Michael had not only noticed that the pressure changed by a certain amount for each 1-km step, but that he had also used his graph to see that he could describe this relationship in terms of steps of .5 km. Michael's observation that "every .5 is 500" is an example of MA4 because it shows that he was reasoning about the rate at which pressure was changing for equal incremental changes in Bob's depth. It also shows that he used his graph as a second model to understand the relationships, illustrating computational thinking.

Conclusion

In response to RQ (a), we believe that this experiment has shown how our design approach can be used to develop integrated STEM+C instructional modules that have a more *balanced focus* (English, 2016) on each of the disciplines. This work supports our assertion that covariational reasoning can serve as a bridge to integrated STEM+C learning. Guided by the task design and questioning, students explored and explained the model's behavior in terms of both rock cycle processes and mathematical relationships. In response to RQ (b), the analysis of students' reasoning showed that they developed a sophisticated understanding of the science content which included identifying the factors that influence the rock cycle and constructing relationships between the involved variables. Students reasoned covariationally at various levels (Carlson et al., 2002) as they interacted with our design. We have also shown that students displayed multivariational reasoning (Kuster & Jones, 2019) and engaged in computational thinking practices (Weintrop et al., 2016). In the future, we plan to continue refining this integrated design framework with the rock cycle module as well as other modules that use covariational reasoning to build conceptual bridges between science and math. We also plan to explore other topics, such as probability and statistics, which might also be able to play the same bridging role that covariation plays in our Bob's Life simulation and investigation design.

Acknowledgments

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INTEGRATED STEM EDUCATION THROUGH GAME-BASED LEARNING

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Science, technology, engineering, and mathematics (STEM) education continues to garner focus and attention from teachers, students, researchers, policymakers, and businesses due to the vast importance of technology in the world. The integration of STEM subjects has the potential to make learning relevant and more engaging for students, which can increase their mathematical knowledge. In this paper, I focus on integrated steM education through open-ended game-based learning within a technological context. Integrated steM education is the integration of STEM subjects with an explicit focus on mathematics. The possibilities for integrating mathematics and technology through open-ended game-based learning has increased in recent years. Recommendations for future work will be discussed.

Keywords: Curriculum Enactment; Middle School Education; STEM

Interest, discussion, and work around technology integration in education continues to grow as advances in technology permeate our daily lives. The way that technology is integrated into the classroom is important in ensuring optimum learning outcomes for students. In game-based learning, technology integration should allow students to engage in more high-level thinking and have new experiences with mathematics that would be difficult without the technology. There is a distinction between how technology is used as an *amplifier* and how it is used as a *reorganizer* of mental activity (Pea, 1985). Technology as an amplifier enables students to perform more efficiently tedious processes that might be done by hand. This does not change what students do or think but does save time and effort and improves accuracy. As a reorganizer, technology is capable of effecting or shifting the focus of students' mathematical thinking or activity. This can enable students to do more higher-level thinking. A recent review of the literature suggests that the potential to integrate technology in a transformative way is not being met. For instance, researchers classified studies based on the ways in which technology has been integrated in mathematics education since 2009. The findings from this work indicated that the majority (61%) of the 139 studies were similar to an amplifier approach in that the technology was used as a direct substitute for traditional approaches with some functional improvement (Bray & Tangney, 2017). This result suggests that although innovative practices undoubtedly exist, the technology that could improve students' learning experience is generally not well implemented in the classroom (Hoyles & Lagrange, 2010). The purpose of this paper is to describe research done with middle school students (ages 11-15) and game-based learning to highlight productive principles for technology integration with mathematics.

Integrated steM Education Framework and Literature Review Integrated STEM Education

Researchers have noted that mathematics is often not emphasized in the integration of STEM subjects (English, 2017; Gravemeijer, Stephan, Julie, Lin, & Ohtani, 2017). In response to this, I have proposed that mathematics teachers and researchers focus on integrated STEM. Integrated STEM is the integration of STEM subjects that has an explicit focus on mathematics (Stohlmann, 2018). It is an effort to combine mathematics with at least one of the three disciplines of science, technology, and engineering, into a class, unit, or lesson that is based on connections between the subjects and open-ended problems. Further, integrated STEM education is an approach that builds on natural connections between STEM subjects for the purpose of (a) furthering student understanding of each discipline by building on students' prior knowledge; (b) broadening student understanding of each discipline through exposure to socially relevant STEM contexts; and (c) making STEM disciplines and careers more accessible and intriguing for students (Wang, Moore, Roehrig, & Park, 2011). There are three main ways in which integrated steM can be implemented by mathematics teachers: engineering design challenges, mathematical modeling with science contexts, and mathematics integrated with technology through open-ended game-based learning (Stohlmann, 2018). Each of the three approaches involves the integration of mathematics with a different science, technology, or engineering (STE) focus. In this paper, I focus specifically on mathematics integrated with technology through open-ended game-based learning.

Game-based Learning

Game-based learning has drawn international interest and has been reported as an effective educational method that can improve students' motivation and performance in mathematics (Byun & Joung, 2018; Wang, Chang, Hwang, & Chen, 2018). Students enjoy playing technology-based games whether it is video games or apps on their phones. However, when used in the mathematics classroom, game-based learning is often not implemented with best practices for teaching mathematics in mind (Byun & Joung, 2018). A meta-analysis was conducted to look at the overall effect size of game-based learning on K-12 students' mathematics achievement. Seventeen studies were identified that had sufficient statistical data from a time frame of the years 2000 to 2014. The overall weighted effect size was 0.37, which is a small effect size. There were 71 authors in the studies reviewed for the meta-analysis, with only five of these authors having a background in mathematics education. This research demonstrates the need for further studies on effective game-based learning approaches and best practices in mathematics education.

For example, most of the games used in the studies involved drill and practice (Byun & Joung, 2018). An example of one popular game includes students solving traditional, non-contextual practice problems in order to get more speed for a race car and attempts to take advantage of students' interest in videogames (Math-Play, 2019). However, in this type of game, students only receive feedback if the answers are correct or incorrect and do not receive support for improving their conceptual understanding. These types of games also emphasize that mathematics is about speed and focus on the memorization of ideas (Bay-Williams & Kling, 2015). Game-based learning for mathematics should move beyond drill and practice.

Another area that requires improvement in the implementation of game-based learning is for students to be able to work collaboratively or competitively. This has been suggested to be more effective than individual gameplay (Hung, Huang, & Hwang, 2014). A study in which this collaboration versus individual play was investigated involved 242 students with an age range of 11 to 15 years. There were four conditions in the study: collaboration and competition, collaboration control, competition control, and control. Overall, the game-based learning improved students' proportional reasoning, but the effects did not differ between conditions (Vrugte et al., 2015).

For game-based learning in integrated STEM, I refer to games in which the mathematics is integrated into the gameplay in a substantial way other than traditional practice problems. When structured well through open-ended problems, technology-based mathematics games may engage students in mathematics and help develop their conceptual understanding.

Methods

This study was structured as a teaching experiment (English, 2003). Nineteen students voluntarily enrolled in a five-week Saturday STEM program at a large urban university. The students were audio-taped and student work was collected including screenshots of the students' work in Desmos. Researcher field notes were also collected. Desmos is an online graphing calculator, but also has a suite of classroom activities available with some of the activities being game-based. In the lessons, students can share ideas and ask questions of one another. The principles that guide the Desmos' team lesson development include the following:

- Use technology to provide students with feedback as they work.
- Use the existing network to connect students, supporting collaboration and discourse.
- Provide information to teachers in real time during class (Danielson & Meyer, 2016, p. 259).

Little research has been conducted on these activities, but they have the potential to enable teachers to develop students' conceptual understanding. Research on how students develop conceptual understanding through technology integration tasks is important to investigate. The specific research question for this paper was the following. *How do students use and develop mathematical vocabulary while playing the Polygraph lines game?*

Results

In this paper I describe results from a Desmos game-based activity called Polygraph lines. I analyzed the data with an interpretative approach by looking at the ways in which students used mathematical vocabulary in the game. In this game, sixteen linear graphs are given. One student selects one of the graphs and the other student asks yes or no questions to determine which graph has been selected. Between games students are shown questions that other students ask. The teacher also is able to view and have a record of all questions asked in each game. Table 1 has the initial questions that were asked by 4 of the groups.

Table 1: Initial questions from polygraph lines

Group	1	2	3	4
Questions	-Does your line go through the origin? -Is your line vertical? -Is your line horizontal? -Is your line in quadrant 2 and 3?	-Does your line have a positive slope? -Does your line pass through the origin? -Does your line have a slope of 0 or undefined? -Does your line have an undefined slope? -Is the slope negative? -Does your line have a slope of 0? -Does your line have a slope greater than -2? -Is the slope of your line greater than or equal to 1?	-Is your slope positive? -Is the y-intercept positive? -Does it cross the origin? -Is your slope steep? -Is your line undefined? -Is your slope negative? -Is your line's linear equation zero or undefined -Is your line horizontal? -Is your line vertical?	-Is your line horizontal? -Is your line vertical? -Does your line intersect 2 quadrants? -Does your line intersect 3 quadrants? -Is your line positive? -Is it negative? -Does it go through the origin? -Does it pass through 3 squares?

After playing the game several times, the students discussed what quality questions to ask and strategies for asking the least amount of questions. Several questions appeared in common in the groups: "Is your slope positive?" "Is the slope negative?" "Is your line horizontal?" "Is your line vertical?" "Does your line go through the origin?" Groups also came up with questions of what quadrants the line crossed through, though not all groups used the term "quadrants." Through playing the game and subsequent discussions, students were able to make use of mathematical vocabulary including slope, positive slope, negative slope, horizontal line, vertical lines, origin, and quadrants. Desmos continues to develop their freely available activities, and further research is warranted on the effect of the games on students' mathematical understanding of linearity and motivation to learn mathematics.

Conclusions

It has been found that the use of puzzles and gamification in mathematics increases students' participation and engagement (Bryne, 2016). The research in this study provides early support for game-based learning done through integration with Desmos. This method can encourage students to develop mathematical understanding in an engaging game-based context. Through investigating technology game-based learning I have developed several important principles that should be incorporated to help make it more likely the game-based learning will be effective. First, the technology integration should allow for the creation of new tasks that would not be possible without the technology or for significant task redesign (Puentedura, 2006). Second, the tasks used should be worthwhile tasks. These tasks have no prescribed methods and there is no perception that there is only one "correct" strategy (Hiebert et al., 1997). Third, the tasks should be aligned with grade-level standards. Fourth, the tasks should enable students to work with multiple representations. Fifth, the technology should provide students feedback. Finally, the tasks should be open-ended and allow for discussion and multiple solutions (Stohlmann, 2019). When structured well, technology-based mathematics games can engage students in mathematics and help develop their conceptual understanding.

Too often middle school students perceive mathematics to be dull, irrelevant, and too difficult (Grootenboer & Marshman, 2016). Further research on game-based learning in the mathematics classroom can help to provide one way to address this problem. Students can play video games for hours on end though with the time going by quickly and the students persevering in problem solving. Bringing more game-based learning into the mathematics classroom has the potential to help more students be successful in mathematics.

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EMBODIMENT AS A ROSETTA STONE: COLLECTIVE CONJECTURING IN A MULTILINGUAL CLASSROOM USING A MOTION CAPTURE GEOMETRY GAME

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The Hidden Village (THV) is a motion-capture video game for investigating how physical movements foster mathematical thinking and proof practices based on principles of embodied cognition. Analysis of the interactions of students in an all-Limited English Proficiency Title 1 high school geometry classroom revealed ways simulated enactment and collaborative gestural co-construction of mathematical ideas can bridge language barriers. These informed a redesign of THV to support both individual and collaborative play, as well as a collection of authoring tools for players to create their own content and upload it to an online database shared by users worldwide. Players, teachers and learners can implement custom directed movements that could foster deeper mathematical understanding and engagement for them and their peers.

Keywords: Technology; Reasoning & Proof; Geometry and Geometrical and Spatial Thinking

New technological interventions for learning mathematics are leveraging the embodied affordances of motion-capture technology to teach proportional reasoning (Abrahamson, 2015), algebraic reasoning (Ottmar et al., 2012), numerical training (Fischer et al., 2015), and geometric angles (Smith et al., 2014). As a design experiment (Brown, 1992), the development of *The Hidden Village* (THV) has been an iterative process of refining and extending its instructional application. THV is designed to help researchers better understand the grounded and embodied nature of geometric proof production by investigating how directed body actions, in combination with verbal prompts (i.e., pedagogical language), help students conceptualize the underlying mathematical ideas for geometric proof practices. Conducted in authentic classrooms, this type of *in situ* research exposes technologies like THV to dynamic environments in which "surprising occurrence[s]" emerge from students' collaboration and co-constructions to become sources for "ontological innovations" in the design process (diSessa & Cobb, 2004, p. 86). We present instances from gameplay of THV v.5, in which students renegotiated how the game was played, using a range of collaborative gestures and discussions to communicate shared understandings and present the newest revision, THV v.6.

Theoretical Background

Mathematicians use particular practices in their formulation of valid proofs. Research suggests that proof "is a richly embodied practice that involves inscribing and manipulating notations, interacting with those notations through speech and gesture, and using the body to enact the meanings of mathematical ideas" (Marghetis, Edwards, & Núñez, 2014, p. 243). Learners may derive new ideas and insights relevant to understanding and solving tasks based on their engagement in physical motions. Prior research shows that students' dynamic gestures reliably predict mathematical

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intuitions and proof validity (Nathan et al., 2018), even when controlling for spatial ability, gender, expertise, prior geometry knowledge, and speech content. Playing THV has been shown to help foster the production of beneficial gestures that promote higher proof performance (Authors, date). As study of collaborative proof has emerged, a growing body of evidence is demonstrating that collaborative gestures, as social extensions of cognition, are relevant to learner-learner interactions in the processes of mathematical sensemaking (Walkington et al., 2019). Collaborative gestures are physically and gesturally taken up by multiple learners. These co-constructive activities often extend cognition by echoing ideas, mirroring each other's reasoning, alternating in co-constructions and jointly operating in the same problem spaces (Walkington et al., 2019; *see* Table 1).

Table 1: Categories of Collaborative Gestures (adapted from Walkington et al., 2019).

Echo	Simple Echo (SE)	One learner makes a gesture and then a second learner makes the same (or a very similar) gesture afterwards.
	Echo & Build (E&B)	The second learner must change or add to the echoed gesture in some way.
Mirror	Simple Mirror (SM)	One learner makes a gesture, and then a second learner makes the same (or a very similar gesture) nearly at the same time.
	Anticipation (A)	One learner is gesturing, and then a second person anticipates a gesture they are about to do (correctly or incorrectly).
Alternate	Alternate & Build (A&B)	One learner gestures their understanding, and then another learner follows up, building upon or extending their reasoning.
	Alternate & Redirect (A&R)	One learner gestures their understanding, and then another learner follows up with a different gesture and reasoning.
Joint	(J)	Multiple learners manipulate mathematical object(s).

Such multimodal discursive practices in communicating mathematics (e.g., Edwards, 2009; Hall, Ma, & Nemirovsky, 2015; Radford, Edwards, & Arzarello, 2009; Roth, 1994, 2001) often externalize representations of students' minds and help establish and maintain *intersubjectivity* in a shared problem space (Nathan & Alibali, 2007). The design of THV draws from the theory of *Gesture as Simulated Action* (GSA; Hostetter & Alibali, 2019), the theory that people gesture because they activate perceptual-motor processes in the brain when they think about--and therefore simulate--the spatial or motoric properties of an idea while speaking and thinking. In this way, gestures can reveal the spatial and motor correlates of abstract and generalizable mathematical thinking. THV also draws on Nathan's (2014) model of *action-cognition transduction*, in which learners' movements serve as inputs that can drive the cognitive system into related cognitive states much like the cognitive system can, reciprocally, direct the motor system to make specific movements in response to one's thoughts and goals. It is this bi-directional process, in which cognitive states give rise to physical actions and vice-versa that THV is designed to demonstrate. As an embodied intervention, THV elicits movements (i.e., directed actions) from its players with the intent of influencing their cognitive processes in ways that fostering mathematical insights in support of the proof process. The action-based intervention from game play not only helps elucidate mathematical ideas for learners (Nathan, 2014; Walkington & Nathan, 2017) by engaging mathematically relevant simulations, but it also offers novel, embodied design opportunities for education researchers and practitioners.

In cases of collaborative gesture, *intersubjectivity* is a key determinant in the amount of collaboration and co-construction between actors. Figure 1 presents a 3D model of collaborative gesture ecology. As individuals collect into groups, intersubjectivity increases as group members echo, mirror, alternate and jointly gesture. Collaborative game play of THV can facilitate transduction to help students communicate their ideas about mathematics.

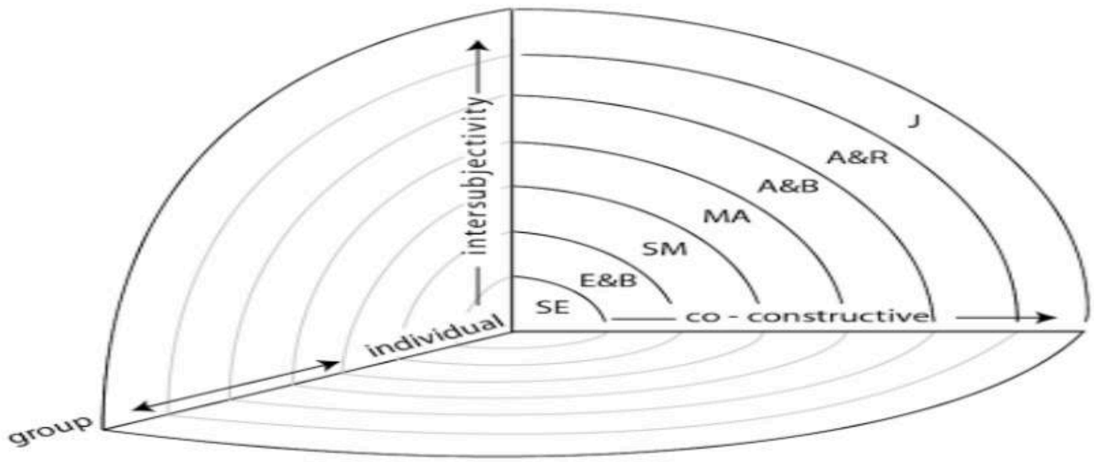

Figure 1: A 3D model of collaborative gesture ecology.

The Hidden Village

THV is a 3D motion-capture video game that delivers interactive math geometry curriculum in which each player mimics movements of in-game characters and then reads a geometry conjecture to determine if it always true or false. Each level of the game is comprised of 6 parts: Players meet members of the hidden village (Panel A), who implore players to perform movements (i.e., mathematically relevant directed actions detected by the MS Kinect; B). Next, players are given a math conjecture (C) and asked to indicate if the conjecture is true/false and why (D), followed by a multiple choice (E) and rewards and game achievement (F; *see* Figure 2).

Figure 2: The Hidden Village Game Play.

For example, in the triangle inequality conjecture (Figure 2, Panel C), a player will have performed the three movements (B) where they experience the arms extended laterally, then at an angle to the midline and then straight in front of the body. This series of movements is repeated 3 times. Next, they read the conjecture (C) and are prompted for an explanation (D). It is here that we video record their spontaneous representational and dynamic co-speech gestures that are hypothesized to contribute to their intuitions, insights and proof production. Then, players choose from among 4 multiple choices (E) before being rewarded by exposing a new portion of the Hidden Village map, a symbol, and energy strings to help power the ship (F).

Methods

Participants and Procedure

Over two days, we observed eight students in an all-LEP (Limited English Proficiency) Title 1 high school geometry classroom as they played THV v.5. Students' languages and ethnicities included Spanish from Central and South America, Arabic and French from North Africa, Hmong from Southeast Asia, and Chinese. Players were grouped as yoked pairs, alternating playing THV and observing game play of their partner.

Coding. Gameplay was audio and video recorded. The video clips were coded for whether participants made individual and collaborative gestures while validating the conjectures. Collaborative gestures were then coded by types identified by prior research (Walkington et al., 2019): echo, mirror, alternate, and joint (*see* Table 1).

Cases of Collaborative Gesture

Students, in light of the variability in English proficiency, dealt with the delivery of the game narrative, instructions, and mathematics in many ways, some unanticipated. In particular, many of the students used their bodies, objects in the room, and the bodies of other students to reason mathematically and formulate their justifications and proofs. To address comprehension and language production challenges, some students turned the dyadic game play into a collective activity where students contributed to each other's successful game play. They translated narratives and conjectures for each other and used directed actions to clarify and ground the math. Gestures traveled from the game to students and then crossed into different student groups as the movements simulated their mathematical ideas.

In this first transcript (*see* Figure 3), the students address the geometry conjecture: *The sum of the lengths of two sides of a triangle is always greater than the length of the third side*. In Figure 3 (below), students (S) engage in a discussion that leads to a series of collaborative gestures (panels A – F). Student 1 (S1, standing), turns to his partner, S2, to discuss the validity of the statement. S1 (*right*) listens to S2 (*left)* and *mirrors* (C) his gesture and then *builds* on the idea (D) in which S1 *anticipates* S2's gesture (E) before finishing *building* his explanation (F).

Figure 3: Embodied Collaboration – Mirror, Alternate and Build Gesturing.

S2: What are you doing? [Giggling]
S2: The sum of the length of two sides of a triangle is always greater than the lengths of the third side.
S1: The sum of the lengths of two sides of a triangle is greater than the length of the third side... No?
S2: Yes, because like the hypotenuse is the like, the greater one.

S1: Oh, yea, yea, yea.
S2: But if they add the other, the other side, then...
S1: The sum of the opposite angles...yea, so it is true.

In this next transcript (*see* Figure 4), S1 (*left*) and S2 consider the false conjecture: *If you double the length and width of any rectangle, then you exactly double the area*. S2 (a student from West Africa) explains the "doubling" of the sides to S1 (a student from Central America). S2's gestures extend a side (panel A), and S1 *echoes* the gesture (B). Then, S1 *builds* (C) upon the imaginary rectangle and that gesture is *echoed* by S2 (D), which is then *mirrored* by S1 (E), at which point they *alternate, build and redirect* gestures (F) as S1 gains insights about the relevant geometric relations.

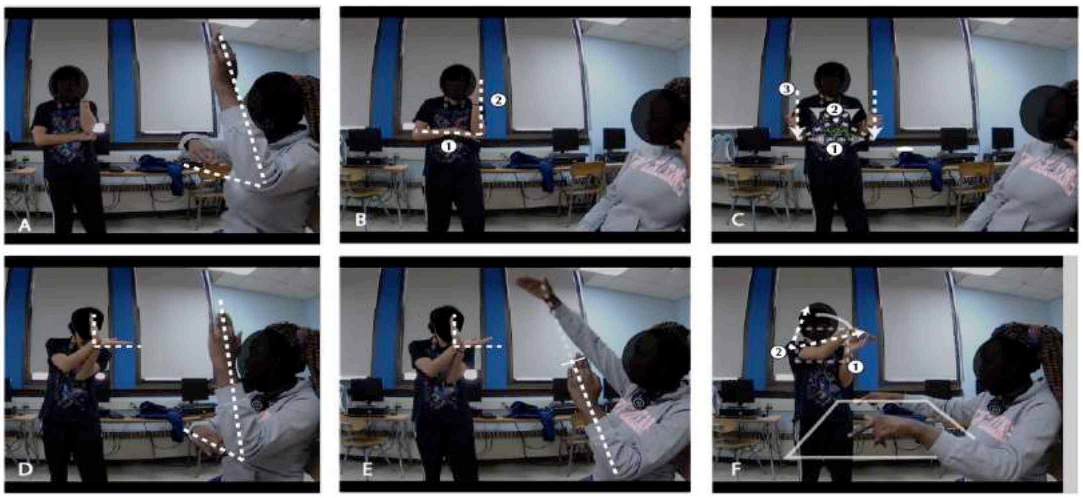

Figure 4: Embodied Collaboration – Echo, Mirror, Build, Alternate & Redirect Gesturing.

S1: So, the width of the rectangle... the area is...
S2: They, they multiplied it by 2 like if... if the length...
S1: So does it, does it have like, like this part?
S2: Let me show you the length – If the length was three
S1: Yeah
S2: They add another 3 and that becomes 6... If this one was 2, they add another 2 and that becomes 4.
S1: Oh...
S2: And they asking if the area, the area is like, like if that was a rectangle, the area is here, what is inside the rectangle?
S1: The angle?
S2: They asking if that would double, like if you...
S1: So... [Giggling]

In the final series (*see* Figure 5), S2 (*right*) asks for help translating words into Spanish and beckons S3 (*left*) over to interpret S1 (*center,* panel A). S3 translates S2's words in Spanish while *echoing* and *mirroring* the arm shape used by S3 (B & C) and replies "*Todo todo todo*" to S2 while adding several small circular motions between them (D). At this point, S3 *mirrors* S1's gestures (E) and then they gesture *jointly* (E) as they use a mutually shared discourse space. Finally, their discussion culminates in their convergence over a nearby tabletop to offload their ideas onto a workspace (F).

Figure 5: Embodied Communication – Echo, Mirror & Joint Gesturing.

S2: [Calls out to other Spanish speaking student for assistance]
S2: [to S3], can you explain to [S1] in Spanish what I mean? That, they double?
S3: Yea… What did it say?
S2: Like, here, this question, they double the length, and they double the, I don't know how to read in English...
S3: Oh.
S2: We talked about it... the width... [S1 and S2 engage in discussion in Spanish (Panels B – F)]

A Game for Collaborative Embodied Co-Creation

In effect, the students' collaborative co-constructive discussions while playing the original 8 conjectures in THV highlighted how varying levels of intersubjectivity combine to surpass language barriers and clarify concepts. Observations of students playing THV proved valuable for informing a redesign of THV. We learned that students may come up with novel movements that will help them understand the mathematical relations more clearly. Based on our observations, in the latest build of THV, students and teachers can now co-create the game characters' movements as ways to foster their own mathematical sensemaking and support the geometric reasoning of their classmates.

The conjecture editor allows players to co-create new levels of the game, including adding new conjectures and multiple-choice responses (*see* Figure 6). This content is stored and accessed via an external online database that allows users to share and play one another's content to support further collaborative, creative play, and learning. Additionally, players can collaboratively co-create new directed actions for a new conjecture using the Pose Editor (Figure 7), which is how THV "learns" to recognize new, user-generated poses. The process of creating and assigning movements is expected to deepen students' understanding of the embodied basis of geometric relations. When forming new poses (see Figure 7), each directed movement is comprised of 2-3 poses (starting, intermediate, and the target) that players design by posing the figure using pivots in the elbows and wrists. Once an individual pose is complete, players confirm or reset the figure. Once players have created all the poses (1), they can preview the sequence of directed action movements (2) in the form of a short GIF-like movie. Then can then set the matching tolerances (3) (i.e., % of allowable error for motion capture detection).

Figure 6: THV v.6 Conjecture Editor. Allows students to author their own content, including designing their own directed actions. Creating a New Conjecture (1) opens the Edit Conjecture (2) portal, where players name their team, name their conjecture, enter keywords, create a PIN, design a new icon, publish the conjecture to the online database repository, create their own directed actions, write the conjecture and devise 4 multiple choices for players to choose from and indicate which is correct.

A study that is currently underway in a set of ethnically and linguistically diverse, mixed ability high school geometry classes will assess how authoring conjectures and poses (i.e., directed actions) contributes to student learning. We hypothesize students making new content for the game will think, act, and talk through the ways that directed actions can foster mathematical insights and proof performance.

Figure 7: THV v.6 Pose Editor. The figure's limbs are posable via mouse movement. Figure can be rotated using a right-click drag or can be reset back to its origin (4). Being able to rotate the figure proved a critical for properly designing 3D directed actions.

Once students or teachers have created a collection of conjectures, they can cluster conjectures together in a game module (currently up to 8 maximum) for others to play (*see* Figure 8). Users can download individual conjectures or entire modules for their own use. As a tool for teaching, learning, and research, the conjectures within a module can be mixed and matched from any of the conjectures in the searchable database. Moreover, the module editor also allows the creator to customize a set of features for the module (e.g., offer hints, change the number of repetitions of the directed movements, etc.). This flexibility allows teachers to contour the game experience for their students. It also makes THV a flexible research tool. By making each feature of the game an option, investigators can create parallel versions of THV that only vary by a single variable, which is ideal for randomized controlled trials.

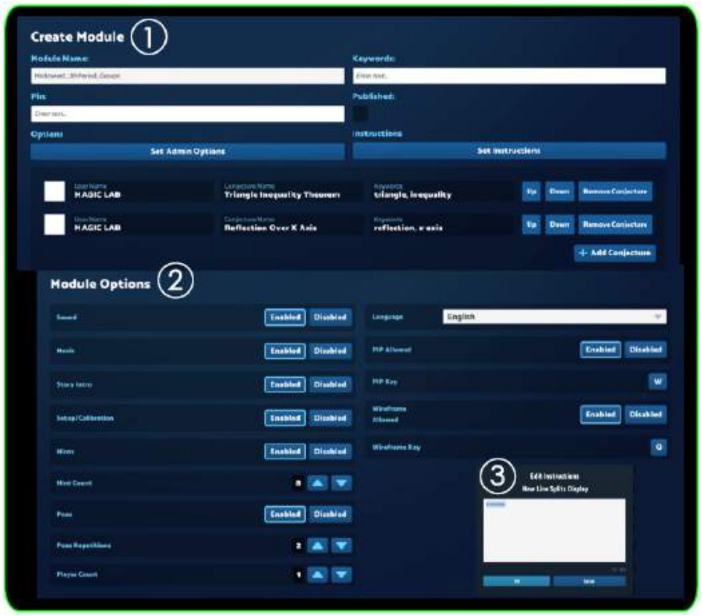

Figure 8: THV v.6 Module Editor. (1) Create a module, name the module, set the pin, customize the module (2) including: turning sounds, music, story, calibrations, hints, poses, number of players, language preferences, and scaffolding features. Players can add keywords (for easier search), publish to the database, and customize player instructions.

Discussion

Embodied principles of learning environment design offer some new opportunities for advancing students' mathematical reasoning. Embodied forms of reasoning offer a kind of Rosetta Stone that can bridge language barriers while supporting deep insights about the generalizable properties of space and shape. The co-construction of directed movements also allow students and teachers to institute more intuitive ways of embodying these mathematical ideas. The emergence of students' collaborative co-constructions inspired insightful ontological innovations in the redesign of THV. Our most proximate future goal is to elucidate how movements can embody spatial relations and transformations in geometry while empowering students to comfortably create their own content and instill confidence for deep, creative mathematics discourse. While THV currently focuses on high school geometry, the platform is applicable to other areas of mathematics that engage body-based forms of reasoning via simulation and metaphor (Lakoff & Núñez, 2000) as a path toward meaningful mathematical reasoning.

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THEORY AND RESEARCH METHODS

RESEARCH REPORTS

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RANKING THE COGNITIVE DEMAND OF FRACTIONS TASKS

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We report on and validate a system for ranking the cognitive demand of mathematical tasks. In our framework, task rankings are determined by the sequences of units and unit transformations students might use to solve each task. Using this framework, we ranked a set of 10 fractions tasks. We then interviewed 12 pre-service teachers to assess the validity of the ranking system. Results validate the task ranking system by demonstrating that increases in task ranking predict increases in the cognitive demand experienced by the pre-service teachers, as evidenced by their responses to the tasks. These results hold implications for instruction that maintains appropriate cognitive demand and future research that models students' mathematics.

Keywords: Cognition; Learning Theory, Number Concepts and Operations; Problem Solving.

In mathematics education, the cognitive demand of mathematical tasks has been categorized in terms of qualitative distinctions, such as procedures without connections and doing mathematics (e.g. Stein, Grover, Henningsen., 1996). In cognitive psychology, cognitive demand is quantified in terms of the number of action schemes a student might need to hold in mind in order to solve the task (Pascual-Leone, 1970). Here, we present a framework that accounts for the cognitive demand of mathematical tasks in terms of the sequences of units and unit transformations students might use to solve fractions tasks. Our framework integrates the math-specific construct of units coordination with the general cognitive construct of working memory.

The purpose of this study is to test a task ranking system based on our integrated framework. This purpose addresses one the major goals of PME-NA, "to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof." We created task rankings based on the hypothesis that longer sequences of units/transformations would induce higher cognitive demand. To test the hypothesis, we applied a simple statistical test from 12 pre-service teachers' (PSTs) responses to 10 ranked fractions tasks. Results confirm that the task's rank predicts the cognitive demand experienced by PSTs, as evidenced by their behavioral (including verbal) responses to the task. Thus, our results validate the task ranking system and its underlying framework.

Theoretical Framework

Piaget characterized mathematics as a coordination of mental actions (e.g. Beth & Piaget, 1966). Mathematics educators who have adopted Piagetian perspectives on mathematical learning have attempted to account for the actions students rely upon to construct mathematical concepts (Simon, Placa, Avitzur, & Kara, 2018; Tzur & Simon, 2004). We are particularly concerned with the mental actions students use to construct and transform units.

Steffe (1992) originally defined units coordination as the distribution of one composite unit (a unit containing units of 1) across each of the units in another composite unit. For example, a student might conceptualize the product 5 times 7 as the distribution of seven units of 1 within each of five units of 1, simultaneously producing five 7s and 35 1s. However, units coordination can be understood more broadly as any coordination of mental actions used to construct or transform units. For example, the unit fraction 1/5 might be constructed by partitioning a whole into five parts; conversely, iterating one of those parts five times reproduces the whole. The coordination of this partitioning action and the corresponding iterating action establish 1/5 as a one-to-five relationship between the unit fraction and the whole (Wilkins & Norton, 2011)

Research on students' mathematics has identified several mental actions that undergird their fractions knowledge (Hackenberg & Tillema, 2009; Steffe & Olive, 2010; Ron Tzur, 1999; Boyce & Norton, 2016; Wilkins & Norton, 2011). In addition to partitioning and iterating, these mental actions include unitizing, disembedding, and distributing, as summarized in Table 1.

Table 1: Mental actions for constructing and transforming

Mental Action	Description	Fractions Example
Unitizing (U_n)	Taking a collection of n items or units, or a continuous span of attention, as a whole unit	Treating a rectangular bar as a whole unit, of 1
Iterating (I_n)	Making *n* identical, connected copies of a unit to form a new unit	Iterating 1/7 of a whole three times to produce 3/7 of the bar
Partitioning (P_n)	Creating *n* equal parts within a whole	Partitioning a whole bar into 15 equal parts
Disembedding (D_n)	Taking *n* parts out of a whole while maintaining their inclusion as part of the whole	Taking one part from a whole that has been partitioned into 9 parts, to make 1/9
Distributing ($T_{m:n}$)	Inserting the *m* units of one composite unit into each of the *n* units in another composite unit to produce a unit of units of units	Inserting three parts within each of the nine parts in 9/9 to make 27 parts in the whole

Working memory is a limited cognitive resource with special relevance in solving mathematical tasks (Bull & Lee, 2014; Swanson & Beebe-Frankenberger, 2004). Here, we adopt Pascual-Leone's definition: "working memory involves the process of holding information in an active state and manipulating it until a goal is reached" (Agostino, Johnson, & Pascual-Leone, 2010, p. 62). In our framing, in the context of solving fractions tasks, working memory involves holding in mind sequences of mental actions used to construct and transform units.

Methods

Task Ranking

We chose to focus on fractions tasks because of the wealth of literature on students' development of fractions knowledge and the mental actions that undergird that knowledge (Boyce & Norton, 2016; Hackenberg & Tillema, 2009; Steffe & Olive, 2010). The literature identifies unitizing, partitioning, iterating, distributing and disembedding as mental actions potentially available to students in solving fraction tasks. We used these five actions along with three types of units (whole units, composite units, and fractional units) as the atoms of fractions knowledge. The fraction tasks we used were modified from Hackenberg and Tillema's (2009) work. We report on a subset of 10 fraction tasks we ranked, listed in Table 2.

In order to determine how cognitively demanding a task might be for a student, we examined results from the literature that reported on students' prior responses. The literature we chose, and Hackenberg & Tillema (2009) in particular, included detailed accounts for the schemes and mental actions students seemed to use in solving the tasks. However, in some cases, we had to break down schemes and advanced ways of operating into the aforementioned atoms—simpler units and actions that undergird students' fractions knowledge. That is, we hypothesized potential solution paths for mathematical tasks using one unit/action at a time, without chunking them into larger structures, such as schemes.

Table 2: Fractions tasks

Task	Rank	Description
3	5	Imagine a cake that is cut into 13 equal pieces. You take 4 pieces. So, how much of the whole cake do you have?
4	7	Imagine you have 1/7 of a whole candy bar. So, could you use that to figure out how long 3/7 of the whole candy bar would be?
5	8	Imagine this [drawing a rectangle] is 5/9 of a whole candy bar. So, how could you make 1/9 of the whole candy bar from what you have?
6	10	Imagine a rectangular cake that is cut into 15 equal pieces. You decide to share your piece of cake fairly with one other person. So, how much of the whole cake would that person get?
7	10	Imagine you share a sub sandwich fairly among 17 people. Now each person shares their piece with two other people (three people total share each piece). So, could you figure out how much one little piece is of the whole sandwich?
8	12	Imagine you are at a party and a cake is cut into nine equal pieces. Two people show up to the party late and you decide to share your piece of cake with them. So, what fraction of the whole cake do the latecomers get together?
9	12	Imagine cutting off 2/5 of 1/3 of a cake. So, how much is that of the whole cake?
10	14	Imagine cutting off 1/4 of 5/6 of a cake. So, how much is that of the whole cake?
11	16/17	Imagine, you need 1/3 of a pound of sugar and all you have are bags of sugar that are 1/7 of a pound. So, how many 1/7 bags do you need?
12	16	Imagine I have 7/9 of a yard of ribbon, but every ninth it changes colors. My friend needs 2/3 of what I have, and she wants all of the colors. So, tell how much of a yard she has.

For example, consider Task 8. To solve this task, a student might start with the whole cake ($whole_1$) and partition it (P_9) into nine pieces (9), then disembed one of those pieces (D_1) to make their piece of cake ($whole_2$). The student could then partition (P_3) that piece into three pieces (3) and disembed one (D_1) of those to make a new piece ($whole_3$). Knowing two pieces are needed, the student could iterate that piece twice (I_2). The student still needs to name the fractional size of the small piece. To generate the relationship between the original whole cake and the small piece, the student could iterate (I_{27}) the small piece to exhaust the original whole. This process is captured in the graph shown in Figure 1. We refer to such graphs as unit transformation graphs.

Since our theory assumes that units and actions count towards the cognitive demand a student experiences when solving a task, both were counted when determining the rank of a task. This total sum of units and actions enumerated the cognitive demand of the task. In the unit transformation graph for Task 8 (Figure 1) there are six units (denoted by circles) and six mental actions (denoted by arrows), together giving a task demand of 12. All tasks were ranked using this same process and the ranks are reported in Table 2. Once all the tasks were ranked, we tested our theoretical ranking system through empirical evidence.

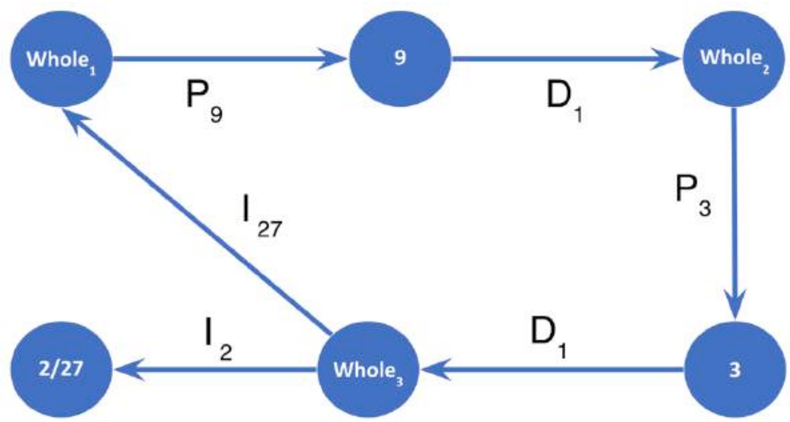

Figure 1: Graph of units and actions for Task 8

Data Collection

Participants were recruited from two sections of a mathematics for elementary school teachers content course taught by the same instructor at a large university in the mid-Atlantic United States. PSTs were selected for this study because they engage in metacognitive skills that enable them to express their thinking well; they commonly practice explaining their mathematics in the mathematics for elementary school teachers course. Moreover, they are mathematically mature enough to have constructed the mental actions required to productively engage with fraction tasks.

Twelve PSTs volunteered to participate in a 75-minute semi-structured clinical interview (Goldin, 2000). Each interview consisted of three parts: a units coordination assessment (Norton et. al, 2015), a working memory assessment (Morra, 1994), and a set of fraction tasks. This paper focuses on the final component of the interview. The PSTs were given the fraction tasks verbally one at a time and asked to solve them, initially without using figurative material. Sometimes follow-up questions were asked to probe a PST's thinking; sometimes PSTs were encouraged to use drawing to support their solution. A subset of tasks from Table 2 was selected for each PST, depending on our assessment of that PST's units coordinating ability and working memory. The tasks were always given in increasing order (top to bottom in Table 2), posing lower ranked tasks before higher ranked tasks. Each interview was video recorded with any written work collected using a Livescribe pen and notebook. The videos were selectively transcribed.

Data Analysis

Data analysis for this report consisted of two phases: coding for cognitive demand the PSTs experienced and a quantitative analysis of the task ranking system from the results of the coded cognitive demand.

Coding cognitive demand. Videos were analyzed for the purpose of coding the cognitive demand of each task, as experienced by each PST. We relied on video recorded behavioral data (including verbalizations) as indicators of this experienced demand.

Videos were analyzed one PST at a time with at least two of the three authors present. Experienced cognitive demand was coded as Low, High, or Over. The Low code was given when the PST was able to solve the task easily and confidently. Behavioral indicators included relaxed posture, providing an answer without verbal rehearsal or giving a fluent rehearsal of solution strategy. When a PST struggled but still had success engaging with the task, we assigned a High code. Behavioral indicators for a High code included asking for the question to be repeated during the solution process, expressing doubt throughout the task, unsure or repeated rehearsal needed to convince themselves of the task solution, and losing track of units during the solution process. The Over code meant that the

PST clearly was unable to assimilate the task or unable to resolve it without significant support from the interviewer or figurative material. This code often provided the easiest behavioral indicators with participants saying things like "Ah, it's just hard to do it in my head. Umm…" or "my brain is confused now."

The following pair of transcripts, from PST G provide an example of the difference between a Low code and a High code. The first transcript is for Task 6 (rank 10), and the second is for Task 8 (rank 12). In her response to Task 6, we see that PST G is quickly successful in solving the task with minimal rehearsal needed. Indeed, she seemed to have an answer ready (one-thirtieth) before saying anything, so that her verbalizations served as explanations to the researcher, rather than a necessary process in generating a solution to the task. The verbal run through of her solution process was succinct and confident.

Researcher: This is the next task [Task 6]. Imagine a rectangular cake that is cut into fifteen equal pieces. You decide to share your piece of cake fairly with one other person. So, how much of the whole cake would that person get?

PST G: [pauses for seven seconds and looks up.] You get one fifteenth of the cake and split that in half. My first thought was one-thirtieth…Of the cake, because…[makes splitting motion with hands in the air.] Splitting that in half, like if you were to split every piece of fifteen in half, then that would be like one thirtieth of the entire cake.

In comparison to Task 6, Task 8 seemed to induce additional cognitive demands for PST G, who required verbal rehearsal of her thought process to determine the answer to this new task. While she was successful in the end, throughout the solution processes there were several times she expressed doubt about a step or result. She would say things like, "Wait, that doesn't seem right," and "I don't know if that'd give you the same answer." She was eventually able to be successful on this task after attempting it twice. The fact that she was able to work through and solve the task despite some expressed doubt meant it did not qualify for an Over coding. However, Task 8 appeared to require substantial mental effort to produce a correct solution, indicating demand was High.

PST G: So, it's split up into nine equal pieces. So, then, you would split one ninth into…Two people come, but you still have a little bit? So, that… So, you would split that up into three. So, then I… Well, I guess you would do one ninth times two thirds to get how much they equal, like how much both their pieces would be. And then whatever that is, I guess it would be… two over… two eighteenths? Wait, that doesn't seem right. [pauses for five seconds] I feel like… I mean, I guess… You take those nine pieces, splitting that one ninth into thirds. But to find out how much two of those thirds are, you'd multiply one ninth by two thirds… Or no. You'd… you'd multiply the one ninth by one third, and then just do that twice? I don't know if that'd give you the same answer.

Researcher: Okay. Uh, let's… Maybe I can help you.

PST G: Okay.

Researcher: If you want me to be your calculator again, I'll do it.

PST G: [begins to draw on table with finger] So, you do one ninth, which divided by three, so you could times it by one third. So, then you'd have one over um… [pauses for five seconds.] Oh wait… [whispers to self] Three times nine, that's twenty-seven. Oh no, one over twenty-seven. And then you multiply that by two… to get two-thirds or to get two parts of the thing… So, then I guess… What's one over twenty-seven times two? Is that just two-twenty-sevenths? Okay.

Researcher: Nice, I like the way you reasoned through it. Yeah.

PST G: Okay. I was like, because I was thinking one over twenty-seven times two over one and I was like I guess that's just two, twenty-sevenths.

Quantitative analysis. The variable we measured was cognitive demand. This ordinal categorical data was coded as Low, High, and Over as described above. To test whether the task ranking system was valid, we only considered instances where a PST experienced a change in cognitive demand between two successive and differently ranked tasks. We excluded any cases where the same cognitive demand was experienced on successive tasks and instances where the demand changed within the same ranked task. For example, if Task 6 (rank 10) was coded as High but Task 7 (rank 10) was coded as Low, we did not count this as a trial. In fact, such instances were common and expected because PSTs might rely on their solution to the first task within a given rank to facilitate their solution to the second task of that rank. After excluding these cases, we were left with 21 trials. The trials are labeled in Table 3 with the number placed in the cell of the higher ranked task where the change occurred. Since our theory assumes an increase in task ranking predicts an increase in experienced cognitive demand, we consider a successful trial to be one where the change in cognitive demand increased for successively given tasks of increasing rank. There are 18 successful trials. The three unsuccessful trials occurring with PSTs F, G, and J, denoted in Table 3 with an asterisk (Trials 7, 11, and 17).

To test the validity of the task ranking system, we asked the following: What is the probability of 18 successes in a sample of 21 trails if the chance of changed cognitive demand is 50%. To answer this question, we used a binomial test with a p-value of 0.05. We assume the independence of observation needed for a binomial test holds across PSTs since each was interviewed separately and any discussion of the interview between PST outside of the interview setting was negligible. We also assume the independence of observation holds within a PTS's interview because of the novelty of the tasks and the exclusion of same ranked tasks from the trials. We analyzed the data using Microsoft Excel (version 15.33).

Results

Table 3 illustrates the cognitive demand of tasks as experienced by each PST. Green indicates Low demand, yellow High, and red Over. Two pairs of tasks, 6-7 and 8-9, have the same rank; if a PST was given both tasks of the same rank, we only consider the first of the same ranked task given to eliminate familiarity with the task as a confounding variable of cognitive demand. At a glance, we can see that the predicted trend of increasing rank with increased cognitive demand did occur. There are three PSTs (F, G, and J) for whom this pattern did not strictly hold outside of same ranked tasks. For PST F, the codes followed the pattern of Over, High, then Over again. PST G had a High code after two Over codes. Lastly, PST J had one High code in the middle of two Low codes before getting coded as Over. We attribute PST F's deviation from the predicted pattern to her initial assimilation of fractions in an unconventional manner (e.g., assimilating "three-sevenths" as one-third of 1/7) before adjusting this understanding in subsequent tasks as parts out of wholes. The switch from Over to High that PST G experienced was a case of persistence in trying to solve a task as she made use of new strategies used on previous tasks. For PST J, she experienced a perturbation with her scheme for "one-ninth" in Task 5 that led to an Over code but was resolved for subsequent tasks.

Table 3: Summary of cognitive demand by task and PST

Task	Rank	A	B	C	D	E	F	G	H	I	J	K	L
12	16							11*		15			
11	16/17												
10	14						8	10	13			20	
9	12												
8	12			3		6	7*	9	12	14	18	19	21
7	10	1											
6	10		2		4	5					17*		

5	8										16		
4	7												
3	5												

With 18 successes out of 21 trials, and a probability of success for a single trial of 50%, we obtained a p-value of 0.0006. This statistical result validates the task ranking system. In particular, it supports the hypothesis that increased length in sequences of units/actions required to solve fractions tasks predicts cognitive demand, as experienced by the PSTs and evidenced by their behavioral responses to the tasks.

Discussion

In validating the task ranking system with a simple statistical test, we have affirmed the hypothesis that informed it. In turn, the affirmation of this hypothesis demonstrates the utility of our framework—a framework that integrates the psychological construct of working memory (Pascual-Leone, 1970) with the mathematics education construct of units coordination (Steffe, 1992). Furthermore, it supports the Piagetian perspective of mathematics as a coordination of actions (Beth & Piaget, 1966), while recognizing students' mental actions as the source of their own mathematical power.

Other mathematics education studies have addressed cognitive demand (e.g., Stein, Grover, Henningsen, 1996) or have identified how students might rely on sequences of mental actions to solve mathematical tasks (Simon, Placa, Avitzur, & Kara, 2018; R. Tzur & Simon, 2004). We used unit transformation graphs to account for both: mental actions constitute the atoms of students' mathematical knowledge, as represented by the circles (unit constructions) and arrows (unit transformations) in our graphs. We enumerated cognitive demand by the number of circles and arrows in each graph. Although this characterization of cognitive demand aligns best with the psychological construct of working memory, it also relates to Stein and colleagues' (1996) categorization.

Stein and colleagues were especially concerned with maintaining high cognitive demand of instructional tasks, where high demand referred to aspirations of engaging their students in "procedures with connections" and "doing mathematics" (Boston & Smith, 2009; Stein et al., 1996). Unit transformation graphs might support such aspirational goals by informing teachers of ways they can help students manage the demands of mathematical tasks without reducing them to the lower categories of "memorization" or "procedures without connections." Within our framework, maintaining such demand would involve facilitating students' coordination of the mental actions involved in a task's solution by providing appropriate figurative supports, such as manipulatives and opportunities for student drawings. Such supports could allow students to offload demands on working memory, especially in long sequences of units/actions, without eliminating the demand for their coordination (cf., Costa et al., 2011).

Prior research has highlighted additional factors that contribute to cognitive demand. For example, Pajares (1994) demonstrated that self-efficacy and math anxiety can moderate the cognitive demands that students experience in response to mathematical tasks. Although we did not take such factors into account in our study, the complexities of teaching necessitate that teachers do. We recognize these complexities and intend unit transformation graphs as a tool teachers and researchers might use to manage them.

Ultimately, we see unit transformation graphs as a means of recognizing and empowering students' mathematics by explicitly accounting for their available mental actions and coordinations thereof. We might expand the program by relying on research that identifies students' mental actions in other domains of mathematics, such as algebra (e.g., Lee & Hackenberg, 2014) and covariation (e.g., Carlson et al., 2002).

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ANALYZING TEACHER LEARNING IN A COMMUNITY OF PRACTICE CENTERED ON VIDEO CASES OF MATHEMATICS TEACHING

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Incorporating video case study of mathematics teaching into professional development (PD) can provide opportunities for teachers to develop new ways of seeing teaching and learning and inform efforts to enact new instructional practices. However, more research is needed to understand how such PD can foster sustained teacher learning about high-quality instruction and materials. In this paper, we share the evolution of our analytic method that aims to reveal how secondary mathematics teachers learn while collectively analyzing video of mathematics teaching. We found that conceptualizing this PD within a community of practice, along with applying analytic tools from frame analysis and professional noticing, helped us recognize and describe the process of teacher learning in this setting. We plan to apply our analytic method to our full dataset to better understand how teacher learning in this context is happening over time.

Keywords: Teacher Education - Inservice / Professional Development, Teacher Knowledge, Research Methods

Teaching is a dynamic endeavor for each teacher; no two learning environments are identical. Each classroom is shaped by teacher and student interactions, including teachers' interpretations of and responses to students' thinking and problem-solving strategies. Teachers' decision making during these interactions provide opportunities for students to develop as problem solvers and effect mathematical thinkers (Schoenfeld, 2017). As educators, we can learn from classroom interactions, personal reflections, and collaborations with others in order to improve our own practice. Analyzing video case studies of mathematics teaching "can help [teachers] develop new ways of seeing teaching and learning and support their efforts to enact new instructional practices" (van Es & Sherin, 2017, p. 1).

However, teachers' opportunities to systematically develop and share ideas about teaching are limited (Ball, Ben-Peretz, & Cohen, 2014). Professional development (PD) has been shown to be key in supporting instructional shifts that deepen students' learning opportunities (Desimone, 2009). In our research, we designed a model of video-based PD for our work within professional learning teams (PLTs) of secondary mathematics teachers, bringing teachers together to collaboratively investigate video of mathematics teaching. One important aspect of our PD model is that it is grounded in the Teaching for Robust Understanding framework (TRU; Schoenfeld, 2017). The TRU framework details dimensions of high-quality instruction that support deep mathematical learning opportunities for students. Our goal is to understand (a) how discussions and activities used in the PD support teacher learning, and (b) the extent of teacher learning about high-quality instruction and instructional materials that can be used within mathematics classrooms.

Submitted to the Theory and Research Methods strand of PME-NA, this proposal describes the analysis process for the NSF-funded project *Analyzing Instruction in Mathematics using the Teaching for Robust Understanding Framework* (project number 1908319), or AIM-TRU. The project team consists of a research group of practitioners and mathematics teacher educators analyzing PLTs' investigation of video cases. Through our analysis process, we aim to demonstrate how our PD model can foster sustained teacher learning about high-quality instruction that contributes to shared professional knowledge across a diversity of school settings. As we worked to understand the impact of our PD model on teacher learning, we drew from and expanded on a collection of complementary conceptual and analytic perspectives, including communities of practice, frame analysis, and professional noticing. Thus, the purpose of this proposal is to share our analytic method and illustrate how we have been using it to understand how teachers learn in a community of practice as they collectively analyze video case materials.

Conceptual and Analytic Perspectives

Our conceptual model relies on understanding learning within a community of practice (CoP; Wenger, 1998). To understand learning in a CoP, we incorporate analytic tools from frame analysis (Bannister, 2015) and apply constructs from professional noticing (Jacobs, Lamb, & Philipp, 2010). In this section we describe each of these frameworks to articulate our process of investigating teacher learning within professional learning teams.

Communities of Practice

Wenger (1998) claims that people learn through their participation in specific communities, called communities of practice (CoPs), consisting of people with whom they interact regularly. CoPs are defined as groups whose members (a) are mutually engaged in an activity, such as analysis of video case studies of mathematics teaching; (b) are connected by a joint enterprise, such as fostering sustained teacher learning about high-quality instruction; and (c) have a shared repertoire of communal resources, such as the TRU framework.

According to Wenger (1998), communities of practice negotiate meaning collectively. This negotiation of meaning is represented by changes in participation which are reified to give form to the meaning through the three dimensions of the CoP outlined above. For our analytic process, we utilize frame analysis to identify these reified changes in participation, which manifest themselves as participants engage with "evolving forms of mutual engagement," "understanding and tuning their enterprise," and "developing their repertoire, styles, and discourses" (Wenger, 1998, p. 95). We consider changes within these three processes as evidence of learning within a CoP.

Frame Analysis

Bannister (2015) linked community participation with tools from frame analysis (Benford & Snow, 2000; Snow & Benford, 1988), to examine how teachers' participation patterns evolve around a community defined problem of practice (PoP). By employing the tools from frame analysis, Bannister sought to understand development within a group of teachers within common planning time to capture teachers' reification patterns and give insights related to member participation. The tools from framing analysis consist of framing tasks (Snow & Benford, 1988): diagnostic framing ("identification of a PoP and the attribution of blame" (p. 200)), prognostic framing ("a proposed solution to the diagnosed PoP that specifies what needs to be done" (p. 199)), and motivational framing ("a call to arms or rationale for engaging in ameliorative or corrective action" (p. 199)).

Bannister (2015) delineated the connections between the key concepts from frame analysis and processes of participation and reification in a CoP (see Figure 1). For example, a group of high school mathematics teachers (a CoP) collaborate weekly (shared repertoire) on developing interventions for struggling students (joint enterprise). As the teachers identify a PoP and specify possible causes (diagnostic framing), and discuss possible solutions (prognostic framing) by interacting each other and sharing their ideas (participation), the framings reify the community's ideas about who the struggling students are and what can be done to help them. Changes in framings within a community help to reify the changes in participation occurring within a CoP. These changes in participation and reification, are in turn, empirical evidence of learning occurring within a CoP.

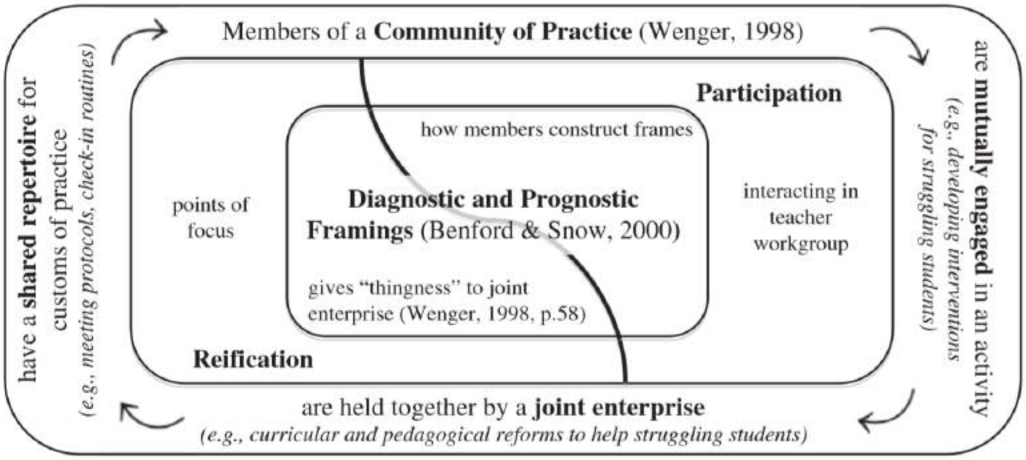

Figure 1: Connections between key ideas from communities of practice and frame analysis. (Bannister, 2015)

Professional Noticing of Children's Mathematical Thinking

To identify a PoP, we use the construct *professional noticing of children's mathematical thinking* to "begin to unpack in-the-moment decision making" (Jacobs et al., 2010, p. 173). Researchers argue that teachers need to first learn to productively attend to pertinent features of an instructional setting and be able to make mention of that which is noticed before they can make responsive instructional decisions (Jacobs et al., 2010; Superfine, Amador, & Bragelman, 2019). Thus, Jacobs et al. (2010) detail the components of professional noticing with three skills: "attending to children's strategies, interpreting children's understandings, and deciding how to respond on the basis of children's understandings" (p. 172). The premise of the framework indicates that in order for teachers to respond to student thinking, the other skills of attending and interpreting are occurring simultaneously to provide the teacher insight and knowledge about how to respond. Moreover, according to Thomas et al. (2015), anticipating how students might respond provides a firm basis of noticing. Therefore, the framework for professional noticing for children's mathematical thinking provides a foundation for us to apply frame analysis techniques to understand the ways in which teachers are learning within a CoP.

Methods

Participants

The participants of the study include eight practicing secondary mathematics teachers and two participant observers from the research team. The teachers volunteered to be part of a Professional Learning Team (PLT) with the goal of analyzing videos of mathematics classrooms to interrogate mathematics teaching and learning. Two of the eight participants are facilitators for the discussions; however, the two research members also engage and probe teacher thinking throughout the PLT meetings.

Context

Teacher participants who elected to participate in this PLT enrolled with the knowledge that they would be analyzing video case studies that are aligned with the TRU framework. Each set of video case materials utilized in PLT meetings was created to demonstrate a teacher implementing a formative assessment lesson (FAL) from the Mathematics Assessment Project. The Mathematics Assessment Project also uses the TRU framework as a way of characterizing powerful mathematics classrooms, defined by a focus on the mathematics, cognitive demand, equitable access to the mathematics, agency, ownership, & identity, and formative assessment (Schoenfeld, 2017). The TRU framework provides a necessary shared repertoire within the PLT for discussing the video case studies.

Participants attended four PLT meetings and analyzed three sets of video case materials. The first PLT session was used for teachers to better acquaint themselves with FALs and the TRU framework with the intent to build a shared repertoire among members. In the next three sessions the participants engaged in a guided analysis of video case materials. Each PLT meeting was two hours in length. The teachers engaged with mathematical content around applying properties of exponents and representing quadratic functions graphically. The facilitators followed a predetermined guide to keep each session consistent throughout the larger project.

Data Collection

Data was collected with the intent of understanding how teachers develop knowledge through engagement with the video case materials. The four PLT meetings were video recorded and each of the recordings were later transcribed to be analyzed. Materials from the PLTs were also collected to cross reference teacher conversation and build knowledge of their understanding. The materials include individual and group generated artifacts, solutions, and responses to question prompts.

Data Analysis

In order for us to understand changes in participation and reification within the CoP, we used frame analysis (Bannister, 2015) as an analytic tool. Our first level of analysis was to reduce the data into episodes of pedagogical reasoning (EPRs). Horn (2005) defines episodes of pedagogical reasoning as

> units of teacher-to-teacher talk where teachers exhibit their reasoning about an issue in their practice. …EPRs are moments in teachers' interaction where they describe issues in or raise questions about teaching practice that are accompanied by some elaboration of reasons, explanations, or justifications. These episodes can be individual, single-turn utterances, such as "I'm not using that worksheet because it bores the kids." Alternatively, these can be multiparty coconstructions over many turns of talk. (p. 215)

Using an EPR as our unit of analysis, we were able to systematically investigate teacher discussions. After collectively identifying each EPR, we analyzed the EPRs to determine what teachers were talking about and the nature by which they were having discussions.

After separating EPRs, we created descriptions or themes to characterize the essence of the conversation. These descriptions detail what the teachers were discussing. For example, two of our identified themes were *understanding one method of student strategies* and *using calculators to evaluate vs. expanding*. To apply frame analysis, we examined each EPR and the previously identified theme to determine the PoP which grounded the teachers' conversations. The PoP occurred when teachers identified an instance as troublesome, challenging, recurrent, unexpectedly interesting, or otherwise worthy of comment (Horn & Little, 2010); they are issues of practice that teachers encounter regularly. We identified the PoP through either explicit mentioning from participants or through prior participation or discussion. While we were stating the PoP in each EPR, we began to discover repetitive language emerging as we defined the PoPs. Therefore, we began to create descriptions of PoPs that could be used across the PoPs and revisited each previously analyzed EPRs to determine if they would also have our newly defined PoP. For example, we analyzed the two EPRs with different themes, labeled above, and determined that in both cases the PoP was *teachers anticipating student solution strategies.*

After identifying the PoP, we analyzed each EPR to understand the nature of the conversation through frame analysis. Incorporating the literature, we organized the definitions of framing tasks as detailed in Table 1.

Table 1: Definitions of Framing Tasks

Framing Task	Definition
Diagnostic Framing	Diagnostic framing is when teachers diagnose a PoP and attribute causality for the problem.
Prognostic Framing	Prognostic framing is when teachers discuss a solution or possible solutions for a PoP diagnosed or implicit from earlier conversations. This implies attribution of causality and solution(s) for the problem.
Motivational Framing	The motivational framing is the rationale for engaging in a particular action to attend to a particular PoP. This rationale should be more than just mentioning what the teacher thinks will change but include justification for why the proposed action will create change. This implies attribution of causality, solution(s) for the problem, and a rationale for why a solution or solutions would actually work.

We described the diagnosis, the prognosis, and/or the motivation with each EPR. The descriptions were generated by using the teachers' exact words or by interpreting the nature of the teachers' conversation. In some EPRs, we saw a progression of conversation from diagnosing the PoP, to prognosing a hypothetical solution, to detailing their motivation behind their prognosis. However, other EPRs were limited to only diagnostic discussions where the teachers did not provide hypothetical solutions to their identified PoP.

During the analysis process, we began to see patterns in our identified PoP and considered layering frame analysis with an additional analytic framework that would help us better understand what the teachers were talking about before we determined how they were talking about it. Through discussions, we realized that the PoPs contained themes aligned to the professional noticing of children's mathematical thinking framework. As a team we revisited the definition of each PoP to align them with the framework as an object of focus for the community's participation and reification. Figure 2 indicates how we visualize the noticing framework inside of a CoP. What and how the teachers notice throughout their engagement with video case studies can be evidence of changes in participation and reification.

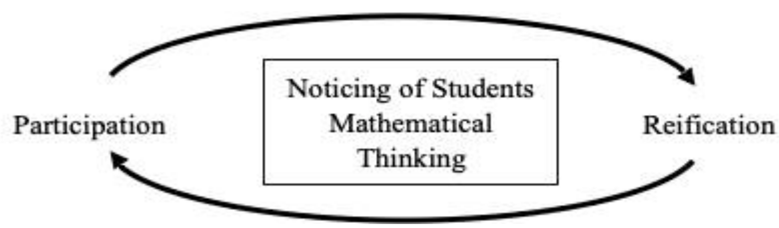

Figure 2: Noticing as a method to analyze changes in participation and reification.

Based on the professional noticing framework, to define our PoP, we categorized each EPRs' problem of practice as either attending, interpreting, or responding to student thinking. In our analysis, we also detailed the subject to what the teachers were either attending, interpreting, or responding. For example, the two EPRs, identified earlier regarding student solution strategies, were now coded as *Attending-Teachers anticipating student solution strategies*. However, using the definitions for attending, interpreting, and responding, we found some EPRs could not be characterized by one of these categories. In these instances, the teachers were not explicitly discussing students' mathematical thinking. However, because the context of their conversations is important for our larger research project, we categorized these as "other" and maintained the original description as the PoP.

Illustrative Example of our Analytic Method

In this section, we provide an illustrative example of our analytic method. We detail the processes of identifying an EPR and the PoP within the EPR, relating the PoP to the noticing framework, and identifying each EPR as a diagnostic, prognostic, and/or motivational frame.

For context, the participating teachers in this example were analyzing the video case of an Applying the Properties of Exponents FAL during the second PLT session (https://tle.soe.umich.edu/MFA/Applying_Properties_of_Exponents_1). While analyzing the video case, the teachers focused on a group of students discussing the problem $2^2 \div 2^3$. Two of the students in the group were trying to convince a third student how the properties of exponents could be used to simplify this expression. The following transcript is a teacher conversation in the PLT meeting about the situation in the FAL.

Louis: I think if I could rewind then ask a question, I might ask them or maybe prompt them
to do is maybe to think about it in terms of a different form. Because, from what I saw, it
was like a back and forth between computation and exponents. And I think if maybe the
students could maybe see it in the expanded form or another way it might, it might
prompt them to think about it in a different way.

Josh: Can I ask how you would go about doing that? Like, what would you ask them to
stimulate that conversation?

Louis: I think I would start at the beginning and I think it was where they were doing the
four divided by eight … it was two to the second divided by two to the third. That's what
it was. And that prompted them to do four divided by eight. And they got some, they got
some validation at the end, like he checked into the calculator and yes, in fact it was two
to the negative one power. So, I guess the question would be like, well why exactly does
that work? And then it would be, what's another way maybe we could write the initial
statement and then maybe that would help them along the way to the other ones.

Jackson: I agree. Have them write it in expanded form and then playing with it that way.

Lisa: I know for me, one of the things I would have, um, and I again, I don't know if this is
the right move, but being that it seems like the boy is a very visual person. Maybe he's
not really listening to these rules because he's not getting, he's not really seeing their
thinking. Uh, I know like I'm a very visual person. I need to see it written to really
understand it. So, I almost want to suggest to the girls like, can you just show, show him
what you're thinking? Can you show it on your whiteboards? And maybe then he'd have
like a better understanding for it.

We identified this teacher conversation as an EPR because it was an incident of teacher-to-teacher talk about a student who had not been building on other students' thinking or reasoning about $2^2 \div 2^3$ and were offering suggestions for improvement. We identified the PoP in this EPR as *providing opportunities to learn in different ways through multiple representations*. Evidence to support this description can be seen, in part, through one teacher's comment: "And then it would be, what's another way maybe we could write the initial statement and then maybe that would help them along the way to the other ones" (lines 13-14). Based on the professional noticing framework, we categorized the PoP as *responding to student thinking* because the teachers used what they learned about the student's understanding of the properties of exponents to pose hypothetical questions to students. As stated earlier, for teachers to respond to student thinking, they must first attend to student thinking. Evidence of teachers attending to student thinking occurs when Louis claims that he saw students using computation and expanding as methods of simplifying exponential functions (lines 2-4). We identified other hypothetical responses to student thinking by the teachers: Louis proposed probing student thinking by asking why their mathematics works (lines 12-14); Jackson suggests having students write the expressions in expanded form (line 15); and Lisa would ask students to make their thinking visible through the use of whiteboards (lines 20-22). In each of these instances, the teachers are sharing questions they would ask students in response to the analysis of their thinking.

We then analyzed the EPR to understand the nature of the conversation through frame analysis. The teachers diagnosed the PoP as *responding to student thinking: providing opportunities to learn in different ways through multiple representations*. They came to this diagnosis by focusing on one student who did not believe that one representation (the properties of exponents) was valid to illustrate the equivalence of expressions. The teachers' determination of the attribution for causality, or what caused the PoP in this EPR, was that the student did not build on other students' thinking. This is evidenced in Lisa's claim, "Maybe he's not really listening to these rules because he's not getting, he's not really *seeing* their thinking" (lines 17-19, emphasis added).

The teachers went on to provide a prognosis for this PoP. In this EPR, the teachers discussed prompting students to consider looking at the expanded form of $4 \div 8$ (line 9) written on the whiteboard (line 21) because it would be helpful for the student, who might be a visual learner, to see and understand the properties of exponents are valid to illustrate the equivalence of expressions. In the end, we coded this EPR as a motivational frame because the teachers went beyond the diagnosis and prognosis of the identified PoP by providing a rationale for their proposed action based on their assessment of the student appearing to be a visual learner (lines 17-22). The summary of our analysis for this illustrative example is organized and presented in Table 2.

Table 2: Sample Analysis

Category	Description
Problem of Practice	Providing opportunities to learn in different ways through multiple representations
Noticing Skill	Responding to student thinking
Diagnostic Framing	The student doesn't believe that one representation (properties of exponents) is valid.
Prognostic Framing	The teacher could prompt students to consider looking at the expanded form of expressions.
Motivational Framing	The student appears to be a visual learner, thus seeing expanded form might assist his belief in properties of exponents.

Discussion

Our analytic method stems from a need to understand how teachers learn in a CoP and, in turn, how to foster sustained teacher learning. In this paper, we elaborated how our analytic method is helpful to understand teacher learning within the PLT during their engagement with video cases. First, frame analysis enabled us to analyze what and how the teachers negotiated meaning in the PLT. Meanwhile, we saw that, during the teachers' interaction, they mostly talked about noticing their students' thinking. Thus, by adding the noticing framework to frame analysis, our team was able to categorize the PoPs in a consistent way. This consistent categorization of PoPs was beneficial to understand how the teachers could start to think about improving their own instruction. Taken together, within our analytic method, we combined frame analysis and professional noticing, which was conducive to our analysis of teacher conversations about FALs. Specifically, frame analysis played a major role while the noticing framework played a supporting role in our analysis process because we utilized the noticing framework after making use of frame analysis. If researchers or teacher educators apply the analytic method in their own context, it might be helpful to utilize an additional framework, like professional noticing, from the beginning of an analysis. By characterizing the PoP in a streamlined manner, patterns related to participation and reification will become more easily evident through analysis. Then, through analysis of the entire PLT data, using the identified PoPs, we will be able to discern if there are reified changes in participation.

Next Steps

In order to better understand teacher learning within PLTs of practitioners and mathematics teacher educators analyzing video cases, we first need to further investigate how teacher conversations and their framings are changing over time. These changes in frame will allow us to identify and describe the reified changes in participation that indicate learning in CoPs. Secondly, we need to look for patterns across EPRs in the remaining sessions of the PLT to verify that this analytic method is robust. Additionally, it will be necessary to determine how to interpret the EPRs we categorized as "other." Our initial understanding of this collection of EPRs is that they are about group dynamics or classroom norms. However, we will need to continue to look at this data, and the PLT sessions to find confirming or disconfirming evidence of this conjecture. Finally, future studies can build on our analytic method by connecting this data to the TRU framework in order to obtain a deeper understanding of our overall research questions around understanding (a) how discussions and activities support teacher learning and (b) the extent of teacher learning about high-quality instruction and instructional materials that can be used within mathematics classrooms.

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AN INVITATION TO CONVERSATION: ADDRESSING THE LIMITATIONS OF GRAPHICAL TASKS FOR ASSESSING COVARIATIONAL REASONING

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We reflect on the limitations of our research group's prior methods for assessing covariational reasoning which primarily used graphical tasks found in extant literature. Graphical tasks dominate the literature on covariational reasoning, and through our use of these tasks we came to question the heavy reliance on them. Our concerns led us to ask the following: (1) What are the limitations of using tasks with graphs to assess covariational reasoning? (2) How can we improve assessment of covariational reasoning to accommodate students with nonnormative graphing schemes? We offer this piece as the beginning of a conversation to develop improved methodologies that attend to the ubiquity of students' nonnormative graphing schemes.

Keywords: Research Methods, Cognition, Precalculus, Calculus

The role of quantitative and covariational reasoning in the teaching and learning of mathematics has received increased attention in recent decades. Researchers have adopted numerous methodologies to investigate said reasoning, and, relatedly, they have developed a number of research-based tasks and instructional settings that afford such investigations. In our previous work, we drew connections between covariational reasoning and units coordination (Boyce et al., 2019). In this paper, we reflect on the relationships between our prior methods for assessing covariational reasoning, including how this influenced our data and claims regarding participants' maximum capacity to reason covariationally across settings. By sharing the analysis of our methods, we hope to spark a larger conversation that is important for a number of reasons.

First, reflection and review of research processes can refine our research and the quality of research in the field as a whole. Second, our research team consists of many newcomers to covariational reasoning research, including three first-year doctoral students—two of whom are the leading authors of this piece. This affords us the opportunity to start a dialogue between both novice and expert researchers concerning the interpretation and use of covariational reasoning frameworks in research. To further help facilitate such a conversation, we invited an expert on quantitative and covariational reasoning (the last author) to participate in crafting this paper. Third, in the sphere of covariational reasoning, there is a new generation of researchers who have started using the covariational reasoning frameworks that Thompson and Carlson (2017) developed over the past 30 years (Gonzalez, 2018; Stevens, 2019). By examining the evolution and contemporary usage of these frameworks, we can adopt a lens by which to identify previous works' contributions and limitations. Fourth, and most relevant to the present work, after reflecting on our methods we contend that extant

[1] Both leading authors contributed equally to the creation of this piece.

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studies have often relied too heavily on graphical items to assess covariational reasoning. We offer specific suggestions on how to improve interview protocols to address this issue.

Previous Study Design and Current Research Questions

Our initial interview protocols for assessing a students' overall ability to reason covariationally were guided by Carlson and Thompson's work (Carlson et al., 2002; Thompson & Carlson, 2017), the Project Aspire covariation tasks (Thompson, 2016; Thompson et al., 2017), and the classic Bottle Problem (Swan, 1985). Project Aspire, led by Thompson, created a validated diagnostic assessment of secondary teachers' mathematical meanings. Their work included drafting a number of covariation items and then piloting them with teachers to determine the best items for reliably assessing covariational reasoning. While the initial covariation item pool included both graphical and non-graphical items, all non-graphical items were discarded for a variety of reasons. Because we drew inspiration from the Aspire instrument and Carlson et al.'s (2002) tasks, and the covariational reasoning literature sparingly highlights the importance of non-graphical tasks, it did not occur to us to include such tasks in our interview protocol.

While retrospectively analyzing our methods and resulting data, we hypothesized that assessing overall covariational reasoning with mostly graphical tasks limited our ability to model students' thinking. This hypothesis stemmed from our observation that some students correctly described relationships between two covarying quantities using words and gestures, but these same students failed to graphically convey their described relationships because of their nonnormative graphing schemes. In hindsight, it makes sense that graphical tasks have limitations given the growing body of evidence that a number of successful undergraduate students use nonnormative graphing schemes (Frank, 2017; Lee et al., 2019; Moore et al., 2019), which often involve meanings for graphs that do not entail covariational reasoning. Consequently, such student graphs do not provide data for a researcher to assess their covariational reasoning beyond an *in-the-moment* absence of it. Indeed, Saldanha and Thompson (1998) conveyed a similar sentiment when stating: "The results of this study lead us to believe that understanding graphs as representing a continuum of states of covarying quantities is nontrivial and should not be taken for granted" (p. 303). Building off Saldanha and Thompson's sentiment, recent researchers' characterizations of students' nonnormative graphing schemes, and our inference of a possible overreliance on graphing tasks to assess covariational reasoning both within and outside of our own work, we ask: (1) What are the limitations of using tasks with graphs to assess covariational reasoning? and (2) How can we improve assessment of covariational reasoning to accommodate students with nonnormative graphing schemes?

Theoretical Framework

Covariational Reasoning

Thompson's research in covariational reasoning stemmed from his interest in how "students conceive situations as composed of quantities and relationships among quantities whose values vary" (Thompson and Carlson, 2017, pp. 424–425). In Thompson's (2011) view, "Quantification is the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship (linear, bi-linear, multi-linear) with its unit" (p. 37). Quantitative reasoning, then, is a person's conception of quantities and the relationships between those quantities. Building off prior quantitative approaches to covariational reasoning (Thompson, 1988, 1993, 1994; Thompson & Thompson, 1992), Carlson et al. (2002) defined covariational reasoning "to be the cognitive activities involved in coordinating two varying quantities while attending to the way they change in relation to each other" (p. 354). For instance, a person can conceive of the volume and height of water in a bottle as varying simultaneously as the bottle is emptied by evaporation.

Carlson et al. (2002) associated mental actions (MAs) with indicative behaviors (both graphical and verbal) in their covariational reasoning framework (see Table 1). Levels of covariational reasoning were then defined in terms of these mental actions. We originally interpreted this framework to mean that someone who exhibits MA2 verbal indicators would also exhibit the corresponding MA2 graphical indicators. Carlson et al. (2002) clarified that a student can exhibit behavior associated with a specific mental action without engaging in the mental action itself, so we suspected someone might be able to perform a graphical behavior indicative of covariational reasoning but be unable to verbalize some covariational relationship. However, it did not occur to us that someone might verbally describe a covariational relationship at a much higher level than their graphical behavior would suggest. We will return to these ideas in our results section.

Table 1: Portion of Carlson et al.'s (2002) Covariational Reasoning Framework

Mental action (MA)	Indicative behaviors
MA1 - Coordinating the value of one variable with changes in the other.	Labeling the axes with verbal indications of coordinating the two variables.
MA2 - Coordinating the direction of change of one variable with changes in the other variable.	Constructing an increasing straight line. Verbalizing an awareness of the direction of change of the output while considering changes in input.
MA3 - Coordinating the amount of change of one variable with changes in the other variable.	Plotting points/constructing secant lines. Verbalizing an awareness of the amount of change of the output while considering changes in the input.

Thompson and Carlson (2017) refined the Carlson et al. (2002) framework to craft a more broadly applicable covariational reasoning framework. Instead of listing indicative behaviors for each level of covariational reasoning, their descriptions focused on students' abilities to envision and anticipate rather than construct, plot, or verbalize an awareness of something. In particular, their descriptions attend to the type of variational reasoning a student uses to envision how each quantity changes as well as the nature of the multiplicative object a student constructs between each quantity's values. Thompson and Carlson followed Saldanha and Thompson's (1998) usage of Piaget's notion of "and" as a multiplicative operator to derive the notion of a multiplicative object. An individual forms a multiplicative object by uniting attributes of two objects to form a new third object. For example, Frank (2017) described how an individual can consider the attributes 'red' (perhaps from an apple) and 'circular' (from a ring) independently, then unite them to construct a single red circle as a multiplicative object. The resulting object is multiplicative because it is simultaneously red and circular. In regard to variational reasoning, Thompson and Carlson built off the variational reasoning research done by Castillo-Garsow (2010, 2012) and adapted the distinctions made by Castillo-Garsow et al. (2013) between discrete, chunky continuous, and smooth continuous variational reasoning.

Thompson and Carlson (2017) outlined two different ways researchers could use their covariational framework. First, the framework levels could be used to characterize a person's covariational reasoning in a specific instance. And second, the framework levels could describe a person's capacity or ceiling for covariational reasoning across settings. For example, a graduate student in mathematics may display a gross coordination of values in describing how the miles driven in their car and the gallons of gas used by the car increased as their road trip continued. Nevertheless, the student may also possess the ability to envision smooth continuous covariation. In this report, we are primarily concerned with the second way of using the framework levels.

Nonnormative Graphing Schemes

Research has recently begun to detail the extent to which students' meanings for graphs diverge from the normative meanings privileged by the mathematical community (Moore et al., 2019). Roughly, there are two key components of a graph: a curve itself and the coordinate system in which the curve is plotted. To attend to the nuanced, nonnormative graphing schemes students use, we consider one construct for each component: (a) the type of *shape thinking* (Moore and Thompson, 2015) students engage in while reasoning about a curve and (b) the quantitative *frame(s) of reference* (Joshua et al., 2015; Lee et al., 2019) students use to construct an underlying coordinate system.

Shape thinking. In order to focus on the meaning an individual has for a graph, Moore and Thompson (2015) introduced the constructs of static shape thinking and emergent shape thinking. Static shape thinking involves attending primarily to the perceptual shape of a graph and inferring associations based on shape, rather than the underlying covariational relationship. On the other hand, emergent shape thinking involves conceiving of a graph as a trace or record representative of the underlying covariational relationship. Thompson (2016) reported that 29 of 111 written responses high school teachers provided to a version of the bouncy ball task were coded as representing static shape thinking (see pp. 449–450). If a nontrivial proportion of teachers are reasoning with static shape thinking, some students likely are as well.

Frames of reference. Joshua et al. (2015) defined a frame of reference as "a set of mental actions through which an individual might organize processes and products of quantitative reasoning" (p. 32). In particular,

> An individual conceives of measures as existing within a frame of reference if the act of measuring entails: 1) *committing to a unit* so that all measures are multiplicative comparisons to it, 2) *committing to a reference point* that gives meaning to a zero measure and all non-zero measures, and 3) *committing to a directionality of measure comparison* additively, multiplicatively, or both. (Joshua et al., 2015, p. 32, emphasis added)

Constructing a complete quantitative structure for a two-dimensional Cartesian coordinate system requires measuring two quantities at once, which is accomplished by simultaneously combining two frames of reference. Ultimately, then, a coordinate system is the result of the many mental actions involved in constructing and combining frames of reference (Joshua et al., 2015). As Lee et al. (2019) found, it is a nontrivial task to coordinate across multiple frames of reference/coordinate systems at once. This suggests that students who do not construct canonical coordinate systems may struggle to interpret the conventional meanings a graph is meant to convey.

Methods

After publishing preliminary results on the relationship between students' units coordination and covariational reasoning (Boyce et al., 2019), our group reflected on the tasks and methods we used to assess students' capacity for covariational reasoning. During this reflective process, we analyzed artifacts from our previous research—including videos of interviews, individual/group coding notes, and meeting memos. For example, we looked at spreadsheets in which two members of the team independently commented on each interview and assigned each student a covariational reasoning level. Over five months, we engaged in sustained group discussion for an hour and a half each week about the theory of covariational reasoning, the literature on graphing and quantitative/covariational reasoning, and how we determined the levels we reported in Boyce et al. (2019).

Prior to each weekly meeting, we spent extensive time considering these issues individually and in small groups. This helped ensure that all group members had a voice and that we were generating several distinct suggestions for improving our covariational reasoning assessment protocol and analysis. As we revised our analysis procedures, we looked back at previous student interviews to determine how these new procedures would affect our past assessments of students' capacity for covariational reasoning.

Results

Limitations of Graphical Tasks

In this section, we discuss our results through an analysis of the actions of a college calculus student named Shania. Specifically, we respond to our first research question by highlighting the limitations of using a graphical task to assess her capacity to reason covariationally. As previously noted, Carlson et al. (2002) remarked that a student might exhibit a behavior associated with a particular mental action or level of covariational reasoning but not reason in a way consistent with that covariation level. They wrote, "Some students have been observed exhibiting behaviors that gave the appearance of engaging in [advanced mental actions] . . . When asked to provide a rationale for their construction, however, they indicated that they had relied on memorized facts to guide their construction" (pp. 361–362). Put more succinctly, graphical activity can lead a researcher to overestimate a student's overall ability to reason covariationally. Shania, as we will demonstrate, is representative of the opposite: her graphing activity may have led us to underestimate her capacity to reason covariationally.

Shania attempted a variation of Thompson's (2016) Bouncy Ball task. The task scenario (with provided graphs shown in Figure 1) follows below:

> A ball is hanging by a 10-foot rubber cord, from a board that is 20 feet above the ground. The ball is given a sharp push downward and is left free to bob up and down. The graph on the left represents the ball's *displacement from its resting point* in relation to the *time elapsed* since the ball was pushed. The graph on the right represents the ball's *total distance* traveled in relation to the *time elapsed*. The information given is in the first second after being pushed. The final graph represents the ball's *displacement from its resting point* in relation to its *total distance traveled* since being pushed.

After Shania read the task, the interviewer asked, "What's happening to the ball based on that description?" The interviewer also provided Shania with a coffee cup to physically illustrate the ball's motion. Following some dialogue to clarify the initial dangling position of the ball relative to the board, Shania gestured up and down with the cup to demonstrate how she believed the bouncy ball would bob up and down after being first pushed down. Already, via gestures, Shania demonstrated signs of reasoning covariationally at the gross coordination of values level. Her movements suggest that she understood how the ball's displacement and the number of seconds elapsed changed together.

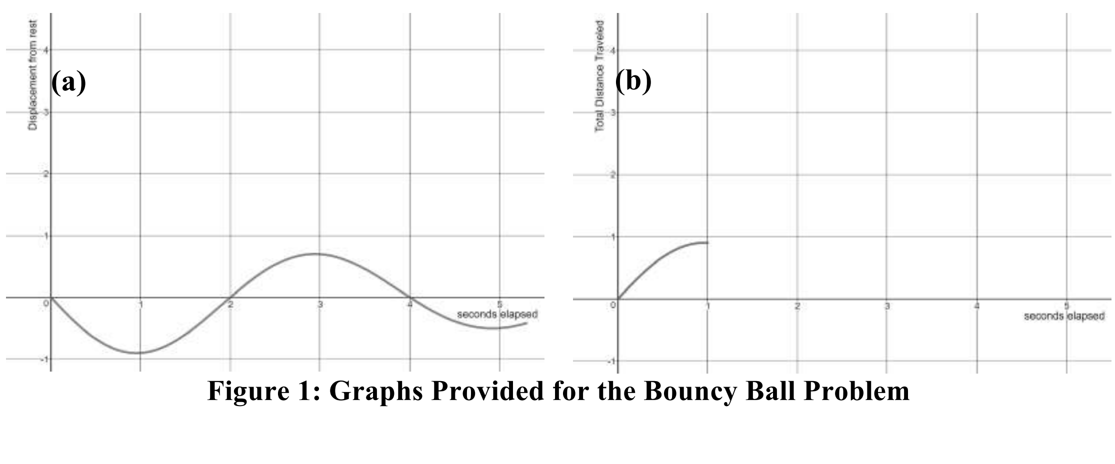

Figure 1: Graphs Provided for the Bouncy Ball Problem

Shania was then prompted to describe the *y*-axis label for the incomplete graph of total distance traveled with respect to time (See Figure 1b). After a brief conversation, the interviewer described how total distance traveled will always be positive. Shania agreed with this description and subsequently stated multiple times without prompting throughout the interview that, "The distance is always increasing." The interviewer then asked Shania to produce the remainder of the incomplete graph. In response, she drew a graph (Figure 2b) that seems to reflect the provided graph over the x-axis on the intervals where the displacement was negative. When describing her graph, Shania said that she thought of it as the derivative of the graph of displacement with respect to time (Figure 1a). But, when she described the relationship between different segments of her graph, she talked about the total distance traveled, rather than the displacement.

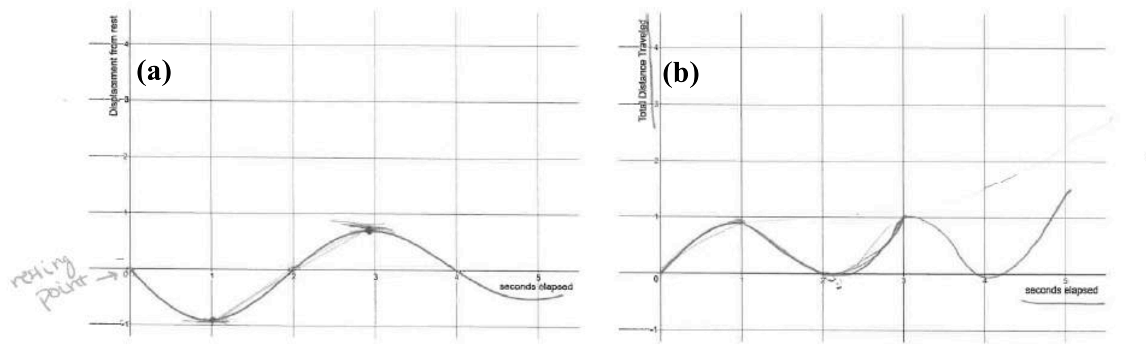

Figure 2: Shania's Work on the Bouncy Ball Task

The interviewer asked Shania why she thought the graph she was asked to produce was the derivative of the first graph. Shania responded, "I kind of saw, like, when we were doing the derivatives in class with sine and cosine and everything and then how sine is the opposite of cosine." The interviewer followed up and asked, "So it wasn't anything about the wording of the question that made you think derivative. Like, it was the memory of an oscillating shape?" To this, Shania responded, "yeah."

The summation of the interactions suggest Shania conflated three different sources to produce her graph: the graphs presented in her calculus course, the shape of the provided graph of displacement with respect to time, and her understanding of how total distance traveled must always be positive. Instead of attending to how the two quantities (total distance traveled and time) simultaneously vary, Shania intended her graph to represent a variety of concepts. We interpret this as significant evidence that Shania created her graph as a record that is intended to communicate a number of specific facts about the ball's movements. In other words, Shania was likely engaged in static shape thinking.

After the interviewer and Shania came to a joint conclusion that the task did not ask for a graph related to the derivative, Shania was given a second opportunity to produce a graph of total distance traveled with respect to time on a new piece of graph paper. Before producing her second graph, the interviewer asked, "What about that graph [Figure 1b] shows us the distance?" Shania responded by saying, "The distance is from here to here [draws an arrow going up parallel to y-axis on the new graph] since it's going up, increasing. And then our seconds were from zero to five." Here, Shania's language indicates that she was engaged in gross coordination of values due to her describing how the total distance traveled increases and how the time increases. Shania's original graph suggests that she did not envision how the two quantities were varying together. Instead, according to Thompson and Carlson's (2017) covariational framework, Shania would have most likely been classified at the precoordination level (a level beneath gross coordination of values).

As evidenced, nonnormative graphing schemes can cause underestimates of a student's capacity to reason covariationally. Frank (2017) demonstrated that in order to construct a graph as a record of quantities covarying, a student must conceive of a multiplicative object. Gross covariational reasoning, however, does not require a student to conceive of a multiplicative object. Thus, a gross covariational reasoner's graphing activity does not provide insight into their maximal capacity for reasoning covariationally. Subsequently, graphing tasks inhibit researchers' ability to assess some students' covariational reasoning capacity.

Improving Covariational Reasoning Assessments

We now transition to our second research question: *How can we improve assessment of covariational reasoning to accommodate students with nonnormative graphing schemes?* A reflective analysis of our own methodologies paired with a review of the literature has led us to one possible improvement that can be made: researchers could provide non-graphical tasks before—or possibly in place of—graphical ones. Although all interviewers in our prior study adhered to a general protocol, we noted in our analysis that one interviewer—the researcher who interviewed Shania—would often ask brief introductory questions to ensure interviewees fully understood the prompt before proceeding to any graphical tasks. Only upon retrospective analysis did the benefits of this practice become apparent. As detailed previously, the language and gestures Shania used to model the trajectory of the bouncy ball with a coffee cup provided strong evidence she was engaged in gross coordination of values. However, when she proceeded to the graphical component of the task, her tendency to engage in static shape thinking obscured evidence of this form of covariational reasoning. Further analysis and group discussion allowed us to see past her static shape thinking in our original study, but without data from the non-graphical introductory questions Shania's interviewer asked, we may have incorrectly assessed her covariational reasoning level.

The case of Shania highlights the importance of providing non-graphical means by which interviewees can engage in covariational reasoning. In line with this suggestion, Moore and Carlson (2012) noted that undergraduate precalculus students' graphical and computational solutions tended to match their emergent image of the problem's context as well as the quantitative structure they constructed for the problem. For example, Shania's gestures provided no evidence that her image of the problem incorporated the damping of the bouncy ball's displacement from rest as time passed. So, in line with Moore and Carlson's findings, it should come as no surprise that her graph of total distance with respect to time did not reflect this damping (see Figure 2b). Moore and Carlson (2012) contended that "It is critical that students first engage in mental activity to visualize a situation and construct relevant quantitative relationships prior to determining formulas or graphs" (p. 48). A simple way of supporting such mental activity could be to have a conversation with an interviewee about the situation and any relevant quantities. Even a short conversation has the added benefit of providing valuable data about the quantities an interviewee is constructing (as we have illustrated in the case of Shania). Another means for supporting the interviewee in constructing mental images of the quantities is to have them model the situation using gestures, like how Shania was instructed to model the trajectory of the bouncy ball using a cup. Of course, there are many other ways to aid interviewees in constructing an image and quantitative structure for the situation. We encourage researchers to be creative and design their pre-graphical tasks in ways that best align with their planned trajectory of tasks and intended methods of analysis.

The Invitation to Conversation

By reflecting on our prior research methods, we identified significant limitations of using predominantly graphical tasks to assess one's capacity for covariational reasoning across settings. Namely, students with nonnormative graphing schemes may be able to reason covariationally at higher levels in non-graphical settings. We suspect the primary difficulty for these students is modeling how two quantities covary in the *normative* Cartesian coordinate system privileged by the mathematical community. After all, even constructing a coordinate system—a prerequisite for creating or reasoning about a graph—requires constructing two frames of reference and then combining them. Not to mention, if the coordinate system a student constructs to model a covariational situation does not align with the normative Cartesian one, they may spend more time resolving perceived contradictions than demonstrating their capacity to reason covariationally. In sum, our analysis has revealed that graphical tasks can add considerable noise to a researcher's data for assessing students' covariational reasoning. If a researcher hopes to use graphs to assess covariational reasoning, we urge them to devise explicit methodologies for reducing or, at the least, acknowledging this noise.

Although we suspect experienced researchers have already begun to develop these types of methodologies (e.g., Johnson, 2015; Stevens & Moore, 2017), they are not yet explicitly outlined in the literature. We believe the community of covariational reasoning researchers, particularly novices, would benefit if experienced researchers shared the fine-grained details of their task design and analysis techniques. After all, such accounts are vital to communicating important considerations for conducting covariational reasoning research that may not yet be salient in the literature, such as the limitations of using graphical tasks. We call on researchers (ourselves included) to consider the role that non-graphical tasks should play in assessing covariational reasoning. To that end, we offer this piece as the beginning of what we hope becomes a larger conversation to develop improved methodologies that accommodate the ubiquity of students' nonnormative graphing schemes.

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FOUR COMPONENT INSTRUCTIONAL DESIGN (4C/ID) MODEL CONFIRMED FOR SECONDARY TERTIARY MATHEMATICS

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Cognitive Load Theory's Four Component Instructional Design (4C/ID) Model has been used in mathematics education but not confirmed as an instructional theory. Using the Factors Influencing College Success in Mathematics (FICSMath) project and confirmatory factor equation modeling, we empirically validated the model and created the 4C/IDMath Model. Instructional experiences of respondents completing the FICSMath survey were mapped to the theoretical components of the 4C/ID Model. The Mathematical Learning Task, Conceptual Understanding, Procedural Fluency, and Practice for Recall Components correspond to the Learning Task, Support, Procedure, and Part Task Components, respectively, from the original 4C/ID Model. The 4C/IDMath Model can be used to guide instruction in secondary precalculus and calculus courses to support transfer of learning to single variable college calculus.

Keywords: Research Methods, Design Experiments, Secondary-Tertiary Transition in Mathematics

Theoretical Perspective

Cognitive load theory (CLT) was introduced in the 1980s as an instructional theory based on well accepted aspects of human cognitive architecture (Sweller, van Merriënboer, & Paas, 2019). A major premise of the theory is that working memory load from cognitive processes is decreased when domain specific schemas are activated from long term memory. Comprehension, schema construction, schema automation, and problem solving in working memory often create high cognitive load. Hence, schemas transported from long term memory into working memory support learning and transfer of learning (Ginns & Leppin, 2019). One of the key developments from CLT has been the Four-Component Instructional Design (4C/ID) Model generated from evolutionary theorizing (Geary, 2008; Ginns & Leppink, 2019). Since its creation, the 4C/ID Model has been successfully applied to instruction that requires the learning of complex tasks. Van Merriënboer, Kester, and Paas (2006) defined a complex task as having many different solutions, real world connections, requiring time to learn, and as creating a high cognitive load. Based on this definition, the instruction and learning of mathematics is a complex task. For example, different solutions are algebraic, analytic, numeric, and graphic. Relative to real world connections, mathematics is one of the domains in the broader science, technology, engineering, mathematics (STEM) field and is regarded as the language of the sciences. Regarding taking time to learn and creating a high load on learner's cognitive systems, mathematics teachers deal with the tension between covering all the required standards and taking the time to teach for understanding. Teachers face challenging decisions about instructional approaches, materials, productive struggle, and the amount of classroom time spent on various standards. Better models for instruction that support transfer of learning could help teachers improve instructional decision making. Although the 4C/ID Model has been used in secondary mathematics education (Sarfo, & Elen, 2007; Wade, 2011), it has never been confirmed as a mathematical instructional theory. The purpose of this research report is to present an empirical confirmation of the 4C/ID Model, using data from the Factors Influencing College Success in Mathematics (FICSMath) project from Harvard University.

FICSMath Project

The Factors Influencing College Success in Mathematics (FICSMath) Project remains the largest and most recent national study of the secondary-tertiary transition in mathematics. Towards the beginning of the 2009 fall semester, college freshmen in single variable calculus courses across the United States (US) responded to questions on the FICSMath survey regarding educational experiences in their last high school mathematics course. Professors secured students' completed surveys until the end of the semester and recorded final grades for each student on their respective survey before returning them to Harvard University. A total of 10,492 surveys were collected, and from this sample 5,985 students had taken either precalculus (n=2,326), or any level of high school calculus (n=3,659) as their most recent high school mathematics course. The 4C/ID Model appears appropriate to use as a theoretical lens through which to view secondary preparation for college calculus because the components of the model explicitly consider instruction to support transfer of learning (van Merriënboer, Kester, Paas, 2006).

The 4C/ID Model

Van Merriënboer and other cognitive load theorists developed the 4C/ID Model in the early 1990s under the premise that instruction for complex tasks should be combined with methods that have been shown to enhance transfer of learning (Van Merriënboer, Kirschner, Kester, 2003;

Van Merriënboer, Clark, & de Croock, 2002). Transfer is required when prior learning must be recalled to support the learning of new tasks. Vertical transfer is required, for example, to transfer knowledge from the high school mathematics to college calculus. The model was not designed specifically for mathematics instruction, but generally for learning environments where complex problems are the basis of instruction and transfer of learning is the goal.

The 4C/ID Model employs human cognitive architecture from cognitive load theory (Sweller, 2008). The assumptions are that working memory is limited in space and duration while there appears to be no limit of either in long-term memory. The three sources of working memory load are assumed to be: (a) extraneous cognitive load coming from how the material is presented during instruction; (b) intrinsic cognitive load coming from element interactivity, or the interaction of the interconnected parts of the content; and (c) germane cognitive load, which sends and hooks new processed and encoded information into long term memory to be connected with existing schemas. Once information has been processed and connected within the learners' schemas in long term memory, it can then be brought back into working memory as a chunk of knowledge to help process more new content. Integration of new content into schemas makes learning more efficient as it lowers the demands on working memory and supports the learning of complex tasks.

Model Components

The 4C/ID Model incorporates four components: Learning Task, Support, Procedure, and Part-Task Components. These come from theorizing how to instruct a complex task to enable working and long-term memory to develop, retain, and recall comprehension, schema construction, schema automation, and problem solving. Figure 1 shows how Van Merriënboer, Kester, and Paas (2006) theorized the model. Each of these components needs attention during precalculus and calculus instruction. The neglect of any one of them could prohibit learning and/or transfer of learning. As such, the components are discussed specifically regarding the instruction of mathematics during the secondary-tertiary transition.

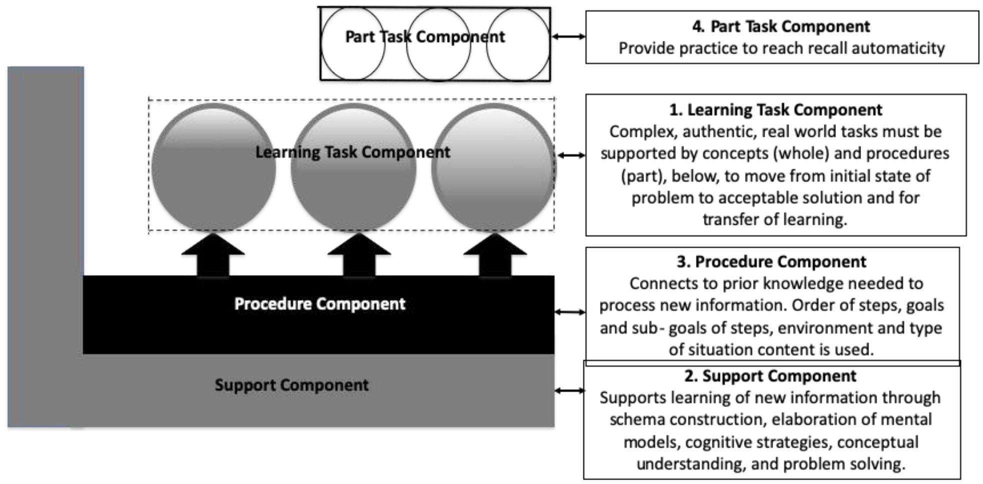

Figure 1: The theorized general 4C/ID Model (modified from van Merriënboer, Clark, De Crook, [2002])

1. The Learning Task Component is modeled to engage learners in meaningful problem-solving tasks. Working with real world problems, often integrated into mathematics to motivate learning (Beswick, 2010), requires mental processes to move from the initial state of the problem to an acceptable solution (van Merriënboer et al., 2003). Engagement in higher-level tasks during mathematics instruction increases students' engagement with mathematical ideas (Boaler and Staples, 2008). Such tasks include high element interactivity, which occurs because of the interacting parts of the mathematics that must be addressed during problem solving. Element interactivity is inherent in secondary preparation for college calculus because of the many interacting mathematical concepts involved in precalculus and calculus problem solving.
2. The Support Component undergirds the Learning Task Component and includes conceptual understanding, reasoning of new information, problem solving, and cognitive assessment (van Merriënboer, Clark, de Croock, 2002).
3. The Procedure Component integrates examples, hooks to previous learning or schemas from long-term memory, which supports the processing of complex ideas. These are important instructional practices in mathematics (Wade, Sonnert, Sadler, & Hazari, 2017; Wade, Cimbricz, Sonnert, Gruver, & Sadler, 2019). This is, first, because mathematics is abstract, and reasoning is required to understand abstract information (Russell, 1999). Another reason is that, when strategies are recalled from long term memory, it is common for mistakes in the problem-solving process to occur. Yet, with guidance, students can learn from their mistakes. This process is referred to as flawed reasoning and is believed to be an important part of learning mathematics (Russell, 1999).
4. The Part Task Component models instruction working towards students developing automaticity. This means that specific tasks from previous learning can occur with little effort, requiring little conscious monitoring and few cognitive resources (Feldon, 2007). The part-task component is included in the 4C/ID Model because there are times that instruction allows repeated practice of information to the point of automaticity. Depending on where the learner is in understanding whole concepts first, this can both benefit or hinder the meaningful learning of mathematics. For example, if a student has not learned the concepts but practices procedures, the result is often what Skemp (2015) refers to as instrumental understanding or rules without reason. The goal is relational understanding, which Skemp

(2015) defined as knowing what to do and why, which requires understanding concepts as well as procedures.

Research Question and Method

Can the 4C/ID Model be empirically validated for mathematics instruction for the secondary-tertiary level using data from the FICSMath Project and confirmatory factor analysis?

Confirmatory Factor Analysis Model

Freshmen respondents in single variable college calculus courses from large, medium, and small 2- and 4-year institutions from across the nation reported instructional experiences from their senior-level high school precalculus or calculus courses (n=5,985). The percent of missing value cases were small (between 1.3% and 4.9%), yet multiple imputation was computed to create a small number of copies of the dataset, with each having missing values suitably imputed. Each complete dataset was analyzed independently and estimates of parameters of interest were averaged across the copies to provide a single estimate (Royston, 2004). In the end, the model reported 6,146 cases, a 2.6% increase from the 5985 respondents included in the model. Then confirmatory factor analysis (CFA) was used to test the extent to which these variables related to the underlying constructs of the 4C/ID Model.

CFA is theory driven, so we began by analyzing the theoretical relationships among the observed and unobserved, or latent, variables (Schreiber, Nora, Stage, Barlow, & King, 2006). The observed variables (Figure 2 rectangles) are intercorrelated secondary instructional experiences reported by single variable college calculus students who completed the FICSMath survey. The unobserved variables (Figure 2 large ovals) are factors that account for correlations among the observed variables (Brown & Moore, 2012) that theoretically aligned to the 4C/ID constructs. We identified instructional experiences that provided: (a) complex mathematical tasks (Learning Task Component, n=4); (b) an overview of whole task mathematical concepts (Support Component, n=17); (c) support for the processing of mathematics, the use of algorithms, and graphing (Procedural Component, n=16); and (d) opportunities for practice (Part-Task Component, n=13). Figure 2 shows the number of observed variables that converged and survived CFA. The loadings to the right of the large ovals show the correlations between the components while loadings to the right of the rectangles show the correlations between the observed variables to each component. The small ovals connected to each rectangle on the left show the errors associated with the observed variables in the model. The loadings to the left of the errors show their correlations while one shows the correlation between the error for emphasis on vocabulary and the support component (discussed later).

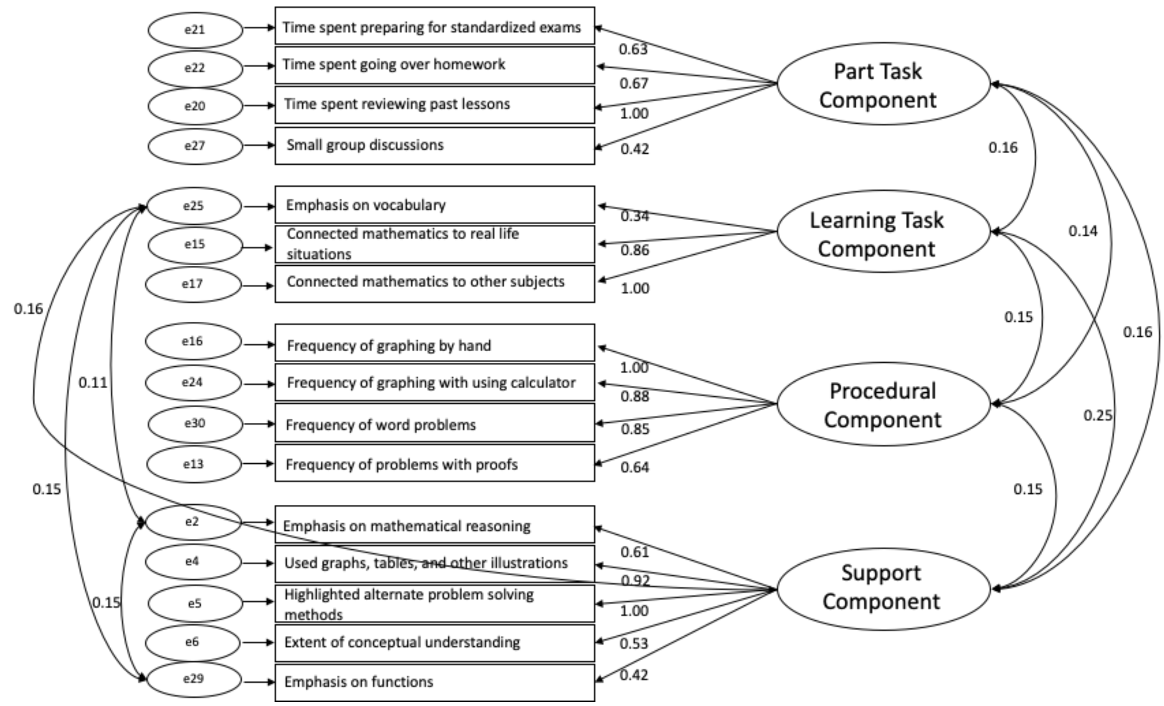

Figure 2: The 4C/ID *Math* Model Confirmed using Confirmatory Factor Analysis.

The large FICSMath sample size (n=5,985) allows assumptions of normality of data and increases the power of a hypothesis test. This large sample size, however, did limit some of the CFA measures that can be reported. For example, the chi square test, Normed Fit Index (NFI), and Tucker Lewis Index (TLI) are typically reported in CFA models, but these are preferable measures for smaller data sets. As shown in Table 1, the Comparative Fit Index (CFI) and the Root Mean Square Error Approximation (REMSA) confirm the components in the 4C/ID Model have meaningful relationships with the observed variables in the FICSMath dataset.

Table 1: Measures of CFA Reported, Accepted Cut-off Scores for Significance, Results of the 4C/ID Math CFA Model with Notes for Clarity.

CFA Measure	Cutoff for Significance	Model Value	Notes
Comparative Fit Index (CFI)	CFI > 0.90	0.907*	Compares the fit of a target model to the fit of a null hypothesis model.
Root Mean Square Error Approximation (RMSEA)	RMSEA < 0.08	0.050*	A parsimony-adjusted index. Values closer to 0 represent a good fit.

*Significant finding. (See Parry (no date); Brown & Moore, 2012).

Figure 3 shows a representation of the confirmed 4C/ID Model, now referred to as the 4C/ID*Math* Model. The constructs have been renamed to align closer with the field of mathematics education. These components are now:

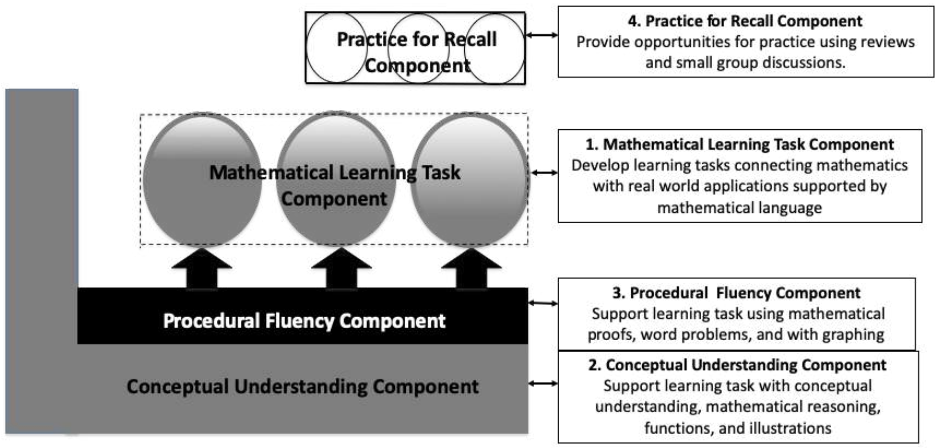

Figure 3: The 4C/ID*Math* Model for Secondary-Tertiary Mathematics (modified from van Merriënboer, Clark, De Crook, [2002])

1. The Mathematical Learning Task Component is the new name for the Learning Task Component. This is where whole tasks should be presented to avoid the transfer paradox. The transfer paradox is described as occurring when instruction breaks apart concepts to minimize the necessary time-on-task. This type of instruction has been shown to have a positive effect on short term retention for performance on tests, but not on transfer of learning (van Merriënboer et al., 2006).
2. The Conceptual Understanding Component is the new name for the Support Component. This name change aligns with what teachers who were identified as teaching for high conceptual understanding on the FICSMath survey concretely did to teach for conceptual understanding (Wade, Sonnert, Sadler, Hazari (2017). This study showed that teaching functions and mathematical reasoning was highly correlated with conceptual understanding.
3. The Procedural Fluency Component is the new name for the Procedure Component. Star (2005) presented thinking flexibly with mathematics as an indicator of deep procedural knowledge. To generate graphs, students must be able to think flexibly across the connections between equations and algorithms to points on various graphing planes. Mathematical proofs require meaningful connections across relevant mathematical relationships, which requires thinking flexibly with those relationships (Williams-Pierce et al., 2017).
4. Practice for Recall Component is the new name for the Part-Task Component. Van Merriënboer, Kester, and Paas (2006) stated that part-task practice may provide additional practice needed to develop knowledge elements that allow the learner to perform routine aspects at a high level of automaticity. In mathematics education, this is better understood simply as practice for recall.

Limitations and Future Work

One weakness of the study may be that the FICSMath Project is from 2009, yet this project remains the most recent national study on secondary preparation for college calculus success. Until the FICSMath project can be replaced by another large-scale national study, the national representation and sample size strength of the project warrants its continued use. Additionally, the 4C/ID*Math* Model is confirmed for students in the secondary-tertiary transition who took either precalculus or calculus as their last mathematics course before entering into single variable college calculus. More research is needed to confirm the model at different levels of mathematical instruction, such as for algebra or geometry. Lastly, how the 4C/ID*Math* Model actually predicts performance in single variable college calculus needs to be investigated. The focus of this paper was to confirm the 4C/ID Model and then modify it to be more user friendly for mathematics teachers.

Discussion

The theoretical perspective of the 4C/ID Model is that instruction of complex tasks should be guided by principles that reinforce learning and transfer of learning. The 4C/ID Model theorizes the Support Component as concepts that structure the learning of complex tasks and the Procedure Component as connecting prior learning, the order of steps and context for use. Both of these components undergird instruction of a complex task, which is represented as the Learning Task Component. The Part Task Component symbolizes the use of automatized information that requires little to no cognitive load in working memory. Van Merriënboer et al. (2006) state the 4C/ID Model was designed to focus instruction on whole tasks and claims breaking apart concepts to minimize time-on-task has a positive effect on short term retention for performance on tests, but not on transfer of learning. This was theorized as the transfer paradox. Skemp (2006) presented similar ideas in mathematics education through relational and instrumental understanding. Relational understanding comes from instruction that focuses on knowing what to do and why while instrumental understanding was conceived as instruction that focused on rules without reason. It was claimed that high school teachers often adopt a two-track strategy of instruction where they spend some time on drill and practice, providing for skills and facts, and some time on developing and integrating understandings (Skemp, 2006). Based on the 4C/ID Model, drill and practice can develop automaticity but does not reinforce learning for transfer. These similarities indicate the 4C/ID Model to be a good fit with mathematics education. The empirical confirmation of the 4C/ID Model using the FICSMath Project resulted in the 4C/ID*Math* Model for secondary-tertiary mathematics instruction. The 4C/ID*Math* Model confirms the importance of generating the learning task first then considering the concepts and procedures needed for learning and transfer of learning. Each of the components for the 4C/ID*Math* Model is discussed below relative to how this model can be used in precalculus and calculus secondary-tertiary mathematics instruction.

1. The Mathematical Learning Task Component represents complex tasks that must be considered as a whole to support transfer of learning. Instruction that breaks apart concepts to minimize time for learning has been shown to have a positive effect on short term retention but not on transfer of learning (van Merriënboer et al., 2006). Considering complex tasks and how to present the many interacting elements as a whole concept first is important, especially in mathematics where transfer of learning is critical. As seen in Figure 2, this component includes the emphasis on (mathematical) vocabulary item. The vocabulary item was originally mapped to the Support Component, but the CFA model was not valid. When the vocabulary item was moved to the Learning Task Component, the error associated with the item was too high. After correlating the vocabulary item error term with the Support Component, the 4C/ID Model converged. This correlation indicates the vocabulary term is essential to both, the Mathematical Learning Task and the Support Component. Tall (2004) stated real world representations require the sophistication of language to support abstract concepts in formal mathematics. After determining what standards and elements are to be instructed, focus should then be placed on the language required to present the content and how to connect the mathematics to real world applications and other subjects.
2. The Conceptual Understanding Component emphasizes conceptual understanding, mathematical reasoning, functions, illustrations, and alternate problem-solving methods necessary to support learning mathematical content. Wade, Sonnert, Sadler, and Hazari (2017) showed mathematical reasoning and emphasis on functions to be part of the construct

that described what teachers did to teach for conceptual understanding. This component aligns well with the field of mathematics education.

3. The Procedural Fluency Component demonstrates hooking previous learning, or schemas, from long-term memory to concepts presented in the Conceptual Understanding Component. Graphing functions and equations require modeling mathematics both by hand and, in the secondary mathematics classroom, the graphing calculator. Most secondary mathematics standardized exams, including AP exams, require the use of a graphing calculator but most single variable college calculus courses do not allow their use in class or on exams. This implies the importance of students understanding the mathematical procedures even if they have a graphing calculator available. Mathematical proofs, independent of the format, require justification from prior learning and are an important part of the secondary-tertiary transition. At the tertiary level, proofs tend to be longer, more complex, and require more mathematical insight than at the secondary level (Selden, 2011). Many students are not well prepared for the types of proofs they will be exposed to in college calculus (Bressoud, 2009). It could be that incorporating more proofs into secondary precalculus and calculus courses may reduce some of the transition struggles for students in college calculus.
4. The Practice for Recall Component illustrates that opportunities for practice using reviews and small group discussions are beneficial for developing automatic recall.

It is our hope that the confirmation of the 4C/ID Model, leading to the 4C/ID*Math* Model, brings this instructional framework into the purview of secondary mathematics teachers and mathematics professors who teach students in the secondary-tertiary transition. Better preparation for single variable college calculus is important because this is the first mathematics course that is commonly required in all STEM majors.

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TEACHER QUESTIONING STRATEGIES IN SUPPORTING VALIDITY OF COLLECTIVE ARGUMENTATION: EXPLANATION ADAPTED FROM HABERMAS' COMMUNICATIVE THEORY

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Facilitating productive mathematical argumentation is challenging; it is critical to develop a specific guiding vision of practices to help teachers learn to teach argumentation. However, what counts as acceptable classroom-based mathematical argumentation remains an open question. In this study, building on Habermas' theory of communicative action, we developed two analytic frameworks to examine questioning strategies used to support the validity of collective mathematical argumentation. Habermas' three components of rationality allowed us to focus on fine-grained rationality components of teacher questioning as well as teachers' intentions of asking these questions. The theory of validity claims was used to capture different forms of validating argumentation. The frameworks may help teachers to be aware of the types of questions that they are asking when aiming at supporting valid argumentation.

Keywords: Research Methods, Classroom Discourse, Reasoning and Proof

Rationale and Purpose

Current research discusses many benefits of incorporating mathematical argumentation in classroom discourse (e.g., Nussbaum, 2008) and emphasizes the essential role of teacher questioning in facilitating collective mathematical argumentation — teacher and students (or a small group of students working independently) working together to determine the validity of a claim (Conner et al., 2014). Studies (e.g., Kazemi & Stipek, 2001; Kosko et al., 2014; Wood, 1999) have highlighted teachers' questioning as a pivotal factor shaping argumentative discourse and as strongly influencing students' engagement in productive mathematical argumentation. However, most of these studies placed more emphasis on documenting current situations or difficulties that teachers had in using questioning to regulate argumentative discourse than on developing effective ways to address some of these difficulties. For example, Sahin and Kulm (2008) analyzed types of questions two teachers used in two sixth-grade classes over two months. They found that the majority of questions teachers posed were factual, even when using a reform-based textbook, which included probing and guiding questions in the teaching guides. Scaffolding argumentation is not an easy task, and it is not clear precisely what actions of the teacher provide the desired results of argumentation. Further, no consensus exists in the field of mathematics education concerning the characteristics of successful argumentative discourse. Some researchers (e.g., Stylianides et al., 2016) have called for more research in the field to design practical tools for use in the classroom to address teachers' difficulties or particular learning goals in orchestrating argumentative discourse.

The goal of this study is to investigate how a beginning secondary mathematics teacher uses *rational questioning* as a didactical tool to support the validity of collective mathematical argumentation according to Habermas's (1984) theory of *validity claims*.

Theoretical Framework

Two concepts from Habermas' theory of communicative action are used in this study to investigate how teachers could support valid argumentative practices with a particular focus on teachers' questioning strategies. The first is Habermas' (1998) perspective on three interrelated components of rationality: epistemic (inherent in the control of validation of statements), teleological (inherent in the strategic choice of tools to achieve the goal of the activity), and communicative (inherent in the conscious choice of suitable means to communicate understandably within a given community). Boero (2006) advocated that Habermas' three components of rationality account for students' rational behavior in proving and argumentative activities. Corresponding to these components, students are expected to strategically choose tools to achieve a goal (teleological rationality) on the basis of specific knowledge (epistemic rationality) and communicate in a precise way with the aim of being understood by the classroom community (communicative rationality). Douek suggested that it was beneficial to develop argumentative discourse along the three components of rationality (i.e., epistemic, teleological, or communicative) and that the teacher should support students to meet the requirements of rationality, thus dialectically forming argumentation (Boero & Planas, 2014). In order to reach such aims, Douek further proposed the idea of using "rational questioning" as a method to “organize the mathematical discussion according to the three components of rationality” (Boero & Planas, 2014, p. 210). Following Douek's idea, we developed a Teacher Rational Questioning Framework (see Table 1) to classify types of rational questioning from teachers' perspectives to engage student participation in argumentation with different kinds of rationality (For more details, see Zhuang & Conner, 2018). We defined *rational questioning* as a question that contains at least one component of rationality. At times, for clarity, we call a question epistemic rational questioning if it contains an epistemic rationality component.

Table 1: Teacher Rational Questioning Framework

Components of Habermas' Rationality	Features	Examples
Epistemic Rationality (*ER*)	The questions intended to allow students to reason and justify their arguments; to clarify/challenge students when they gave unclear or incorrect responses.	Can you tell me why?
Teleological Rationality (*TR*)	The questions intended to allow students to show or reflect on the strategic choices that they used to achieve their arguments or ideas; to point students towards the specific means or tools.	How did you figure that out?
Communicative Rationality (*CR*)	The questions intended to allow students to communicate or reflect on the steps involved in their reasoning and arguments to ensure that their ideas can be understandable in the given community; to point students towards the correct use of mathematical terminology.	How would we write this correctly mathematically?

In terms of validation of argumentation, we adapted Habermas (1984) theory of *validity claims*, which proposed that three forms of validity claims exist: to truth, to rightness, and to sincerity (see Table 2). By adapting Habermas' theory of validity claims to collective argumentation in mathematics classrooms, we identified three parallel dimensions of validating argumentative practice (For more details, see Zhuang & Conner, unpublished).

Table 2: Validity Claims and Corresponding Validating Argumentative Practice

Validity Claims	Features	Corresponding Validating Argumentative Practice
Truth	Concern the way things are in the external world of objects and spatiotemporal entities, thus constituting a constative (fact-stating) speech act.	Argumentation results in correct mathematical conclusions (T). The truth of an argumentation was judged by the researcher's perspective according to shared mathematical theorems, axioms, and principles in the given mathematical classroom community.
Rightness	Concern the way things are in the social world of shared duties, norms, values, thus constituting a regulative speech act.	Argumentative practices conforming to the social norms (N-S) and sociomathematical norms (N-M) (Yackel & Cobb, 1996) in a given classroom social context.
Sincerity	Concern the way things are in the subjective world consisting of personal self-understandings, thoughts, intentions, feelings, thus constituting an expressive speech act.	We assume when the students engaged in the argumentation, they satisfied the sincerity of argumentation unless there is clear evidence demonstrating that the argumentative discourse deteriorates into oppositional or confrontational talk and interpersonal conflicts spill over into the intellectual content.

Further, Habermas (1984) argued that the acceptance of valid argumentation not only links to the referred mathematical objective world, to norms, but also to the use of language. If a speaker cannot present comprehensible and accepted language, then there is no way to establish a shared understanding through communication. This concern about the fundamental use of language gives rise to a new dimension for validating argumentation, which focuses on communicative validity of argumentation and the participants' intentions on reaching a shared understanding within an argumentative practice, that is: Argumentation is communicated by using appropriate mathematical language and representations with participants' intentions to reach a consensus or a shared understanding (*C*).

In this study, we adapted two developed frameworks on the basis of Habermas' theory of communicative action to investigate how rational questioning supports the validity of collective argumentation in a 9th-grade algebraic mathematics classroom.

Data and Methods

The participant, Jill (a pseudonym), was in her third year of teaching and was purposefully selected based on her good understanding of argumentation and willingness to support student engagement in argumentation. We video recorded two consecutive days of Jill's instruction per month, which translated into eight classes a semester. The primary data sources in this study included video recordings and transcriptions of two consecutive days of Jill's instruction, focused on factoring and expanding binomials with integer coefficients.

Each lesson was first divided into multiple argumentation episodes. An argumentation episode was located by identifying the final claim of an argument and the accompanying data, warrants, and data/claims supporting the final claim the collective attempted to establish. Therefore, if there were arguments or claims that supported or refuted the initial argument, these arguments were viewed as connected with each other and included in an episode of argumentation. The next step was analyzing all teacher questions within each chosen argumentation episode in order to identify and categorize rational questioning based on our Teacher Rational Questioning Framework (see Table 1). Each rational question was also categorized according to the valid argumentation analytic framework (see Table 2) to explore how teachers used rational questioning to support the validity of argumentative practices. The classification of teacher questioning started with developed frameworks, but we kept an open mind by using a grounded theory approach (Glaser & Strauss, 1967) to ensure the inclusion of additional themes that were not included initially in the framework. A simple enumerative approach was finally used to quantify rational questioning in order to explicate the patterns that emerged from the open-coding process.

An Example of Using Habermas' Frameworks

As an illustration, let us consider an argumentation episode on the second day where students had reviewed the greatest common factor and expansion of binomials with form $(x \pm a)(x \pm b)$ on the first day of the lesson. During this episode, the students were learning about factoring trinomials with integer coefficients in a small group:

Given $x^2 + \_\_\_ x + 12$, *what are the possible values for the blank?*

1 T: **All right, what do we think?** (Questioning without a rational component: *N*).
2 S1: It's six or nine.
3 T: Six or nine.
4 S2: Yup.
5 T: **Tell me why.** (*ER:* contains epistemic rational component).
6 S1: Tell her why S2.
7 S2: Why do I have to tell her. Oh. Um, okay, so couldn't, couldn't like...
8 T: **Hang on. I want to hear 6 or 9 explanations first.** (*N*)
9 S2: Oh gosh. Could you say the 9 explanation and I say 6 explanation?
10 T: **Tell me the 6 explanation.** (*ER*)

Interpretation. At the beginning of this episode, both students provided incorrect answers. Instead of giving direct corrective feedback, the teacher challenged students' arguments by asking them to provide an explanation of incorrect answers (Lines 5, 8 and 10). Thus, we coded these three questions as rational questioning that contains epistemic rational components (*ER*). These questions also illustrated that the teacher has a special role to play in trying to develop classroom social norms (*N-S*) to address expectations for student participation in argumentative practices through ongoing negotiations. In this context, students were expected to provide warrants, reasons, or backings to justify their claims. Thus, we coded these rational questions as facilitating the validity of argumentative practices in regard to classroom social norms.

11 S2: Okay. So, 6 times 2 is 12.
12 T: Yes, 6 times 2 is 12. That's true.
13 S2: Yeah, and then 6 might not work, 6 wouldn't work.
14 T: **Why not? Talk to me about why 6 might not work.** (*ER*)
15 S2: Because 6 plus 2 is 8 and you have to have 12 and so because [mumbling]
16 T: **Hang on, hang on. You are saying things that are on the very right track.**
17 **Think through it.** (*ETCR:* contains all three rational components).

Interpretation. Through the explanation of her arguments, S2 noticed that 6 was incorrect and worked towards the correct answer 8. However, she lacked the confidence to further articulate the justification in her thinking. At this point, the teacher encouraged her to explain her reasoning (Lines

16 to 17) which revealed again that students are expected to provide reasons to justify their claims (*N-S*).

S2: Okay. 6 plus 2 is 8 but yeah do not even know where 12 like, how are you supposed to like, do you know what I am saying it's like
S1: You can put 8 in here. That's the point.
S2: Yeah.
S1: So we are trying to find the line, what goes on the line up here.
S2: Yeah.
S1: You can put 8 but 6 times 2 is 12 and then 6 plus 2 is 8.
S2: So it's 8.
S3: Yeah.
T: **Yes. You are thinking about it in the right way. You said 8. That's okay. That's why I want you to think about it. Now does that make sense?** (*ETCR*)
S2: Yeah.

Interpretation. The student-student interactions (Lines 18 to 24) illustrated that pushing students to justify why their arguments hold served to support students to understand that the acceptable claims are based on mutual understanding and agreement on epistemic reasons. The question "Now does that make sense?" showed Jill's intention to provide students with opportunities to make sense of other students' epistemic, teleological and communicative requirements of argumentative practices (*ETCR*). It also pushed students to be able to learn from each other which promotes their productive disposition towards mathematics to reach a consensus or a shared understanding in argumentation (*C*).

T: **So does anyone come up with another number besides 8 that could go there? Anybody come up...S4, why could you do 7?** (*ER*)
S4:Oh gosh.
T: **You said you could do it. Why?** (*ER*)
S4:4 plus 3 is 7 and 4 times 3 is 12.
T: Very good. **Is there anything else?** (*TR*)

Interpretation. When S4 came up with answer 7, which was different than others, the teacher intentionally called on her to explain why this could work (Lines 30 to 31, *ER*), which established the expectations for students in the class to share their thinking, ideas, and solutions, even if they have answers that differ from other students' answers (*N-S*). At this point, the teacher's directive question served to help students understand what counts as mathematically different solutions (*N-M*). The final claims of this argumentation are 8 and 7 could work while 6 cannot. Notice that the answers provided here do not include all possible values. At the end of this episode, the teacher asked students to keep thinking of any other possible values that might exist (Line 35, *TR*). From the researcher's point of view, all the rational questions used in this episode also support the requirement of truth argumentation (*T*). Therefore, rational questioning can support multiple forms of validating argumentation.

Results

Based on our definition of argumentation episodes, Jill's two consecutive days of lessons contained 23 argumentation episodes. Within argumentation episodes, Jill asked 136 questions, and 81% (110/136) of questions involved rational questioning. According to our analysis, Jill used a variety of combined forms of rational questioning: some questions included two or three components of rationality and others only involved one. For the purpose of this study, we looked across epistemic, teleological, and communicative rational questioning and examined how different types of rational questioning were related to these components of validity of argumentative discourse. Table 3 gives the numbers in each type of rational questioning and how it related to engagement in valid argumentation.

Table 3: Rational Questioning Supports Validity of Collective Argumentation

Type of Rational Questioning	Number of Questions	Truth	Norms	Communication
Epistemic	46	27	45	9
Teleological	70	54	31	15
Communicative	32	22	17	15

Note. The number of questions in each category is not discrete; a question might be categorized in several categories.

The content of these lessons included multiple problem-solving mathematical activities to teach students how to factor and expand binomials with integer coefficients (see Figure 1), and the teleological (i.e., producing strategies to achieve the aim of the activity) was the most common rationality component among all rational questioning, in which over 60% (70/110) of rational questioning contained a teleological component. Most of Jill's teleological rational questions (54/70) were strategically goal-oriented to support students achieving the *truth* of arguments in regard to filling in an area model (e.g., "Alright now I have the inside of my area model filled out. How do I get the outside?") and finding the greatest common factor in each row and column of the area model so as to solve the problem (e.g., "What is the greatest common factor of the bottom row?"). Jill also intentionally used some teleological rational questioning to encourage other students to join the discussion: "Okay at this point we have two empty boxes. Somebody else, I want you to tell me how we find what goes into those two empty boxes that we have." By continuing to ask other students to respond to particular students' answers, Jill developed *norms* that every student in the class was expected to pay attention to what other students say and be ready to share solutions. On a few occasions, Jill wanted students to be able to use precise mathematics language to *communicate* their ideas and communicated this by asking them to be more specific about their solutions.

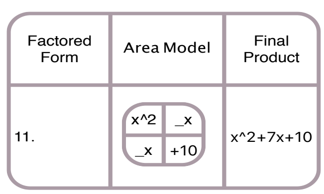

Figure 1: An Example of Task in Lesson One

Epistemic rationality followed as the second most common component of rational questioning (46/110). Jill used most of her epistemic rational questioning (45/46) to encourage her students to justify why their arguments hold (e.g., "You are correct; it's not three, but why?") or challenge her students to provide reasons for their arguments, especially when they gave incorrect answers. Epistemic rational questioning presses students to provide evidence to support the claims that contribute to the development of another *norm*: when engaging in argumentation, constructing a claim is not enough, you are expected to provide your reasoning for the claim. Through analysis we also noticed that not all epistemic rational questioning resulted in correct responses (27/46 prompted correct answers). Jill used sequences of epistemic rational questioning to make students' implicit ideas more explicit and help students to revise their incorrect answers (as shown in the example above). In this way, the teacher also supports the development of students' ability to form comprehensive and acceptable speech acts (i.e., *communication*) based on mutual understanding and agreement. This result provides empirical evidence to support Frank and colleagues' (2009) contention that a single specific question is not enough to elicit a complete explanation or justification; sequences of questions that concentrate on students' explanations are required.

Based on our analysis, 29% (32/110) of the rational questioning contained communicative components. In this unit, Jill's goal for students was apparently less focused on communicative rationality than on teleological and epistemic rationality. Most of the communicative rational questions (22/32) served to introduce graphic representation (the area model) to help students reason and to pull out correct answers (*truth*). For example, she asked, "If this is an area model, what could I call this [point at length] and what could I call this [point at width]?" Sometimes Jill intentionally asked students to rewrite a mathematical expression so that they could easily find the greatest common factor (e.g., "x^4, how do I rewrite this one?"). Occasionally, Jill wanted to highlight her expectations for students to use correct mathematical representations and ensure their representations can be understood in the given classroom community (i.e., *norms and communication*). An example of this type of question would be as follows: "Have I actually finished...I need to write it in the factor form. So tell me what to write."

Conclusions and Implications

Drawing on two different concepts from Habermas' theory of communicative action, in this study we developed two frameworks focused on teacher questioning strategies to facilitate valid argumentative practices. Our definition and classification of *rational questioning* came from Habermas' three components of rational behavior. Many researchers (e.g., Boero, 2006; Cramer, 2015) applied this construct as a tool to analyze students' participation in argumentation; our study shows it could also be used to analyze teachers' ways of dealing with argumentation in the classroom. Habermas' theory of validity claims provides a tool to develop an analytic framework to capture "validation" of an argumentative discourse according to the three forms of validity claims. The two analytic lenses from Habermas' theory provide us with a more comprehensive perspective to shed light on teacher questioning that supports collective mathematical argumentation. Habermas' threefold perspective on epistemic, teleological and communicative rationality helps us to identify fine-grained rationality components of teachers' questions and how teachers' questioning is constrained in relation to the three components of rational behavior; the teacher's use of rational questioning to control the validation of argumentation is seen through Habermas' theory of validity claims.

Classroom-based argumentative discourse is a form of collaborative discussion, and classroom discussions are complex, messy (Frank et al., 2007), and sometimes the argumentation may not happen in the intended way. In order to facilitate productive collective mathematical argumentation, it is critical to understand what constitutes successful argumentative practice. We view rational questioning as a teaching intervention to enrich different levels of argumentation and help students to meet the requirements of rationality, thus dialectically forming productive collective argumentation. Our analytic framework for valid argumentation supports the analysis of classroom instruction related to argumentation and identifies different forms of valid argumentation. It considers students as mathematics learners to participate in argumentation throughout the grades and emphasizes the validity of argumentation as context-dependent. For future research, we should continue to find effective ways to support students' participation in appropriate local acceptance criteria for argumentation and study the role of teachers in regulating valid collective argumentation.

More importantly, in this study, we investigated the effectiveness of classroom-based rational questioning as a didactical tool to support validating argumentation, which responds to the call from the field to use theoretical ideas to design practical tools for teachers to use in the classroom. Our results indicated that although epistemic rational questioning may not always elicit correct or complete reasoning, it served as a way for teachers to set social norms that students were expected to provide reasons to justify their claims when engaging in argumentative activities. Through leading students to work towards a specific method or foreground a particular piece of mathematics for consideration, teleological rational questioning worked well in constructing correct claims. Further, by calling on a particular student to share a different solution, the class worked on what counted as a mathematically different solution, which facilitated the establishment of sociomathematical norms. Communicative rational questioning contributed to the development of students' communicative competencies by asking students to make sure their representations were correct and to use appropriate mathematical terminology to communicate ideas. Questioning focused on communicative rationality also cultivated norms that students were expected to ensure their use of mathematical language and representation can be understood in the given mathematical classroom community. Teacher questioning is one of the most frequently used ways of orchestrating students' reasoning and a key factor in promoting argumentation (Kosko et al., 2014). The fine-grained analysis of teacher questioning in regard to Habermas' three components of rationality served as a method to help us understand how collective argumentation could be initiated and sustained, which thereby contributes to the construction of the culture of rationality in argumentative discourse. This study only focused on general types of rational questioning; it will be interesting to examine what combinations of components of rational questioning appeared to be more supportive and which are less supportive of the validity of argumentation.

In summary, the findings of this study have implications for both theory and professional development in mathematics education. This study illustrates how an important theoretical construct from outside mathematics education can be interpreted and flexibly adapted to offer a new and promising perspective into the study of discursive practices that are related to mathematical argumentation. The frameworks provide a new perspective to understand the roles of teacher questioning in supporting mathematical argumentation. As for professional development, the types of questions provide information about how a beginning mathematics teacher used questions to support mathematical argumentation. In addition, the work of this study contributes to illustrate the link between theoretical and classroom-based research and can be applied in teacher professional programs as a means to develop teachers' awareness about using rational questioning to support the rationality and validity of argumentative discourse.

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THEORY AND RESEARCH METHODS:

BRIEF RESEARCH REPORTS

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PSEUDO-EMPIRICAL, INTERNALIZED, AND INTERIORIZED COVARIATIONAL REASONING

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In this brief theoretical report, I describe the process of the construction of units coordinating structures as the result of a non-linear progression from pseudo-empirical to internalized to interiorized mental activity, and I propose the utility of a parallel distinction between pseudo-empirical, internalized, and interiorized levels of covariational reasoning.

Keywords: Cognition, Number Concepts and Operations, Rational Numbers

Research with middle grades students suggests that the attributes of a quantity conceptualized by a student can lead to assimilation to schemes involving different units coordinating structures (Boyce & Norton, 2017). Similarly, I expect that when analyzing students' covariational reasoning, a critical aspect of their reasoning is *how* they assimilate attributes of objects as measurable and *how* they assimilate quantities as co-varying. In this paper, I propose utility of adopting distinctions between students' schemes for coordinating units (pseudo-empirical, internalized, and interiorized) to levels of covariational reasoning (Thompson & Carlson, 2017). I begin by providing background on scheme theory (von Glasersfeld, 1995).

Scheme Theory

A scheme consists of three parts: recognition of a situation, operations (mental actions), and an expected result (von Glasersfeld, 1995). Following von Glasersfeld (1995), I distinguish three types of schemes based on their activity: *pseudo-empirical, internalized, and interiorized* schemes. The "empirical" part of a pseudo-empirical scheme refers to an individual's need for an external object of attention to act upon; the activity portion of the scheme requires sensory-motor experience of an external transformation. What makes it "pseudo"-empirical rather than empirical is that the object acted upon is figurative material; the result of the scheme is not about the object itself. With an *internalized* scheme, perception of an act of transformation is still required, but the transformation can involve completely imagined representations (i.e., mental imagery). Representations of the results of internalized schemes can still involve external representations, but actions with these external representations involve communicating internalized reasoning rather than being a necessary aspect of one's reasoning. Both internalized and pseudo-empirical schemes involve mental activities that are experienced temporally; as part of a flow of experience of perceiving an object, acting upon it (mentally), perceiving the resulting object, and conceiving the results of the action. In contrast, an *interiorized* scheme does not require either internal or external representations for mental activity. Interiorized schemes are *anticipatory*, in the sense that the recognition of a situation, the mental actions, and the expected result of the actions of an interiorized scheme are experienced as synchronous, reversible, and necessary.

Although the process of interiorization is prefaced by stages of pseudo-empirical and internalized activity, constructing more advanced schemes is dependent upon individuals' lived experiences rather than following a strictly linear process, via psychological processes of *perturbation*, *abduction*, *assimilation*, *accommodation*, and *reflective abstraction*. Perturbation is the experience of a lack of stability or reliability of one's current schemes; often accompanied with emotive experiences of uncertainty or confusion (Piaget, 1970). Perturbation can be momentary and is most often closely tied to social interactions (communication with others about their mathematical reasoning can be viewed as a process of introducing and resolving perturbations involving interpreting others' semiotics; Steffe & Thompson, 2000). Perturbation can also be prolonged and invoke a powerful *intellectual need* for resolution (Harel, 2013) and involve internalized communication (Sfard 2007).

Abduction is a logical process of forming a hypothesis that, if true, would be experienced as satisfying an observation (Norton, 2008; Prawat, 1999). The process of assimilation is the result of a successful abduction of a modification of recognition of a situation or a modification of the recognition of a result that resolves a perturbation (most commonly an expansion of the recognition template, so that a scheme applies more broadly, von Glasersfeld, 1995). Typically accommodations involve a curtailing that is the reverse of the most common form of assimilation. Processes of assimilation and accommodation are thus intertwined, as assimilations lead to accommodations that lead to assimilations (von Glasersfeld, 1995).

Schemes can be thought of as recursive in the sense that the output of a scheme can become part of the activity of another scheme. I use the term *meta-scheme* to refer to processes that act on schemes (cf., Piaget, 1970). I thus consider the processes of assimilation and accommodation as meta-schemes. Schemes for internalization and interiorization of schemes are also meta-schemes whose input is a scheme itself. For the vast majority of situations, meta-schemes are enacted without meta-cognitive awareness, but learners also develop a meta-scheme of *reflective abstraction*. Reflective abstraction begins with reasoning about prior experiences, via *re-presentations* (mental recordings of prior experiences). Via processes of abduction, perturbation, assimilation and accommodation, these re-presentations can become successively more abstract. Reflective abstraction is thus an accommodation of an individual's meta-schemes to include more awareness, control, and flexibility. Due to limitations of working memory, reasoning about successively more abstract re-presentations of mental objects both requires and necessitates interiorizations of systems of mental actions on those objects as *conceptual structures*, which are systems of interiorized operations (Piaget, 1970; Norton & Bell, 2017).

Units Coordinating Structures

A units coordinating structure defines and regulates relationships between transformed *units* as possible, logically necessary, and reversible (Boyce & Norton, 2017). Here a *unit* refers to a size, and transformations include operations of partitioning and iterating as well as composing (putting one unit inside another unit) and disembedding (removing a copy of a unit from within a composite unit without modifying the composite unit). Such operations are constructed by students as part of their process of constructing sequences of counting numbers and reorganized to apply to fractions (Norton & Wilkins, 2012) and integers (Ulrich, 2015).

Individuals' schemes for rational number are thus characterized in part by their levels of units (Steffe & Olive, 2010), where the number of levels of the structure refer to the nestedness of reversible coordinations. The iterative fraction scheme requires assimilation with a units coordinating structure relating three levels of units (e.g., four 1/4 units within one and nine 1/4 units within 9/4). Assimilation of fractional situations with three levels of units allows a student who has constructed an interiorized iterative fraction scheme to *anticipate* iteration of an amount determined by partitioning before actually carrying out the partitioning with internalized or physical objects.

Need for Distinguishment of Covariational Reasoning Levels by Pseudo-Empirical, Internalized, or Interiorized Mental Activity

Note that with one exception, the descriptions of the covariational reasoning levels (depicted in Figure 1) refer to forming mental imagery, which I associate with internalized schemes. I propose theorized distinguishment of pseudo-empirical and interiorized covariational reasoning. I contend that to understand learners' development of covariational reasoning across levels requires understanding their pseudo-empirical, internalized, and interiorized reasoning within levels. For instance, whereas internalized continuous covariational reasoning (both smooth and chunky) might require assimilation with three levels of units, perhaps students assimilating with two levels of units can construct pseudo-empirical schemes for smooth continuous covariational reasoning. Consideration of this additional lens can help to inform the field of more specific learning trajectories and support the design of tasks engendering *perturbation*, *abduction*, *assimilation*, *accommodation*, and *reflective abstraction* that result in students' construction of more powerful covariational reasoning.

Framing mental imagery associated with levels of covariational reasoning as internalized activity of covariational reasoning schemes allows for other representations of internalized actions (such as imagery of zooming in or out on the graph of an emergent trace (Ellis, Ely, Singleton, & Tasova, 2018) that may require the same levels of units interiorized. More generally, it allows for identifying learning trajectories within and across levels of covariational reasoning that extend beyond descriptions of internalized mental activities to include focus on how students act upon standard and non-standard representations of graphs (Frank, 2018; Paoletti & Moore, 2017) and equations (Stevens, 2019) as part of analyses of covariational reasoning.

Level	**Description from (Thompson & Carlson, 2017, p. 440)**	**Proposed Distinguishment by Pseudo-empirical, Internalized, or Interiorized Reasoning**
Smooth continuous covariation	The person envisions increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously	Interiorized: The person anticipates smooth and continuous covariation between two quantities without necessarily forming mental imagery. Pseudo-empirical: The person evokes reasoning about a smooth and continuous representation without envisioning covariation between two quantities.
Chunky continuous covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.	Interiorized: The person anticipates chunky and continuous covariation between two quantities without necessarily forming mental imagery. Pseudo-empirical: The person evokes reasoning about a chunky and continuous representation without envisioning covariation between two quantities.
Coordination of values	The person coordinates the values of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x, y).	Interiorized: The person anticipates correspondence between two variables' values without necessarily forming mental imagery of their pairing. Pseudo-empirical: The person anticipates forming a new representation of discrete correspondences by which to reason about changes.

Gross coordination of values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.	Interiorized: The person anticipates binary correspondences between two variables' changes without forming mental imagery. Pseudo-empirical: The person identifies and reasons about representations of binary correspondences between two variables' changes.
Pre-coordination of values	The person envisions two variables' values varying, but asynchronously—one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.	Interiorized: The person anticipates an asynchronous sequence of binary changes in values without forming mental imagery. Pseudo-empirical: The person identifies and reasons about representations of an asynchronous sequence of binary changes in values.
No coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.	Internalized: The person forms an image of one variable varying. Pseudo-empirical: The person reasons about variation in one variable by relying on a representation.

Figure 1. Covariational Reasoning Level Descriptions

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LEARNING FROM TEACHING: A NEW MODEL OF TEACHER LEARNING

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The development of a model that explains how teachers learn from teaching is critical for informing the design of quality professional development, which in turn can support teachers' effectiveness and student learning. This article reports the authors' effort to develop a model that brings together critical findings from existing research to unpack when and under what conditions teachers learn from teaching. Grounded in evidence drawn from research relating to teachers' learning and practice, the authors build a rationale for the Learning from Teaching (LFT) model, introduce each component of the model and propose two conditions that increase the likelihood of teachers' learning from their own teaching.

Keywords: Learning Theory, Mathematical Knowledge for Teaching, Teacher Education, Teacher Knowledge

Most people would agree that teachers continue to learn and improve their teaching throughout their career. Yet, when and under which conditions teachers learn from teaching are not clearly identified. Reviews of professional development programs pinpoint different attributes of professional learning opportunities that result in changes in teacher practices and improvements in student learning (cf. Blank, las Alas, & Smith, 2008; Borko, Jacobs, & Koellner, 2010; Darling-Hammond, Hyler, & Gardner, 2017; Desimone, 2009; Garet et al., 2011, 2016; Kennedy, 2016; Piasta, Logan, Pelatti, Capps, & Petrill, 2015; Santagata, Kersting, Givven, & Stigler, 2011). For instance, Darling-Hammond and colleagues (2017) identified in their review of professional development studies that the content focus was a characteristic of effective programs, whereas Kennedy (2016) found that programs with a content focus did not seem effective.

We suggest that this cycle of conflicting findings about what makes professional development effective can be interrupted by the development of a testable model of how teachers learn from teaching. Without such a foundational model that seeks to explain the key mechanisms underlying teachers' *learning from teaching*, researchers will continue to conduct assessments of teacher learning from various perspectives that yield conflicting findings. In alignment with our argument, Kennedy (2016) noted in her recent review that *"Education research is at a stage in which we have strong theories of student learning, but we do not have well-developed ideas about teacher learning"* (p. 973).

Thus, our intentions of the present article are (a) to contribute to the literature by bringing attention to the importance of developing a model of how teachers learn from teaching and (b) to share our theoretical Learning from Teaching (LFT) model that is informed by prior research and can be tested in future research. We conceptualize teachers' *learning from teaching* as adjusting, adding to, or changing instructional practices.

The Learning from Teaching (LFT) Model

Our model considers how teachers and students co-create the teaching context that shapes teachers' learning process (see Figure 1). Central to this model is that the *temporal links (i.e., time interval) between teaching actions and evidence of student learning* influences what can be learned from teaching. For instance, the model suggests that a teacher who does not attempt to capture students' learning (through formative or summative assessments) for a week will be unlikely to learn from his or her teaching because it will be challenging to pinpoint which teaching actions contributed to students' learning. We also identify teachers' problem-solving skills as the key mechanism for their learning. We argue that without problem-solving skills, teachers cannot learn from their teaching because they will not be able to identify what teaching action is causing students to learn or struggle.

Teachers and Students Co-Create the Teaching Context

As shown in the first part of the figure, our model highlights how characteristics of individual teachers and their students will co-create the teaching context. This teaching context will shape what teachers can learn from their teaching. What we suggest here is that each individual teacher has a somewhat different teaching context and encounters different teaching moments that influence the teacher's learning environment. Therefore, understanding how teachers, their students, and other contextual factors simultaneously create a potential learning environment that could be different for individual teachers is crucial.

This dynamic and yet individualized teaching context includes instances of teaching actions and evidence of students' learning. While many scholars focus on either teaching actions (e.g., improving the cognitive demand of tasks) or students' thinking (identifying instances of students' mathematical thinking as key to productive classroom discussions; Leatham, Peterson, Stockero, & Van Zoest, 2015), both are included in the LFT model.

Temporal Links Between Teaching Actions and Evidence of Student Learning

In the next part of the LFT model, we consider how the temporal links between teaching actions and evidence of student learning play a key role in whether teacher learning occurs. If the time interval between the teaching actions and evidence of student learning is too great, it becomes a difficult task for teachers to identify which of their actions is leading to student learning. Our argument is both supported by research suggesting that formative assessment, which includes teachers' informal assessment of students' learning throughout a lesson, can lead to student learning (Black & William, 1998) and data driven research (e.g., Farrell & Marsh, 2016a; 2016b). To illustrate our point, consider a dramatized example of two teachers who have identical teaching contexts (identical students, the same levels of knowledge and skills, identical beliefs, and the same teaching materials). Teacher A is not collecting any information on his students' understanding through questions or observations and is not frequently inviting students to share their ideas to reveal their thinking. In contrast, Teacher B is frequently "collecting data" from her students through observations, student participations, or questions to see whether her students are on track. Thus, we propose that because the time distance between the teaching actions and student input is longer for Teacher A, it becomes challenging for him to pinpoint what his students do or do not learn and identify what part of his instruction could potentially have contributed to this outcome. As illustrated in Figure 1, when the time distance between the teacher's actions and student learning narrows, the number of potential links decreases, which in turn helps the teacher identify how his or her teaching interacts with the students' learning.

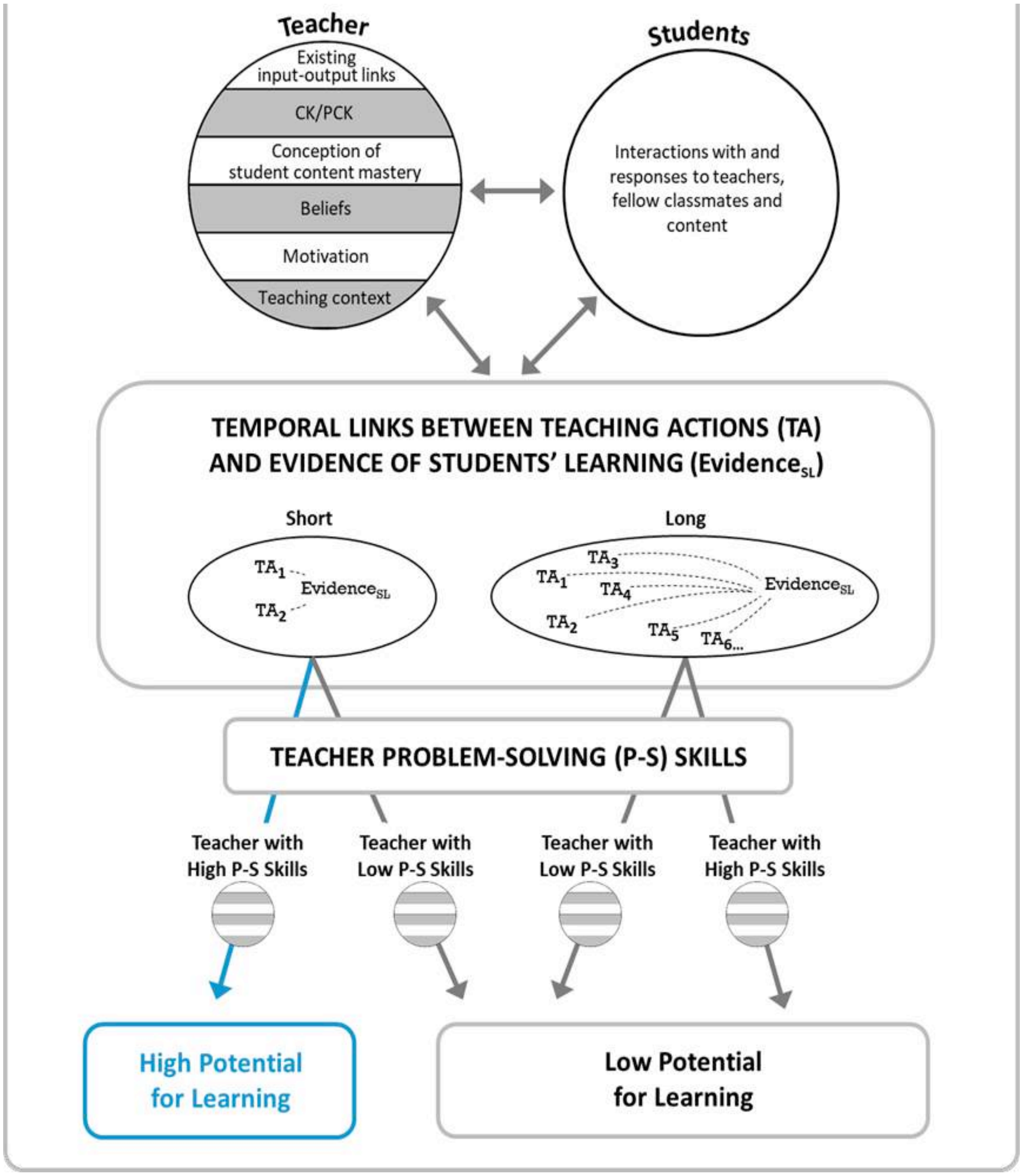

Figure 1. The Learning from Teaching (LFT) model.

Teachers' Problem-Solving Skills

We argue that problem-solving skills are key to teacher learning, and those who have developed problem-solving skills can learn on their own from teaching (Franke, Carpenter, Levi, & Fennema, 2001). As for any sorts of problems, dealing with them effectively requires developing a systematic approach to problem solving. That is why we have turned to one of the most successful strategies developed by Polya (2004) to help students develop problem-solving skills. According to Polya, problem solving involves four phases: (1) understanding the problem (why students learned or did not learn, what contributed to this outcome, what data we must have to find a solution, what other factors we need to take into consideration); (2) devise a plan (of all the potential strategies, knowing which one is more likely to lead to a correct solution); (3) execute the plan; and (4) look back (identifying whether the strategy was the right one and what can be generalized from this experience to other similar situations).

Understanding the problem is one of the first and most vital steps in solving any problem. It requires teachers to identify “the unknown, the data, the condition” (p. 28, Polya, 2004). Consider a teacher who wants to know whether his or her students have achieved the learning goal. What is *unknown* is what contributed to students’ learning or confusion. The *data* are the temporal links created during teaching or additional data, such as exit tickets, gathered on student learning. The *condition* is whether other factors in the teaching context and the available data are sufficient to determine what students learned or did not learn.

Devising a plan is the long journey that takes place after understanding the problem; it involves many unsuccessful trials. Indeed, this is why we created different learning paths, depending on teachers’ problem-solving skills. Teachers with strong problem-solving skills may think of a similar situation with similar unknowns and analyze how the current problem is related to similar problems solved before.

The third phase, *carrying out the plan*, is testing what is determined to be the reason for student learning. Executing the plan requires paying attention to the steps involved in the plan. For instance, if the plan is to use a specific manipulative (e.g., base-10 blocks) to help students understand the concept they are struggling with (e.g., the place-value system), then attending to the fact that mathematical ideas and representations (base-10 models) are clearly linked is the step required for correct execution of the plan.

The final step is *looking back*, which allows teachers to reexamine both the strategy and the result (e.g., whether modeling with base-10 blocks helped students understand what each digit means in the base-10 system). Checking whether the solution is supported by all the data collected helps teachers learn to analyze their teaching systematically to determine what works. Finally, good problem solvers generalize what is learned from a particular problem to solve similar problems by looking back at the same problem. Thus, we propose that only teachers with good problem-solving skills may change or adapt their existing conceptions because they collect data, devise a strategy, and evaluate their strategy by using evidence and reasoning.

Summary of the LFT Model

The LFT model suggests that teacher learning from teaching is situated in the teacher’s dynamic teaching environment and is jointly created by teachers and their students. Learning from teaching depends on the time distance between teaching actions and student learning evidence as well as on teachers’ problem-solving skills. In particular, two conditions increase the likelihood of teachers’ learning from teaching: (1) shortening the temporal links between teaching actions and evidence of student learning, because this limits the amount of potential actions the teacher can select to explain a certain outcome and (2) problem-solving abilities, because these allow teachers to use the information on hand systematically to find an answer to how particular teaching actions are linked to student learning. Teachers with problem-solving skills can work on the problem of teaching systematically and eventually find a correct answer to what is helping students learn or causing them to struggle.

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DESIGN AND VALIDATION OF TWO MEASURES: COMPETENCE AND SELF-EFFICACY IN MATHEMATICAL MODELING

DISEÑO Y VALIDACIÓN DE INSTRUMENTOS DE MEDICIÓN DE COMPETENCIAS Y AUTOEFICACIA DE MODELIZACIÓN MATEMÁTICA

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In this article, we share the design and validation processes of two instruments measuring aspects of the mathematical modeling process – one that measures competency and one that measures students' self-efficacy to do modeling. The study evaluates both instruments to establish their validity and reliability, using classical test theory.

Keywords: Measurement and Evaluation, Post-Secondary Education, Advanced Mathematical Thinking

Despite calls over recent decades to increase the number of graduates in STEM fields, these numbers have not grown sufficiently. Learning to apply knowledge in these majors implies integration of experiences leveraging design theories, scientific inquiry, technological literacy, and mathematical thinking (Kelley & Knowles, 2016). These goals can be realized through mathematical modeling. Modeling is of utmost importance for students pursuing STEM majors because modeling skills are of critical import to solving society's problems – whose solutions have global consequences. Today's students also take great interest in solving them (Eccles & Wang, 2016; Su, Rounds, & Armstrong, 2009). Further, research suggests that learning mathematics through modeling, as a pedagogical approach, has potential to increase student interest, proficiency in mathematics, robustness of mathematical knowledge, and self-efficacy for doing mathematics (Czocher, 2017; Czocher, Melhuish, & Kandasamy, 2019; Lesh, Hoover, Hole, Kelly, & Post, 2000; Rasmussen & Kwon, 2007; Sokolowski, 2015). Taken together, these factors are positively associated with persistence in mathematics and therefore in majors with high mathematics requirements. One aspect of incorporating more modeling in undergraduate mathematics classrooms is being able to demonstrate the efficacy of instructional interventions by measuring gains in students' modeling skills. This information would help refine programmatic innovations that focus on augmenting students' modeling experiences. Despite the need, there are presently no validated, reliable instruments to measure students' modeling skills available for undergraduates. In this article, we share two such instruments and their psychometric properties: one for modeling competencies and one for self-efficacy to carry out those competencies.

Conceptual Framework

For this project, we adopt a view of mathematical modeling as a cognitive process of rendering a non-mathematical problem about a real-world phenomenon of interest, such as those common to STEM fields, as a well-posed mathematical problem to be solved. It is a cyclic process realized as a suite of mathematical activities and cognitive processes (e.g., Kaiser, 2017). The mathematical problem can be expressed as an equation, a graph, a table, etc. The modeler solves the mathematical problem and interprets its solution in terms of the real-world context. The modeler validates and verifies each step of the process, evaluating whether the model correctly represents the situation and whether the solution makes sense (Czocher, 2018). Table 1 summarizes the conceptual framework,

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

called a mathematical modeling cycle (MMC) (Blum & Leiss, 2007; Czocher, 2016; Maaß, 2006), and also defines the competencies that constitute the modeling process.

Table 1 Modeling competencies

Competency	Description
Understanding	Forming an idea of the real World problema or identifying a real world phenomenon worth investigating
Structuring	Identifying (ir)relevant quantities and variables; making assumptions to simplify the problem
Mathematizing	Expressing relations among the variables using a mathematical representation
Working mathematically	Solving the mathematical problema, using techniques learned in mathematics classes
Interpreting	Interpreting the mathematical results with reference to the context of the real world problem
Validating	Evaluating whether the model represents the situation; verifying the analysis; establishing limitatinos

We operationalize self-efficacy *about* a task as an individual's self-assessed capacity to successfully carry it out (Bandura, 2006; Betz & Hackett, 1983; Hackett & Betz, 1989). In this study, self-efficacy is always evaluated with reference to a specified task. We operationalize the construct *self-efficacy for mathematical modeling* as an individual's self-assessed capacity to successfully carry out the interrelated competencies of the mathematical modeling process. In this way, we can, for example, consider a student's self-efficacy to identify the most important variables involved in estimating the spread of smart homes in the 21st century. The conceptual frameworks are compatible and we used them together to guide the design of the modeling self-efficacy and modeling competency scale items.

Methods

This study has a quantitative nature and is situated within the development of the two instruments, with the purpose of establishing evidence in support of their validity and reliability. The population under study was university STEM majors in the United States. The modeling self efficacy (MSE) instrument went through four rounds of design and testing. In each field test, we used a sample of STEM majors who participated in an international modeling competition called SCUDEM[1], which focuses on modeling with differential equations. In the first round of field testing, there were 6 related items for students to report their self-efficacy for the modeling competencies. In the second round, we created an additional item asking about *establishing limits* (a competency of *validating*, see Table 2, item 6) and we clarified previous items. We used pre- and post- forms of the MSE to measure change in students' self-efficacy from before to after competing. We found gains of moderate effect size $d = 0.545$ ($t(92) = -6.663, p < 0.001$). In the third round, we created a new item targeting *working mathematically* (Table 2, Item 4). Previously, this competency was excluded because it is traditionally the focus of mathematics instruction, and is complementary to modeling. In the third round, we measured statistically significant positive gains in self-efficacy for those participants who answered both the pre- and post-survey ($t = 4.202, df = 51, p < 0.001$). The final round was carried out concurrently with field testing of the Modeling Competency Questionnaire (MCQ), detailed be low. In each round, we carried out a principal component analysis (Abdi & Williams, 2010) to estimate variance and calculated Cronbach's α as a measure of internal consistency. Summary statistics are in Table 3. Our analyses, together with the instrument's construction based in theories of mathematical modeling, suggest that the MSE is unidimensional with high internal consistency, fase, content, and construct validity.

[1] The annual SCUDEM challenge is hosted by SIMIODE, https://www.simiode.org/scudem. SIMIODE is a professional organization of educators who advocate teaching differential equations from a modeling perspective.

Table 2 Final MSE instrument.

Rate your level of confidence by recording a number from 0 to 100 using the scale given below 0 10 20 30 40 50 60 70 80 90 100 Cannot do at all Moderately can do Highly certain can do.	Competencies
Create a differential equation model for the spread of smart home appliances in the United States during the twenty-first century.	Mathematize
In (1) identify the important variables leading to a reasonably accurate prediction.	Identify variables
In (1) make simplifying assumptions to reduce the number of important variables.	Make assumptions
In (1) select an appropriate numerical, graphical, or analytic technique to solve the resulting differential equation	Work mathematically
In (1) consult appropriate resources to check whether your model was reasonable.	Validate
In (1) list the real-life and mathematical limitations of your model.	List limitations
In (1) create a short presentation to convince a smart appliance manufacturer that they could rely on your model to develop their business plan.	Communicate findings
Given a differential equation which describes the rate of formation of material A, $A'(t) = \alpha A(t)^{\beta}$ and a data set of observations for time, t, amount of material A at each time t, you could estimate the parameters α and β.	Estimate parameters

Tabla 3 Summary of analysis of MSE

Round	N	Varience (ACP)	α	Round	N	Variance (ACP)	α
1	38	62.5%	0.822	3	198	61.5%	0.908
2	276	67.1%	0.917	4	226	69.0%	0.935

Design and testing for the MCQ was carried out in three rounds (feasibility, difficulty, and discrimination) with distinct samples drawn from a large, southwestern university in the United States. We imposed four restrictions on the design: (1) items should be drawn from authentic and relevant contexts (e.g., radioactive decay or analysis of a recycling program), (2) items should draw on knowledge from STEM content or everyday knowledge, (3) items should target the aspects of the modeling competencies, and (4) distractor choices should capture decisions and justifications common to students' reasoning. We created 118 multiple choice items belonging to 9 real-world situations, selected from instructional and research materials from STEM education. Mathematics content included arithmetic, algebra, calculus, and differential equations. For each item, we created one correct answer and four distractors that would appear reasonable to the students but would not help to model the situation. To establish content and construct validity, we invited two mathematicians who teach differential equations to STEM students and three mathematics education researchers who specialize in teaching and learning of mathematical modeling to evaluate the items for appropriateness, correctness, and aptness to the MMC. In the first round, 14 students answered the MCQs and gave us reasoning to justify their choices. We eliminated items that did not make sense to the student. In cases where a student selected a distractor but had sensible reasoning, we modified the item. In the second round, 78 students answered 63 items, distributed among two forms that balanced contexts and competencies. For each item, we calculated the mean difficulty. The majority (76%) of the items had moderate difficulty ($0.20<p<0.70$). We eliminated items outside this range as either too difficult or too easy, restructuring some of the too-difficult items. To analyze distractor efficiency, we calculated the proportion of students that selected each option. At least 5% of the students selected each of the 253 distractors (one item had 5 distractors). For 17 items, a distractor was selected more frequently than the correct answer. These items were flagged as potentially strong discriminators among students with varying levels of modeling competencies. After restructuring problematic items, we chose 30 items (15 items for each of 2 forms). The two forms were administered to a sample of $n = 314$ volunteers who participated in the SCUDEM competition, $n = 135$ responded to Form 1 and $n = 139$ responded to Form 2. For each item, we calculated the mean difficulty. Form 1 had mean difficulty 0.359 ($SD = 0.126$), with $0.177 < p < .0595$. Form 2 had mean difficulty 0.369 ($SD = 0.129$), with $0.147 < p < 0.580$. Four items were too difficult. We conducted another analysis of distractors and concluded that they were functioning adequately. We used point-biserial correlations (rPBIS) to conduct discrimination analysis. Only one item from Form 1 had a negative rPBIS; the remaining items had rPBIs > 0.20. We report the statistics Revelle's Omega Total (ω_T) as an estimate of internal consistency. This selection is appropriate in cases, like the MCQ, where the instrument is multidimensional and when those multiple dimensions contribute to the construct under investigation (Revelle & Zinbarg, 2009). Using the software package '`userfriendlyscience`' in R, we obtained $\omega_T = 0.59$ y $\omega_T = 0.63$, respectively, for Forms 1 and 2. The scales are approaching traditional estimates of 0.7.

Discussion

In this article, we have shared two instruments measuring mathematical modeling competencies and modeling self-efficacy. We also documented their design processes and their psychometric properties. The instruments are aligned with theories of mathematical modeling and have gone through several rounds of field testing. Future research will move into Item Response Theory as a means for constructing and calibrating parallel versions of the modeling competence questionnaire for use as pre/post or group comparison measures. In this way, the instruments can help to evaluate innovative educational interventions aimed at augmenting students' modeling skills. With such information, instructors, researchers, and academic units can improve modeling experiences for students and provide evidence of their efficacy. We are cautious but optimistic that the instruments can meet this goal as the evidence presented here suggests that both are reliable and valid for that purpose.

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DISEÑO Y VALIDACIÓN DE INSTRUMENTOS DE MEDICIÓN DE COMPETENCIAS Y AUTOEFICACIA DE MODELIZACIÓN MATEMÁTICA

DESIGN AND VALIDATION OF TWO MEASURES: COMPETENCE AND SELF-EFFICACY IN MATHEMATICAL MODELING

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En este artículo, describimos el diseño y la validación de dos instrumentos – uno que mide la autoeficacia y otro que mide las competencias del proceso de la modelización matemática. La investigación consiste en la evaluación de ambas para establecer la validez y la confiabilidad utilizando técnicas de teoría clásica de validación.

Palabras clave: Valoración y Evaluación, Educación Postsecundaria, Modelización

A pesar de que una de las metas de la educación postsecundaria ha sido el aumentar el número de los graduados universitarios en las carreras de ciencia, tecnología e ingeniería, estos números no se han incrementado suficientemente. Para adquirir conocimiento útil, es necesario que el aprendizaje se fundamente en la combinación de la práctica y teoría de diseño, indagación científica, y en el pensamiento matemático (Kelley & Knowles, 2016). A través de la modelización matemática, se pueden lograr estas metas. Las habilidades de modelización matemática son de suma importancia al cursar carreras universitarias que requieren técnicas aplicadas a las matemáticas. También son importantes para resolver problemas sociales cuyas soluciones conllevan consecuencias mundiales y tangibles. La posibilidad de resolver los problemas sociales llama la atención de los estudiantes (Eccles & Wang, 2016; Su, Rounds, & Armstrong, 2009). Además, las investigaciones empíricas sugieren que aprender matemáticas a través de la modelización es beneficiosa para obtener una autoeficacia y un conocimiento matemático más robusto (Czocher, 2017; Lesh, Hoover, Hole, Kelly, & Post, 2000; Rasmussen & Kwon, 2007; Sokolowski, 2015). La modelización matemática, guiada por las innovaciones educativas, aumenta el interés, la competencia, y la autoeficacia de los estudiantes hacia las matemáticas (Czocher, Melhuish, & Kandasamy, 2019). Conjuntamente, esos factores también están asociados positivamente con la perseverancia en los campos disciplinarios que requieren las matemáticas. Para evaluar atentamente las intervenciones educativas y mostrar su eficacia, es necesario medir el aprendizaje. Esto ayuda a refinar programas que se enfoquen en las habilidades de la modelización matemática. A pesar de su necesidad tangible no existen instrumentos válidos ni confiables para evaluar las habilidades de modelización de los estudiantes universitarios. Aquí, compartimos dos instrumentos de medición y sus propiedades psicométricas: uno de competencias de modelización y otro de autoeficacia en realizarla.

Marco de Referencia

En este trabajo de investigación, se plantea el supuesto de que la modelización matemática es un proceso iterativo y cíclico que puede ser conceptualizado como un conjunto de actividades matemáticas y procesos cognitivos (e.g., Kaiser, 2017). El proceso comienza con un problema de la vida real – como los que son comunes en los estudios de ciencia, ingeniería o en la vida cotidiana – y desemboca en un problema matemático. El problema matemático se puede expresar como una ecuación, un gráfico, o una tabla de valores. El modelador resuelve el problema matemático y desde la solución matemática él interpreta el significado de los resultados al problema original planteado. El modelador valida y verifica cada etapa del proceso para evaluar si el modelo representa correctamente la situación real y si la solución tiene sentido (Czocher, 2018). La Tabla 1 presenta el marco de referencia que se denomina "ciclo de modelización matemática" (CMM) (Blum & Leiss, 2007; Czocher, 2016; Maaß, 2006) y define las competencias que constituyen el proceso de modelización matemática. Definimos la autoeficacia de realizar una tarea como la confianza de una persona en sí misma y en su capacidad para lograr resolver la tarea exitosamente (Bandura, 2006; Betz & Hackett, 1983; Hackett & Betz, 1989). En esta investigación, la autoeficacia siempre es evaluada con referencia al objetivo de la tarea. Definimos el constructo *autoeficacia de modelización matemática* como la confianza de una persona en sí misma y en su capacidad de realizar las actividades interrelacionadas que constituyen el proceso de modelización. De esta manera podríamos medir la autoeficacia de un estudiante para identificar las variables más importantes involucrados en estimar la propagación de hogares inteligentes en el siglo 21. El CMM y la autoeficacia son compatibles, y los utilizamos en conjunto para guiar el diseño de los ítems.

Tabla 1 Competencias de modelización.

Competencia	Descripción
Comprender	Formación de una idea de lo que debe ser el problema o identificación de un fenómeno de la vida real que merece investigación
Establecer estructura	Identificar los factores y cantidades reales relevantes y la información que se puede ignorar; imponer restricciones o supuestos para simplificar el problema
Matematizar	Expresar las relaciones entre las cantidades en una representación matemática
Analizar	Resolver el problema matemático, usando técnicas aprendidas en la clase de matemáticas
Interpretar	Observar y entender los resultados matemáticos desde el contexto del problema real
Validar	Examinar si el modelo representa la situación; verificar el análisis; establecer limitaciones

Metodología

La investigación es de naturaleza cuantitativa y se enmarca dentro de un estudio de desarrollo para establecer evidencia en apoyo de la validez y la confiabilidad de los instrumentos. La población bajo estudio consistió en estudiantes universitarios que estudian carreras en ciencias, tecnología, ingeniería, y matemáticas. A continuación, se documenta el diseño de los ítems. La evaluación del instrumento de autoeficacia se realizó en cuatro rondas de pruebas. En cada prueba empírica, usamos una muestra de estudiantes universitarios inscritos en un concurso internacional de modelización basado en lo que se llama SCUDEM (por sus siglas en inglés). El concurso se lleva a cabo cada año y es parte de una organización de capacitación que apoya a los profesores de matemáticas a quienes les gustaría enseñar los conceptos de ecuaciones diferenciales desde una perspectiva de aplicaciones y modelización matemática. En la primera ronda, eran 6 ítems relacionados con la autoeficacia de modelización. En la segunda, creamos un ítem (Tabla 2, ítem 6) y modificamos los ítems anteriores para mejorar su claridad. En la segunda ronda, también medimos el cambio de autoeficacia antes y después de participar en el concurso y constatamos una ganancia de efecto moderato, $d = 0.545$ $(t(92) = -6.663, p < 0.001)$. En la tercera, creamos un item nuevo de análisis matemático (Tabla 2, Ítem 4). Previamente fue excluido porque el enfoque eran las actividades complementarias de modelización. En la tercera ronda el instrumento midió el cambio positivo de autoeficacia ($t = 4.202$, $df = 51, p < 0.001$) de los participantes que contestaron las preguntas antes y después de participar en el concurso. En cada ronda de validación, realizamos un análisis de los componentes principales (Abdi & Williams, 2010), calculamos el Chronbach's α para estimar la consistencia interna, y medimos el cambio de autoeficacia antes y después de participar en el concurso. La Tabla 3 resume los resultados. Este análisis, en conjunto con su construcción basado en la teoría de modelización matemática, indica que el instrumento de autoeficacia es unidimensional con coherencia interna alta y tiene validez de diseño y de constructo.

Tabla 2 El instrumento final de autoeficacia.

Indica tu nivel de confiabilidad en cada uno de los escenarios siguientes, elegiendo un numero de 0 a 100 usando la siguiente escala: **0 10 20 30 40 50 60 70 80 90 100** **No puedo hacer. Tengo dudas… Con certeza elevada.**	**Competencias**
Crear un modelo de ecuaciones diferenciales para estimar la propagación de hogares inteligentes en el siglo 21.	Matematizar
En (1), identificar las cantidades importantes que aseguran una predicción razonablemente precisa.	Establecer estructura
En (1), establecer los supuestos que reducen la cantidad de factores importantes.	Establecer estructura.
En (1), elegir un método apropiado de tipo numérico, gráfico ó analítico para resolver la ecuación diferencial que resulta de (1).	Analizar
En (1) consultar a los recursos apropiados para verificar si el modelo matemático es razonable.	Validar
En (1) enumerar las limitaciones del modelo matemático, incluyendo restricciones de la vida real y restricciones matemáticas.	Validar
En (1), crear una presentación breve para persuadir un fabricante de aparatos inteligentes que podrían depender en tu modelo matemático para fomentar un plan de negocios.	Comunicar
Proporcionando una ecuación diferencial que modela la tasa de formación del material A, $A'(t) = \alpha A(t)^{\beta}$ y los datos de observaciones en tiempo t, la cantidad de material A por cada punto de tiempo t, podría estimar los parámetros α y β.	Establecer estructura

Tabla 3 El resumen del análisis del instrumento de autoeficacia de modelización

Ronda	N	Varianza (ACP)	α	Ronda	N	Varianza (ACP)	α
1	38	62.5%	0.822	3	198	61.5%	0.908
2	276	67.1%	0.917	4	226	69.0%	0.935

La evaluación del instrumento de competencias de modelización se realizó en tres rondas de pruebas con muestras distintas de una universidad de más de 40,000 estudiantes en los EEUU: viabilidad, dificultad, y discriminación. Para diseñar el instrumento tomamos en cuenta cuatro restricciones: (1) los ítems parten de contextos auténticos y relevantes (por ejemplo, la desintegración radioactiva o un programa de reciclaje). (2) Los ítems evocan conocimientos de matemática, ciencia, ingeniería, y sentido común. (3) Los ítems abordan aspectos de las competencias. Por ejemplo, un ítem aborda la competencia de establecer la estructura que se requiere al utilizar la habilidad de identificar cantidades importantes. (4) Los distractores son basados en las decisiones y justificaciones comunes al pensamiento de estudiantes actuales. Elaboramos 118 ítems de tipo selección múltiple (ISM) que pertenecen a 9 situaciones de la vida real elegido de materiales de cursos de matemáticas, física, biología, química e ingeniería. El contenido matemático incluye aritmética, álgebra, cálculo diferencial e integral y ecuaciones diferenciales. Por cada ítem, elaboramos una respuesta correcta y cuatro distractores que parecieran razonables a los estudiantes pero que no ayudaran a modelizar la situación. Para establecer la validez de contenido y la validez de los constructos, invitamos a revisar los ítems a dos investigadores matemáticos que se enfocan en la investigación de ecuaciones diferenciales y tres profesores universitarios de matemáticas que se especializan en realizar investigaciones sobre el aprender y enseñar la modelización. Aplicamos los cambios que sugirieron los expertos y eliminamos los ítems que resultaron no válidos. En la primera ronda, 14 estudiantes nos dieron su razonamiento para justificar sus elecciones. En el caso de que un estudiante eligiera un distractor y su razonamiento tuviera sentido, el ISM fue ajustado. Eliminamos los que no tenían sentido para los estudiantes. En la segunda ronda, 78 estudiantes contestaron 63 ISM en 2 versiones, equilibrando ítems de acuerdo a las distintas competencias de modelización. Por cada ISM, calculamos la dificultad media. La mayoría (76%) de los ISM tenían dificultad moderada (0.20<p<0.70). Eliminamos los ítems que eran demasiado fáciles (p>0.7) y restructuramos los ítems que fueron demasiado difíciles (p<0.20). Para analizar la eficacia de los distractores, calculamos la proporción de los estudiantes que eligieron cada opción. De los 253 distractores (62 ítems contaban con 4 distractores y 1 contaba con 5), El 5% de los participantes eligieron la mayoría de estos distractores. En 17 de los ítems, los distractores fueron elegidos más frecuentemente que las respuestas correctas. Estos fueron identificados de acuerdo con su potencial de discriminar entre estudiantes de distintas habilidades o como ítems que necesitaban ser reestructurados. Después de reestructurar los ISM según el análisis de distractores, elegimos 30 ítems (2 versiones de 15 ítems). Las dos versiones se administraron a una muestra de $n = 314$ voluntarios que participaron en el concurso SCUDEM, incluyendo $n = 135$ que contestaron a la versión 1 y $n = 139$ que contestaron a la versión 2. Por cada ISM, calculamos la dificultad media. La versión 1 obtuvo dificultad media de 0.359 ($SD = 0.126$), con $0.177 < p < .0595$. La versión 2 obtuvo dificultad media de 0.369 ($SD = 0.129$), con $0.147 < p < 0.580$. Cuatro ítems eran demasiado difíciles. Se realizó un análisis de detractores y concluimos que los distractores funcionaban adecuadamente. Para realizar el análisis de discriminación, usamos la correlación point-biserial (rPBIS por sus siglas en ingles). Un solo ítem de la versión 1 tenía rPBIS negativo. El resto tenían rPBIS > 0.20. Reportamos la estadística Revelle's Omega Total (ω_T) para estimar la consistencia interna. La selección fué apropiada en casos donde el instrumento era multidimensional y cuando múltiples dimensiones contribuían a predecir el constructo bajo investigación (Revelle & Zinbarg, 2009). Usando el paquete de software 'userfriendlyscience' del programa R, obtenemos $\omega_T = 0.59$ y $\omega_T = 0.63$ para la versión 1 y la versión 2, respectivamente. Las escalas se acercan al estimado tradicional 0.7.

Discusión

En este artículo, presentamos dos instrumentos de medición, uno de autoeficacia de modelización y uno de competencia de modelización. Así mismo, documentamos los procesos de construcción y diseño y las propiedades de ambos. Los instrumentos se alinean con las teorías de modelización y han pasado múltiples rondas de pruebas. Se planea emplear la Teoría de Repuesta al Ítems para componer versiones que sean paralelas para medir los cambios positivos de las competencias de modelización de los estudiantes con el propósito de evaluar programas educativos que se enfoquen en ensenar la modelización. Con esta información investigadores y docentes pueden mejorar las experiencias de modelización o proporcionar evidencia de su éxito. Estamos cautas pero optimistas que los instrumentos alcancen este objetivo ya que la evidencia expuesta aquí sugiere que los instrumentos son confiables y válidos para su propósito.

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GESTURE IN PROOF AND LOGICAL REASONING

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Twelve doctoral students in mathematics took part in clinical interviews during which they were asked about their experiences with teaching, learning and doing proof. They were also asked to work together to find a proof for an unfamiliar conjecture. The students' discourse, including gesture, was analysed from the perspective of embodied cognition. In particular, a potential continuity between mathematical and everyday discourse was investigated, with a particular focus on epistemic conditionals, that is, "if-then" statements.

Keywords: Embodiment and Gesture; Reasoning and Proof; Advanced Mathematical Thinking; Cognition

Objectives of Study

In recent decades, researchers have investigated how the body in implicated in mathematical teaching and learning, challenging the paradigm that cognition is amodal and abstract, based solely "in the head." In addition, attention to embodiment has broadened the focus within mathematics education research beyond written symbols, images, and oral speech to include modalities such as gesture and other bodily movements (Edwards, Ferrara, & Moore-Russo, 2014; Hall & Nemirovsky, 2012). The purpose of this paper is to examine mathematical proof and logical reasoning from the perspective of embodied cognition (Edwards, 2011; Varela, Thompson, & Rosch, 1991), using data collected from clinical interviews with 12 doctoral students in mathematics.

The analysis presented here is based on the principle of cognitive continuity; that is, the proposition that there are not multiple different kinds of thinking, even within a domain like mathematics, but rather all thought is ultimately founded in embodied, physical experience (Johnson, 2012; Lakoff, & Núñez, 2000; Varela, Thompson, & Rosch, 1991). The implication is that even with "advanced" mathematical thinking, like that involving proof and logic, connections can be made with more everyday kinds of thinking and basic human experiences. As Johnson states, "we do not have two kinds of logic, one for spatial-bodily concepts and a wholly different one for abstract concepts. There is no disembodied logic at all. Instead, we recruit body-based, image-schematic logic to perform abstract reasoning" (Johnson, 2012, p. 181). The research reported her aims to delineate one way in which elements within the abstract domain of mathematical proof are connected to similar ones in everyday discourse.

Following Hanna (1990), we take proof to be:

> [A] finite sequence of sentences such that the first sentence is an axiom, each of the following sentences is either an axiom or has been derived from preceding sentences by applying rules of inference, and the last sentence is the one to be proved. (Hanna, 1990, p. 6)

Although this definition is appropriate for the end product of a process of proving, we also frame proof and proving as a specialized type of discourse, built on simple logical elements and constrained by agreements on validity generated within the mathematical community. In the current research, the specific focus is on logical statements that take the form of "if-then" statements; these statements can be seen as the building blocks of proofs. The central research question is whether the physical gestures that accompany these "if-then" statements when talking about proof are similar to those accompanying "if-then" statements in non-mathematical contexts. If so, then this would provide support for the notion of cognitive continuity between these two contexts.

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Theoretical Perspective

The research was carried out utilizing the theoretical perspective of embodied cognition, making use of tools from cognitive linguistics and gesture studies. The theory of embodied cognition focuses on the bodily basis of thinking, that is, "on the ways in which complex adaptive behavior emerges from physical experience in biologically-constrained systems" (Núñez, Edwards, & Matos, 1999, p. 49; see also Varela, Thompson, & Rosch, 1991). Here, we focus not on specific mathematical content, for example, algebra or analysis, but on the mechanisms used by mathematicians to test and establish logical truth. Under Johnson's continuity principle, we propose that deductive proof and logic are constructed using the same basic conceptual building blocks as more mundane thought (Johnson, 2007).

Within an embodied cognition framework, mathematics is not seen as a transcendental, formal collection of rules and patterns, unrelated to everyday thinking and experience, but instead, as a human intellectual product, one which develops both historically as a discipline over time, and ontologically as it is constructed by an individual learner. It is socially-constructed, but not in an arbitrary way, being both constrained and enabled by the biological capabilities and physical situatedness of human beings. Embodiment does not deny the influence of social interaction and culture; rather it grounds it in shared biological constants (Hall & Nemirovsky, 2012; Nuñéz, Edwards, & Matos, 1999). As stated by Hall and Nemirovsky (2012), "We think of concepts (in mathematics but also in other domains) as forms of modal engagement in which bodies incorporate and express culture" (p. 212).

Prior Research on Proof

Prior research has been fruitful in its examination of the learning and teaching of proof, whether addressing the understandings and misunderstandings of novices, productive instructional practices and tools, or the thinking of advanced mathematicians (a selection of recent work can be found in Lin, Hsieh, Hanna & deVilliers, 2009). The current research builds on this foundation, particularly in seeing proof as a form of socially constructed knowledge and a specific form of discourse (Balacheff, 1991; Sfard, 2001). The current analysis adds the lens of embodiment and gesture studies in analyzing this discourse.

Prior Research on Conditional Statements

From the point of view of cognitive linguistics, mathematical or logical deductions ("if-then" statements) belong to a linguistic category known as conditionals (Dancygeir & Sweetser, 2005). Specifically, "if-then" statements represent the type called epistemic conditionals, because they reference a reasoning process, rather than a prediction or statement of fact. Two examples of epistemic conditionals are: "If the car is in the driveway, he must be home" and "If x is even, then x/2 is an integer" (p. 17). These kinds of conditionals involve what Danceygeir and Sweetser call a "metaphoric 'compulsion'" (p. 20) in which the speaker is "forced" to draw the given conclusion, either based on inductive reasoning ("the car is almost always in the driveway when he is home") or deductive logic (the mathematical definition of "even"). An analysis of how this metaphoric "compulsion" is grounded in early embodied experiences, providing a physical basis for the later construction of the notion of proof, can be found in Edwards (2017, 2019).

In addition to the linguistic analysis of explicit conditional statements by Danceygeir and Sweetser, recent research by Sweetser has examined gestures associated with spoken conditionals. In a study involving 402 video clips of talk shows, Sweetser and Smith (2015) found that conditionals were generally accompanied by a particular hand motion, specifically a movement along a transverse axis through gesture space, starting on the speaker's left and moving toward the speaker's right. The current analysis examined the gestures of mathematical doctoral students to see whether they also reflected this characteristic motion when orally stating epistemic conditionals. If so, then this would constitute evidence of the continuity between everyday uses of conditionals and their use in mathematical proof.

Methods

The research took the form of a qualitative study similar in format to a task-based clinical interview, recorded on audio and videotape. The participants, pairs of doctoral students in mathematics, were first interviewed about their specializations in mathematics, their experiences with teaching proof, and their ideas about whether there are different kinds of proofs. They were then presented with the conjecture below on a sheet of paper, and asked to work together to find a proof for it.

> Let f be a strictly increasing function from [0, 1] to [0, 1]. Prove that there exists a number a in the interval [0, 1] such that $f(a)=a$.

They were given 40 minutes to try to find a proof, during which the researcher left the room so that the participants could work without feeling self-conscious about being observed. During the third part of the interview, the students were asked to evaluate a visual "proof." The results presented here were drawn from the first part of the interview.

Participants

The participants were 12 doctoral students in mathematics, 9 men and 3 women, attending a research university in the United States. They were placed in pairs for the interviews based on their availability and schedules. They all knew each other as fellow students in the doctoral program, and two of the women, who worked together as a pair, were good friends. The time they had spent in the doctoral program ranged from less than a year to almost four years, and all had had experience in teaching undergraduate mathematics courses, although this experience did not involve much teaching of proof.

Context

The interviews took place in a small unused office with a blackboard at one end. The participants sat on chairs in front of the blackboard, facing the interviewer and the video camera. They were asked to use only the blackboard while working on the proof.

Data Collection

All sessions, lasting from 60 to 90 minutes each, were recorded on videotape and via digital audio, with the camera oriented to capture both the blackboard and the students as they sat or stood in front of it. A total of 6 hours and 55 minutes of video and audio were collected.

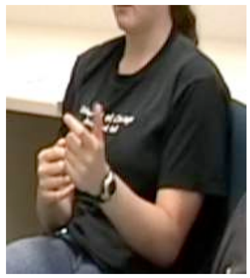
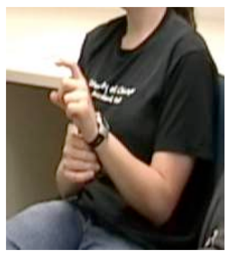
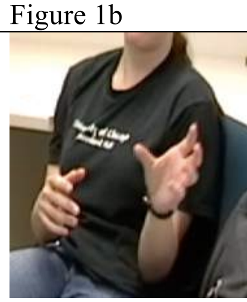
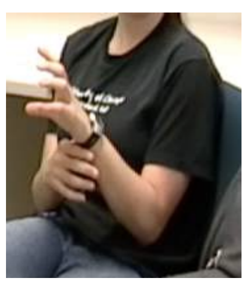
Analysis

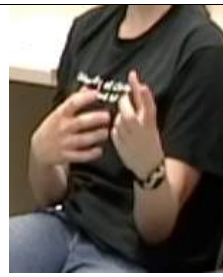
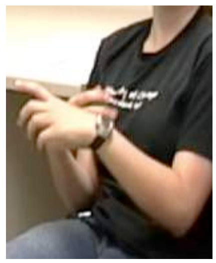
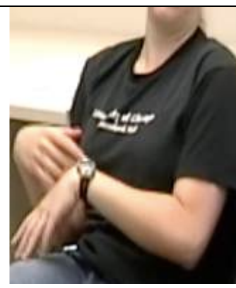
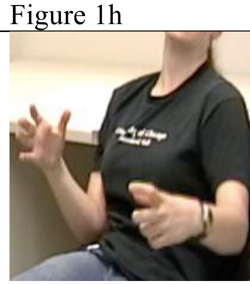
The audiotapes were transcribed and annotated with brief notations of gestures as well as time spans of the use of different modalities by the participants. Specific segments of discourse containing gestures of interest were analyzed in more detail, utilizing the concurrent speech, written symbols, and drawn graphs to develop plausible interpretations consistent the context and with other research into gesture (Alibali, Boncoddo, & Hostetter, 2014; McNeill, 1992; Perrill & Sweetser, 2004).

Results

The analysis of the doctoral students' gestures when making conditional statements did indeed reveal the presence of the same left-to-right transverse gesture previously identified in non-mathematical contexts. Although the use of epistemic conditionals in speech was found throughout the video data, most instances occurred while the students were actively working on finding a proof; thus, their hands were often occupied with chalk or they were pointing to inscriptions on the board, meaning that "if-then" statements were often not accompanied by gestures. However, the transverse gesture did occur regularly in the data, approximately once in every ten instances in which an epistemic conditional was uttered. This occurred primarily when the students were talking to the interviewer, explaining a proof.

The example shown in Figure 1 illustrates three instances of this gesture form. In this example, the epistemic conditional that the student is expressing can be summarized as follows: "If you have a scalar function and a vector function, then the rule for finding their product is the same as the rule for finding the product of two scalar functions."

AC: Well, I guess, so, the other day they were trying to prove that, um, if you have some scalar function of T Int: Uh huh AC: —and some vector function of T,	 Figure 1a  Figure 1b	Left hand starts in horizontal C-shape ("bracket") facing upward on left side of body Left to right motion with left hand along transverse axis, ending in middle of body, with C-shape turning vertical
Int: Uh huh AC: —that the derivative of their product...	 Figure 1c  Figure 1d	Left to right motion with left hand along transverse axis, with left hand open and facing outwards. Left hand begins on left side of body and ends in middle of body.
is the same...		Rapid left to right motion with left hand along transverse axis. Left hand starts in loose horizontal C-shape ("bracket") facing upward on left side of body and ends in pointing gesture to the right.

	 Figure 1e  Figure 1f	
AC: ...product rule essentially that you know from just, you Int: (talking over): Uh huh.	 Figure 1g  Figure 1h	A complex motion in which the left hand begins by pointing downward, then is moved in a circle twice around the right hand while saying “you know,” ending up open and facing the speaker
AC: know from like scalar functions	 Figure 1i	Left hand moves to right and finishes in horizontal C-shape (“bracket”) on left side of body. This is the same shape and location as when the phrase “scalar function” was initially uttered.

Note: Underlined speech indicated the stroke or emphasized portion of the gesture

Figure 1: Student’s discourse about scalar functions

The sequence of gestures accompanying the student's speech is very rich, taking into account characteristics including hand shape and orientation, hand location, and movement of the hands through space. Consistent with other conditionals used in non-mathematical contexts, the sequence includes left-to-right motion along the transverse axis; in fact, this transverse motion occurs three different times, as shown in the pairs of figures above:

- Figure 1a – b: A relatively small left-to-right motion of the left hand, as AC begins by saying, "If you have some scalar function of T and some vector function of T." This sequence also includes a change in orientation of the left hand; when holding it on the left, AC uses an upward-opening (horizontal) C-shape as if "bracketing" or "holding" a scalar function. As she moves her hand to the right, she rotates her wrist so that when she says, "vector function," the C-shape is now vertical. She thus uses both hand shape and hand location to gesturally distinguish the two different kinds of functions.
- Figure 1c – d: A wider left-to-right motion of the left hand, as AC says, "the derivative of their product." In this case, the hand shape stays the same throughout, open and facing outward.
- Figure 1e – f: After saying "derivative of their product," AC pauses briefly, then makes a very rapid left-to-right motion of her left hand while saying, "is the same," starting with a horizontal C-shape and ending with a right-facing point.

As can be seen above, in addition to an overall left-to-right movement that occurs three times during the sequence, gestures are also used to mark or indicate specific mathematical objects, in a scheme that Calbris (2008) calls "two-entity opposition." Two-entity opposition occurs when either two locations in space or the two hands are used to denote or "mark" two related but distinct entities. In Figure 1, this happens when AC uses a horizontal "bracket" held to her left when saying "scalar functions" and then a vertical bracket held to her right when saying "vector functions." The terms "derivative" and "product" have the same hand shape but are marked by left and right hand locations, indicating two-entity opposition.

The discourse segment ends with AC discussing a "product rule" while using an iterative circular gesture during a pause in speech. This pause and rhythmic circular gesture may indicate that the participant is searching for her next words (Lucero, Zaharchuk, & Casasanto, 2014). She compares this product rule to a presumably familiar rule for scalar functions. Interestingly, the final gesture of the sequence, associated with the words "scalar function" has an identical shape and location as the gesture used the first time the words were uttered. This is an example of using specific hand shapes and locations in gesture space to "hold" a referent in discourse (Calbris, 2008; McNeill, 1992).

Discussion

Calbris (2008) has stated that in gesture space, the transverse axis can represent logico-temporal concepts, such as cause and effect, or before and after:

> A path in space or time is depicted by a left-to-right movement. But give that body symmetry allows this axis to account for splitting in two as well as two-entity oppositions, it can be used to oppose past and future, or precedence and successor, by locating the past on the left side and the future on the right side. (Calbris, 2008, p. 43)

In the current case, and in the research by Sweetser and Smith (2015), the transverse axis is used to indicate the premise followed by the conclusion of a conditional "if-then" statement.

The transverse axis of the body has been also called "the axis of reading and writing, pointing to the right in the Western world" (Calbris, 2008, p. 28). In this case, the motion of AC's gestures is consistent both with the placement of the "cause" (premise) on the left and the "effect" (conclusion) on the right, as well as the left-to-right order in which premise and conclusion are generally written in English. In the example given above, the left-to-right motion along the transverse axis is thus consistent both with how "if-then" statements are written in English, and with prior research and theory identifying this gestural motion with logical and conditional statements.

Taken in conjunction with related research (Edwards, 2010, 2011, 2017), we would argue that the examples above provide further evidence that proof and its building blocks, statements of logical deduction, are not abstract elements of disembodied rationality. Instead, we argue, these sophisticated forms of discourse make use of metaphorical mappings related to motion, and are supported by conceptual metaphors grounded in physical experiences.

Mathematical proof is thus seen as a specialized cultural product and a specific form of discourse, with particular constraints that distinguish it from everyday speech and make it more powerful for the purpose of exploring structure and patterns. Yet the form that this discourse takes is not arbitrary, but rather is grounded in embodied human experience. As shown above, there exists a continuity between the gestural grounding for the logical conditionals used in proof and those used in non-mathematical contexts. This kind of analysis is relevant to mathematics education because the conceptual sources that students draw from in constructing new mathematical knowledge may not correspond to the more sophisticated intra-mathematical sources that their instructors use (c.f., Núñez, Edwards & Matos, 1999). For example, students who are beginning to learn about formal logic often "import" expectations about conditionals from everyday speech, assuming that "if A, then B" implies "if not A, then not B" (Evans, Newstead, & Byrne, 1993). A better understanding of the cognitive roots of mathematical thinking may help in designing corrective instruction in such situations.

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MAKING MEANING OF LEARNING TRAJECTORIES AMIDST MULTIPLE METAPHORS

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In this theoretical report we focus on the issue of communicating learning trajectories (LTs) to researchers. There is great variation in the body of work on LTs including how researchers communicate what a LT entails, and the kinds of metaphors employed for making meaning of LTs. We elaborate possible affordances and limitations of different metaphors for LTs including "a garden path" and "growing flowers." This work has implications for how LTs are taken up by researchers, and also how LTs are leveraged to inform student-centered teaching practices.

Keywords: Learning Trajectories, Metaphors, Representations and Visualization

Learning trajectories are theoretical tools that elaborate transitions in students' processes of learning and goal-directed instructional supports. As a construct, learning trajectories can guide the design and study of teaching and learning by establishing predictions of how teaching and learning will unfold in tandem. Imagining, testing, and analyzing learning trajectories is core to the work of some design-based researchers, as are inquiries into how students learn mathematical ideas in response to instructional supports. Despite the integral nature of learning trajectories to some design-based research, the work of communicating such complex tools is challenging. Indeed, the domain of research on learning trajectories has grown and diversified in the approaches and theoretical orientations researchers take (e.g., see reviews by Empson, 2011; Lobato & Walters, 2017; and Fonger, Stephens et al. 2018).

This variation in theoretical orientation parallels what Simon (2009) articulated as a growth in the variety of theories of learning (and we might add, local instructional theories) being employed and developed in the field of mathematics education writ large. Amidst this growth, Simon recounts several challenges, including the need for researchers to engage in ongoing conversations to make theoretical choices transparent. In response to this challenge, we shed light on the practice of evoking visual metaphors for learning trajectories by considering the variation in both what learning trajectories communicate and how they are communicated in the research community.

Background and Major Issue

Variation in Learning Trajectories Research

One common definition for the construct of a learning trajectory (LT) is that it is a conceptual tool that links goals, instructional activities, and processes of students' learning (Simon, 1995). Learning trajectories vary widely in how they may (or may not) give evidence of students' mathematics with linked descriptions of the mathematical goals, learning activities, and/or the teacher moves that may engender such ways of understanding (cf. Lobato & Walters, 2017). As Empson (2011) elaborates, there has been a tendency for research reports in mathematics education to separate descriptions of learners' conceptions as evidenced on specific mathematical tasks, from the contexts, teaching, and tools, that might have engendered such learning (e.g., Steffe & Olive, 2010; Clements & Sarama, 2009). This focus on articulating students' conceptual trajectories on sets of tasks is a general trend in mathematics education research. Consider, for example, Hackenberg's (2014) research that details three epistemic algebra students' learning trajectories of fraction schemes and operations. In introducing these LTs, Hackenberg describes some ideas as to how the interviewer's questioning may have incited particular shifts in the students' reasoning on particular tasks. However, she

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

acknowledges that many open questions remain regarding the role of instructional support in such shifts in students' learning processes.

Despite this apparent trend in how LT are taken up, not all LTs are communicated with this same orientation and emphasis. Some LTs more explicitly attend to things like teacher moves or other contextual features influencing student learning as a way to developing local instructional theory. Stephan and Akyuz's (2012) research, for example, leverages LTs as a tool to build an instructional theory. In their approach, instructional supports for students' understanding of addition and subtraction included classroom mathematical practices as engendered through tools, gestures, imagery, and taken as shared activities.

Evoking Metaphors as a Tool for Communicating Learning Trajectories Research

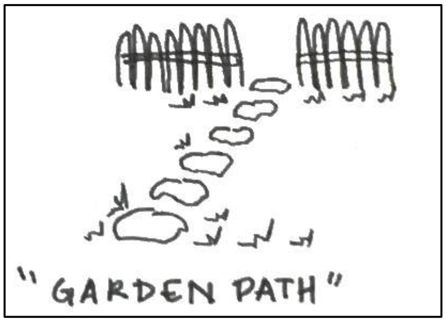
The challenge of conveying the simultaneity of change in students' learning and goal-directed instructional supports (as we believe learning trajectories as a research construct are primed to engender) may be a challenge of conceptual metaphor and related figural representations. Metaphors are essential to theory building and scientific inquiry (cf. Sfard, 1998). "Because metaphors bring with them certain well-defined expectations as to the possible features or target concepts, the choice of metaphor is a highly consequential decision. Different metaphors may lead to different ways of thinking and to different activities. We may say, therefore, that we live by the metaphors we use" (*ibid*., p. 5). Taking learning trajectories as the 'target concept' of our inquiry and theory building, we seek to understand the issue of communicating learning trajectories through an examination of the metaphors and meanings people make of them.

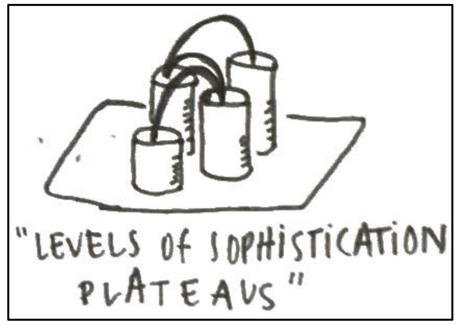
Different learning trajectories represent different things. Hence it is not always clear what types of information a particular learning trajectory might offer. Given this challenge, our inquiry into learning trajectories research is guided by the question of "What metaphors do researchers leverage in communicating learning trajectories?" We hypothesize that there is close link between what is communicated as a learning trajectory, and the conceptual and/or visual metaphor evoked in research.

Visual Metaphors for Learning Trajectories

Learning Trajectories Are Like "A Garden Path"

Sarama (2018) evoked the metaphor of a "garden path" to conceptualize learning trajectories, in which there are stepping stones that act as a developmental path to lead students through a gate—the goal state. From this view, learning trajectories must be interpreted by teachers and realized through social interaction around mathematical tasks. Indeed, the tight coupling of developmental progressions in children's thinking and sequences of tasks is evident in Sarama's research program (e.g., Clements & Sarama, 2014). Yet notice in Figure 1a, how the teacher, social interaction, and instructional activities are absent from the visual metaphor itself. Relatedly, Battista (2004) evoked the metaphor of "levels of sophistication plateaus" to characterize the "cognitive terrain" of learning processes for a learning trajectory (p. 186-187). In Battista's (2011) research, the sequence of tasks and ordered levels of sophistication in students' understanding are central to both what is conveyed in a learning trajectory and how it is communicated. Figure 1b offers a visual depiction of Battista's metaphor of "plateaus" of levels of understanding, which are complemented by narrative descriptions of related task type and instruction in his research.





(a) **(b)**

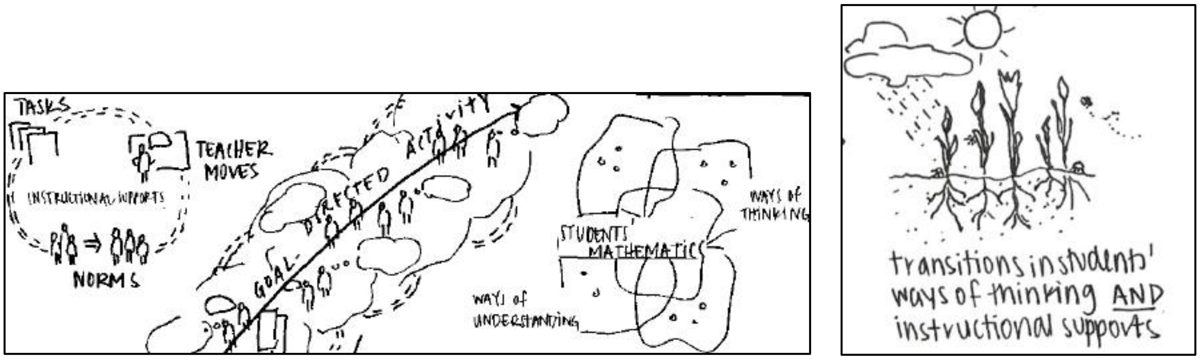
Figure 1. A visual depiction of (a) Sarama's (2018) "garden path" metaphor and (b) Battista's (2004) "levels of sophistication plateaus" metaphor for a learning trajectory.

It is notable that the aforementioned visual metaphors (garden path and plateaus, cf. Figure 1) evoke a sense of capturing the shifts or changes in how students understand mathematical ideas as measured by student outcomes on mathematical tasks. In these "conceptual trajectories" (cf. Empson, 2011), the role of instructional supports is not necessarily captured in the metaphor itself. Said otherwise, these approaches to learning trajectories afford great insight into the nature of students' conceptions and mathematical reasoning that are possible given certain task situations, with possible descriptive connections to instructional supports. However, the nature, character, and nuances of the instructional intervention, teacher-student relationships, time, and place, are often masked in the levels and path metaphors for learning trajectories. As a field, advancing understanding of student cognition and related curricular supports (e.g., sequences of task progressions) remain important, valued research agendas. Indeed, recent research has indicated that teacher's knowledge of students is an important predictor for improving student learning outcomes (Hill and Chen, 2018). However, learning and instruction are complex inter-related processes, with learning as a function of teaching (cf. Empson, 2011). If a goal of learning trajectories research is to convey a progression from lesser to greater sophistication in students' mathematical learning processes toward desired goal states, these metaphors provide little guidance for how instruction might engender change in students' learning that goes beyond a coupling of knowledge of students and related mathematical tasks.

Learning Trajectories Are Like "Growing a Garden"

In some of our own work on learning trajectories (Ellis et al., 2016; Fonger, Ellis, & Dogan, 2019; Fonger, Ellis, & Dogan, forthcoming), we frame learning trajectories as a networked relationship between transitions in students' ways of thinking and related instructional supports. In Figure 2a we offer a visual depiction of how students' mathematics (their ways of thinking and ways of understanding, ala Harel, 2014), transitions together with instructional supports (tasks, teacher moves, and norms) as guided by goal-directed activity (arrow). In our work, this conceptualization of learning trajectories was supported by evoking the metaphor of "growing a garden." Depicted in Figure 2b, the growing a garden metaphor captures the complex interplays between how a plant grows in response to environmental conditions such as soil, sun, and water. Evoking this metaphor for a learning trajectory accounts for changes in students' conceptions of mathematical ideas (i.e., the growth of a flower) together with the nature of instructional supports including but not limited to teacher moves, task design features, norms, student discourse, and student activity with artifacts and tools (i.e., the environmental conditions).

With these depictions, we intend to capture a more nuanced model of both what change in students' conceptions might look like, and how the learning environment (of which tasks are just one part) supported such change. We argue for treating learning trajectories as more than "just a bunch of flowers;" learning trajectories convey the transitions in the growth of the flower in relation to the supporting environment that evoked such change. This interpretation of a growing garden metaphor seems well-aligned to other research on learning trajectories (e.g., Stephan & Akyuz, 2012) that communicates the complexity of the interplay between the teaching and learning.



(a) **(b)**

Figure 2. Visualized Metaphors for Learning Trajectories as (a) a "growing garden", (b) Accentuating Transitions in Students' Mathematics and Instructional Supports

Discussion and Conclusion

To address the issue of what and how learning trajectories are communicated, we leveraged visual metaphors as a tool to make sense of two different approaches to learning trajectories research. In some approaches, learning trajectories are conceptualized as "a garden path", or "levels" wherein processes of students' learning and/or development as conceptual trajectories is foregrounded. Said otherwise, in a "garden path," characterizations of students' conceptual trajectories on sequenced sets of tasks is prominent. In another approach, learning trajectories are conceptualized as an interactive system of "growing a garden" wherein representations of goal-directed learning and instructional supports are taken together.

We invite conversation about the metaphors evoked in learning trajectories research as a way to address the issue of both what and how learning trajectories are communicated. We see a great need and opportunity to enrich the body of literature on learning trajectories (and learning progressions). In this report we argue that the use of interdisciplinary visual metaphors are productive for studying and representing learning trajectories. By articulating the metaphor(s) that guide our research, the products of learning trajectories research can become more explicit expressions of our theoretical assumptions about learning and teaching. Moreover, for researchers and practitioners concerned with learning trajectory based instruction (cf. Stzajn, Confrey, Wilson, Edgington, 2012), we hypothesize that by making theories of teaching (e.g., local instructional theory) more explicit in our communication of metaphors for learning trajectories, the field might improve the potential for learning trajectories to inform practice.

In close, by attending more seriously to the metaphors researchers evoke to communicate learning trajectories, the affordances and constraints of different approaches to learning trajectories becomes clearer. We intend for such elaborations to support the knitting together of a tapestry of research on learning trajectories that does not pit one approach against another, but that instead pushes for greater specification in the power of learning trajectories to advance research and inform practice as theoretical tools for research.

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DIFFERENCES IN STUDENTS WITH LEARNING DISABILITIES' (LD) UNITS CONSTRUCTION/COORDINATION AND SUBITIZING

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This theoretical discussion provides insight into an intersect of the mathematics education, cognitive psychology, and special education fields. To examine this intersect, the authors focus on how students identified with a learning disability develop actions on material when constructing and coordinating units. This theoretical frame considers results from several case studies in special education and cognitive learning fields, focusing on young students' number development, set in their subitizing activity and units construction/coordination. These results provide context and illustrate critical importance to their actions in light of neural differences and differences in their rate of development for future number and operation construction.

Keywords: Learning Theory; Number Concepts and Operations; Special Education; Cognition

Before children construct arithmetic units, they construct pre-numerical units, evident through a reliance on external representations, such as touching items while counting aloud or flashing four fingers sequentially or with manipulatives in patterned spatial arrangements (MacDonald & Wilkins, 2019). Children construct and reflect on their pre-numerical units to form internal or arithmetic units (Steffe & Cobb, 1988). To internalize units, children would need to step away from a reliance on perceptual material towards material that can stand in for the perceptual units they have constructed. These new pre-numerical units are described as figurative units and evidenced with fingers or counting words. Steffe (2017) estimated that about 40% of first graders do not yet use figurative units when counting and unitizing; this population remains at about 5-8% by third grade. By remaining reliant upon perceptual units, children are not yet able to develop mental operations grounded in their conceptual understandings. These same students are sometimes also identified as having a mathematics learning disability (LD) (Butterworth, 2011). Clements et al. (2013) explain that many children evidence precursors for an LD but are not yet identified, preventing them from receiving targeted mathematics interventions.

To consider how interventions could best be designed, we need to begin leveraging information pertaining to how young children construct pre-numerical units instead of focusing on deficits students with LD evidence (Butterworth, 2011). The purpose of this theoretical commentary is to shift from a deficit model towards a progressive model. In particular, we discuss students' actions and their possible progressions when *subitizing* (a quick apprehension of the numerosity of a small set of items - Kaufman, Lord, Reese, & Volkmann, 1949) and constructing units to determine how students' actions with visual patterns can best support early mathematics development. This, the aim of this theoretical commentary is to *examine aspects of number abstraction processes through students' subitizing activity and/or units construction/coordination.*

Theoretical Framework

To frame this theoretical commentary and consider this aim in the context of special education and mathematics education, we draw broadly from an intersection of cognition and learning and radical constructivist paradigms. In particular, we consider concepts that inform these paradigms: executive functioning (Clements & Sarama, 2019) and units construction and coordination (Norton & Boyce, 2015).

Executive Functioning

Executive functioning is evidenced through several processes that young children develop throughout their early childhood years (birth to third grade) (Clements & Sarama, 2019). These processes assist children in their ability to self-regulate their learning of mathematics and have been found to positively correlate with children's mathematics achievement (e.g., Best et al., 2011; Clements et al., 2016; Viterbori et al., 2015). In this commentary, we focus on attentional shifting and updating working memory.

*Attentional shiftin*g can explain mathematics strategy development through use of attentional mechanisms (Clements & Sarama, 2019). Attentional shifting is when children are able to shift their attention from perceived material to new perceptual material when developing problem solving strategies. This is evident when young children conceptually subitize (relying on conceptual processes when subitizing). For example, when a four-year-old child is shown five items arranged in a patterned spatial arrangement typical to the face of a die, MacDonald and colleagues (2016; 2019; under review) found children typically subitize two sets of two and one set of one. To segment and unitize, students would need to subitize two and associate this with a verbal word for two. By their attentional shifting between twos and ones, while attending to new information (what warrants attention) and not to other visual material (distractors), children are developing additive strategies (Clements & Sarama, 2019). Many students with LD experience attention differences compared to their peers that contribute to differences when learning mathematics.

When an individual manipulates and maintains information relevant for problem solving, Clements et al. (2013) explain this characterizes students' ability to *update their working memory*. Children engage in this when given multi-step problems, which require their working memory to be engaged and then updated with additional information. For example, when a five-year-old child is shown five items arranged in the same orientation described earlier, and then two additional items are added to this spatial arrangement, this child would be required to hold on to the five items while adding two additional items (possibly combining subitizing and counting). If a child considers the set of five items in isolation to the adding of two items, this child may struggle to construct units for these items, resulting in a counting all strategy. Students with LD struggle with some of these executive functions, pressing them to learn "tricks" grounded in procedural knowledge as they realize their peers are developing more sophisticated strategies for number and operation tasks (Hunt et al., 2019; see also Hunt & Silva, 2020).

Units Construction and Coordination

Units coordination and construction refers to the number of levels and type of units children can construct and bring into a situation (Norton & Boyce, 2015). Prior to units coordination, children use counting to construct *pre-numerical units* in their activity. For instance, children first rely on manipulatives (perceptual units) to construct a pre-numerical unit and determine the total amount through their counting activity. When pressed to step away from the perceptual units, children construct pre-numerical units with finger patterns (figurative units), pointing/tapping (motor units), and/or number words (verbal units) (Steffe & Cobb, 1988). Progressions from perceptual units towards verbal units provide evidence of children transitioning towards internalized actions (imagined external activity). When children interiorize units, these units are considered *arithmetic units* and allows children to operationalize number through their coordination of units (e.g., five is three away from two).

MacDonald and Wilkins (2019) found that one preschool student's subitizing related to her pre-numerical units construction. When developing conceptual processes to assist in her subitizing (e.g., two, two, and one is five; two and three is five), this preschool student constructed perceptual units and then figurative units to evidence her reasoning. Moreover, this student's units were represented with parallel actions (e.g., picking up two manipulatives simultaneously, flashing three fingers). Thus, counting and subitizing activity has been found to inform students of conceptual material that promotes their pre-numerical units construction.

The Intersect of Special Education, Subitizing, and Executive Functioning

Butterworth (2011) found students with LD encode numerosity information differently when subitizing compared to their normal achieving peers. In particular, Butterworth draws from decades of research to explain how young children typically develop *numerosity codes,* where individuals use a particular region of their brain to process sets of items over time and space. Fundamental to number understanding, numerosity codes have been found to evidence deficits in students identified with a LD and explains different types of subitizing activity (Butterworth, 2011). Butterworth explains that this neural difference fundamentally explains why students with LD rely mainly on rudimentary reasoning and strategy development with number (see also Hunt & Silva, 2020).

Hunt et al. (2016) compared findings from clinical interviews involving 21 upper elementary age students with LD with 23 students identified with a mathematics difficulty. Findings evidenced nuances to students with LD's partitioning (partitioning with no regard to equal parts, partitioning with regard to "halves", partitioning with regard to equal parts). When comparing students with LD to students with mathematics difficulties, Hunt et al. (2016) found that 30% of students with LD were able to partition with no regard to equal parts (10%) or with regard to "halves" (20%). Comparatively, students with mathematics difficulties did not rely on such rudimentary partitioning activity. Moreover, 70% of students with LD and 100% of students with mathematics difficulties partitioned with regard to equal parts. These differences suggest some students with LD partition in a very similar way to students experiencing mathematics difficulties, but may be developing their partitioning at a different rate than their peers. These different types of partitioning may also explain working memory differences that students with LD experience when given other tasks that do not provide external representations when working with complicated mathematics concepts. If a student with LD has not yet begun partitioning with equal parts, then solving symbolic fraction tasks may be too much for their working memory to manage. For instance, when solving tasks that only represent fractions as symbols, students may need to consider each symbol as a separate item (e.g., ¾ is considered as a 3 and a 4).

Given these different rates of development, students with LD may evidence seemingly puzzling ways of reasoning that, from a developmental and psychological stance, actually makes sense. For example, Hunt et al. (2019) found that one third grade student, Gina, relied only upon ways of solving number problems using procedures that she could not explain or make sense of. Interestingly, Gina was not perturbed when differences between her procedural number knowledge and physical actions did not align. Yet, when given novel rational number tasks for which she had no procedures for, Gina more readily connected her conceptual knowledge with her actions. Hunt et al. (2019) argued that students with LD are able to develop the same conceptual knowledge as their normal achieving peers, but may be doing so at a different rate. This is important because noticing differences between procedures and physical actions would not be a goal for Gina if procedures were not yet connected in her long-term memory and would make connections back to conceptual understanding difficult.

Conversely, another student, Stu, (Hunt et al., 2016) was also able to anticipate which strategies to use because he was engaged in a platform that supported him to successfully develop equi-partitioning (mental segmenting to form equal parts). In fact, he developed anticipatory types of strategies that allowed him to utilize mental actions so he was not dependent on his physical actions/material to solve problems. Opportunities to develop and abstract the actions that bring about number and rational number affords students with LD opportunities to access tasks that only represent number through symbols.

When considering how these differences evidence themselves in students' subitizing activity, we consider findings from Koontz and Berch (1996) who found elementary age students with LD had significantly slower response times when matching small (two and three) dot arrangements to number words. These findings suggest that young children with LD struggle to update their working memory because items were not processed in a parallel manner and these children struggled to inhibit distracting visual information. When considering this in relation to shifting attention it seemed young children were not yet able to inhibit distracting perceptual material and then shift their attention to primary perceptual material.

Finally, MacDonald et al.'s (under review) findings echo some of these findings, as one first-grade student, Diego, relies mainly on his unitization and iteration actions when solving subitizing and units construction tasks. Findings further indicate that Diego relies heavily on perceptually clustered items when unitizing and not yet able to construct figurative units. These findings also echo Butterworth's (2011) discussion, as he describes an unfinished amount of research examining relationships between students with LD's use of fingers and their numerosity code development.

Conclusion

To date, the research base remains an unfinished work when considering if and/or how students with mathematics difficulties develop separate, or different, understandings of part whole and what features of their diverse cognitive backgrounds (e.g., working memory or attentional processes) might interact with development (e.g., Hunt et al., 2016; Hunt et al., 2019a, 2019b; Hunt & Silva, 2020; Lewis, 2014; Lewis, 2017). In the absence of a convergence of evidence in the research literature, present research efforts document elements of students' diverse cognitive background thought to interplay with children's mathematical learning from an early point in their lives (Compton, Fuchs, Fuchs, Lambert, & Hamlett, 2012). These factors are then used in a predictive sense to explain "learning difference" as variations in certain norms that predict performance over time (Vukovic, 2012).

These findings suggest that students with LD are constructing and coordinating units with partitioning/segmenting activity at a different rate than other students. In fact, we wonder if these different rates of development relate to differences in their development of parallel processing activity, attentional mechanisms, and/or working memory resources. These developmental differences might evidence themselves in their subitizing activity and prevent them access to particular tasks in their early childhood years. To consider these differences more closely, we need to begin adopting new questions and theoretical perspectives, which allow us to work with intersections of mathematics education, special education, and cognitive psychology. Moreover, we need to consider how early actions students construct can explain these differences and inform intervention design that aligns with a wide variety of elementary age students' mathematics development.

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PUTTING THE "M" BACK INTO STEM: CONSIDERING HOW UNITS COORDINATION RELATES TO COMPUTATIONAL THINKING

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This theoretical commentary examines theory driven discussions in Science, Technology, Engineering, and Mathematics (STEM) fields and mathematics fields. Through this examination, the authors articulate particular parallels between spatial encoding strategy theory and units coordination theory. Finally, these parallel are considering pragmatically in the Elementary STEM Teaching Integrating Textiles and Computing Holistically (ESTITCH) curriculum where STEM and social studies topics are explored by elementary students. This commentary concludes with questions and particular directions our mathematics education field can progress when integrating mathematics in STEM fields.

Keywords: STEM/STEAM; Elementary School Education; Learning Theory; Interdisciplinary Studies

Computational thinking (CT) has recently been making a larger presence in elementary classrooms, yet it is still not yet clear how CT relates to young children's mathematical reasoning or even how it can be defined. Feldon (2019) explains "computational thinking has been characterized as a foundational competency, akin to reading and arithmetic" (p. 1). Given this characterization, the instructional technology field has yet to define CT (Feldon, 2019; Grover & Pea, 2013, 2018). Margulieux (2019) examined findings that suggest relationships between students' spatial reasoning and their Science, Technology, Engineering, and Mathematics (STEM) achievement when outlining particular theories that explain CT achievement. Pragmatic delineation of CT in the K-12 standards of the Computer Science Teachers Association (CTSA Task Force, 2011, p. 10) broadly characterize CT as a "problem-solving methodology" that draws from reasoning present in mathematics education, such as "abstraction, recursion, and iteration." These learning constructs and types of reasoning echo K-12 mathematics reasoning, effective mathematics practices, and mathematics learning objectives. Thus, *the purpose of this brief research report is to consider theoretically how CT reasoning (framed through spatial reasoning) relates to mathematics reasoning (framed through units construction and coordination).*

To frame this theoretical commentary, we first draw from spatial encoding strategy theory to explain how students' engagement with visual and mental representations may explain CT achievement. Second, we draw from the units coordination learning theory to determine how young children may be drawing from mathematics reasoning in elementary grade levels. From this theoretical framing, we consider particular parallels between these theories to determine multifaceted mathematics reasoning, as integrated in CT activities. Through an integrated STEM-driven curriculum grounded in social studies, titled, Elementary STEM Teaching Integrating Textiles and Computing Holistically (ESTITCH) (Hawkman et al., under review), we frame the pragmatic aspects of these integrated activities, and we delineate parallels between particular CT and mathematics reasoning, objectives, and practices. Moreover, we include social studies topics to determine how

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social artifacts leverage CT and mathematics engagement and what is gained with such an interdisciplinary instructional approach.

Theoretical Framework

This theoretical discussion is set in an emergent perspective paradigm (Cobb & Yackel, 1996), meaning we examine individuals’ construction of mental objects and actions before considering the meaning gained through their engagement with social artifacts. Therefore, we begin by articulating theories framed with cognition learning science paradigms (Attkinson & Shiffrin, 1971; Baddeley, 1994; Clements & Sarama, 2019) and radical constructivist paradigms (Glasersfeld, 1995; Norton & Boyce, 2015), before drawing on this emergent perspective (Cobb & Yackel, 1996) within the context of STEM curricula. This framework begins by discussing spatial encoding strategy theory (cognition learning paradigm) before drawing from units construction and coordination learning theory (radical constructivist paradigm).

Spatial Encoding Strategy Theory

Through a review of the literature and drawing specifically from Parkinson and Cutts’ (2018) findings, Margulieux (2019) proposed a spatial encoding strategy theory to explain the cognitive mechanisms related to individuals’ spatial skills and STEM achievement. Margulieux explains that both the encoding of mental representations and the identification of landmarks (non-verbal representations) help individuals develop strategies and spatial skills (e.g., orientation, relations, and visualization). *Encoding* (making sense of) mental representations is best characterized in the cognition learning sciences where (1) individuals “chunk” information to act on in their working memory (limited memory capacity – Baddeley, 1994) and (2) individuals draw from attentional mechanisms (a component of executive functioning processes – Clements & Sarama, 2019) to determine what feature of a representation warrants attention (Attkinson & Shiffrin, 1971). For instance, when young children are asked to use text or symbols to solve problems in STEM fields (e.g., develop a code to move a LEGO® robot), they would need to map their anticipated results to a mental model that they can manipulate (Parkinson & Cutts, 2018). Prior to this experience, we argue young children would need physical experiences to form this model.

Moreover, Margulieux (2019) proposes individuals’ mental representation construction partially depends upon individuals’ development of non-verbal representations. For instance, when individuals chunk encoded information of mental representations, they are required to determine critical features and relationships of non-verbal representations (Margulieux, 2019). Thus, for individuals to encode mental representations successfully, they need to engage with/construct non-verbal representations that form these mental models.

Units Construction and Coordination Theory

Units coordination and construction refers to the number of levels and type of units children can construct and bring into a situation (Norton & Boyce, 2015). We utilize units construction and coordination learning theories to frame students’ actions and establish transitions from their construction of *pre-numerical units* (physical material representing number) towards arithmetic units coordination. Children begin counting when first constructing pre-numerical units with which to use as material for future activity (Steffe & Cobb, 1988). These units are first constructed through children’s external activity before becoming internalized (imagined activity) and then interiorized (automaticity).

To transition from pre-numerical units construction to arithmetic units coordination, children engage in one of four actions: unitizing, partitioning, iterating, and disembedding. Once students can count on they are next able to *unitize* (taking an item, or collection of items, as a whole unit that can be further acted upon) and *iterate* (making copies of a unit) units to construct number sequences (1, 2, 3, 4, 5, 6 …) (Norton, 2016; Steffe & Cobb, 1988). Once number sequences are constructed, students are able to *partition* (break into equally sized parts) these number sequences with which to count on from (Norton, 2016; Steffe & Cobb, 1988). To coordinate two levels of units, students would need to both iterate and partition (reversible actions), but would not yet be able to use them simultaneously (Norton, 2016). For instance, through counting, students could unitize two composite units (e.g., 3 and 12) where there are able to iterate three in a "count by" sequence (e.g., three, six, nine, twelve).

Once students coordinate all three levels of units and are able to do so in an anticipatory manner, they compose reversible actions and develop what Piaget (1970) described as logico-mathematical actions (operations). These operations allow students to construct number as a mental object with which to *disembed* a whole into parts while remaining cognizant of the whole (i.e., 12 is understood as 4 sets of 3) (Norton, 2016; Steffe & Cobb, 1988).

The CTSA (2011) articulate objectives grounded in some of these actions "abstraction, recursion, and iteration" (p. 10). For instance, as children iterate units, they construct sequences and are more readily able to abstract these sequences. Moreover, through children's composition of reversible actions (e.g., iterating and partitioning), they are able to recursively make sense of activity in STEM fields, providing them strategy development for future success.

Intersection of CT and Mathematics Reasoning within Social Studies Activities

Much of the mathematics education literature (Sarama & Clements, 2009) has found relationships between young children's spatial reasoning and mathematics development. By considering Margulieux's (2019) spatial encoding strategy theory, we argue that children's "chunking" of features from representations occurs in CT and in mathematics activities. By setting these activities in Social Studies, we posit children are using social artifacts to determine what warrants attention, which provides culturally responsive learning opportunities. Thus, we first consider parallels between one of the two CT learning objectives (see table 1) before considering how these might evidence themselves in the ESTITCH curriculum where integration of social studies and STEM provide meaning to students' units coordination.

In table 1, we outline relationships between two CT learning objectives and how they relate to corresponding elementary mathematics objectives and practices. For instance, when considering children's ability to decompose systems of computational thinking tasks, we propose their reasoning would be similar when they apply properties of operations, generate patterns, and evaluate expressions. To meet both sets of objectives, we posit they would need reason abstractly and attend to precision. In particular, students would be required to have two or three levels of interiorized units (dependent on type of operation) and would be required to determine critical features of a visual representation that relates to the goal of the task.

On day five, part 2 in the ESTITCH curriculum, students use stories centered on immigration, migration, and forced relocation to determine what landmarks are present in their own histories and how might they be used to form a timeline. Through their timeline development, they create circuits coded to represent these landmarks and proportional length/time to represent relationships between these landmark events. Through these activities, students are representing time and length in a scaled model, which presses them to generalize particular patterns abstractly and attend to precision of these events. Moreover, students are constructing units based on features of cultural artifacts they value to coordinate in a linear format. These integrated activities are powerful because they draw from cultural artifacts that children can connect to their mental representations of experiences and development of relationships between units that form these relationships.

Table 1: Intersection of Computational Thinking, Mathematics Standards and Mathematical Practices

Computational Thinking	*Operations and Algebra*	*Mathematical Practices*
Decomposition: Break down a task into minute details.	Apply properties of operations as strategies to multiply and divide (3.OA.B.5). Generate a pattern that follows a given rule. Identify features of the pattern not explicit in the rule itself (4.OA.C.5).	Reason abstractly and quantitatively (MP2) Attend to precision (MP6).
Pattern Generalization and Abstraction: Filter out information to solve a certain type of problem and generalize information.	Identify arithmetic patterns and explain them using properties of operations (3.OA.D.9). Write simple expressions, and interpret numerical expressions. Analyze patterns and relationships (5.OA.A.2).	Look for and make use of structure (MP7). Use appropriate tools strategically (MP5).

To emphasize the mathematics in this unit of study, an educator could have students construct visual models of decimals to represent time in such a proportional manner. For instance, if ten meter sticks represented one whole unit (one second), students could explore proportional relationships with smaller portions of a second with base-ten blocks (one centimeter in length) to explore coding with milliseconds. This type of precursor activity allows students opportunities to develop proportional relationships with physical models before requiring them to draw from mental models of the same relationships (MacDonald et al., 2018).

Conclusion

By considering the intersection of students' reasoning associated with STEM and mathematics fields, we are more able to emphasize mathematical reasoning in curricula development while utilizing theories that focus on students' mathematics reasoning. Moreover, as theory and associated curricula begins to emerge in the STEM fields, more questions surrounding theory and curricula need to be considered. For instance, how do STEM activities afford and/or constrain students' mathematics reasoning? What trajectories in STEM are present for young children in prekindergarten classrooms as they transition to elementary classrooms? How might students with particular learning disabilities evidence STEM reasoning development in elementary classrooms and how might this relate to their access to mathematics in schools? Only through such multi-faceted theoretical frameworks and questions will our field continue to progress in a technology-driven society.

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WHOSE PROBLEM, WHOSE PRACTICE? NEGOTIATING THE FOCUS OF RESEARCH-PRACTICE PARTNERSHIPS WITHIN SCHOOL MATHEMATICS

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At the core of productive research-practice partnerships is a mutual commitment to addressing problems of practice, which must be jointly negotiated, working through differences in perspective, status, and authority across partners. But what is less clear is how to account for and navigate the influence of broader accountability policies in the framing of those problems, as their application can lead to "manufactured" problems specified around state standardized testing outcomes. In this conceptual paper, we reflect on recent encounters with school district personnel, some of whom we were interested in fostering research-practice partnerships, to describe an ethical dilemma of whether and how, in our position as researchers, to invite potential partners to take up more "authentic" problems of practice.

Keywords: research-practice partnerships, design experiments, equity and diversity, systemic change

Late in 2017, the lead author and a small team of researchers were meeting with leaders of a small, rural district in the midwest U.S. in hopes of initiating partnership work around a common problem of interest. That fall, they had conducted a series of interviews with leaders and teachers, attempting to learn about the mathematics-related problems with which the district was wrestling and how individuals in the district framed those problems (Benford & Snow, 2000). From the outset, the researchers had expressed interest in developing a partnership rooted in district challenges rather than imposing ideas stemming from their own research agenda (Penuel, Fishman, Cheng, & Sabelli, 2011). At this meeting they were sharing what they had learned through the interviews, inviting district personnel's responses, and listening for opportunities to endorse and offer help with a challenge of mutual interest. Toward the end of the near-hourlong meeting, the lead author (CM) invited the group to take a step back and articulate, in broad terms, their aspirations for mathematics learning in the district, to which a school administrator (A) responded:

CM: Thinking broadly, what- what are your goals for the children of [your community] in mathematics?
A: Short term, for at least *my* building, by the time they get to fifth grade, their fifth-grade scores are not- not good. Like, at *all*.
CM: So, Ok-
A: And a lot of it was number sense even starting in the third grade, so, like, when third graders are struggling with number sense, that's obviously coming from us too, so- I mean, that's not very narrowed down, but-
CM: But, I want to suggest that you, kind of, immediately went to *test* scores.
A: Well, because data shows if there's progress or not.

In this exchange, the administrator suggested that the grades 3-5 intermediate school may be insufficiently supporting students in developing number sense ("that's obviously coming from us too"), but what was most troubling for the administrator—and what defined the broader mathematics goal—was improvement on state standardized test scores. Munter, failing to hide his disappointment with that framing, lamented that the administrator "immediately went to test scores" when invited to think broadly about children's mathematics learning.

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Negotiating Problems of Practice

Over the last decade, there has been increasing attention to and support for research-practice partnerships (RPPs) in education as a means of bridging the "research-practice divide" (Coburn & Stein, 2010), as evidenced by grant competitions (e.g., the Institute of Education Science's Researcher-Practitioner Partnerships in Education Research program; the Spencer Foundation's Research-Practice Partnership Grants Program) or special issues in research journals (e.g., Herenkohl & Herenkohl, 2019; Penuel, Cole, & O'Neill, 2016; Penuel & Hill, 2019). One frequently employed definition for such partnerships is that they are "long-term, mutualistic collaborations between practitioners and researchers that are intentionally organized to investigate problems of practice and solutions for improving district outcomes" (Coburn, Penuel, & Giel, 2013, p. 2). Indeed, a consistent commitment across the growing body of work through/on RPPs in education is that they begin with *problems of practice* "as encountered by participants in an activity system, rather than with researchers' goals for the improvement of teaching and learning" (Penuel, 2014, p. 100).

Increasingly, there are also accounts in the literature of the messiness and tensions that can arise as researchers and practitioners negotiate the problems they might collaboratively address (e.g., Desimone, Wolford, & Hill, 2016; Henrick, Muñoz, & Cobb, 2016; Penuel, Coburn, & Gallagher, 2013; Vakil, McKinney de Royston, Nasir, & Kirshner, 2016). Penuel and colleagues (2013), in particular, have highlighted the roles that status (e.g., from university affiliation or role as practitioner) and authority (e.g., from positions of leadership in a district) play as problems are identified and defined and as decisions about which solution strategies to pursue are made. And Vakil et al. (2016) have stressed how hierarchical relationships between researchers and practitioners can persist, especially when dimensions of researchers' positionalities are left unexamined. Such work helps to foreground the ethical issues that underly the formation and work of research-practice partnerships. As Bang and Vossoughi (2016) have reminded us, "we must take seriously the question of 'Who does the design and why?' (Engeström, 2011, p. 3)," the answers to which are "also deeply bound up with the *how and where* of design, demanding a focus on process and the genesis of relations as well as the places within which they are made, live, and unfold" (p. 179, italics in original).

Whose Problems? Whose Practice?

In some ways, the episode with which we began this paper matches descriptions of partnership negotiation in the literature, as the researchers and practitioners were likely entering the conversation with different values, interests, and perspectives, and differences in authority and status were undoubtedly shaping the interactions. But we have come to view the question of "Who does the design and why?"—and concomitantly, "Who does the *problem framing* and why?"—as likely implicating entities not at the negotiating table. Specifically, for us, the school administrator's pointing to "fifth grade scores" as a primary object of concern invokes a much broader set of reforms and accompanying discourse that are rooted in neoliberal principles (Croft, Roberts, & Stenhouse, 2016) and reduce the role of K-12 students to data production for the benefit of adults' regimes (McDermott, 2013). And, although a host of local stakeholders are arguably complicit in building and maintaining this system, more broadly, it is imposed by legislators, policymakers, and other "silent partners" who, with *their* authority, impose constraints within which school leaders and teachers identify goals and problems.

Given the potential influence of this broader frame, we can ask whether the problems of practice that district leaders articulate are *real* or *manufactured*. When we say "real," we refer to challenges that exist in pursuit of an extensive aim of education, such as the pursuit of truth (Dewey, 1938) or the practice of freedom (Freire, 1970), which seek to prepare students for a "whole life, not simply an economic existence" (Noddings, 2007, p. 26). And by "manufactured" (Berliner, 1995) we refer to challenges that exist only as a result of the imposition of standardized test-based metrics of proficiency and "adequate progress" (U.S. Department of Education, 2002; 2015), which offer no direct benefits to the laborers "at the bottom"—public school children (Munter & Haines, 2019). Based on experiences like the opening scenario, we became curious which kinds of challenges are most salient in school districts, and whether there might be patterns in relation to community contexts.

Patterns in Problems of Practice

To answer those questions, we investigated the mathematics-related challenges identified by mathematics leaders across the U.S. state of Missouri. We sampled 50 districts, ranging from rural to urban contexts, and in each one interviewed the district leader most directly responsible for mathematics instruction (Munter, Nguyen, & Quinn, 2020). In each interview, we invited leaders to describe their biggest mathematics-related challenges, what they perceived to be the cause(s), and what, if any, initiatives they were pursuing in response. Two of the leaders reported that their districts did not have any mathematics-related challenges, and nine described initiatives they were pursuing, but without specifying any problems motivating their efforts. For the present discussion, we focus on the other 39, among whom three main types of problems emerged: student outcomes (n=30), student experiences (n=6), and equity (n=3).

As we have alluded to above, we found some of these problems of practice to be *real* (challenges that exist in pursuit of expansive educational aims) and others manufactured (challenges that exist only as a result of the imposition of standardized test-based metrics of proficiency). Of the 30 leaders who articulated problems related to student outcomes, two pertained to internally established indicators of course taking patterns; all of the other 28 were focused on externally established indicators, primarily state standardized test scores. It is possible, of course, that some districts were facing challenges with ensuring all of their students were being supported in learning mathematics. However, their identification of low test scores as, itself, the problem and not merely a symptom, suggests that the bulk of district leaders we interviewed articulated manufactured problems of practice.

Of the remaining nine leaders, three articulated challenges related to issues of equity and six described challenges related to how students experience school mathematics. Because they look past the surface of standardized test scores, it is possible that these problems of practice are more genuine. For example, six leaders centered students' classroom experiences, with one describing a challenge of providing a cohesive school experience, and five describing challenges related to supporting student engagement (i.e., enjoying mathematics, seeing its usefulness, or engaging in sensemaking). For example, one small metropolitan district is working on supporting student engagement by adopting a workshop model, with the intention that through collaborative activities, mathematics experiences will be more authentic for students. And the three leaders' descriptions of problems related to equity may signal a concern about real problems in differences in opportunity (Da Silva, Huguley, Kakli, & Rao, 2007; Flores, 2007). However, it is also possible that even these problems are, to some extent, manufactured. For example, a district may be concerned about student engagement or cohesion only as a means of improving state test scores. Similarly, leaders' equity concerns may be defined by and limited to "achievement gaps"—narratives that are framed by externally established, standardized test-based metrics and often reinforce deficit narratives of students of color (Martin, 2009).

Additionally, as reported elsewhere (Munter et al., 2020), we found that leaders from districts outside of metropolitan areas were more likely to describe problems of practice related to outcomes. This suggests that the "*where* of design" (Bang & Vossoughi, 2016) indeed matters, as small, rural, districts may be suffering the differential impact of accountability policies, and more often attending to manufactured problems.

An Ethical Dilemma

It is at this point in our consideration that we reach an ethical dilemma. In our position as university-affiliated researchers, with the hierarchy and status that accompanies whatever "academic expertise" we are perceived to have (Penuel et al., 2013, p. 247), if we are convinced that the problems with which a great number of our potential partners are concerned are manufactured (i.e., that they are accepting a frame from the broader neoliberal agenda), who are we to tell them that their problem is not real? After all, even if the problem *is* manufactured, how they experience its implications are likely very real. Then again, who are we to withhold those insights? If we "honor" the problem they have identified, we risk (being complicit in) perpetuating a system of sorting and ranking that does not enrich the lives of students, particularly those who are most vulnerable in institutions of schooling. And, if we simply walk away from a potential partnership because we are unwilling or uninterested in taking up what we view to be a manufactured problem, whatever assistance we might have had to offer is squandered.

We have not reached a clear resolution of this dilemma. We do, however, take inspiration from Freire (1970) in assuming that the question is not *whether* to engage in fostering partnerships, but *how* to do so in liberating ways—which may require further reflection and analysis with respect to how to ethically pursue problems of practice that are increasingly "real," including how we can respectfully acknowledge and affirm practitioners' very real institutional pressures and demands while advocating for problem frames that more explicitly target inequity and students' experiences in school mathematics.

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THEORY AND RESEARCH METHODS:

POSTER PRESENTATIONS

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

FALA FRAMEWORK: A LEARNING PROGRESSION FOR NOVICE TEACHERS' USE OF FORMATIVE ASSESSMENT

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Keywords: Learning Trajectories, Assessment and Evaluation, Teacher Knowledge

Formative assessment has been identified as one way that teachers can gather critical information about a student's level of understanding in order to make informed instructional adaptations that meet the needs of all students (NCTM, 2000; Shepard et al., 2005). Over several decades, research has shown the potential of formative assessment to effectively improve student achievement (Black & Wiliam, 1998; Kingston & Nash, 2011). Despite its potential, issues in preparing teachers to implement formative assessment practices has kept its potential from being realized (Schoenfeld, 2015) and many teachers have limited understanding of its use (Shepard et al., 2005).

In order to better understand how to prepare teachers to use formative assessment, a trajectory describing how teachers develop formative assessment knowledge and practice is needed. The Formative Assessment Levels of Appropriation (FALA) framework evolved from a larger qualitative research study on the evolution of formative assessment knowledge and practice of novice teachers from teacher preparation through their third year of teaching. Grounded in activity theory, the FALA framework describes the levels of appropriation (Grossman et al., 1998) for the five aspects of formative assessment defined by Black & Wiliam (2009). Table 1 provides an example of the framework for one aspect of formative assessment—clarifying intentions and criteria for success.

Table 1: Levels of Appropriation for Clarifying Intentions and Criteria for Success

Level 1	Level 2	Level 3	Level 4	Level 5
No conceptual understanding and/or not used in the classroom	Identifies learning goals/criteria by name, but does not understand purpose of learning goals/criteria (e.g., equates with standards) and does not engage students with them.	Understands that learning goals/criteria as targets for performance and may share them with students, but does not impact student learning and instructional decisions.	Understands learning goals/criteria and their relationship to student learning progressions. Makes connections to instructional decisions but may have limited experience applying to practice.	Understands the purpose of learning goals/criteria in shaping student learning and frequently makes instructional adaptations based on student progression towards learning goals. Evidence of student engagement with goals/criteria.

To create each level of appropriation, a typological analysis approach (Hatch, 2002) was used to code course data over three semesters of a course on the use of formative assessment for teaching mathematics. The data was first chunked by individual assignment posts and coded by its reference to one of the five aspects. Next, data for each aspect was read through and examples of the five levels were coded based on Grossman & Smagorinsky's (1998) general definitions of each level of appropriation. All levels were re-read and summarized as one to two sentence generalizations. A secondary coder worked with the primary researcher on coding a sample of the data until an inter-rater reliability level of 81% as met (Miles & Huberman, 1984).

In: Sacristán, A.I., Cortés-Zavala, J.C. & Ruiz-Arias, P.M. (Eds.). (2020). *Mathematics Education Across Cultures: Proceedings of the 42nd Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, Mexico. Cinvestav / AMIUTEM / PME-NA. https:/doi.org/10.51272/pmena.42.2020

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A THEORY-METHODOLOGY FRAMEWORK FOR CONCEPTUALIZING A CULTURALLY RESPONSIVE MATHEMATICS/TEACHER EDUCATION

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Keywords: Culturally Responsive Pedagogy; School Mathematics; Theory-Methodology

Overview

The research described in this presentation asks the question of how school mathematics and mathematics teacher education might be reframed through critical and culturally responsive pedagogies through the introduction of a new theory-methodology framework. By synthesizing perspectives offered by Ethnomathematics (EM), Critical Mathematics (CM), Indigenous Education (IE), Language Diversity (LD) and Equity-based (E-b) approaches to research in mathematics education, a new (disruptive) form of culturally responsive pedagogy (CRdP) is being conceptualized. CRdP is *pedagogically* informed by the EM-CM-IE-LD-E-b collective; it is *theoretically* informed by Nancy Fraser's three-dimensional approach to social justice and participatory parity; and it is *methodologically* informed by discourse analysis. Grounded in a conceptual approach to responding to the study's question, the research (and this presentation) offers a comprehensive approach to challenging teacher education practices and navigating toward socio-economic, cultural and political justice/parity for all mathematics learners.

Theory-Methodology Framework

CRdP is being conceptualized by building on existing and highly relevant research in the field of mathematics education— research which views the education of mathematics teachers through diverse theoretical/philosophical lenses, including those of Culturally Responsive Pedagogy (CRP) (see, for example, Aguirre & Zavala, 2013; Nicol et al, 2013); Ethnomathematics (EM) (e.g., Presmeg, 1998; Rosa & Orey, 2011); Critical Mathematics (CM) (e.g., Ernest et al, 2016); Indigenous Education (IE) (e.g., Lunney Borden & Wiseman, 2016; Sterenberg, 2013); Language Diversity (LD) (e.g., Barwell, 2018; Chronaki & Planas, 2018) and Equity-based (E-b) (e.g., Gutiérrez, 2012; Herbel-Eisenmann et al, 2012) approaches. The socio-critical work of Nancy Fraser (Fraser, 2009; Lingard & Keddie, 2013; Meaney et al., 2016) is central to the research, with the design of CRdP framed through the three dimensions of Fraser's theory of social justice: distribution (socio-economic justice), recognition (cultural justice), and representation (political justice), revealing how school mathematics may be reframed through the EM-CM-IE-LD-E-b collective. As a means of grounding Fraser's three dimensions in a research method, I draw on a critical discourse analytic (CDA) approach (Wodak & Meyer, 2009). The six-question CDA framework that I have developed to interrogate research texts and to construct CRdP, along with initial results of its application, will be outlined in this presentation.

Significance of the Work

In acknowledging the culture of mathematics, of mathematics classrooms and of students' lives and communities, the research is an innovative and productive response to the call to educate teachers in critical and culturally responsive pedagogies. In doing so, the research serves to strengthen communication and connections between the fields of teacher education, mathematics education research, and curriculum cultural revitalization in multi-cultural contexts.

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A CONCEPTUALIZED FRAMEWORK FOR ASSESSING FACTORS THAT MAINTAIN AND LOWER COGNITIVE DEMAND DURING TASK ENACTMENT

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The National Council of Teachers of Mathematics (NCTM, 2014) has called for the use of cognitively demanding tasks. When using such tasks it is not only important that a task start out as cognitively demanding, but also that high cognitive demand is maintained during the task enactment; unfraternally, cognitive demand is often not maintained (Stein & Lane, 1996). To better understand this phenomenon, Stein, Grover, & Henningsen (1996) identified seven factors that help maintain, and six factors that lower cognitive demand during task enactments (Fig. 1). These factors have been used by numerous researchers who looked to understand what the maintenance of cognitive demands looks like (e.g. Hong & Choi, 2018; Lunt, 2011). But, the studies that use these factors don't discuss *how* they use the factors. They don't explain if they were just looking for the existence of the factors, or if they were coding the degree to which factors were applicable. As such, there is no reliable way to measure or compare what the maintenance of cognitive demand looks like within and across studies. The Instructional Quality Assessment (IQA; Boston, 2012) is a tool that could be used to help us to do this, but the IQA takes all 13 factors related to the maintenance of cognitive demand, and puts them on a single rubric with a four point scale. As such, it does not provide much detail about what each of the individual factors that maintains or lowers cognitive demand looks like during a task enactment.

To address the need for a tool that can provide a detailed analysis of how cognitive demand is maintained during task enactments, I conceptualized the **R**eorganized Factors that **U**ndermine or **K**eep Cognitive Demand (RUK). Looking at each factor individually, I found that many of the factors that lower and maintain cognitive demand are similar, and can be considered two ends of a continuum. This is true for nearly every factor, as can be seen by the continuums of the RUK (Fig. 1). For each continuum, the RUK provides a four point scale (available by contacting the author) to aid in the quantification of the factors that lower and maintain cognitive demand. By viewing these factors as two ends of a continuum, the RUK provides an efficient way to create a detailed analysis of what the maintenance of cognitive demand looks like during a task enactments. Additionally, the RUK provides a medium that can be used in subsequent research to allow *how* cognitive demand is maintained to be compared across different studies.

Factors that that maintain cognitive demand	Factors that lower cognitive demand	Factors combined on the RUK continuum
1) Tasks were built on students' prior knowledge	1) Task is inappropriate for the students	To what extent were students prepared to engage with this task?
2) Tasks were of the appropriate amount of time	2) Students are given too much or too little time to work on a task	To what extent was the amount of time students were given to work on this task appropriate?
3) High-level performance was modeled	3) The focus of the tasks shifts to finding a correct answer	To what extent are solution strategies discussed and important mathematical ideas and concepts uncovered?
4) The teacher sustained pressure for explanation and meaning	4) Lack of accountability	To what extent were students held accountable for explaining their thinking/reasoning?
5) Tasks had proper scaffolding	5) Challenges become nonproblems	To what extent did the teacher or more capable peers give away solution strategies in an attempt to help other?
	6) Classroom management problems	To what extent do classroom management issues occur during this task?
6) Student self-monitoring		To what extent can students provide evidence for their claims or explain their thinking?
7) The teacher drew conceptual connections		To what extent did the teacher draw conceptual connections?

Figure 1: Factors that Maintain and Lower Cognitive Demand (Derived from the work of Stein, Grover, & Henningsen, 1996) and their relationship to the RUK.

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BUILDING A ROBOT: MAKING MATHEMATICS VISIBLE IN A NON-FORMAL STEM LEARNING ENVIRONMENT

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Keywords: Research Methods, STEM/STEAM, Informal Education

Emerging research has shown how student engagement with educational robots (e.g., Dash) can be associated with application and learning of mathematical concepts (e.g., Zhong & Xia, 2020). Yet, less is known about student learning in non-formal contexts (e.g., after-school programs) in contrast to formal learning environments (Pattison et al., 2016). The purpose of this proposal is to describe our varying, yet complementary analytical perspectives in understanding the complex nature of mathematical learning in a non-formal STEM environment. The project studied here – to design, build, and test an electronic vehicle – was not developed with an explicit mathematical goal(s) or objective(s). Thus, we intend to make the mathematical learning process visible through our emerging analytical perspective. Our proposed poster in the Theory and Research Methods strand will address a gap in the literature regarding how robotics can foster the application and learning of mathematics in a non-formal STEM learning environment (Karim et al., 2015), as well as address the PME-NA theme of looking across different cultures of mathematics and other disciplines (e.g., engineering). For context, data consisted of 24 days of audio/video recordings of two students, each equipped with a chest-mounted camera (i.e., GoPro) to capture their individual and collaborative points of view.

Our units of analysis are mathematical moments, or spontaneous experiences to engage with and explore mathematical concepts (Cunningham, 2015), that emerged as students collaborated in a non-formal STEM learning environment. To observe the growth of mathematical understanding in these moments, the Pirie-Kieren theory (1994) was used as an analytical tool. This theory perceives understanding as a dynamic, leveled but nonlinear, recursive process and describes eight levels of actions for mathematical understanding. By tracing the participants' growth of understanding along these potential levels, we can ascertain a global sense of the mathematical thinking and learning occurring during these moments. To complement the Pirie-Kieren theory and to capture details of the learning process, we apply additional frameworks to reveal perseverance pathways as the students navigated mathematical obstacles (DiNapoli, 2018) and their emotions as in-the-moment affective states (Middleton et al., 2017). We will detail our analysis of a mathematical moment through these three frameworks to make visible the learning that occurred in this non-formal STEM learning environment (e.g., the concept of variable through programming and engineering activity).

We argue that the overlay of multiple complementary perspectives on the same piece of data helps make visible the mathematics occurring in spaces where it may be hard to see otherwise, as well as provide triangulation of claims about student cognition and affect. This approach of aligning different views on an activity may be productive for other researchers as well. Ultimately, this work is a step towards understanding how non-formal environments can support mathematics learning and points of connection between non-formal and formal spaces.

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