Perceptual and Number Effects on Students' Solution Strategies in an Interactive Online Mathematics Game

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Abstract

We investigated the effects of proximal grouping of numbers, problem-solving goals to make 100, and prior knowledge on students' solution strategies in an online mathematics game. Logistic regression on 857 problem-level data points from 227 middle-school students showed that students were more likely to use productive solution strategies on addition and multiplication problems when proximity supported number grouping, 100 was the problem-solving goal, and students had high prior knowledge. Furthermore, when proximity and number goals did not support problem-solving, students with low prior knowledge were less likely to use productive solution strategies on multiplication problems than students with high prior knowledge. Findings demonstrate that the effects of perceptual and number features on solution strategies vary by students' prior knowledge.

Keywords
Mathematical solution strategies; Perceptual grouping; Individual differences; Mathematical structure

Introduction

Mathematics problems can often be solved with several different strategies. For example, the equation $3(2+x)=18$ can be solved with a three-step standard strategy (distribute 3 into the parentheses, subtract 6 from both sides, divide both sides by 3) or a two-step efficient strategy (divide both sides by 3, subtract 2 from both sides). Although efficient and flexible problem solving is a primary goal in mathematics education (NGA Center & CCSSO, 2010), students often apply standard procedures without noticing important patterns in problem structures that afford more efficient strategies (Carpenter et al., 1980; Star & Rittle-Johnson, 2008). Prior studies have demonstrated that perceptual and conceptual problem structures (Alibali et al., 2018; Geary et al., 2004; Landy & Goldstone 2010; Lemaire & Callies, 2009), and students' prior knowledge (Siegler, 1988), impact problem-solving performance. In this study, we examined the effects of proximal grouping of numbers, problem-solving goals of making 100, and students' prior knowledge on subsequent solution strategies (Figure 1) within a mathematical game in which students transform expressions to reach an equivalent goal (Figure 2).

Factors That May Affect Students' Solution Strategies

Proximal grouping of numbers. Empirical work suggests that algebraic reasoning is grounded in perceptual processes (Alibali & Nathan, 2012; Goldstone et al., 2010). For instance, students use *proximity* as a perceptual cue to group symbols (e.g., 2 + 3×4; Landy & Goldstone, 2007); and students solve expressions incorrectly when the symbols are spaced in a manner that is incongruent with the order of operations (e.g., 2+3 × 4; Braithwaite et al., 2016). Here, we operationalize proximal grouping as the location of numbers within the problem (e.g., 3+5+4 vs. 3+4+5) rather than the physical spacing of numbers and operations. Drawing on the Gestalt laws of perceptual organization (Hartmann, 1935), stimuli that are proximal in terms of spatial location are more likely to be grouped together than are stimuli that are distal. Thus, we hypothesize that students are more likely to make a productive first step towards the goal when the numbers to be added are adjacent to each other as opposed to far apart.

Problem-solving goals of making 100. The mathematical system in the United States has an underlying base-10 structure. Base-10 structure knowledge is a key predictor of students' mathematics performance (Geary, 2006; NCTM, 2000; NRC, 2001), and is related to advanced problem-solving strategies (Laski et al., 2014). Adults are faster and more accurate on addition problems summing to 10 (Aiken & Williams, 1973; Krueger & Hallford, 1984) and problems in which answers are multiples of 5 (Campbell, 1995). State mathematics standards tend to focus on students' facility in making 10 and 100 (NGA Center & CCSSO, 2010). Furthermore, curricular activities often focus on addition and subtraction with 10 or 100 (University of Chicago School Mathematics Project, 2014; Pearson Education, Inc, 2008). Given this focus on 10s and multiples of 10s, we anticipate that 100 will be easier to create combinations for as compared to other numbers and hypothesize that students will be more likely to use productive solution strategies if the goal is to make 100.

Prior knowledge. To use efficient solution strategies, students need to have knowledge of underlying concepts and procedures (Star & Rittle-Johnson, 2008). Middle-school students with higher mathematics achievement are more likely to solve algebraic equations using fewer steps compared to students with lower achievement (Newton et al., 2019). Furthermore, sixth graders were more likely to use more efficient strategies that involve fewer steps during algebraic equation solving after receiving instruction on multiple strategies, suggesting that mathematical knowledge affects strategy choices (Star & Rittle-Johnson, 2008). Here, we hypothesize that students with higher prior algebraic knowledge will be more likely to make a productive first step than students with lower prior algebraic knowledge.

The Present Study

The goal of this study was to understand how proximal grouping of numbers, problem solving goals of making 100, and prior knowledge affect students' solution strategies in an interactive online mathematics game. Our three research questions were:

1) Do proximal grouping, goals of making 100, and prior knowledge uniquely influence students' solution strategies?
2) Do these factors interact to affect students' solution strategies?
3) Are these effects consistent across addition and multiplication?

Method

Participants

A sample of 227 students from six middle schools in the Southern U.S. was included in analyses, drawn from a larger randomized control trial examining the efficacy of an online mathematical game. Of 227 students (56% male), most (96%) were in sixth grade, and the rest (4%) were in seventh grade. Over half of the sample was Asian (53%), followed by White (36%), Hispanic (4%), and other ethnicities (7%).

Materials

This study analyzed log data collected in an interactive online mathematics game where students explored algebraic notations by performing mouse or touch-based gestures to move symbols according to mathematical principles. On each problem, students saw two mathematical expressions—a start state, which is transformable, and a goal state, which is perceptually different but mathematically equivalent to the start state (Figure 2). The game objective is to transform the start state into the goal state.

Problem structure. For both addition and multiplication, we designed a quartet of problems in the game that varied on the proximal grouping of numbers in the start state and making 100 in the goal state (Table 1). For instance, transforming 44+56+a+72+28 into 100+a+100 involves proximal grouping in the start state and making 100 in the goal state (i.e., 44 and 56 are adjacent and make 100). Transforming 47+33+b+52+68 into 99+b+101 involves non-proximal grouping of numbers and making non-100 (i.e., 47 and 52 are far apart and make 99). Dummy variables were created for analyses (proximal grouping = 1; making 100 = 1).

Measures

Prior knowledge. Prior algebra knowledge was measured with 11 items, each scored as correct (1) or incorrect (0), selected from two previously validated measures (Rittle-Johnson et al., 2011; Star et al., 2014). An example item is 5(y-2)=-3(y-2)+4, solving for y.

Solution strategies. Solution strategies were measured by whether or not students made a productive mathematical transformation to reach the goal state. We coded students' first mathematical transformation (i.e., "first step") to measure their productivity because their first step impacts the subsequent steps in reaching the goal state. Specifically, a first step was productive if it moved students closer to the goal state of the problem. For example, transforming the start state "47+33+b+52+68" into "99+33+b+68" is productive because the student moved closer to the goal state by combining 47 and 52 to make 99 (Table 2). However, transforming the start state into " 80+b+52+68" by combining 47 and 33 is not productive because "80" is not related to any numbers in the goal state. We coded first steps as productive (1) or non-productive (0). The intraclass correlation coefficient of the coding was .92, indicating excellent reliability.

Data Analysis Plan

We performed binary logistic regression analyses because the outcome (productivity of solution strategies) was binary. We considered hierarchical binary logistic regression modeling because problem-level data were nested within student-level data. However, the result of the unconditional models showed that there were no significant variations among students for both addition and multiplication problems. Thus, we conducted one-level binary logistic regression analyses using problem-level data. The number of problem-level data points included was 857 for addition and 871 for multiplication problems. We used IBM SPSS Statistics 25 for analyses.

Results

Before performing data analyses, we computed frequencies for students' solution strategies for each problem (Table 3; note that not all students completed each problem).

Main Effects for Addition Problems

Model 1.1 with proximal grouping, making 100, and prior knowledge predicting the probability of making a productive first step on addition problems indicated that the three predictors were significantly associated with students' productivity, and explained 35.7% of the variance in the probability of making a productive first step (Table 4). The log odds of making a productive first step was positively related to proximal grouping (B = 3.163, $p < .001$), making 100 (B = 2.363, $p < .001$), and prior knowledge (B = .127, $p = .017$). Students were more likely to make a productive mathematical transformation when numbers to be combined were adjacent to each other, the goal was to make 100, and when the students had higher prior knowledge.

Interaction Effects for Addition Problems

Next, we tested an interaction term of proximal grouping by making 100 (Model 1.2 in Table 4), as well as interaction terms of proximal grouping by prior knowledge and making 100 by prior knowledge (Model 1.3 in Table 5). The results indicated no statistically significant interactions between these variables. Finally, we tested a three-way interaction among the three predictors (Model 1.4 in Table 5), and this interaction was also not statistically significant (Figure 3).

Main Effects and Interactions for Multiplication Problems

We repeated the analyses for multiplication problems to test if these effects replicate across operations. Model 2.1 containing three predictors as main effects explained 26.9% of the variance in the outcome variable (Table 6). The log odds of making a productive first step was positively related to proximal grouping ($B = 2.436$, $p < .001$), making 100 ($B = 1.148$, $p < .001$), and prior knowledge ($B = .139$, $p = .001$).

For Model 2.2 (Table 6), the interaction term (proximal grouping by making 100) was added, and the model explained 28.1% of the variance in the probability of making a productive first step. Students were significantly less likely to make a productive first step when the proximal grouping *and* number goals did not support problem solving compared to the other three types of problems. Next, we added two interaction terms with prior knowledge to the model (Model 2.3 in Table 7). The making 100 by prior knowledge interaction was significant ($B = -.265$, $p = .003$), whereas the proximal grouping by prior knowledge interaction was not significant. Students with low prior knowledge were significantly less likely to make a productive first step when the number goal was non-100, whereas the effect of number goal was not significant for students with high prior knowledge. Finally, we tested a three-way interaction among proximal grouping, making 100, and prior knowledge (Model 2.4 in Table 7). The result showed a significant three-way interaction ($B = -.239$, $p = .001$). Students with low prior knowledge were less likely to make a productive first step when the numbers to be combined were not adjacent and the number goal was to make non-100 compared to students with high prior knowledge (Figure 3).

Discussion

This study examined the effects of perceptual (proximal grouping of numbers) and conceptual (problem-solving goals to make 100) problem structures, and prior knowledge on middle-school students' solution strategies within an interactive online mathematics game. For both addition and multiplication problems, students were more likely to make a productive mathematical transformation when numbers to combine were adjacent, the goal was to make

100, and when the students had a higher level of prior knowledge. The three-way interaction among proximal grouping, making 100, and prior knowledge was only significant for multiplication problems. Specifically, students with low prior knowledge were less likely to make a productive first step when the proximal grouping and number goals did not support problem solving compared to students with high prior knowledge.

Although this study only examined students' first mathematical transformation as a measure of productive problem solving, the findings revealed the ways in which perceptual features and numbers influence students' solution strategies. Future studies could investigate these effects with other measures of solution strategies (e.g., number of steps made, sequence of transformations), and the relations between problem structure and other student behaviors in the game, such as pause time before solving.

These results suggest it may be helpful to teach students to notice important patterns in problem structures and build upon their familiarity with 100 to use it as an anchor for decomposition in multi-digit problem solving, which may promote uses of more efficient solution strategies. Overall, results extend past work demonstrating the effects of perceptual features and numbers within experimental settings into a digital context.

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Table 1

The Design of Problem Structure

Making 100	Proximal Grouping: Yes (1)	Proximal Grouping: No (0)
Yes (1)	P10: (S) 44+56+a+72+28→ (G) 100+a+100 P24: (S) 25*4*b*50*2 → (G) 100*b*100	P7: (S) 11+55+y+89+45 → (G) 100+y+100 P32: (S) 10*20*a*10*5 → (G) a*100*100
No (0)	P13: (S) 15+87+c+62+32 → (G) 102+c+98 P30: (S) 8*12*d*3*32 → (G) 96*d*96	P14: (S) 47+33+b+52+68 → (G) 99+b+101 P26: (S) 4*6*c*24*16 → (G) 96*96*c

Note. S = Start state. G = Goal state.

Table 2

Examples of Productivity of Students' Solution Strategies

Start state	Goal state	Productive first steps	Non-productive first steps
47+33+b+52+68	99+b+101	• 99+33+b+68 • 47+b+52+101 • 33+b+99+68 • 47+101+b+52 • 47+52+33+b+68[a] • 47+b+52+33+68[a]	• 80+b+52+68 • 47+33+b+120

[a] Transformations that involved commuting (i.e., moving numbers to be added closer together) were considered productive in bringing the student closer to the goal state.

Table 3

List of Problems by Problem Structure and Frequencies of Students' Solution Strategies

Problem structure	Number (%) of students with a productive first step	Number (%) of students with a non productive first step
Problem 10 (*n* = 219): S: 44+56+a+72+28 G: 100+a+100		
Problem 7 (*n* = 213): S:		

11+55+y+89+45 G: 100+y+100	Making 100	(95.8%) 9 (4.2%) 217 (98.2%)
Problem 13 (*n* = 221): S: 15+87+c+66+32 G: 102+c+98	Proximal Grouping, Making non-100	4 (1.8%) 137 (67.2%) 67
Problem 14 (*n* = 204): S: 47+33+b+52+68 G: 99+b+101	Non-Proximal Grouping, Making non-100	(32.8%)
Problem 24 (*n* = 217): S: 25*4*b*50*2 G: 100*b*100	Proximal Grouping, Making 100	209 (96.3%) 8 (3.7%) 181
Problem 32 (*n* = 211): S: 10*20*a*10*5 G: a*100*100	Non-Proximal Grouping, Making 100	(85.5%) 30 (14.2%) 215
Problem 30 (*n* = 222): S: 8*12*d*3*32 G: 96*d*96	Proximal Grouping, Making non-100	(96.8%) 7 (3.2%) 134 (60.6%)
Problem 26 (*n* = 221): S: 4*6*c*24*16 G: 96*96*c Proximal Grouping, Making 100	Non-Proximal Grouping, Making non-100 218 (99.5%) 1 (0.5%) 204	87 (39.4%)

Non-Proximal Grouping,

Note. S = Start state. G = Goal state.

Table 4

The Main Effects of Three Predictors and the Interaction Effect (Proximal Grouping × Making 100) on Students' Solution Strategies for Addition Problems (*N* = 857 problems)

Predictors Model 1.1 Model 1.2 B SE Exp(B) *p* B SE Exp(B) *p*

Intercept -.082 .366 .921 .823 -.099 .367 .906 .788

Proximal grouping

(ref.: non-proximal grouping) 3.163 .474 23.646*** .000 3.311 .528 27.402*** .000

Making 100

(ref.: making non-100) 2.363 .356 10.622*** .000 2.440 .374 11.469*** .000

Prior knowledge .127 .053 1.135* .017 .127 .053 1.136* .017

Proximal grouping ×

Making 100 - - - - -1.039 1.183 .354 .380

$R^2_{Nagelkerke}$ =.359

Model Statistics $R^2_{Nagelkerke}$ =.357 $\chi^2(3) = 155.721$, $\chi^2(4) = 156.365$, $p < .001$
$p < .001$

$^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$

Table 5

The Interaction Effects of Proximal Grouping, Making 100, and Prior Knowledge on Students' Solution Strategies for Addition Problems (N = 857 problems)

Predictors	Model 1.3				Model 1.4			
	B	SE	Exp(B)	*p*	B	SE	Exp(B)	*p*
Intercept	-.573	.417	.564	.170	-.159	.372	.853	.669
Proximal grouping (ref.: non-proximal grouping)	5.890	1.666	361.535***	.000	3.469	.551	32.099***	.000
Making 100 (ref.: making non-100)	3.964	1.036	52.664***	.000	2.518	.380	12.403***	.000
Prior knowledge	.205	.063	1.227**	.001	.135	.054	1.144*	.013
Proximal grouping × Prior knowledge	-.403	.208	.668	.053	-	-	-	-
Making 100 × Prior knowledge	-.252	.143	.777	.079	-	-	-	-
Proximal grouping × Making 100 × Prior knowledge	-	-	-	-	-.261	.135	.770	.053

$^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$

Model Statistics	$R^2_{Nagelkerke} = .372$ ($\chi^2(5) = 162.704$, $p < .001$)	$R^2_{Nagelkerke} = .363$ ($\chi^2(4) = 158.409$, $p < .001$)

Table 6

The Main Effects of Three Predictors and the Interaction Effect (Proximal Grouping × Making 100) on Students' Solution Strategies for Multiplication Problems (*N* = 871 problems)

Predictors	Model 2.1				Model 2.2			
	B	SE	Exp(B)	*p*	B	SE	Exp(B)	*p*
Intercept	-.368	.295	.692	.212	-.476	.300	.621	.113
Proximal grouping (ref.: non-proximal grouping)	2.436	.289	11.433***	.000	3.032	.410	20.743***	.000
Making 100 (ref.: making non-100)	1.148	.220	3.153***	.000	1.399	.244	4.050***	.000
Prior knowledge	.139	.042	1.149**	.001	.142	.042	1.153**	.001
Proximal grouping × Making 100	-	-	-	-	-1.571	.581	.208**	.007
Model Statistics	$R^2_{Nagelkerke} = .269$ ($\chi^2(3) = 145.984$, $p < .001$)				$R^2_{Nagelkerke} = .281$ ($\chi^2(4) = 153.106$, $p < .001$)			

$^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$

Table 7

The Interaction Effects of Proximal Grouping, Making 100, and Prior Knowledge on Students' Solution Strategies for Multiplication Problems (*N* = 871 problems)

Predictors	Model 2.3				Model 2.4			
	B	SE	Exp(B)	*p*	B	SE	Exp(B)	*p*
Intercept	-1.134	.391	.322**	.004	-.596	.310	.551	.055
Proximal grouping								

(ref.: non-proximal grouping) 3.506 .793 33.328*** .000 3.014 .386 20.372*** .000

Making 100

(ref.: making non-100) 2.813 .620 16.654*** .000 1.397 .240 4.045*** .000 Prior knowledge .261 .059

1.298*** .000 .162 .044 1.176*** .000

Proximal grouping ×

Prior knowledge -.168 .115 .846 .144 - - - -

Making 100 ×

Prior knowledge -.265 .090 .767** .003 - - - - Proximal grouping ×

Making 100 × Prior knowledge

- - - - -.239 .075 .787** .001

*$p < .05$, **$p < .01$, ***$p < .001$

$R^2_{Nagelkerke} = .284$

($\chi^2(4) = 154.942$, $p < .001$)

Model Statistics $R^2_{Nagelkerke} = .287$ ($\chi^2(5) = 156.297$, $p < .001$)

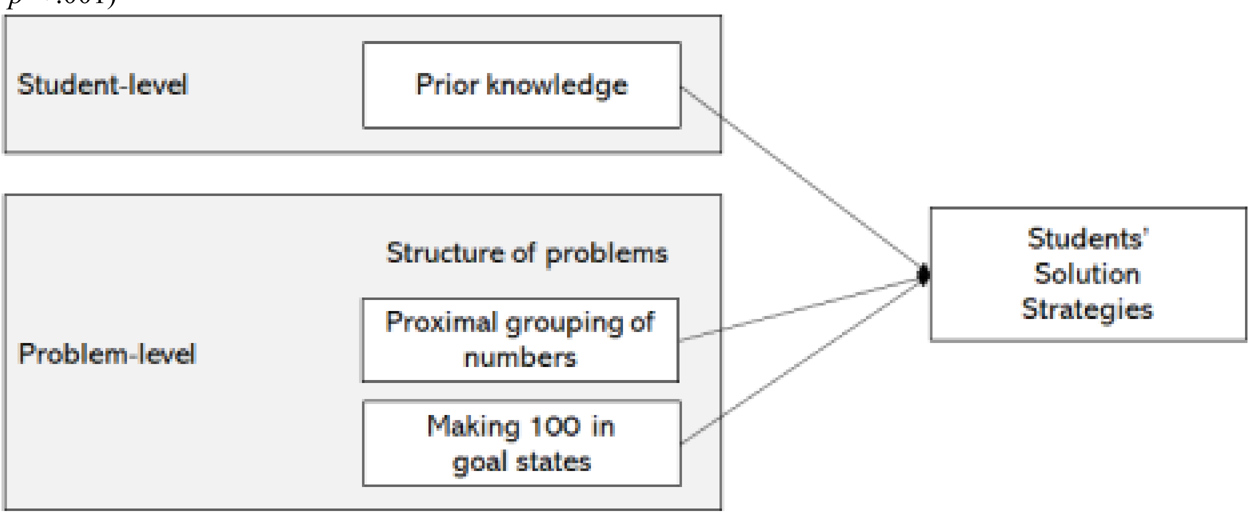

Figure 1. Logic model linking student-level and problem-level predictors to the outcome variable of interest (students' solution strategies), for both addition and multiplication problems.

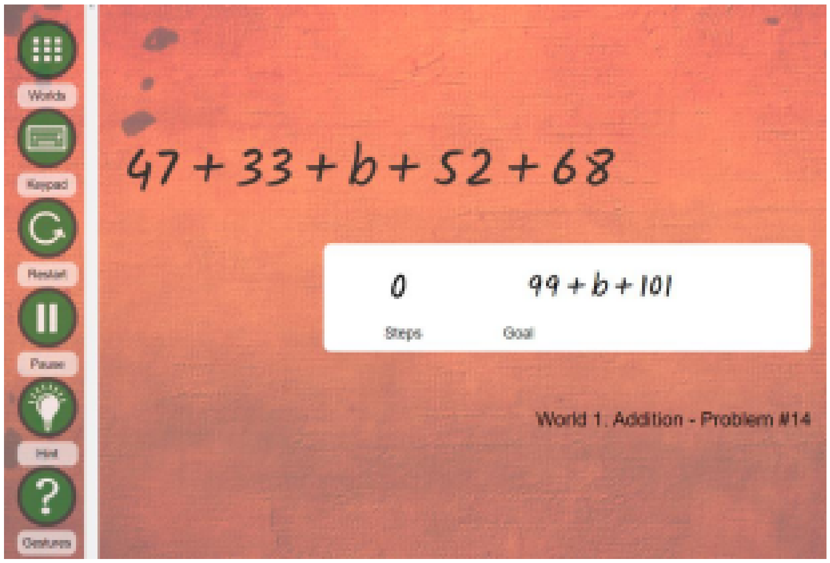

Figure 2. An example of a problem consisting of a start state (47+33+b+52+68) and a goal state (99+b+101)

Figure 3. Interaction Plots for Productivity of First Steps (Note that students were divided into two groups by a mean-split of the prior knowledge score(M=7))