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Show the Flow: Visualizing Students' Problem-Solving Processes in a Dynamic Algebraic Notation Tool

JI-EUN LEE
Worcester Polytechnic Institute
jlee13@wpi.edu

ARAVIND STALIN
Worcester Polytechnic Institute
astalin@wpi.edu

VY NGO
Worcester Polytechnic Institute
vngo@wpi.edu

KATIE DRZEWIECKI
Worcester Polytechnic Institute
kcdrzewiecki@wpi.edu

CINDY TRAC
Worcester Polytechnic Institute
ctrac@wpi.edu

ERIN OTTMAR
Worcester Polytechnic Institute
erottmar@wpi.edu

New educational technologies that utilize students' interaction data and visualizations provide a means to expand our understanding of learning processes. In this study, we apply two advanced and novel data visualization techniques, called the *Indivisualizer* and a *Sankey diagram*, to explore how middle school students (N = 343) solved problems in a game-based dynamic algebraic notation tool. Specifically, this study 1) illustrates the application of a set of data visualization techniques using clickstream data collected from the tool and explores the detailed step by step information about students' algebraic problem-solving processes that these visualizations provide, and 2) examines the associations between the productivity of initial solution strategies, prior knowledge, and efficiency of problem-solving. The results indicate a large variation in the types of strategies that students use to solve the problems, with some approaches being more efficient than others. Moreover, the productivity of initial solution strategies and prior knowledge significantly predict the efficiency scores in the game, indicating that noticing the structure of the equations plays a significant role in problem-solving. The findings suggest that these visualizations can be used both in research and practice to reveal and unpack our understanding of variability in mathematical problem-solving strategies and cognition.

Keywords

Mathematics learning; Problem-solving strategies; Data visualization; Sankey diagram

INTRODUCTION

Teachers' knowledge of their students' learning progress and problem-solving processes is one of the important factors contributing to classroom practices, and ultimately, what students learn in mathematics (Asquith et al., 2007). However, teachers often have difficulties monitoring students' learning progress and identifying their learning patterns or misconceptions (Asquith et al., 2007; Sadler et al., 2013). In response to these issues, advances in educational technologies and data analytics have brought opportunities to explore and analyze students' learning progress and processes at a more fine-grained level (Bienkowski et al., 2012). In particular, *data visualizations* (also called visual data analytics), which is a method of discovering and understanding users' patterns in large datasets through visual representation, have enabled teachers and researchers to identify complex data on student learning in an easier and faster way (Caprotti, 2017; Ganley & Hart, 2017; Vieira et al., 2018).

However, despite extensive interest in data visualizations in educational research, little work has employed advanced data visualization techniques on students' mathematical learning processes (Papamitsiou & Economides, 2014; Vieira et al., 2018). Moreover, few studies have made strong connections between findings of data visualizations with other types of data, such as students' performance, and further, learning theories. Part of this gap is a result of a lack of educational technologies that record moment-by-moment mathematical derivations or afford opportunities to explore students' mathematical ideas.

Over the past several years, Weitnauer, Landy, and Ottmar (2016) have designed and developed a new dynamic algebraic notation tool, Graspable Math (GM, https://graspablemath.com), which helps students develop their conceptual and procedural learning in algebra (Chan et al., 2022; Hulse et al., 2019; Ottmar et al., 2015). The tool allows students to dynamically manipulate and transform numbers, symbols, and mathematical expressions using various touch or mouse-based gesture-actions. Using the data collected in GM, students' problem-solving processes can be represented as a series of time-based steps that form the mathematical derivation as they transform mathematical expressions and equations. Thus, in the present study, we present visualizations of students' algebraic problem-solving processes and solution strategies in GM using two advanced and novel data visualization techniques, called a *Sankey diagram* and *Indivisualizer*, and investigate how we can use these visualizations to provide meaningful and comprehensive information to researchers and teachers. Specifically, the aims of this study are to 1) examine variation in individual students' algebraic problem-solving processes using data visualization, 2) demonstrate the entire students' overall algebraic problem-solving processes through data visualization, and 3) visualize students' productivity of initial solution strategies and examine whether the productivity of initial solution strategies influences the efficiency of algebraic problem-solving.

LITERATURE REVIEW

Students' algebraic problem-solving and efficiency of strategies

Understanding and evaluating multiple solution strategies (here, a solution strategy refers to a way to solve the problem), then selecting an efficient one are core competencies in algebra (Lynch & Star, 2014; Star & Rittle-Johnson, 2008). For instance, to solve the equation $4(x-1) = 36$, a student could start solving the problem using a conventional approach by distribution (i.e., $4x-4 = 36$) or a non-conventional method by dividing both sides by 4 (i.e., $x-1 = 9$). Both strategies may lead to a correct answer but selecting the most efficient strategy that involves fewer steps and computations (e.g., dividing both sides by 4) would lead to solving the problem more quickly and accurately. Moreover, a number of studies have reported that a broad knowledge about solution strategies was positively associated with students' gains in conceptual knowledge, procedural knowledge, flexibility, as well as further learning in mathematics (Heinze et al., 2009; Rittle-Johnson & Star, 2007).

Despite its importance, students often struggle with the adaptive use of efficient solution strategies in algebra (Star et al., 2015). Many students tend to implement one strategy, for example, left-to-right problem-solving procedures, rather than applying the most efficient solution strategy after evaluating multiple possible strategies based on their conceptual knowledge (Robinson & Dubé, 2013; Siegler & Araya, 2005).

Several studies have examined the student or teacher-related factors that influence students' low proficiency in algebraic problem-solving. First, they have found that the strategies that students use in their solution processes are influenced by their understanding of two core algebraic concepts, equivalence and variables (Bush & Karp, 2013; Knuth et al., 2005). However, many middle school students have misconceptions about these concepts; particularly, students tend to hold an operational view of the equal sign (i.e., the equals sign indicates computing) rather than as a symbol indicating an equivalence, which leads to difficulty in recognizing the underlying structure or important features of equations (Stephens et al., 2013). Another critical factor that contributes to students' efficiency of problem-solving is their prior knowledge of algebraic methods (Khng & Lee, 2009; Rittle-Johnson et al., 2009). For instance, one experimental study (Rittle-Johnson et al., 2009) found that students who had high prior knowledge were more likely to learn algebra equation solving by comparing different solution strategies than students with low prior knowledge.

Moreover, while it is important for teachers to grasp students' understanding of algebraic concepts or problem-solving processes, they often have difficulty identifying students' strategies, misconceptions, or obstacles to solving problems (Asquith et al., 2007). In terms of research perspective, many of the methodological approaches that have been used, such as analyzing those processes or strategies on paper by hand-coding or think-aloud method (i.e., conducting one-on-one interviews to ask students to explain their thinking), have notable limitations when scaling, including the time and cost of coding students' work.

Part of these challenges are due to a lack of efficient technology tools or methods to record, code, and visualize student problem-solving processes or strategies. New innovative educational technology tools that automatically log student behaviors and problem-solving processes could provide a means to more efficiently monitor and research students' mathematical strategies at scale, leading to a more robust understanding of students' mathematical understanding and flexibility. Providing information on the process of students' use of strategies for a given mathematical task through these technology tools would help teachers and researchers perceive students' general mathematical thinking processes as well as their individual differences, in turn, enable them to support students' learning accordingly (Lynch & Star, 2014).

Expanding data visualizations for education research using Sankey diagrams

Data visualizations are commonly used in education research to help teachers or researchers monitor students' learning or performance (Caprotti, 2017; Ganley & Hart, 2017; Vieira et al., 2018). However, many of these visualizations tend to focus only on displaying the correctness of students' answers, usage patterns, or collaborative behaviors, rather than students' learning paths or problem-solving processes (Park & Jo, 2015; Vieira et al., 2018).

Moreover, while understanding students' problem-solving processes is critical to support their learning (Pape & Smith, 2002), limited work has used data visualizations to demonstrate how problem-solving strategies vary across several students. For instance, one study (Liu et al., 2017) investigated students' problem-solving processes (e.g., debugging activities) using the data collected in a web-based game for computational thinking. They classified students' moves (productive vs. unproductive) in the game and mapped them to conceptual and problem-solving skills. However, the study did not use any means to visualize the variation in students' problem-solving behaviors. Another study (Ruipérez-Valiente et al., 2015) used several data visualizations (e.g., heatmap, line graph) to explore individual students' skill progress in physics, chemistry, and mathematics courses in the Khan Academy platform, but it did not provide a holistic visualization of students' overall skill progresses.

Visualizing individual students' problem-solving strategies could provide a means for informing our understanding of students' mathematical knowledge and provide insight into where and how students make decisions. Beyond individual variability, it is also critical to gain a larger understanding of how a large group of students solved problems. For example, at a practical level, when researching the mathematical strategies that students use when solving equations, it would be beneficial to know what the most common initial approaches to solving a problem are. However, for large amounts of students, without data visualizations that aggregate and present this information at once, this task becomes extremely laborious.

To simultaneously visualize several students' overall problem-solving processes as well as the variability across the students, *Sankey diagrams* can be used. A Sankey diagram is a type of flow diagram which depicts a flow and its quantities (or frequencies) from one set of values to another using the width of lines (Riehmann et al., 2005). This type of diagram enables us to visually identify different paths involved, as it depicts all the existing paths in a large process (Tiwari, 2017). Moreover, they can clearly display the proportions and variety of paths within an event, allowing users to quickly identify both dominant and minority pathways (Lee & Tan, 2017).

A number of studies have utilized Sankey diagrams in education research, in particular, to visualize college students' academic pathways, such as change of major, completion, or dropout (Askinadze et al., 2019; Basavaraj et al., 2018; Heileman et al., 2015; Horvath et al., 2018; Morse, 2014; Oran et al., 2019). For example, one study (Lee & Tan, 2017) used Sankey diagrams to visualize graduate students' idea development and flow within the discourse on an online discussion tool. Another study (Wang et al., 2017) created Sankey diagrams to model the paths undergraduate students took when solving a Python assignment. However, some studies also noted the limitations and challenges of implementing Sankey diagrams (Askinadze et al., 2019; Wang et al., 2017). Due to cognitive load, it could be challenging to visualize a process with a large number of paths and nodes clearly and intuitively. Another important limitation is that users need to interpret the diagrams to find useful patterns; thus, the interpretation and the effective use of visualizations might greatly vary between users.

Despite some challenges, data visualizations, in particular, Sankey diagrams, allow researchers and educators to examine paths and variability in data, quickly identify areas of interest, and drill down into specific areas. However, limited work has implemented advanced data visualization techniques to explore students' problem-solving processes or solution strategies in mathematics learning contexts (Vieira et al., 2018). Thus, in the study described here, we present two new ways to visualize both individual students' and entire students' algebraic problem-solving processes and use of different solution strategies (individual visualizations and Sankey diagrams) using log data from a dynamic instructional technology, GM, and examine if our findings support the results of previous studies.

METHODS

Sample

Our sample (N = 343) was drawn from a randomized controlled study conducted in 2019 that examined the efficacy of a gamified version of GM (Chan et al., 2022). The initial sample consisted of 355 students from six middle schools located in the Southern U.S. who were assigned to the GM condition; however, we excluded 12 students who did not attempt the selected two problems for the study. Of the 343 students included for further analyses (54% male, 43% female, 3% not reported), most students (96%) were in sixth grade, and the remaining students (4%) were in seventh grade (who were generally 11 to 13 years old). In terms of instruction-level, 85% of the students were in advanced math classes, 8% were in on-level classes, and 7% were in support classes.

From Here to There!: A Gamified Version of GM

From Here to There! (FH2T, https://graspablemath.com/projects/fh2t) is a gamified version of the GM tool that was developed based on several learning theories (perceptual learning, embodied cognition, gamification) to improve students' conceptual understanding, procedural learning, and flexibility in algebra (Chan et al., 2022; Hulse et al., 2019; Ottmar et al., 2015). In this game, math symbols are reified as movable physical objects so that students can dynamically manipulate and transform numbers or mathematical expressions on the screen using various gestures.

The goal of the game is to transform an expression into the mathematically equivalent target goal using algebraically permissible actions. More specifically, each problem in the game consists of two mathematically equivalent mathematical expressions, a start state (e.g., 9×4) and a goal state (e.g., 3×6×2) (See Figure 1). Students must transform the starting expression into the target goal state using gesture-actions, such as moving, tapping, splitting, or decomposing numbers or expressions. The students receive three clovers if they solve the problem in the most efficient way (i.e., with the minimum required number of steps to reach the goal state), and the number of clovers is deducted if they exceed the fewest steps possible.

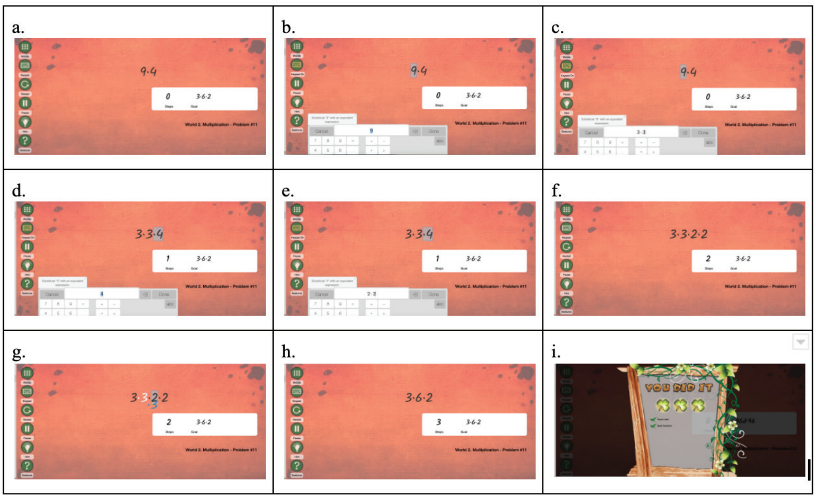

Figure 1. A sample problem (problem A) and students' actions in the gamified version of GM, From Here to There!.

FH2T includes 14 worlds (a total of 252 problems) that cover a variety of mathematical concepts, such as addition, multiplication, and the order of operations, with gradually increasing difficulty. Among the 252 problems, this study focuses on two problems in World 2-Multiplication, problem A (start state: 9×4, goal state: 3×6×2) and problem B (start state: 6×10, goal state: 2×15×2). These two problems were selected for two reasons. First, these two problems were intentionally designed to compare the effect of problem structure (i.e., factoring square numbers vs. factoring non-square numbers) on students' solution strategies. Second, there was greater variability in students' problem-solving processes (e.g., number of steps) on these two problems compared to other problems in that concept.

Research procedure

The students took a pretest prior to playing the game to measure their prior knowledge of algebra. After completing the pretest, they played the game individually using their own devices for four 30-minutes sessions (over four weeks) during their regular math classes. The students played the game at their own pace so that they completed a different number of problems in the game when the study ended.

As students solved problems in FH2T, the GM system automatically recorded detailed log files of all students' clickstream data with timestamps, such as the number of attempts, the number of steps, and all mathematical expressions and touch- or mouse-based recordings of students' actions for each problem. We pre-processed the data retrieved from the GM database and created individual and aggregated visualizations to identify the variation in students' solution strategies. Then, we hand-coded the productivity of each problem-solving approach using the information that the visualizations provided. The data pre-processing and visualization processes are described in more detail in the following section. Finally, we examined whether or not the productivity of first steps and prior knowledge predicted the efficiency of problem-solving.

Measures

Prior knowledge

Students' prior knowledge was measured with 11 items adopted from two previously validated measures (Rittle-Johnson et al., 2011; Star et al., 2014). It consisted of four sub-constructs: conceptual knowledge (4 items), procedural knowledge (3 items), flexibility (2 items), and mathematical equivalence (2 items). Each item was scored as correct (1) or incorrect (0). An example item (procedural knowledge) is "Solve the equation for n, 12n + 3 = 14n + 15 – 8n". The Kuder–Richardson 20 coefficient for 11 items was .68, indicating an acceptable level of reliability.

Productivity of initial solution strategy

Using the Indivisualizers and Sankey diagrams (described below), we coded each initial pathway for productivity using the information the visualization provided. Here, productivity refers to whether or not a student made an appropriate mathematical transformation; in other words, an action that brings the student closer to the target goal of the problem in the game. Among the transformations, we used students' first mathematical transformation (i.e., first step) to measure the productivity of their initial solution strategies, as we hypothesized that it would impact their subsequent transformations as well as the overall efficiency of problem-solving (i.e., solving a problem with fewer steps and computations) in the game. Table 1 lists examples of productive and non-productive first steps for problem A and problem B.

Table 1
Examples of productivity of students' initial solution strategies

Problem	Problem type	Productive first steps (1)	Non-productive first steps (0)
Problem A Start state: 9×4 Goal state: 3×6×2	Factoring square numbers	• 3×3×4 • 9×2×2	• 36 • 4×9 • (3+6)×4
Problem B Start state: 6×10 Goal state: 2×15×2	Factoring non-square numbers	• 2×3×10 • 3×2×10 • 6×2×5 • 6×5×2	• 60 • (5+1)×10 • 6×(5+5)

For instance, for problem A, we coded the transforming "9×4" into "3×3×4" as a productive first step because the student decomposed 9 to make 3, which is the number in the goal state of the problem (3×6×2). Contrary to this, transforming the start state into 36 was coded as a non-productive first step because this action did not bring the student closer to the target goal of the problem. We hand-coded productivity as productive (1) or non-productive (0). The intraclass correlation coefficient of the coding for these two problems was .92, indicating excellent reliability. These productivity codes were then applied to the larger data set.

Efficiency of problem-solving

The efficiency of problem-solving is an indicator of performance in the game and refers to how efficiently a student solves a problem. We computed the efficiency scores by dividing the fewest steps possible to reach the goal state for each problem by the number of steps made by the students for that problem. Thus, higher scores indicate more efficient problem-solving that involves fewer steps and computations.

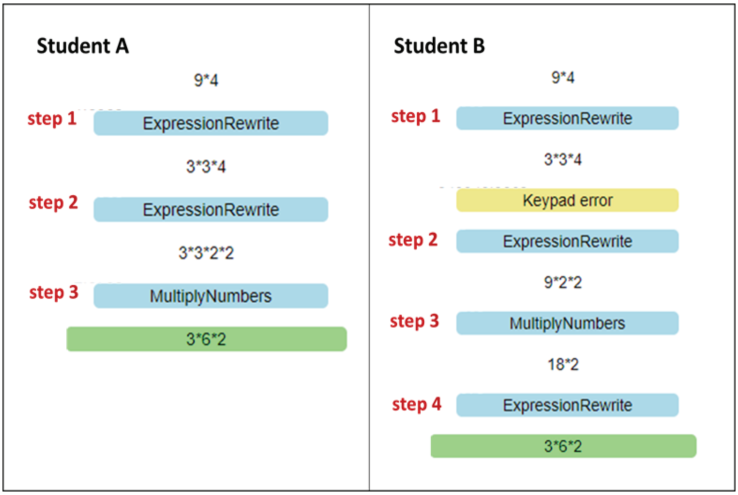

Figure 2. Examples of individual student's problem-solving process.

For example (See Figure 2), if a student reaches the goal state using three steps (Student A in Figure 2), the efficiency score for this student is equal to 1 (i.e., 3÷3). On the other hand, for a student who solved the problem using four steps (Student B in Figure 2), the efficiency score is equal to 0.75 (i.e., 3÷4). Finally, Table 2 summarizes the variables, operational definitions of the variables used in the study, and how we measure them.

Table 2
Summary of the variables and measures included in the study

Variables		Operational definitions	Measures
Predictors	Productivity	Whether or not student made an appropriate mathematical transformation towards the goal state on their first steps	Productive step = 1, Non-productive step = 0
	Prior knowledge	Students' knowledge of algebra before the intervention	The sum of the correctness on 11 items (correct = 1, incorrect = 0), Ranges between 0 and 11
Outcome variable	Efficiency scores- in-game performance	How efficiently a student solves a problem (i.e., using a procedure that involves fewer steps and fewer computations)	The fewest steps possible to solve the problem / the number of steps made Ranges between 0 and 1

Data visualizations and analyses

In order to examine variation in individual students' problem-solving processes in the game, we created individual visualizations for each student (called "*Indivisualizers*") (Manzo, 2020). Indivisualizers are semi-automatically created using HTML and implemented into a web-based dashboard (http://fh2tresearch.com). The Indivisualizer displays a student's expression transformation process between the start state and the goal state in FH2T. The metrics included in the Indivisualizer consist of the time to each action (in seconds), each state in the derivation, where errors and resets occurred, and the mathematical and gesture-actions used to initiate each step.

More specifically, each column shows the student's steps until they reset the problem or reach the goal state, and the colors of the boxes represent different events. If the student successfully reached the goal state, the last column represents the final try that resulted in success with the goal state in a pale green box. The light blue box represents the student's actions leading to the transformation, and the numbers above the light blue box indicate the time (in seconds) taken between transformations. The red box indicates that the student hit the reset button to retry the problem. Lastly, the yellow box represents the errors made by the student, which include: shaking error (i.e., performing a mathematically invalid operation), snapping error (i.e., dragging a number does not lead to a valid transformation), and keypad error (i.e., entering an expression on the keypad that is not equivalent to the expression). The mathematical and gesture-actions used to initiate each step can be inferred by comparing consecutive problem states.

While Indivisualizers provide a means to visualize individual students' detailed step-by-step information about their problem-solving processes, it does not provide a holistic view of entire students' overall problem-solving processes. Thus, we created Sankey diagrams in order to visualize students' overall algebraic problem-solving processes in the game. The data pre-processing to create Sankey diagrams was performed in the following steps: 1) select related features (e.g., student id, trial id, expr_ascii (i.e., the whole math expression on the screen in ASCII code)) from the raw data retrieved from the GM database, 2) remove students who did not attempt the selected two problems for this study, 3) select data on first attempts if a student attempted the problem more than once. To generate the Sankey diagrams, we used "plotly.js," which is one of the Javascript data visualization libraries. We considered various tools to create Sankey diagrams (e.g., SAS Visual Analytics, Tableau) and selected Javascript because of its greater flexibility in comparison with other tools. Sankey diagrams consist of two main components: *nodes* and *links*. In our study, each node represents the steps (i.e., transformations of equations) made by the students in the game, and the thickness of a link (i.e., paths in the diagram) indicates the number of students who made that mathematical transformation.

Lastly, to explore how the productivity of the initial solution strategies influences the efficiency of problem-solving, we created colored Sankey diagrams using the productivity codes (blue=productive, red=not productive). We also performed simple and multiple regression analyses to examine the association between prior knowledge, productivity, and efficiency using IBM SPSS Statistics 25.

RESULTS

Visualizations of Individual Students' Problem-Solving Processes

First, we created Indivisualizers on the web-based dashboard to trace an individual student's problem-solving process. Figure 3 shows two examples of Indivisualizers depicting individual students' problem-solving processes for problem A (square-numbers problem). As noted earlier, each column shows the student's steps until they reset the problem or reach the goal state, and the colors of the boxes represent different events. As shown in Figure 3, student A made a productive first step (i.e., 3*3*4) that brought the student closer to the target goal state (i.e., 3*6*2). This student successfully solved the problem using the minimum required steps to complete the problem (3 steps with an efficiency score of 1 [i.e., 3÷3]) without making any errors. In contrast, student B took a non-productive first step by factoring 4 in the start state in the wrong way (i.e., 9*(2+2) instead of 9*4*4) and made several errors (keypad error, shaking error, snapping error). The student then reset the problem to the initial state and made a productive first step (9*2*2) in the second trial. The student solved the problem using seven steps in total, with an efficiency score of 0.43 (i.e., 3÷7). As such, Indivisualizers allow users to easily compare the changes in individual students' solution strategies across multiple consecutive tries as well as variability across different students.

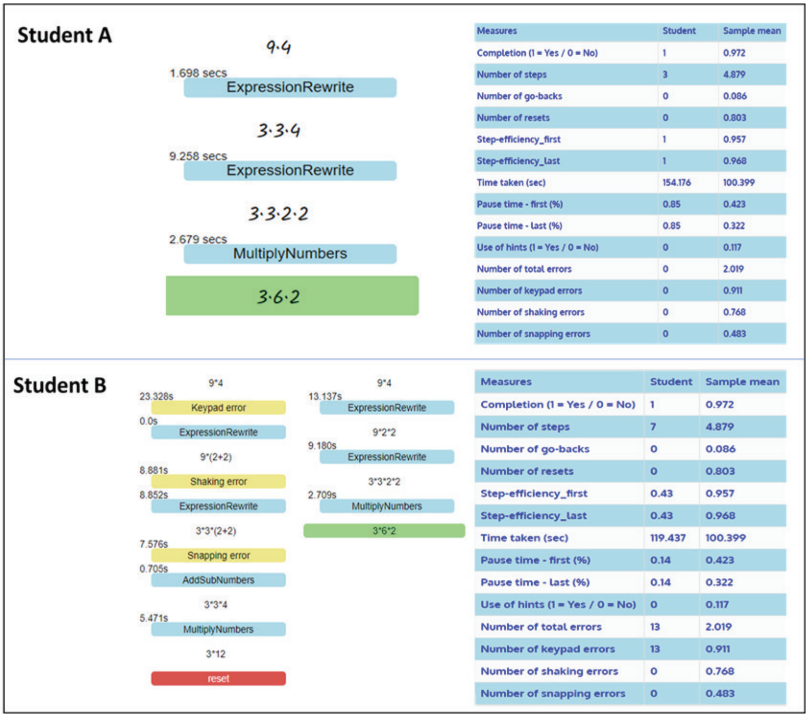

Figure 3. Example of Indivisualizer depicting two students' problem-solving processes for the square numbers problem.

Visualizations of Students' Overall Problem-Solving Processes

Second, we explored the entire students' overall problem-solving processes using Sankey diagrams. Figure 4 presents the students' problem-solving processes for the square numbers problem (problem A, turning "9×4" into "3×6×2"). As mentioned earlier, each node in the diagram represents a different mathematical expression made by students, and the thickness of each path represents the number of students who made that expression.

Figure 4 shows the full Sankey diagram for the problem. A large number of paths indicates that there was a large variation in students' problem-solving processes. Specifically, as indicated by the thickest line on the left, the most prominent problem-solving process made by the students was 9×4 (factoring 9) → 3×3×4 (factoring 4) → 3×3×2×2 (multiplying 3 and 2) → 3×6×2. Thus, many students first attended to the left sides of the start expression and the goal state by breaking 9 down into 3 and 3, then moved to the right side of the starting expression. Relatively few students first attended the right sides of the problem equation first (e.g., 9×4 (factoring 4)→ 9×2×2 (factoring 9)→ 3×3×2×2 (multiplying 3 and 2)→ 3×6×2).

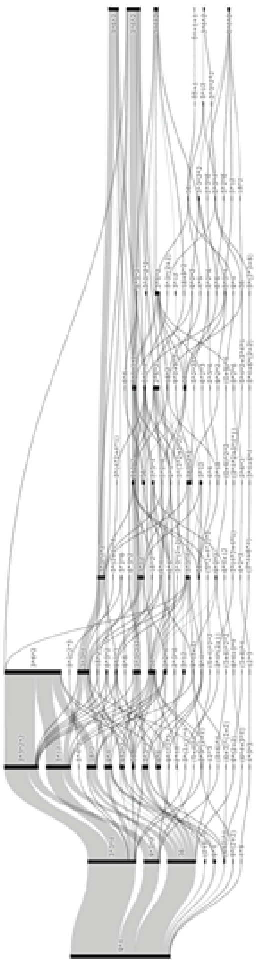

Figure 4. A Sankey diagram showing entire students' problem-solving processes for the square numbers problem (for full image: http://tiny.cc/mp-dzsz).

The second most prominent first step made by the students was creating 36. These students simply combined two numbers in the goal state (9 and 4) rather than attending to the structure of the equation or the numbers in the goal state. While some of these students successfully solved the problem with the fewest steps possible to reach the goal state (e.g., 9×4 (multiplying 9 and 4)→ 36 (factoring 36)→3×12 (factoring 12) →3×6×2), most of them took more than three steps and did not solve the problem in the most efficient way.

Next, we created another Sankey diagram to examine the students' problem-solving processes for the non-square numbers problem (problem B, turning "6×10" into "2×15×2") (See Figure 5). As shown in Figure 5, there was a much larger variation in students' problem-solving processes in problem B compared to problem A. There was no single dominant problem-solving pathway, and relatively fewer students solved the problem in the most efficient way, indicating that the factoring non-square numbers problem was more challenging for the students than the square problem. Specifically, the most prominent pathway was 6×10 (factoring 6) → 2×3×10 (factoring 10) → 2×3×5×2 (multiplying 3 and 5) → 2×15×10. Similar to problem A, many students first attended to the left sides of the goal state expression and the target goal expression, then moved to the right side of the expression. The most prominent first step was making "60" by multiplying two numbers (6 and 10) in the start state, but there was greater variation in their second steps. Some students who made 60 on their first step reached the goal state with the fewest steps possible by successfully breaking 60 down into "2 and 30" or "30 and 2". However, many of them did not solve the problem in the most efficient way, indicating that factoring 60 into smaller numbers was an obstacle point to the students.

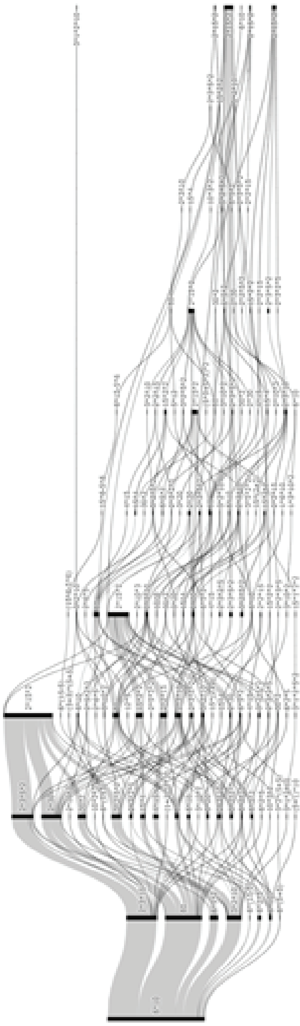

Figure 5. A Sankey diagram showing entire students' problem-solving processes for the non-square numbers problem (for full image: http://tiny.cc/kvdzsz).

PRODUCTIVITY OF STUDENTS' INITIAL SOLUTION STRATEGIES AND EFFICIENCY OF PROBLEM-SOLVING

Before exploring the productivity of initial solution strategies using Sankey diagrams, we computed frequencies for the productivity of students' first steps (See Figure 6). As shown in Figure 6, 64% (n = 220) of the 343 students made productive first steps for the factoring square numbers problem (problem A). Relatively fewer students (n = 204, 59%) made productive first steps for problem B, indicating that factoring non-square numbers seemed to be more challenging for the students.

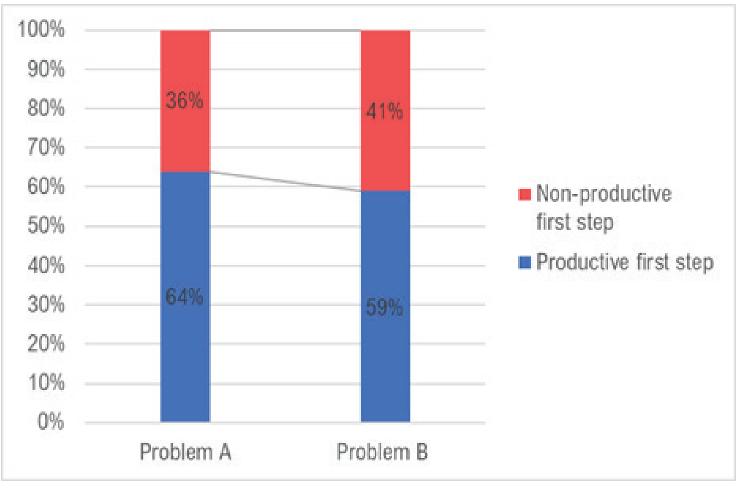

Figure 6. Frequencies of productivity of first steps by the problem.

We then computed descriptive statistics and correlation coefficients of the variables included in the study (See Table 3).

Table 3

Descriptive statistics and correlations

	1	2	3	4	5
1. Prior knowledge	-	-	-	-	-
2. Productivity of first steps: Problem A	-.009	-	-	-	-
3. Productivity of first steps: Problem B	-.033	.373***	.-	-	-
4. Efficiency: Problem A	.182**	.202***	-.043	-	
5. Efficiency: Problem B	.161**	-.089	.131*	.227***	-
M	6.360	.641	.595	.808	.791
SD	2.433	.480	.492	.247	.240

$^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$

We conducted a Pearson correlation analysis for the pair of continuous variables (e.g., prior knowledge - efficiency) and a point-biserial correlation analysis for the pairs of dichotomous and continuous variables (e.g., productivity - efficiency). As presented in Table 3, the productivity of the first steps for "problem A" had a significant positive association with its efficiency ($r_{pb}(341) = .202$, p < .001). The productivity of the first steps for "problem B" was also positively associated with its efficiency ($r_{pb}(341) = .131$, $p = .015$). The students' prior knowledge did not show statistically significant relationships with the productivity of both problems. However, students' prior knowledge was positively associated with efficiency scores of both problems.

Next, we created a colored-Sankey diagram to visualize the productivity of students' initial solution strategies for problem A and their subsequent solutions (See Figure 7). Note that colors in the diagram represent the productivity of students' first steps (blue: productive first steps, red: non-productive first steps).

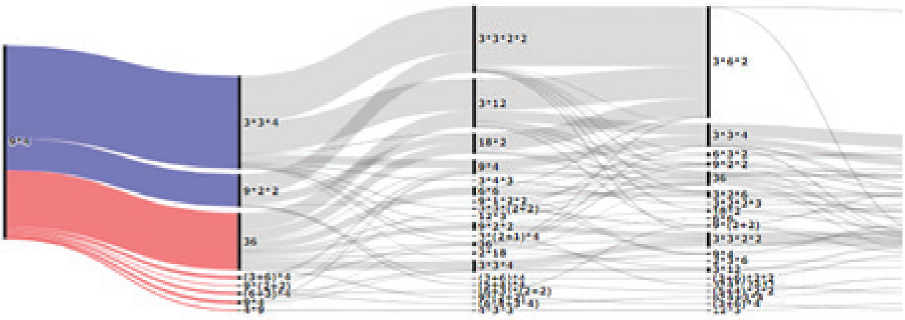

Figure 7. Section of the Sankey diagram showing productivity of initial solution strategies for the square numbers problem (for full image: http://tiny.cc/pd51tz).

As shown in Figure 7, more than half of the students made productive first steps for this problem (e.g., 3×3×4, 9×2×2). Most students who made productive first steps solved the problem with the fewest possible steps (3 steps) to reach the goal state (e.g., 9×4 → 3×3×4 → 3×3×2×2 → 3×6×2). However, the students who made non-productive first steps (e.g., 36, (3+6)×4) tended to exceed the fewest possible steps to complete the problem (3 steps) and did not solve it in the most efficient way. Thus, as we hypothesized, noticing the underlying structure of the problem equation and how to transform it played a critical role in students' subsequent transformations as well as their overall efficiency in algebraic problem-solving.

We created another colored-Sankey diagram to visualize the productivity of students' initial solution strategies for problem B (See Figure 8) and the subsequent solution pathways.

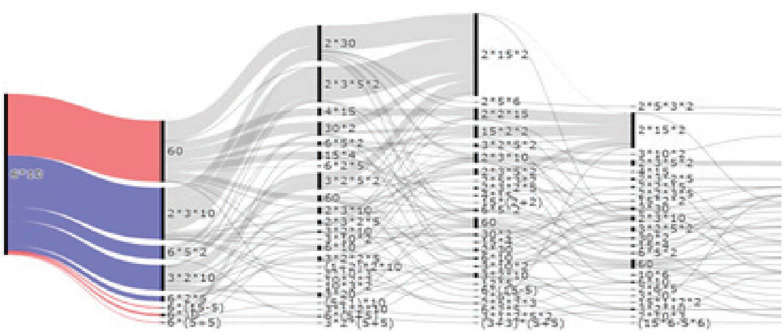

Figure 8. Section of the Sankey diagram showing students' problem-solving processes for the non-square numbers problem (for full image: http://tiny.cc/wd51tz).

As presented in Figure 8, more than half of the students made productive first steps (e.g., 2×3×10, 6×5×2). Specifically, most students who made the numbers included in the goal state (e.g., 2) or the factors (e.g., 3, 5) of the numbers in the goal state on their first steps solved the problem in the most efficient way. However, many students who did not attend to the structure or the feature of the problem and simply combined two numbers in the start state (i.e., 6, 10) failed to solve the problem in the most efficient way. Thus, similar to problem A, students' first mathematical transformation played a significant role in students' subsequent problem-solving processes as well as the overall efficiency of students' problem-solving.

Lastly, we examined whether or not the productivity of first steps and prior knowledge predicted the efficiency of problem-solving (Table 4). Note that one student who did not have data on prior knowledge was excluded from further analyses.

Table 4
Results of the regression analyses predicting the efficiency of problem-solving

Variable	*B*	*SE*	β	*t*	*p*	R^2	R^2 *Change*
Square-numbers problem (problem A)							
Model 1.1						.041	-
(Constant)	.742	.022		33.946***	<.001		
Productivity of first steps	.104	.027	.202	3.803***	<.001		
Model 1.2						.074	.034
(Constant)	.623	.040		15.531***	<.001		
Productivity of first steps	.105	.027	.204	4.194***	<.001		
Prior-knowledge	.019	.005	.183	3.511**	.001		
Non-square numbers problem (problem B)							
Model 2.1						.017	-
(Constant)	.753	.020		37.205***	<.001		

Productivity of first steps	.064	.026	.131	2.430*	.016		
Model 2.2						.044	.027
(Constant)	.648	.039			<.001		
Productivity of first steps	.066	.026	.136	2.561*	.011		
Prior-knowledge	.016	.005	.165	3.107**	.002		

$^{*}p < .05$, $^{**}p < .01$, $^{***}p < .001$

First, a simple linear regression analysis was conducted to predict the students' efficiency scores for problem A (Model 1.1 in Table 4). The model explained 4.1% of the variance in the efficiency scores ($F(1, 340) = 14.460$, $p < .001$), and the productivity of the first steps significantly predicted the outcome variable ($B = .104$, $p < .001$). The students' predicted efficiency score was equal to 0.742 + 0.104*(productivity).

Next, we added the students' prior knowledge to Model 1.1 and performed a multiple linear regression analysis (See Model 1.2 in Table 4). The addition of prior knowledge contributed 3.4% additional variance and the model explained 7.4% of the variance in the efficiency scores ($F(2, 339) = 13.634$, $p < .001$). Further, we examined the individual predictors, and the results indicated that both the productivity of first steps ($B = .105$ $p < .001$) and prior knowledge ($B = .019$, $p = .001$) significantly predicted the efficiency scores. The students' predicted efficiency score was equal to 0.623 + 0.105*(productivity) + 0.019*(prior knowledge).

We then repeated the analyses for problem B, the non-square numbers problem. Results of the simple linear regression analysis (Model 2.1 in Table 4) indicated that there was a statistically significant effect between the productivity and the efficiency score for problem B ($F(1, 340) = 5.904$, $p = .016$), with an R^2 of 0.017. The productivity of the first steps significantly predicted the efficiency scores ($B = .064$, $p = .007$), and the predicted efficiency score for problem B was equal to 0.753 + 0.064*(productivity). Next, the students' prior knowledge was added to Model 2.1, and a multiple linear regression analysis was conducted. The addition of prior knowledge contributed 2.7% additional variance and the model explained 4.4% of the variance in the efficiency scores ($F(2, 339) = 7.855$, $p < .001$). In terms of the individual predictors, both the productivity of first steps ($B = .066$, $p = .011$) and prior knowledge ($B = .016$, $p = .002$) significantly predicted the efficiency scores. The students' predicted efficiency score for problem B was equal to 0.648 + 0.066*(productivity) + 0.016*(prior knowledge).

DISCUSSION

While understanding students' problem-solving processes is critical to support their algebra learning, limited work has explored detailed step-by-step information about their mathematical problem-solving processes through data visualizations (Pape & Smith, 2002; Vieira et al., 2018). This study presented visualizations of students' algebraic problem-solving processes and solution strategies using two novel data visualization techniques, Indivisualizers and Sankey diagrams, and investigated how we could use these visualizations to provide meaningful and comprehensive information to researchers and teachers.

First, we visualized variation in individual students' problem-solving processes using Indivisualizers. The individual visualizations showed that these could provide information to researchers and teachers on specific students' problem-solving processes, including where and when particular students make errors and reset problems. These visualizations also provide information about response time and students' mathematical strategies. Using Indivisualizers, teachers and researchers can easily compare the changes in an individual student's mathematical transformation across multiple consecutive attempts to reach the goal state.

Second, we visualized the entire students' overall algebraic problem-solving using Sankey diagrams. The results showed that Sankey diagrams could provide a lot of information to help teachers and researchers understand overall students' algebraic problem-solving processes. Specifically, they presented the variability of problem-solving processes, the most common pathways used, the productivity of those pathways, the obstacle points of problem-solving (e.g., factoring a non-square number into small numbers), and the efficiency of different solution strategies. In particular, the diagrams revealed that many students tended to attend to the left side of the equation first and then move to the right side of the equation, which was consistent with findings of other studies that showed students tended to use left-to-right problem-solving procedures when solving algebraic problems (Robinson & Dubé, 2013; Siegler & Araya, 2005).

Therefore, our findings support the idea of previous research that Sankey diagrams can be used as a useful method to quickly identify different paths involved in a process, the proportion of each path, and the variability across the students (Lee & Tan, 2017; Tiwari, 2017). Further, as noted by Lynch and Star (2014), visualizing students' problem-solving processes for a given mathematical task would help teachers and researchers perceive students' individual differences and their mathematical thinking processes.

Lastly, we visualized the productivity of initial solution strategies and examined the association between the productivity of initial solution strategies and the efficiency of problem-solving. In particular, we built colored Sankey diagrams to distinguish between productive first steps versus non-productive first steps. One of the limitations of Sankey diagrams is that it is challenging to visualize a process with a large number of different paths and nodes clearly and intuitively (Askinadze et al., 2019; Wang et al., 2017). Colored-Sankey diagrams can intuitively visualize productive or non-productive pathways to solve algebraic problems in the game, which would help teachers or researchers easily interpret the diagrams and use the information provided more effectively.

Lastly, we examined whether or not the productivity of students' first steps and their prior knowledge influenced the overall efficiency of problem-solving. The results indicated that productivity and prior knowledge significantly predicted the efficiency of problem-solving for both the square-numbers problem and the non-square numbers problem.

Taken together, noticing the underlying structure of the problem equation and how to transform it on their first step played a critical role in students' subsequent transformations as well as their overall efficiency of algebraic problem-solving, which was aligned with previous literature (Stephens et al., 2013). The findings are also consistent with those of other studies, which found that students' prior knowledge was positively associated with the efficiency of problem-solving (Khng & Lee, 2009; Rittle-Johnson et al., 2009). Thus, our findings suggest that algebra instruction may need to focus more on correcting students' misconceptions about equivalence (i.e., the equals sign indicates computing) and teaching students to notice the underlying mathematical structure of expressions.

Implications for Teaching and Practice

This work has clear implications for practice as students' strategies in educational technologies are often invisible to teachers. In combination with GM, the algebraic notation system, teachers could use these visualizations (Sankey diagrams, Individualizers) in several ways to inform their instruction. First, the visualizations provide evidence of students' work while problem-solving, something that teachers often ask students to include on paper but is often absent from most online mathematics learning tools. Teachers could use information from visualizations in the classroom in several ways: 1) using the locations of common errors from the individual visualizations to foster discussions about common misconceptions and mistakes made by students, 2) showing Sankey diagrams to students to demonstrate the use of multiple valid pathways and have discussions about why they would choose one path or another, 3) having discussions about what makes a particular approach more productive than another, then presenting similar problems to have students practice those conceptual and flexibility skills. The Indivisualizer can be used synergistically with the Sankey diagrams, as Sankey diagrams display information across all students for their first attempts, whereas the Indivisualizer displays information regarding all tries for only one student at a time.

Researchers can also use these visualization techniques to support hand-labeling and coding and to inform new methods and territories. Finally, it is important to note that this data was quickly and easily collected from 6th and 7th graders in authentic classrooms as students solved problems in GM so that the visualizations represent students' real work and problem-solving processes. Presenting this information in real-time could transform the ways that teachers approach instruction about flexibility and efficient algebraic problem-solving.

Limitations and Future Directions

A number of important limitations need to be considered. First, although we found that Sankey diagrams could provide a lot of information to help teachers and researchers understand students' algebraic problem-solving processes, the interpretation and the effective use of visualizations might vary between users depending on their visual literacy skills (Askinadze et al. 2019; Wang et al., 2017). There is a large amount of information presented to users at once, and concerns about cognitive load are definitely present (Feldon, 2007). Thus, understanding teachers' perceptions, decisions, or ability to use and interpret these visualizations to inform instruction is an important next step.

Second, the current study examined the relationship between the productivity of initial solution strategies and the overall efficiency of problem-solving using only two multiplication problems in the game. Future research needs to replicate analyses using a larger sample of problems or more complex and ill-structured problems. Moreover, the prediction models with two predictors (prior knowledge, the productivity of first steps) explained a relatively small amount of variance in the efficiency of problem-solving. Thus, it is suggested that the effect of other student behaviors (e.g., time taken to solve a problem, pause time before problem-solving, number of errors) needs to be investigated in future studies.

This work also has usefulness for future work in learning analytics and math education and cognition, particularly by providing a means to facilitate coding of student mathematical strategies and behaviors. For example, it is possible to use educational data mining techniques or machine learning algorithms to create a detection system that could automatically identify the productivity of students' solution strategies. In this way, it could help reduce the time required for hand-coding the vast amount of data and training human coders, thus accelerating the progress of research (Mu et al., 2012). The automated detection system also could predict the efficiency of problem-solving and further generate these visualizations in real-time. Our team is currently investigating the feasibility of using these techniques to predict students' in-game performance in real-time.

CONCLUSION

This work presents two advanced and novel ways (Indivisualizer and Sankey diagram) to visualize students' problem-solving processes and use of strategies as they solve mathematical problems in a dynamic algebraic notation tool. These visualizations provide a proof of concept of how we can integrate more qualitative and strategy data into data analytics and begin to unpack the complex learning processes that students use when solving algebraic problems. Our findings suggest that these visualizations can be used both in research and practice to reveal and unpack our understanding of variability in mathematical problem-solving strategies and cognition.

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