"It has the same numbers, just in a different order": Middle School Students Noticing Algebraic Structures Within Equivalent Equations

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In this paper, we explore student solutions to a free response mathematical assessment task which had opportunities for students to notice structural properties in the context of number systems. In total, 308 students aged between 10 years to 13 years participated in the study. Their responses were analysed to determine whether they noticed algebraic structures in a task using equivalent equations. Findings indicate that students were able to recognise equation pairs that drew on both the associative and distributive properties. A limited number of students were able to notice the general structure and draw on number properties to support their claims, moving beyond using algorithmic thinking.

Introduction

In recent years, there has been an increased recognition of the importance of developing algebraic reasoning in primary and middle school classrooms. This has included a focus on growing patterns and the development of functional thinking, supporting student capability to engage in the generalisation process and the unification of arithmetic and algebra as a unified curricula strand (Chimoni et al., 2018; Fonger et al., 2018; Hunter & Miller, 2021). Across these different areas, positioning students to notice mathematical structure is a key aspect of supporting students to make sense of mathematics, understand operations as mathematical objects, and both engage in algebraic transformational activity and make sense of transformations (Kieran, 2018; Schifter, 2018). We draw on Schifter's (2018, p. 310) definition of mathematical structure as "behaviors, characteristics, or properties that remain constant across specific instances". Of interest, is students' capacity to work flexibly with numbers and equations and to notice relationships and mathematical structure. In this paper, we explore student solutions to a free response mathematical assessment task which had opportunities for students to notice structural properties in the context of number systems. We examine the responses of students within the age band of 10 years to 13 years old in relation to whether they noticed algebraic structures in a task using equivalent equations. We also investigate the properties that they identified, and the explanations provided by the students. This paper contributes to previous research in relation to interrogating student capability to use structure and relationships when working with equations.

Literature review

Noticing Mathematical Structures Across Equations

The ability to notice and identify structure is central to mathematics and one's ability to work with generalised forms. Mason et al. (2009) articulated that mathematical structure is "the identification of general properties which are instantiated in particular situations as relationships between elements" (p. 10). In the context of number systems, structure is typically referred to as the properties of arithmetic such as commutativity, associativity, and

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distributivity properties, identity law, and an understanding of inverse relationships; and these properties are generalisable (Kieran et al., 2016). Warren (2003), further expands this definition of structure in number systems to include: "(i) relationships between quantities (for example, are the quantities equivalent, is one less than or greater than the other); (ii) group properties of operations (for example, is the operation associative and/or commutative?; do inverses and identities exist?); (iii) relationships between the operations (for example, does one operation distribute over the other?); and (iv) relationships across the quantities (for example, transitivity of equality and inequality)" (p. 123–124). However, in the teaching of arithmetic, it appears that students are rarely given the opportunities to focus on and build an appreciation of these structures when forming generalisations (Arcavi et al., 2017).

In primary school mathematics, considerable attention is given to providing students with opportunities to engage with a range of equations across number systems. Despite this, little attention is given to the structure of these equations, with the majority of the focus being on teaching the procedures of how to calculate (Schifter, 2018). It can be argued that students frequently do not notice the structural differences between each of the equations to form generalities. Schifter argues that a "consequence of such absence is the lack of salience of the operations in students' minds. The operations are interpreted as instructions to perform a set of steps rather than as objects, each with its own set of characteristics and properties" (p. 325). Moving away from a procedures/operations focus on arithmetic, to one that examines addition, subtraction, multiplication and division properties, gives space for these to be stand-alone mathematical objects (Kieran, 1989; Slavit, 1999). This is where students demonstrate relational understanding and reasoning (Schifter, 2018). However, research conducted by Arcavi et al. (2017), demonstrates that when students' have a compulsion to calculate numerical answers it presents a barrier for them to recognise the patterns and mathematical structures. This presents a challenge to shift students' attention from "calculating" to that of noticing the underlying structures of operations to see these as mathematical objects, which is essential for algebra.

Making the shift from arithmetic to algebraic thinking across number properties and equations requires students to understand equivalence and equality. Research that examines this shift typically focuses on equality and how students understand the equal sign. Matthews et al. (2012) highlight that tasks typically used in this research fall into the following categories: (i) solving open equations, such as $9 + 4 = * + 6$ (e.g., McNeil, 2007; Powell & Fuchs, 2010); (ii) equations focusing on true or false statements (e.g., Molina & Ambrose, 2006; Seo & Ginsburg, 2003); (iii) students verbally articulating what the equal sign means (Knuth et al., 2006; Seo & Ginsburg, 2003); and, finally, though typically not as common as the other tasks mentioned (iv) advanced relational reasoning items, such as asking children to solve the equation $24 + 57 = * + 55$ without having to add $24 + 57$ (e.g., Blanton, et al., 2011; Carpenter et al., 2003). We note that much of this research has informed the initial underpinnings of how to begin to support students to make this shift from arithmetic to algebraic thinking. However, to date, there appears to be little research that focuses on how young students articulate the structures and relationships they notice when considering how two equations are equivalent and which generalised number properties they are drawing from.

Methodology

This study was exploratory in nature and used a qualitative case study design. We were interested in examining student solution strategies to a free-response mathematical assessment task that involved structural properties and relationships in the context of number systems. Our research aligned with better understanding the advanced relational structures students notice and the reasoning they provide to articulate these relations. In particular, the study addresses the following research questions:

1) *What do students notice in a task involving algebraic structures with equivalent equations?*
2) *How do students notice and explain structural properties and relationships in the context of number systems?*

Participants

The participants were 308 middle school Year 7-8 students aged between 10 years and 13 years old. The data were collected from two low decile schools. Decile ratings in New Zealand are based on census data of households with school-aged children and use household measures such as income, government assistance, occupation, and education with a low decile rating indicating that students live in low socio-economic communities. The students were from a range of ethnic groupings with most being Pacific nations ethnic grouping (55%), followed by Māori (23%), and Pakeha/European (13%), and included 151 male students and 157 female students.

Data Collection

The students were given a free-response task developed by the first author. This consisted of a set of equations (see Figure 1), follow-up prompts and two blank pages for the students to comprise their response. The equations were designed to be matching pairs which could be identified through noticing structural properties and relationships in the context of number. This included aspects such as the associative property of addition and multiplication, the distributive property of multiplication, and exponents, with an over-arching focus on equivalent relationships. Each pair of equations was developed to match a specific property or relationship. The prompts following the number sentences were provided to position the students to notice, describe, explain, and generalise the structural properties without the need for calculation.

76 x 15 = 37 + 43 + 40 + 36 = 99 ÷ 3 ÷ 3 =

7 x 86 = 99 ÷ 9 =

6^3= (70 x 5) + (70 x 10) + (6 x 10) + (6 x 5) =

37 + 40 + 36 + 43 = 12 x 22 = 6 x 6 x 6 =

(7 x 90) - (7 x 4) = 4 x 66 =

Look at the number sentences above:

- Describe what patterns you can find
- Why do your patterns work?
- Do they work with other numbers?

Figure 1. Number sentence task

The task also aligned with the New Zealand Curriculum (NZC) (MoE, 2007) elaborations for Level 4 which relates to Year 7–8 students. The NZC outlines an expectation that students are both expected to generalise properties of multiplication and division and describe these using appropriate mathematical terminology and/or symbols. Additionally, students should be able to use the properties when operating on numbers. There is also a developing expectation at these year levels that students should be able to understand and use mathematical notation.

The item related to exponents and mathematical notation was included to address this expectation.

The task was administered by the classroom teacher and completed by all students independently and individually during their regular mathematics lesson. Students were provided with adequate time to complete it. The students were advised that the assessment task was not a test but an opportunity for them to show what they knew in mathematics. Student responses on the two blank pages were collected and wholly scanned for analysis by the research team.

Data Coding and Analysis

In the first instance, all student responses were coded as either: (i) identifying structure and relationships in the task; or (ii) as not identifying structure or relationships in the task. Those responses that were coded as student being able to identify structure and/or relationships, required an explicit response that included the identification of one or more mathematical structures in the number sentences. These explicit responses may have been represented as drawing arrows to represent matched equations, re-writing the equations together, or providing a more detailed written description (see Figure 2). Responses coded as no identification of structure or relationships were those where the student did not explicitly identify any mathematical structures in the number sentences.

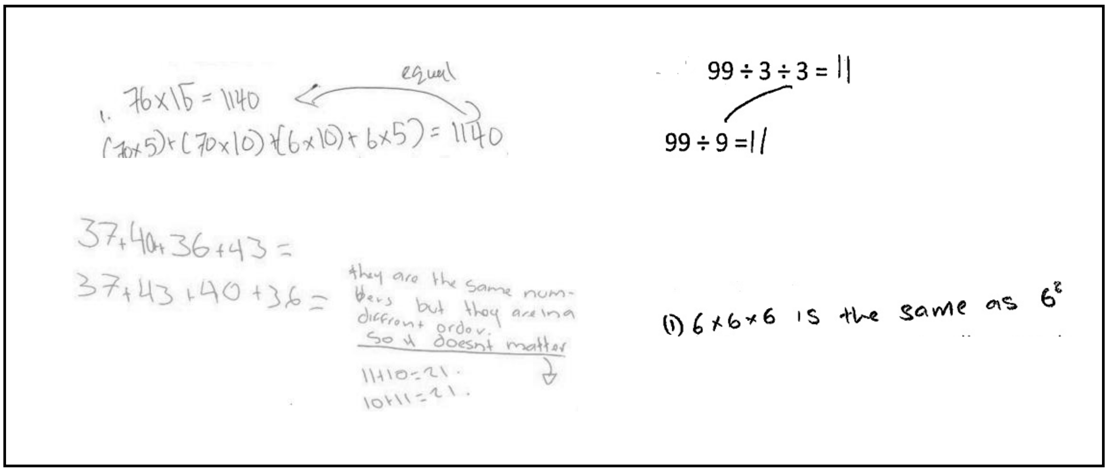

Figure 2. Student responses coded as identifying structure.

The dataset of student responses that were coded as identifying structure and relationships were then re-analysed. The second phase of analysis consisted of three aspects. First, the responses were analysed in relation to the pairs of equation (based on mathematical structures and relationships) that were identified. Second, the response for each pair of equations were classified as calculation or relational dependent on whether the student had calculated the answers when identifying the mathematical structure or had used relational strategies without calculation. Finally, student explanations were analysed qualitatively to identify themes and to examine the differing levels of sophistication in the explanations that were provided.

Results and Discussion

Identifying Structural Properties in the Context of Number Systems

From our initial analysis, we found that the majority of students (n = 204 or 66%) treated the task as a computation activity and solved one or more of the equations without responding to the prompts in relation to the patterns. This aligned with past findings from research (Arcavi et al., 2017). Instead, the students recorded calculations and solution strategies for the equations without referring to the structural properties. In contrast, a relatively small proportion of students (*n* = 104 or 34%) identified structural properties and described these and the patterns that were evident in the task.

The second layer of analysis re-examined the responses from the 34% of students who identified structural properties. In the first instance, we examined the equations and properties that were identified by the students as shown on Table One.

Table 1
Number of Students Identifying Each Pair of Equations Representing Structural Properties (N = 104)

Equation pair	Property	Number of students who identified the pair
37 + 43 + 40 + 36 = 37 + 40 + 36 + 43 =	Associative	87% (*n* = 90)
6^3 = 6 x 6 x 6 =	Exponents	69% (*n* = 72)
99 ÷ 3 ÷ 3 = 99 ÷ 9 =	Associative	63% (*n* = 65)
12 x 22 = 4 x 66 =	Associative	34% (*n* = 35)
76 x 15 = (70 x 5) + (70 x 10) + (6 x 10) + (6 x 5) =	Distributive	30% (*n* = 31)
7 x 86 = (7 x 90) – (7 x 4) =	Distributive	19% (*n* = 20)

As illustrated on the table, we found that most commonly, the students were able to identify the associative property as represented in addition. Many students also recognised the connection between mathematical notation (y^3) (powers) and expanded multiplicative relationships. Less than half of the students were able to recognise the structural relationships between multiplication equations. Multiplication equations represented, using the distributive property, were particularly challenging for these students to recognise, with only 30% of students recognising this pair to equations.

How do Students Notice and Explain Structural Properties and Relationships in the Context of Number Systems?

This section of the findings will use a sub-set of the related equations and student responses to illustrate the ways in which students noticed and explained structural properties and relationships in the context of number systems.

Associative property in addition. We begin with a focus on the associative property in addition (37 + 43 + 40 + 36 = 37 + 40 + 36 + 43) given that this was the property most identified by students. Many of the students (*n* = 68/90) undertook calculations in responding to this aspect of the task. In some cases, the student calculated both equations separately and then recognised that they were equivalent and identified the relationship. In other examples, the student undertook one calculation and then proceeded to write the same sum for both equations.

In this case, it appeared that the student was able to recognise equivalence in the equations by using knowledge of the associative property. A smaller group of students (*n* = 22/90) did not record any sum or calculations and appeared to identify the equivalence of the equations by recognising the associative property. For example, one student drew an arrow between the equations without recording any calculations and wrote: "37 + 43 + 40 + 36 = is the same as 37 + 40 + 36 + 43 = but just the 40 and the 43 are swapped."

The sophistication of the explanations provided by the students varied with some students (*n* = 34/90) providing no explanation or simply writing their workings to show how they had solved the equations. Other students (*n* = 15/90) also provided limited explanation of the associative property and connected the equations by identifying that the answers were the same or the equations were equal. The largest group of student responses (*n* = 37/90) began to explain the relationship between the two equations using informal mathematical language and related this to the specific example provided in the task:

> Student: The same numbers, they are just jumbled up.
>
> Student: It has the same numbers, just in a different order.

Finally, a small group of students (*n* = 4/90) began to provide an explanation that moved beyond the specific to provide general examples of the structure of the associative property. Interestingly, several of these students gave examples using informal language that referred to the commutative property: "they are the same numbers but in a different way so it doesn't matter like 11 + 10 = 21 and 10 + 11 = 21." Other students provided explanations with further examples of the associative property: "they are the same numbers but just mixed up but because it's addition they will add up as the same number, 1 + 5 + 6 is the same as 5 + 6 + 1."

Associative property in multiplication. In contrast to recognition of the associative property in addition, students appeared to have greater difficulty in recognising the associative property in multiplication. Most students (*n* = 29/35) undertook a calculation and then identified the equations as related. It was evident in the student responses that they typically calculated the solution for each equation rather than only completing one calculation as many did for the addition equivalence equations. A small group of students (*n* = 6/35) did not record any calculations with most of these students simply drawing an arrow or circle to connect the equations.

Similar to responses related to the addition equations and associative property, there was variation in student explanations. Overall, for this aspect of the task, many students (*n* = 17/35) provided no explanation or only recorded their solution strategy for each equation. It was evident that students found it difficult to construct an explanation with an additional group of students (*n* = 9/35) giving a limited explanation indicating that the answers were the same or equal. A further group of students (*n* = 9/35) provided explanations related to the associative property of why the specific equations in the task were equivalent:

> Student: The two equations have a pattern because 4 and 12 are both a common multiple of three.
>
> Student: They equal the same number. Also the 66 is 3 times bigger than the 22 and the 12 is also 3 times bigger than the 4.

One of these students provided an explanation that included a further example of the associative property; however, this was directly related to the example in the task: "Look, 4 x 6, 8 x 33, 12 x 22, that's the pattern."

Distributive property in multiplication. The relationship developed from the distributive property in multiplication appeared to be difficult for the students to identify in regards to equivalence. This final section examines the student responses to the equations 7 x 86 = (7 x 90) – (7 x 4) as this was the pair of equations least often identified as equivalent by the students.

Analysis of the student responses revealed similar findings to the previous pairs of equations that have been discussed, with most students (*n* = 16/20) undertaking a calculation to identify that the equations were equivalent. Interestingly, in parallel with the responses to the associative property of addition, a number of students calculated the product of one equation, generally (7 x 90) – (7 x 4) and wrote the same product for both equations or used one calculation to connect the equations. We assume that in this case, the students were able to recognise the relationship of the distributive property without undertaking both calculations. For example, one student solved the first equation and then wrote below: "7 x 86 is the same as (7 x 90) – (7 x 4) because 7 x 90 is rounded up from 7 x 86 and subtracted by 7 x 4". A small group of students (*n* = 4) did not record any calculations and appeared to use their understanding of the distributive property to identify the equivalence of both equations.

In relation to student explanations, most students (*n* = 8/20) again either provided no explanation or simply recorded calculations. A small group of students (*n* = 5/20) gave a simple explanation referring to the equations having the same answer. Finally, the other group of students (*n* = 7/20) provided explanations linked to their understanding of the distributive property as represented in the specific equations in the task:

Student: (7 x 90) – (7 x 4) is the same as 7 x 86 because 90 – 4 = 86

This section has illustrated the ways in which the students in this study noticed and explained structural properties and relationships in the context of number systems.

Conclusion and Implications

In both curriculum documents (MoE, 2007) and research studies (Chimoni et al., 2018; Fonger et al., 2018; Schifter, 2018), there has been increasing emphasis placed on early algebra and the need to facilitate students to work flexibly with numbers and notice relationships and mathematical structure. The aim of this exploratory study was to begin to address the gap in the literature in relation to better understand which number properties students use to notice the general structure of equivalent equations. Prior to this study, the majority of research has typically focused on tasks that require students to identify true and false equations statements, solve open-ended tasks, and or verbalise their understanding of the equal sign (Matthews et al., 2012).

Findings from this study indicate that students noticed equivalent equations underpinned by the associative and distributive properties, by matching paired equations. It was apparent that it was easier for students to notice equations underpinned by the associative property than it was to recognise the distributive property. While students noticed the pairs, relatively few students appeared to approach the task in structural way. Students typically performed calculations, rather than seeing the equations as mathematical objects (Kieran, 1989; Slavit, 1999; Schifter, 2017). This meant that there was an ongoing focus on arithmetic solutions rather than engaging in algebraic thinking by generalising the common structures across the equation pairs. This may have been due to the fact that the task included an equal sign and students interpreting this as "calculate" or "find the answer." Many of the responses provided by the students, demonstrated that they could provide a simple explanation. It may be that, if in fact students were interviewed, the opportunity to verbalise their response with accompanying gestures may have given more insight into their thinking. Implications from this study include that there needs to be a greater focus on supporting students to notice the structure of number properties across different equations and then to use this to form generalities. It is evident, future research is needed in this field to more deeply understand how students notice algebraic structures within equivalent equations.

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