Key Shifts in Thinking in the Development of Mathematical Reasoning

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This symposium will draw on the evidenced-based learning progressions for multiplicative thinking, algebraic reasoning, geometrical reasoning, and statistical reasoning presented at previous MERGA conferences (see references by symposium authors in the papers that follow). The four papers will consider key shifts in thinking identified within each progression, without which students' progress may be seriously constrained.

Paper 1: *A Disposition to Attend to Relationships: A Key Shift in the Development of Multiplicative Thinking*
[Dianne Siemon]

This paper draws on multiple data sources to better understand the shift from additive to multiplicative thinking, which is crucial to all further participation in school mathematics.

Paper 2: *Key Shifts in Students' Capacity to Generalise: A Fundamental Aspect of Algebraic Reasoning*
[Max Stephens, Lorraine Day, & Marj Horne]

This paper will elaborate five levels of algebraic generalisation and two key understandings based on an analysis of students' responses to RMFII algebraic reasoning tasks.

Paper 3: *Cognitive Flexibility and the Coordination of Multiple Information in Geometry and Measurement*
[Rebecca Seah & Marj Horne]

This paper analyses students' solutions to problems in geometry and measurement situations in order to identify key components needed to nurture reasoning.

Paper 4: *Facilitating the Shift to Higher-order Thinking in Statistics and Probability*
[Rosemary Callingham, Jane Watson, & Greg Oates]

Students have difficulty moving from concrete representations and procedural mathematical statistics to context-based appreciation of data. This paper examines the barriers to this shift to higher-order thinking based on the Statistical Reasoning Learning Progression.

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), p. 41–57. Launceston: MERGA.

Key Shifts in Students' Capacity to Generalise: A Fundamental Aspect of Algebraic Reasoning

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This paper will elaborate five levels of algebraic generalisation based on an analysis of students' responses to Reframing Mathematical Futures II (RMFII) tasks designed to assess algebraic reasoning. The five levels of algebraic generalisation will be elaborated and illustrated using selected tasks from the RMFII study. The five levels will be matched against the eight zones identified in the RMFII study supported by its Rasch analysis. We identify two shifts where students' capacity to generalise appear difficult to navigate. The first being where students move from noticing and describing regularities to formalising these regularities into verbal or symbolic expressions. The second is where students use their understanding of equivalence based on relational thinking to write and recognise equivalent algebraic expressions.

Key ideas implicit in the idea of generalisation as they relate to the algebraic reasoning tasks of RMFII have been presented by authors such as Love (1986) and Mason (1996), who suggested that the generalisation of a pattern, at its core, rests on the capability of noticing something *general in the particular*. Kieran (2007), however, noted that this feature alone may not be sufficient to characterise the *algebraic generalisation* of patterns, arguing that, in addition to seeing the general in the particular, students need to be able to express their generalisation algebraically, drawing on *explicit* reasoning in terms of justification and explanation. These points are directly relevant to the tasks used by RMFII to assess algebraic reasoning in which students were invited to explain their reasoning. Kieran's ideas will feature clearly in the third, fourth and fifth levels of a progression for algebraic generalisation advanced in this paper.

These five levels were enumerated in a previous paper (Stephens et al., 2021). They are: Working with particular instances; Noticing and describing regularities and patterns; Forming expressions—either verbal or symbolic; Using equivalence to examine different expressions of the same relationships and expressions; and Explicit generalised reasoning where students move between the particular to the general and vice versa, are able to identify and describe what varies and what stays the same, and work confidently with generalised expressions including their representation in different forms.

The research in RMFII developed an effective evidence-based learning progression with associated tasks for students' algebraic reasoning (Day et al., 2017). Nearly all tasks are graduated (multi-part) and designed to elicit progressive levels of students' algebraic generalisation, which is a key element of algebraic reasoning. Assessment tasks of this kind are helpful for classroom teachers to focus on the key shifts in students' thinking in order to foster their capability in this area. This paper will firstly show how the existing RMFII tasks, supported by Rasch modelling, align with and illustrate our five-level categorisation of algebraic generalisation. Secondly, the paper will show teachers of mathematics in the middle school years the importance of having *all* students progress at least to the third level of algebraic reasoning.

Drawing on the Rasch modelling (Bond & Fox, 2015) that was used in RMFII to rank the task item difficulty of scored responses across eight zones of algebraic reasoning, the Learning Progression for Algebraic Reasoning (LPAR) is related to the five levels of algebraic

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generalisation. In our recent paper (Stephens et al., 2021), several of the RMFII tasks were used to illustrate and validate the five levels of algebraic generalisation and in this paper one task, the Relational Thinking task, is used to exemplify how the LPAR zones relate to the levels of generalisation (Table 1). The Relational Thinking task (ARELS) is comprised of seven task items (ARELS1-ARELS7). The coding in the right column refers to the task items enumerated in Table 1, and to the score obtained for that item. For example, in Table 2, ARELS4.3 refers to the fourth relational thinking task item for which a score of 3 has been obtained.

Table 1
Relational Thinking Task Items and Rubrics

Item no.	Task item	Task item rubric Score
ARELS1	What numbers would go in these boxes to make a true number sentence (the numbers may be different). Explain your reasoning. $\square + 521 = 527 + \square$	0 No response or irrelevant response 1 Incorrect response but suggest the difference of 6 is recognised in some way (e.g., *add 6 to the right hand side)* 2 Two correct numbers given (e.g., 13 and 7; 527 and 521) but little/no reasoning. 3 Two correct numbers given where the number on the left is 6 more than the number on the right (e.g., 100 and 94) with reasoning that reflects the relationship between 521 and 527 (difference of 6).
ARELS2	Find a different pair of numbers that would make the number sentence above true.	0 No response or irrelevant response 1 A different and correct pair.
ARELS3	Describe how you could find **all** possible pairs of numbers that would make this a true number sentence.	0 No response or irrelevant response 1 Incomplete attempt based on previous answers (e.g., *add 2 more to both*). 2 Statement regarding the difference of 6 (e.g., *number on the left must be six more than the number on the right)* or expression showing the difference (e.g., a + 6, and a)
ARELS4	What numbers would go in these boxes to make a true number sentence (the numbers may be different). $\square - 521 = \square - 527$ Explain how you worked it out.	0 No response or irrelevant response 1 Incorrect answer (possibly due to errors in calculation) but recognises relationship between 521 and 527 (difference of 6). 2 Two correct numbers given (e.g., 613 and 619) but little/no reasoning, may include some calculations. 3 A pair of correct numbers given where the number on the right is 6 more than the number on the left (e.g., 600 and 606) with reasoning that reflects the relationship between 521 and 527 (difference of 6).
ARELS5	Find another pair of numbers that would make the number sentence above true.	0 No response or irrelevant response 1 A different and correct pair.
ARELS6	Describe how you could find all possible pairs of numbers that would make this a true number sentence.	0 No response or irrelevant response 1 Incomplete attempt based on previous answers (e.g., *add 10 to both*). 2 Statement regarding the difference of 6 (e.g., *number on the right must be six more than the number on the left*) or an expression showing the difference (e.g., a and a + 6)
ARELS7	What can you say about the relationship between *c* and *d* in this equation? $c \times 2 = d \times 14$	0 No response or irrelevant response 1 Specific solution provided (e.g., *c must be 7 and d must be 1 to make it a true number sentence)* or a general statement *(*e.g., *c is bigger than d*) 2 Statement correctly describes relationship (e.g., *c is 7 times the number d)*

The first level of our classification of algebraic generalisation is working with particular instances, where students find solutions to simple equivalence situations or extending simple growing patterns. For example, in the Relational Thinking task the first part of the task asks students to find two numbers that make the number sentence true (ARELS1), and the second part of the task asks the students to identify a second pair of numbers that also make the statement true (ARELS2).

The second level of our classification of algebraic generalisation is noticing and describing regularities, where students are asked to notice regularities among a sequence of particular cases. In these cases, students attend to quantities that stay fixed and those that vary (Radford, 2006; Rivera, 2013) within the context of the task. This is an important level as the next three algebraic generalisation levels rely upon being able to notice regularities.

Forming expressions, either verbal or symbolic is the third level of algebraic generalisation, which extends the noticing of regularities to expressing these regularities, as constants and variables in formulae that may be articulated verbally or using symbolic language. To obtain all three marks for the ARELS1 task item, students have to provide two correct numbers as well as demonstrate reasoning that showed the difference of six relationship.

Establishing and using equivalence enables students to be able to recognise that generalisations may be represented by different symbolic expressions. Students should be able to show that different expressions can generate the same number where the same variables are used and/or algebraic simplification can be used to show equivalence. It is important for students to be able to distinguish situations where although two expressions may look different from each other, they are in fact equivalent.

The final level of our classification of algebraic generalisation is explicit generalised reasoning. This is where students can move flexibly between the particular and the general and vice versa. Students at this level can identify and describe variables and constants and work confidently with generalised expressions. The Relational Thinking task item (ARELS7) asked students to comment on the relationship between c and d in the equation $c \text{ x } 2 = d \text{ x } 14$. To answer this successfully, students need to understand the equivalent relationship between two product expressions, and to generalise a relationship explicitly between the two variables c and d, using appropriate mathematical language.

Table 2
RMFII Zones and Levels of Generalisation Reported in Stephens et al. (2021)

Item no.	RMFII Zone	Level of algebraic generalisation
ARELS1.1	Zone 1	Level 1: Working with particular instances.
ARELS1.2	Zone 2	Level 1: Working with particular instances.
ARELS1.3	Zone 6	Level 3: Forming expressions – verbally or symbolically.
ARELS2.1	Zone 3	Level 2: Noticing and describing regularities.
ARELS3.1	Zone 5	Level 2: Noticing and describing regularities.
ARELS3.2	Zone 6	Level 4: Using equivalence.
ARELS4.1	Zone 3	Level 2: Noticing and describing regularities.
ARELS4.2	Zone 4	Level 2: Noticing and describing regularities.
ARELS4.3	Zone 7	Level 4: Using equivalence.
ARELS5.1	Zone 5	Level 2: Noticing and describing regularities.
ARELS6.1	Zone 6	Level 3: Forming expressions – verbally or symbolically.
ARELS6.2	Zone 6	Level 4: Using equivalence.
ARELS7.1	Zone 5	Level 2: Noticing and describing regularities.
ARELS7.2	Zone 7	Level 5: Explicit generalised reasoning.

From the examination of the Relational Thinking task, coupled with the analysis of three other RMFII tasks (Stephens et al., 2021) where several responses were located in Zone 8, it appeared that two of the key shifts in students' ability to generalise are difficult for students to navigate. The first of these key shifts is where students move from Level 2 noticing and describing regularities to Level 3 where they formalise this noticing and describing to correctly form algebraic expressions, either verbally or symbolically. This is demonstrated by noticing and describing regularities appearing in Zones 3 and 4 and the beginning of Zone 5 in the LPAR (Table 2), while Level 3, which formalises this in verbal and symbolic algebraic expressions does not appear until Zone 6. The second of the key shifts, which students find difficult to negotiate, is moving from Level 3 to Level 4 drawing on students' understanding of equivalence based on relational thinking and the writing and recognition of equivalent algebraic expressions. This level is evident in Zones 5, 6 and 7 of the LPAR (Table 2).

As these two key shifts are somewhat problematic for students, it is important that teachers provide multiple opportunities for students to identify regularities, identify variables and constants, form and communicate expressions, and use equivalence. One way for teachers to do this is to utilise rich tasks, such as Garden Beds from maths300 (maths300.com), that provide opportunities for students to demonstrate all forms of generalisation. By using several rich tasks within different contexts, teachers can ensure that students are being exposed to these critical steppingstones in the generalisation process. The RMFII Teaching Advice (Day et al., 2018) includes references to rich tasks from well-known sources such as maths300, reSolve (resolve.edu.au) and nrich (nrich.maths.org) at each of the LPAR Zones, which provide teachers with tasks that will assist them to progress students in their algebraic learning journeys.

The algebraic generalisations exemplified in this paper require students to become proficient in using appropriate combinations of language, algebraic representation, and mathematical justification. These forms of reasoning and proof are applicable across many problem-solving situations and explicitly generalised algebraic reasoning will be necessary for students' continuing study of mathematics. Just as important, this paper has drawn attention to assisting all students to navigate successfully Levels 3 and 4 where they learn to form correct algebraic expressions either verbally or symbolically, and subsequently become able to recognise and work with equivalent expressions. Navigating these two key shifts appears essential for students to be able to reason algebraically.

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