Key Shifts in Thinking in the Development of Mathematical Reasoning

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This symposium will draw on the evidenced-based learning progressions for multiplicative thinking, algebraic reasoning, geometrical reasoning, and statistical reasoning presented at previous MERGA conferences (see references by symposium authors in the papers that follow). The four papers will consider key shifts in thinking identified within each progression, without which students' progress may be seriously constrained.

Paper 1: *A Disposition to Attend to Relationships: A Key Shift in the Development of Multiplicative Thinking*
[Dianne Siemon]

This paper draws on multiple data sources to better understand the shift from additive to multiplicative thinking, which is crucial to all further participation in school mathematics.

Paper 2: *Key Shifts in Students' Capacity to Generalise: A Fundamental Aspect of Algebraic Reasoning*
[Max Stephens, Lorraine Day, & Marj Horne]

This paper will elaborate five levels of algebraic generalisation and two key understandings based on an analysis of students' responses to RMFII algebraic reasoning tasks.

Paper 3: *Cognitive Flexibility and the Coordination of Multiple Information in Geometry and Measurement*
[Rebecca Seah & Marj Horne]

This paper analyses students' solutions to problems in geometry and measurement situations in order to identify key components needed to nurture reasoning.

Paper 4: *Facilitating the Shift to Higher-order Thinking in Statistics and Probability*
[Rosemary Callingham, Jane Watson, & Greg Oates]

Students have difficulty moving from concrete representations and procedural mathematical statistics to context-based appreciation of data. This paper examines the barriers to this shift to higher-order thinking based on the Statistical Reasoning Learning Progression.

2022. N. Fitzallen, C. Murphy, V. Hatisaru, & N. Maher (Eds.), *Mathematical confluences and journeys* (Proceedings of the 44th Annual Conference of the Mathematics Education Research Group of Australasia, July 3–7), p. 41–57. Launceston: MERGA.

A Disposition to Attend to Relationships: A Key Shift in the Development of Multiplicative Thinking

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This paper draws on numerous data sources to better understand the shift from additive to multiplicative thinking in years 4 to 9. Research studies that have used the Scaffolding Numeracy in the Middle Years assessment tasks have found that while students can be supported to move through the early and upper zones of the Learning and Assessment Framework for multiplicative thinking, it has been difficult to move students through Zone 4 at the same rate. A closer examination of item responses at this level reveal that a disposition to notice and work with relationships between quantities may explain this phenomenon.

Access to multiplicative thinking has long been recognised as critical to success in school mathematics in the middle years and beyond (e.g., Harel & Confrey, 1994; Hilton et al., 2016; Lamon, 1993; Siemon et al., 2006). However, many students at this level do not have access to this critical capacity (Brown et al., 2010; Siemon, 2019) suggesting that the transition from additive to multiplicative thinking is more complex than previously recognised (e.g., Clark & Kamii, 1996; Van Dooren et al., 2010; Vergnaud, 1983).

Research studies that have used the *Scaffolding Numeracy in the Middle Years* (SNMY) assessment tasks have found that while students can be supported to move through the early and upper zones of the Learning and Assessment Framework (LAF) for multiplicative thinking (Siemon, 2016, 2019), this appears not to be the case for Zone 4, which is where students are starting to use multiplicative thinking on a more consistent basis (see Figure 1 for examples). This and the fact that the proportion of students in Zone 4 is typically higher than in any other zone confirms the difficulty of acquiring multiplicative thinking, but it also prompts the question, "What can be learnt about the barriers to multiplicative thinking from a closer analysis of student responses to tasks that span Zone 4?"

Solves more familiar multiplication and division problems involving two-digit numbers (e.g., *Butterfly House c* and *d, Packing Pots c, Speedy Snail a*). Tend to rely on additive thinking, drawings and/or informal strategies to tackle problems involving larger numbers and/or decimals and less familiar situations (e.g., *Packing Pots d*, *Filling the Buses a* and *b*, *Tables & Chairs g* and *h*, *Butterfly House h* and *g, Speedy Snail c, Computer Game a, Stained Glass Windows a* and *b*). Tends not to explain their thinking or indicate working. Able to partition given number or quantity into equal parts and describe part formally (e.g., *Pizza Party a* and *b*), and locate familiar fractions (e.g., *Missing Numbers a*). Beginning to work with simple proportion, for example make a start, represent problem, but unable to complete successfully or justify their thinking (e.g., *How Far a*, *School Fair a* and *b*).

Figure 1. Rich text description of Zone 4 (Siemon et al., 2006).

Approach

The *Stained Glass Windows* task (Figure 2) was selected for analysis as the item difficulties ranged from Zone 3 to Zone 7 and the setting, while accessible, did not conform with the more familiar multiplicative models implicit in problems such as *Packing Pots* (i.e., equal groups or arrays). It was also selected because the context invited additive thinking, which tested the extent to which students could see past that to the underlying multiplicative structure (e.g., Vergnaud, 1983), which was hinted at in the task stem. These same criteria were met by another

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task, *Canteen Capers*, which involved lunch order options given two choices of rolls, four choices of filling, and three choices of drink. The first item required students to identify the number of options for a roll with a specified filling and a drink item (2 x 3). The second item required them to determine if everyone in a class of 26 children could have a different lunch order made up of a roll, filling, and drink. In both cases students were asked to explain their reasoning using as much mathematics as they could.

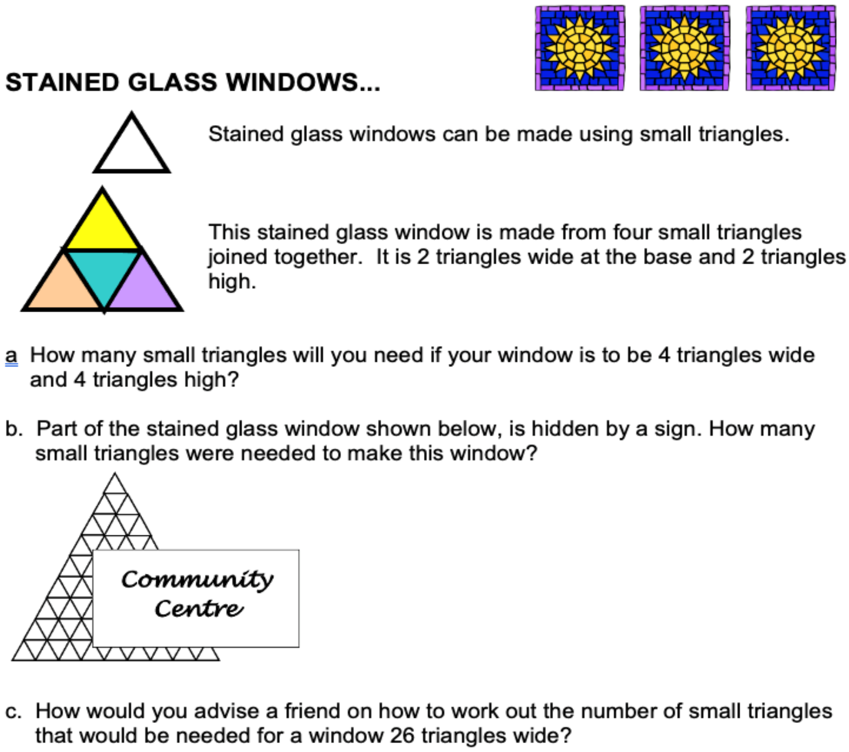

Figure 2. Stained Glass Windows task from SNMY Assessment Option 1 (Siemon et al., 2006).

Data sets from four different projects are used in the analysis reported here. That is, the SNMY project (Siemon et al., 2006a), the *Reframing Mathematical Futures Priority* project (Siemon, 2016), the *Reframing Mathematical Futures II* project (Siemon et al., 2018), and the *Growing Mathematically—Multiplicative Thinking* project (Callingham & Siemon, 2021). The student populations across the four projects ranged from Year 4 to Year 9 of whom approximately 65% were from low socio-economic backgrounds.

A total of 11,775 students (67% in Years 7 or 8) responded to the Stained-Glass Windows task and 4985 students (83% in Years 7 or 8) to the Canteen Capers task. Student responses were marked by project schoolteachers using partial credit scoring rubrics and entered into a deidentified spreadsheet which was forwarded to the research team for analysis.

Analysis and Discussion

Table 1 shows the proportion of students scoring a 1, 2, or 3 on items a, b, and c of the two tasks with the last entry for each item indicating the proportion of students providing a multiplicative response. The very low proportion of students evidencing either an additive or a multiplicative response to both problems is at odds with the suggestion that strategy usage is impacted by the numbers involved or the extent of the challenge (Downton & Sullivan, 2017; Larsson et al., 2017). It is undoubtedly the case that “some students use strategies that are only as complex as they need” (Downton & Sullivan, 2017, p. 303). However, the proportion of students providing a correct answer supported by additive reasoning (i.e., a score of 2 on items a and b of Stained Glass Windows and item a of Canteen Capers) is surprisingly low, given that the majority of the students were from Years 7 or 8.

Table 1
Proportion of Students Scoring a 1, 2, or 3 on Each Item of Each Task

	Stained Glass Windows (*n* = 11,775)			Canteen Capers (*n* = 4985)	
Score	A	b	c	a	B
1	22.9%	29.5%	11.3%	22.2%	29.8%
2	28.5%	13.8%	22.4%	22.6%	24.9%
3	13.7%	18.5%		17.8%	

An insight into why this might be the case is afforded by the item difficulties shown in Table 2 for the Stained Glass Windows task. On the ordered list of item difficulties produced by the Rasch analysis a score of 3 on item a (sgwa3) was located towards the top of the scale in Zone 7. However, the item difficulties associated with recognising and using the same relationship in items b and c (i.e., sgwb3 and sgwc2) were located in Zone 6, which suggests that noticing the rule is harder than applying the rule despite the strong suggestion of the rule in the stem (2 x 2) and the likelihood that 4 and 16 would be recognised as square numbers.

Table 2
Scoring Rubrics for Stained Glass Windows by Item Difficulty (LAF location)

Item	Rubric (item difficulty code)	Score	Zone
A	Incorrect based on inaccurate drawing and/or counting of triangles, or correct with little/no explanation (sgwa1)	1	3
	Correct (16 triangles), with evidence of additive reasoning based on drawing and counting (sgwa2)	2	4
	Correct (16 triangles), with evidence of multiplicative reasoning based on 4 x 4 (sgwa3)	3	7
B	Incorrect based on inaccurate drawing and/or counting of triangles, or correct (81 triangles) with little/no explanation (sgwb1)	1	3
	Correct (81 triangles), with evidence of additive reasoning based on drawing and counting, or inappropriate use of area formula (e.g., L x W) (sgwb2)	2	4
	Correct (81 triangles), with evidence of multiplicative reasoning based on pattern (e.g., 9 by 9) (sgwb3)	3	6
C	Advice based on additive thinking (e.g., “2 less each time you go up”) (sgwc1)	1	5
	Correct, advice based on rule (e.g., 26 x 26) (sgwc2)	2	6

A similar phenomenon is observed for the Canteen Capers task where the item difficulties ranged from Zone 2 to Zone 8. Recognising and providing a multiplicative explanation for part a (e.g., “It’s 6 because for each roll she could have one of the 3 drinks”) was located in Zone 8. For item b, determining that there were enough different options for each child in a class of 26 on a systematic basis that suggested use of 2 x 4 x 3, was located in Zone 6. Again, this suggests that noticing the relationship was harder than applying it.

There are a number of possible explanations for the difficulty of these items that warrant further investigation. One is the absence of a familiar multiplicative model, which is known to facilitate multiplicative understanding and calculation (Larsson et al., 2017). However, the fact that multiplicative thinking is elicited by these tasks despite this suggests that something more is needed to support the shift from additive to multiplicative thinking, particularly as models connected to solution strategies can invoke instrumental responses (Skemp, 1976) making it difficult to discern multiplicative thinking.

Apart from the obvious need to offer a broader range of multiplicative tasks and contexts that are not readily connected to students' existing models of multiplication (e.g., Downton & Sullivan, 2017), the analysis here suggests that the "something more" is a disposition to attend to relationships between quantities in ways that look for generalities rather than particulars. In other words, it is about an alertness to and appreciation of mathematical structure (e.g., Mason et al., 2009) and multiplicative structure in particular (e.g., Mulligan, 2002; Vergnaud, 1983).

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