SECOND-YEAR MATHEMATICS

BRESLICH

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FELIX KLEIN

FELIX KLEIN, probably the most eminent German mathematician of his time, was born at Düsseldorf in 1849 and is still living, though he retired from professional life in 1912. He studied at Bonn, Germany, where at the age of seventeen he became assistant to the renowned physicist, Plücker, in the Physical Institute. He took his Doctor's degree at eighteen years of age, then went to Berlin, and a little later to Göttingen. Here he assisted in editing Plücker's works.

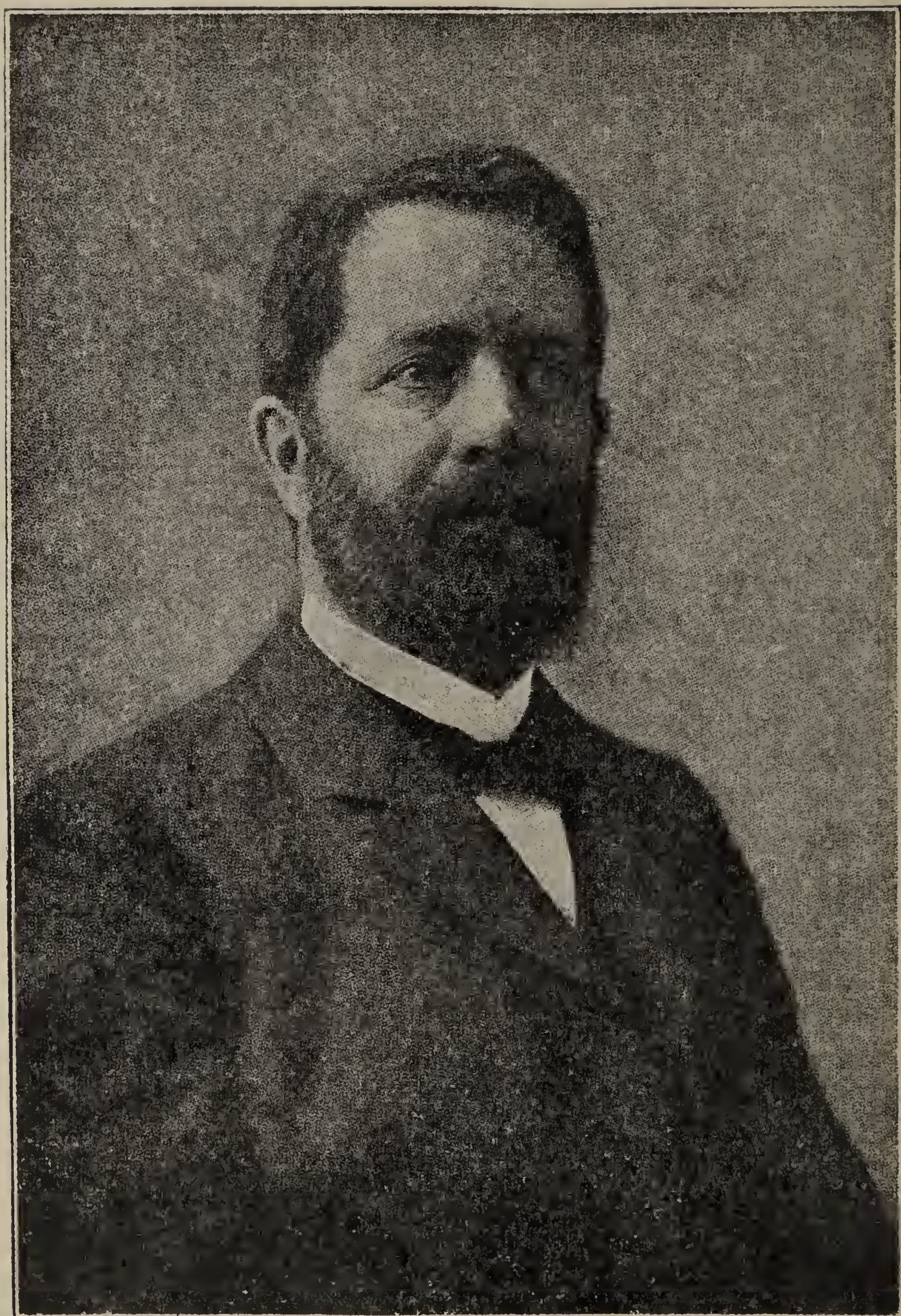
Klein entered the Göttingen university faculty in 1871. The next year he became professor of mathematics at Erlangen, and afterward held professorships at the Technical Institute of Munich (1875–80) and at the universities of Leipzig (1880–86) and of Göttingen (1886–1912). He was sent to the World's Fair at Chicago in 1893 by the Prussian government, to represent the university interests of the nation.

Klein's pupils are found in most of the leading universities of the United States. No one else in Germany has exerted so great an influence on American mathematics. He has been a tireless worker himself, both in the science and in the improvement of the teaching of mathematics. He was made president of the International Commission on the Teaching of Mathematics, in 1908, by the Fourth International Congress of Mathematicians held that year in Rome, Italy.

His contributions to mathematics are extensive, but they cannot even be enumerated here. It is scarcely too much to assert that Klein has led the main movements for advancement of mathematical teaching since the beginning of the present century. This applies not only to university teaching but to secondary (high-school) teaching as well.

In his *Teaching of Geometry*, p. 69, Professor David Eugene Smith of Teachers' College, Columbia University, says of Klein: "He has the good sense to look at something besides good mathematics: (1) he insists upon the psychological point of view; (2) he demands careful selection of subject-matter; (3) he insists on reasonable correlation with practical work; (4) he looks with favor upon the union of plane and solid geometry; (5) he favors also the union of algebra and geometry."

Some of Klein's best interpreters have said of his reformatory movement that Klein's main idea is to make "functional thinking in its geometrical form" the distinguishing mark of secondary-school work in mathematics.



F. Klein

Second-Year Mathematics *for* Secondary Schools

BY

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EDITOR'S PREFACE

This book by its copyright purports to be a second edition of a former text of the same title by other authors. It is this in the sense that it carries forward through the second high-school year a reconstructed form of the union mathematics of a first-year text. It allies itself with the former work also in that it places chief emphasis on plane geometry.

The older *Second-Year Mathematics* was an attempt to furnish a concrete contribution to the problem of introducing greater homogeneity and continuity into the secondary mathematical subjects from year to year. In this particular also this book resembles the earlier text.

In a very real sense, however, this volume is a new contribution, with its own plans and purposes. Its primal aim is to furnish a gradually progressive continuation of the form of reconstructed mathematics of the text *First-Year Mathematics,* by Mr. Breslich himself. It aims definitely to teach *how to study* as well as the *content* of a second unit of secondary mathematics. It accomplishes this through the nature and form of the material, through explicit exhibits and formulated tests of sound and unsound reasoning, through study-helps, directions for working, and systematic chapter summaries.

It seeks neither to eliminate nor to curtail inherent mathematical formalism, but to fill forms and technique with the meanings that flow from a well-balanced treatment of related material drawn from the kindred elementary subjects. A unique feature of the book is the attractive presentation of a considerable body of associated solid geometry. This is an economy and is in accord with modern educational precepts.

The cordial response from the best sources that Mr. Breslich's *First-Year Mathematics* has met in the first year after its publication proves his first text to be generally adaptable to classroom conditions, and augurs that the present book will be found to work smoothly under average conditions. An examination of the context is sufficient to convince the open-minded reader that the educational results of this book will greatly surpass those of the text it is displacing, as well as those of any standard text treating plane geometry as a separate subject.

G. W. Myers

Chicago, Ill.
August, 1916

AUTHOR'S PREFACE

In planning the work of the second year the author has kept in mind the following facts:

1. *Through the second year the combined type of material of the mathematics taught in the first year is to be carried forward, the emphasis here being shifted to geometry.*

2. *The operations and laws of arithmetic are to be reviewed wherever opportunity is offered or occasion warrants,* as in the evolution of formulas, in the introduction of new algebraic topics, and in problems of calculation.

3. *The algebraic ground gained in the first year is to be held and the field extended at least as far as is customary with the algebra before the third year.*

A firm hold is kept on algebra by the employment of algebraic notation and by the continued application of the equation to geometrical matters. New algebraic topics are developed when opportunity and need arise. Thus, elimination by comparison and by substitution, so frequently needed in proofs and in the solution of exercises, is taught very early. The solution of the quadratic equation by means of the formula, the operations with fractions, and factoring are all reviews or further extensions of topics begun in the first year.

4. *The study of plane geometry is to be completed.*

In the first-year course the student has gained a thorough understanding of the fundamental notions of geometry. Accordingly, in the second year, methods of a more formal character are introduced *from the start.* But even before this, the advantage of the reasoning process over the process of measuring has been *recognized.*

Mathematical fallacies and optical deceptions are now used to make the need of logical proof still more apparent.

A definite aim is to give the pupil something of the secret of geometrical strategy, i.e., some skill in attacking, taking possession of, and exploiting a geometrical difficulty. With this in view, methods of proof are discussed and emphasized, not once for all, but throughout the course.

To cultivate versatility and system, students are taught to choose between various methods of proof, and always to follow some definite plan and not to trust to the chance of stumbling upon a proof. To this end many model proofs are given. With other proofs statements or reasons that are more or less apparent to the student are omitted, in order that he may also acquire the habit of independent thought and that his powers of argumentation may be strengthened.

The custom of dividing the subject of geometry into a few "books" has been abandoned as being of only traditional or historical value. The course is, however, divided into a number of short chapters, each dealing with one or but a few central topics. This arrangement is far better adapted to the study of high-school students than is the traditional grouping into "books," since the aims and purposes of the short chapters are easily seen. It is found to be more economical of the student's time and energy than the old method.

5. *The student should receive training in both plane and solid geometry.*

Many theorems of solid geometry that are closely related to corresponding theorems in plane geometry are proved in the second year, thus furnishing the student appropriate exercise in both two- and three-dimensional thinking.

A real advance is thus made in the study of solid geometry before plane geometry is completed. The work in solid geometry includes the theorems on lines and planes in space and on diedral angles.

6. *The study of trigonometry begun in the first year is to be continued.*

It is a distinct educational loss that the strong appeal that trigonometry has for high-school pupils should not be utilized earlier than is customary. Moreover, trigonometric methods here often replace algebraic and geometric methods, giving the student the opportunity to see some of the advantages of trigonometry over algebra and geometry.

In addition to the foregoing aims the following are included: (*a*) the application of three trigonometric functions (sine, cosine, and tangent) to the solution of the right triangle and to a number of practical problems; (*b*) the development of some of the fundamental relations between these important functions.

7. *No topical treatment of the theory of limits is intended.*

Such a treatment is believed not to belong to the early years of the high-school course. However, the question of the existence of incommensurable lines and numbers is raised, examples of these are given, and the notion of the limit of a sequence is developed.

8. *Since the usefulness of a study always appeals very strongly to a beginner, this phase is emphasized throughout the course.*

The importance and the significance of geometrical facts in the affairs of everyday life are impressed upon the pupil. This wins his sanction of the worth of the study to himself more fully than any other sort of appeal that the teacher of geometry can make.

9. *The plan of introducing definitions whenever needed and not before, which is used in the first-year course, has been followed also in the second year.*

After definitions are introduced they are continually used, in order that the pupil may acquire mastery through use.

The material as arranged in this course opens to the student a broader, richer, more useful, and therefore more alluring field of ideas, and lays a more stable foundation for future work, than does any separate treatment. A great saving of the student's time is effected by developing arithmetic, algebra, geometry, and trigonometry side by side. This union of subjects also makes unnecessary the long and tiresome reviews usually given at the beginning of each subject, and gives place for frequent incidental reviews leading immediately to an extension of the subject.

Often a high-school pupil fails rightly to esteem a high-school subject because he cannot discern its bearing either on what has preceded or on what is to follow. But, having experienced the closeness of the relation between the subjects he does not lose sight of the familiar fields while he is obtaining an outlook into neighboring and more remote ones. There is thus an economy resulting both from accomplishing more work in less time and from the performance of tasks that are intelligently motivated.

The book contains exercises in sufficiently large numbers to allow the instructor some choice in case he wishes to reduce the scope of the course. Problems and theorems which may be omitted are marked with the symbol ‡. These problems may be taken either in the course by the stronger pupils or at the end of the course by all. If taken at the end of the course, they will give the student ample drill and review of the right sort.

"Second-Year Mathematics" may be used successfully in classes that have had only algebra during the first year.

The syllabus at the beginning of the book gives all the theorems and axioms taught in *First-Year Mathematics*, indicating the order in which they were given. This furnishes an effective introduction to the formal geometry of the second year, especially so if it is taught by the *syllabus method.* It helps the student very materially in overcoming the difficulties usually encountered in beginning demonstrative geometry, and at the same time it gives him the opportunity of availing himself of all the advantages of the correlation of algebra, geometry, and trigonometry in the second year.

The author desires to render acknowledgment to Professor Charles H. Judd for his numerous suggestions and criticisms. His recent book on *The Psychology of High-School Subjects* has been of invaluable service in planning this course.

The encouragement, interest, and advice of Principal F. W. Johnson, of the University High School, have been a very substantial help in bringing about the publication of this course.

The author is also indebted to his colleagues, Messrs Raleigh Schorling, Horace C. Wright, and Harry N Irwin, who have read and criticized in detail every chapter of the book.

The portraits of Fermat and Gauss which are used as inserts in the text have been taken from the "Philosophical Portrait Series," published by the Open Court Publishing Company, Chicago.

Ernst R. Breslich

Chicago, Ill.
September, 1916

CONTENTS

STUDY HELPS FOR STUDENTS[1]

The habits of study formed in school are of greater importance than the subjects mastered. The following suggestions, if carefully followed, will help you make your mind an efficient tool. Your daily aim should be to learn your lesson in *less* time, or to learn it *better* in the *same* time.

1. Make out a definite daily program, arranging for a definite time for the study of mathematics. You will thus form the habit of concentrating your thoughts on the subject at that time.

2. Provide yourself with the material the lesson requires; have on hand textbook, notebook, ruler, compass, special paper needed, etc. When writing, be sure to have the light from the left side.

3. Understand the lesson assignment. Learn to take notes on the suggestions given by the teacher when the lesson is assigned. Take down accurately the assignment and any references given. Pick out the important topics of the lesson before beginning your study.

4. Learn to use your textbook, as it will help you to use other books. Therefore understand the purpose of such devices as index, footnotes, etc., and use them freely.

5. Do not lose time getting ready for study. Sit down and begin to work at once. Concentrate on your work, i.e., put your mind on it and let nothing disturb you. Have the will to learn.

[1] These study helps are taken from *Study Helps for Students in the University High School.* They have been found to be very valuable to students in *learning* how to study and to teachers in *training* students how to study effectively.

6. As a rule it is best to go over the lesson quickly, then to go over it again carefully; e.g., before beginning to solve a problem read it through and be sure you understand what is given and what is to be proved. Keep these two things clearly in mind while you are working on the problem.

7. Do individual study. Learn to form your own judgments, to work your own problems. Individual study is honest study.

8. Try to put the facts you are learning into practical use if possible. Apply them to present-day conditions. Illustrate them in terms familiar to you.

9. Take an interest in the subject. Read the corresponding literature in your school library. Talk to your parents about your school work. Discuss with them points that interest you.

10. Review your lessons frequently. If there were points you did not understand, the review will help you to master them.

11. Prepare each lesson every day. The habit of meeting each requirement punctually is of extreme importance.

CHAPTER I

ASSUMPTIONS, THEOREMS, AND CONSTRUCTIONS GIVEN IN FIRST-YEAR MATHEMATICS

To the Student

In the first-year course the student has become familiar with a number of geometric truths. In the second-year course these are used to establish other truths. A complete list of the geometric assumptions and theorems of the first course is given below. Future references will be made to this list, to save the student the inconvenience of looking them up in *First-Year Mathematics.*

The numbers in the parentheses refer to the sections in *First-Year Mathematics* in which the statements were given for the first time.

Classes which did not use *First-Year Mathematics* as the text of the first year may use this list as a syllabus, the students working out the proofs under the teacher's direction and in the order indicated by the numbers in the parentheses. For this book, however, these truths play the part of assumptions.

Assumptions

1. Through two points one and only one straight line can be drawn. (20)

2. A straight line two of whose points lie in a plane, lies entirely in the plane. (204)

3. The shortest distance between two points is the straight line-segment joining the points. (21)

4. Two straight lines intersect in one and only one point. (25)

5. A line-segment or an angle is equal to the sum of all its parts. (33)

6. A segment or an angle is greater than any of its parts, if only positive magnitudes are considered. (34)

7. If the same number is added to equal numbers, the sums are equal. (35)

8. If equals are added to equals, the sums are equal. (36)

9. If the same number or equal numbers be subtracted from equal numbers, the differences are equal. (41)

10. The sums obtained by adding unequals to equals are unequal in the same order as are the unequal addends. (42)

11. The sums obtained by adding unequals to unequals in the same order, are unequal in the same order. (43)

12. The differences obtained by subtracting unequals from equals are unequal in the order opposite to that of the subtrahends. (44)

13. If equals be divided by equal numbers (excluding division by 0), the quotients are equal. (78)

14. If equals be multiplied by the same number or equal numbers, the products are equal. (80)

Angles

15. All right angles are equal. (118)

16. Equal central angles in the same or equal circles intercept equal arcs. (124)

17. In the same or equal circles equal arcs are intercepted by equal central angles. (125)

18. A central angle is measured by the intercepted arc. (126)

19. If two angles have their sides parallel respectively they are equal or supplementary. (197)

20. If the sum of two adjacent angles is a straight angle, the exterior sides are in the same straight line. (177)

21. The sum of all the adjacent angles about a point, on one side of a straight line, is a straight angle. (179)

22. The sum of all the angles at a point just covering the angular space about the point is a perigon. (180)

23. If two lines intersect, the opposite angles are equal. (183)

Angles of a Triangle

24. The sum of the angles of a triangle is 180°. (112), (198)

25. The sum of the exterior angles of a triangle, taking one at each vertex, is 360°. (115)

26. An exterior angle of a triangle equals the sum of the two remote interior angles. (118), (199)

27. If the angles of one triangle are respectively equal to the angles of another, the triangles are similar. (233)

28. The base angles of an isosceles triangle are equal. (280)

29. An equilateral triangle is equiangular. (281)

30. If two angles of a triangle are equal the triangle is isosceles. (281)

31. The acute angles of a right triangle are complementary angles. (184)

32. In a right triangle whose acute angles are 30° and 60°, the side opposite the 90°-angle is twice as long as the side opposite the 30°-angle. (185)

33. If two sides of a triangle are unequal, the angles opposite to them are unequal, the greater angle lying opposite the greater side. (281)

34. If two angles of a triangle are unequal the sides opposite to them are unequal, the greater side lying opposite the greater angle. (281)

Perpendicular Lines

35. The shortest distance from a point to a line is the perpendicular from the point to the line. (285)

36. At a given point in a given line one and only one perpendicular can be drawn to the line. (176)

From a given point one and only one perpendicular can be drawn to a given line.

37. All points on the perpendicular bisector of a line-segment are equidistant from the endpoints of the segment. (281)

38. If a point is equidistant from the endpoints of a line-segment, it is on the perpendicular bisector of the segment. (283)

39. If each of two points on one line is equidistant from two points of another line the lines are perpendicular. (283)

Parallel Lines

40. Parallel lines are everywhere equally distant. (192)

41. One and only one parallel can be drawn to a line from a point outside the line. (194)

42. If two lines are cut by a transversal making the corresponding angles equal, the lines are parallel. (195)

43. Two lines perpendicular to the same line are parallel. (195)

44. Two lines are parallel if two alternate interior angles formed with a transversal are equal. (195)

45. Two lines are parallel if the interior angles on the same side formed with a transversal are supplementary. (195)

46. Two lines parallel to the same line are parallel to each other. (195)

47. If two parallel lines are cut by a transversal the corresponding angles are equal; the alternate interior angles are equal; the interior angles on the same side are supplementary. (196)

Proportional Line-Segments

48. A line parallel to one side of a triangle divides the other two sides into corresponding parts having equal ratios. (244)

49. A line bisecting an angle of a triangle divides the side opposite that angle into parts whose ratio is equal to the ratio of the other sides. (245)

50. A line dividing two sides of a triangle into corresponding parts having the same ratio, is parallel to the third side of the triangle. (246)

Areas and Volumes

51. The area of a square is equal to the square of one side. (140)

52. The area of a rectangle equals the product of the base by the altitude. (141)

53. The volume of a rectangular parallelopiped equals the product of the length by the height by the width. (145)

54. The volume of a cube is equal to the cube of one edge. (146)

55. The area of a parallelogram is equal to the product of the base by the altitude. (163)

56. The area of a triangle is equal to one-half the product of the base by the altitude. (164)

57. The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases. (166)

Proportionality of Areas

58. In a proportion the product of the means is equal to the product of the extremes. (259)

59. The areas of two rectangles are in the same ratio as the products of their dimensions. (260)

60. Two rectangles having equal bases are in the same ratio as the altitudes. (261)

61. Two rectangles having equal altitudes are in the same ratio as the bases. (262)

62. The areas of parallelograms are in the same ratio as the products of the bases and altitudes. (263)

63. The areas of triangles are in the same ratio as the products of the bases and altitudes. (264)

64. The areas of parallelograms having equal bases are in the same ratio as the altitudes. (265)

65. The areas of triangles having equal bases are in the same ratio as the altitudes. (266)

Congruent Triangles

66. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other. (*s.a.s.*) (274)

67. Two triangles are congruent if two angles and the side included between their vertices in one triangle are equal respectively to the corresponding parts in the other. (*a.s.a.*) (275)

68. If three sides of one triangle are equal, respectively, to the three sides of another triangle, the triangles are congruent. (*s.s.s.*) (283)

69. Two right triangles are congruent if the hypotenuse and one side of one are equal respectively to the hypotenuse and a side of the other. (285)

Similar Triangles

70. Two triangles are similar if the ratios of the corresponding sides are equal. (236)

Loci

71. The perpendicular bisector of a segment is the locus of all points equidistant from its endpoints. (284)

72. The bisector of an angle is the locus of points which are equidistant from the sides. (304)

Tangents

73. The radius drawn to the point of contact of a tangent is perpendicular to the tangent. (308)

74. A line perpendicular to a radius at the outer endpoint is tangent to the circle. (309)

Theorem of Pythagoras

75. In a right triangle the sum of the squares on the sides including the right angle is equal to the square on the hypotenuse. (402)

CHAPTER II

METHODS OF PROOF

Logic

76. Reasoning. In the first-year course we studied some of the laws of algebra and became acquainted with a number of useful geometric facts. The truth of many of these facts was found and verified by measurement, of others, especially toward the end of the course, by a process of reasoning, or proof.

In our everyday life we reason whenever we infer one truth from another. Thus, from the general truths that metals are good conductors of heat and that aluminium is a metal, we infer that aluminium is a good conductor of heat. Or, if we accept as true the statement that iron is the most useful metal and that iron is the cheapest metal, we may infer that the most useful metal is also the cheapest metal.

In every branch of knowledge there are employed certain principles and forms of thought by means of which all persons must think and reason. Logic treats of these principles. Moreover, it helps us to avoid the fallacies which may arise from neglecting the correct rules of thinking. In particular, it points out why it is absurd to make such an inference as that all Europeans are Frenchmen from the known fact that all Frenchmen are Europeans.

False reasoning. By incorrect reasoning, some of the ancient Greek philosophers pretended to prove that motion was impossible. "For," they said, "a moving body must move either in the place where it is, or where it is not; now it is absurd to hold that a body could be where it is not; and if it moves, it cannot be in a place where it is; therefore it cannot move at all."

The student is probably familiar with the following absurdity:

> No dog has 9 tails.
> One dog has 1 more tail than no dog.
> Therefore, one dog has 10 tails.

Thus, to know how to use the rules of correct reasoning is valuable also in that it enables us to point out weak places in an incorrect argument, and to replace incorrect reasoning by sound reasoning in our own work.

Geometrical Fallacies

77. Likewise the reasoning used in geometry and algebra follows certain laws. The importance of exercising great care in a geometric proof may be illustrated by two of the well-known puzzles of geometry, viz.:

1. **Theorem:** *Every triangle is isosceles.*

Given any triangle, as ABC, Fig. 1.

To prove that ABC is isosceles.

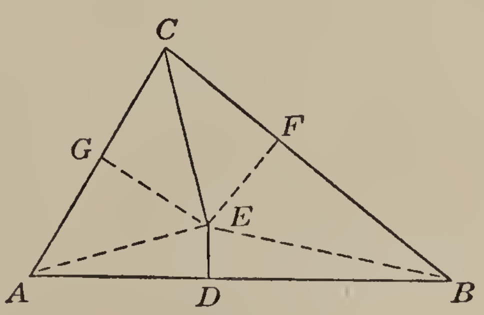


Fig. 1

Proof: Let DE be the perpendicular bisector of AB and let CE be the bisector of angle C, meeting DE at E.

From E draw EA and EB.

Draw EG perpendicular to AC, and EF perpendicular to CB.

Then $\triangle ADE \cong \triangle BDE$ (§ 69).

Hence, $AE = BE$

(since corresponding sides of congruent triangles are equal).

$\triangle CEG \cong \triangle CEF$ (§ 67).

Hence, $EG = EF$ and $CG = CF$. Why?

Therefore $\triangle AEG \cong \triangle BEF$

(hypotenuse and a side, § 69).

Hence, $GA = FB$.

Since $\underline{CG = CF,}$

it follows that $CG + GA = CF + FB$,

or $CA = CB$.

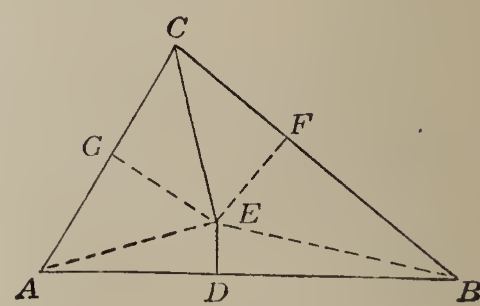


Fig. 1

Therefore the triangle ABC, although known not to be isosceles, would seem to have been proved to be isosceles.

Make a careful construction of Fig. 1, and discover the error in the demonstration.

2. *To show geometrically that* $64 = 65$.

Draw two right triangles having the sides including the right angle equal to 3 and 8, respectively (Fig. 2).

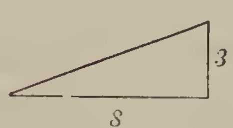


Fig. 2

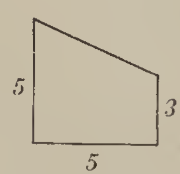


Fig. 3

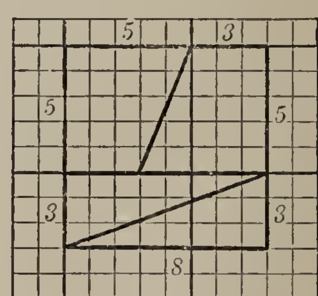


Fig. 4

Draw two quadrilaterals (Fig. 3) having one pair of opposite sides parallel and equal to 3 and 5, respectively, and the third side perpendicular to the parallel sides and equal to 5. Placing the triangles and quadrilaterals as in Fig. 4, a square is obtained whose area is equal to $8 \times 8 = 64$. If now, they are placed as in Fig. 5, a rectangle is formed whose area is equal to $13 \times 5 = 65$.

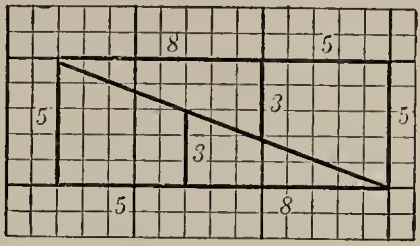


FIG. 5

Hence, $64 = 65$!

Make a careful construction and discover the error.

78. Need for proof. In both the fallacies in § 77 the difficulty has come from assuming that what looks to be *nearly* true is *exactly* true. The moral is, of course, things that look correct cannot always be relied upon as correct. The word "intuition" is used to designate the sort of reasoning that draws its conclusions from direct appearances.

The following exercises are illustrations of the danger of going astray even in geometry through *too ready* a reliance on intuition.

EXERCISES

1. Compare the segments a and b, Fig. 6, as to length by looking at the figure. Then measure each segment.

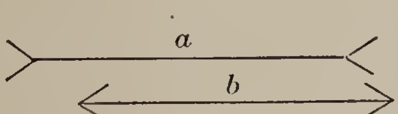


FIG. 6

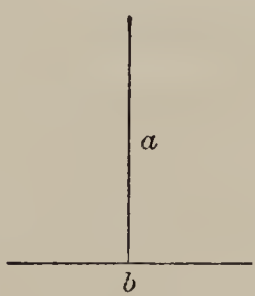


FIG. 7

2. Compare, as in Exercise 1, the segments a and b in Fig. 7. Test by measuring.

3. Are the lines *AB* and *CD* in Fig. 8 parallel? Answer the question, then test by measuring the distances between the lines.

FIG. 8

4. Are the lines *AB* and *CD*, Figs. 9 and 10, in the same straight line? Test with a ruler.

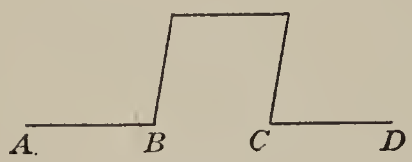


FIG. 9

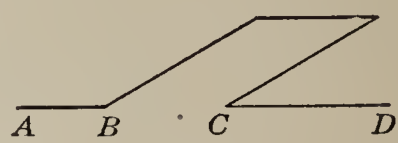


FIG. 10

5. Are the lines *AB* and *CD*, Fig. 11, straight lines? Test with a ruler.

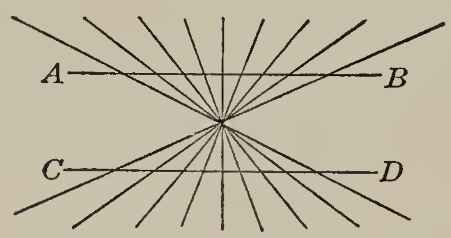


FIG. 11

6. Count the number of blocks in Fig. 12. Continue to look at the figure and you will see either one more or one less.

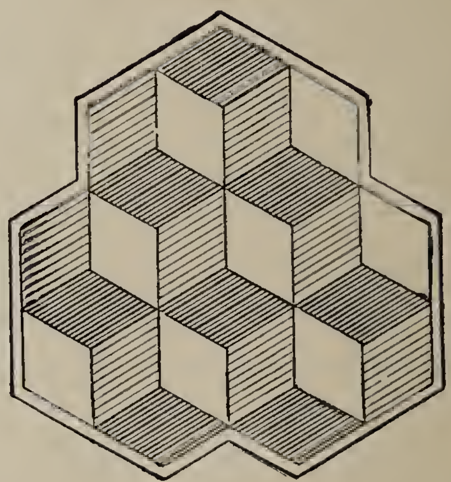


FIG. 12

79. Methods. There is no *one specific* method by which *all* theorems or problems may be attacked or proved. However, certain *general directions and methods* as to the way of attacking problems and proving theorems may be stated. A knowledge of these methods is of greatest importance as they will keep the student from groping about blindly for a proof, wasting his time and energy. Several methods of proof are discussed in this chapter, others are considered in chapter IV.

80. General directions. Hypothesis. Conclusion.

1. Read the problem carefully, get it clearly in mind, and keep it in mind while at work on it. Most problems need at least *two* readings.

2. If the problem is a geometric theorem or exercise, draw carefully a *general* figure. Thus, if the theorem refers to a triangle, draw a triangle with unequal sides, not an equilateral, or isosceles, or right triangle. This will keep you from committing the error of proving a theorem only for a *special case.*

3. Write down what is given (**the hypothesis**) and what is to be proved (**the conclusion**), referring all statements to the figure.

4. If a proof does not readily suggest itself to you, think of all the things you have learned that are like the problem you are trying to work out, e.g., recall the theorems that seem like the task before you.

Thus, if you are to prove two angles equal, ask the question: Under what conditions are two angles equal? If you wish to prove two lines parallel, the question should be: When are two lines parallel? Then select the theorem that seems to you most promising or suitable, until you find something that brings you to your goal. It is a good plan to review and summarize the theorems and problems that have been established previously. Keep up this practice until it becomes a *habit,* and you will acquire the art of selecting very quickly the theorem that is needed to prove a new theorem or problem.

5. The conclusion may sometimes be obtained by drawing lines, not given in the figure, as described by the hypothesis. Thus, if $AC = CB$, Fig. 13,

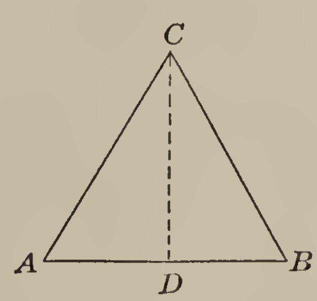


Fig. 13

we may prove that $\angle A = \angle B$, by drawing the bisector of angle C and then proving $\triangle ADC \cong \triangle BDC$.

81. Method of proof by superposition. This method was used in proving some of the theorems on congruent triangles, §§ 66, 67. It consists in placing one figure over another and then showing that all parts of the one coincide with the corresponding parts of the other. This method, although practical when the elements involved in the proof are few and specific, is not considered a good theoretical test by the mathematician. For, the axioms validating superposition are usually not given in full detail. The result is that the student is in danger of drawing rashly the conclusion which is to be established by the superposition of the one figure upon the other. The method is used only in a few cases.

82. Method of congruent triangles. When trying to prove that lines or angles are equal, it is sometimes possible to show that they are corresponding parts of congruent triangles. It may be necessary to draw helping lines to obtain the congruent triangles, of which the lines or angles to be proved equal are corresponding parts. The following proof will illustrate the method:

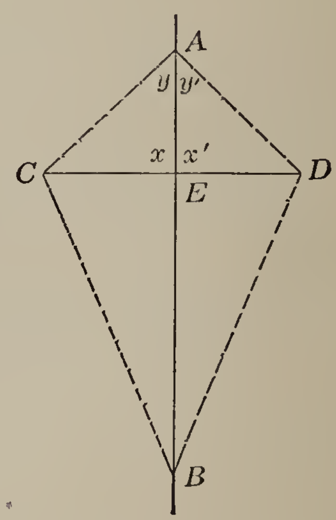


FIG. 14

83. Theorem: *If each of two points on a given line is equally distant from two given points, the given line is the perpendicular bisector of the segment joining the given points.*

Given the line AB, Fig. 14, and the points C and D such that

$$AC = AD, \quad CB = BD$$

To prove $\quad x = x', \quad CE = ED$

Preliminary discussion: We know that $x = x'$, if $\triangle CAE \cong \triangle DAE$.

However, since we only know that $AE = AE$, that $CA = AD$ and therefore that $\angle ACE = \angle ADE$ (§ 28), we do not have the required parts to show that $\triangle CAE \cong \triangle DAE$.

Hence, we shall first prove $y = y'$, by proving that $\triangle ACB \cong \triangle ADB$.

Proof:

STATEMENTS	REASONS
$AC = AD$, $CB = BD$	by hypothesis.
$AB \equiv AB$	common to both triangles ACB and ADB.
Therefore, $\triangle ACB \cong \triangle ADB$	*s.s.s.* (68)
Hence, $y = y'$	corresponding parts of congruent triangles are equal.
$AE \equiv AE$	common.
$AC = AD$	by hypothesis.
Therefore, $\triangle ACE \cong \triangle ADE$	*s.a.s.* (67)
Hence, $x = x'$, and $CE = ED$	corresponding parts of congruent triangles are equal.

84. Symbols for "therefore" and "since." The symbol $\therefore$ means **therefore**, and $\because$ means **since.**

85. Conventional treatment of a theorem. The formal demonstration of a theorem consists of three main parts: the **hypothesis, conclusion,** and **proof.** In writing a proof a reason must be given for each step. This means that each statement must be based upon (1) a definition, (2) the hypothesis, (3) an axiom, or (4) a theorem which has been proved previously.* The last step in the proof must be the same as the conclusion.†

86. Reviews. It is a good plan to review *daily* for a time after passing them, the proofs of theorems previously established. This may be done by simply recalling the figure, the method of proof used, and the principal steps, i.e., a sort of sketch or outline of the proof. Thus, in a few minutes a day the student will accomplish easily what will be a most difficult task if left until the end of a chapter, or until the day before an examination.

* **Hippocrates** (b. about 470 B.C.) introduced the method of "reducing" one theorem to another that has been previously proved. See W. W. R. Ball, *A Short Account of the History of Mathematics*, 5th ed., p. 39, hereafter referred to as Ball.

† The processes of proving theorems were developed by the Greeks. Greece was indebted to Egypt for its beginnings in geometry. However, the Egyptians carried geometry no farther than was necessary for the practical needs of life. They may have *felt* the truth of some theorems; but the Greeks formulated these geometric truths into scientific language and subjected them to proof (see Ball, pp. 16–19). The Greeks also recognized that it is impossible to prove *everything* in geometry and that *some* simple statements have to be assumed.

Euclid (about 300 B.C.) used the term *common notion* in the sense in which in modern mathematics we use *axiom*, i.e., a general statement admitted to be true without proof. Thus, the statement: "If equals are added to equals, the sums are equal" is an *axiom* because it holds in mathematics in general, i.e., in arithmetic, algebra, or geometry.

In modern mathematics, a statement referring to geometry only and admitted to be true without proof, is called a *postulate*. Thus, the statement "Two points determine a straight line" is a postulate. Some textbook writers use the word *axiom* or *assumption* to denote postulates as well as axioms.

Moreover, just as we assume unproved propositions, we have undefined terms, such as points, lines, etc. Pasch (1881) recognized the obvious impossibility of defining everything in geometry.

87. Inductive method. Mathematical facts can often be *discovered* by considering enough special cases to enable the student to recognize the general law underlying these cases. The method may be illustrated by the following example:

EXAMPLE OF INDUCTIVE METHOD

Problem: *It is known that the sum of the angles of a triangle is 180°. What is the sum of the angles of a quadrilateral, pentagon . . . . , etc., or of any polygon?*

To find the sum of the angles of a polygon, divide it into triangles by means of diagonals drawn from one

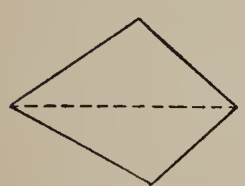


FIG. 15

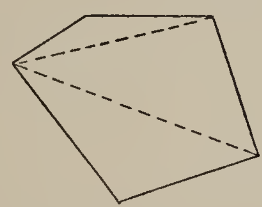


FIG. 16

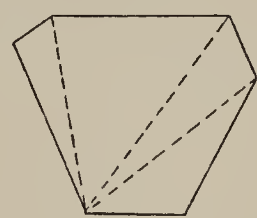


FIG. 17

vertex to the others. Thus, a quadrilateral may be divided into two triangles, Fig. 15; a pentagon into three triangles, Fig. 16; a hexagon into four triangles, Fig. 17, etc. The table below gives the sum of the angles in the various cases.

How does the number of triangles in each polygon compare with the number of sides? Hence, how does the sum of the angles compare with the number of sides?

What seems to be the sum of the angles of an n-gon?

Make the table complete by filling out the blank spaces.

Number of sides of polygon	3	4	5	6	7	10	15	n
Number of triangles	1	2	3	4	5			
Sum of angles.	180°	$2 \times 180°$	$3 \times 180°$	$4 \times 180°$	$5 \times 180°$			

It is seen that *the inductive method suggests mathematical facts, but does not prove them.* Hence, having found that the sum of the interior angles of an n-gon would seem to be $(n-2)\ 180°$, it still remains to be *proved* that this is true. This may be done as follows:

88. Theorem: *The sum of the interior angles of a polygon having n sides is $(n-2)180°$, or $(n-2)$ straight angles.*

Given the polygon $ABCD\ \ldots\ldots$, Fig. 18, having n sides.

To prove that the sum of the interior angles, S, is given by the equation:

$$S=(n-2)180°.$$

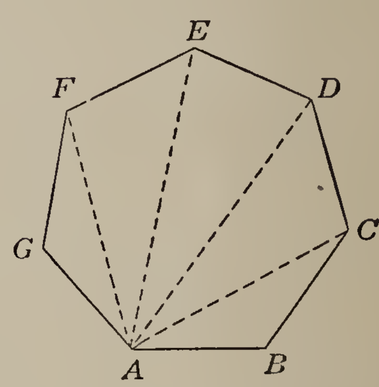


FIG. 18

Proof: Draw diagonals from A to the other vertices.

This divides the polygon into $(n-2)$ triangles. Why?

The sum of the angles of the triangles of the polygon is $(n-2)180°$. Why?

The sum of the angles of these triangles is equal to the sum of the angles of the polygon. Why?

Hence, the sum of the angles of the polygon is $(n-2)180°$. Why?

EXERCISES

1. Using the formula $S=(n-2)180°$, find the sum of the interior angles of hexagon, octagon, decagon, $2n$-*gon*.

2. The sum of the angles of a polygon is $1800°$. Find the number of sides.

89. Theorem: *The sum of the exterior angles of a polygon, one exterior angle at each vertex being taken, is 360°, or 2 straight angles.*

Given the polygon $ABCD$ . . . . , etc., Fig. 19, having n sides and the exterior angles a, b, c, d . . . . , etc.

To prove that $a+b+c+d$ . . . . $=360°$, or 2 straight angles.

Preliminary discussion:

How is an exterior angle related to the adjacent interior angle?

How may we find the sum of the exterior and interior angles?

Knowing the sum of the interior angles to be $(n-2)180°$, how may we find the sum of the exterior angles?

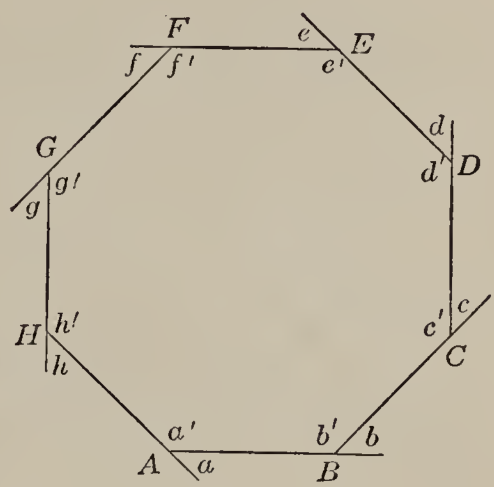


FIG. 19

Proof:

$$a+a'=180° \qquad \text{Why?}$$
$$b+b'=180° \qquad \text{Why?}$$
$$c+c'=180°, \qquad \text{Why?}$$
. . . . etc.

$\therefore a+b+c+\ldots a'+b'+c'+\ldots = n\cdot 180° \qquad =(180n)°$ Why?

$\because \qquad a'+b'+c'+\ldots = (n-2)180° = (180n)° - 360°$ Why?

$\therefore a+b+c\ldots. \qquad = 360°$ Why?

Show that the sum of the exterior angles of a polygon is independent of the number of sides of the polygon.

EXERCISES

1. Prove that any interior angle of a regular polygon is $\frac{n-2}{n}180°$.

2. If two angles of a quadrilateral are supplementary, show that the other two are supplementary.

3. How many right angles are contained in the sum of the angles of a polygon having n sides?

4. How many sides has a polygon the sum of whose angles is 36 right angles? 18 straight angles? 720°?

5. Show that the sum, S, of the interior angles of a polygon is a *function* of the number of sides.

6. What is the sum of the vertex angles a, b, c, d, and e of the five-point star, Fig. 20?

7. If the sum of the interior angles of a polygon is twice the sum of the exterior angles, how many sides are there in the polygon?

8. How many diagonals may be drawn in a polygon having 4 sides? 5 sides? 6 sides?

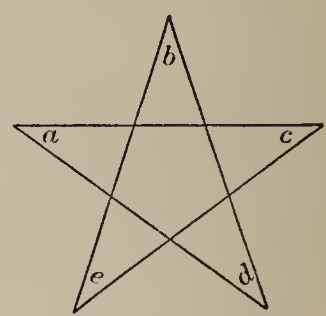


FIG. 20

9. Show that in an n-gon $(n-3)$ diagonals may be drawn from *one* vertex.

10. Show that in an n-gon $\frac{n(n-3)}{2}$ diagonals may be drawn.

11. Show that the number of diagonals, N, that may be drawn in a polygon is a function of the number of sides, n.

90. Algebraic method. This method is used when the numerical value of a magnitude is to be found or when a relation between several magnitudes is to be proved.

First, the relations between the magnitudes are expressed in algebraic symbols. The required magnitude is then found by a process of elimination. The following problem illustrates the method.

EXAMPLE OF ALGEBRAIC METHOD

Prove that the bisectors of two supplementary adjacent angles are perpendicular to each other.

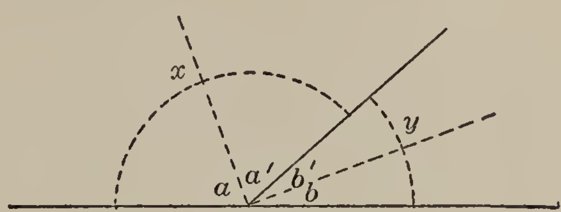


FIG. 21

Given that x and y, Fig. 21, are adjacent angles and that $x+y=180$.

Also, $a=a'$ and $b=b'$.

To prove that $a'+b'=90$.

Proof: $a+a'+b'+b=180$. Why?

Thus, we have a relation between a, a', b and b'.

Since the conclusion contains only a' and b', we must eliminate a and b from the equation $a+a'+b'+b=180$.

$$a=a' \qquad \text{Why?}$$

and

$$b=b' \qquad \text{Why?}$$

Then, a and b may be eliminated by substituting a' and b' for a and b, respectively.

This gives $a'+a'+b'+b'=180$

Collecting terms, $2a'+2b' = 180$.

Hence, $a'+b'= 90$ Why?

91. In the preceding proof a and b were eliminated by **substitution.** Methods of elimination will be discussed in the next chapter.

EXERCISES

Prove the following exercises:

1. If two angles of one triangle are equal to two angles of another, the third angles are equal and the triangles are mutually equiangular.

2. Find the angle formed by the bisectors of the acute angles of a right triangle.

3. One base angle of an isosceles triangle is $\frac{1}{3}$ of the vertex angle. Find the angles of the triangle.

Summary

92. The chapter has shown the *value* of logic in supporting correct reasoning and detecting fallacies, the *danger* in depending upon intuition alone as a means of proof, and the *need* for a logical proof.

93. The meaning of the following terms has been taught: *hypothesis, conclusion, proof.*

94. The following methods of proof have been illustrated: *superposition, the method of congruent triangles, the inductive method,* and *the algebraic method.*

95. Some general directions have been given for attacking, or proving, problems and theorems (§ 80). The importance of systematic reviews has been emphasized (§ 86). In the study helps (p. xix) the student will find some valuable suggestions as to the way he may study effectively.

96. The following theorems have been proved:

1. *The sum of the interior angles of a polygon is $(n-2)$ straight angles.*

2. *The sum of the exterior angles of a polygon is $360°$.*

3. *If each of two points on a given line is equally distant from two given points, the given line is the perpendicular bisector of the segment joining the given points.*

CHAPTER III

METHODS OF ELIMINATION. PROBLEMS AND EXERCISES IN TWO UNKNOWN NUMBERS

97. Elimination. In the first-year course we learned how to eliminate literal numbers by **addition or subtraction.** In § 90 we have seen that in a system of equations magnitudes may be eliminated by substituting equal magnitudes for them. In future work we shall have occasion frequently to eliminate numbers. There are various methods of elimination, and we should be able to select that method which for any particular problem is most advantageous. We shall accordingly review briefly what we know about elimination, and then study other methods.

98. Elimination by addition or subtraction. The solution of the following system (pair) of equations will recall the method of eliminating by addition or subtraction.

ILLUSTRATIVE PROBLEM

Let $$9x - 8y = 1$$
and $$15x + 12y = 8$$

The problem is to find the values of x and y.

Multiplying the first equation by 3 and the second by 2, we have

$$27x - 24y = 3$$
$$30x + 24y = 16$$

By adding the equations the y-terms are eliminated, and we obtain

$$57x = 19$$
$$\therefore x = \tfrac{1}{3}$$

Substituting this value for x in either one of the given equations, as $9x - 8y = 1$, we get

$$3 - 8y = 1$$
$$8y = 2$$
$$y = \tfrac{1}{4}$$
$$\left.\begin{aligned} \therefore x &= \tfrac{1}{3} \\ \text{and } y &= \tfrac{1}{4} \end{aligned}\right\} \text{ is the solution of the system.}$$

Thus, to eliminate by addition or subtraction we proceed as follows:

1. *By multiplying one or both equations by the proper numbers the coefficients of one of the unknown numbers are made numerically the same in both equations.*

2. *One of the unknowns is then eliminated by adding or subtracting the equations according as the coefficients of this unknown have unlike signs, or like signs.*

EXERCISES

Solve the following systems, eliminating by addition or subtraction:

1. $\begin{cases} 27x - 5y = 26 \\ 18x + 7y = 131 \end{cases}$

2. $\begin{cases} 7m + 5n = 81 \\ 9m - 2n = 62 \end{cases}$

3. $\begin{cases} 2.7a + 3.5b = 2.4 \\ 2.7a - 3.5b = .3 \end{cases}$

4. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = \frac{7}{5} \\ \frac{3}{2}x + \frac{2}{3}y = 3\frac{1}{5} \end{cases}$

99. Graphical method of solving a system of equations. The pupil will recall that every linear equation in two variables, as x and y, may be represented graphically by a straight line. To graph a linear equation in two variables, we may graph two, preferably three, solutions of the equation and draw the straight line passing through the three points corresponding to the solutions.

The solution of a system of two linear equations consists of the x- and y-distances (co-ordinates) of the point of intersection of the two straight lines. In Fig. 22, line AB represents the equation $9x-8y=1$ and line CD represents $15x+12y=8$. The point of intersection, P, represents the solution $x=\frac{1}{3}$, $y=\frac{1}{4}$, in the sense that the x- and y-distances of P represent the values of x and y that satisfy $9x-8y=1$ and $15x+12y=8$, *simultaneously*.

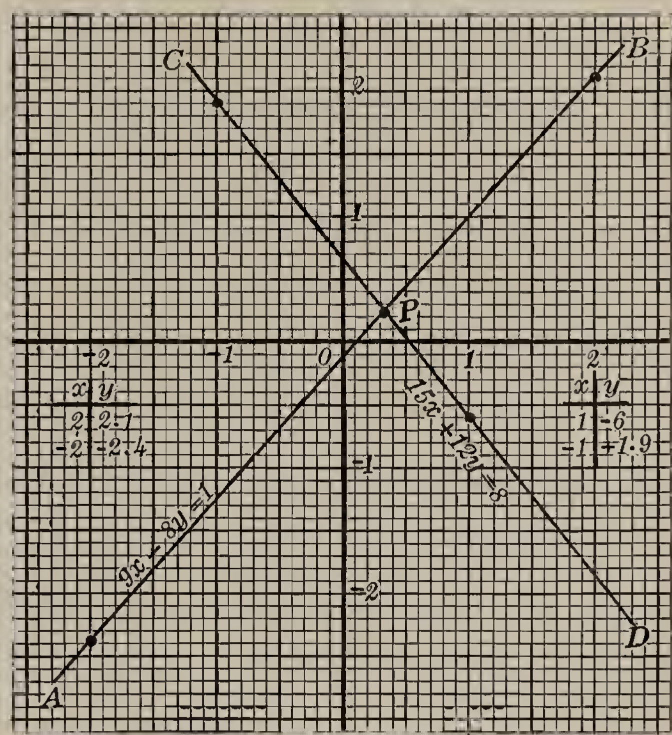


FIG. 22

EXERCISES

Solve graphically the following systems:

1. $\begin{cases} 3x-4y=14 \\ 5x+2y=32 \end{cases}$

‡3. $\begin{cases} x-2y+4=0 \\ x+y=5 \end{cases}$

2. $\begin{cases} 9x+6y=51 \\ 4x+3y=24 \end{cases}$

‡4. $\begin{cases} 3x-2y=9 \\ 2x-3y=4 \end{cases}$

‡ All problems marked ‡ are not essential, and may be omitted at the discretion of the teacher.

100. Elimination by substitution. This method is most advantageous when one of the unknown numbers is easily expressed in terms of the other.

For example, if $x-2y=7$, it follows that $x=7+2y$. Why?

The following problem will illustrate the method:

ILLUSTRATIVE PROBLEM

Solve the following system of equations eliminating *by substitution:*

$$7w-2z=46 \dots\dots (1)$$
$$w+z=13 \dots\dots (2)$$

Solving equation (2) for z, $z=13-w$

Substituting $13-w$ for z in equation (1),

$7w-2(13-w)=46$. This eliminates z.

Hence, $w=8$. Why?

and $z=5$. Why?

$\therefore$ The solution is $\begin{cases} w=8 \\ z=5 \end{cases}$

Thus, *to solve a system of equations, eliminating by substitution, express one of the unknown numbers in terms of the other by solving one equation. Then substitute the result in the other equation, and solve the equation thus obtained.**

PROBLEMS AND EXERCISES

Solve the equations obtained from the following problems by the method of elimination by substitution:

1. One of the base angles, x, of an isosceles triangle is equal to twice the vertex angle, y. Find all the angles of the triangle.

2. The difference of two numbers is 14, and the sum is 100. What are the numbers?

* This method of solving equations was first used by Isaac Newton (1642–1727).

3. A man invests one part of \$3,200 at 6 per cent and the other part at 5 per cent. If his annual income is \$180, how much did he invest at each rate?

Solve the following systems of equations:

4. $\begin{cases} 9R-2r=44 \\ 6R-\ \ r=31 \end{cases}$

‡5. $\begin{cases} x+2y=17 \\ 3x-\ \ y=2 \end{cases}$

6. $\begin{cases} 8x+5y=44 \\ 2x-\ \ y=\ 2 \end{cases}$

‡7. $\begin{cases} 9x=2y+84 \\ 7x+\ \ y=73 \end{cases}$

‡8. $\begin{cases} 7x=63-4y \\ x=5y-30 \end{cases}$

9. $\begin{cases} 2x=\ y+\frac{3}{4} \\ 6x+6y=5 \end{cases}$

10. The angles r and $3s$ are supplementary and $r-s=20°$. Find r, s, and $3s$.

11. The angles x and y are complementary and the difference is 10°. Find x and y.

12. The sides of an equiangular triangle are denoted by $x+3y$, $2x-y$, and 14. Find x and y.

13. The angles of an equiangular triangle are denoted by $7x+2y$, $3(3x-2y)$, and 60. Find x and y.

‡14. The three sides of an equiangular triangle are denoted by $7x+2y$, $5(x+2y)$, and $8x-3y+2$. Find x and y.

15. A man bought two pieces of vacant property, one at \$42 a foot, the other at \$56 a foot. Altogether he had 140 ft., and paid for it the sum of \$6,780. Find the number of feet of ground he bought at each price.

16. From two kinds of coffee selling at 30 cents and 35 cents, respectively, a grocer wishes to get a mixture of 20 pounds to be sold at 32 cents a pound. How many pounds of each kind must he use?

17. A leaves town three hours before B, traveling at a rate of $2\frac{1}{2}$ mi. an hour. B travels at 4 mi. an hour. When and where does B overtake A?

‡**101. Elimination by comparison.** This method works well if one of the unknowns has the same coefficient in both equations of the system, as with

$$\begin{cases} 4x=3 \\ 4x=5+2y \end{cases}$$

The value of x to be determined here being the same for both equations, it follows that $5+2y=3$. Why?

Hence, $y=-1$.

EXERCISES

Solve the following systems, eliminating by *comparison*, doing as much of the work as you can without the pencil.

1. $\begin{cases} p+w=12 \\ p-w=-4 \end{cases}$

2. $\begin{cases} x+y=3 \\ x=5-3y \end{cases}$

3. $\begin{cases} x=my+n^2 \\ x=ny+m^2 \end{cases}$

4. $\begin{cases} \frac{x}{5}+y=5 \\ \frac{x}{5}=\frac{1}{5}+\frac{3y}{5} \end{cases}$

5. $\begin{cases} 2F+3K=7.5 \\ 6F+3K=-2 \end{cases}$

6. $\begin{cases} x+\ y=560 \\ x-3y=0 \end{cases}$

PROBLEMS LEADING TO EQUATIONS IN TWO UNKNOWNS

102. Problems about work. Solve the following, doing all you can orally:

1. If the time required to do a piece of work is 10 days, what part of it is done in 1 day? In 3 days? In 10 days?

2. If the entire time is x days, what part of the work is done in 1 day? In 3 days? In x days?

3. If the time is y days, what part of the work is done in 1 day? In 3 days? In y days?

4. If A works 3 days on a piece of work and B, 2 days, they do $\frac{14}{15}$ of it. But if A works 2 days and B, 3 days, they do $\frac{9}{10}$ of it. In how many days could each one do it, working alone?

Letting x and y denote the number of days required by A and B, respectively, then $\frac{1}{x}$ and $\frac{1}{y}$ will denote the parts A and B, respectively, can do in one day.

Whence, $$\frac{3}{x}+\frac{2}{y}=\frac{14}{15} \qquad (1)$$

and, $$\frac{2}{x}+\frac{3}{y}=\frac{9}{10} \qquad (2)$$

These equations are not *linear* in x and y, but are linear in $\frac{1}{x}$ and $\frac{1}{y}$. They should not be cleared of fractions, but $\frac{1}{x}$ or $\frac{1}{y}$ should first be eliminated, thus,

$$\frac{6}{x}+\frac{4}{y}=\frac{28}{15}$$
$$\frac{6}{x}+\frac{9}{y}=\frac{27}{10}$$

Subtracting, $$-\frac{5}{y}=-\frac{25}{30}$$

Whence, $$\frac{1}{y}=\frac{1}{6}$$

and, $$y=6$$

Substituting in equation (1), $$x=5.$$

Solution: $$\begin{cases} x=5 \\ y=6 \end{cases}$$

EXERCISES

Solve the following systems of equations without clearing of fractions, and check them:

1. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{8}{15} \\ \frac{1}{x}-\frac{1}{y}=\frac{2}{15} \end{cases}$

2. $\begin{cases} \frac{3}{x}-\frac{7}{y}=5 \\ \frac{2}{x}-\frac{5}{y}=3 \end{cases}$

3. $\begin{cases} \frac{10}{x}-\frac{9}{y}=\frac{1}{20} \\ \frac{4}{x}+\frac{3}{y}=\frac{1}{6} \end{cases}$

4. $\begin{cases} \frac{7}{x}+\frac{4}{y}=\frac{11}{30} \\ \frac{5}{x}-\frac{6}{y}=-\frac{3}{28} \end{cases}$

‡5. $\begin{cases} \frac{6}{x}+\frac{5}{y}=2 \\ \frac{12}{x}-\frac{25}{y}=3 \end{cases}$

6. $\begin{cases} \frac{4}{x}+\frac{3}{y}=5 \\ -\frac{6}{x}+\frac{5}{y}=-17 \end{cases}$

‡7. $\begin{cases} \frac{5}{x}+\frac{6}{y}=\frac{24}{143} \\ \frac{13}{x}-\frac{11}{y}=\frac{1}{30} \end{cases}$

‡8. $\begin{cases} \frac{7}{x}+\frac{9}{y}=\frac{22}{105} \\ \frac{15}{x}-\frac{21}{y}=-\frac{4}{21} \end{cases}$

9. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{7}{12} \\ \frac{4}{x}-\frac{3}{y}=0 \end{cases}$

10. $\begin{cases} \frac{2}{x}-\frac{7}{y}=2 \\ \frac{3}{y}=6 \end{cases}$

‡11. $\begin{cases} \frac{11}{x}-\frac{2}{y}=\frac{1}{6} \\ \frac{2}{x}+\frac{3}{y}=\frac{31}{24} \end{cases}$

‡12. $\begin{cases} \frac{6}{x}+\frac{3}{y}=\frac{13}{60} \\ \frac{9}{x}+\frac{20}{y}=\frac{7}{12} \end{cases}$

MISCELLANEOUS PROBLEMS

103. Solve the following problems:

1. A boy is 17 months 5 days older than his sister. After 21 days he is twice as old as his sister. How old is each?

2. Two kinds of coffee, one at 32 cents a pound, the other at 25 cents a pound are to be mixed in the ratio 3:2. How many pounds of each must be taken to make a mixture to cost $8.00?

3. The sides of a rectangle are to each other as 3:8. Find the lengths of the sides if the perimeter is 283.

4. Two sums are invested at 3 per cent and $3\frac{1}{2}$ per cent, respectively, bringing an annual income of $52.60. If the first sum is invested at $3\frac{1}{2}$ per cent and the second at 3 per cent, the annual income is $52.70. What are the two sums?

5. Three times the reciprocal of the first of two numbers and 4 times the reciprocal of the second are together equal to 5. Seven times the reciprocal of the first less 6 times the reciprocal of the second is equal to 4. What are the numbers?

Summary

104. In this chapter the processes of solving linear equations in two unknowns graphically and by eliminating magnitudes have been extended.

The following processes have been studied for the first time:

(1) Elimination by substitution.
(2) Elimination by comparison.

CHAPTER IV

QUADRILATERALS. PRISMATIC SURFACE. DIEDRAL ANGLES

Parallelograms

105. Parallelogram. *A quadrilateral having both pairs of opposite sides parallel is a parallelogram.* (See Fig. 23.)

Fig. 23

106. Uses of the parallelogram. Some designs are based upon the parallelogram. A designer sometimes constructs tile flooring, Fig. 24, from a network of parallelograms. Give other examples of designs based upon the parallelogram.

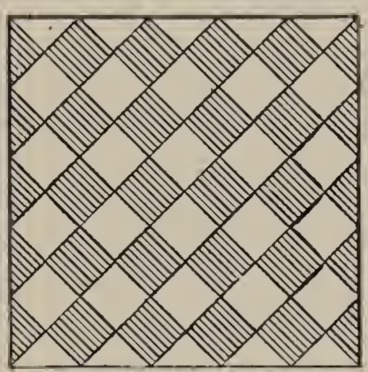


Fig. 24

Tops of desks and tables, blackboards, windows, walls, picture frames, etc., are examples of parallelograms.

Constructions with the *parallel ruler*, Fig. 25, which is used to draw parallel lines, are based upon a property of parallelograms. (See § 124).

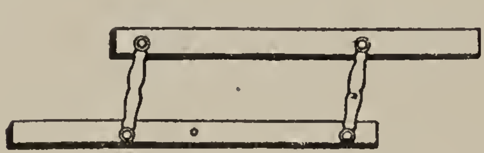


Fig. 25

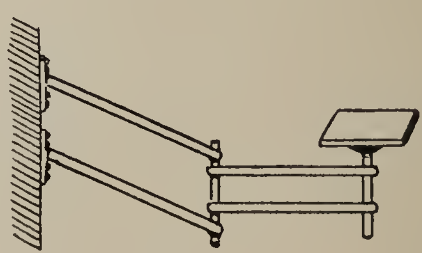


Fig. 26

The same principle is used in the construction of the adjustable shelf, Fig. 26, which remains in horizontal position as it is moved to and from the wall.

Surveyors make use of a property of the parallelogram to lay off parallel lines, Fig. 27, or to extend a line beyond an obstacle, Fig. 28 (see § 125).

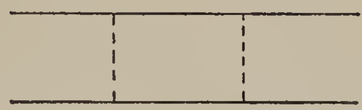


Fig. 27

Fig. 28

The use of the parallelogram in physics may be seen from the following problem:

The wind drives a steamer northeastward with a force which would carry it 12 miles per hour, and the engine is driving it southward with a force which would carry it 15 miles per hour. What distance will it travel in an hour and in what direction?

Let AB, Fig. 29, represent in magnitude and direction the 12-mile rate toward the northeast and AC the 15-mile rate southward; then it is shown by experiments that the rate and direction in which the boat actually moves may be represented by a line-segment as follows:

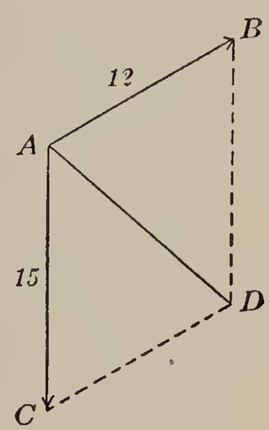


Fig. 29

Construct a parallelogram as $ABDC$ on AB and AC as adjacent sides and draw a line-segment from A to the opposite vertex, D. The diagonal line AD is the required segment. Before we can solve this problem we must know how to construct the parallelogram from these given parts.

107. Construction of parallelograms. *To construct a parallelogram having given two adjacent sides and the included angle.*

Given two segments, a and b and angle x, Fig. 30.

Required to construct a parallelogram having two adjacent sides equal to a and b, and including an angle equal to x.

Construction: Suppose $a = 1.5$ in., $b = 1$ in., and $x = 45°$.

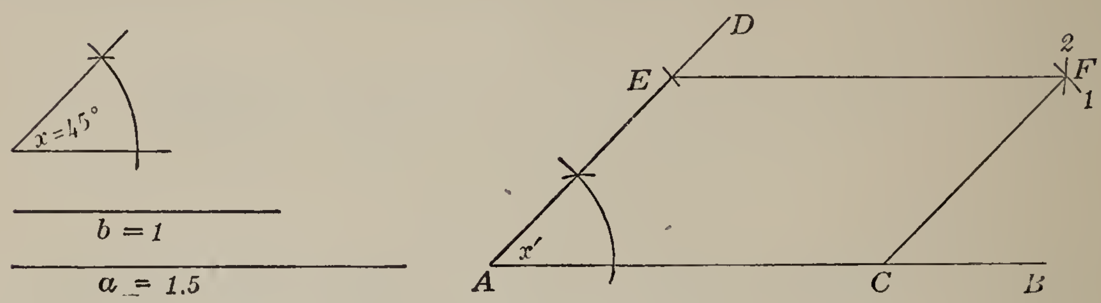


FIG. 30

Draw a line as AB.

On AB, lay off $AC = a$.

At A, construct line AD making with AB angle x' equal to angle x.

On AD lay off $AE = b$.

With C as center and radius equal to b, draw arc 1.

With E as center and radius equal to a, draw arc 2 intersecting arc 1 at F.

Draw EF and CF.

$ACFE$ is the required parallelogram.

The proof that $ACFE$ is a parallelogram is based upon one of the properties of parallelograms to be studied later in this chapter (see § 124).

EXERCISES

1. Construct a parallelogram having $a = 2$ in., $b = 1.5$ in., and $x = 50°$, and compare it with the parallelograms constructed by other members of the class.

2. How do two parallelograms, having two sides and the included angle equal respectively, seem to compare in size and shape?

3. Prove that *two parallelograms are congruent, if two adjacent sides and the included angle of one are equal, respectively, to the corresponding parts of the other.*

Proof by superposition: Apply one of the parallelograms to the other and show that they can be made to coincide throughout.

4. On squared paper construct the parallelogram of the steamer problem in § 106, and find the solution by measuring the diagonal AD, and $\angle CAD$, Fig. 29.

5. Represent graphically a force of 20 lb. acting northeast and a force of 30 lb. acting northwest upon the same body. What single force has the same effect upon the motion of the body as the two forces together? In what direction will the body move?

108. Before taking up the study of the parallelogram, we shall recall some facts about parallel lines. At the same time we shall discuss and exemplify two important *methods of proof*, that were not given in chapter II.

109. Indirect method of proof. The indirect method may be illustrated by proving the following theorem:

Theorem: *If each of two lines is parallel to a third line, they are parallel to each other.*

Given $AB \parallel EF$, $CD \parallel EF$, Fig. 31.

To prove $AB \parallel CD$.

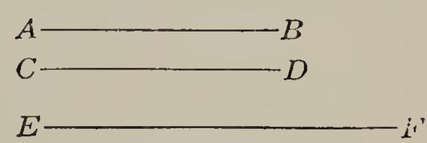


FIG. 31

Proof: Assume AB not parallel to CD.

Then AB and CD intersect at some point, as P, if far enough extended. For, two intersecting lines have one point in common (§ 4).

But $PA \parallel FE$, by hypothesis,

and $PC \parallel FE$, by hypothesis.

Hence, there are two lines parallel to FE passing through the point P.

This is impossible. For, through a point outside of a given line only one line can be drawn parallel to the given line (§ 41).

Therefore the assumption that AB is not parallel to CD is wrong, and $AB \parallel CD$.

It is thus seen that the indirect method of proof consists of the following *four* steps, numbered I, II, III, and IV:

I. *Make an assumption which denies the conclusion of the theorem.*

Thus, if you are to prove that $a=b$, assume that $a \neq b$, or if AB is to be proved parallel to CD, assume AB *not* parallel to CD.

II. *By correct reasoning show that the assumption leads to an absurdity.*

In the preceding theorem the absurdity is the statement that two lines can be drawn parallel to the same line passing through a given point.

III. *It then follows that the assumption is wrong.*

For, if we start right, correct reasoning cannot lead us to a wrong conclusion. To reach a correct conclusion in a course of reasoning, *two things* are necessary, and only two, namely: (1) the *premises* from which the reasoning starts—in geometry, we call them the *assumptions—must be correct*, and (2) the reasoning *must be sound.*

If a certain conclusion is known to be incorrect, the assumption from which the reasoning starts is incorrect or the reasoning is faulty, or both. If a conclusion is incorrect and the reasoning is sound, the assumption *must be* incorrect.

IV. *Hence, the conclusion is correct and the theorem is proved.*

110. Theorem: *Two lines that are perpendicular to the same line are parallel.*

Given $AB \perp EF$, $CD \perp EF$, Fig. 32.

To prove $AB \parallel CD$.

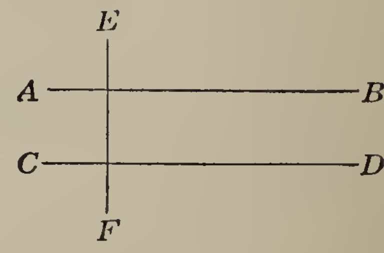


Fig. 32

Proof (indirect method):

Assume AB *not* parallel to CD.

Then AB and CD intersect at some point, P. Why?
Hence, $PA \perp EF$ and $PC \perp EF$. Why?
This is impossible. Why?

Therefore, the assumption that AB is not parallel to CD is wrong, and $AB \parallel CD$.

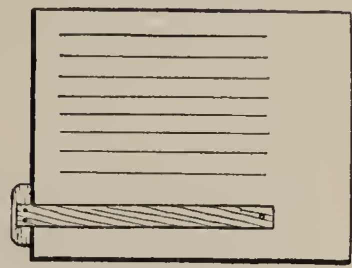


FIG. 33

EXERCISES

1. Show that this theorem affords a very simple way of drawing parallel lines by means of a T-square (Fig. 33).

2. State the conditions which make two lines parallel to each other.

3. Draw two parallel lines, using § 110.

4. Point out in the classroom two lines not in the same plane, but perpendicular to the same line. Are these lines parallel?

111. The method of proof used in §§ 109–110 is commonly known as a *reductio ad absurdum*,* or a *reduction to an absurdity*, which means that from assuming the negation of the conclusion of the theorems in question we are (by correct reasoning) led to a statement which is contradictory to known, or accepted, facts. It is a powerful method of proof, used not only in geometry but in everyday life.

* Eudoxus of Cnidus (408 B.C.), founder of the School at Cyzicus, and a contemporary of Plato, used the *reductio ad absurdum* method. In his *Short Account of the History of Mathematics* (5th ed.), Ball says (p. 39) that while the principle of the *reductio ad absurdum* had been used occasionally before, Hippocrates of Chios (b. about 470 B.C.) drew attention to it as a legitimate mode of proof, capable of numerous applications. In this sense Hippocrates may be regarded as having introduced the method.

112. Method of analysis.* The following example will illustrate the *method of analysis:*

Theorem: *If two alternate interior angles, formed by two lines and a transversal are equal, the lines are parallel.*

Given AB, CD, and the transversal EF; $a = a'$, Fig. 34.

To prove $AB \parallel CD$.

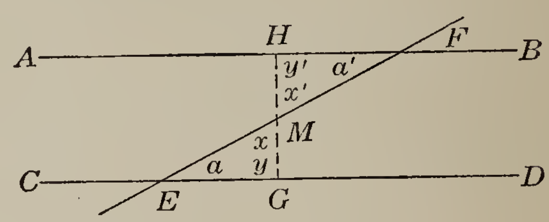


Fig. 34

Preliminary discussion: To prove $AB \parallel CD$, we may begin by asking the general question: When are two lines parallel?

Thus we know that AB is parallel to CD, if both lines are perpendicular to the same line (§ 110).

This suggests drawing a line, as GH, perpendicular to one of the given lines and then proving it to be perpendicular to the other.

GH is perpendicular to AB, if $\angle GHF$ is a right angle.

We may show $\angle GHF$ to be a right angle by showing that it is equal to the right angle HGE.

This will be true, if we can show $\triangle MHF \cong \triangle MGE$.

In triangles MHF and MGE, we know that $x = x'$ and $a = a'$, which is not sufficient to make the triangles congruent. But by taking M, so that $EM = MF$, we will have the third part which is necessary to make $\triangle MHF \cong \triangle MGE$.

Being able to prove $\triangle MHF \cong \triangle MGE$, it may be possible so to reverse the steps in this discussion as to prove the lines AB and CD to be parallel. This may be done as follows:

* Plato (429–348 B.C.) is said to have formulated this method of proof.

Proof: Bisect EF at M.

Through the middle point, M, of EF, draw MG perpendicular to CD and prolong GM to meet AB at H.

Prove that $\triangle EGM \cong \triangle MHF$ (*a.s.a.*, § 67).

$\therefore \quad y = y'$ Why?

But, $y = \text{rt. } \angle$ Why?

$\therefore \quad y' = \text{rt. } \angle$ Why?

$\therefore$ AB and CD are both perpendicular to GH Why?

$\therefore$ $AB \parallel CD$ Why?

It is seen that the method of analysis consists of the following four steps:

I. Ask the question: "Under what conditions is the conclusion true?" Select from the answers the one you think you can establish to be true. Thus, when AB is to be proved parallel to CD the question should be, "When are two lines parallel?" The answer will be: "We have proved previously that two lines are parallel, (1) if they are perpendicular to the same line, or (2) if they are parallel to the same line." From these two possibilities, select the one you think you can prove to be true. The conclusion is true, if the truth of this second fact is established.

II. Repeat the same process of reasoning with the second fact, thus: This second fact is true, if a certain third fact can be proved.

III. Continue this type of reasoning until you deduce a fact that is *known to be true.*

IV. Starting from this *known* fact, reverse the process, proving every statement, until the conclusion is reached.

113. Proof by analysis. Step IV, § 112, is the **proof** of the theorem. The preliminary reasoning in steps

I, II, and III is called the **analysis**. The purpose of the analysis is to enable the student to discover the known fact from which to start, and to learn how to arrange the proof. In the demonstration of a theorem only the *proof* is given.

114. Converse of a theorem. A theorem is said to be the **converse** of another theorem, if the hypothesis and conclusion of one are, respectively, the conclusion and hypothesis of the other.

State the converse of the following:

1. *If two sides of a triangle are equal, the angles opposite them are equal.*

2. *In a circle equal arcs are subtended by equal chords.*

Are the converses of the following statements true?

If two angles are right angles they are equal.

If two parallelograms have equal bases and altitudes, they are equal.

All righteous people are happy.

If two angles of a triangle are equal the sides opposite them are equal.

Thus, because a theorem is true, it does not follow that the converse is true. Since some converses are true and some are not, a proof is necessary before the converse can be accepted as true.

115. Methods used to prove the converse of a theorem. Two methods are used most frequently to test the truth of the converse of a theorem.

1. If the steps of the proof of the *original* theorem are reversible, use this proof as analysis and retrace it, step by step, until the hypothesis is reached. Since the hypothesis of the original theorem is the conclusion of the converse, this proves the converse.

2. The indirect method.

116. Theorem: *If two parallel lines are cut by a transversal, the alternate interior angles are equal.* (Converse of the theorem in § 112.)

Given $AB \parallel CD$. AB and CD cut by EH, Fig. 35.

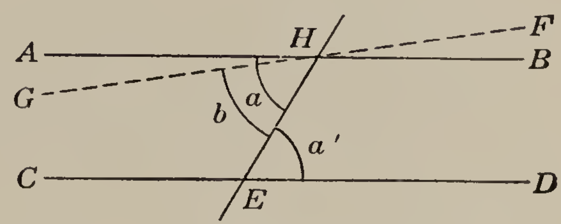


FIG. 35

To prove $a = a'$.

Proof (indirect method):

Suppose $a \neq a'$

Draw GF making $b = a'$.

Then, $GF \parallel CD$ Why?

But, $AB \parallel CD$ Why?

It is impossible that both GF and AB are parallel to CD. Why?

Therefore, the assumption that $a \neq a'$ is wrong, and $a = a'$.

EXERCISE

Prove that if one of two parallel lines is perpendicular to a third line the other is also.

117. Properties of parallelograms. In § 106 it was seen that some of the properties of parallelograms could be applied in a number of ways. We will now prove the following:

If a quadrilateral is a parallelogram—

1. *A diagonal divides it into congruent triangles;*

2. *The opposite sides are equal;*

3. *The opposite angles are equal;*

4. *The consecutive angles are supplementary;*

5. *The diagonals bisect each other.*

118. Theorem: *A diagonal divides a parallelogram into two congruent triangles.*

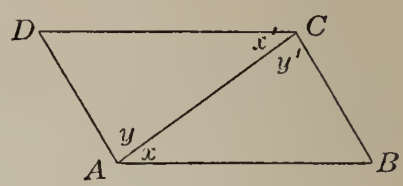


Fig. 36

Given the parallelogram $ABCD$ with the diagonal AC, Fig. 36.

To prove that

$$\triangle ABC \cong \triangle ADC.$$

Analysis: What conditions are sufficient to make two triangles congruent? In what ways can we prove two angles equal, and which of them may be used to prove $x = x'$?

Proof:

STATEMENTS	REASONS
$AC \equiv AC$	Common
$DC \parallel AB$	Since $ABCD$ is a parallelogram by hypothesis.
$\therefore x = x'$	If two parallel lines are cut by a transversal the alternate interior angles are equal.
$AD \parallel BC$	By hypothesis.
$\therefore y = y'$	Alternate interior angles formed by parallel lines and a transversal are equal.
$\therefore \triangle ABC \cong \triangle ADC$	*a.s.a.*

119. Theorem: *The opposite sides of a parallelogram are equal* (method of congruent triangles).

Use § 118.

120. Theorem: *The opposite angles of a parallelogram are equal.*

Use § 118 to prove $\angle D = \angle B$, Fig. 36. Then draw diagonal DB to prove $\angle A = \angle C$.

121. Theorem: *The consecutive angles of a parallelogram are supplementary.*

Notice that the consecutive angles are interior angles on the same side, formed by two parallels cut by a transversal. Use § 47.

122. Theorem: *The diagonals of a parallelogram bisect each other.*

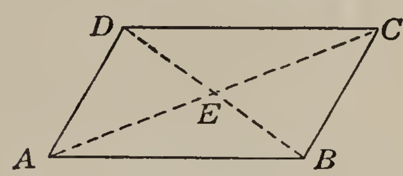


FIG. 37

Given the parallelogram $ABCD$ with the diagonals AC and BD, Fig. 37.

To prove that $AE = EC$, $DE = EB$.

Analysis: How may two line-segments be proved equal?

Which of these ways seems the most promising to prove $AE = EC$?

Proof: Prove $\triangle DEC \cong \triangle AEB$.

AE then equals EC, and BE equals ED. Why?

EXERCISES

1. Prove that if one of the angles of a parallelogram is a right angle, all the angles are right angles.

2. Prove that if two adjacent sides of a parallelogram are equal, all the sides are equal.

3. Prove that parallels intercepted between parallels are equal.

4. Prove that parallels are everywhere equally distant.

5. One pair of opposite sides of a parallelogram is denoted by x^2+x and $6(3-x)$ and the other pair by y^2-y and $3(5-y)$. Find x and y, and the lengths of the sides.

‡**6.** Two opposite angles of a parallelogram are denoted by x^2+6 and $7(x+2)$. Find x and all the angles of the parallelogram.

123. Conditions under which a quadrilateral is a parallelogram.

In the following it will be proved that a quadrilateral is a parallelogram—

1. *If the opposite sides are parallel;*
2. *If the opposite sides are equal;*
3. *If one pair of opposite sides are equal and parallel;*
4. *If the opposite angles are equal;*
5. *If the diagonals bisect each other.*

124. Theorem: *If the opposite sides of a quadrilateral are equal, the quadrilateral is a parallelogram.*

Given the quadrilateral $ABCD$, Fig. 38, having $AB = DC$, $AD = BC$.

To prove $AB \parallel DC$, $AD \parallel BC$.

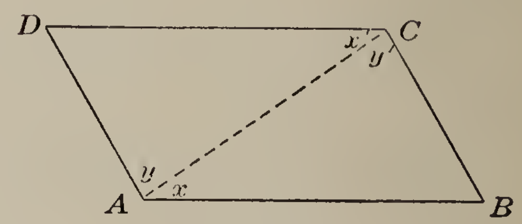


FIG. 38

Proof: Draw AC.

Prove $\triangle ABC \cong \triangle ADC$. (*s.s.s.*)

Then $x = x'$, and $y = y'$ Why?

Hence, $AB \parallel DC$ and $AD \parallel BC$. Why?

125. Theorem: *If one pair of opposite sides of a quadrilateral are equal and parallel, the quadrilateral is a parallelogram.*

Given the quadrilateral $ABCD$, Fig. 39, having $AB = DC$, and $AB \parallel DC$.

To prove that $ABCD$ is a parallelogram.

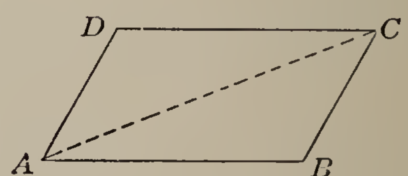


FIG. 39

Proof: Prove $\triangle ABC \cong \triangle ADC$. (*s.a.s*).

Then $AD = BC$. Why?

Use the theorem of § 124 to prove that $ABCD$ is a parallelogram.

126. The proof of the following theorem is a good example of the **algebraic method** of proof:

Theorem: *If the opposite angles of a quadrilateral are equal, the quadrilateral is a parallelogram.*

Given the quadrilateral $ABCD$, Fig. 40, having $a=c$, $b=d$.

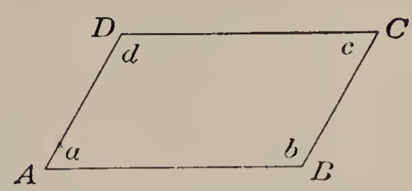


FIG. 40

To prove $AB \parallel DC$, $AD \parallel BC$.

Analysis: Under what conditions are two lines parallel? What relations are known between, a, b, c, and d? How may we obtain from these relations a relation which will show that $AB \parallel DC$?

Proof:

STATEMENTS	REASONS
$a+b+c+d=360$	Why?
$a=c$	Why?
$b=d$	Why?
Hence, $a+d+a+d=360$	By eliminating b and c.
$2a+2d=360$	Combining like terms.
$a+d=180$	Why?
Hence, $AB \parallel DC$	Why?

Similarly, prove that $AD \parallel BC$.

127. *If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.*

Draw the diagonals AC and BD, Fig. 41.

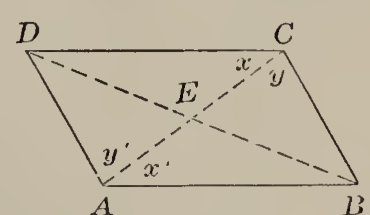


FIG. 41

Prove $\triangle DEC \cong \triangle AEB$.

Then $DC=AB$.

Similarly, show that $AD=BC$.

Hence, $ABCD$ is a parallelogram. Why?

128. Classification of quadrilaterals. Quadrilaterals may be classified as follows:

Parallelogram. A quadrilateral having two pairs of opposite sides parallel is a **parallelogram.**

Rhomboid. A parallelogram whose angles are oblique is a **rhomboid.**

Rhombus. An equilateral rhomboid is a **rhombus.**

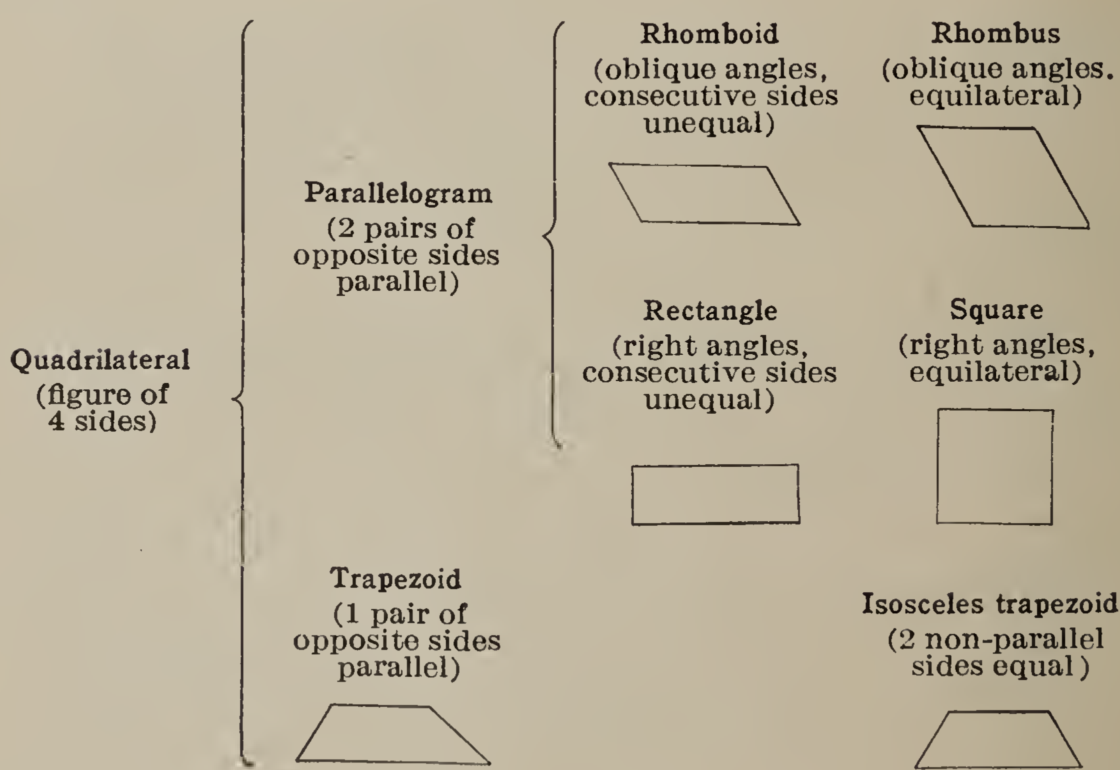
Rectangle. A parallelogram whose angles are right angles is a **rectangle.**

Square. An equilateral rectangle is a **square.**

Trapezoid. A quadrilateral having one pair of opposite sides parallel is a **trapezoid.**

Isosceles trapezoid. If the two non-parallel sides are equal the trapezoid is **isosceles.** The parallel sides of the trapezoid are the **bases.**

The same classification is represented in the following table:



EXERCISES

Prove the following:

1. If two *consecutive* sides of a rectangle are equal, all the sides are equal.

2. If the diagonals of a parallelogram are equal, the figure is a rectangle.

3. The diagonals of a rectangle are equal.

4. The diagonals of a square are equal.

5. The diagonals of a rhombus bisect each other perpendicularly.

6. The diagonals of a square bisect each other perpendicularly.

7. If the diagonals of a parallelogram bisect each other perpendicularly, the figure is a rhombus, or a square.

8. A circle may be circumscribed about a rectangle, or a square.

9. If the angles of a parallelogram are bisected by the diagonals, the figure is a rhombus, or a square.

‡10. If the midpoints of two opposite sides of a parallelogram are joined to a pair of opposite vertices, Fig. 42, a parallelogram is formed.

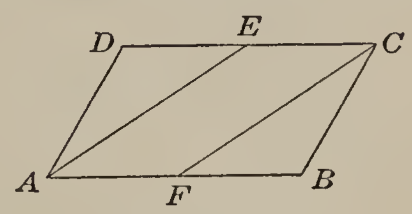


FIG. 42

‡11. In the parallelogram, Fig. 43, $AE=BF=CG=DH$. Prove that $EFGH$ is a parallelogram.

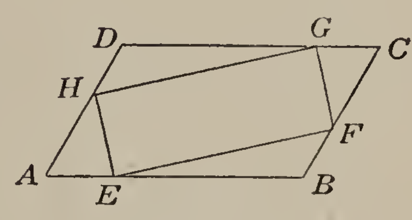


FIG. 43

‡12. The perpendiculars to a diagonal of a parallelogram from the vertices not on the diagonal are equal, Fig. 44, i.e., $DE=BF$.

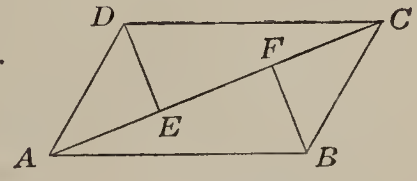


FIG. 44

13. If two points on the same side of a line are equally distant from the line, the line passing through the two points is parallel to the given line.

‡**14.** The bisectors of two opposite angles of a parallelogram are parallel.

‡**15.** The bisectors of the angles of a parallelogram form a rectangle.

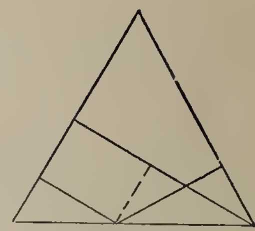


Fig. 45

‡**16.** The bisectors of the angles of a rectangle form a square.

‡**17.** The sum of the perpendiculars from a point on the base of an isosceles triangle to the two equal sides is equal to the altitude to one of these sides. Fig. 45.

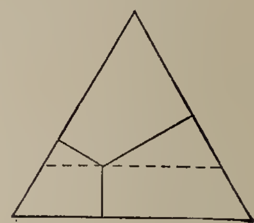


Fig. 46

‡**18.** The sum of the perpendiculars from a point within an equilateral triangle to the three sides is equal to the altitude. Fig. 46.

Constructions

129. Make the following constructions:

1. Given a side and the diagonal of a rectangle, construct the rectangle.

2. Given a side and an angle of a rhombus, construct the rhombus.

3. Given the diagonal of a square, construct the square.

4. Given the diagonals of a rhombus, construct the rhombus.

Problems

130. Solve the following problems algebraically:

1. The diagonals of a rectangle are denoted by x^2-x and $2(2x+7)$. Find both values of x and the diagonals.

2. The diagonals of a parallelogram divide each other so that the segments of one are x^2+x and $2(5x-7)$, and of the other, t^2+2t and $8(3-t)$. Find x, t, and the lengths of the diagonals.

‡3. The diagonals of a rhombus divide each other so that the parts of one diagonal are denoted by x^2 and $3(2x+9)$, and of the other by y^2 and $2(y+4)$. Find x, y, and both of the diagonals.

‡4. Two of the four angles that the diagonals of a rhombus make with each other are given by x^2-10 and $10(2x-11)$. Find x and the four angles.

Quadratic Equations

131. Solve the following equations, using either the method by factoring or by completing the square:

1. $x^2=5x-4$

2. $x^2+1=2(x+18)$

3. $x^2-x=3x-4$

4. $x^2-4=x+16$

The Trapezoid

132. Prove the following:

1. If the two angles at the ends of a base of a trapezoid are equal, the trapezoid is isosceles.

Draw $CE \parallel DA$, Fig. 47.
Prove $CE=CB$.

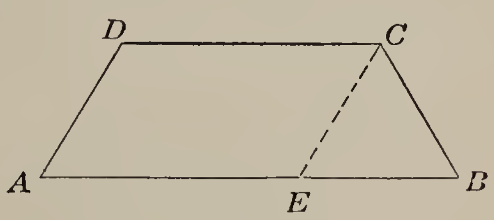


FIG. 47

2. If the non-parallel sides of a trapezoid are equal, the angles at the ends of a base are equal.

‡3. Prove that the diagonals of an isosceles trapezoid are equal.

The Kite

133. The kite. A quadrilateral having two pairs of adjacent sides equal, is a **kite**.

Thus, $ABCD$, Fig. 48, is a kite if $AD=DC$, and $AB=BC$.

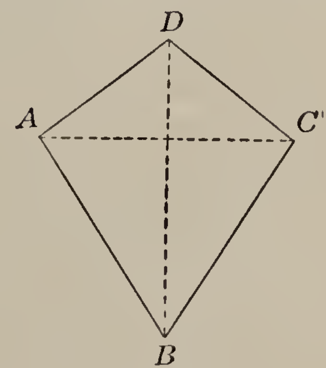


FIG. 48

EXERCISES

Prove the following:

1. The diagonals of a kite are perpendicular to each other.

2. One pair of opposite angles of a kite are equal, i.e., $\angle A = \angle C$.

Symmetry

134. Axis of symmetry. A line is called an **axis of symmetry** of a figure if it is the perpendicular bisector of all line-segments joining corresponding points of the figure. Thus, AE, Fig. 49, is the *axis of symmetry* in $DCBAB'C'D'$.

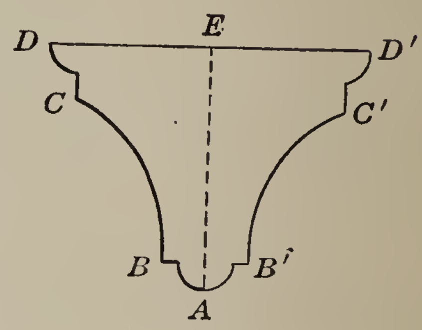


FIG. 49

EXERCISES

1. What is the axis of symmetry of a line-segment?

2. Draw the axis of symmetry of a given angle.

3. Draw an axis of symmetry in a given equilateral triangle.

Prove the following:

4. The diagonal BD of the kite, Fig. 50, is an axis of symmetry.

5. The point of intersection E of the diagonal DB, Fig. 50, and the bisector of $\angle A$ (axis of symmetry of $\angle A$), is equidistant from AD and AB and therefore may be taken as a center of a circle inscribed in the kite.

Use Exercise 4.

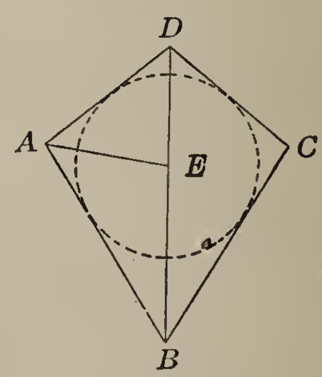


FIG. 50

6. The perpendicular bisector of a base of an isosceles trapezoid is an axis of symmetry.

Prove by superposition.

7. The point of intersection of the perpendicular bisectors of one of the bases and of one of the non-parallel sides of an isosceles trapezoid is equidistant from the 4 vertices. Hence, a circle can be circumscribed about an isosceles trapezoid.

8. Each diagonal of a rhombus is an axis of symmetry, Hence, a circle can be inscribed in any rhombus.

Loci*

135. Solve the following problems:

1. Where must the center of a wheel lie while the wheel rolls along a straight track?

2. Find the place (locus) of a point in a plane having a fixed distance from a given line.

3. Find the locus of a point in a plane equidistant from two parallel lines.

Surfaces

136. Prismatic surface. Prism. Given a polygon *ABCD* . . . , Fig. 51. A straight line *AA′*, not in the

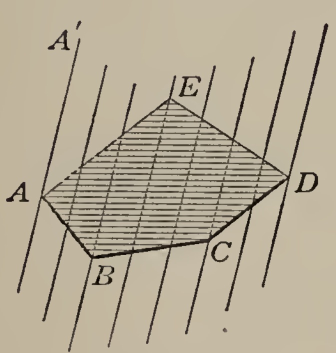


Fig. 51

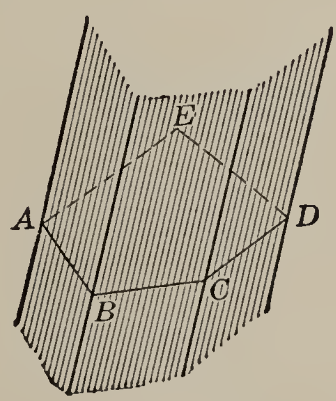


Fig. 52

plane of the polygon, moves always remaining parallel to its first position *AA′*, and always touching the polygon. *AA′* is said to generate a **prismatic surface,** Fig. 52.

* *Loci* is the plural of *locus.*

Let the polygon $ABCD$ . . . . , Fig. 52, move, to a position, $A'B'C'D'$ . . . . , Fig. 53, always remaining parallel to its first position, points A, B, C, ... moving along the straight lines AA', BB', CC' . . . , respectively. The figure thus formed is a **prism.**

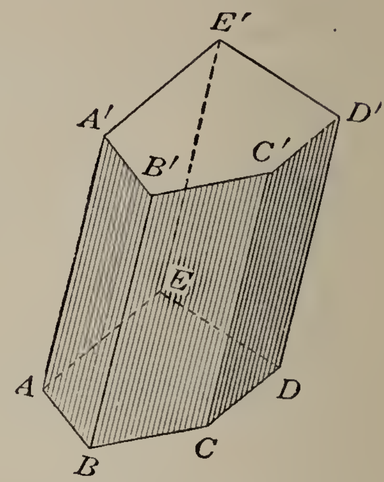


Fig. 53

137. Bases of prism. Lateral surface. The parallel polygons $ABCD$ . . . . and $A'B'C'D'$ . . . . are the **bases** of the prism. The portion of the prismatic surface between the bases is the **lateral surface.**

138. Lateral faces. The quadrilaterals of which the lateral surface is composed are the **lateral faces** of the prism.

In the classroom point out a prism and indicate its bases and the lateral faces.

EXERCISES

1. Show that the lateral faces of a prism are parallelograms.

2. Show how to generate according to the method of § 136 a triangular prism, i.e., a prism whose base is a triangle, Fig. 54.

3. Show how to form a parallelopiped using as a base a parallelogram, Fig. 55.

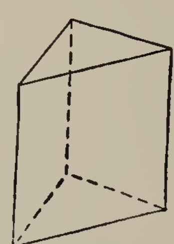


Fig. 54

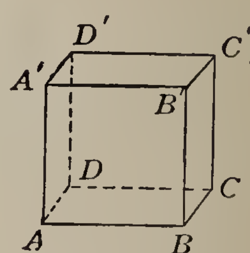


Fig. 55

4. What must be the position of the generating line AA', Fig. 55, with reference to the plane of the base $ABCD$, in order that all the lateral faces of the parallelopiped be rectangular?

Lines and Planes in Space

139. Determination of a plane. In the first-year course the pupil has become acquainted with the following important solids of geometry: the cube, parallelopiped, prism, cone, pyramid, cylinder, and sphere (§§ 203–13). In the study of these solids, he has learned the meaning of such terms as plane, surface, lines perpendicular to planes, parallel planes, etc. Illustrate these terms on the cube.

The pupil has seen that several planes may pass through (contain) the edge of a solid, or through two given points. Illustrate this with a cube, or by using the hinges of a door as the two points and the door as the plane.

When a plane passes through two points in space, it is possible to let the plane rotate about the straight line determined by these points, so that *any number* of planes may be passed through the line. However, the position of a plane is fixed, if besides making it pass through a given straight line, we make it pass through a *fixed point* not on the given line.

The conditions which determine the position of a plane in space are as follows:

1. *A straight line and a point not in that line.*

2. *Three points not in the same straight line.*

For, if two of the points are joined by a straight line, condition (1) is satisfied.

3. *Two intersecting straight lines.*

For by taking one of the lines and a point on the other (not the point of intersection), condition 1 is satisfied.

4. *Two parallel straight lines.*

For two parallel lines lie in the same plane and there exists but one plane containing one of the parallel lines and a point on the other (condition 1).

EXERCISES

1. Illustrate each of the 4 conditions named above on a cube.

2. Illustrate the same facts by using lines and points in the classroom.

140. Relative positions of two straight lines. Two straight lines in space may have the following relative positions:

1. *They may intersect,* produced if necessary.

2. *They may be parallel.*

3. *They may not be parallel and not intersect.*

EXERCISES

1. Illustrate these three possibilities by selecting the proper edges of a cube.

2. Find other illustrations in the classroom.

141. Relative positions of a straight line and a plane. A straight line and a plane may have the following relative positions:

1. The straight line may intersect the plane, produced if necessary.

2. The straight line may be parallel to the plane, i.e., have no point in common with the plane, however far produced.

3. The straight line may have two points in common with the plane and therefore lie entirely within the plane.

Illustrate each of these cases on the cube and on lines and planes in the classroom.

142. Representation of a plane in space. A plane is conveniently represented by a plane figure, such as a rectangle, parallelogram, etc., Fig. 56. However, the figure indicates only the *position* of the plane, the plane itself being regarded as indefinite in extent.

FIG. 56

143. Theorem: *If two planes intersect, the intersection is a straight line.*

Given two intersecting planes P and Q, Fig. 57.

To prove that P and Q intersect in a straight line.

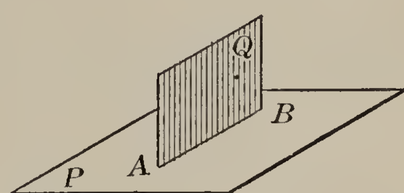


FIG. 57

Proof (indirect method):

Suppose the intersection, AB, of planes P and Q not to be a straight line.

Then it will be possible to find three points on AB not in the same straight line.

Since these three points lie on the intersection, they must be in both planes, P and Q.

Therefore P and Q must coincide. Why?

This contradicts the hypothesis in which P and Q are understood to be two *different* planes. Hence, the assumption, that the intersection of P and Q is not a straight line, is wrong.

Therefore the intersection of P and Q is a straight line.

Diedral Angles

144. Diedral angles. Two intersecting planes form a **diedral angle,** Fig. 58. The planes are the **faces** and the line of intersection is the **edge** of the diedral angle. Point out diedral angles in the classroom and on the cube.

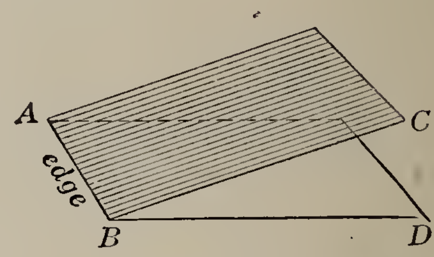


Fig. 58

A diedral angle is named by two points in the edge and an additional point in each face.

Thus, the diedral angle in Fig. 58 is denoted *C–AB–D*. Sometimes it is sufficient to name only two points on the edge, as *AB*.

145. Size of diedral angles. A diedral angle may be formed by rotating a plane about a line in the plane. The size of the diedral angle, therefore, depends upon the amount of rotation, not upon the extension of the faces.

146. Plane angle. If at a point in the edge of a diedral angle two lines are drawn perpendicular to the edge, one in each face, the angle formed is the **plane angle** of the diedral angle.

Thus, *ABC*, Fig. 59, is the plane angle of *P–QR–S*.

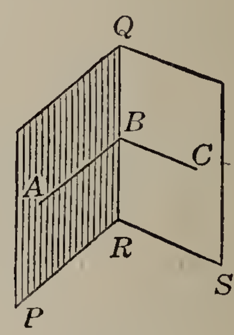


Fig. 59

A plane angle may be drawn at any point of the edge.

It will be shown in § 380 that all plane angles of a diedral angle are equal.

147. Classification of diedral angles. A diedral angle is said to be *right, straight, acute, obtuse, reflex, oblique,* according as the *plane angle* is right, straight, etc. Diedral angles are *adjacent* if they have a common edge and a common face between them. Thus, A–BC–D and D–BC–E, Fig. 60, are adjacent diedral angles.

Two diedral angles are *complementary* or *supplementary* according as the plane angles are complementary or supplementary.

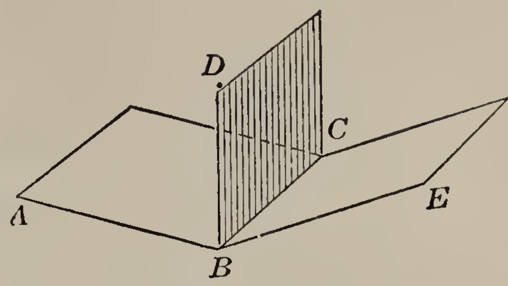


FIG. 60

148. Perpendicular planes. Two planes are **perpendicular** to each other, if they form a right diedral angle.

Point out perpendicular planes on the cube; in the classroom.

Summary

149. The chapter has taught the meaning of the following terms:

parallelogram	kite	plane angle of a diedral angle
rhomboid	prismatic surface	
rhombus	prism	perpendicular planes
rectangle	base of prism	analysis
square	lateral surface	converse of a theorem
trapezoid	lateral faces	axis of symmetry
isosceles trapezoid	diedral angle	locus

150. The following theorems have been proved:

1. *Two parallelograms are congruent, if two adjacent sides and the included angle of one are equal, respectively, to the corresponding parts of the other.*

2. *A parallelogram may be constructed if two adjacent sides and the included angle are given.*

3. *Two lines perpendicular to the same line are parallel.*

4. *If two alternate interior angles formed by two lines and a transversal are equal, the lines are parallel.*

5. *If two parallel lines are cut by a transversal, the alternate interior angles are equal.*

6. *If a quadrilateral is a parallelogram—*
   1. *A diagonal divides it into congruent triangles;*
   2. *The opposite sides are equal;*
   3. *The opposite angles are equal;*
   4. *The consecutive angles are supplementary;*
   5. *The diagonals bisect each other.*

7. *A quadrilateral is a parallelogram if—*
   1. *The opposite sides are parallel;*
   2. *The opposite sides are equal;*
   3. *One pair of opposite sides are equal and parallel;*
   4. *The opposite angles are equal;*
   5. *The diagonals bisect each other.*

8. *If two planes intersect, the intersection is a straight line.*

151. Quadrilaterals have been classified as follows:

$$\text{Quadrilaterals}\begin{cases}\text{Parallelograms}\begin{cases}\text{Rhomboid—Rhombus}\\ \text{Rectangle—Square}\end{cases}\\ \text{Trapezoid—Isosceles Trapezoid}\end{cases}$$

152. The following methods of proof have been taught: (1) the indirect method, (2) the method of analysis.

153. Quadratic equations were solved by factoring, or by completing the square.

154. Each of the following conditions determines the position of a plane in space:

1. A straight line and a point not in that line;
2. Three points not in the same straight line;
3. Two intersecting straight lines;
4. Two parallel straight lines.

CHAPTER V

PROPORTIONAL LINE-SEGMENTS

Uses of Proportional Line-Segments

155. Measurement of line-segments. To measure a line-segment is to find how many times it contains another line-segment, called the **unit-segment.** The number of times a segment b is contained in a segment a is the **numerical measure** of a in the unit b, or the **numerical measure** of a with respect to b.

156. Ratio of two segments. The ratio of the numerical measures of two segments, both being measured with the same unit, is the **ratio of the two segments.** Another method of finding the ratio of two segments is given in § 162.

157. Proportion. An equation of two equal ratios, as $\frac{4}{6}=\frac{2}{3}$, $\frac{1}{5}=\frac{3}{15}$, $\frac{a}{b}=\frac{c}{d}$, is called a **proportion.** Four *magnitudes* are said to be in *proportion,* if their numerical measures are proportional.

Thus, if the ratio of the rectangles, Fig. 61, is $\frac{3}{5}$ and if the ratio of the altitudes is also $\frac{3}{5}$, the rectangles are proportional to the altitudes.

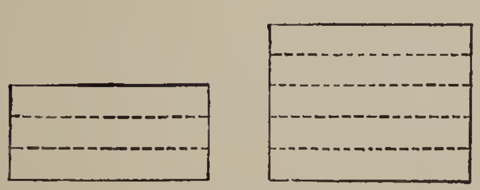


Fig. 61

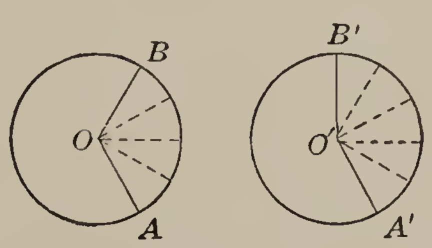


Fig. 62

Show that the central angles, AOB and $A'O'B'$, in Fig. 62, are proportional to the intercepted arcs.

158. Uses of proportional line-segments. Fig. 63 represents a pair of **proportional compasses,** used to make scale drawings of given figures. By making $OB' = \frac{2}{3}OB$ and $OA' = \frac{2}{3}OA$ and by opening the compass so that AB equals a given line-segment, we obtain $A'B'$ equals $\frac{2}{3}AB$. This fact follows from one of the principles of proportional line-segments (§ 167).

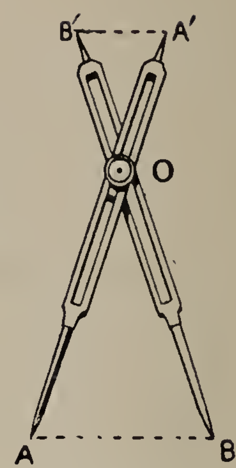


Fig. 63

The **pantograph,** Fig. 64, is used to draw figures to definite scales, and to enlarge or to reduce maps, drawings, designs, etc. The instrument consists of four pointed bars, making BB_1A_2A a parallelogram. According to the principles of proportional line-segments, if $\frac{OB}{OB_1}$ is made equal to $\frac{B_1A_2}{B_1A_1}$, points O, A, and A_1 must fall in a straight line, making $\frac{OA}{OA_1} = \frac{OB}{OB_1}$. Keeping point O fixed, point A is made to describe figure (a). The pencil at A_1 will then describe figure (b), which is figure (a) *magnified* to the scale OB_1 to OB.

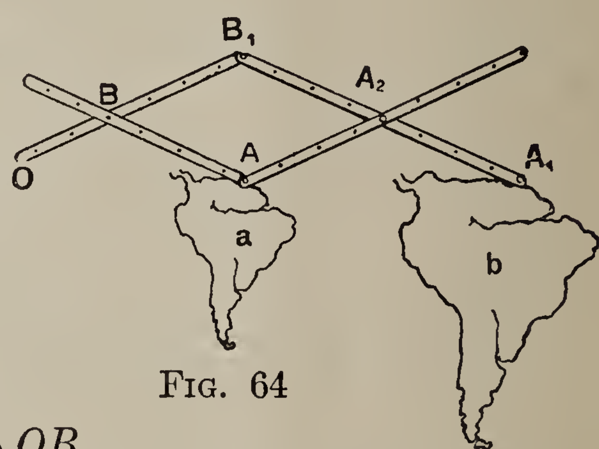


Fig. 64

The **diagonal scale,** Fig. 65, is another instrument whose construction is based upon principles of propor-

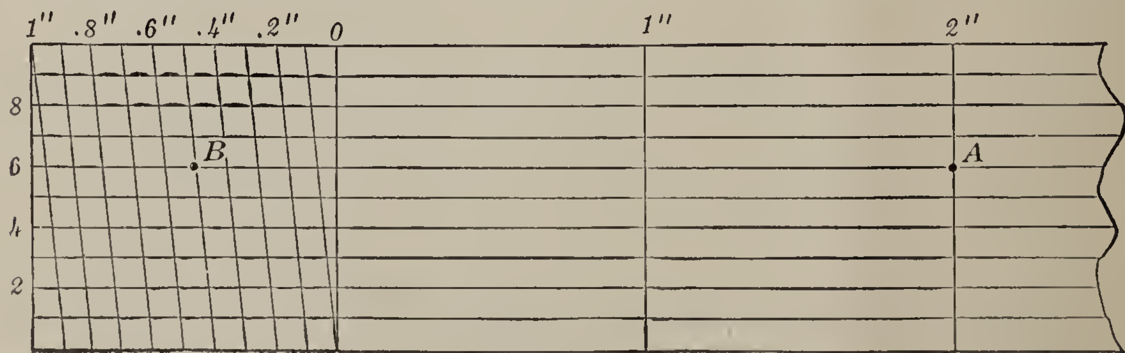


Fig. 65

tional line-segments. By means of it lengths may be measured to hundredths of an inch.

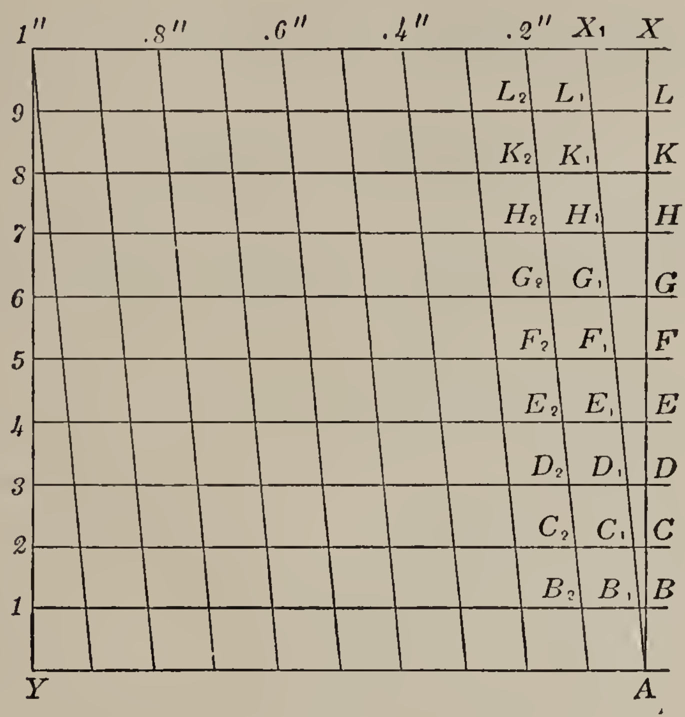


FIG. 66

Fig. 66 represents part of Fig. 65, enlarged.

By § 167, $\frac{BB_1}{XX_1}=\frac{AB}{AX}=\frac{1}{10}$

Hence, $BB_1=\frac{1}{10}XX_1$

Similarly, $CC_1=\frac{2}{10}XX_1$

and $DD_1=\frac{3}{10}XX_1$, etc.

Since $XX_1=\frac{1}{10}AY=\frac{1''}{10}$,

we have $BB_1=.01''$, $CC_1=.02''$, $DD_1=.03''$, etc.

Likewise, $BB_2=.1''+.01''=.11''$

$CC_2=.1''+.02''=.12''$

$DD_2=.1''+.03''=.13''$, etc.

EXERCISES

1. What is the length of AB, Fig. 65?

2. Draw a line-segment. Measure it to hundredths of an inch by using the diagonal scale, Fig. 65.

Proportional Segments

159. Theorem:* *A line that bisects one side of a triangle, and is parallel to a second side, bisects the third side.*

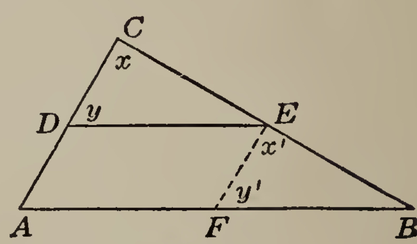


FIG. 67

Given $\triangle ABC$, $CD=DA$, $DE \parallel AB$, Fig. 67.

To prove $CE=EB$.

Analysis: How may two line-segments be proved to be equal?

Draw $EF \parallel CA$.

CE will equal EB, if $\triangle DEC \cong \triangle FBE$.

Proof: $AFED$ is a parallelogram. Why?

$\therefore FE=AD$. Why?

Show that $FE=CD$.

Show that $x=x'$, $y=y'$.

$\therefore \triangle DEC \cong \triangle FBE$. Why?

$\therefore CE=EB$. Why?

EXERCISES

1. Prove that CD, DA, CE, and EB, Fig. 67, are proportional.

2. Prove that $DE=\frac{1}{2}AB$. State this fact in form of a theorem.

3. If $CD=3$, $DA=3$, $CE=4$, and $EB=x-2$, find x and the length of EB.

* This theorem is probably to be credited to Eudoxus (408–355 B.C.).

4. If $CD=1$, $DA=1$, $CE=\frac{3x-1}{4}-\frac{4x-5}{5}$, $EB=\frac{7x+5}{10}-4$, find x and the lengths of CE and EB.

160. Theorem:* *If three or more parallel lines intercept equal segments on one transversal they intercept equal segments on every transversal.*

Proof (method of congruent triangles): Draw helping lines $a'' \parallel a$, $b'' \parallel b$, etc., Fig. 68.

Prove $\triangle\mathrm{I} \cong \triangle\mathrm{II} \cong \triangle\mathrm{III}$, etc.

Then $a'=b'=c'$, etc. Why?

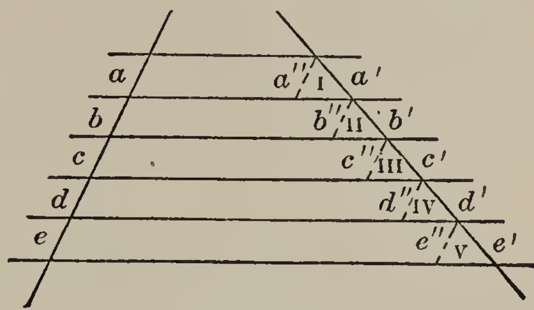


FIG. 68

EXERCISES

1. Prove segments $a, b, c, \ldots\ldots, a', b', c', \ldots\ldots$, Fig. 68, proportional.

2. A line drawn through the midpoint of one of the non-parallel sides of a trapezoid, parallel to the bases, bisects the other side. Prove. Apply § 160.

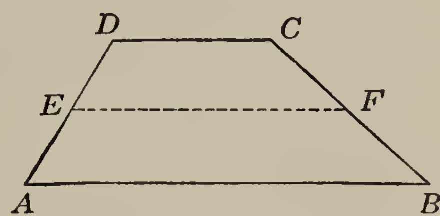


FIG. 69

161. Median of a trapezoid. The segment joining the midpoints of the non-parallel sides of a trapezoid is the **median** of the trapezoid.

EXERCISE

Prove that the median of a trapezoid equals one-half the *sum* of the bases.

Show that $m_1=\frac{1}{2}b_1$, $m_2=\frac{1}{2}b_2$, Fig. 70;
$\therefore\ m=m_1+m_2=\frac{1}{2}(b_1+b_2)$.

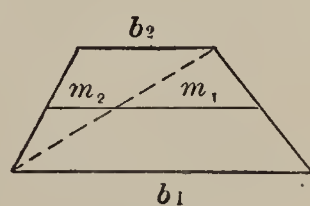


FIG. 70

* This theorem is attributed to Archimedes (287–212 B.C.).

162. Ratio of line-segments. The ratio of two line-segments may be found by means of the compass as follows:

Let AB and CD, Fig. 71, be two segments whose ratio is to be found.

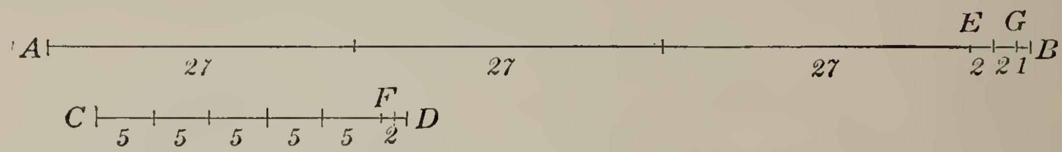


FIG. 71

Let us assume that AB and CD contain a common unit of measure. It will be shown in § 165 that there are line-segments that have no common unit of measure. To find the common unit, proceed as follows:

Lay off the smaller segment CD on the larger AB as often as possible, leaving a remainder EB, which is less than CD.

Lay off EB on CD, leaving a remainder FD, which is less than EB.

Lay off FD on EB, leaving a remainder GB.

Lay off GB on FD, leaving no remainder.

The last remainder, GB, is a common unit of measure of AB and CD.

Using GB as unit, show that $AB=86$, and $CD=27$.

Therefore $\dfrac{AB}{CD}=\dfrac{86}{27}$.

163. Theorem: *If two parallels cut two intersecting transversals the segments intercepted on one transversal are proportional to the corresponding segments on the other.*

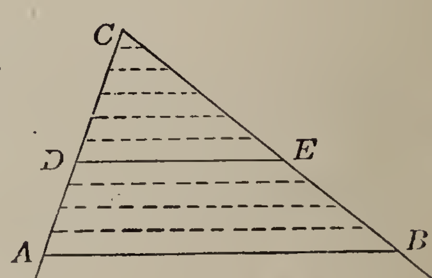


FIG. 72

Given $AB \parallel DE$ and AD intersecting BE at C, Fig. 72,

To prove

$$\frac{CD}{DA}=\frac{CE}{EB},\ \frac{CD}{CA}=\frac{CE}{CB},\ \frac{DA}{CA}=\frac{EB}{CB}.$$

Proof: To find the ratio $\frac{CD}{DA}$, lay off the smaller segment, DA, on the larger, CD, as often as possible. If there is a remainder, lay it off on AD. If there is still a remainder lay it off on the preceding remainder, etc. We will assume that after laying off these remainders a definite number of times there is no remainder. Then the *last remainder* is a *common unit* of CD and DA.

Let this common unit be contained in CD and DA, m and n times, respectively.

Then $\frac{CD}{DA}=\frac{m}{n}$, $\left(\text{in Fig. 72, } \frac{CD}{DA}=\frac{6}{4}\right)$.

To find the value of $\frac{CE}{EB}$ proceed as follows:

Draw lines parallel to AB passing through the points of division of CA.

These lines will divide CE and EB into m and n parts, respectively. Why?

Moreover, these parts are equal to each other. Why?

Hence, $\frac{CE}{EB}=\frac{m}{n}$ Why?

$\therefore \quad \frac{CD}{DA}=\frac{CE}{EB}$ Why?

Similarly, we may prove $\frac{CD}{CA}=\frac{CE}{CB}$, $\frac{DA}{CA}=\frac{EB}{CB}$.

EXERCISES

1. Prove that a line parallel to one side of a triangle divides the other two sides proportionally. (Apply § 163.)

‡**2.** Prove the theorem* in § 163, using Fig. 73.

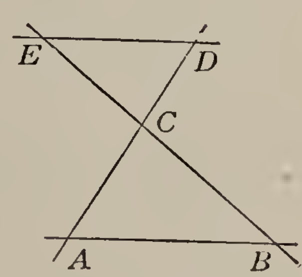


FIG. 73

* This form of the theorem is attributed to Archimedes.

164. Commensurable magnitudes. In the proof of the theorem of § 163 it was assumed that a common unit for CD and DA could be found. Two magnitudes which have a common unit of measure are said to be **commensurable.**

165. Incommensurable magnitudes. Not all magnitudes have a common unit of measure. Magnitudes not having a common unit of measure are said to be **incommensurable.**

EXAMPLE

The side and diagonal of a square are incommensurable segments. This may be seen as follows:

Since $AB < AC$, Fig. 74, AB may be laid off on AC, leaving a remainder B_1C.

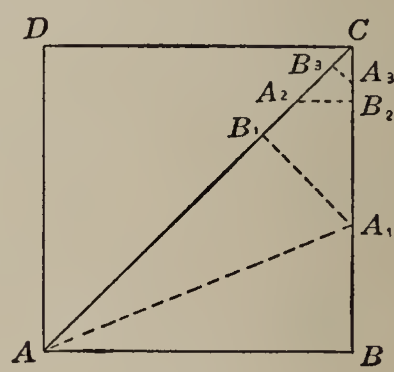


FIG. 74

Draw $B_1A_1 \perp AC$ at B_1. Prove, by congruent triangles, that $B_1A_1 = A_1B$.

Prove $B_1C = B_1A_1 = A_1B$.

Since $B_1C < A_1C$, it follows that B_1C may be laid off on A_1C leaving a remainder, as B_2C.

Thus, B_1C may be laid off on BC twice, leaving a remainder B_2C.

In the same way it may be shown that B_2C may be laid off on B_1C twice, leaving a remainder, B_3C; that B_3C may be laid off twice on B_2C, leaving a remainder, etc.

In each case the process is a repetition of the preceding case, only with smaller segments. Since in each case there is a remainder, the process may be kept up indefinitely.

Hence, *no common unit of AB and AC can be found.*

It will be seen later, § 254, that the ratio $\frac{AC}{AB} = \sqrt{2}$, is an *irrational* number, i.e., a number which cannot be expressed exactly in terms of integers, or of fractions whose terms are integers.

‡ **166. The incommensurable case of the theorem in § 163.** Since several theorems that have not yet been proved are needed for proof of the incommensurable case, only an outline of the proof will be given. The method of proof is *indirect*.

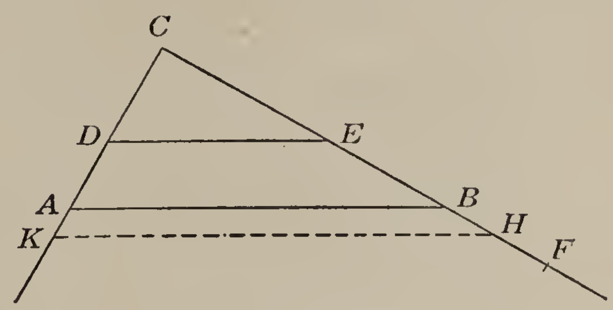


FIG. 75

Assume $\frac{DA}{CD} \neq \frac{EB}{CE}$, Fig. 75.

Then either $\frac{DA}{CD} > \frac{EB}{CE}$, or $\frac{DA}{CD} < \frac{EB}{CE}$.

Let $\frac{DA}{CD} > \frac{EB}{CE}$.

Select a point F on the extension of EB, making EF long enough to give $\frac{DA}{CD} = \frac{EF}{CE}$ . . . .(1)

It is possible to determine a point H between B and F, making CE and EH *commensurable*.

Draw $HK \parallel BA$

Then, $\frac{DK}{CD} = \frac{EH}{CE}$

Since, $DA < DK$,

it follows that $\frac{DA}{CD} < \frac{EH}{CE}$ . . . .(2)

Comparing (2) and (1), we have $\frac{EF}{CE} < \frac{EH}{CE}$

$\therefore \quad EF < EH$

This is impossible and the assumption that $\frac{DA}{CD} > \frac{EB}{CE}$ is wrong.

Similarly, we may prove that $\frac{DA}{CD}$ is not *less* than $\frac{EB}{CE}$.

Hence, $\frac{DA}{CD} = \frac{EB}{CE}$.

167. Theorem:* *If a number of parallel lines are cut by two transversals, the segments of one transversal are proportional to the corresponding segments of the other.*

Show that $\frac{a}{b}=\frac{a'}{b'}$ (§ 163)

Draw $DE \parallel AB$

Prove that $\frac{b''}{c''}=\frac{b'}{c'}$

But $b=b''$ and $c=c''$.

Why?

$\therefore \frac{b}{c}=\frac{b'}{c'}$, etc.

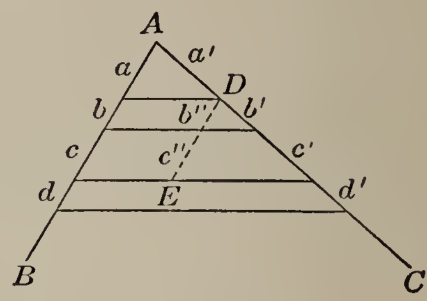


FIG. 76

EXERCISE

Prove that *if two given lines are cut by two parallel lines the segments of the parallel lines are proportional to the corresponding segments of the given lines.*

We are to prove $\frac{CD}{CA}=\frac{DE}{AB}$, Fig. 77.

Draw $DF \parallel CB$.

Then, $\frac{CD}{CA}=\frac{BF}{BA}=\frac{DE}{AB}$. Why?

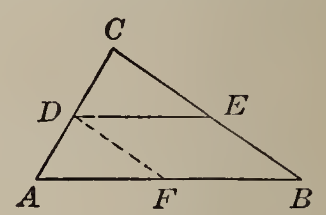


FIG. 77

168. Theorem: *Two lines that cut two given intersecting lines, and make the corresponding segments of the given lines proportional, are parallel* (converse of § 163).

Given $\frac{CD}{DA}=\frac{CE}{EB}$, Fig. 78,

To prove $AB \parallel DE$

Proof (indirect method):

Suppose AB *not* parallel to DE.

Draw $AF \parallel DE$

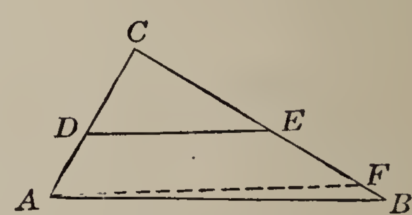


FIG. 78

* This theorem was first proved by Archimedes.

Then, $\frac{CD}{DA}=\frac{CE}{EF}$ Why?

But, $\frac{CD}{DA}=\frac{CE}{EB}$ Why?

$\therefore \frac{CE}{EF}=\frac{CE}{EB}$ Why?

$\therefore CE \cdot EB = CE \cdot EF$ Why?

$\therefore EB = EF.$ Why?

This is impossible. Why?

Therefore the assumption that AB is not parallel to DE is wrong and $AB \parallel DE$.

EXERCISES

Prove the following:

‡**1.** If $\frac{CA}{CD}=\frac{CB}{CE}$, Fig. 78, then $DE \parallel AB$.

2. *The line joining the midpoints of two sides of a triangle is parallel to the third side.*

Prove that two sides are divided proportionally. Then apply § 168.

‡**3.** Prove § 168, using Fig. 79.

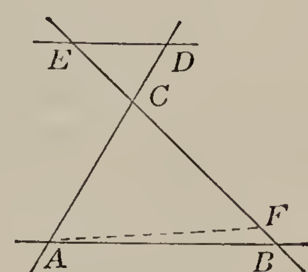


FIG. 79

4. In Fig. 80 $a \parallel b$ and $\frac{x}{y}=\frac{x'}{y'}$. Prove that $c \parallel b$.

5. The median of a trapezoid is parallel to the bases.

6. The quadrilateral whose vertices are the midpoints of the sides of a triangle and one vertex of the triangle is a parallelogram.

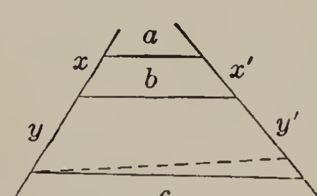


FIG. 80

7. The midpoints of the sides of a quadrilateral, Fig. 81, may be taken as vertices of a parallelogram.

Fig. 81

Draw the diagonal. Use Exercise 2.

169. Summary of the more important theorems in proportional segments in §§ 159–68:

I

1. A line bisecting one side of a triangle and parallel to a second side, bisects the third side, Fig. 82.

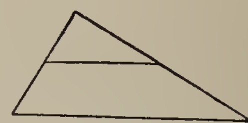


Fig. 82

2. If the segments intercepted by parallel lines on one transversal are equal, then the segments intercepted on every transversal are equal, Fig. 83.

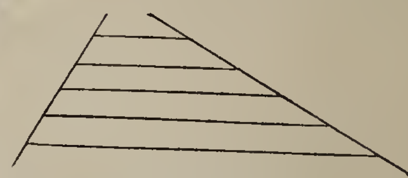


Fig. 83

3. The line drawn through the midpoint of one of the non-parallel sides of a trapezoid parallel to the bases, bisects the other side, Fig. 84.

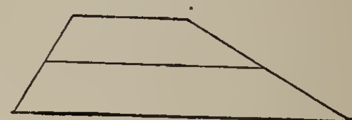


Fig. 84

II

4. A line parallel to one side of a triangle divides the other two sides proportionally, Fig. 85.

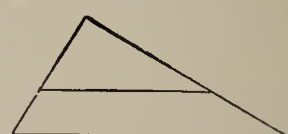


Fig. 85

5. If a number of parallels cut two transversals the segments of one transversal are proportional to the corresponding segments of the other, Fig. 86.

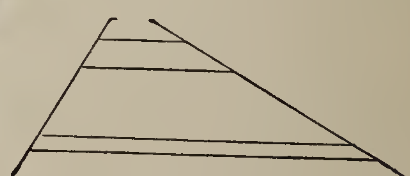


Fig. 86

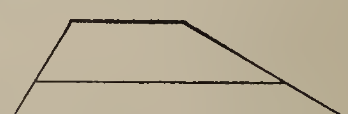


Fig. 87

6. A line parallel to the base of a trapezoid divides the two non-parallel sides proportionally, Fig. 87.

7. Two lines that cut two intersecting lines making the corresponding segments proportional are parallel, Fig. 88.

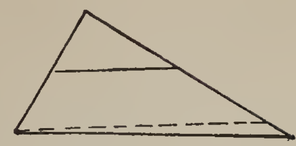


FIG. 88

8. The line joining the midpoints of two sides of a triangle is parallel to the third side, Fig. 89.

III

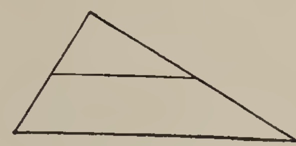


FIG. 89

9. The median of a trapezoid is parallel to the bases, Fig. 90.

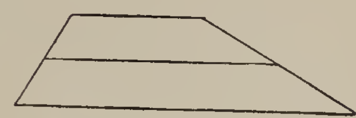


FIG. 90

EXERCISES

‡ **1.** If the segment joining the midpoints of two opposite sides of a quadrilateral and a diagonal bisect each other, the quadrilateral is a parallelogram.

2. Prove that the medians and diagonals of a parallelogram meet in a common point (Fig. 91).

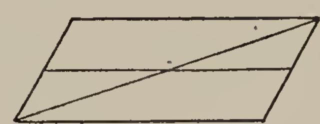


FIG. 91

170. Theorem: *The bisector of an interior angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.*

Given $\triangle ABC$, $x=y$, Fig. 92,

To prove $\frac{AD}{DB}=\frac{AC}{CB}$.

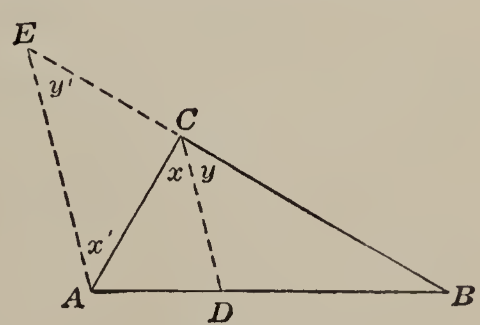


FIG. 92

Proof: Extend BC

Draw $AE \parallel DC$

Then, $\dfrac{AD}{DB}=\dfrac{EC}{CB}$ Why?

$x=x'$ Why?

$y=y'$ Why?

$x=y$ Why?

$\therefore\ x'=y'$ Why?

$\therefore\ EC=CA$ Why?

$\therefore\ \dfrac{AD}{DB}=\dfrac{AC}{CB}$ Why?

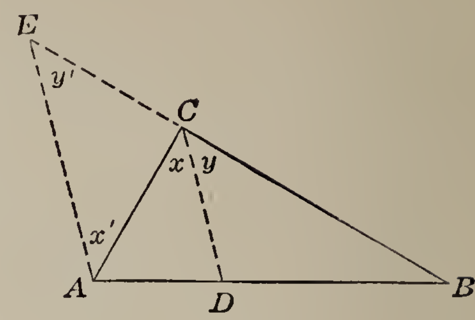


FIG. 92

EXERCISES

1. Prove that a line passing through the vertex of a triangle and dividing the opposite side into segments proportional to the other two sides, bisects the angle included between those sides (converse of § 170).

To prove this exercise, use the proof in § 170 as an *analysis*.

2. In Fig. 92, $AC=8$, $CB=10$, $AB=9$. Find the lengths of AD and DB.

3. In Fig. 92, $AC=5$, $CB=4$, $DB=3$. Find the lengths of AD and AB.

4. If $CA=8$, $CB=16$, and $AB=12$, Fig. 92, find AD and DB.

171. External division of a segment. A point P on a segment AB divides AB into the segments AP and PB, Fig. 93. Considering the direction AB as **positive,** and the direction BA as **negative,** then $(+AP)+(+PB)=(+AB)$.

FIG. 93

FIG. 94

If P is on the extension of AB, Fig. 94, then AP is positive, and PB is negative, nevertheless the statement $(AP)+(PB)=(+AB)$ still holds good. Because of this equation, AP and PB are called parts of AB, and AB is said to be divided **externally** by P. Thus, in *external* as in *internal* division of AB the two parts are measured one from A to P, and the other from P to B.

172. Theorem: *The bisector of an exterior angle of a triangle divides the opposite side externally into segments that are proportional to the other sides.**

Given $\triangle ABC$, $x=y$.

To prove $\dfrac{AD}{DB}=\dfrac{AC}{CB}$.

The proof is practically the same as in § 170.

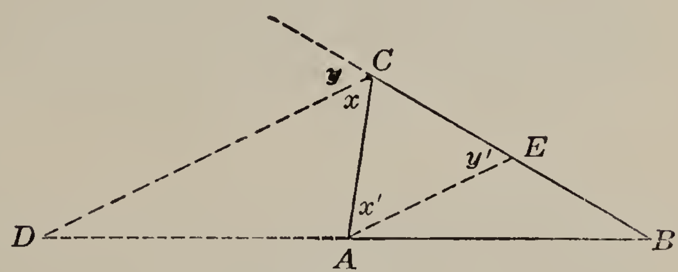


FIG. 95

173. Harmonic division. If a segment is divided internally and externally in *the same ratio* it is said to be divided **harmonically.**

EXERCISE

Prove that the bisector of an interior angle of a triangle and the bisector of the exterior angle at the same vertex divide the opposite side harmonically.

Problems of Construction

174. Fourth proportional. In a proportion, as $\dfrac{a}{b}=\dfrac{c}{d}$, d is the **fourth proportional to** a, b, and c.

* Pappus of Alexandria recognized this theorem, though the Pythagoreans were the first to deal with the harmonic division of lines (Tropfke, *History of Elementary Mathematics* [in German], Vol. II, p. 82).

EXERCISES

1. *To construct the fourth proportional to three given segments.*

Given the segments a, b, and c.

Required to construct the fourth proportional to a, b, and c.

a) **Algebraic solution:** Let x be the fourth proportional. Find the values of a, b, and c by measuring and substitute them in the proportion $\frac{a}{b}=\frac{c}{x}$.

Solve this equation for x.

Construct a segment whose measure is x.

This is the required fourth proportional.

b) **Geometric solution:** On one of two intersecting lines, as AB, lay off $AD=a$, $DE=b$, Fig. 96.

On the other, as AC, lay off $AF=c$.

Draw DF.

Draw $EG \parallel DF$.

Then FG is the required fourth proportional. Prove by § 163.

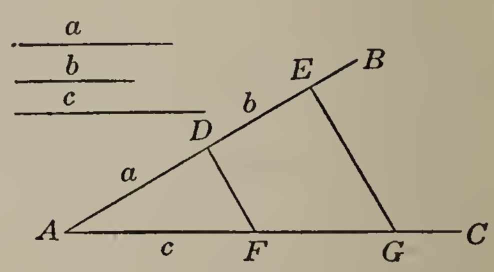


FIG. 96

To test the correctness of the construction, measure the four segments to two decimal places and see if these four numbers are proportional.

2. Find by means of an equation the fourth proportional to 1, 2, and 8.

3. Solve for x: $\frac{x}{57}=\frac{4}{13}$.

175. Third proportional. In a proportion, as $\frac{a}{b}=\frac{b}{c}$, c is called the **third proportional** to a and b.

EXERCISE

Construct the third proportional to two segments: (a) algebraically; follow the instructions of (a), exercise 1, § 174; (b) geometrically, as in (b), § 174.

PHOTOGRAPH OF 500-DRAW SPAN, SHOWING CHANNEL OPEN

CONSTRUCTION OF RAILWAY BRIDGE IN SIERRA LEONE, WEST AFRICA

Point out the uses of mathematical forms in bridge and trestle construction, using the structures shown above as illustrations.

176. To divide a segment in a given ratio. To divide a segment AB *internally* in the ratio $\frac{m}{n}$ means to find a point, P, *on* AB so that $\frac{AP}{PB}=\frac{m}{n}$. To divide AB *externally* in the ratio $\frac{m}{n}$ means to find a point, P', *on the extension* of AB so that $\frac{AP'}{P'B}=\frac{m}{n}$ (see § 171).

EXERCISES

1. *To divide a segment internally in the ratio* $\frac{m}{n}$.

Let AB (Fig. 97) be the given segment. Draw a line AC through A and lay off $AD=m$ and $DE=n$. Draw EB. Through D draw $DF \parallel EB$. Then F divides AB internally in the ratio $\frac{m}{n}$.

Test the correctness of the construction by measuring the segments. Give proof.

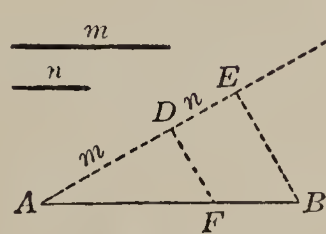


FIG. 97

2. Show how to divide a segment internally in the ratio $\frac{m}{n}$, using § 170.

3. *To divide a segment externally in the ratio* $\frac{m}{n}$.

Draw $AD=m$ (Fig. 98), $DE=n$. Join E to B and draw $DF' \parallel EB$. Then $\frac{AF'}{F'B}=\frac{m}{n}$. Prove.

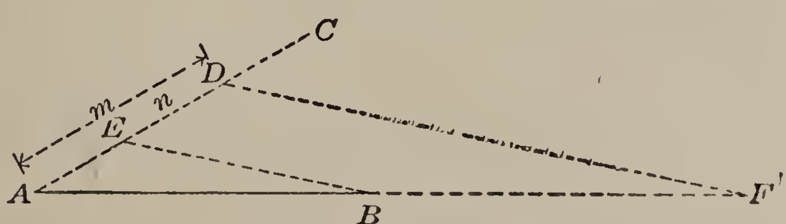


FIG. 98

4. Show how to divide a segment externally, using § 172.

5. A segment $AB=18$ is divided internally, or externally, at a point P. What is the ratio $\frac{AP}{PB}$ for $AP=2$? 3? 6? 9? 20? 30?

6. *To divide a given segment, AB, into segments proportional to several given segments, x, y, and z.*

On a line, as AC (Fig. 99), lay off x, y, z successively and join B to the last point of division D. Draw parallels to BD at the points of division. Then $\frac{x'}{y'}=\frac{x}{y}$. Why?

$\frac{y'}{z'}=\frac{y}{z}$. Why?

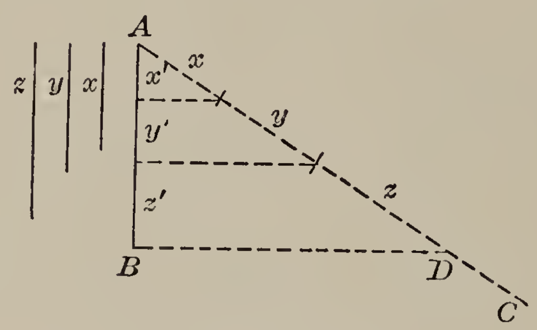


FIG. 99

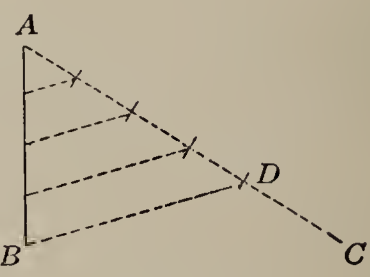


FIG. 100

7. *To divide a segment into equal parts* (Fig. 100).

The construction is the same as in exercise 6, using equal segments instead of x, y, and z.

Lines and Planes in Space

177. Line perpendicular to a plane. If a straight line intersects a plane and is perpendicular to every straight line passing through the point of intersection and lying in the plane, it is said to be **perpendicular to the plane.**

Show that the vertical edge of a door is perpendicular to the floor of the classroom.

Show that an edge of a cube is perpendicular to one of the faces.

178. Theorem: *Two planes perpendicular to the same line are parallel.*

Given planes P and Q perpendicular to AB.

To prove $P \parallel Q$.*

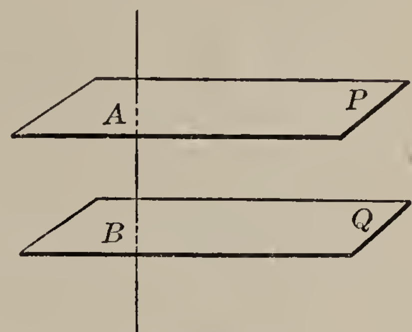


Fig. 101

Proof (indirect method):

Suppose P is not parallel to Q.

Then P, if far enough extended, meets Q in some point, C.

Imagine CA and CB drawn.

Then CA lies wholly in plane P. Why?

CB lies in plane Q. Why?

$\therefore$ CA and CB are both perpendicular to AB (§ 177).

This is impossible, as only one perpendicular can be drawn from a point to a line.

Therefore, the assumption is wrong and P is parallel to Q.

179. Theorem: *If two parallel planes are cut by a third plane, the intersections are parallel.*

Given plane $P \parallel$ plane Q, plane R intersecting planes P and Q in AB and CD respectively; Fig. 102.

To prove $AB \parallel CD$.

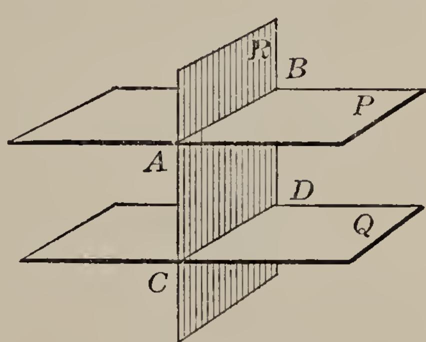


Fig. 102

* When proving theorems involving lines and planes in space, the student will find it helpful to think of lines and planes in the classroom as representing the conditions of the theorem. Thus, the ceiling, the floor and the line of intersection of two walls will illustrate the conditions of this theorem.

Proof (indirect method):
Assume AB not parallel to CD.
Since AB and CD lie in plane R they would meet, if far enough extended, at some point E.
Then E, being on both lines AB and CD would lie in both planes P and Q. Why?

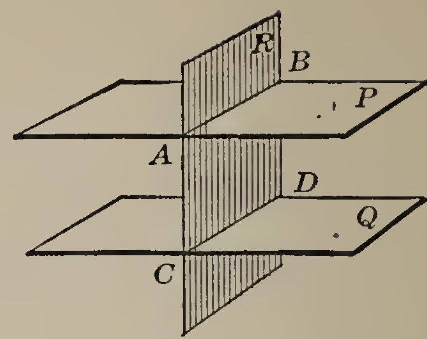


FIG. 102

This contradicts the hypothesis that $P \parallel Q$.
Therefore the assumption is wrong and $AB \parallel CD$

180. Theorem: *Parallel line-segments intercepted by parallel planes are equal.*

Prove $ACDB$, Fig. 103, a parallelogram.

Then $AB = CD$. Why?

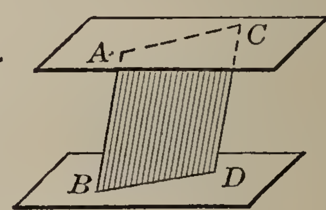


FIG. 103

181. Theorem: *If three or more parallel planes are cut by two transversals, the corresponding segments of the transversals are in proportion.*

Given planes $P \parallel Q \parallel R$, cut by AB and CD, Fig. 104.

To prove $\dfrac{AE}{EB} = \dfrac{CF}{FD}$.

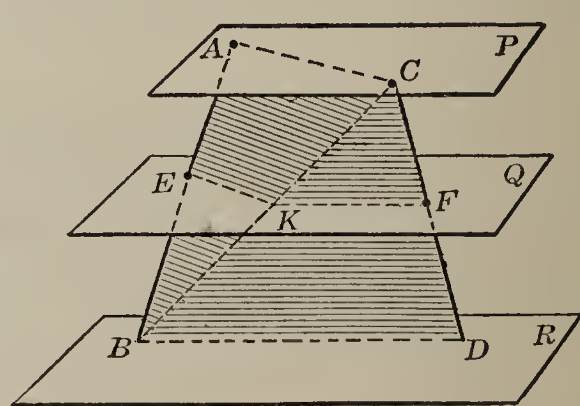


FIG. 104

Proof: Draw CB, cutting Q in K. Pass planes through AB and BC, and BC and CD, cutting planes P, Q, and R in AC, EK, KF, and BD.

$P \parallel Q$ Why?

$\therefore\ AC \parallel EK$ Why?

$$Q \parallel R \qquad \text{Why?}$$

$$\therefore \quad KF \parallel BD \qquad \text{Why?}$$

$$\therefore \quad \frac{AE}{EB} = \frac{CK}{KB} \qquad \text{Why?}$$

$$\text{and} \quad \frac{CK}{KB} = \frac{CF}{FD} \qquad \text{Why?}$$

$$\therefore \quad \frac{AE}{EB} = \frac{CF}{FD} \qquad \text{Why?}$$

Summary

182. The chapter has taught the meaning of the following terms:

proportion
diagonal scale, pantograph, proportional compasses
median of a trapezoid
commensurable and incommensurable magnitudes
internal and external division of a segment, harmonic division
fourth proportional, third proportional
line perpendicular to a plane

183. The following theorems have been proved:

1. *A line bisecting a side of a triangle and parallel to a second side bisects the third side.*

2. *If three or more parallel lines intercept equal segments on one transversal, they intercept equal segments on every transversal.*

3. *If two parallels cut two intersecting transversals, the segments intercepted on one transversal are proportional to the corresponding segments on the other.*

4. *If a number of parallels cut two transversals the segments intercepted on one transversal are proportional to the corresponding segments on the other.*

5. *Two lines that cut two given intersecting lines and make the corresponding segments of the given lines proportional, are parallel.*

6. *The line joining the midpoints of two sides of a triangle is parallel to the third side.*

7. *The bisector of an interior (exterior) angle of a triangle divides the opposite side internally (externally) into segments that are proportional to the adjacent sides.*

8. *Two planes perpendicular to the same line are parallel.*

9. *If two parallel planes are cut by a third plane the intersections are parallel.*

10. *Parallel segments intercepted by parallel planes are equal.*

11. *If three or more parallel planes are cut by two transversals, the corresponding segments of the transversals are in proportion.*

184. The following constructions have been taught:

1. *To construct the fourth proportional to three given line-segments.*

2. *To construct the third proportional to two segments.*

3. *To divide a segment in a given ratio, internally and externally.*

4. *To divide a segment into parts proportional to several given segments.*

5. *To divide a segment into equal parts.*

CHAPTER VI

PROPORTION. FACTORING. VARIATION

Fundamental Theorems

185. In the first-year course we saw the importance of proportions in the solution of problems. In chapter v we made a study of proportional line-segments. It is one of the purposes of this chapter to study the **properties** of proportions.

186. Theorem: *In a proportion the product of the means is equal to the product of the extremes.*

Proof: Multiply both members of the equation $\frac{a}{b}=\frac{c}{d}$ by bd.

The preceding theorem is important because it is a convenient **test of proportionality,** and also because it suggests a simple way of **clearing of fractions** such equations as are proportions.

EXERCISES

Using the theorem in § 186, work the following exercises:

1. Which of these statements are proportions?

$$\frac{15}{9}=\frac{10}{6};\quad \frac{8}{15}=\frac{4}{7};\quad \frac{6}{18}=\frac{7}{21};\quad \frac{4}{7}=\frac{12}{20}.$$

2. Clear the following equations of fractions, but do not solve them:

$$\frac{4}{x}=\frac{20}{3};\quad \frac{180-x}{x}=\frac{5}{14};\quad \frac{2+x}{3-x}=\frac{5+x}{8-x};$$

$$\frac{x-y}{10}=\frac{7}{x^2+xy+y^2};\quad \frac{8}{u+v}=\frac{u^2-uv+v^2}{2};$$

$$\frac{x+2}{x-4}=\frac{x+6}{x-7};\quad \frac{x-1}{x^2+3x+4}=\frac{x+2}{2x^2-4x+7}.$$

3. Solve the equation $\frac{x}{2}=\frac{12}{8}$.

187. The following exercises show that proportions may be obtained from equations that express equality of products.

EXERCISES

1. The statements below are different arrangements of the four factors in the equation $8 \cdot 7=14 \cdot 4$. Some of them are equations, others only appear to be equations. Apply the *test of proportionality* and point out which statements are proportions.

1. $\frac{8}{14}=\frac{4}{7}$ **3.** $\frac{4}{7}=\frac{8}{14}$ **5.** $\frac{8}{7}=\frac{14}{4}$ **7.** $\frac{8}{4}=\frac{7}{14}$

2. $\frac{7}{4}=\frac{14}{8}$ **4.** $\frac{14}{8}=\frac{7}{4}$ **6.** $\frac{7}{14}=\frac{8}{4}$ **8.** $\frac{8}{14}=\frac{7}{4}$

2. Exercise 1 shows that proportions are formed from the numbers 4, 7, 8, and 14 only when they are taken in a certain order. From what place in the equation $8 \cdot 7=14 \cdot 4$ must the **means** be taken to form a proportion? The **extremes**?

3. Write four proportions from $3 \cdot 28=4 \cdot 21$. Apply in each case the test of proportionality.

4. Write four proportions from $a \cdot 12b = 3a \cdot 4b$, and *test*.

Exercises 1 to 4 illustrate the following theorem:

188. Theorem: *If the product of two factors is equal to the product of two others, proportions may be formed by taking as means the factors of either product, and as extremes the factors of the other product.*

Given $ad = bc$.

To prove that $\frac{a}{b} = \frac{c}{d}$.

Proof: Divide both members of the equation $ad = bc$ by bd.

EXERCISES

1. Let $ad = bc$. Prove that the following statements are proportions:

$$\frac{a}{c} = \frac{b}{d} \qquad \frac{b}{a} = \frac{d}{c} \qquad \frac{c}{a} = \frac{d}{b} \qquad \frac{c}{d} = \frac{a}{b}$$

2. Form proportions from:

1. $5a - 10b = 4x^2 - 3xy$
2. $16a^2 - 2axy = ax + ay - az$
3. $(x+y)^2 = m^2 - 2mx + x^2$
4. $K^2 + 8K + 16 = b^2 - 6b + 9$
5. $16a^2 - 25b^2 = 36 - 25y^2$
6. $a^2 - b^2 = c^2 - d^2$
7. $p^4 - 16 = a^2 - 64$
8. $ax + ay + az = br + bs + bt$
9. $5m^2 + 10mn - 15n^2 = 9a^2 - 4ab - 13b^2$
10. $6x^2 + 13x + 2 = a^2 + 2a + 1$
11. $x^2 - 5x + 6 = y^2 + 3y - 28$

Factoring

189. Review: In arithmetic we have tests of divisibility by which we can tell when 2, 3, 5, 9, 11, etc., are divisors of a number. Likewise, in the course of the first year we have learned how to recognize factors of certain polynomials. This work may be summarized as follows:

Polynomials: *Common monomial factor,* as $ax+ay$. In this case the common factor is one of the factors of the polynomial. The other factor is found by dividing the polynomial by the common factor.

Thus, $$ax+ay=a(x+y).$$

Factor the following:

1. $3x+3y$
2. $cx^2+dx^3+fx^4$
3. $5a^3b+24a^4c-10a^5d$
4. $3a^2b-12ab^2$
5. $8x^3y^3+4x^2y^3$
6. $3x^2y^2-2xy-3xy^3$
7. $15a^3x-10a^3y+5a^3z$
8. $32a^3b^3-ab^6$

Binomials: *The difference of two squares,* as x^2-y^2.

The factors are: the difference of the square roots of these squares and the sum of the square roots.

Thus, $$x^2-y^2=(x-y)(x+y).$$

Factor the following:

1. $1-144x^2y^2$
2. $\frac{x^2}{y^2}-1$
3. x^6-25y^2
4. $(b+c)^2-a^2$
5. $(a+b)^2-(c+d)^2$
6. $9a^2-16$
7. $16a^2-25$
8. $1-r^4$
9. $4x^2-9y^2$
10. a^4-b^4
11. x^6-y^6

Trinomials: (1) *Trinomial Squares*, as $x^2+2xy+y^2$ and $x^2-2xy+y^2$.

In each case we have two equal factors, i.e., the sum of the square roots of the square terms if the sign of the remaining term is $+$, and the difference of the square roots of the square terms if the sign of the remaining term is $-$.

$$\text{Thus, } \mathbf{x^2+2xy+y^2=(x+y)^2}$$
$$\text{and } \mathbf{x^2-2xy+y^2=(x-y)^2}.$$

Factor the following:

1. $4m^2-12am+9a^2$

2. $a^2-8a+16$

3. $9+30x+25x^2$

4. $36x^2+25y^2-60xy$

5. $25+80r+64r^2$

6. $c^2-16c+64$

7. x^4+30x^2+225

8. $121a^2+198ay+81y^2$

(2) *Trinomials of the form* ax^2+bx+c. The factors are found by trial.

Thus, for factors of $3x^2+17x+10$ we have as one of the various possibilities: $\left\{\begin{matrix}3x+2\\x+5\end{matrix}\right.$. Multiplying, we find that $3x^2+17x+10=(3x+2)(x+5)$.

Factor the following:

1. $2x^2+11x+12$

2. $8c^2+4bc-12b^2$

3. $3x^2-17x+10$

4. $11a^2-23ab+2b^2$

5. $8y^2-31y+21$

6. $5x^2-38x+21$

7. $7k^2+123k-54$

8. $5m^2-29mn+36n^2$

190. Further extension of factorable polynomials.

EXERCISES

1. Multiply as indicated and make a rule by which we may find by inspection the products of polynomials like the following:

1. $(x+y)(x^2-xy+y^2)$
2. $(x-y)(x^2+xy+y^2)$
3. $(a+b)(a^2-ab+b^2)$
4. $(a-b)(a^2+ab+b^2)$
5. $(a+2b)(a^2-2ab+4b^2)$
6. $(3a-b)(9a^2+3ab+b^2)$
7. $(2a+3b)(4a^2-6ab+9b^2)$
8. $(3a^2+5b^2)(9a^4-15a^2b^2+25b^4)$
9. $(7a^3-4b^2)(49a^6+28a^3b^2+16b^4)$
10. $(2a^2b^2-3c^2)(4a^4b^4+6a^2b^2c^2+9c^4)$

2. Make a rule for factoring the sum of two cubes.

3. Make a rule for factoring the difference of two cubes.

4. Factor $64a^3+27b^3$.

The expression is the sum of two cubes, $64a^3=(4a)^3$ and $27b^3=(3b)^3$. Therefore, *one factor is the sum of the cube roots* of $64a^3$ and $27b^3$, i.e. $(4a+3b)$.

The other factor is obtained from the first factor as follows: *square the first term,* $(4a)^2=16a^2$, *subtract the product of the two terms,* $-(4a)(3b)=-12ab$, *add the square of the second term,* $(3b)^2=9b^2$.

Hence, $64a^3+27b^3=(4a+3b)(16a^2-12ab+9b^2)$.

5. Factor $8x^3-125y^3$.

Show by multiplying that—

$$8x^3-125y^3=(2x-5y)(4x^2+10xy+25y^2).$$

Explain how the factor $4x^2+10xy+25y^2$ may be formed from the terms of the factor $2x-5y$.

6. Form proportions from $x^2-y^2=m^3+n^3$. (Apply § 188.)

7. Form proportions from $p^3-v^3=a^2-b^2$.

Factor the following expressions, doing as many as you can mentally:

8. a^3+b^3
9. a^3-b^3
10. $8x^3-y^3$
11. m^3+27n^3
12. $8c^3-d^3$
13. $343+x^3$
14. x^3+64
15. $x^3+\frac{1}{8}$
16. ax^3-8ay^3
17. $216-27a^3$
18. $125x^3+8y^3$
19. $27a^3+64b^3$
20. $512c^3-27d^3$
21. K^3l^3+343
22. $729a^6+216c^6$
23. $(a+b)^3+c^3$
24. $(m+n)^3-a^3$
25. $(w+3)^3-t^3$
26. $(5m-n)^3+c^3$
27. $(s+2t)^3+27x^3$

Proportions Obtained from Given Proportions

191. Proportions may be obtained from other proportions in various ways, as is shown in the following:

EXERCISES

1. Using a length equal to 2 centimeters as a unit, measure to two places of decimals AB, DB, DA, EB, EC, and BC, Fig. 105, and show by dividing that—

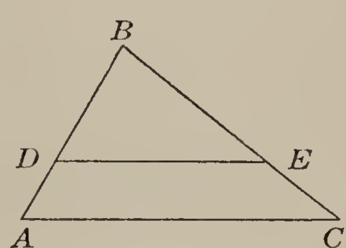


FIG. 105

1. $\dfrac{BD}{DA}=\dfrac{BE}{EC}$
2. $\dfrac{BD}{BE}=\dfrac{DA}{EC}$
3. $\dfrac{DA}{DB}=\dfrac{EC}{EB}$
4. $\dfrac{BA}{BD}=\dfrac{BC}{BE}$
5. $\dfrac{BA}{DA}=\dfrac{BC}{EC}$

2. Apply the test of proportionality to the following:

1. $\dfrac{4}{7}=\dfrac{12}{21}$
2. $\dfrac{4}{12}=\dfrac{7}{21}$
3. $\dfrac{7}{4}=\dfrac{21}{12}$
4. $\dfrac{4+7}{4}=\dfrac{12+21}{12}$
5. $\dfrac{4+7}{7}=\dfrac{12+21}{21}$
6. $\dfrac{7-4}{4}=\dfrac{21-12}{12}$

3. What change in the position of the terms of proportion **1**, exercise 2, will transform it into proportion **2**? Equation **2** is said to be obtained from **1** by *alternation.*

4. What change will transform proportion **1** into **3**? Equation **3** is said to be obtained from **1** by *inversion.*

5. What change will transform proportion **1** into **4**? Equation **4** is said to be obtained from **1** by *addition.*

6. What change will transform proportion **1** into **5**?

7. What change will transform proportion **1** into **6**? Equation **6** is said to be obtained from **1** *by subtraction.*

192. Alternation. When, by interchanging the means, or by interchanging the extremes of a given proportion a second proportion is formed, it is said to be obtained from the given proportion by **alternation.**

EXERCISES

1. Apply alternation to the proportion $\frac{15}{27}=\frac{10}{18}$.

2. Apply alternation to $\frac{a}{b}=\frac{c}{d}$.

3. Show that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{b}{d}$ and $\frac{d}{b}=\frac{c}{a}$.

Apply first the theorem of § 186, then of § 188.

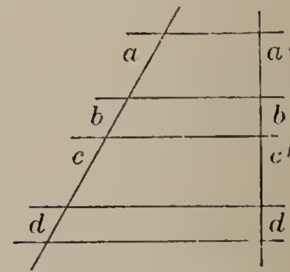


Fig. 106

4. Show that if $\frac{a}{b}=\frac{a'}{b'}$ and $\frac{b}{c}=\frac{b'}{c'}$, etc., Fig. 106, then $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$, etc.

5. If two equilateral polygons, Fig. 107, have the same number of sides, the corresponding sides are in proportion. Prove.

Show that $\frac{a}{b}=1$ and $\frac{a'}{b'}=1$

$\therefore \quad \frac{a}{b}=\frac{a'}{b'}$ Why?

$\therefore \quad \frac{a}{a'}=\frac{b}{b'}$ Why?

Similarly, $\frac{b}{b'}=\frac{c}{c'}=\frac{d}{d'}$, etc.

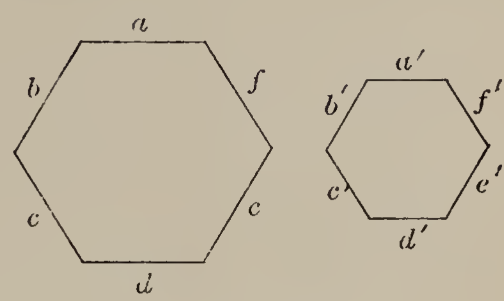


FIG. 107

193. Inversion. By inverting the ratios of a given proportion a second proportion is formed, which is said to be obtained from the given proportion by **inversion.**

See § 191, exercise 2, **1** and **3**.

EXERCISES

1. Apply inversion to $\frac{a}{b}=\frac{c}{d}$.

2. Prove that if $\frac{a}{b}=\frac{c}{d}$ then $\frac{b}{a}=\frac{d}{c}$.

Apply the theorems of §§ 186, 188.

194. Antecedent. Consequent. In a ratio as $\frac{AB}{CD}$, or $\frac{a}{b}$, AB and a are the **antecedents** and CD and b the **consequents.**

195. Addition. Subtraction. Theorem: *In a proportion the sum (or difference) of the terms of one ratio is to the antecedent, or consequent, as the sum (or difference) of the terms of the other ratio is to its antecedent or consequent.*

Thus, from $\frac{8}{5}=\frac{16}{10}$, we obtain the proportions

$$\frac{8+5}{5}=\frac{16+10}{10} \text{ and } \frac{8-5}{5}=\frac{16-10}{10}.$$

The resulting proportion is said to be obtained from the given proportion by **addition** if the sum is taken, and by **subtraction** if the difference is taken.

Given $\frac{a}{b}=\frac{c}{d}$.

To prove that $\frac{a+b}{b}=\frac{c+d}{d}$.

Analysis:

1. Assume $\frac{a+b}{b}=\frac{c+d}{d}$
2. Then $(a+b)d=(c+d)b$ Why?
3. $\therefore\ ad+bd=cb+bd$ Why?
4. $\therefore\ ad=cb$ Why?
5. $\therefore\ \frac{a}{b}=\frac{c}{d}$ Why?

The proof is obtained by reversing the steps in the preceding analysis as follows:

Proof:

1. $\frac{a}{b}=\frac{c}{d}$ Why?
2. $ad=cb$ Why?
3. $ad+bd=cb+bd$ Why?
4. $(a+b)d=(c+d)b$ Why?
5. $\frac{a+b}{b}=\frac{c+d}{d}$ Why?

Notice the method used in obtaining this proof. First, we assume the conclusion to be true.

Then, by correct reasoning, we deduce a known fact, e.g., the hypothesis. The steps being reversible, we start from this known fact and get the conclusion by reversing the steps.

This last part is the *proof* of the theorem.

Similarly, prove that $\frac{a-b}{b}=\frac{c-d}{d}$.

196. Theorem: *In a proportion the sum of the terms of one ratio is to their difference as the sum of the terms of the other ratio is to their difference.*

Thus, if $\frac{7}{3}=\frac{21}{9}$, it follows that $\frac{10}{4}=\frac{30}{12}$.

EXERCISE

Use the method of analysis, as in § 195, to prove that if $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

197. Addition and subtraction. The proportion $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ is said to be formed from $\frac{a}{b}=\frac{c}{d}$ by **addition and subtraction.**

EXERCISES

Apply addition and subtraction to the following proportions:

1. $\frac{2m+3n}{2m-3n}=\frac{2s+3t}{2s-3t}$

$$\frac{2m+3n+2m-3n}{2m+3n-2m+3n}=\frac{2s+3t+2s-3t}{2s+3t-2s+3t}, \text{ or } \frac{4m}{6n}=\frac{4s}{6t}, \text{ or } \frac{m}{n}=\frac{s}{t}$$

2. $\frac{a}{b}=\frac{s}{t}$

3. $\frac{x-a+b}{x+a-b}=\frac{a-b-x}{a+b+x}$

4. $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=3$

Solve for x:

5. $\frac{2+\sqrt{x}}{2-\sqrt{x}}=\frac{\sqrt{x+5}+\sqrt{x}}{\sqrt{x+5}-\sqrt{x}}$

Apply addition and subtraction and then solve.

198. Theorem: *If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

Thus, from $\frac{2}{3}=\frac{4}{6}=\frac{8}{12}= \ldots$, it follows, according to this theorem, that $\frac{2+4+8 \ldots}{3+6+12 \ldots}=\frac{2}{3}$.

Given $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}= \ldots$

To prove that $\frac{a+c+e+g+ \ldots}{b+d+f+h+ \ldots}=\frac{a}{b}=\frac{c}{d}= \ldots$

Proof:

$$\begin{cases} \frac{a}{b}=\frac{a}{b} & \text{Why?} \\ \frac{c}{d}=\frac{a}{b} & \text{Why?} \\ \frac{e}{f}=\frac{a}{b} & \text{Why?} \\ \frac{g}{h}=\frac{a}{b}, \text{ etc.} & \text{Why?} \end{cases}$$

$$\therefore \begin{cases} ab=ab & \text{Why?} \\ cb=ad & \text{Why?} \\ eb=af & \text{Why?} \\ gb=ah, \text{ etc.} & \text{Why?} \end{cases}$$

Adding $(a+c+e+g \ldots\ldots)b=a(b+d+f+h \ldots\ldots)$

$$\therefore \frac{a+c+e+g \ldots\ldots}{b+d+f+h \ldots\ldots}=\frac{a}{b}=\frac{c}{d}, \text{ etc.} \qquad \text{Why?}$$

EXERCISES

Prove the following exercises:

1. If $\frac{a}{b}=\frac{c}{d}$, it follows that $\frac{a^2}{b^2}=\frac{c^2}{d^2}$; $\frac{a^3}{b^3}=\frac{c^3}{d^3}$; $\frac{\sqrt{a}}{\sqrt{b}}=\frac{\sqrt{c}}{\sqrt{d}}$.

2. If $\frac{a}{b}=\frac{c}{d}$, it follows that $\frac{2a}{3b}=\frac{2c}{3d}$; and $\frac{ma}{nb}=\frac{mc}{nd}$.

Prove the following by the method of analysis:

3. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{3a+b}{b}=\frac{3c+d}{d}$

4. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a}{c}=\frac{a+5b}{c+5d}$

5. If $\frac{x}{y}=\frac{s}{t}$, then $\frac{3y+2t}{4y}=\frac{3x+2s}{4x}$

6. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a+2c}{b+2d}=\frac{c+3e}{d+3f}=\frac{e+5a}{f+5b}$

7. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a+c}{b+d}=\frac{a^2d}{b^2c}$

Analysis: Assume $\frac{a+c}{b+d}=\frac{a^2d}{b^2c}$

Then	$ab^2c+b^2c^2=a^2bd+a^2d^2$	Why?
Since,	$bc=ad$	Why?
and	$b^2c^2=a^2d^2$	Why?
	$\therefore \quad ab^2c=a^2bd$	Why?
	$\therefore \quad bc=ad$	Why?

The *proof* is obtained by retracing the steps in the analysis.

8. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^2+c^2}{a^2-c^2}=\frac{ab+cd}{ab-cd}$

9. If $\frac{a}{b}=\frac{b}{c}$, then $\frac{a^2+b^2}{a}=\frac{b^2+c^2}{c}$

10. If $\frac{a}{b}=\frac{b}{c}$, then $\frac{a+c}{b^2+c^2}=\frac{a-c}{b^2-c^2}$

11. If $\frac{a}{b}=\frac{b}{c}$, then $\frac{a^2+ab}{a}=\frac{b^2+bc}{c}$

12. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a+c}{b+d}=\frac{c+e}{d+f}=\frac{e+a}{f+b}$

199. We have seen in §§ 192–98 that from a proportion, as $\frac{a}{b}=\frac{c}{d}$, the following proportions may be obtained:

1.	$\frac{a}{c}=\frac{b}{d}$ and $\frac{d}{b}=\frac{c}{a}$,	by alternation
2.	$\frac{b}{a}=\frac{d}{c}$,	by inversion
3.	$\frac{a+b}{a}=\frac{c+d}{c}$ and $\frac{a+b}{b}=\frac{c+d}{d}$,	by addition
4.	$\frac{a-b}{a}=\frac{c-d}{c}$ and $\frac{a-b}{b}=\frac{c-d}{d}$,	by subtraction
5.	$\frac{a+b}{a-b}=\frac{c+d}{c-d}$,	by addition and subtraction
6.	$\frac{a+c}{b+d}=\frac{a}{b}=\frac{c}{d}$	by § 198: the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

EXERCISES

1. Divide 40 into parts that are in the ratio of 3:5.

2. Divide 44 into parts in the ratio of 2/3:4/5.

3. Divide m into parts in the ratio of $a:c$.

4. The denominator of a fraction is 5 greater than the numerator, and the value of the fraction is 2/3. Find the fraction.

5. The value of a fraction is 2/3. If 3 is added to both terms the value becomes 7/10. Find the fraction.

The required fraction is of the form $\frac{2x}{3x}$. Why?

Then $\frac{2x+3}{3x+3}=\frac{7}{10}$. Why? Solve.

6. The value of a fraction is 2/5. If 5 be added to the denominator and subtracted from the numerator, the value becomes 3/10. Find the original fraction.

‡**7.** Solve each of the following if the value of the original fraction is 5/7:

1. If 1 be added to both terms the value of the fraction becomes 8/11. Find the original fraction.

2. If 1 be subtracted from both terms the value becomes 7/10. Find the original fraction.

3. If 1 be added to the numerator and subtracted from the denominator the value becomes 4/5. Find the original fraction.

4. If 1 be subtracted from the numerator and added to the denominator the value becomes 7/11. Find the original fraction.

8. Find the values of x and y from $\frac{1}{x}=\frac{2}{y}=\frac{3}{7}$.

Relation between Proportion and Variation

200. Direct Variation. When two variables change values but have always the same ratio, each is said to *vary directly* as, or to *vary as*, the other.

Thus, the number y is said to vary directly as x, if the ratio $\frac{y}{x}$ remains constant, x and y both changing, or varying. The equation

$$\frac{y}{x}=c$$

expresses algebraically, and is equivalent to, the statement that y **varies directly as** x

Show that y is a *function of* x.

201. Relation between direct variation and proportion. Let y vary directly as x and let x_1, y_1; x_2, y_2; x_3, y_3, etc., be corresponding values of x and y.

Since $y=cx$, it follows that $\frac{y}{x}=c$ and that $\frac{y_1}{x_1}=c$, $\frac{y_2}{x_2}=c$, $\frac{y_3}{x_3}=c$, etc.

Therefore, $\frac{y_1}{x_1}=\frac{y_2}{x_2}$

From this equation we can determine any one of the four numbers x_1, y_1, x_2 and y_2, if the other three are given.

EXERCISES

1. The area of a rectangular piece of land of given width varies directly as the length. If the area of a piece 30 ft. long is 2100 sq. ft., what must be the length of a strip containing 10500 sq. feet?

Since the area varies directly as the length,

$$\frac{A_1}{A_2}=\frac{l_1}{l_2}$$

But $A_1=2100$,

$A_2=10500$,

and $l_1=30$

Hence, $$\frac{2100}{10500}=\frac{30}{l_2}.$$

Solve this equation for l_2.

2. The cost of silk of a certain grade varies as the number of yards. If 35 yd. of silk cost \$61.25, find the cost of 90 yards.

3. One hundred feet of copper wire of a certain size weighs 35 pounds. What is the length of wire weighing 175 pounds?

4. If y varies as x, and if $y=80$ when $x=10$, what is the value of y when $x=18$?

202. Inverse variation. When two numbers so vary as to leave the product of any value of one by the corresponding value of the other *constant*, then one is said to **vary inversely** as the other.

The equation

$$xy = c$$

expresses algebraically, and is equivalent to, the statement that the variable y **varies inversely** as the variable x.

Show that y is a *function of* x.

203. Relation between inverse variation and proportion. Let y vary **inversely** as x and let x_1, y_1; x_2, y_2; x_3, y_3; etc., be corresponding values of x and y.

Since $xy = c$, it follows that $x_1y_1 = c$, $x_2y_2 = c$, $x_3y_3 = c$, etc.

Hence, $x_1y_1 = x_2y_2$.

From this we may obtain the proportion $\frac{x_1}{x_2} = \frac{y_2}{y_1}$.

If any three of the four numbers x_1, x_2, y_1, and y_2 are given, the fourth may be found from this proportion.

EXERCISES

1. The volume of air in a bicycle pump varies inversely as the pressure on the piston. If the volume is 16 cu. in., when the pressure is 18 lb., what is the pressure when the volume is 2 cubic inches?

2. The pressure of steam in an engine cylinder varies inversely as the volume. When the pressure is 100 lb. per sq. in. the volume is 50 cubic inches. What will be the pressure per sq. in. when the volume is 75 cubic inches?

3. If x varies inversely as y and if $x = \frac{2}{3}$ when $y = \frac{3}{4}$, find the value of y when $x = 1\frac{1}{2}$.

204. Historical note. Like many other mathematical topics, proportion was long *used* before men *comprehended* its principles. The two forms of proportion that have been studied for over two thousand years are proportion applied to *numbers*, and proportion applied to *line-segments* and *areas*.

Proportion as applied to numbers is one of the oldest mathematical topics. In the oldest known mathematical writing, the

Book of Ahmes (see Cajori, p. 11), written by an Egyptian scribe 1700 B.C., proportion is one of the important subjects. The ancient Chaldeans, Phoenicians, Hindus, Chinese, and Greeks all gave it an important place in their books. The Greeks, Arabs, Hindus, Moors, Romans, and other European peoples of the dark and mediaeval ages, that made any pretensions to learning, all emphasized the doctrine of proportion. Mediaeval geometries and mercantile arithmetics made it a major theme. Indeed, until fifty years ago the "single rule of three" and the "double rule of three," which meant *simple proportion* and *compound proportion*, made up most of advanced arithmetic.

The principles of proportionality as applied to *line-segments* and to *areas* were first studied by the Greeks. They drew their beginnings from Egypt and, perhaps, Babylon. Thales of Miletus (640–546 B.C.) used proportionality, perhaps without knowing it. The Pythagoreans (after 529 B.C.) employed it more extensively. Archytas of Tarentum (428–347 B.C.) extended the theory greatly. Plato (429 – 348 B.C.) was well versed in it, and Eudoxus of Cnidos (408–355 B.C.) greatly perfected the form of the doctrine. *Euclid's Elements* (300 B.C.) devotes the fifth and a part of the sixth book to the doctrine of proportionality as applied to line-segments and areas, a form of the doctrine believed to be due to Eudoxus.

Every nation and people that has acquired any standing in mathematics has given great attention to this doctrine. It was once the most practical part of all geometry, and some of the most practical subjects and topics of mathematics are still based on it.

Summary

205. The chapter has taught the meaning of the following terms:

alternation
inversion
addition applied to a proportion
subtraction applied to a proportion
addition and subtraction
direct variation
inverse variation
antecedent
consequent

206. The following theorems have been proved:

1. *In a proportion the product of the means equals the product of the extremes.*

2. *If the product of two factors equals the product of two others, proportions may be formed by taking as means the factors of one product and as extremes the factors of the other product.*

3. *Proportions may be obtained from other proportions by alternation, by inversion, by addition, by subtraction, by addition and subtraction.*

4. *If two or more ratios are equal, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

207. The following expressions may be factored:

I. *Polynomials:* having a common factor, as $ax+ay$.

II. *Binomials:* which are the difference of two squares, as x^2-y^2;
the difference of two cubes, as a^3-b^3;
the sum of two cubes, as a^3+b^3.

III. *Trinomials:* which are perfect squares, as $x^2 \pm 2xy+y^2$;
which are of the form ax^2+bx+c.

208. The relation between **variation** and **proportion** has been shown.

CHAPTER VII

SIMILAR POLYGONS

Uses of Similar Triangles

209. Similar triangles and polygons. We saw in our work of the first year that similar triangles have the following two important properties:

(1) the ratios of the corresponding sides are equal and
(2) the corresponding angles are equal.

The same properties are possessed by similar polygons. For this reason **similar polygons** are defined as *polygons having the corresponding sides proportional and the corresponding angles equal.* Hence, the statement: polygon $ABCDEF \backsim A'B'C'D'E'F'$, Fig. 108, may be expressed symbolically by the two following statements:

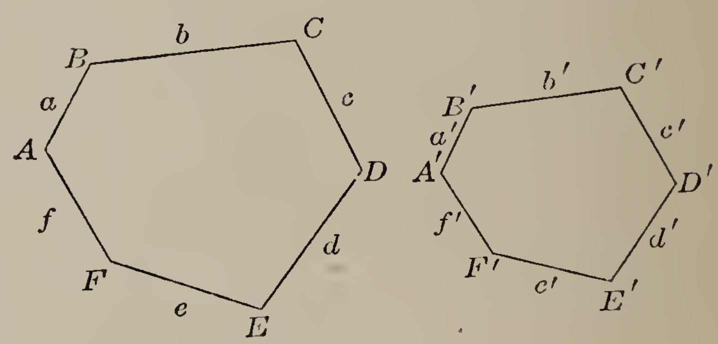


FIG. 108

$$\begin{cases} 1.\ \dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}=\dfrac{d}{d'}=\dfrac{e}{e'}=\dfrac{f}{f'} \\ 2.\ \angle A=\angle A',\ \angle B=\angle B',\ \angle C=\angle C', \text{ etc.} \end{cases}$$

210. Uses of similar triangles. Many problems may be solved by the aid of similar triangles, as may be seen from the following exercises.

EXERCISES

1. To find the height of a chimney.

Let AC, Fig. 109, represent the shadow of the chimney AB, and $A'C'$ the shadow of a vertical stick $A'B'$.

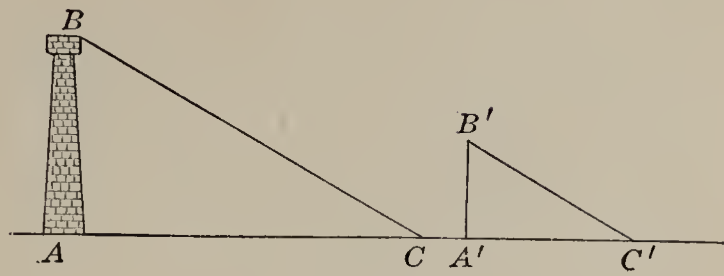


Fig. 109

Assuming rays of sunlight to be parallel, show that $\angle C = \angle C'$.

Since triangles ABC and A'B'C' have two angles equal respectively, they can be shown to be similar (§ 217).

Hence, $\frac{AB}{A'B'} = \frac{AC}{A'C'}$ Why?

and $AB = AC \cdot \frac{A'B'}{A'C'}$ Why?

Using this equation as a formula, find the height of a chimney whose shadow is 108 ft., if at the same time the shadow of a 4-ft. vertical stick is 9 ft. long?

2. To determine the distance across a river.

Sighting across the river with telescope A, Fig. 110, place in the line of sight vertical rods, as at B and C. Take readings of rods at E and D. Depress the telescope sighting at C and take the reading at F. From the readings compute the length of DF and EC.

Fig. 110

$$EC \parallel DF \qquad \text{(See § 373.)}$$

$\therefore$ Triangles AFD and ACE are similar.

For, *a line parallel to a side of a given triangle forms with the other two sides a triangle similar to the given triangle* (§ 214).

$$\text{Hence, } \frac{AE}{AD} = \frac{EC}{DF}$$

$$\therefore \quad AE = \frac{AD \cdot EC}{DF}, \text{ which}$$

expresses AE in terms of the known lengths AD, EC, and DF.

The length, ED, may be found by subtracting AD from AE.

3. To determine an inaccessible distance.

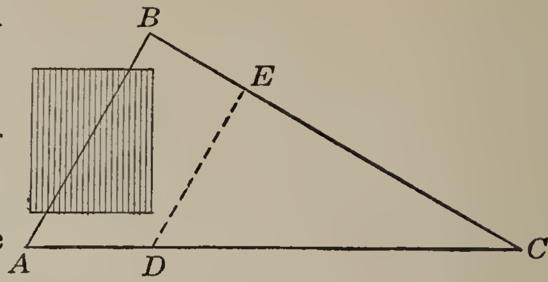


Fig. 111

Let AB, Fig. 111, be the distance to be measured.

From a point C, chosen conveniently, measure BC and AC.

Mark point D on AC.

On BC determine point E so that $\frac{CD}{CA}=\frac{CE}{CB}$. Measure DE.

Triangles CDE and CEB may be shown to be similar.

For, *two triangles are similar if the ratio of two sides of one equals the ratio of two sides of the other, and the angles included between these sides are equal* (§ 218).

$$\text{Hence,} \quad \frac{AB}{DE}=\frac{AC}{DC}$$

$$\text{and} \quad AB=\frac{DE \cdot AC}{DC}$$

Thus DE, AC, and DC being known, AB may be found as the quotient of the product $DE \cdot AC$ and DC.

‡**211.** *To find graphically the quotient of two arithmetical numbers.*

There are a number of instruments for performing mechanically the processes of multiplication, division, and extraction of roots. Fig. 112 is a device based upon similar triangles for finding the **quotients** of arithmetical numbers

Let OA be the dividend-line and OB the line of divisors.

To divide 42 by 72, let the side of a large square represent 10.

Lay off $OC=42$ and from C lay off vertically $CD=72$.

Stretch the string fastened at O so that it passes through D, meeting the *quotient-line*, FQ, at E.

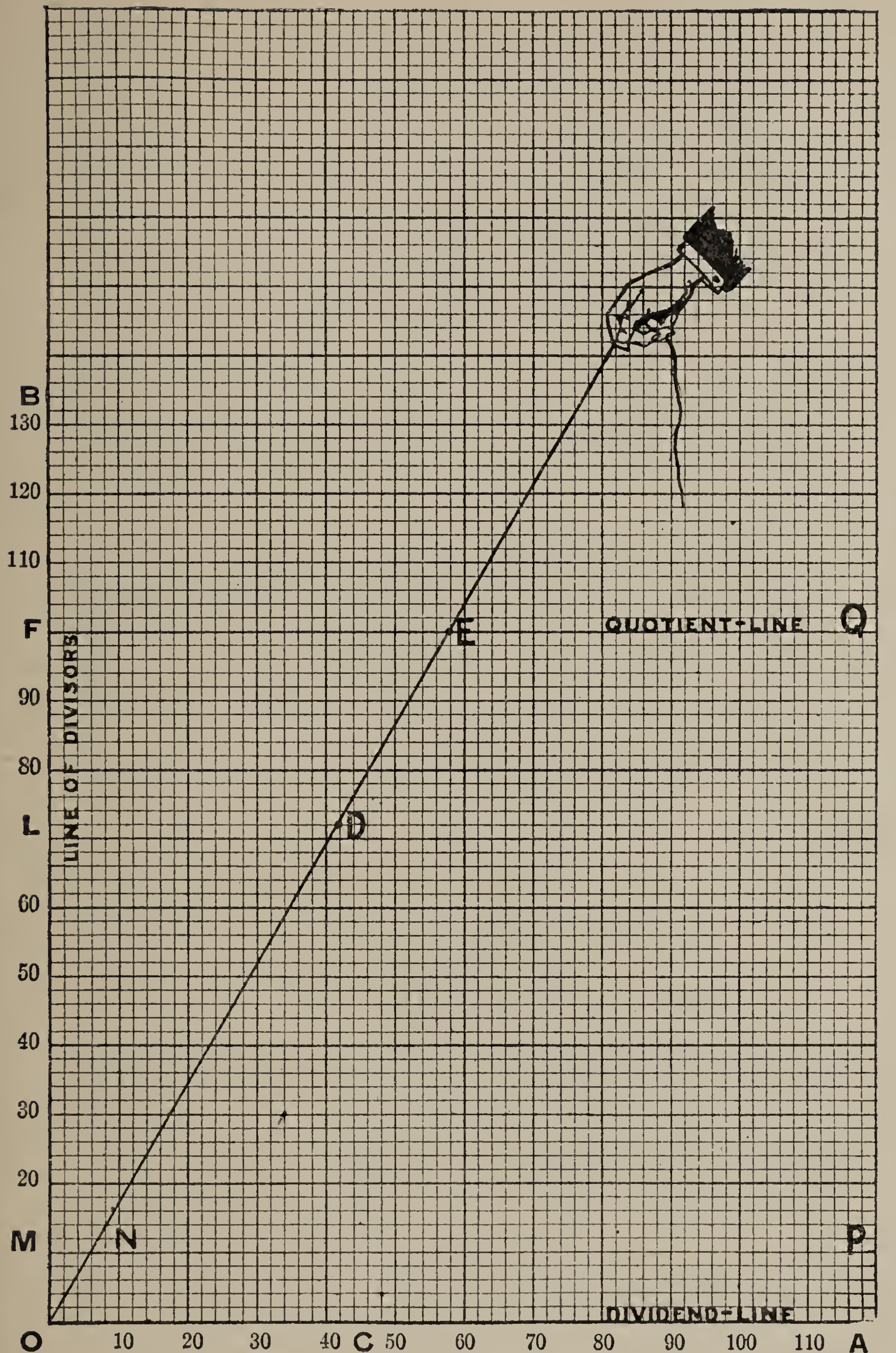

B
130
120
110
F
90
80
L
60
50
40
30
20
M
O
10
20
30
40
C
50
60
70
80
90
100
110
A
LINE OF DIVISORS
E
QUOTIENT-LINE
Q
D
N
P
DIVIDEND-LINE

Fig. 112

Then $\frac{FE}{100}$ represents the quotient $\frac{42}{72}$.

For, triangles OFE and OLD are similar.

Therefore, $\frac{FE}{LD}=\frac{OF}{OL}$

Hence, $\frac{FE}{OF}=\frac{LD}{OL}$ Why?

and $\frac{FE}{100}=\frac{42}{72}$

Since $FE=58$ approximately, it follows that $\frac{42}{72}=.58$, approximately.

EXERCISES

1. Using Fig. 112 find the following quotients approximately to two decimal places: $\frac{76}{125}$, $\frac{64}{88}$, $\frac{45}{96}$, $\frac{57}{79}$.

2. Find the quotient $\frac{42}{72}$, using MP as quotient-line.

Since $\triangle OMN \backsim \triangle OLD$,

it follows that $\frac{MN}{LD}=\frac{OM}{OL}$.

$\therefore \frac{MN}{OM}=\frac{LD}{OL}$ Why?

$\frac{MN}{10}=\frac{42}{72}$

Hence, the quotient $\frac{42}{72}$ could be obtained by taking $\frac{1}{10}$ of MN which is .6 approximately.

In a similar way find the quotient $\frac{68}{22}$.

3. In a freshman class of 130 pupils taking mathematics, 21 obtained a grade of A, 29 a grade of B, 35 a grade of C, 27 a grade of D, and 18 failed. What per cent of pupils in the class received a grade of A? of B? of C? of D? What per cent failed?

Suppose x per cent of the pupils receive an A grade.

Then, $$21 = \frac{x}{100} \cdot 130$$

Hence, $$\frac{x}{100} = \frac{21}{130}$$

From Fig. 112, we find $\frac{21}{130} = .16$, approximately.

Therefore, approximately 16 per cent receive an A grade.

212. Construction of similar polygons.

Let $ABCDEF$, Fig. 113, be a given polygon.

To construct a polygon similar to $ABCDEF$.

Construction: Draw diagonals from one vertex, as B, to the other vertices, and extend them.

From any point on BA, as A', draw $A'F' \parallel AF$.

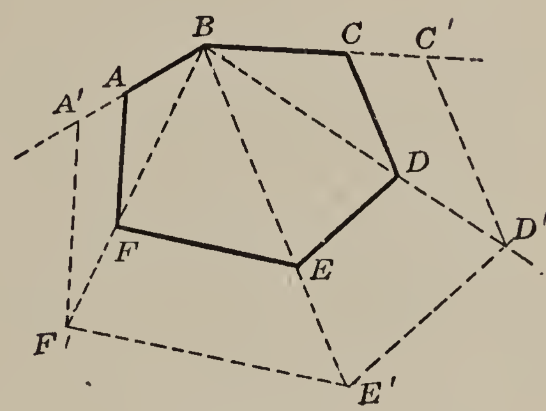


Fig. 113

Draw $F'E' \parallel FE$, $E'D' \parallel ED$ and $D'C' \parallel DC$.

Then $A'BC'D'E'F'$ is the required polygon.

Proof: Prove $\angle D = \angle D'$, $\angle E = \angle E'$, etc.

Show that $$\frac{CD}{C'D'} = \frac{BD}{BD'} = \frac{DE}{D'E'}, \text{ etc.} \quad (\S\ 214)$$

Hence, $$\frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EF}{E'F'}, \text{ etc.} \qquad \text{Why?}$$

213. Homologous parts. Corresponding sides of similar polygons are **homologous** sides.

Corresponding angles, diagonals, altitudes, and medians are **homologous** angles, diagonals, altitudes, and medians.

EXERCISES

1. Show that congruent polygons are similar.

2. Show that polygons similar to the same polygon are similar to each other.

Theorems on Similar Figures

214. Theorem: *A line parallel to one side of a triangle forms with the other two sides a triangle similar to the given triangle.*

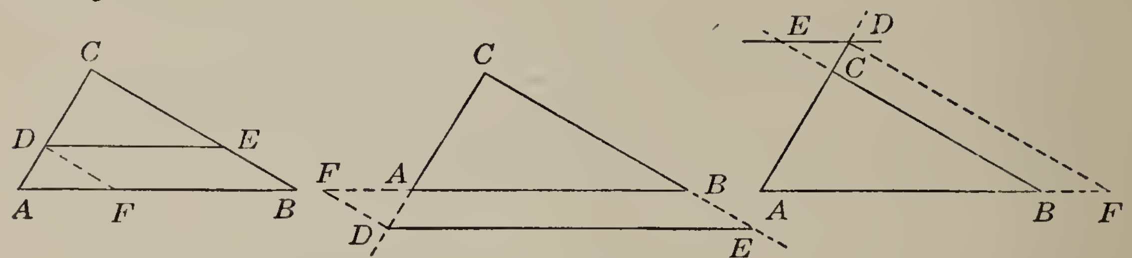


FIG. 114

Given $\triangle ABC$, and $DE \parallel AB$, Fig. 114,

To prove that $\triangle DEC \backsim \triangle ABC$.

Analysis: What conditions must be satisfied to make two triangles similar? More definitely, what must be shown for triangles ABC and DEC?

Proof: Prove that the angles of $\triangle DEC$ are respectively equal to the angles of $\triangle ABC$.

Since $DE \parallel AB$ Why?

$\therefore \dfrac{CD}{CA} = \dfrac{CE}{CB}$ Why?

Draw $DF \parallel BC$.

Then $\dfrac{FB}{AB} = \dfrac{DC}{AC}$. Why?

Quadrilateral $DFBE$ is a parallelogram. Why?

$\therefore DE = FB$ Why?

Substituting for FB its equal, DE,

$$\frac{DE}{AB} = \frac{DC}{AC}$$

This may be written $\dfrac{CD}{CA} = \dfrac{DE}{AB}$

Hence, $\dfrac{CD}{CA} = \dfrac{CE}{CB} = \dfrac{DE}{AB}$ Why?

$\therefore \triangle ABC \backsim \triangle DEC$ Why?

215. Conditions sufficient to make triangles congruent. In geometry we have seen the importance of congruent triangles in proving theorems and solving problems. The definition of *congruent* triangles contains **six** conditions, viz.:

1. The equality of the corresponding angles,

$$\angle A=\angle A',\ \angle B=\angle B',\ \angle C=\angle C'.$$

2. The equality of the corresponding sides,

$$a=a',\ b=b',\ c=c'.$$

However, it was shown that we do not need to establish all of these conditions to prove two triangles congruent and that the following conditions are **sufficient:**

1. Two sides and the angle included between them in one triangle equal respectively to the corresponding parts of the other triangle.

2. Two angles and the side between their vertices equal, respectively.

3. Three sides equal, respectively.

Thus, the problem of proving two triangles congruent is greatly simplified.

216. Conditions sufficient to make two triangles similar.

The definition of *similar* triangles contains **five** conditions, viz.:

1. The equality of the corresponding angles, or—

$$\angle A=\angle A',\ \angle B=\angle B',\ \angle C=\angle C'.$$

2. The proportionality of the corresponding sides, or $\frac{a}{a'}=\frac{b}{b'}$, $\frac{b}{b'}=\frac{c}{c'}$, from which it follows that $\frac{c}{c'}=\frac{a}{a'}$.

As in the case of congruent triangles it is not necessary to show that all five of these conditions are satisfied to make two triangles similar. It will be shown that any one of the following three conditions is **necessary and sufficient:**

1. The equality of two pairs of corresponding angles.

2. The proportionality of two pairs of corresponding sides, and the equality of the included angle.

3. The proportionality of the corresponding sides.

217. Theorem: *Two triangles are similar if two angles of one are respectively equal to two angles of the other.*

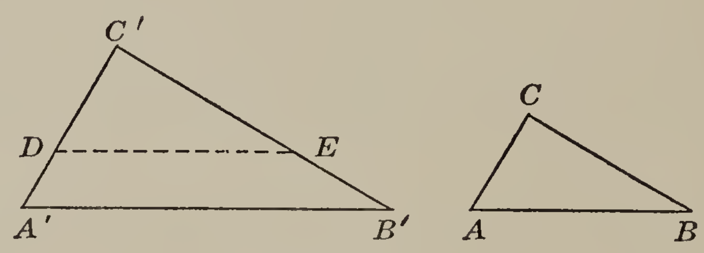


FIG. 115

Given ⚠ ABC and $A'B'C'$, with $A = A'$ and $C = C'$. Fig. 115.

To prove that $\triangle ABC \backsim \triangle A'B'C'$.

Proof:

STATEMENTS		REASONS
On $C'A'$ lay off $C'D = CA$		
Draw	$DE \parallel A'B'$	
Then,	$\triangle DEC' \backsim \triangle A'B'C'$	§ 214
	$\triangle DEC' \cong \triangle ABC$	*a.s.a.*
$\therefore$	$\triangle ABC \backsim \triangle A''B'C'$	Why?

EXERCISE

Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

218. Theorem: *Two triangles are similar, if the ratio of two sides of one equals the ratio of two sides of the other and the angles included between these sides are equal.*

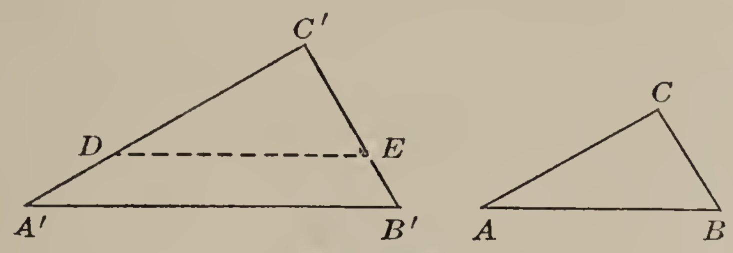


FIG. 116

Given $\triangle ABC$ and $A'B'C'$, with $C = C'$ and $\frac{CA}{C'A'} = \frac{CB}{C'B'}$, Fig. 116.

To prove that $\triangle ABC \backsim \triangle A'B'C'$.

Proof:

STATEMENTS	REASONS
On $C'A'$ lay off $C'D = CA$	By construction
On $C'B'$ lay off $C'E = CB$	By construction
Then, $\frac{C'D}{C'A'} = \frac{C'E}{C'B'}$	Why?
$\therefore\ DE \parallel A'B'$	Why?
$\therefore\ \triangle DEC' \backsim \triangle A'B'C'$	Why?
But $\triangle DEC' \cong \triangle ABC$.	*s.a.s.*
$\therefore\ \triangle ABC \backsim \triangle A'B'C'$	Why?

EXERCISES

1. Two right triangles are similar if the ratio of the sides including the right angle of one, is equal to the ratio of the corresponding sides of the other.

2. Lines drawn joining the midpoints of the sides of a triangle form a triangle which is similar to the first triangle.

3. Two isosceles triangles are similar, if an angle in one is equal to the corresponding angle in the other.

219. Theorem: *Two triangles are similar if the corresponding sides are in proportion.*

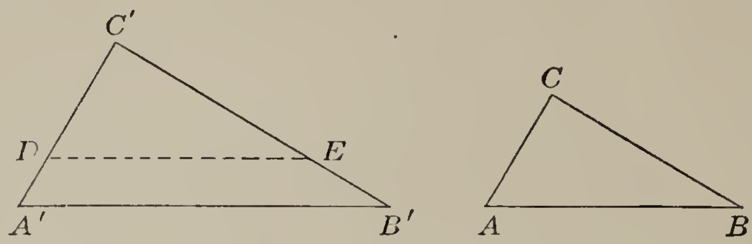


FIG. 117

Given ∆ ABC and $A'B'C'$, with $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CA}{C'A'}$, Fig. 117,

To prove that $\triangle ABC \sim \triangle A'B'C'$.

Proof:

STATEMENTS	REASONS
On $C'A'$ lay off $C'D=CA$	
On $C'B'$ lay off $C'E=CB$	
Then $\frac{C'D}{C'A'}=\frac{C'E}{C'B'}$	Why?
$\therefore \triangle DEC' \sim \triangle A'B'C'$	(§ 218)
$\therefore \frac{C'D}{C'A'}=\frac{DE}{A'B'}$	Why?
But $\frac{C'D}{C'A'}=\frac{AB}{A'B'}$	Why?
$\therefore \frac{DE}{A'B'}=\frac{AB}{A'B'}$	Why?
$\therefore DE=AB$	Why?
$\therefore \triangle DEC' \cong \triangle ABC$	s.s.s.
$\therefore \triangle ABC \sim \triangle A'B'C'$	Why?

PROBLEMS AND EXERCISES

Prove the following exercises:

1. Two triangles are similar if the corresponding sides are parallel, or perpendicular.

For, if the sides of the angles are parallel or perpendicular, each to each, the angles are either equal or supplementary.

Thus, (1) $A=A'$, or (2) $A+A'=2$ right angles
(3) $B=B'$, or (4) $B+B'=2$ right angles
(5) $C=C'$, or (6) $C+C'=2$ right angles

Show that the three equations (2), (4), and (6) cannot all be true at the same time.

Show that two of the equations (2), (4), and (6) cannot both be true at the same time.

Hence, at least two of the equations (1), (3), and (5) must be true and the triangles are mutually equiangular.

Apply § 217.

‡**2.** Two parallelograms are similar if an angle in one is equal to an angle in the other and the including sides are proportional.

‡**3.** Two rectangles are similar if the ratio of two consecutive sides of one is equal to the ratio of the corresponding sides of the other.

4. *The perimeters of similar triangles are to each other as any two homologous sides.*

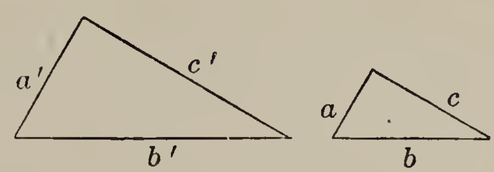


FIG. 118

Since the triangles, Fig. 118, are similar,

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'} \qquad \text{Why?}$$

$$\therefore \quad \frac{a+b+c}{a'+b'+c'}=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'} \qquad \text{Why?}$$

5. *The perimeters of similar polygons are to each other as any two homologous sides.*

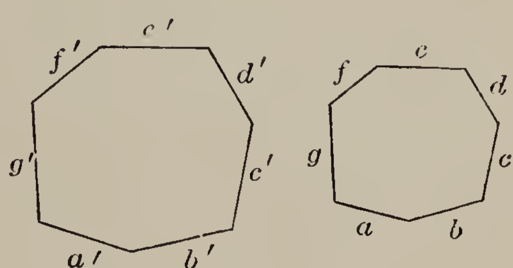


FIG. 119

Since the polygons, Fig. 119, are similar

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}, \text{ etc.} \qquad \text{Why?}$$

$$\therefore \quad \frac{a+b+c+\text{etc.}}{a'+b'+c'+\text{etc.}}=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}, \text{ etc.} \qquad \text{Why?}$$

6. The homologous altitudes of similar triangles are to each other as the homologous sides, and as the perimeters.

Prove

$\triangle ADC \backsim \triangle A'D'C'$, Fig. 120.

Then $\frac{b}{b'}=\frac{h}{h'}$ Why?

But $\frac{b}{b'}=\frac{a}{a'}=\frac{c}{c'}$ Why?

$\therefore \frac{h}{h'}=\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$ Why?

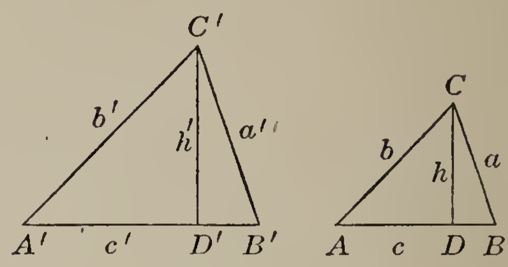


Fig. 120

‡7. The altitudes of a triangle are inversely proportional to the sides to which they are drawn.

Prove $\triangle DBC \backsim \triangle ABE$, Fig. 121.

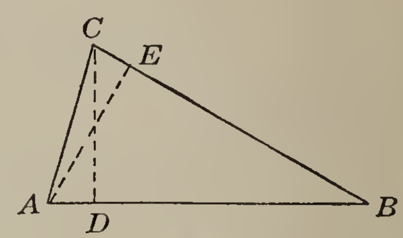


Fig. 121

‡8. The homologous medians of two similar triangles are to each other as any two homologous sides, and as the perimeters.

‡9. The bisectors of homologous angles of similar triangles are to each other as two homologous sides, and as the perimeters.

‡10. The length of the shadow cast by a 4-ft. vertical rod is $5\frac{1}{2}$ feet. At the same time the length of the shadow cast by a spire is 220 feet. How high is the spire?

‡11. A man at a window sees a point on the ground in line with the top of a post and window-sill. He finds that the point is 2 ft. 8 in. from the foot of the post, and that the post is 3 ft. high and $24\frac{1}{2}$ ft. from a point just under the window. How high is the window from the ground?

‡12. A boy wishes to know how far it is from the shore of a lake at A to an island, B, Fig. 122. At C, 20 yd. from A on the line BA, he lays off $CD \perp CB$ and $CD = 60$ rods. At A he constructs a line perpendicular to AB meet-

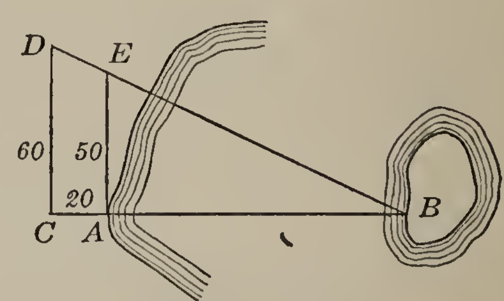


Fig. 122

ing DB at E. By measuring he finds $AE=50$ rods. Find the required distance.

‡**13.** The line joining the midpoint of one of the bases of a trapezoid to the point of intersection of the diagonals bisects the other base.

14. The lengths of the sides of a triangular piece of land are approximately 125 rd., 54 rd., and 112 rods. A drawing is made of it, the longest side of which is 3 feet. What are the lengths of the other sides of the triangle in the drawing?

15. The non-parallel sides of a trapezoid of bases 18 and 60 and of altitude 6 are produced until they meet. What are the altitudes of the triangles on the bases of the trapezoid?

‡**16.** The base of a triangle is 72 in., and the altitude is 12 inches. Find the upper base of the trapezoid cut off by a line parallel to the base and 8 in. from it.

‡**17.** Two sides of a triangle are 14 in. and 3.5 in. and the included angle is 75°. Two sides of another triangle are 20 in. and 5 in. and the included angle is 75°. Show that the triangles are similar.

18. The perimeter of a triangle is 15 cm., and the sides of a similar triangle are 4.5 cm., 6.4 cm., and 7.1 centimeters. Find the lengths of the sides of the first triangle.

19. The perimeters of two similar triangles are x^2+3x+2 and 16, and a pair of homologous sides are respectively $3x$ and 8. Find the value of x.

‡**20.** The perimeters, p and p', of two similar triangles, and

p	p'	a	a'
x^2+1	x	$2\frac{1}{2}$	1
$3K^2+9K$	27	35	4
4	y^2-2y+1	1	4

a pair of homologous sides, a and a', are expressed in the table above. Find the values of x, y, and K.

220. Theorem: *Similar polygons may be divided by homologous diagonals into triangles similar to each other and similarly placed.*

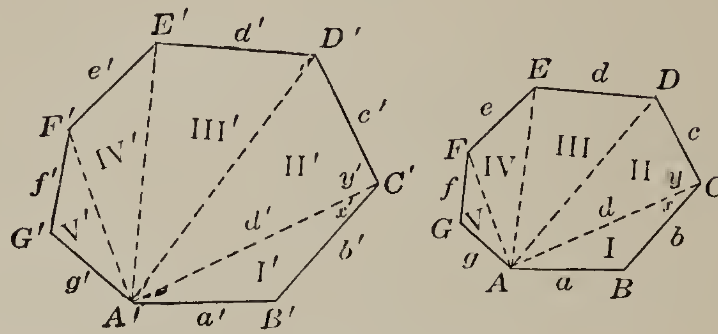


FIG. 123

Given polygon $ABCD$, etc., $\backsim$ polygon $A'B'C'D'$, etc., Fig. 123, with diagonals drawn from A and A'.

To prove $\triangle\text{I} \backsim \triangle\text{I}'$, $\triangle\text{II} \backsim \triangle\text{II}'$, etc.

Proof:

	STATEMENTS	REASONS
	$\frac{a}{a'} = \frac{b}{b'}$	Why?
	$B = B'$	Why?
$\therefore$	$\triangle\text{I} \backsim \triangle\text{I}'$	Why?
$\therefore$	$x = x'$	Why?
$\because$	$C = C'$	Why?
$\therefore$	$y = y'$	Why?
	$\frac{b}{b'} = \frac{d}{d'}$	Why?
	$\frac{b}{b'} = \frac{c}{c'}$	Why?
$\therefore$	$\frac{c}{c'} = \frac{d}{d'}$	Why?
$\therefore$	$\triangle\text{II} \backsim \triangle\text{II}'$, etc.	Why?

Summary

221. The following theorems were proved in this chapter:

1. *A line parallel to one side of a triangle forms with the other two sides a triangle similar to the given triangle.*

2. *Two triangles are similar if two angles of one are respectively equal to two angles of the other.*

3. *Two triangles are similar, if the ratio of two sides of one equals the ratio of two sides of the other and the angles included between these sides are equal.*

4. *Two triangles are similar if the corresponding sides are in proportion.*

5. *The perimeters of similar polygons are to each other as any two homologous sides.*

6. *Similar polygons may be divided by homologous diagonals into triangles similar to each other and similarly placed.*

222. It was shown how to construct a polygon similar to a given polygon.

223. Quotients of arithmetical numbers may be found mechanically by means of squared paper and a string.

CHAPTER VIII

RELATIONS BETWEEN THE SIDES OF TRIANGLES. THEOREM OF PYTHAGORAS AND ITS GENERALIZATIONS. QUADRATIC EQUATIONS. RADICALS.

Similarity in the Right Triangle

224. Theorem: *The perpendicular to the hypotenuse from the vertex of the right angle divides a right triangle into parts similar to each other and to the given triangle.*

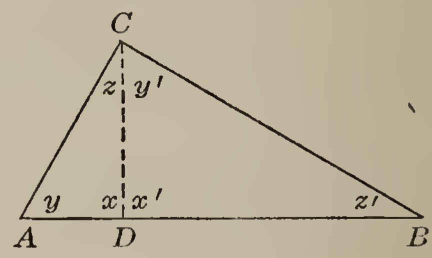


FIG. 124

Given $\triangle ABC$ with the right angle C, and $CD \perp AB$, Fig. 124.

To prove

$$\triangle ADC \backsim \triangle BDC \backsim \triangle ABC.$$

Proof:

$x = x'$ Why?

$y = y'$ Why?

$\therefore \quad \triangle ADC \backsim \triangle BDC$ Why?

Prove that $\triangle$s ADC and ABC are mutually equiangular and therefore similar.

Similarly, prove $\triangle BDC \backsim \triangle ABC$.

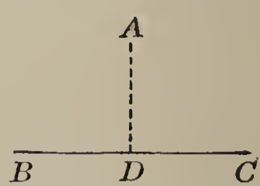


FIG. 125

225. Projection of a point. The **projection of a point** upon a given line is the **foot** of the perpendicular drawn from the point to the line. Thus, point D, Fig. 125, is the projection of point A upon BC.

226. Projection of a segment. To project a **line-segment**, as AB, Fig. 126, upon a line, as CD, drop perpendiculars to CD from the endpoints of the segment AB. Then EF is the projection of AB upon CD.

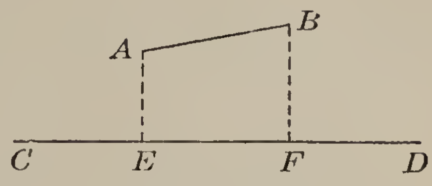


FIG. 126

In general, the **projection of a given segment** upon a line is the segment of the line whose endpoints are the projections of the endpoints of the given segment.

EXERCISES

1. In each of the following figures name the projection of AB upon CD, (Figs. 127–29.)

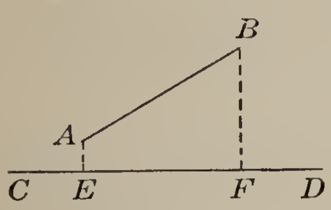


FIG. 127

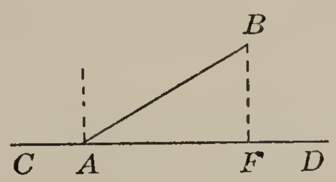


FIG. 128

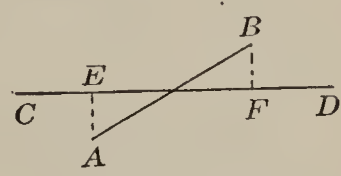


FIG. 129

Draw a figure in which the segment is equal to the projection.

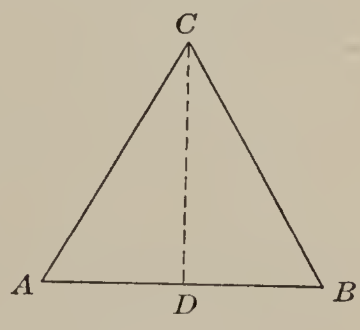


FIG. 130

2. In triangle ABC, Fig. 130, name the projection of AC upon AB; of BC upon AB.

3. In triangle ABC, Fig. 130, project BC upon AC; AB upon BC.

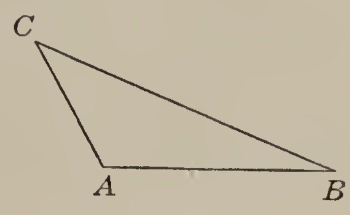


FIG. 131

4. Draw an obtuse triangle, as ABC, Fig. 131. Project AB upon BC; AC upon AB; BC upon AB; AB upon AC.

227. Mean proportional. In the proportion $\frac{a}{b}=\frac{b}{c}$, b is a **mean proportional** between a and c.

EXERCISES

1. Find a mean proportional between 4 and 9.

Denoting the mean proportional by x, we have $\frac{4}{x}=\frac{x}{9}$.

$\therefore\ x^2=4\cdot 9$ Why?

$\therefore\ x=\pm\sqrt{4\cdot 9}$

$\therefore\ x=\pm 2\cdot 3$ Why?

or $x=+6, -6.$ Check both results.

2. In triangle ABC, Fig. 132, find the projection of the median, m, upon AB.

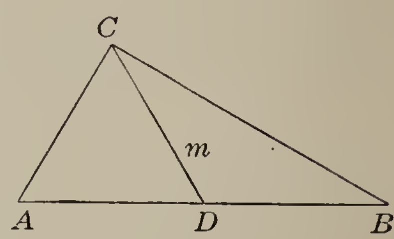


FIG. 132

228. Radical. An indicated root of a number is a **radical**. Thus, $\sqrt{5}$, $\sqrt[3]{x}$, $\sqrt[4]{16}$, $\sqrt{a+b^2}$ are radicals.

229. Simplification of radicals. In computing the value of a radical it is often of advantage to change the form of the number under the radical sign. The following examples illustrate this:

$$\text{I.}\ \begin{cases}\sqrt{25\cdot 16}=5\cdot 4, \text{ for } (5\cdot 4)(5\cdot 4)=25\cdot 16.\\ \sqrt{36\cdot 9}=6\cdot 3, \text{ for } (6\cdot 3)(6\cdot 3)=36\cdot 9.\end{cases}$$

Thus the values of $\sqrt{25\cdot 16}$, $\sqrt{36\cdot 9}$, etc., are found by extracting the square roots of the factors separately and then multiplying the results.

In general, *the square root of a product, as ab, may be found by taking the square roots of the factors, as a and b, and then taking the product of these square roots.* This may be stated briefly in the form of an equation, thus,

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$$

This principle enables us to obtain *by inspection* the square roots of some large numbers, as is shown by the following examples:

$$\text{II.}\begin{cases}\sqrt{3136}=\sqrt{4\cdot 784}=\sqrt{4\cdot 4\cdot 196}=\sqrt{4\cdot 4\cdot 4\cdot 49}\\ \qquad =2\cdot 2\cdot 2\cdot 7=56.\\ \sqrt{4225}=\sqrt{5\cdot 845}=\sqrt{5\cdot 5\cdot 169}=5\cdot 13=65\end{cases}$$

The principle explained above may be applied to advantage even when the number under the radical sign is not a square. For example:

III. $\sqrt{50}=\sqrt{5\cdot 5\cdot 2}=5\sqrt{2}$.

Knowing the square root of 2 to be $1.414+\ldots\ldots\ldots$, it follows that $\sqrt{50}=7.070+\ \ldots\ldots$

Similarly, $\sqrt{8a^3}=\sqrt{4a^2\cdot 2a}=2a\sqrt{2a}$

and $\sqrt{108}=\sqrt{9\cdot 12}=\sqrt{9\cdot 4\cdot 3}=6\sqrt{3}$

EXERCISES

1. Reduce the following radicals to the simplest form:

1. $\sqrt{75}$	5. $\sqrt{128a^2b^2}$	9. $\sqrt{a^2+2ab+b^2}$
2. $\sqrt{27}$	6. $\sqrt{162x^2y^2}$	10. $\sqrt{4a^2-20ab+25b^2}$
3. $\sqrt{a^5b^3}$	7. $\sqrt{243ab^2}$	11. $\sqrt{9a^3-9a^2b}$
4. $\sqrt{20x^2y}$	8. $\sqrt[3]{16}$	12. $\sqrt{(a+b)(a^2-b^2)}$

2. Find the mean proportionals between 2 and 18; 10 and 90; 8 and 200; 20 and 180.

3. Find the mean proportionals between a^2 and b^2; c^2 and d^2.

4. Find the mean proportional between $x^2+2xy+y^2$ and $x^2-2xy+y^2$.

5. Show that the mean proportional between a and b is the square root of the product of a and b.

230. Theorem: *In a right triangle, the perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse.*

That is, we are to prove $\frac{m}{h}=\frac{h}{n}$, Fig. 133.

To prove this proportion, use the principle that in similar triangles the sides opposite equal angles are *homologous* sides and are therefore *proportional.*

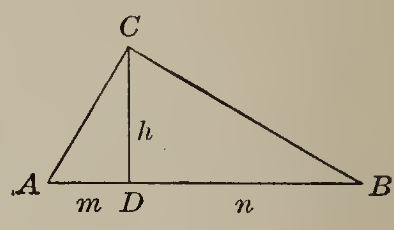


FIG. 133

231. Section 230 affords a way of finding *geometrically* the mean proportional between two segments (see problem 1, below).

Problems of Construction

1. *To construct a mean proportional between two segments.*

Given the segments m and n, Fig. 134,

Required to construct the mean proportional between m and n.

Construction: On a line, as AB, lay off $AC=m$, $CD=n$.

Draw $CE \perp AB$.

Draw the circle AFD on AD as a diameter, meeting CE at F.

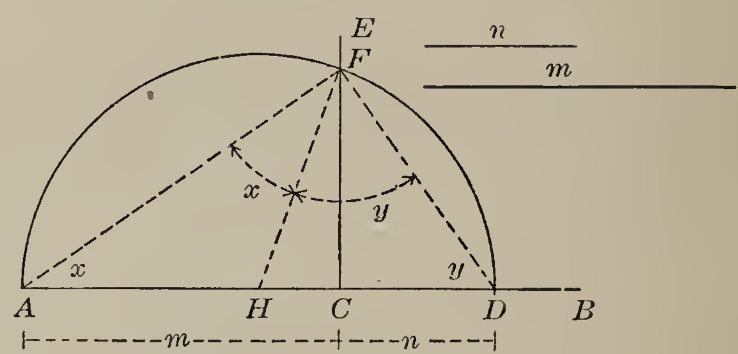


FIG. 134

Then FC is the mean proportional between m and n.

Proof: Draw AF, DF, and the median HF.

Show that $\angle AFH = \angle A = x$

Show that $\angle HFD = \angle D = y$

Then $2x+2y=180°$ Why?

$\therefore \angle AFD = 90°$ Why?

$\therefore \frac{m}{FC}=\frac{FC}{n}$ Why?

2. *Construct a square equal to a given rectangle.*

Let a and b be the dimensions of the given rectangle, Fig. 135.

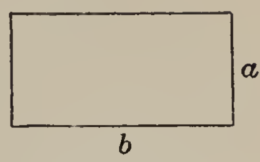


Fig. 135

Construct the mean proportional between a and b.

On the mean proportional between a and b as a side, construct a square.

Prove that the area of this square is equal to the area of the given rectangle.

3. *Construct the square root of a number.*

1. To find the square root of 2, lay off on squared paper two factors of 2, as 2 and 1, Fig. 136, in the same way as m and n, problem 1. (Use the scale $1 = 2$ cm.)

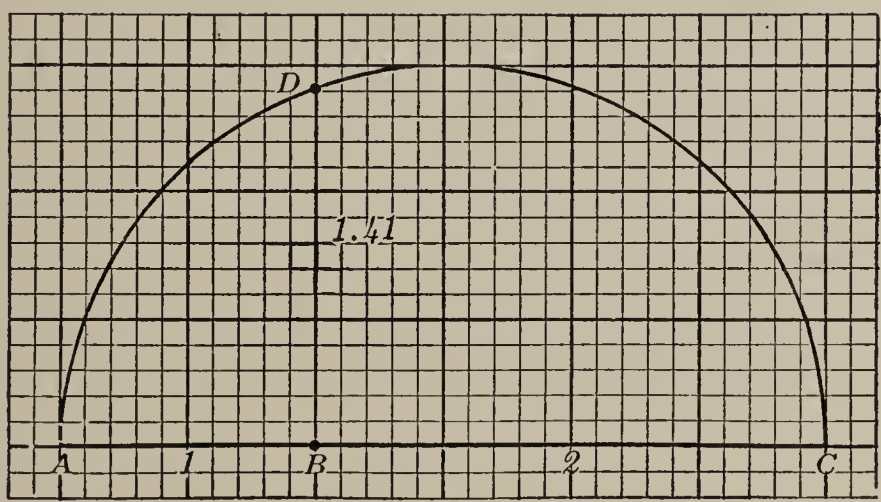


Fig. 136

The mean proportional, BD, between AB and BC, represents graphically the required square root of 2. Why?

Measure BD to two decimal places.

Check by extracting the square root of 2 to two decimal places.

2. Find geometrically the square root of 6; of 5; of 8.

EXERCISE

A perpendicular to a diameter of a circle at any point, extended to the circle, is the mean proportional between the segments of the diameter. Prove.

Relations of the Sides of a Right Triangle

232. Theorem: *In a right triangle either side of the right angle is a mean proportional between its projection upon the hypotenuse and the entire hypotenuse.*

We are to prove $\frac{m}{a}=\frac{a}{c}$ and $\frac{n}{b}=\frac{b}{c}$, Fig. 137.

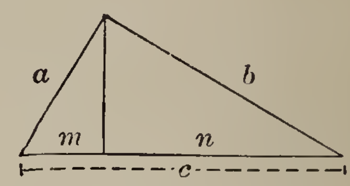


FIG. 137

To prove this, apply the principle that homologous sides of similar triangles are in proportion.

This theorem enables us to obtain a proof for one of the most important theorems of geometry:

233. Theorem of Pythagoras. *The square of the hypotenuse in a right triangle is equal to the sum of the squares of the sides of the right angle.*

Proof:	$\frac{m}{a}=\frac{a}{c}$	Why?
and,	$\frac{n}{b}=\frac{b}{c}$	Why?
	$\therefore\ a^2=mc$	Why?
and	$b^2=nc$	Why?
	$\therefore\ a^2+b^2=(m+n)c$	Why?
or	$a^2+b^2=c^2$	

The last four steps in this proof suggest the following geometric illustrations:

The equation, $a^2=m\cdot c$, means that the square on BC, Fig. 138, is equal to a rectangle of dimensions m and c, as $BEFD$. (Notice that the sides of the rectangle $BEFD$ are m, the projection of BC on AB, and BE which is equal to c, the length of the hypotenuse AB.)

Similarly, $b^2=n\cdot c$ means that the square on AC is equal to a rectangle as $FHAD$, having the dimensions equal to n, the projection of BC on AB, and BH which is equal to the hypotenuse, c.

Hence, the sum of the squares on AC and BC is equal to the sum of these two rectangles, or to the square on the hypotenuse.

This illustration may be used as an outline of Euclid's proof of the theorem of Pythagoras given in § 462.

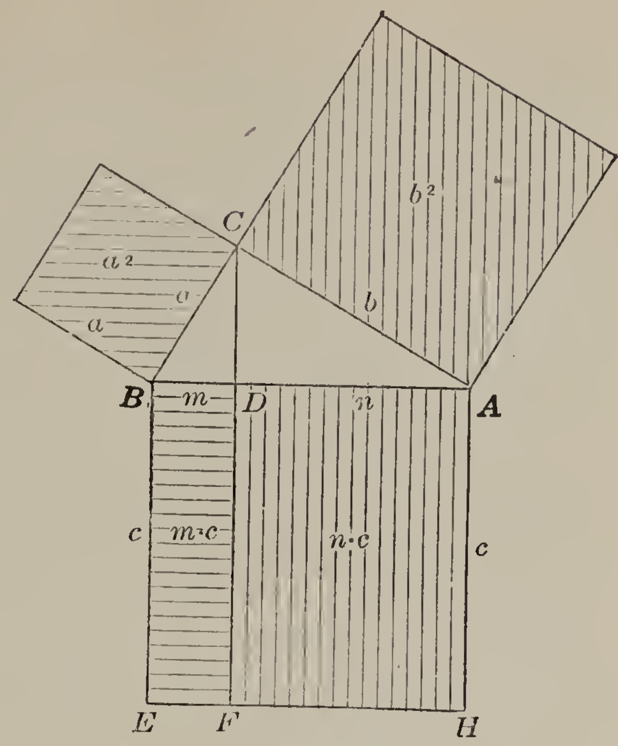
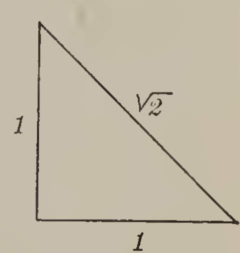


FIG. 138

234. Historical note: It is said that Pythagoras, jubilant over his great accomplishment of having found a proof of the theorem, sacrificed a hecatomb to the muses who inspired him. The invention was well worthy of this sacrifice, for it marks historically the first conception of **irrational numbers.** It is believed that Pythagoras showed the existence of irrational numbers, by showing that the hypotenuse of a certain isosceles right triangle is equal to $\sqrt{2}$ (See Figure.)



His followers found much pleasure in finding special sets of integral values of a, b, c satisfying the equation $a^2+b^2=c^2$, the simplest set being 3, 4, and 5. Such numbers are called *Pythagorean numbers.* The question naturally arose later whether there existed any sets of integral values of a, b, and c that would satisfy the equations $a^3+b^3=c^3$, $a^4+b^4=c^4$, etc., in general, $a^n+b^n=c^n$ for $n>2$.

The great mathematician Fermat, who lived 1601–65, states among his notes the theorem that the equation $x^n+y^n=z^n$ is not satisfied by a set of integral numbers for x, y, z, and n except for $n=2$. He also makes the statement that he has discovered a really wonderful proof for the theorem. Unfortunately, he gives not the least suggestion as to the nature of his proof. The theorem is very simple, but it has been impossible to this day to find a proof, although a price of 100,000 marks ($20,000) has been offered by a German society to the fortunate person who first gives a complete proof of the theorem, or who shows by a single exception that the theorem is not true. (See Ball's *Mathematical Recreations*, 4th ed., 1905, pp. 37–40.)

EXERCISES

1. In triangle ABC, Fig. 139, $\angle ACB$ is a right angle and $CD \perp AB$. $AD=2$, $DB=30$. Find the lengths of AC and CB.

2. The radius of a circle is 12.5, Fig. 140. Find the projection of the chord AC upon the diameter AB passing through one of the end-points of the chord.

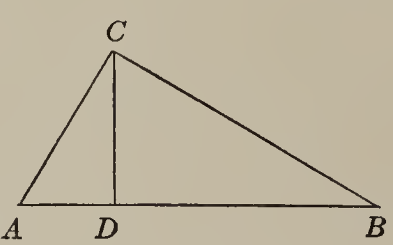


Fig. 139

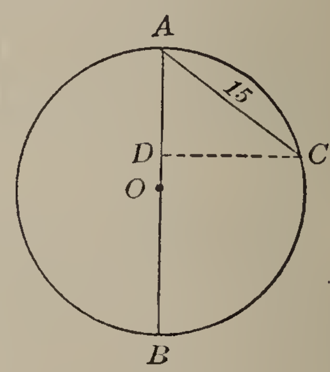


Fig. 140

3. In the right triangle ABC, Fig. 141, find c, m, n, and h, if $a=12$ and $b=5$.

Find b, m, n, and h, if $a=8$ and $c=10$.

Find a, b, c, and h if $m=9\frac{3}{5}$ and $n=5\frac{2}{5}$.

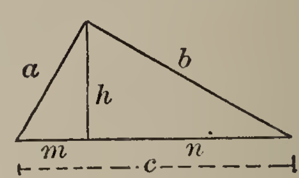


Fig. 141

4. Compute the dimensions of the section of the strongest beam that can be cut from a cylindrical log.

Let the circle, Fig. 142, represent a cross-section of the log. Then the dimensions of the strongest beam are computed as follows:

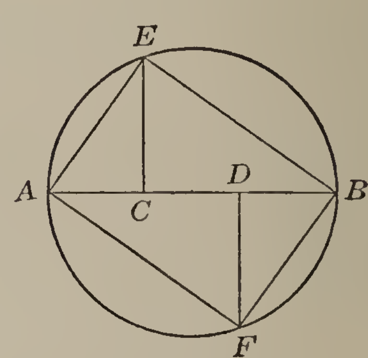


Fig. 142

Trisect the diameter AB at C and D (§ 176, exercise 7).


FERMAT,
né en 159.. mort en 1665

PIERRE DE FERMAT

PIERRE DE FERMAT was born near Toulouse in 1601 and died at Castres in 1665. The great mathematical historian Cantor and others have called Fermat "the greatest French mathematician of the seventeenth century," and this was a century of great French mathematicians. He was the son of a leather merchant and was educated at home. He studied law at Toulouse and in 1631 became a councilor of the Parliament of Toulouse. He is said to have performed the duties of his office with scrupulous accuracy and fidelity. He loved mathematical study, made it his avocation, spending most of his leisure on it. His disposition was modest and retiring. He published very little—only one paper—during his lifetime. Though his vocation was that of a lawyer and parliamentarian, his celebrity rests upon what he accomplished in his avocation.

Notwithstanding the fact that Fermat published very little, he exerted a great influence on the mathematicians of his age through a continual correspondence which he carried on with them. The mathematical discoveries upon which his fame rests were made known to the world through his correspondence or through the notes on his results that were found after his death, written on loose sheets of paper, or scribbled on the margins of books he had annotated while reading. A part of these notes and Fermat's marginal notes, found in his copy of Diophantus' *Arithmetic*, were published after his death by his son, Samuel. As Fermat's notes do not seem ever to have been intended for publication, it is often difficult to estimate when his discoveries were made, or whether they were really original.

Most of his proofs are lost, and probably some of them were not rigorous. He seems to have worked carelessly, or at least unsystematically, for one of his marginal notes on an important theorem that still awaits proof, notwithstanding the facts that several of the world's greatest mathematicians have tried their wits upon it and that the Paris Academy of Sciences has on two occasions at least, in 1850 and 1853, offered to the world a prize of 3,000 francs for a complete proof of it, is the remark: "I have found for this a truly wonderful proof, but the margin is too small to hold it."

The theorem referred to in the foregoing remarks is called the "greater Fermat theorem," or the "last Fermat theorem" (see § 234 of this book). Several of Fermat's theorems have been proved by later mathematicians, but they have required good mathematical ability. Some are still awaiting mathematical genius.

For fuller information about Fermat see Ball's *History*, pp. 293–301; Cajori's *History*, pp. 173 and 179–82; or *Historical Introduction to Mathematical Literature*, by G. A. Miller, published by Macmillan.

Erect $CE \perp AB$ and $DF \perp AB$.

Draw the quadrilateral $AFBE$.

This is the required section of the strongest beam.

If the diameter of the log is 15 in., compute AE and AF.

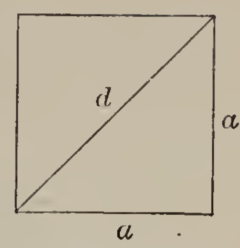


FIG. 143

5. Prove that the diagonal of a square is equal to the product of the side by the square root of 2, Fig. 143.

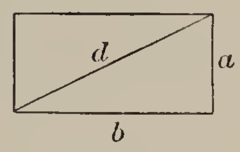


FIG. 144

6. Prove that the diagonal of a rectangle is equal to the square root of the sum of the squares of two consecutive sides, Fig. 144.

7. Express the altitude of an equilateral triangle in terms of the side, Fig. 145.

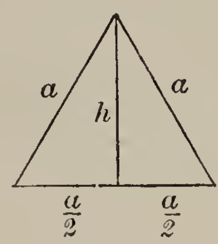


FIG. 145

8. Fig. 146 represents a circular window. The radius of the largest circle is 6. Find the radius, x, of the smallest window.

The sides of the right triangle ABC are 3, $x+3$ and $6-x$ respectively. Why?

$$\therefore \quad (x+3)^2=(6-x)^2+9$$
$$x^2+6x+9=36-12x+x^2+9$$
$$\therefore \quad x=2$$

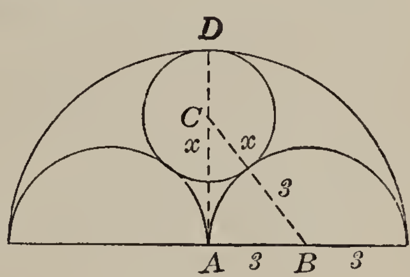


FIG. 146

Quadratic Equations*

235. Summary of methods of solving quadratic equations. In the preceding course quadratic equations have been solved by the following *three* methods:

(1) By graph.

(2) By factoring.

(3) By completing the square.

* See historical note, § 238.

The graphical method exhibits to the eye the solutions of the equation and enables one to determine the solutions approximately.

The method of factoring is brief, but fails when we are unable to factor the trinomial.

The method by completing the square always gives the exact results. The objection to it is the length of the process.

For this reason another method will be developed which is not only brief, but which can be applied to *any* quadratic equation.

All quadratic equations in one unknown may be arranged in the **normal form**

$$\boldsymbol{ax^2+bx+c=0,}$$

where a stands for the coefficient of x^2 when all terms in x^2 have been combined into one; b denotes the coefficient of x, and c is the constant, i.e., the term or the sum of terms not containing x.

Thus, in $5x^2+3x-4=0$, $a=5$, $b=3$, $c=-4$.

EXERCISES

Arrange each of the following equations in the normal form, $ax^2+bx+c=0$, and determine the values of the coefficients a, b, and c:

1. $x^2+4x-5=0$

2. $y^2-2y=11$

3. $c^2=4c+1$

4. $a^2=7a-7$

Change the following equations to the normal form:

5. $ax^2+bx=b+ax$

6. $2y^2+4ay+2ab=-by$

7. $2z^2+ab=2az+bz$

8. $s^2+a^2=2as-2$

236. Solution of the equation $ax^2+bx+c=0$. Since every quadratic equation may be changed to the normal form $ax^2+bx+c=0$, we may obtain a solution of *every* quadratic equation by solving $ax^2+bx+c=0$. Thus, we shall derive a formula, by means of which the solution of *any* quadratic equation may readily be found.

Give reasons for every step in the following solution:

$$ax^2+bx+c=0$$
$$ax^2+bx=-c$$
$$x^2+\frac{b}{a}x=-\frac{c}{a}$$

Completing the square on the left side by adding $\frac{b^2}{4a^2}$, to both sides of the equation we have—

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{c}{a}$$

or $$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$$

or $$x^2+\frac{bx}{a}+\frac{b^2}{4a^2}=\frac{b^2-4ac}{4a^2}$$

Whence, $$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

and $$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

or $$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$$

Whence, $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

237. General quadratic formula. The values of x in the equation—

$$ax^2+bx+c=0$$

have been found to be

$$x_1=\frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$x_2=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

These are the **general quadratic formulas.** They may be combined into a single formula thus,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

EXERCISES

By means of the quadratic formula solve the following equations. In these equations consider a, b, and c as *knowns* and all other letters as *unknowns:*

1. $3x^2+5x-2=0$

Here $a=3$, $b=5$, $c=-2$.
Substituting these values in the formula,

$$x=\frac{-5\pm\sqrt{25+24}}{6}=\frac{-5\pm7}{6}=\frac{1}{3}, \text{ or } -2.$$

2. $2x^2+5x+2=0$
3. $6x^2-11x+5=0$
4. $2r^2-r-6=0$
5. $2x^2+x=15$
6. $1.4x^2+5x=2.4$
7. $1\frac{1}{5}x^2+x-11.2=0$
8. $.6x^2-1.4x=3.2$
9. $x^2+\frac{3}{2}x=1$

‡10. $r^2-9r-36=0$

11. $t^2+15t=-44$

‡12. $x^2-72=6x$

‡13. $3m^2=6-7m$

14. $6+11x=-18x^2-20$
15. $14y^2+2y=28y-10y^2+5$

‡16. $11R^2-10R=24-10R^2$

17. $6p^2-13p=10p-21$

‡18. $8l^2-12l+3=0$

19. $s^2-2as+a^2+2=0$

20. $t^2-3abt+2a^2b^2=0$

21. $a-y^2=(1-a)y$

22. $cy^2+ly+r=0$

‡23. $y^2+my+n=0$

24. $8y^2+8cy+2c^2=-19c^2-6y^2$

‡25. $28b^2=-17by+3y^2$

26. $12m^2-16am-3a^2=0$

27. $ax^2+(b-a)x-b=0$

28. $2y^2+(4a+b)y=-2ab$

‡29. $2z^2-(2a+b)z+ab=0$

Solve the following problems:

30. The diagonal of a rectangle, Fig. 147, is 17 inches. One of the sides is 7 in. longer than the other. Find the length of each side.

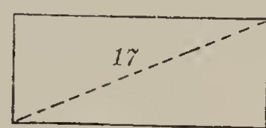


FIG. 147

31. The diagonal of a rectangle is 8 units longer than one side and 9 units longer than the other. How long is the diagonal?

32. A ladder 33 ft. long leans against a house. The foot of the ladder is 14 ft. from the house. How far from the ground is the point of the house touched by the top of the ladder?

33. The diagonal of a rectangle, Fig. 148, is 26. The distance from the vertex to the diagonal is 12. Find the segments into which the perpendicular divides the diagonal.

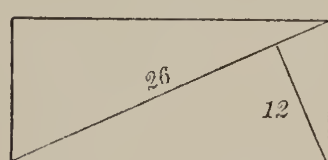


FIG. 148

34. The height, y, to which a ball thrown vertically upward, with a velocity of 100 ft. per second, rises in x seconds is given by the formula $y=100x-16x^2$. In how many seconds will the ball rise to a height of 144 feet?

Make a graph of the function $100x-16x^2$ and by means of this graph interpret the meaning of the solutions of the equation.

238. Historical note: To solve a pure quadratic equation, such as $x^2=25$, is merely to extract a square root. A way of extracting square roots of numbers has been known since the dawn of history. Early mathematical students did what amounted to solving a pure quadratic long before they even thought about quadratic equations.

But no one could have written the tenth book of *Euclid's Elements* (300 B.C.) without a good knowledge of ways of solving quadratic equations. Since this tenth book contains most of Euclid's original work, it may safely be assumed that Euclid had this knowledge. He solved no quadratics algebraically, but he proved geometrical theorems that amounted to such solutions. Euclid was a Greek and Greek geometers did not like calculatory processes like solving quadratics, because they did not think practical numerical calculating scientific work. Plato (429–348 B.C.) had said calculating is a childish art beneath the dignity of a philosopher.

The great skill of Archimedes (287–212 B.C.) in difficult calculations, makes men think that he also must have known how to solve quadratics algebraically, but his writings contain nothing about it.

Heron of Alexandria (first century B.C.) was a scientific engineer and surveyor and he solved correctly numerous quadratic equations. In his *Geometria* he solves a problem leading to a quadratic, which in modern symbolism, is—

$$\frac{11}{14}d^2+\frac{29}{7}d=S$$

in which S is a given number and d is the diameter of a circle. He gives correctly a rule which in modern form is—

$$d=\frac{\sqrt{154S+841}-29}{11}$$

Thus by Heron's time the algebraic rule had become entirely dissociated from geometry, and was known and studied for itself, without any connection with geometrical theorems of area or of lines. It had taken centuries, however, to bring about this separation from geometry.

The next important appearance of the solution of the quadratic equation is in the *Arithmetic* of Diophantus (third and fourth centuries A.D.). He distinguishes three normal forms, viz.—

1. $ax^2+bx=c$ 2. $ax^2=bx+c$ 3. $ax^2+c=bx$

As the Greeks knew no negative numbers, the three forms had to be kept separate for treatment, and of course they could not handle the form—

$$x^2+px+q=0,$$

for it requires a knowledge of both negative and complex numbers, which neither antiquity nor later times until the seventeenth century B.C. was able to comprehend.

The union of the three normal forms into one was first accomplished by the Hindus. The rule of Brahmagupta (b. 598 A.D.), which was assumed as known by his predecessor Aryabhatta (b. 476 A.D.), expressed in modern form, was—

$$ax^2+bx=c, \text{ whence } x=\frac{\sqrt{ac+\left(\frac{b}{2}\right)^2}-\frac{b}{2}}{a}$$

the agreement of which with Diophantus' first form perhaps suggests a Greek origin of Hindu algebraic knowledge.

A later Hindu scholar, Cridhara, introduced a slight improvement by changing the form to the following:

$$x=\frac{\sqrt{4ac+b^2}-b}{2a}$$

The eastern Arab Alkarchi (about 1010 A.D.), who was the greatest Arabian algebraist, introduced the higher degree equations of quadratic form—

$$ax^{2n}+bx^n=c;\ ax^{2n}=bx^n+c;\ \text{and } ax^{2n}+c=bx^n,$$

and solved them by reducing them to the three principal cases.

Mediaeval European mathematicians before Cardan (1501–76), still unable to construe the significance of negative number, continued to split up the solution of quadratics into numerous special cases, often including as many as 24 special cases each with its special rule of reckoning. Finally, Cardan succeeded in gaining the correct insight into negative number, and the Italian school of thinkers attacked the imaginary. Through the work of this school it became possible to supply the lacking form—

$$x^2+px+q=0, \text{ for the cases of } p>0 \text{ and } q>0.$$

The Generalization of the Theorem of Pythagoras

239. In the right triangle ABC, Fig. 149, imagine the angle ABC to decrease, leaving the lengths of the sides

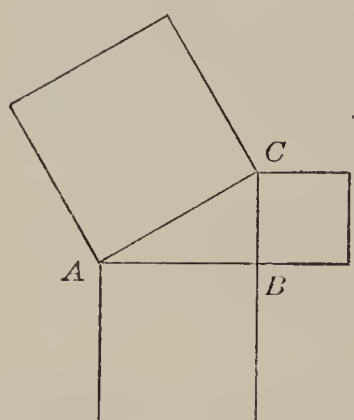


Fig. 149

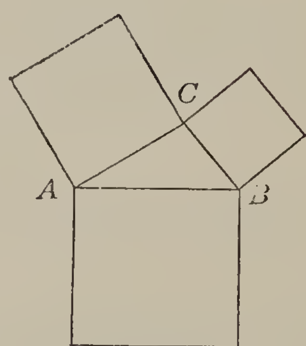


Fig. 150

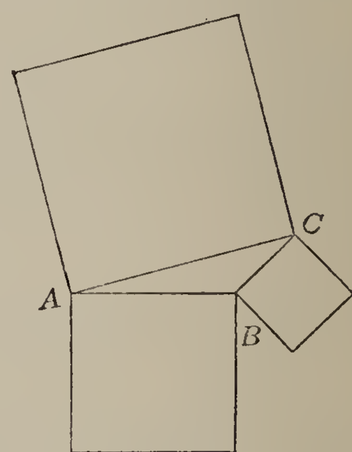


Fig. 151

AB and BC unchanged. Then the squares on AB and BC are not changed in size, but as the distance between the endpoints A and C, of AB and BC, decreases, the square on AC decreases, Fig. 150. Therefore in a triangle the square on the side opposite an *acute* angle is *less* than the sum of the squares on the other two sides.

In a similar way, by increasing the right angle ABC, Fig. 149, as in Fig. 151, we find that the square on the side opposite the *obtuse* angle B is *greater* than the sum of the squares on the other two sides.

The following two theorems will show by *how much* the square on one side of a triangle differs from the sum of the squares on the other two sides.

240. The square on the side opposite an acute angle.

Let $\angle B$ be an acute angle of triangle ABC, Fig. 152.

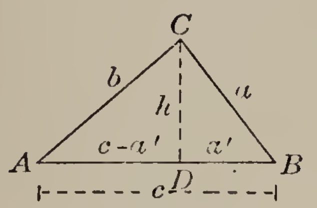


FIG. 152

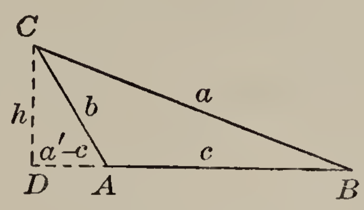


FIG. 153

Draw CD perpendicular to AB. Denote the projection of a on c by a'.

Then $\qquad b^2 = h^2 + (c - a')^2.\qquad$ Why?

And $\qquad a^2 = h^2 + a'^2.$

Subtracting, $b^2 - a^2 = (c - a')^2 - a'^2 = c^2 - 2ca' + a'^2 - a'^2.$

Therefore $\qquad b^2 - a^2 = c^2 - 2ca'.$

Solving for b^2, $b^2 = a^2 + c^2 - 2ca'.$

This shows that the product $2ca'$ is the amount by which $a^2 + c^2$ exceeds b^2.

Hence, we have proved the following theorem:

Theorem: *In a triangle the square on the side opposite an acute angle is equal to the sum of the squares of the other two sides, diminished by two times the product of one of these two sides and the projection of the other upon it.*

EXERCISES

1. Find a', Fig. 152, when a, b, and c are respectively 2, 4, 5; 7, 10, 8.

2. Prove the theorem in § 240, using Fig. 153.

241. The square on the side opposite an obtuse angle.

Theorem: *In a triangle the square on the side opposite an obtuse angle is equal to the sum of the squares on the other two sides, increased by two times the product of one of them and the projection of the other upon it.*

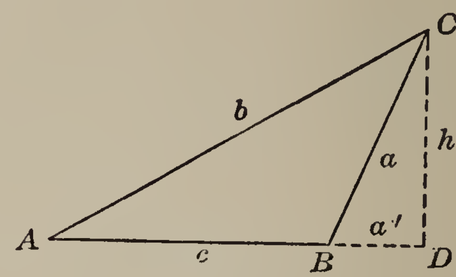


FIG. 154

Given $\triangle ABC$ with $\angle ABC$ obtuse, Fig. 154.

To prove $b^2 = a^2 + c^2 + 2ca'$

Proof: $b^2 = h^2 + (c + a')^2$ Why?

$a^2 = h^2 + a'^2$ Why?

Therefore $b^2 - a^2 = c^2 + 2ca' + a'^2 - a'^2$ Why?

Hence, $b^2 = a^2 + c^2 + 2ca'$.

EXERCISE

The side opposite an obtuse angle is b, and c' is the projection of c upon a, Fig. 155.

Find a' and c' when a, b, and c are respectively

1. 5, 15, 12
2. 6, 12, 8

‡3. 7, 11, 8

4. s^2-1, s^2+2, $2s$

and in each case compare $2a'c$ with $2ac'$

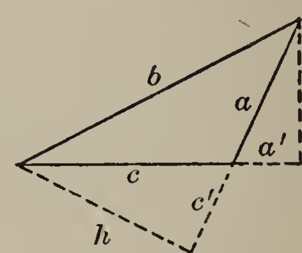


FIG. 155

Summary

242. The chapter has taught the meaning of the following terms:

projection of a point	quadratic formula
projection of a segment	radical
mean proportional	reduction of radical to simplest form

243. The following theorems were proved:

I. Theorems expressing relations between the **sides of a triangle**:

1. *In a right triangle the square of the hypotenuse is equal to the sum of the squares of the sides of the right angle.*

2. *In a triangle the square on the side opposite an* **acute** *angle is equal to the sum of the squares of the other two sides* **diminished** *by two times the product of one of these two sides and the projection of the other upon it.*

3. *In a triangle the square on the side opposite the* **obtuse** *angle is equal to the sum of the squares on the other two sides,* **increased** *by two times the product of one of them and the projection of the other upon it.*

II. Theorems on **mean proportionals**:

1. *In a right triangle the perpendicular from the vertex of the right angle to the hypotenuse is the mean proportional between the segments of the hypotenuse.*

2. *In a right triangle either side of the right angle is the mean proportional between its projection upon the hypotenuse and the entire hypotenuse.*

3. *A perpendicular to a diameter of a circle at any point, extended to the circle, is the mean proportional between the segments of the diameter.*

III. Similarity in the right triangle:

The perpendicular to the hypotenuse from the vertex of the right angle divides a right triangle into parts similar to each other and to the given triangle.

244. The following constructions were taught:

1. To construct a mean proportional between two segments.

2. To construct a square equal to a given rectangle.

3. To construct the square root of a number.

245. Quadratic equations may be solved **by graph, by factoring, by completing the square,** and **by the formula:**

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

where a, b, c are the coefficients in the equation—

$$ax^2+bx+c=0.$$

246. The following principle is useful in reducing radicals to the simplest form:

The square root of a product may be found by taking the square root of the factors and then taking the product of these square roots.

In symbols the principle may be stated thus:

$$\sqrt{ab}=\sqrt{a}\,\sqrt{b}$$

CHAPTER IX

TRIGONOMETRIC RATIOS. RADICALS. QUADRATIC EQUATIONS IN TWO UNKNOWNS

Trigonometric Ratios

247. Finding angles and distances. The theorem of Pythagoras, the fact that two right triangles are similar if an acute angle of one equals an acute angle of the other, and the principle that the acute angles of a right triangle are complementary, enable us to work out a method for finding unknown angles and distances.

These principles are the basis of **trigonometry,** a subject which is useful not only in the study of more advanced mathematics, but also in all the exact sciences.

EXERCISES

1. Show that all right triangles having an acute angle of one equal to an acute angle of the other, are similar.

2. On squared paper draw a right triangle having an angle of 30°. Measure the sides to two decimal places and find the ratio of the side *opposite* the angle of 30° to the *hypotenuse.*

3. Prove that this ratio is the same for all right triangles, having an angle of 30°.

4. In the triangle of exercise 2, find approximately to two decimal places the ratio of the side *opposite* the angle 60° to the *hypotenuse.* Compare your result with the results obtained by other members of the class.

5. Prove that this ratio is constant for all right triangles that have an angle of 60°.

6. In a right triangle having an angle of 45°, find the ratio to two decimal places of the side *opposite* the angle 45° to the *hypotenuse.*

7. Prove that this ratio is constant for all right triangles having an angle of 45°.

8. Draw with a protractor an angle of 40°, Fig. 156. From points on either side of the angle as A_1, A_2, A_3, draw perpendiculars to the other side. Measure A_1C_1, and A_1O and find their ratio.

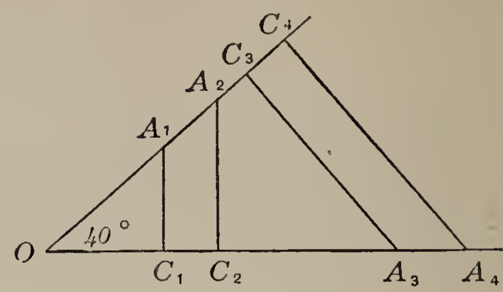


Fig. 156

9. Prove that the ratio of the side opposite the angle 40° to the hypotenuse is the same for all triangles of Fig. 156.

Exercise 9 illustrates the fact that the *ratio* of the sides, Fig. 156, remains *constant* as the *lengths* of the sides *vary*.

The constant *ratio of the opposite side to the hypotenuse*, as in Fig. 156, is called *the sine of angle 40°*.

248. Trigonometric ratios of an angle. Let angle A, Fig. 157, be a given angle. From any point, as B, on either side of the angle draw a perpendicular to the other side. Thus, a right triangle is formed, as ABC.

In this triangle, the ratio of the *side opposite* the vertex of $\angle A$ to the hypotenuse is the *sine of angle A** (written: sin A),

$$\text{i.e., } \sin A = \frac{a}{c}.$$

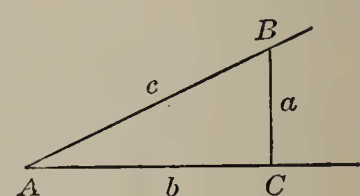


Fig. 157

* The word "sine" is a shortened form of the latin *sinus*, which is the translation of an Arabic word meaning a "bay," or "gulf." Albert Girard (1595–1632), a Dutch mathematician, was the first to use the abbreviations "sin," and "tan" for "sine" and "tangent" (Ball, p. 235). Ball (p. 243) says the term "tangent" was introduced by Thomas Finck (1561–1646) in his *Geometriae Rotundi* of 1583. The same historian says (p. 243) the term "cosine" was first employed by E. Gunter in 1620 in his *Canon on Triangles*, and that the abbreviation "cos" for "cosine" was introduced by Oughtred in 1657. These contractions, "sin," "cos," and "tan," did not however come into general use until the great Euler reintroduced them in 1748. The word "cosine" is an abbreviation for "complementary sine."

to 90° are tabulated in the table on p. 140. Compare your results for exercises 1 and 2 with the corresponding values given in the table.

250. Values of the trigonometric ratios found by means of the table. The table on p. 140 gives approximately to 4 places the values of the ratios for angles containing an integral number of degrees from 1° to 90°. This is quite sufficient for our purposes.

Where greater accuracy is required, tables are available which give the values of the trigonometric ratios of angles containing fractions of degrees.

EXERCISE

From the table find the values of the following ratios:

sin 2°	cos 11°	tan 20°
sin 42°	cos 63°	tan 85°

State your results in the form of equations.

251. Trigonometric functions. Examine the table, p. 140, and notice how the values of the trigonometric ratios change as the angle changes from 1° to 90°. Since a change in the angle produces a *corresponding* change in the ratio, the trigonometric ratios are also called **trigonometric functions.**

From the table, obtain the changes of sin A as A increases from 0° to 90°.

Similarly, obtain the changes of cos A.

Having given the *value* of a single *function* of an angle the values of the other functions and the number of degrees in the angle may be determined in various ways. If a table of trigonometric functions is available, they may be looked up in the table. An algebraic method is given in § 262. The following is a graphical method.

252. Graphical method of finding the values of the functions of an angle when one of them is known.

EXERCISES

1. Given the sine of an angle equal to $\frac{1}{2}$, find the values of the other functions and the number of degrees in the angle.

Draw a right angle, A, Fig. 159.

On one side of the angle lay off $AB=1$.

With B as center and radius equal to 2 draw a circle arc meeting AC at C.

Measure AC and find the values of the cosine and tangent of angle C.

With a protractor find the number of degrees in $\angle C$.

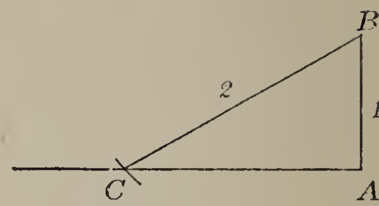


Fig. 159

2. Find the number of degrees in an angle whose sine is $\frac{7}{8}$; .2; .75. Also find the values of the other functions.

3. Find the angle and the values of the other two functions if $\cos B=0.6$; if $\tan A=\frac{4}{3}$.

Exact Values of the Functions of 30°, 45°, and 60°.

253. Values of the functions of 30° and 60°. Since angles of 30°, 45°, and 60° are used in a large number of problems, the student should *remember* the exact values of the functions of these angles, as found in the following exercises:

EXERCISES

1. To construct a right triangle containing an angle of 30°, draw an equilateral triangle, Fig. 160, and divide it into two congruent triangles by drawing the altitude to one side.

Show that the acute angles of triangle ADC are 60° and 30°.

Show that the hypotenuse is twice as long as the side opposite the 30°— angle.

Hence, if AD be denoted by x, AC must be $2x$.

Show that $CD=x\sqrt{3}$.

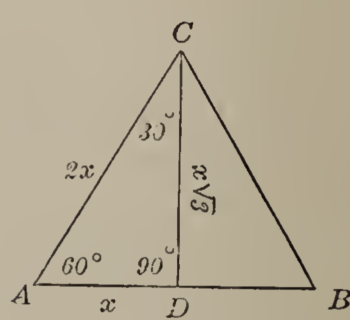


Fig. 160

2. Find the value of sin 30°, using Fig. 160.

$$\sin 30° = \frac{x}{2x} = \frac{1}{2} \qquad \text{Why?}$$

3. Find the value of sin 60°.

$$\sin 60° = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2} = \tfrac{1}{2}\sqrt{3}$$

4. Find the value of cos 30°.

5. Find the value of cos 60°.

6. Find the value of tan 30°.

$$\tan 30° = \frac{x}{x\sqrt{3}} = \frac{1\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}} = \frac{\sqrt{3}}{3}$$

7. Find the value of tan 60°.

254. Rationalizing the denominator. In exercise 6 the fraction $\frac{1}{\sqrt{3}}$ was changed to $\frac{1}{3}\sqrt{3}$ by multiplying numerator and denominator by $\sqrt{3}$. This does not change the **value** of the fraction but changes the denominator to a *rational number.* This process is called *rationalizing the denominator.* The object of the rationalizing process is to obtain a form of the fraction more easily calculated arithmetically.

EXERCISES

Rationalize the denominators in the following fractions:

1. $\frac{1}{\sqrt{2}}$

2. $\frac{\sqrt{5}}{\sqrt{2}}$

3. $\frac{6}{\sqrt{a}}$

4. $\frac{12}{7\sqrt{3}}$

5. $\frac{\sqrt{6}-\sqrt{3}}{2\sqrt{3}}$

6. $\frac{\sqrt{10}-\sqrt{2}}{2\sqrt{3}}$

7. $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}$ 8. $\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{z}}$ 9. $\dfrac{3}{2-\sqrt{3}}$

To rationalize the denominator in $\dfrac{3}{2-\sqrt{3}}$ multiply the numerator and the denominator by $2+\sqrt{3}$. Thus,

$$\frac{3}{2-\sqrt{3}}=\frac{3(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}=\frac{6+3\sqrt{3}}{4-3}=6+3\sqrt{3}.$$

10. $\dfrac{5}{2+\sqrt{5}}$ 12. $\dfrac{4}{\sqrt{2}-1}$ 14. $\dfrac{3-\sqrt{2}}{5+\sqrt{2}}$

11. $\dfrac{6}{3-\sqrt{5}}$ 13. $\dfrac{7}{3+2\sqrt{5}}$ 15. $\dfrac{1+\sqrt{3}}{2-\sqrt{3}}$

In the following rationalize the denominator and then find the approximate values of the fractions to two decimal places:

16. $\dfrac{8+\sqrt{6}}{8-\sqrt{6}}$ 18. $\dfrac{5+\sqrt{3}}{3-\sqrt{2}}$

17. $\dfrac{4+3\sqrt{5}}{4-3\sqrt{5}}$ 19. $\dfrac{1}{3-\sqrt{2}}+\dfrac{1}{2+\sqrt{3}}$

‡20. Find the value of x satisfying the equation

$$5x=\sqrt{3}(1+2x)$$

and express it as a fraction with a rational denominator.

255. Exact values of the functions of 45°. To construct an angle of 45°, draw an **isosceles right** triangle, Fig. 161.

EXERCISES

1. In the isosceles right triangle ABC, Fig. 161, show that $A=C=45°$.

2. Denoting the equal sides of triangle ABC, Fig. 161, by x, show that $AC=x\sqrt{2}$.

3. Find the values of the functions of 45°, giving all results with rational denominators.

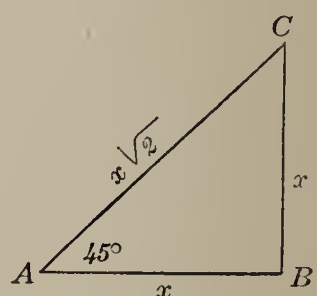


FIG. 161

256. Summary of the exact values of the functions of 30°, 45°, and 60°. The following is a simple device for memorizing these values. For the sake of symmetry, let $\frac{1}{2}$ be written in the form $\frac{1}{2}\sqrt{1}$, then

$$\sin 30° = \tfrac{1}{2}\sqrt{1},\ \sin 45° = \tfrac{1}{2}\sqrt{2},\ \sin 60° = \tfrac{1}{2}\sqrt{3}.$$

The values of the cosine are the same as above, but in reverse order, thus:

$$\cos 60° = \tfrac{1}{2}\sqrt{1},\ \cos 45° = \tfrac{1}{2}\sqrt{2},\ \cos 30° = \tfrac{1}{2}\sqrt{3}$$

This may be conveniently arranged in the form of a table:

Angle / Function	30°	45°	60°
Sine.	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
Cosine.	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$

It will be seen in § 262 that it is not necessary to memorize the values of the tangent-function because they are easily computed from a simple relation existing between the trigonometric functions. However, before making a study of these relations, we shall take up some of the **practical applications** of the functions.

Application of the Trigonometric Functions

257. Determination of a triangle. We know that all *right* triangles in which the following parts are equal, each to each, are congruent:

1. The two sides including the right angle.

2. A side and one acute angle.

3. The hypotenuse and one of the other sides.

In other words, if in a right triangle **two parts** in addition to the **right angle** are given (*at least one* being a side), the triangle is completely *determined*, and may be *constructed* from these given parts. The unknown parts may be *computed* by the methods of scale drawing, or by using the sine, cosine, and tangent of the angles, as will be seen in the following exercises:

EXERCISES

1. The rope of a flagpole is stretched out so that it touches the ground at a point 20 ft. from the foot of the pole, and makes an angle of 73° with the ground. Find the height of the flagpole.

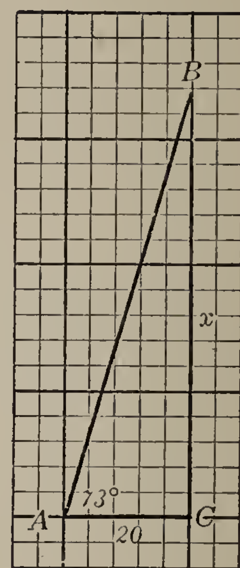


FIG. 162

I. **Graphical solution:** With a ruler and protractor, draw the right triangle, ABC, Fig. 162, to scale. By measurement, x is found to represent 66 ft. approximately.

II. **Trigonometric solution:** Using the tangent of $\angle A$, we have:

$$\frac{x}{20} = \tan 73° = 3.2709, \text{ from the table on p. 140.}$$

Therefore $x = 20 \times 3.2709 = 65.418$

The result, 65.418, is misleading, as it gives the impression that the length of BC has been determined accurately to *three* decimal places. This is impossible since the length of AC, i.e., *20*, from which 65.418, was derived by multiplication had not been determined even to the first decimal place. Hence, the decimal .418 has no meaning and should be discarded. The length of BC is said to be 65 ft., *approximately.*

2. A balloon is anchored to the ground by a rope 260 ft. long, making an angle A of 67° with the ground. Assuming the rope line to be straight, what is the height of the balloon?

Use the sine of angle A.

3. A kite-string 300 ft. long, Fig. 163, is fastened to a stake at A. The distance from the stake to a point C directly under the kite B is $102\frac{1}{2}$ feet. Find the height of the kite, supposing the kite-string to be straight.

Find the angle of elevation of the kite from the stake.

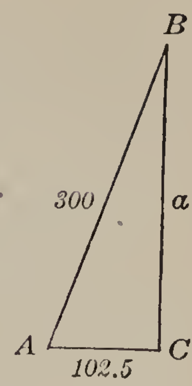


FIG. 163

I. **Graphical solution:** Draw the right triangle ABC to scale and measure a and A.

II. **Trigonometric solution:**

$$\cos A = \frac{102.5}{300} = .3$$

From the table, p. 140, $\cos 72° = .3090$

and $\cos 73° = .2924$

$\therefore$ the angle of elevation of the kite is about 72° or 73°.

Since $\frac{a}{300} = \sin\ 72° = .9511$, from p. 140,

therefore $a = 300 \times .9511 = 285$.

III. **Algebraic solution:** The value of a may also be obtained from the equation

$$a = \sqrt{300^2 - 102.5^2} \qquad \text{Why?}$$

4. A vertical pole, 8 ft. long, casts on level ground a shadow 9 ft. long. Find the angle of elevation of the sun.

Use the tangent ratio.

5. The angle of elevation of an aeroplane at a point A on level ground, is 60°. The point C on the ground directly under the aeroplane is 300 yd. from A. Find the height of the aeroplane.

6. What is the angle of elevation of the top of a hill $500\sqrt{3}$ ft. high, at a point in the plain whose shortest distance from the top of the hill is 1,000 feet?

7. What is the angle of elevation of a road that rises 1 ft. in a distance of 50 ft. measured on the road?

‡**8.** A road makes an angle of 6° with the horizontal. How much does the road rise in a distance of 100 ft. along the horizontal?

9. On a tower is a search-light 140 ft. above sea-level. The beam of light is depressed (lowered) from the horizontal, through an angle of 20°, revealing a passing boat. How far is the boat from the base of the tower?

‡**10.** A boat passes a tower on which is a search-light 120 ft. above sea-level. Find the angle through which the beam of light must be depressed from the horizontal, so that it may shine directly on the boat when the boat is 400 ft. from the base of the tower.

11. From the top of a cliff 150 ft. high, the angle of depression of a boat is 25°. How far is the boat from the top of the cliff?

12. When an aeroplane is directly over a town C the angle of depression of town B, $2\frac{1}{4}$ miles from C, is observed to be 10°. Find the height of the aeroplane.

‡**13.** From an aeroplane, at a height of 600 ft., the angle of depression of another aeroplane, at a height of 150 ft. is 39°. How far apart are the two aeroplanes?

14. Two persons, 1,200 ft. apart, observe an aeroplane directly over the straight line from one to the other. One person finds the angle of elevation of the aeroplane to be 35°; the other, at the same time, from his position, finds it to be 55°. Find the height of the aeroplane.

‡**15.** On the top of a tower stands a flagstaff. At a point A on level ground, 50 ft. from the base of the tower, the angle of elevation of the top of the flagstaff is 35°. At the same point A, the angle of elevation of the top of the tower is 20°. Find the length of the flagstaff.

16. A boy wishes to determine the height HK of a factory chimney. He places a transit first at B and then at A and measures the angles x and y. The transit is on a tripod $3\frac{1}{2}$ ft. from the ground. A and B are two points in line with the chimney and 50 ft. apart. What is the height of the chimney if the ground is level and if $x=63°$ and $y=33\frac{1}{2}°$, Fig. 164?

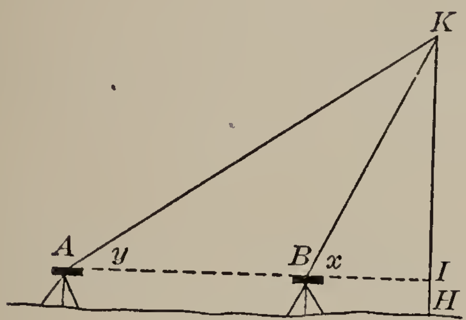


FIG. 164

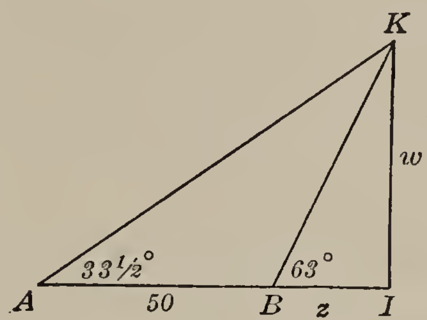


FIG. 165

In Fig. 165 $$w/z=\tan 63°=1.9626 \qquad (1)$$

$$\frac{w}{50+z}=\tan 33\tfrac{1}{2}=.6620 \qquad (2)$$

(1) and (2) are **simultaneous equations** in which w and z are the unknown numbers. To eliminate w, by substitution, we have

$$w=1.9626z \text{ (from (1))} \qquad (3)$$

By substituting (3) in (2),

$$\frac{1.9626z}{50+z}=.6620 \qquad (4)$$

Find the value of z in (4). Substitute this value of z in (3), thus obtaining the value of w.

Show how the height of the chimney could be readily found by measuring shadow-lengths, without using angles. One method would thus furnish a check on the other.

17. From a point A on the south bank of a river flowing due east the angle of elevation of the top of a tree on the north side is 45°. At a point B, 70 yd. south of A, the angular elevation is 30°. Find the width of the river.

18. At a window 20 ft. from the ground, the angle of depression of the base of a tower is 15°, and the angle of elevation of the top of the tower is 37°. What is the height of the tower?

‡**19.** Village B, Fig. 166, is due north of village C. An army outpost is located at a point A, 8 miles due west of C. B bears 60° east of north from A. An areoplane is observed to fly from C to B in a quarter of an hour. Find the average horizontal speed of the aeroplane.

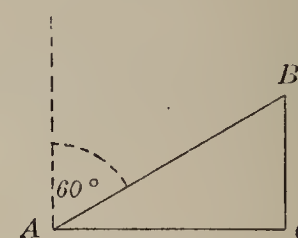


Fig. 166

‡**20.** To measure the width of a river flowing due east, a man selects a point A from which a tree at C, on the other side bears 60° east of north. He then walks east from A until he finds a point B from which C bears 30° west of north. AB is found to be 300 yards. Find the width of the river, CH, Fig. 167.

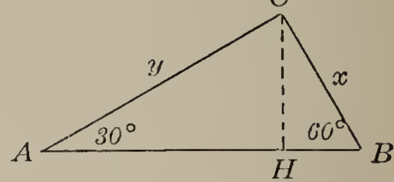


Fig. 167

Show that $x = 150$, and $y = 260$.

‡**21.** Two aeroplanes start from city C at the same time. Aeroplane A flies south at the average rate of 15 mi. an hour. Aeroplane B flies west. At the end of $\frac{3}{4}$ of an hour, aeroplane B is observed to bear $51\frac{1}{2}°$ west of north from aeroplane A. How far apart are the aeroplanes at the time of observation? What is the average speed of aeroplane B?

‡**22.** A balloon is directly over a straight road. The angles of depression of two buildings on the road are 34° and 64°. If the buildings are 65 yd. apart, how high is the balloon?

‡**23.** From a lighthouse, situated on a rock, the angle of depression of a ship is 12°, and from the top of the rock it is 8°. The height of the lighthouse above the rock is 45 feet. Find the distance of the ship from the rock.

‡**24.** From an aeroplane the angles of depression of the top and bottom of a flagpole 55 ft. high, are 45° and 67°, respectively. Find the height of the aeroplane.

258. Problems on isosceles triangles. Problems on isosceles triangles may be solved by using the two right triangles into which an altitude line from the vertex-angle of an isosceles triangle divides the triangle.

PROBLEMS

1. The distance from a cannon to a straight road is 7 miles. If the range of the cannon is 10 mi., what length of the road is commanded by the cannon?

Show that BAC, Fig. 168, is an *isosceles* triangle, and that AH bisects BC. In the right triangle ABH, find the length of BH.

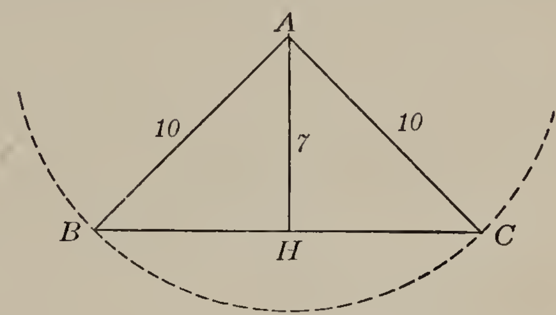


Fig. 168

2. The arms of a pair of compasses are opened to a distance of 6.25 cm. between the points. If the arms are 11.5 cm. long, what angle do they form?

In the isosceles triangle ABC, Fig. 169, draw the altitude AH.

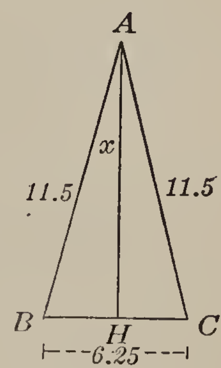


Fig. 169

3. A pair of compasses is opened to an angle of 50°. What is the distance between the points if the arms are 12.5 cm. long?

Draw the altitude of the isosceles triangle.

‡**4.** A cannon with a range of 11 mi. can shell a stretch of 13 mi. on a straight road. How far is the cannon from the road?

‡**5.** A clock pendulum, 20 in. long, swings through an angle of 6°. Find the length of the straight line between the farthest points which the lower end reaches.

6. A clock pendulum is 25 in. long. Through what angle does the pendulum swing if the distance between the farthest points which the lower end reaches is 6 inches?

7. Two firemen are playing a stream of water on the wall of a burning building from a fire-hose which throws water 120 feet. The distance on the ground from the firemen to the wall is 100 feet. What is the greatest distance on the wall which can be reached by the water?

Relations of Trigonometric Functions

259. Important relations between the *sine, cosine,* and *tangent* of an angle can be shown by simple formulas.

EXERCISES

1. Prove that if A is any acute angle

$$(\sin A)^2+(\cos A)^2=1$$

In Fig. 170

$$\sin A=\frac{a}{c} \qquad (1)$$

$$\cos A=\frac{b}{c} \qquad (2)$$

Squaring (1) and (2),

$$(\sin A)^2=\frac{a^2}{c^2} \qquad (3)$$

$$(\cos A)^2=\frac{b^2}{c^2} \qquad (4)$$

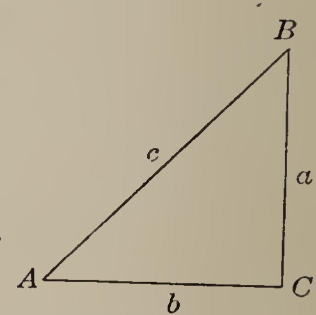


Fig. 170

Adding (3) and (4),

$$(\sin A)^2+(\cos A)^2=\frac{a^2+b^2}{c^2} \qquad (5)$$

$$\because \quad a^2+b^2=c^2$$

$$\therefore \quad (\sin A)^2+(\cos A)^2=1 \qquad (6)$$

$(\sin A)^2$ is usually written $\sin^2 A$; similarly $(\cos A)^2$ and $(\tan A)^2$ are written $\cos^2 A$ and $\tan^2 A$.

2. In Fig. 170 prove that $\sin^2 B+\cos^2 B=1$.

3. Using the formula $\sin^2 x+\cos^2 x=1$, show that

$$\sin x=\sqrt{1-\cos^2 x}\,* \qquad (1)$$

and

$$\cos x=\sqrt{1-\sin^2 x} \qquad (2)$$

* We shall not use the double sign before the radical because we have found no meaning for a negative sine or cosine of angles.

4. From Fig. 170 show that

$$\tan A = \frac{\sin A}{\cos A},$$

and
$$\tan B = \frac{\sin B}{\cos B}$$

260. Trigonometric identities. The two fundamental relations

$$\sin^2 A + \cos^2 A = 1 \tag{1}$$

$$\tan A = \frac{\sin A}{\cos A}, \tag{2}$$

are true for any value of A. They are therefore called *identities*, and are sometimes written thus,

$$\sin^2 A + \cos^2 A \equiv 1; \ \tan A \equiv \frac{\sin A}{\cos A}$$

261. Symbol of identity. The symbol, $\equiv$, is read *is*, or *is identical to*.

EXERCISES

1. In Fig. 171 show that—

1. $\sin A = \cos B$ 2. $\cos A = \sin B$,

i.e., that *the sine of an angle is the cosine of the complement of the angle.*

2. In Fig. 171 show that

$$\tan A = \frac{1}{\tan B},$$

i.e., *the tangent of an angle equals the reciprocal of the tangent of the complement.*

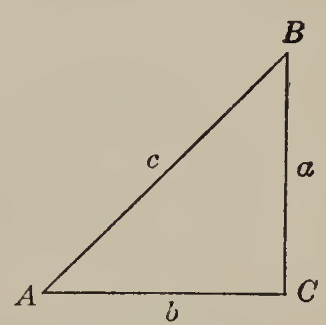


FIG. 171

262. Given the value of one function, to find algebraically the values of the others. The exercises on p. 154 show that the two fundamental identities $\sin^2 A + \cos^2 A \equiv 1$, and $\tan A = \dfrac{\sin A}{\cos A}$, may be used to find the values of *two* of the functions, if the value of the *third* function is known.

EXERCISES

In the following exercises find the values of two of the functions when—

1. $\tan B = \frac{3}{4}$

Solution:

$$\tan B = \frac{\sin B}{\cos B} = \tfrac{3}{4}. \quad \text{Why?} \qquad (1)$$

$$\text{and} \quad \sin^2 B + \cos^2 B = 1. \qquad (2)$$

Equations (1) and (2) may be solved as simultaneous equations in the two unknowns, sin B and cos B. Sin B may be eliminated by substitution, as follows:

From (1) $\sin B = \frac{3}{4} \cos B$ (3)

Substituting (3) in (2), $\frac{9}{16} \cos^2 B + \cos^2 B = 1$ (4)

Clearing (4) of fractions, $9 \cos^2 B + 16 \cos^2 B = 16$ (5)

$25 \cos^2 B = 16$ (6)

$\cos^2 B = \frac{16}{25}$ (7)

$\cos B = \frac{4}{5}$ (8)

From equation (3), $\sin B = \frac{3}{5}$

2. $\cos B = \frac{1}{3}$

3. $\sin B = \frac{1}{4}$

4. $\tan B = \frac{4}{3}$

5. $\sin B = 0.5$

6. $\cos B = m$

7. $\sin B = \frac{1}{2}\sqrt{2}$

8. $\cos B = \frac{1}{2}\sqrt{2}$

9. $\sin B = \frac{1}{2}\sqrt{3}$

10. $\cos B = \frac{1}{2}\sqrt{3}$

11. $\tan B = \sqrt{3}$

12. $\tan B = \frac{1}{3}\sqrt{3}$

13. $\tan B = s$

263. Exercises 1 to 13, § 262, illustrate one of the uses of the fundamental trigonometric relations, $\tan A = \frac{\sin A}{\cos A}$ and $\sin^2 A + \cos^2 A = 1$. The study of other trigonometric relations is postponed until we have had a good review of the principles of the operations with arithmetic and algebraic fractions. These principles are reviewed and extended in chapter x.

Quadratic Equations in Two Unknowns

264. In exercise 1, § 262, we have solved the system of equations:

$$\begin{cases} \sin B = \frac{3}{4}\cos B \\ \sin^2 B + \cos^2 B = 1. \end{cases}$$

In this system of equations sin B and cos B are considered as unknowns. To solve the system, sin B was eliminated by substituting $\frac{3}{4}$ cos B in place of sin B in the second equation. This is the *general method* of solving a system of equations in two unknowns, of which one is *linear* and the other *quadratic.*

Many problems lead to quadratic equations in *two* unknowns. The following problem will illustrate further the method of solution to be used in solving a **system of equations** in two unknowns, when *one* equation is of the *first* degree and the *other* of the *second.*

ILLUSTRATIVE PROBLEM

In the right triangle ABC, Fig. 172, with sides 3 and 4, to construct a line through C so that the perimeters of the two new triangles formed may be equal.

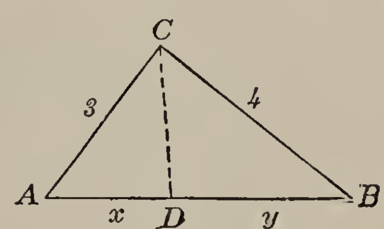


FIG. 172

Analysis: Consider the problem solved and let CD be the required line through C. The position of D evidently is determined by determining AD.

Solution: Denoting the length of AD by x, and the length of DB by y,

$$3 + x + CD = 4 + y + CD. \quad \text{Why?}$$

Hence, $$x - y = 1 \quad \text{Why?} \qquad (1)$$

$$\overline{AB}^2 = (x+y)^2 = 3^2 + 4^2 = 25 \quad \text{Why?}$$

Hence, $$x^2 + 2xy + y^2 = 25 \qquad (2)$$

The values of x and y are the solutions of the system of equations (1) and (2).

Solving (1) for x and substituting in (2),

$$(1+y)^2+2(1+y)y+y^2=25 \qquad (3)$$

$$y^2+y-6=0$$

$$\therefore \begin{cases} y_1=2 \\ x_1=3 \end{cases} \text{ and } \begin{cases} y_2=-3 \\ x_2=-2 \end{cases}$$

The values $x=-2$, $y=-3$ satisfy equations (1) and (2) but do *not* satisfy the **conditions of the problem.** Therefore this solution is disregarded and 3 and 2 are the required values of x and y, respectively.

From the preceding solution it is seen that a system of equations in two unknowns, when one of the equations is of the first and the other of the second degree, may be solved as follows:

Solve the linear equation for one of the unknowns, x or y, and substitute that value in the second-degree equation. This will lead to a second-degree equation in the other unknown, y or x, as (3), which is then to be solved.

The values of y or x, thus found, may then be substituted in the first-degree equation, as (1), to determine the corresponding values of the other unknown.

Notice that the method of solving the system of equations above is the method of *elimination by substitution.*

EXERCISES

1. Construct a right triangle whose perimeter is 30 and whose hypotenuse is 13.

2. In the right triangle ABC, Fig. 173, the perimeters of ACD and BCD are equal. $CB=4$, and $DB=2$. Find AC and AD.

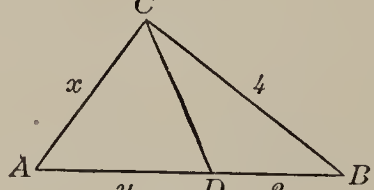


FIG. 173

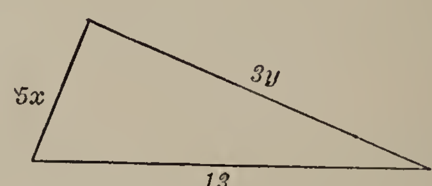


FIG. 174

3. In the right triangle of Fig. 174, with sides $5x$, $3y$, and 13, $x+y=5$. Construct the triangle.

4. Solve $\begin{cases} x^2+y^2=25 \\ y-x=1 \end{cases}$

5. Sclve $\begin{cases} r^2+4s^2=25 \\ r+2s=7 \end{cases}$

6. Solve $\begin{cases} m^2+mn+n^2=63 \\ m-n=3 \end{cases}$

‡7. Solve $\begin{cases} 2m^2-mn+3n^2=54 \\ m+n=7 \end{cases}$

Quadratic Equations Solved by the Graph

265. Problems which lead to quadratic equations in two unknowns may be solved by means of the graph, as follows:

EXERCISES

1. Solve $x^2+y^2=25$ and $y-x=1$ by the graph.

By assuming values for x, and solving $x^2+y^2=25$ for y, we have the following solutions of the equation $x^2+y^2=25$:

$\begin{cases} x=0 \\ y=\pm 5 \end{cases}$ $\begin{cases} x=3 \\ y=\pm 4 \end{cases}$ $\begin{cases} x=4 \\ y=\pm 3 \end{cases}$ $\begin{cases} x=5 \\ y=0 \end{cases}$ $\begin{cases} x=-3 \\ y=\pm 4 \end{cases}$ $\begin{cases} x=-4 \\ y=\pm 3 \end{cases}$ $\begin{cases} x=-5 \\ y=\ \ 0 \end{cases}$

Plotting these solutions, Fig. 175, we find that the graph of $x^2+y^2=25$ is a **circle** whose center is at the *origin* and whose *radius* is $\sqrt{25}$, or 5.

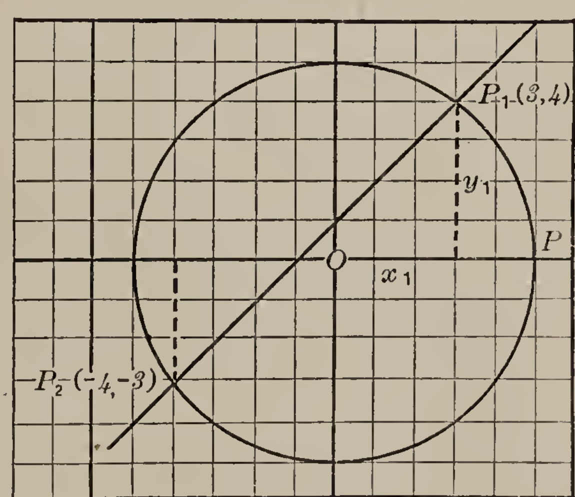


FIG. 175

That $x^2+y^2=25$ is a circle may be shown as follows:

The equation expresses the fact that the sum of the squares of the co-ordinates of any point on the graph of the equation is 25, e.g.,

$$OP_1^2=x_1^2+y_1^2=25$$

Hence, $\qquad OP_1=5.$

Moreover, a line every point of which has the same distance from a given point is a circle. Therefore the graph of $x^2+y^2=25$ is a circle whose radius is 5.

The points of intersection of this circle and the straight-line graph of equation $y-x=1$, are $P_1(3, 4)$, and $P_2(-4, -3)$.

Thus $\begin{cases} x=3 \\ y=4 \end{cases}$ $\begin{cases} x=-4 \\ y=-3 \end{cases}$ are the required values of x and y.

2. In triangle ABC, Fig. 176, $AB=5$, $CD=\frac{12}{5}$, angle $C=90°$ Construct the triangle.

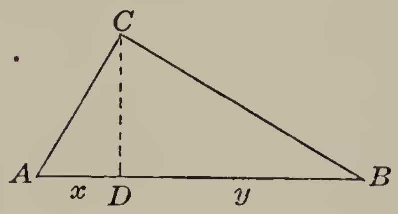


FIG. 176

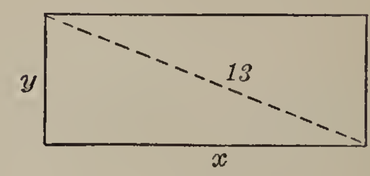


FIG. 177

3. The perimeter of the rectangle, Fig. 177 is 34. Find the dimensions.

4. Solve by eliminating by substitution, and verify by graphing:

1. $\begin{cases} x^2=58-y^2 \\ y=10-x \end{cases}$

2. $\begin{cases} x^2+y^2=40 \\ x-3y=0 \end{cases}$

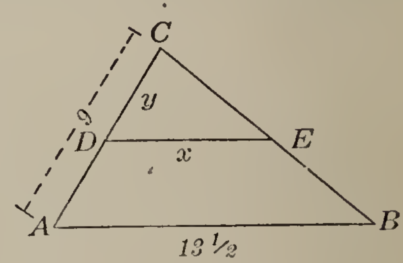


FIG. 178

5. In triangle ABC, Fig. 178, draw DE parallel to AB so that DE is the *mean proportional* between AC and DC.

6. Solve the following systems:

1. $\begin{cases} xy=18 \\ x-2y=0 \end{cases}$

2. $\begin{cases} 3xy-5y+1=0 \\ x-2y=0 \end{cases}$

3. $\begin{cases} x^2+xy+y^2=7 \\ x+4y=-1 \end{cases}$

‡4. $\begin{cases} x^2+4y^2=32 \\ 5x+6y=8 \end{cases}$

5. $\begin{cases} 2r^2-rs=6s \\ r+2s=7 \end{cases}$

‡6. $\begin{cases} 2a^2+b^2-33=0 \\ 2a=9-b \end{cases}$

7. $\begin{cases} x^2-y^2=25 \\ x-y=10 \end{cases}$

‡8. $\begin{cases} x^2-y^2=9 \\ x+y=9 \end{cases}$

‡9. $\begin{cases} x^2+y^2=25 \\ x+y=1 \end{cases}$

‡10. $\begin{cases} 2a+b=5 \\ 3a^2-7b^2=5 \end{cases}$

Summary

266. The trigonometric ratios sine, cosine, and tangent have been defined.

267. The value of a trigonometric ratio of a given angle may be found (1) from the table, (2) graphically.

268. The exact values of the sine of angles of 30°, 45°, and 60° are $\frac{1}{2}$, $\frac{1}{2}\sqrt{2}$, and $\frac{1}{2}\sqrt{3}$ respectively.

The exact values of the cosine of the same angles are $\frac{1}{2}\sqrt{3}$, $\frac{1}{2}\sqrt{2}$, and $\frac{1}{2}$, respectively.

The value of the tangent is found from the relation

$$\tan A = \frac{\sin A}{\cos A}.$$

269. Many problems in distances, which may be solved graphically, can be solved simply by calculating by the aid of trigonometric functions.

270. The following fundamental trigonometric identities have been proved:

$$\sin^2 A + \cos^2 A \equiv 1$$

$$\tan A \equiv \frac{\sin A}{\cos A}$$

271. If the value of one function is given the values of the other functions may be found, (1) from the table, (2) graphically, (3) algebraically, using the identities in § 270.

272. A system of equations in two unknowns, when one equation is of the first degree and the other of the second, may be solved by the method of *elimination by substitution.*

273. The irrational denominator of a fraction may be rationalized by multiplying the numerator and the denominator by the same number.

CHAPTER X

THE CIRCLE

Review and Extension of the Properties of the Circle

274. Gothic arch. One of the uses of the circle in designs is illustrated in Fig. 179. It represents the so-called equilateral **Gothic arch,** frequently found in modern architecture. Its most common use is in church windows.

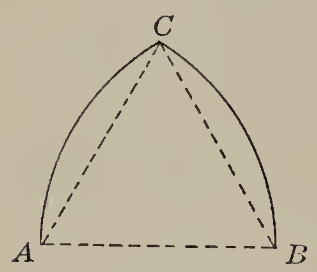


FIG. 179

ABC is an equilateral triangle and arcs AC and CB are drawn with centers at A and B, respectively, and radius AB.

EXERCISES

1. In Fig. 180 three Gothic arches are joined with a circle. Construct this figure with ruler and compass.

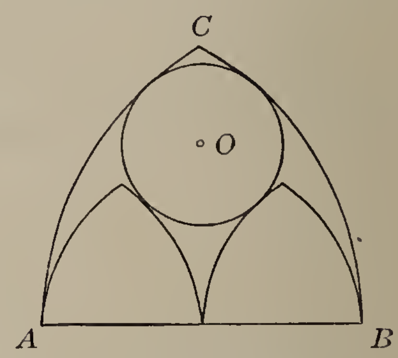


FIG. 180

To find the center of circle O use A and B as centers and radius equal to $\frac{3}{4}AB$. In exercise 3, § 289, we shall learn to prove that the circles in this figure are tangent to each other in pairs.

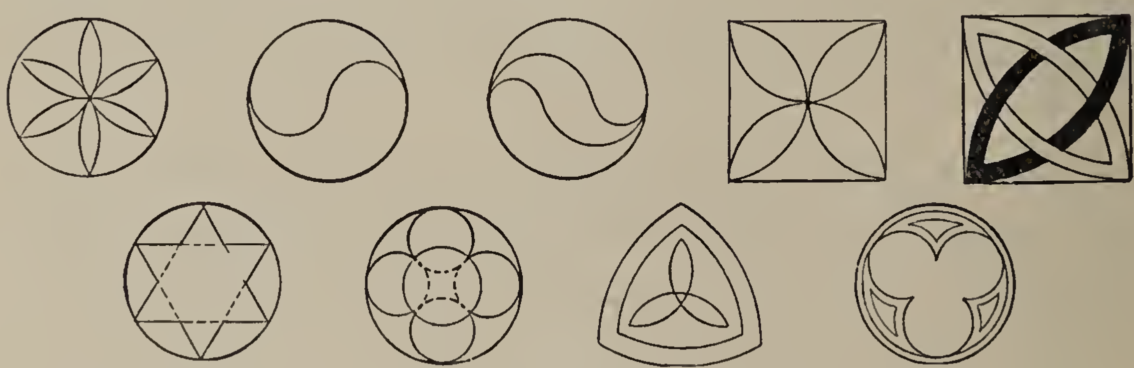


FIG. 181

2. Study the designs in Fig. 181 and construct them, using ruler and compass.

DRYBURGH ABBEY—CLOISTER DOOR

Courtesy of Walter Sargent

GOTHIC DOOR AND WINDOW

3. Compare the distances from the center of a circle to several points taken anywhere within, upon, or outside of, the circle with the length of the radius.

Exercise 3 shows that *a point is within, upon, or without a circle according as its distance from the center is less than, equal to, or greater than, the radius.*

275. Concentric circles. Draw several circles having the same center but unequal radii. Circles having the same center are called **concentric** circles.

EXERCISE

On notebook paper draw two circles having equal radii. If one of the circles is cut out and laid upon the other making the centers coincide, the circles should coincide. See if you can make one of your circles coincide with the other.

If they do not coincide, what seems to cause the failure of coincidence?

In general, *two circles having equal radii are equal,* and *equal circles have equal radii.*

276. Semicircle. Major arc. Minor arc. Cut a circle from paper. Fold it along a *diameter.* How do the two parts of the circle compare as to size?

This shows that *a diameter divides a circle into two equal parts.** Each of these parts is called a **semicircle.** If a circle is divided into unequal parts, one is called the **major arc,** the other the **minor arc.**

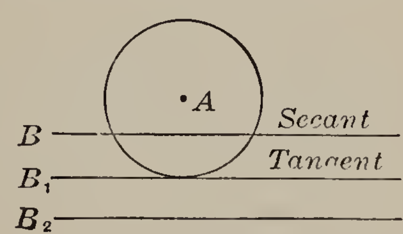


Fig. 182

277. Secant. Tangent. Draw a circle as A, Fig. 182. Move a ruler B across the circle and notice the different positions of the edge, as B, B_1, B_2, etc. How many points may a circle and a straight line have *in common?*

* According to Proclus this theorem is a discovery of Thales.

A straight line intersecting a circle in two points is a **secant.**

A line touching a circle in *only one* point is a **tangent.**

278. Number of points common to two circles. By moving one circle over another, Fig. 183, show that two

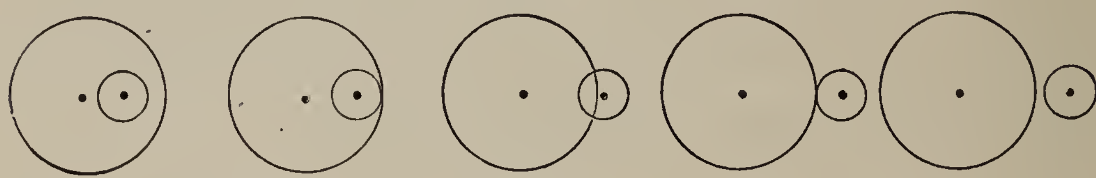


FIG. 183

circles must intersect in *two* points, or touch in *one* point, or have *no* point in common.

279. Chord. A segment *joining* two points of a circle is a **chord,** Fig. 184.

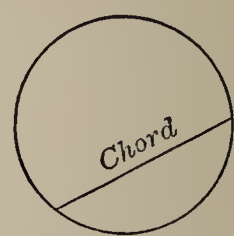


FIG. 184

280. Symbol for arc. The symbol $\frown$ means **arc.** Thus, arc AB may be written $\overset{\frown}{AB}$.

Draw two equal circles. Lay one circle upon the other, making the centers coincide. If $\overset{\frown}{AB}$ on one circle is equal to $\overset{\frown}{CD}$ on the other, they can be made to coincide.

How do the chords AB and CD compare?

In a given circle construct two equal arcs.

281. Theorem: *In the same or equal circles equal central angles intercept equal arcs, and equal arcs are intercepted by equal central angles.*

For if the arcs are made to coincide the central angles coincide, and *conversely.*

282. Subtending chord. The chord joining the end-points of an arc **subtends** (stretches under or across) the arc.

283. Theorem: *In the same or equal circles equal arcs are subtended by equal chords; and conversely, equal chords subtend equal arcs.*

The truth of the theorem is easily shown by the method of superposition.

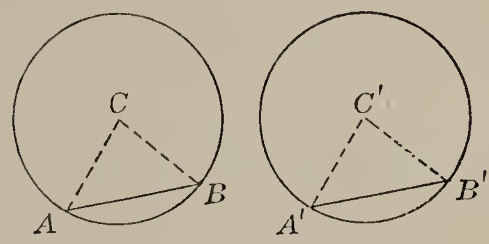


FIG. 185

To prove the converse, draw CA, CB, $C'A'$, and $C'B'$, Fig. 185.

Prove $\triangle ABC \cong \triangle A'B'C'$.

Then, $\angle C = \angle C'$ Why?

$\overset{\frown}{AB} = \overset{\frown}{A'B'}$ Why?

Diameters, Chords, and Arcs

284. Theorem: *A line drawn through the center of a circle perpendicular to a chord, bisects the chord and the arcs subtended by the chord.*

Given $\odot O$* and CD drawn through the center O, intersecting the chord AB at E; also $CD \perp AB$, Fig. 186.

To prove $\overline{AE} = \overline{EB}$; $\overset{\frown}{AD} = \overset{\frown}{DB}$; $\overset{\frown}{AC} = \overset{\frown}{CB}$.

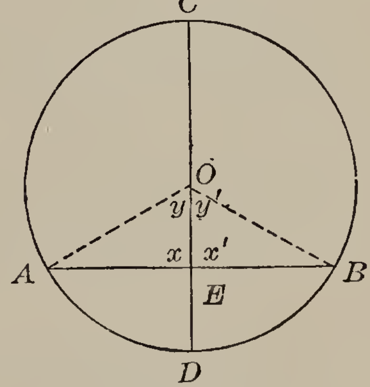


FIG. 186

Proof (method of congruent triangles):

Draw AO and OB.

Prove $\triangle AEO \cong \triangle BEO$

Hence, $\overline{AE} = \overline{EB}$ Why?

and $y = y'$ Why?

Show that $\overset{\frown}{AD} = \overset{\frown}{DB}$

and $\overset{\frown}{AC} = \overset{\frown}{CB}$

* The symbol $\odot$ O means the circle whose center is O. The symbol Ⓢ means *circles*.

The theorem above is one of a group of theorems involving the following conditions:

1. A line passes through the center.
2. A line is perpendicular to a chord.
3. A chord is bisected by a line.
4. A minor arc is bisected.
5. A major arc is bisected.

By taking *as hypothesis* any two of these five conditions and *as conclusion* one of the remaining three we can form a number of theorems. Some of these are stated among the following exercises:

EXERCISES

1. A diameter that bisects a chord is perpendicular to the chord and bisects the subtended arcs. Prove.

2. A line bisecting a chord and one of the subtended arcs passes through the center, is perpendicular to the chord, and bisects the other subtended arc. Prove.

Prove $\triangle ACE \cong \triangle BCE$, Fig. 187.

Then $CD \perp AB$. Why?

Hence, CD passes through O. For the perpendicular bisector of a segment contains all points equidistant from its endpoints (§ 71).

Show that $\overline{AD} = \overline{DB}$

$\therefore \quad \overset{\frown}{AD} = \overset{\frown}{DB}$. Why?

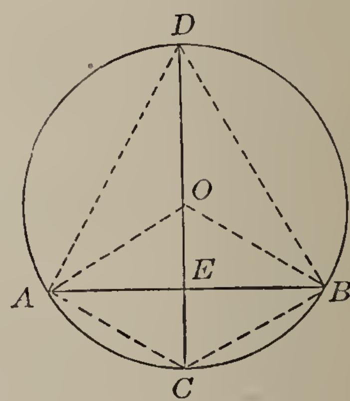


FIG. 187

3. The line-segment joining the mid-points of the arcs into which a chord divides a circle is a diameter, bisects the chord, and is perpendicular to the chord. Prove.

‡**4.** A diameter bisecting an arc is the perpendicular bisector of the chord subtending the arc. Prove.

5. The perpendicular bisector of a chord passes through the center of the circle and bisects the subtended arcs. Prove.

‡**6.** A line perpendicular to a chord and bisecting one of the subtended arcs passes through the center of the circle, and bisects the chord and the other subtended arc. Prove.

7. A diameter that bisects a chord bisects the central angle between the radii drawn to the endpoints of the chord.

8. Bisect a given arc.

9. Given a circle, find the center.

10. Given an arc, find the center and draw the circle.

11. Draw a circle through three points not lying in the same straight line.

12. Show that the perpendicular bisectors of the sides of an inscribed polygon meet in a common point.

13. Circumscribe a circle about a triangle.

‡**14.** Through a point within a circle draw a chord that will be bisected by the point.

15. Draw a circle that will pass through two given points and have a given radius.

16. If a circle is divided into 3 equal parts, and the points of division are joined by chords, an equilateral triangle is formed. Prove.

17. If the endpoints of a pair of perpendicular diameters of a circle are joined consecutively, what kind of polygon is formed? Prove.

‡**18.** Show that the perpendicular to a tangent at the contact-point passes through the center of the circle.

19. Construct a tangent to a circle at a given point of the circle.

‡**20.** To a given circle draw a tangent that shall be parallel to a given line.

285. Theorem: *In the same, or in equal circles, equal chords are equally distant from the center; and, conversely, chords equally distant from the center are equal.*

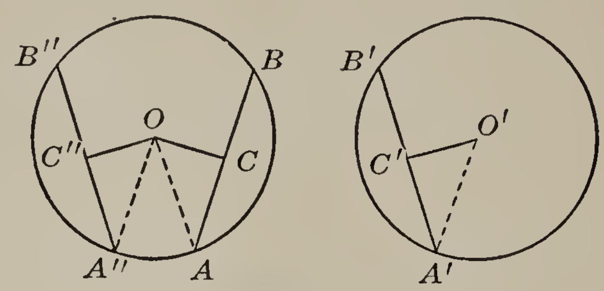


FIG. 188

Given $\odot O = \odot O'$, $\overline{AB} = \overline{A'B'} = \overline{A''B''}$

$OC \perp AB$, $O'C' \perp A'B'$, $OC'' \perp A''B''$, Fig. 188.

To prove $OC = OC'' = O'C'$.

Proof (method of congruent triangles):

Draw OA, OA'' and $O'A'$.

Prove that $AO = A'O' = A''O$.

Prove $\triangle AOC \cong \triangle A''OC'' \cong \triangle A'O'C'$.

Then, $OC = OC''$, and $OC = O'C'$.

Conversely, If $OC'' = O'C' = OC$, prove that

$$\overline{AB} = \overline{A'B'} = \overline{A''B''}.$$

Prove $\triangle OAC \cong \triangle OA''C'' \cong O'A'C'$.

Then, $AC = A'C' = A''C''$ and $AB = A'B' = A''B''$.

EXERCISES

1. If two intersecting chords make equal angles with the line joining their common point to the center, the chords are equal. Prove.

2. In a circle the distances from the center to two equal chords are denoted by—

1. $x^2 + 3x$ and $4(15 - x)$
2. $x(x+4)$ and $3(2x+5)$
‡3. $x(x-3)$ and $4(3x-9)$
‡4. $3x^2 + 4x$ and $12(1-x)$

Find x and the distances from the center to the chords.

286. Theorem: *The arcs included between two parallel secants are equal; and, conversely, if two secants include equal arcs, and do not intersect within the circle, they are parallel.*

I. **Given** circle O and $AB \parallel CD$, cutting the circle at A and B, and at C and D, respectively, Fig. 189,

To prove $\overset{\frown}{AC} = \overset{\frown}{BD}$.

Proof: Draw $OE \perp AB$, and prolong it to meet CD.

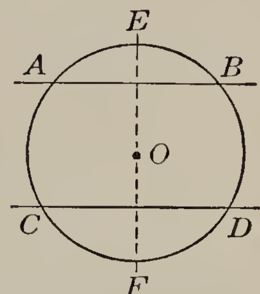


FIG. 189

Then, $OE \perp CD$. Why?

$\overset{\frown}{CE} = \overset{\frown}{DE}$ Why?

$\overset{\frown}{AE} = \overset{\frown}{EB}$ Why?

$\therefore \overset{\frown}{AC} = \overset{\frown}{BD}$ Why?

II. **Conversely, given** $\overset{\frown}{AC} = \overset{\frown}{BD}$, Fig. 189, AB and CD not intersecting within the circle,

To prove $AB \parallel CD$.

Proof: Draw $OE \perp AB$ and prolong it to F.

Prove $\overset{\frown}{EC} = \overset{\frown}{ED}$. Why?

Then, $EF \perp CD$ Why?

$\therefore AB \parallel CD$ Why?

III. Prove the theorem with one of the lines, as AB, tangent to the circle, as in Fig. 190.

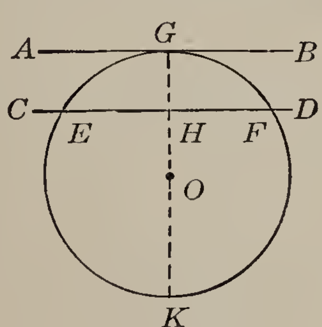


FIG. 190

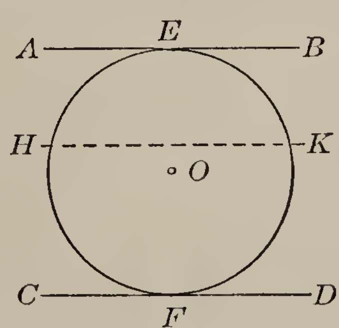


FIG. 191

IV. Prove the theorem with both parallels tangent to the circle, as in Fig. 191.

Draw $HK \parallel AB$ and apply Case III.

287. Theorem: *The line joining the centers of two intersecting circles bisects the common chord perpendicularly.*

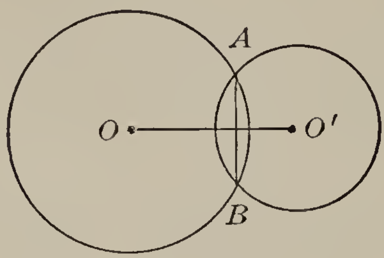


Fig. 192

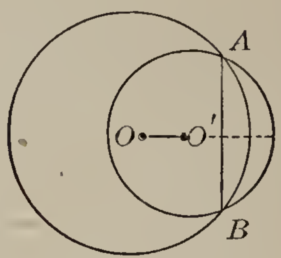


Fig. 193

Let O and O' be the intersecting circles, Figs. 192, 193. Let AB be the common chord.

To prove $OO' \perp AB$.

To prove this apply § 39.

Tangent Circles

288. Tangent circles. Two circles are said to be **tangent to each other** if both are tangent to the same line at the same point. This point is the **point of tangency,** or the **point of contact** of the circles.

If the tangent circles lie wholly without each other they are **tangent externally,** Fig. 194.

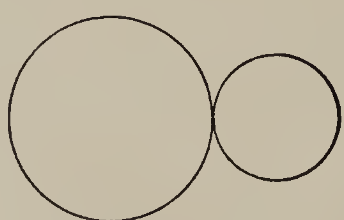


Fig. 194

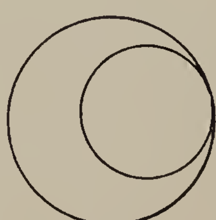


Fig. 195

If one of the tangent circles lies within the other they are **tangent internally,** Fig. 195.

289. Theorem: *If two circles are tangent to each other, the centers and the point of tangency lie in a straight line.*

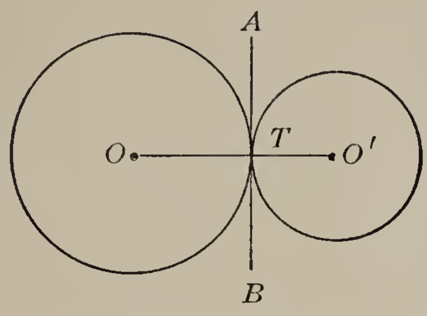


Fig. 196

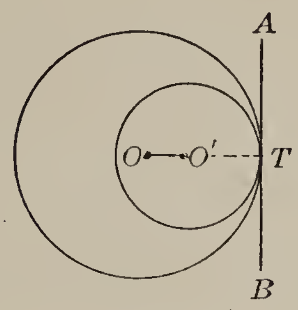


Fig. 197

I. Let O and O' be the centers of two circles tangent externally, T being the point of tangency, Fig. 196.

To prove O, O', and T lie in a straight line.

Prove that OTO' is a straight angle.

Then OT and $O'T$ are in a straight line. Why?

II. Prove the theorem for the case shown in Fig. 197.

EXERCISES

1. Draw a circle tangent to a given circle at a given point. How many such circles can be drawn?

2. Draw a circle through a given point and tangent to a given circle.

3. If the distance between the centers of two circles is equal to the sum of their radii the circles are tangent externally. Prove.

4. The distance between the centers of two tangent circles is $2\frac{1}{2}$ inches. The radius of one is $\frac{3}{4}$ inch. Draw the two circles.

‡**5.** The radii of three circles are 1 in., $1\frac{1}{2}$ in., and $\frac{3}{4}$ in., respectively. Draw the circles tangent to each other externally.

6. Construct a circle with a given point as center and tangent to a given circle.

7. To construct a circle having a given radius and tangent to two given circles.

‡**8.** With the vertices of a triangle as centers construct three circles tangent to each other. (See Fig. 198.)

Show algebraically that one of the radii is equal to half the perimeter diminished by one of the sides.

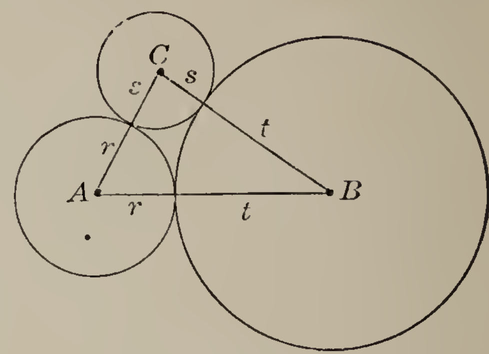


FIG. 198

290. Historical note: The part of the theory of the circle that deals with chords, tangents, and secants is older than the time of Euclid. Most of it was probably first worked out by the Pythagoreans. It is well known that Archytas of Tarentum (430–365 B.C.) at a certain point in his construction of the problem of doubling a cube, assumed a knowledge of the theorem that the angle between a tangent and the contact-radius is a right angle. The first use of the theorem of the equality of the two tangents to a circle from an outside point of which we have knowledge is with Archimedes (287–212 B.C.). Heron (first century B.C.) is the first to give it place as an independent theorem.

The converse theorem, that the center of the circle lies on the bisector of the angle between two tangents is first met with in the seventh book of the *Synagoge* of Pappus about the end of the third century A.D. Archimedes is said to have written an entire work on the tangency of circles. The so-called *taction-problem* of Apollonius was to draw a circle which should fulfil *three* conditions, viz., go through a given point, be tangent to a given straight line, and a given circle. In the fourth book of his *Synagoge* Pappus studied the problem to draw a circle tangent externally to three given circles and treated another interesting problem "through three points of a straight line to draw three other straight lines that should form an inscribed triangle within a given circle." This problem has more recently given rise to varied generalizations.

Summary

291. The meaning of the following terms was taught:

concentric circles
arc
semicircle
major arc
minor arc
secant
tangent
chord
subtending chord
tangent circles

The following symbols were introduced: $\overline{}$ for chord, $\frown$ for arc, $\odot$ for circle, Ⓢ for circles.

292. The truth of the following theorems has been shown:

1. *A point is within, upon, or without, a circle according as its distance from the center is less than, equal to, or greater than, the radius.*

2. *Circles having equal radii are equal, and equal circles have equal radii.*

3. *A diameter divides a circle into equal parts.*

4. *In the same or equal circles equal central angles intercept equal arcs,* and *equal arcs are intercepted by equal central angles.*

293. The following theorems have been proved:

1. *In the same or equal circles equal arcs are subtended by equal chords; and, conversely, equal chords subtend equal arcs.*

2. If any two of the following conditions are taken as hypothesis the remaining three are true:

(1) A line passes through the center.

(2) A line is perpendicular to a chord.

(3) A chord is bisected by a line.

(4) A minor arc is bisected.

(5) A major arc is bisected.

3. *In the same or equal circles equal chords are equally distant from the center;* and, conversely, *chords equally distant from the center are equal.*

4. *The arcs included between two parallel secants are equal;* and, conversely, *if two secants include equal arcs, and do not intersect within the circle, they are parallel.*

5. *The line joining the centers of two intersecting circles bisects the common chord perpendicularly.*

6. *If two circles are tangent to each other, the centers and the point of tangency lie in a straight line.*

7. Two arcs are equal if one of the following conditions holds:

(1) The subtending chords are diameters.

(2) The central angles intercepting the arcs are equal.

(3) The subtending chords are equal.

(4) The arcs are intercepted by parallel chords, secants, and tangents.

8. Two chords are equal if one of the following conditions holds:

(1) The chords subtend equal central angles.

(2) The chords subtend equal arcs.

(3) The chords are equally distant from the center.

CHAPTER XI

MEASUREMENT OF ANGLES BY ARCS OF THE CIRCLE

294. Units of angular measure. In all preceding work angles have been measured by comparing them with such angular units as *degree, minute, second, right angle,* and *straight angle.* Thus, the measure of an angle is 45, if it contains 45 *degrees;* the measure of the same angle is $\frac{1}{2}$, if the *right angle* is used as unit; or it is $\frac{1}{4}$, if the *straight angle* is the unit of measure.

In the following it will be shown that, *if the sides of an angle touch or intersect a circle, it is possible to measure the angle in terms of the arcs intercepted* by the sides of the angle.*

EXERCISES

1. From cardboard cut a right angle. Move it so that the sides always pass through two fixed points, as A and B, Fig. 199. This may be done by letting the sides always touch two pins stuck into the paper at A and B. Mark the position of the vertex for various positions of the angle. How does the vertex move?

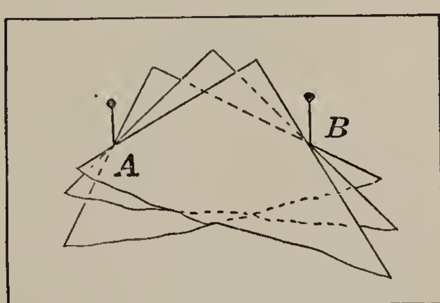


FIG. 199

2. Repeat exercise 1 with an acute angle; with an obtuse angle.

3. Draw a semicircle. Join various points of the semicircle to the endpoints of the diameter forming angles whose vertices lie on the circle. With a protractor measure these angles. How do they compare in size?

* Note the difference between the words "intercept" and "intersect." The former means "to hold between" and the latter "to cut, or to cross."

4. Draw a circle. With a chord cut off an arc greater than a semicircle and join various points of the arc to the endpoints of the chord. By measuring, compare the angles having the vertices on the arc.

5. Repeat exercise 4, using an arc less than a semicircle.

295. Inscribed angle. An angle whose vertex is on a circle and whose sides are chords is an **inscribed angle.**

EXERCISES

1. Exercises 4 and 5, § 294, illustrate the fact that all inscribed angles intercepting the same arc are equal, and that the angle is acute, right, or obtuse according as the intercepted arc is *less* than, *equal* to, or *greater* than, a semicircle.

How does an inscribed angle vary as the arc increases from a short length to the length of the circle?

2. Show how a carpenter's square may be used to test the accuracy of a semicircular groove. (See Fig. 200.)

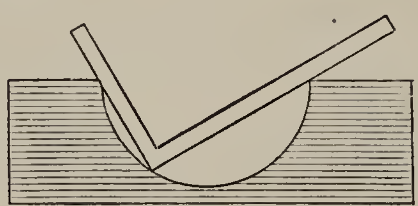


FIG. 200

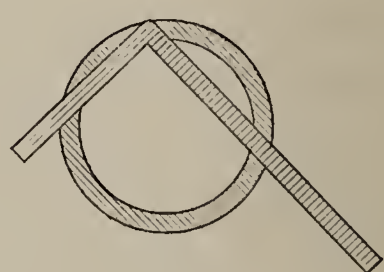


FIG. 201

3. Show how a carpenter's square may be used to find where a ring must be cut so that the two parts are equal. (See Fig. 201.)

4. The circle in Fig. 202 represents a region of dangerous rocks to be avoided by ships passing near the coast AB. Outside of the circle there is no danger. Show that the ship S is out of danger as long as angle ASB, found by observations made from the ship, is *less than* the known angle ACB.

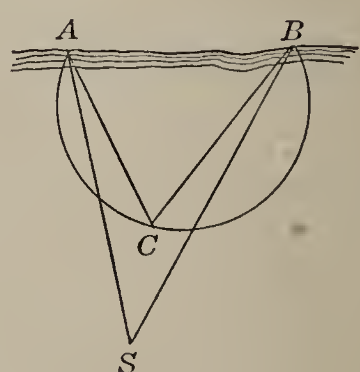


FIG. 202

296. If two lines intersect and also cut or touch a circle, the various positions may be illustrated as in Figs. 203–209.

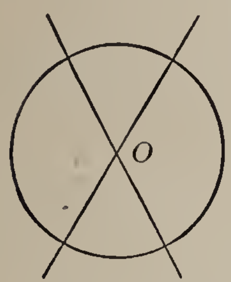


FIG. 203

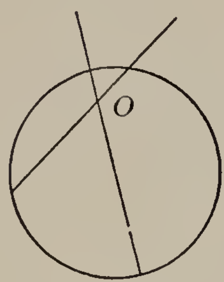


FIG. 204

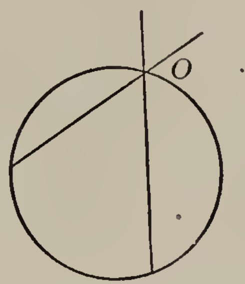


FIG. 205

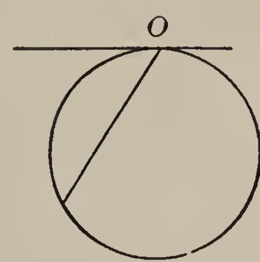


FIG. 206

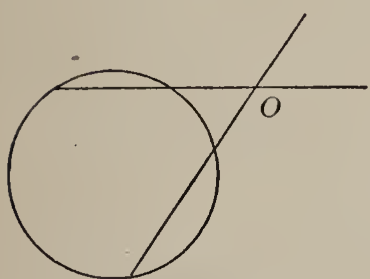


FIG. 207

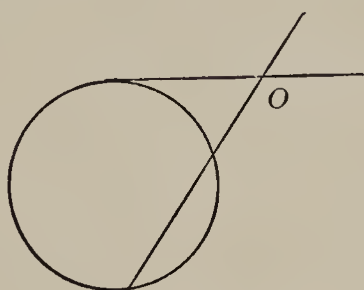


FIG. 208

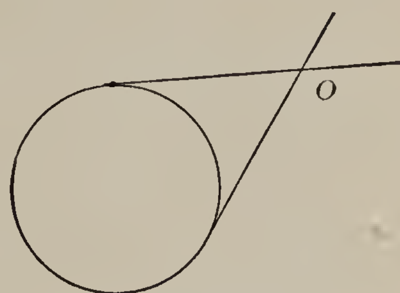


FIG. 209

In Fig. 203 the lines intersect at the center of the circle, i.e., the angle is formed by *two radii.*

In Fig. 204 the lines intersect within the circle, not at the center, i.e., the angle is formed by *two chords.*

Moving the intersecting lines until the vertex of the angle is on the circle, the angle becomes an *inscribed angle,* Fig. 205.

Leaving one side of the angle, Fig. 205, fixed and turning the other until it is *tangent* to the circle, Fig. 206 is obtained. In this figure the angle is formed by a *tangent* and a *chord.*

Fig. 207 shows the lines intersecting *outside* of the circle, the angle now being formed by *two secants.*

Rotating the sides of the angle about O, Fig. 207, until they became tangent to the circle, Figs. 208 and 209 are obtained.

Some of the following theorems show how, in each of the Figs. 203 to 209, the measure of the angle formed by the two intersecting lines may be expressed *in terms of the intercepted arc* or *arcs*.

297. Measure of a central angle. Let $\angle AOB$, Fig. 210, be a *central angle* and let it be divided into equal parts. Taking one of these as a unit, the number of equal parts is the **measure** of the angle. What is the measure of $\angle AOB$?

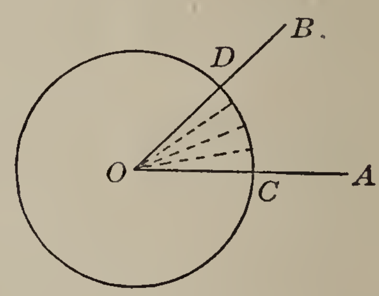


FIG. 210

Show that $\overset{\frown}{CD}$ is divided into equal parts.

Taking as a unit one of the equal parts of $\overset{\frown}{CD}$, what is the measure of $\overset{\frown}{CD}$?

In general, if the measure of a central angle is m, the measure of the intercepted arc is also m. Why?

Briefly, we may say a central angle has the same measure as the intercepted arc, or—

A central angle is measured by the intercepted arc.

EXERCISES

1. Draw a central angle. With a protractor find the number of degrees, integral or fractional, contained in the angle. How many arc-degrees are there in the intercepted arc? What is a measure of the intercepted arc?

2. Draw a circle and mark off an arc. Find the number of arc-degrees contained in it. What is the measure of the arc?

3. Using ruler and compass only, divide a circle into 2 equal arcs, 4 equal arcs, 8 equal arcs.

4. Using ruler and compass only, construct arcs of 90°, 45°, 60°, 30°, 15°, 75°, 105°, 165°.

How may an angle of 90° be trisected? An angle of 45°?

5. Divide a circle into three arcs in the ratio 1:2:3.

Find algebraically the number of degrees in each. Then use the protractor to draw the arcs.

6. A circle is divided into 4 arcs in the ratio 1:4:6:7. Find the number of degrees contained in each arc.

7. The length of a circle is 63 inches. A central angle intercepts an arc 7 in. long. How many degrees does the angle contain?

8. *In the same or equal circles two central angles have the same ratio as the arcs intercepted by their sides.*

To show this, let the measures of the angles be m and n, respectively.

Show that the measures of the intercepted arcs are also m and n respectively.

Then each ratio is $\frac{m}{n}$. Why?

9. In Fig. 211, AB is a diameter. The number of degrees in $\angle AOC$ is denoted by x^2+4x and in $\angle BOC$ by $3x^2+12x$. Find the values of x and the number of arc-degrees in arcs AC and CB.

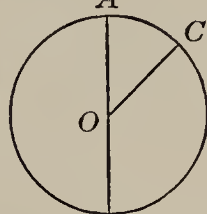


FIG. 211

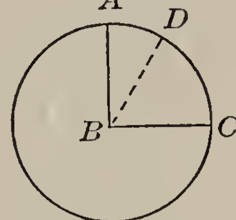


FIG. 212

‡10. In Fig. 212 $\angle ABC$ is a right angle. $\angle ABD = (2x^2-3)°$, and $\angle DBC = (10x^2-15)°$. Find the values of x and the number of degrees in arcs AD and DC.

298. Measure of an inscribed angle. Draw an inscribed angle, as ABC, Fig. 213. With a protractor measure angle ABC. Find the number of arc-degrees in $\overset{\frown}{AC}$.

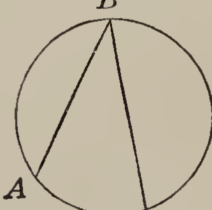


FIG. 213

How does the measure of the inscribed angle compare with the measure of the arc?

The following theorem shows how to find the measure of an inscribed angle in terms of the intercepted arc:

Theorem: *An inscribed angle is measured by one-half the arc intercepted by its sides.*

Let ABC, Fig. 213, be an inscribed angle intercepting $\overset{\frown}{AC}$.

To prove that ABC is measured by $\frac{1}{2}\overset{\frown}{AC}$.

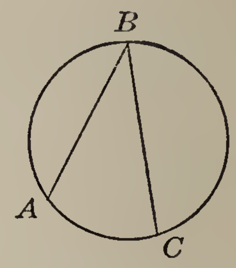


FIG. 213

In proving the theorem three cases are considered:

Case I. The center of the circle lies *on one side* of the angle, Fig. 214.

Proof: Draw the radius CD.

Denote the measures of ABC, ADC, and $\overset{\frown}{AC}$ by x, y, and x', respectively, and show that $\angle BCD = x$.

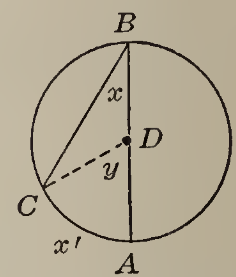


FIG. 214

Hence, we have the relation—

$$x + x = y \qquad \text{Why?}$$

Solving for x, $\quad x = \frac{1}{2}y$

But, $\quad y = x' \qquad$ Why?

$$\therefore \quad x = \frac{1}{2}x' \qquad \text{Why?}$$

Case II. The center of the circle lies *within* the angle, Fig. 215.

Proof: Draw the diameter BD.

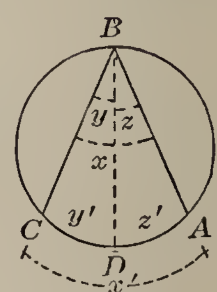


FIG. 215

$$x = y + z \qquad \text{Why?}$$

$$y = \frac{y'}{2} \qquad \text{Case I}$$

$$z = \frac{z'}{2} \qquad \text{Case I}$$

$$\therefore \quad x = \frac{y' + z'}{2} \qquad \text{Why?}$$

or $\quad x = \frac{1}{2}x' \qquad$ Why?

Case III. The center of the circle lies *outside* of the angle, Fig. 216.

Proof: Draw the diameter BD.

$$z = x + y \quad \text{Why?}$$

$$\therefore \quad x = z - y \quad \text{Why?}$$

But $$z = \frac{z'}{2} \quad \text{Why?}$$

and $$y = \frac{y'}{2} \quad \text{Why?}$$

$$\therefore \quad x = \frac{z' - y'}{2} \quad \text{Why?}$$

or $$x = \tfrac{1}{2}x'$$

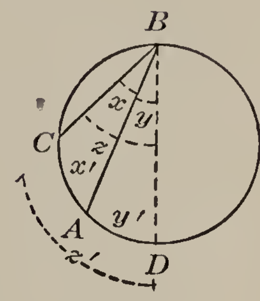


Fig. 216

299. Segment of a circle. The portion of a plane included between a chord and the arc it subtends is a **segment** of the circle. The shaded part ABC, Fig. 217, is a segment of circle O.

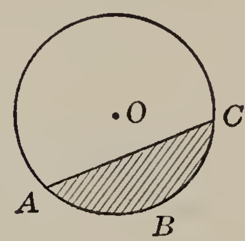


Fig. 217

EXERCISES

Prove the following exercises:

1. All angles inscribed in the same segment of a circle are equal.

2. All angles inscribed in a semicircle are right angles.*

3. All angles inscribed in a segment smaller than a semicircle are greater than a right angle.

4. All angles inscribed in a segment greater than a semicircle are less than a right angle.

5. Two chords AB and CD, Fig. 218, intersect within a circle. Show that $\triangle AEC$ and BED are mutually equiangular and therefore similar.

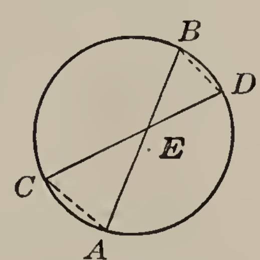


Fig. 218

* Perhaps known and used by Thales; first proved by the Pythagoreans.

‡**6.** (*Mathematical puzzle*). Find the error in the proof of the following theorem: From a point not on a given line two perpendiculars may be drawn to the line.

In the two intersecting circles O and O', Fig. 219, diameters AB and AC are drawn from A, one of the points of intersection of the circles.

Draw CB intersecting the circles in points D and E.

Draw AE and AD.

$\angle AEC$ is a right angle, being inscribed in a semicircle.

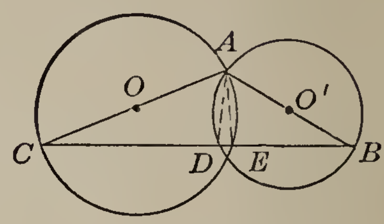


FIG. 219

$$\therefore \quad AE \perp CB$$

Similarly, $\angle ADB$ is a right angle.

$$\therefore \quad AD \perp CB$$

7. An inscribed triangle is a triangle whose vertices lie on a circle. Two angles of an inscribed triangle are 82° and 76°. How many degrees are there in each of the three arcs subtended by the sides?

8. Two circles intersect at points A and B, Fig. 220. AC and AD are diameters. Prove that C, B, and D lie in the same straight line.

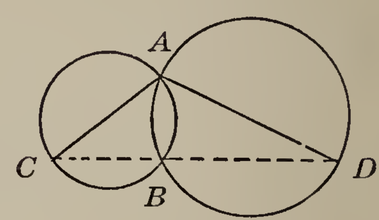


FIG. 220

300. Theorem: *An angle formed by a tangent and a chord passing through the point of contact is measured by one-half of the intercepted arc.*

Let CD, Fig. 221, be tangent to circle O, and let AB be a chord of the circle, drawn from the point of contact.

To prove that $\angle ABC$ is measured by one-half of $\overset{\frown}{AB}$.

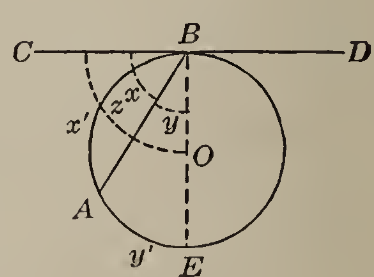


FIG. 221

Proof: Draw the diameter BE.

Denoting the measures of $\measuredangle ABC$, EBA, and EBC by x, y, and z, respectively, and the measures of arcs BA, AE, and BAE by x', y', and z', we have the following relations:

$$z=x+y. \qquad \text{Why?}$$

$$\therefore \quad x=z-y. \qquad \text{Why?}$$

But, $$y=\tfrac{1}{2}y'. \qquad \text{Why?}$$

and, $$z=\tfrac{1}{2}z', \text{ since } z=90° \text{ and } z'=180°.$$

$$\therefore \quad x=\tfrac{1}{2}z'-\tfrac{1}{2}y'=\tfrac{1}{2}(z'-y'). \qquad \text{Why?}$$

$$\therefore \quad x=\tfrac{1}{2}x'. \qquad \text{Why?}$$

EXERCISES

1. A triangle ABC, Fig. 222, is inscribed in a circle and $\angle A=57°$, $\angle B=66°$. Tangents are drawn at A, B, and C forming the circumscribed triangle $A'B'C'$. Find the angles A', B', and C'.

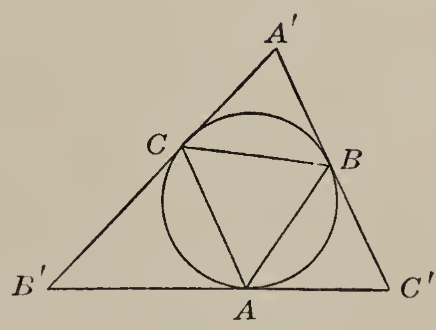


FIG. 222

2. Two angles of a circumscribed triangle $A'B'C'$ are 70° and 80°, Fig. 222. Find the number of degrees in each of the three angles of the inscribed triangle ABC.

3. The vertices of an inscribed quadrilateral divide the circle into arcs in the ratio 3:4:5:6. Find the angles of the quadrilateral.

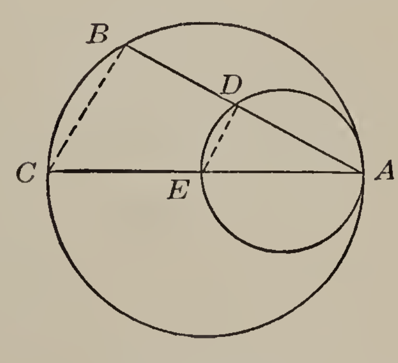


FIG. 223

4. From the point of tangency, A, Fig. 223, of two circles tangent internally two chords are drawn meeting the circles in B, C, D, and E. Prove $BC \parallel DE$.

Draw the common tangent.

5. Prove that the tangents drawn from a point to a circle are equal, Fig. 224.

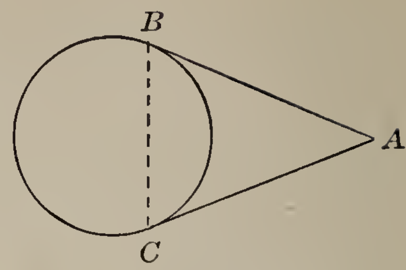


Fig. 224

Problems of Construction

301. Make the following constructions:

1. *Upon a given line-segment as a chord construct a segment of a circle in which the inscribed angles are equal to a given angle.*

Given the line-segment a and an angle equal to x, Fig. 225.

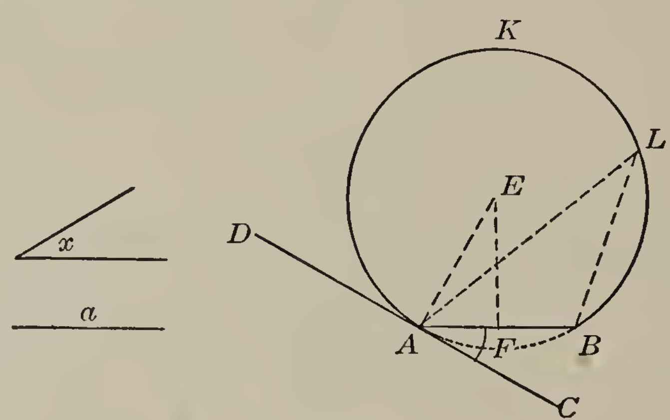


Fig. 225

To construct upon a as a chord a segment of a circle in which an angle equal to x may be inscribed.

Construction: Draw $AB = a$.

At A, on AB, construct $\angle CAB = x$.

Draw $AE \perp DC$.

Draw FE, the perpendicular bisector of AB. It will meet AE as at E. Why?

With E as center and radius EA draw a circle. This circle must pass through B. Why?

AKB is the required segment.

Proof: Let $\angle ALB$ be any angle inscribed in segment AKB.

Then	$\angle ALB = \frac{1}{2}\overset{\frown}{AB}$.	Why?
	$\angle BAC = \frac{1}{2}\overset{\frown}{AB}$.	Why?
$\therefore$	$\angle ALB = \angle BAC$.	Why?
$\therefore$	$\angle ALB$ is equal to x.	Why?

Test the accuracy of the construction with the protractor.

2. Make the construction of problem 1, using a given obtuse angle.

3. On a given line-segment, construct a segment of a circle containing an inscribed angle of 60°: of 30°; of 120°; 45°; 135°; using ruler and compass only.

4. *From a point outside of a circle to construct a tangent to the circle.*

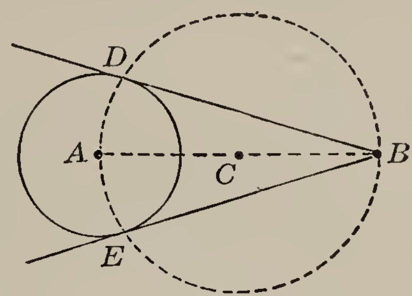


Fig. 226

Let A be the center of the given circle and B the given point outside of the circle, Fig. 226.

To construct a tangent to circle A from B.

Construction: Find the midpoint of AB.

Draw a circle having AB as diameter, cutting circle A at D and E.

Draw BD and BE.

BD and BE are the required tangents.

Proof: Draw AD and show that $\angle ADB$ is a right angle. Then BD is tangent to circle A. Why?

‡**5.** Euclid's method of solving problem 4, as given in Book III, Theorem 17 of his *Elements*, was as shown in Fig. 227.

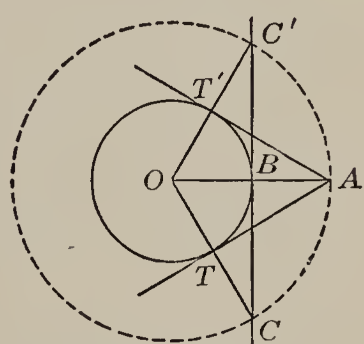


Fig. 227

The given circle is O and the given point, A.

A concentric circle is drawn through A. O and A are joined with OA. Where OA cuts the given circle, at B, erect CC' perpendicular to OA. Connect C and C' with O. Join the crossing points, T and T', with A. AT and AT' are the required tangents. Prove.

Tropfke says the mode of construction of problem 4 first occurred in 1583 in Thomas Finck's well-known and valuable geometrical work, entitled *Geometriae rotundi.* Some elementary geometries of the eighteenth century followed Finck's construction, and some followed Euclid's. Which do you prefer and why?

6. *To draw a common tangent to two circles exterior to each other.*

The number of common tangents to two circles depends upon the position of the circles. If one circle is entirely outside of the other, Fig. 228, there are four common tangents, i.e., two external tangents, AB and CD, and two internal tangents, EF and GH.

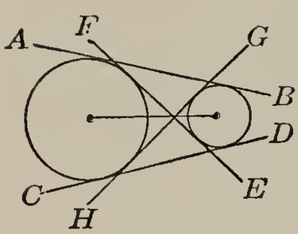


Fig. 228

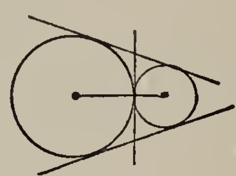


Fig. 229

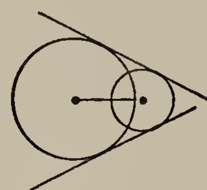


Fig. 230

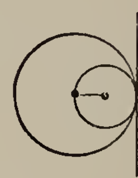


Fig. 231

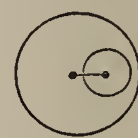


Fig. 232

If the circles are tangent to each other externally, there are two external and one internal tangent, Fig. 229.

If the circles intersect, two external tangents can be drawn, Fig. 230.

If the circles are tangent internally, there exists only one external tangent, Fig. 231.

No common tangent exists if one circle lies entirely within the other, Fig. 232.

Notice that in every case the line passing through the centers of the circles is an **axis of symmetry** of the figure.

Let A and A', Fig. 233, be the center of two circles exterior to each other.

I. *It is required to draw the common internal tangents.*

Construction: Draw AA'.

Divide AA' into segments having the same ratio as the radii (§ 176), and let B be the point of division.

From B construct BC tangent to circle A (problem 4).

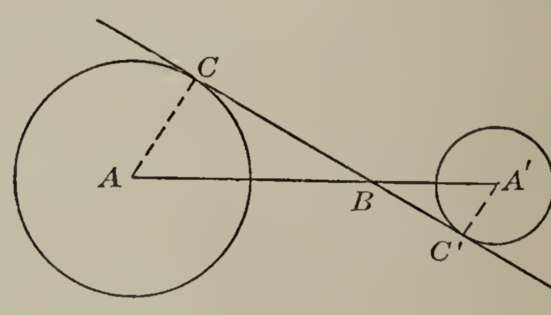


Fig. 233

BC is one of the required internal tangents.

Proof: Draw AC. Draw $A'C' \perp CB$.

If it can be proved that $A'C'$ is equal to the radius of circle A', then BC is tangent to circle A' (§ 74).

Denoting the radii of circles A and A' by R and R',

$$\frac{AB}{BA'} = \frac{R}{R'}, \text{ by construction.}$$

Prove $\triangle ABC \backsim \triangle A'BC'$

$$\therefore \frac{AB}{BA'} = \frac{AC}{A'C'} = \frac{R}{A'C'}. \qquad \text{Why?}$$

$$\therefore \frac{R}{R'} = \frac{R}{A'C'}. \qquad \text{Why?}$$

Prove that $A'C' = R'$.

$\therefore$ BC is tangent to circle A'. Why?

Show how to construct the other *common internal tangent.*

II. *To draw the external tangents.*

Construction: Draw AA', Fig. 234.

Divide AA' externally in the ratio of the radii at the point B (§ 176).

Draw BC tangent to circle A.

BC is one of the required external tangents.

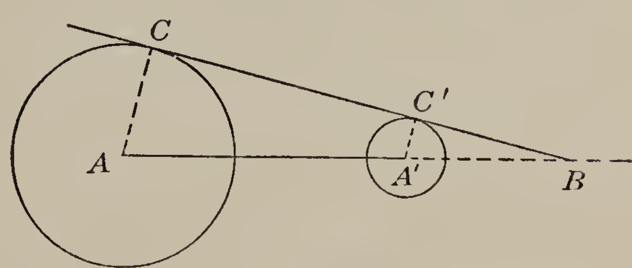


Fig. 234

Show how to construct the other *external tangent.*

The proof is the same as for Case I.

In §§ 302, 303 we find two illustrations of external and internal tangents common to two circles.

302. Circular motion. Circular motion may be transmitted by means of a belt running over two pulleys, Figs. 235, 236.

Two pulleys whose radii, R and r, are 12 in. and 5 in., respectively, are fastened to parallel shaftings and are connected by a belt, Fig. 235.

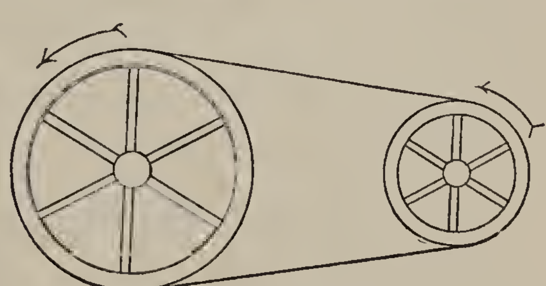


Fig. 235

The distance, a, between the centers of the pulleys is 32 inches. Make a drawing to the scale 1 to 16.

Find the length, l, of the belt from the formula

$$l=\pi(R+r)+2a.$$

In Fig. 236 the pulleys are connected by a crossed belt. Find the length of the belt by means of the formula,

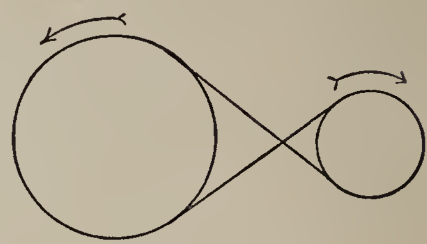


FIG. 236

$$l=2\sqrt{(R+r)^2+a^2}+\pi(R+r).$$

Notice that the pulleys, as connected in Fig. 236, turn in *opposite* directions.

303. Lunar eclipse. A lunar eclipse occurs when the moon passes through the earth's shadow. If the moon is within the dark part of the shadow, Fig. 237, the eclipse

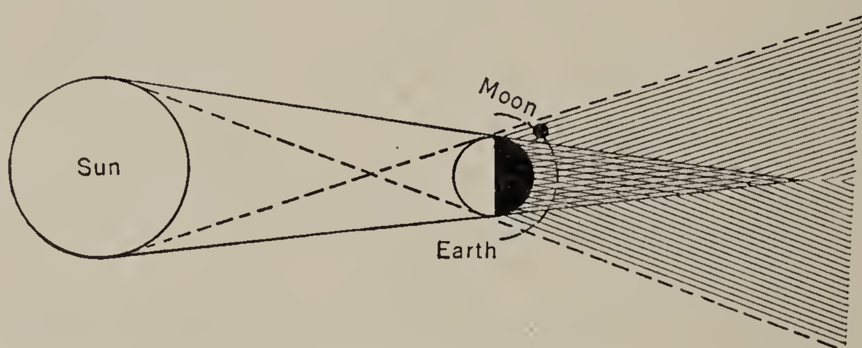


FIG. 237

is said to be *total*. This part is included between the earth and the two *external tangents* common to the earth and the sun. If the moon is in the half-light region which is determined by the common internal tangents the eclipse is said to be *partial*.

Find the length of the earth's shadow, taking the distance from Earth to Sun as 93,000,000 mi., the diameter of the Sun as 866,500 mi., and the diameter of Earth as 8,000 miles.

304. Theorem: *If two chords intersect within a circle, either angle formed is measured by one-half the sum of the intercepted arcs.*

Draw AD, Fig. 238.

Show that $x=y+z$

$y=\frac{1}{2}y'$

$z=\frac{1}{2}z'$

$\therefore\ x=\frac{1}{2}(y'+z')$

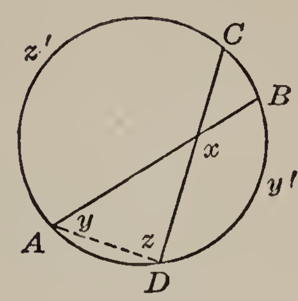


FIG. 238

305. Theorem: *If two secants meet outside of a circle the angle formed is measured by one-half the difference of the intercepted arcs.*

Draw AD, Fig. 239.

Show that $y=x+z$

and $x=y-z$

Complete the proof as in § 304.

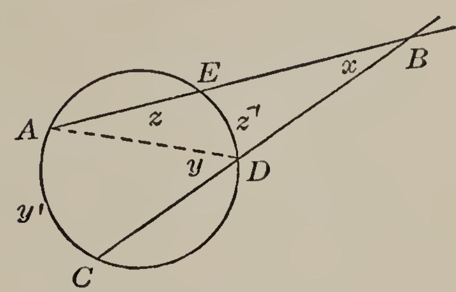


FIG. 239

306. Theorem: *The angle formed by a tangent and a secant meeting outside of a circle is measured by one-half the difference of the intercepted arcs.*

Draw CD, Fig. 240.

Then $y=x+z$

and $x=y-z$

$y=\frac{1}{2}y'$

$z=\frac{1}{2}z'$

$\therefore\ x=\frac{1}{2}(y'-z')$

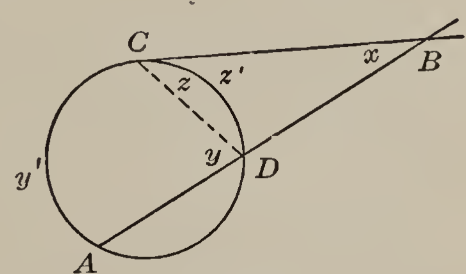


FIG. 240

307. Theorem: *The angle formed by two tangents to a circle is equal to one-half the difference of the intercepted arcs.*

Show that

$y=x+z$, Fig. 241.

$x=y-z=\frac{1}{2}(y'-z')$

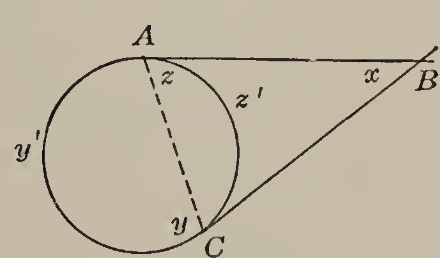


FIG. 241

EXERCISES

1. The arcs and angle being denoted as in Fig. 242, find x and y.

2. Find x and y, Fig. 243, the arcs and angle between the secants being as indicated in the figure.

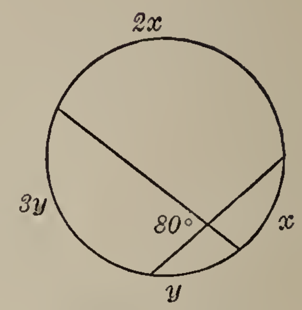


FIG. 242

3. When two tangents to a circle make an angle of 60° into what arcs do they divide the circle?

‡4. Into what arcs do two tangents at right angles to each other divide the circle?

5. Two tangents include two arcs of a circle, one of which is four times the other. How many degrees in the angle they form?

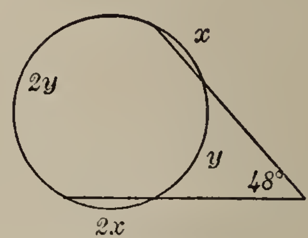


FIG. 243

‡6. The angle between two secants intersecting outside of a circle is 76°. One of the intercepted arcs is 243°. Find the other.

7. The points of tangency of a circumscribed quadrilateral divide the circle into arcs in the ratio of 7:8:9:12. Find the angles of the quadrilateral.

8. Two tangents to a circle from an outside point form an angle of 70°. What part of the circle is the larger arc included by the points of tangency?

9. The angle between two secants is 30°, Fig. 244. The number of degrees in arc DE is represented by $\dfrac{6x^2+29x+30}{2x+3}$, in the arc BC, by $\dfrac{2x^2-7x-15}{x-5}$. Find x and the number of degrees in each of the two arcs.

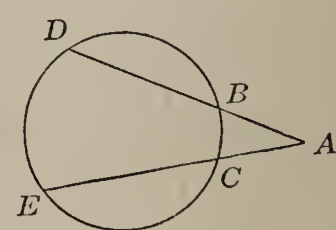


FIG. 244

Reduce the fractions to lowest terms.

‡**10.** In Fig. 245 $\angle AED$ is 60°, arc BC is represented by $\frac{x^2+8x+15}{x+3}$; arc AD, by $\frac{x^2+12x-45}{x+15}$. Find the number of degrees in each of the two arcs.

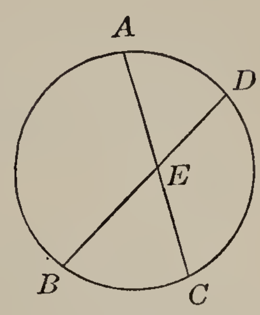


FIG. 245

11. Prove that *the sum of the three angles of a triangle is two right angles.*

In Fig. 246, let ABC be any triangle. Circumscribe a circle about it.

The three inscribed angles are measured by one-half the sum of the three arcs AB, BC, and CA.

But the sum of the three arcs AB, BC, and CA is the entire circle.

∴ One-half the circle, or 180°, is the measure of the sum of the three angles of the triangle.

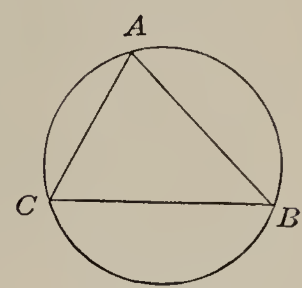


FIG. 246

‡**12.** In laying a switch on a railway track a "frog" is used at the intersection of two rails to allow the flanges of the wheels moving on one rail to cross the other rail. Show that the angle of the frog, a, Fig. 247, made by the tangent to the curve and the straight rail DE, is equal to the central angle FOB, of the arc BF.

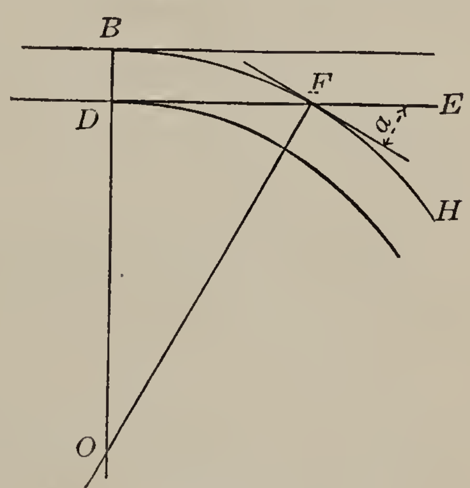


FIG. 247

MISCELLANEOUS EXERCISES

308. A limited number of the exercises below may be worked:

1. Prove that the circles described on any two sides of a triangle as diameters intersect on the third side.

2. A circle described on one of the two equal sides of an isosceles triangle as a diameter, cuts the base at its middle point.

3. Prove that if a circle is circumscribed about an isosceles triangle, the tangents drawn through the vertices form an isosceles triangle.

4. A point moves so that the angle made by the two lines that connect it with two fixed points, C and D, is always the same. Find the *locus* of the point.

5. Prove that a parallelogram inscribed in a circle is a rectangle.

6. Two lines, Fig. 248, are drawn through the point of tangency of two circles touching each other externally. If the lines meet the circles in points A, B, C, and D, prove $AB \parallel CD$.

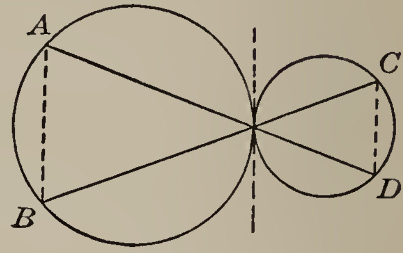


FIG. 248

7. Two circles intersect at points A and B. A variable secant through A cuts the circles in C and D. Prove that the angle CBD is constant for all positions of the secant.

8. Two circles are tangent to each other externally, and a line is drawn through the point of contact terminating in the circles. Prove that the radii to the extremities of the line are parallel.

9. Given two diagonals of a regular inscribed pentagon intersecting within it. Find the number of degrees in the angle between them.

10. In triangle ABC the altitudes BD and AE are drawn. Prove $\angle ABD = \angle AED$.

Draw a semicircle on AB as diameter.

11. One side of a triangle is fixed in length and position, and the opposite angle is given. The other two sides being variable, find the *locus* of the movable vertex.

12. Two circles are tangent externally at P. A tangent common to the two circles touches them at points A and B. Prove $\angle APB = 90°$.

13. Two circles are tangent externally. A line through the point of tangency intersects the circles at A and B, respectively. Prove that the tangents at A and B are parallel.

14. Three circles, Fig. 249, touch each other at A, B, and C. Lines AB and AC meet the third circle at E and D. Prove that E, O, and D lie in the same straight line.

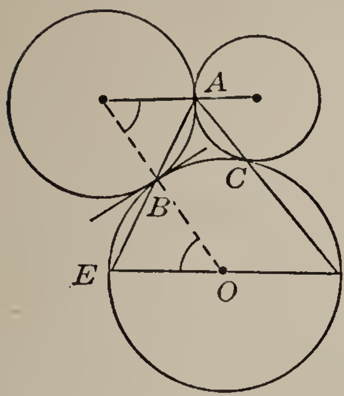


FIG. 249

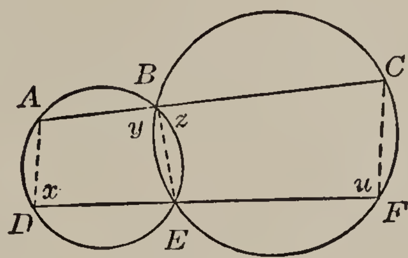


FIG. 250

15. In Fig. 250 AC and DF are drawn through the points of intersection of two circles. Prove that $AD \parallel CF$.

Prove $\quad x+y=180,\ u+z=180$

$\therefore \quad x=z,\ y=u$

$\therefore \quad x+u=180$

16. Prove that the common external tangent AB, Fig. 251, to two circles that are tangent externally is a mean proportional between the diameters of the circles.

Prove that $AE=EB=EF$ is a mean proportional between CF and FD.

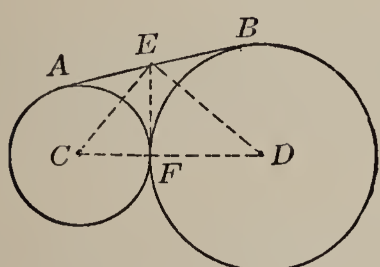


FIG. 251

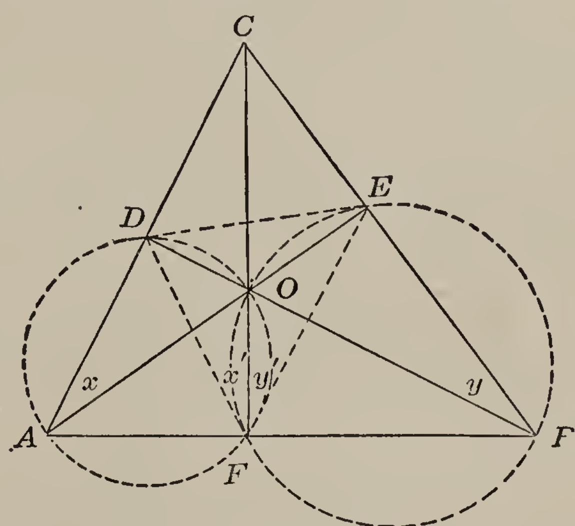


FIG. 252

17. Triangle DEF, Fig. 252, is formed by joining the feet of the altitudes of $\triangle ABC$. Prove that the altitudes bisect the angles of $\triangle DEF$.

Show that $x=y$, both being complements of $\angle ACB$.

Draw circles on AO and BO as diameters.

Show that $x'=x$ and $y'=y$.

$\therefore \quad x'=y'$.

18. Prove that a line from the center of a circle to the point of intersection of two tangents bisects the angle between the tangents.

Summary

309. The chapter has taught the meaning of the following terms:

inscribed angle
segment of a circle
inscribed and circumscribed polygons

310. The following theorems were shown to be true:

1. *A central angle is measured by the intercepted arc.*

2. *In the same or equal circles two central angles have the same ratio as the intercepted arcs.*

311. The following theorems were proved:

1. *An inscribed angle is measured by one-half the arc intercepted by the sides.*

2. *An angle formed by a tangent and a chord passing through the point of contact is measured by one-half of the intercepted arc.*

3. *If two chords intersect within a circle either angle formed is measured by one-half the sum of the intercepted arcs.*

4. *If two secants meet outside of a circle the angle formed is measured by one-half the difference of the intercepted arcs.*

5. *The angle formed by a tangent and a secant meeting outside of a circle is measured by one-half the difference of the intercepted arcs.*

6. *The angle formed by two tangents to a circle is equal to one-half the difference of the intercepted arcs.*

312. The following constructions were taught:

1. *Upon a given line-segment as a chord construct a segment of a circle in which the inscribed angles are equal to a given angle.*

2. *From a point outside of a circle to construct a tangent to the circle.*

3. *To draw the common external and internal tangents to two circles exterior to each other.*

CHAPTER XII

PROPORTIONAL LINE-SEGMENTS IN CIRCLES

313. A railroad surveyor wishes to determine the radius of a **circular railway curve** ABC, Fig. 253. He measures the chord AC, and BD, the part of the perpendicular bisector of AC intercepted by AC and arc ABC. If $AC = 200$ ft. and $BD = 6$ ft., how may the radius be determined?

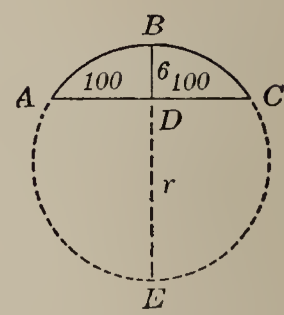


FIG. 253

If we can establish a relation between AD, DE, DC, and DB, the problem will easily be solved.

To find this relation, draw a circle, Fig. 254, and a chord AC intersecting chord BE, as at D. Measure to two decimal places the segments AD, DE, DC, and DB and compare $AD \cdot DC$ with $ED \cdot DB$.

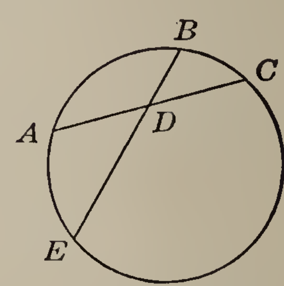


FIG. 254

Note the approximate equality of the products of the segments of each of the two chords.

To what is the difference, if any, probably due? This illustrates the following theorem:

314. Theorem: *If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other.*

State the hypothesis and the conclusion. Then prove the theorem as follows:

Proof: Draw BC and AE, Fig. 255.
Prove $\triangle ADE \backsim \triangle BDC$.

Show that $\dfrac{AD}{DB}=\dfrac{DE}{DC}$.

$\therefore \quad AD \cdot DC = DE \cdot DB.$ Why?

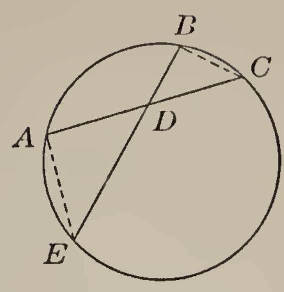


FIG. 255

EXERCISES

1. Solve the problem of § 313 by applying the theorem in § 314.

2. Using the theorem in § 314, *construct a square equal to a given rectangle.*

In a circle large enough draw a chord equal to the sum of two consecutive sides of the given rectangle.

Draw a radius to the point of division. What chord through this point is bisected at the point?

3. Show how exercise 2 may be used to find geometrically the square root of a number. Using this method, find the square roots of 6; 5; 10.

4. The segments of two intersecting chords are $x+5$ and $x-6$ of the one, and $x+2$ and $x-5$ of the other. Find x and the length of each chord.

5. A chord of a circle DC, Fig. 256, cuts the chord AB at the midpoint E. ED is 4 in. longer than EC and $AB=16$ inches. Find the lengths of ED and EC approximately to $\frac{1}{100}$ inch.

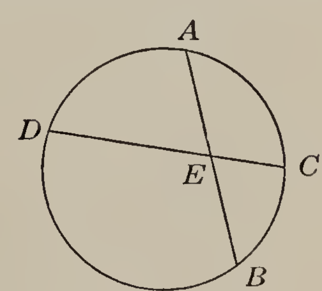


FIG. 256

6. The segments of intersecting chords are given below. Find x.

	First Chord		Second Chord	
1.	$x-4$	$x+8$	$x+3$	$x-4$
2.	$x+2$	$x+6$	$x-4$	$x+18$
‡3.	$2x+5$	$x+1$	$x+2$	$3x+2$
‡4.	$2x+2$	$3x-5$	$x+1$	$x+5$

‡**7.** The distance between two points, A and B, on a railroad curve is $2a$ ft., and the distance from the midpoint of the chord AB to the midpoint of the curve is b feet. Find the radius.

‡**8.** Find the radius of the circle in exercise 7 if $a=100, b=4$; $a=150, b=5.6$.

9. How far in one direction can a man see from the top of a mountain 2 mi. above sea-level?

Let AB, Fig. 257, represent the height of the mountain and let AD be the required distance. Assuming the diameter of the earth to be 8,000 mi., the value of AD may be found if we establish a relation between AB, AD, and AC.

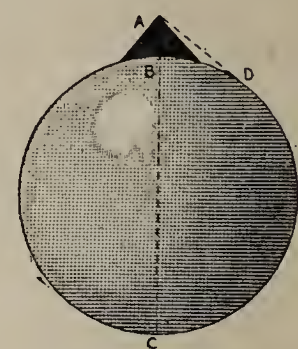


FIG. 257

The following theorem expresses this relation:

315. Theorem: *If from a point without (outside of) a circle a tangent and secant be drawn, the tangent is a mean proportional between the entire secant (to the concave arc) and its external segment.*

State the hypothesis and the conclusion.

Proof: Draw DB and DC, Fig. 258.

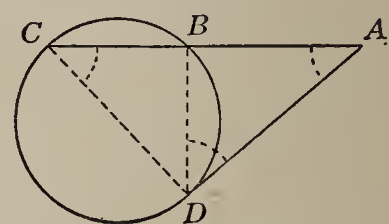


FIG. 258

Show $\triangle ABD \backsim \triangle ACD$.

$$\therefore \quad \frac{AC}{AD}=\frac{AD}{AB}. \qquad \text{Why?}$$

EXERCISES

1. Using the theorem § 315, solve exercise 9, § 314.

2. If two adjacent sides of a rectangle are given, show how the theorem in § 315 may be used to construct other equivalent rectangles.

3. Using the theorem in § 315, show how to *construct a square equal to a given rectangle.*

4. Show how the theorem in § 315 may be used to find geometrically the square root of a number.

5. Prove by means of the theorem in § 315 that the two tangents from an external point to a circle are equal.

6. A tangent and a secant are drawn from the same point outside of a circle. The secant measured to the concave arc is three times as long as the tangent, and the length of its external segment is 10 feet. Find the length of the tangent and secant.

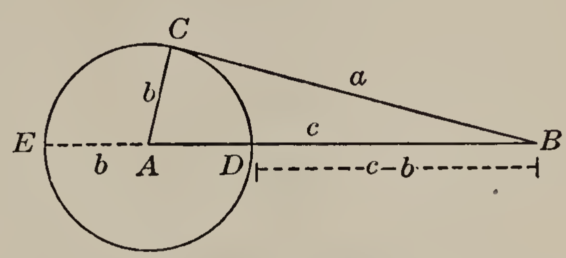


FIG. 259

7. Using Fig. 259, prove that *the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides.*

Let ABC be a right triangle having $\angle C = 90°$.
Show that $BE \cdot BD = \overline{BC}^2$.
Hence, $(c+b)(c-b) = a^2$, or, $c^2 = a^2 + b^2$.

8. *To divide a line-segment into two parts so that the longer part is a mean proportional between the whole segment and the shorter part.*

Let AB be the given line-segment, Fig. 260.

To find the point C, such that $\frac{AB}{AC} = \frac{AC}{CB}$.

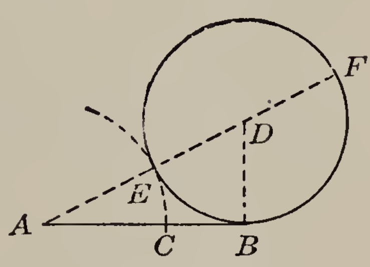


FIG. 260

Construction: Draw $BD \perp AB$ at B, making $BD = \frac{AB}{2}$.

With D as center and radius DB, draw circle D.
Draw AD cutting circle D at E and F.
On AB lay off $AC = AE$.
C is the required point.

This may be proved as follows:

Proof: $\frac{AF}{AB}=\frac{AB}{AE}$. Why?

$\therefore \frac{AF-AB}{AB}=\frac{AB-AE}{AE}$. (§ 195)

$\therefore \frac{AF-EF}{AB}=\frac{AB-AC}{AC}$. Why?

$\therefore \frac{AE}{AB}=\frac{CB}{AC}$. Why?

$\therefore \frac{AC}{AB}=\frac{CB}{AC}$. Why?

$\therefore \frac{AB}{AC}=\frac{AC}{CB}$. Why?

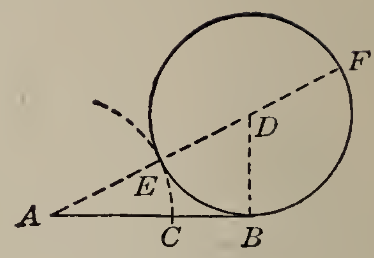


FIG. 260

Problem 7 will be used in the construction of a regular inscribed decagon (10-side), § 443.

316. Mean and extreme ratio.* A line-segment is divided into **mean and extreme ratio** if the longer part is a mean proportional between the segment and its shorter part.

* The current method of dividing a line in extreme and mean ratio is, according to an Arabian commentator, due to Heron of Alexandria. The theorem for dividing the line has been called by various names. Plato called it "The Section"; Lorentz (1781) called it "Continued Division."

Campanus (last half of the twelfth century) called continued division "a wonderful geometrical performance." Paciolo (1445–1514) gave it even higher esteem by writing an entire work dealing with problems in continued division and gave his work the title: *Divine Proportion.*

The peculiar mysticism of later times seized upon Paciolo's idea and went still beyond him. Ramus (1515–72) associated the divine trinity with the three segments of a continued division. Kepler (1571–1630) created a complete symbolism for his *sectio divina* ("divine section"). In the middle of the nineteenth century there arose a sort of amateurish natural philosophy that sought to subtilize mathematical laws in every branch of study. A kind of universal validity was fantastically ascribed to this continued division, and it was now christened "Golden Section."

This "Golden Section" was held to be not only the criterion for all metrical relations in nature, but it was also regarded as the "principle of beauty" in painting, architecture, and the plastic arts, as well. (Tropfke, *Geschichte der Elementar-Mathematik,* Band II, S. 99–103.)

317. Theorem: *If from a point without a circle two secants are drawn to the concave arc, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*

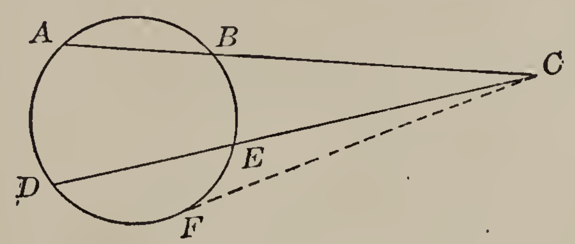


FIG. 261

Proof: From C draw CF, Fig. 261, tangent to the circle.

Show that $CA \cdot CB = \overline{CF}^2$ and $CD \cdot CE = \overline{CF}^2$.

EXERCISES

1. Two secants to the same circle from an outside point are cut by the circle into chords that are to their external segments as $\frac{5}{3}$ and $5(=\frac{5}{1})$. The first secant is 8 ft. long. Find the length of the second secant.

2. The following exercises relate to two secants from an external point as in exercise 1. Find the length of the second secant.

	Ratios of Segments of First Secant	Ratios of Segments of Second Secant	Length of First Secant
1.	5:2	3:1	28 ft.
2.	3:1	5:2	28 ft.
‡3.	4:1	5:4	625 ft.
‡4.	4:1	4:3	25 ft.
‡5.	7:2	7:3	36 ft.

‡**3.** Two lines drawn through the common points of two intersecting circles, Fig. 262, meet the circles in A, B, C, and D, E, F, respectively. Prove $AD \parallel CF$.

Show that $\frac{GA}{GC} = \frac{GD}{GF}$.

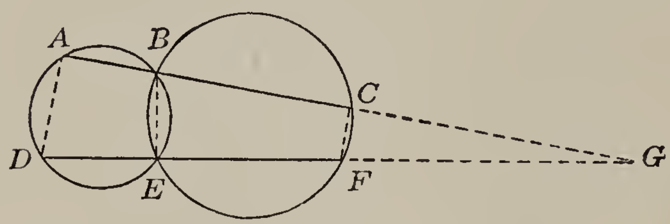


FIG. 262

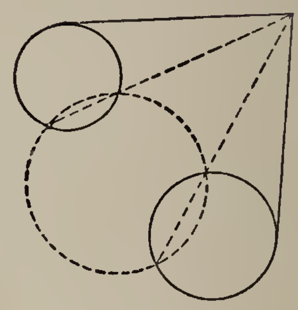


FIG. 263

4. Show how to find a point such that the tangents to two given circles are equal (see Fig. 263).

5. Determine a point A without a circle so that the sum of the length of the tangents from A to the circle shall be equal to the distance from A to the farthest point of the circle.

Summary

318. The following theorems were proved:

1. *If two chords of a circle intersect, the product of the segments of one is equal to the product of the segments of the other.*

2. *If from a point without a circle a tangent and secant be drawn the tangent is a mean proportional between the entire secant to the concave arc and the external segment.*

3. *If from a point without a circle two secants be drawn to the concave arc, the product of one secant and its external segment is equal to the product of the other secant and its external segment.*

319. The following construction was taught:
To divide a segment into mean and extreme ratio.

CHAPTER XIII

THE OPERATIONS WITH FRACTIONS. FRACTIONAL EQUATIONS

320. In future work we shall need considerable skill in working with fractions, which occur in many problems. It is the purpose of this chapter to review and extend our knowledge of the operations with fractions.

Addition and Subtraction of Fractions

321. Adding and subtracting fractions that have the same denominator.

EXERCISES

1. Show from Fig. 264 that

$$\frac{3}{8}+\frac{4}{8}=\frac{7}{8}.$$

2. Show from a figure that

$$\frac{7}{9}-\frac{4}{9}=\frac{3}{9}.$$

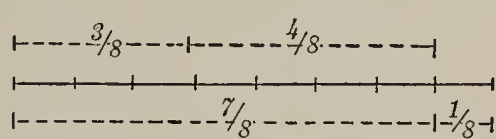


FIG. 264

3. Make a rule for adding and subtracting fractions having the same denominator and, using this rule, combine each of the following expressions into a single fraction:

1. $\frac{2}{7}+\frac{5}{7}$

2. $\frac{1}{5}+\frac{3}{5}$

3. $\frac{1}{3}+\frac{4}{3}-\frac{2}{3}$

4. $\frac{1}{13}+\frac{5}{13}-\frac{3}{13}+\frac{7}{13}$

5. $\frac{7}{9}-\frac{4}{9}-\frac{8}{9}+\frac{6}{9}$

6. $\frac{7}{2}-\frac{5}{2}-\frac{3}{2}-\frac{11}{2}$

7. $\frac{a}{c}+\frac{b}{c}$

8. $\frac{a}{c}-\frac{b}{c}$

9. $\frac{m}{x}+\frac{n}{x}+\frac{k}{x}$

10. $\frac{r}{y}-\frac{s}{y}+\frac{t}{y}-\frac{v}{y}$

11. $\frac{3}{a+b}+\frac{5}{a+b}-\frac{7}{a+b}$

12. $\frac{1}{16m}-\frac{19x}{16m}-\frac{31y}{16m}-\frac{43x}{16m}$

13. $\frac{c-ax}{4s}+\frac{c+3ax}{4s}$

14. $\frac{3a}{5b}-\frac{3a-8b}{5b}$

15. $\frac{10x}{17a}+\frac{12x}{17a}-\frac{5x}{17a}$

16. $\frac{5a-b}{x-y}-\frac{2a-3b}{x-y}$

7. $\left(\frac{61a}{17b}+\frac{48c}{17b}\right)-\left(\frac{49a}{17b}+\frac{57c}{17b}\right)$

18. $\frac{17y+18z}{27x}-\frac{22y+11z}{27x}$

19. $\left(\frac{29x}{10}-\frac{13y}{10}\right)+\left(\frac{33x}{10}-\frac{19y}{10}\right)$

20. $\frac{2x+15y}{6K}+\frac{9x-8y}{6K}-\frac{7x-3y}{6K}$

21. $\frac{23a+8b}{12x}-\frac{19a-28b}{12x}-\frac{17b-8a}{12x}$

322. Adding and subtracting fractions having different denominators.

EXERCISES

1. Reduce $\frac{2}{3}$ and $\frac{3}{5}$ to fifteenths.

2. Reduce $\frac{2}{7}$ and $\frac{3}{4}$ to fractions having the same denominator.

3. Add $\frac{5}{6}$ and $\frac{7}{8}$.

$$\frac{5}{6}+\frac{7}{8}=\frac{5\cdot 4}{6\cdot 4}+\frac{3\cdot 7}{3\cdot 8}=\frac{5\cdot 4+3\cdot 7}{6\cdot 4}=\frac{82}{6\cdot 4}=\frac{41}{24}.$$

4. Subtract $\frac{4}{7}$ from $\frac{8}{9}$.

$$\frac{8}{9}-\frac{4}{7}=\frac{7\cdot 8}{7\cdot 9}-\frac{9\cdot 4}{9\cdot 7}=\frac{7\cdot 8-9\cdot 4}{9\cdot 7}=\frac{20}{63}.$$

323. Exercises 1 to 4, § 322, show that *fractions with different denominators are added* (or *subtracted*) *by first changing the form so that all have the same denominator. The sum* (or *the difference*) *of the numerators is then written over the common denominator and the resulting fraction reduced to its lowest terms.*

EXERCISES

In the following exercises change to one fraction each of the indicated sums and differences, giving as many as you can mentally. Reduce all results to lowest terms by dividing numerator and denominator by common factors.

1. $\frac{4}{15}+\frac{7}{20}-\frac{5}{9}-\frac{3}{4}+\frac{11}{18}$

2. $\frac{2}{3}-\frac{3}{8}+2\frac{3}{4}-1\frac{7}{12}$

3. $\frac{11}{14}-\frac{3}{4}-\frac{5}{3}+\frac{16}{21}$

4. $x+\frac{5x}{22}-\frac{7x}{33}+\frac{x}{6}$

5. $2\frac{2}{3}x+4\frac{7}{10}x-8x+3\frac{5}{18}x+\frac{14}{15}x$

6. $\left(\frac{3}{4a}+\frac{7}{5a}\right)-\left(\frac{7}{6a}-\frac{5}{18a}-\frac{2}{5a}\right)$

7. $\frac{a}{cx}+\frac{b}{cy}$

8. $\frac{1}{c}-\frac{b}{ca}$

9. $\frac{x}{12ab}-\frac{z}{6b}$

10. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$

11. $\frac{a}{x^2y}+\frac{b}{xy^2}$

12. $\frac{a+b}{x}+\frac{a-b}{3x}$

13. $\frac{a-2b}{3x}-\frac{4a-5b}{5x}$

14. $\frac{1}{a^3}-\frac{1}{a^2}+\frac{1}{a}$

15. $\frac{2}{xy}-\frac{3y^2}{xy^3}+\frac{xy^5+y^3}{x^2y^6}$

16. $\frac{a}{a-1}-\frac{ab}{a(a-1)}$

‡17. $\frac{1}{x-1}-\frac{1}{2(x-1)}$

18. $\frac{1}{2x-3y}+\frac{x+y}{4x^2-6xy}$

19. $\dfrac{b}{a+2b}+\dfrac{ab}{3ad+6bd}$

20. $\dfrac{a}{x}+b$

Put $b=\dfrac{b}{1}$

21. $\dfrac{x}{y}-a$

‡**22.** $5a+\dfrac{2a}{3}$

23. $\dfrac{a}{bc}+\dfrac{b}{ac}+\dfrac{c}{ab}$

‡**24.** $\dfrac{1}{x-y}-\dfrac{1}{y}$

25. $\dfrac{5x-4y+3z}{x}+\dfrac{2x+3y-4z}{3y}$

‡**26.** $\dfrac{7a+3b-4c}{a}-\dfrac{2b+4a-3c}{b}$

27. $\dfrac{x}{x^2y}-\dfrac{4}{x^2y^2}+\dfrac{z}{xy^2}$

28. $\dfrac{2}{3}-\dfrac{5}{a+b}$

29. $\dfrac{a+b}{a}-\dfrac{a}{a-b}$

30. $\dfrac{(a+b)^2}{4ab}-1$

31. $\dfrac{x}{x^2-1}+\dfrac{x+3}{x-1}-\dfrac{x-3}{x+1}$

‡**32.** $\dfrac{3a}{c+d}-\dfrac{a}{c-d}-\dfrac{2ac}{c^2-d^2}$

33. $\dfrac{1}{b}-\dfrac{1}{a+b}$

34. $\dfrac{1}{a+b}-\dfrac{1}{a-b}$

35. $\dfrac{1}{2a+3b}-\dfrac{1}{3a+2b}$

36. $\dfrac{1}{x^2+y^2}-\dfrac{1}{x^2-y^2}$

37. $\dfrac{1}{a+b}-\dfrac{1}{a+b+c}$

38. $\dfrac{3x}{4a(x+y)}-\dfrac{5y}{3a(x+y)}-\dfrac{7z}{2ax+2ay}$

39. $\dfrac{9}{2x+4y}-\dfrac{7}{3x+6y}+\dfrac{2}{5x+10y}$

40. $\dfrac{4}{(a+1)(a+2)}-\dfrac{3}{(a+2)(a+3)}-\dfrac{2}{(a+3)(a+1)}$

41. $\dfrac{x+4}{(x+2)(x+3)}-\dfrac{x+2}{(x+3)(x+4)}+\dfrac{x}{(x+4)(x+2)}$

‡**42.** $\dfrac{x+y-z}{(x+z)(y+z)}+\dfrac{x-y+z}{(x+y)(y+z)}-\dfrac{x+y+z}{(x+y)(x+z)}$

43. $\frac{5x+7m}{4x+12m}-\frac{25x}{6x+18m}+\frac{7x}{2x+6m}$

44. $\frac{5x}{x+y}+\frac{xy}{x^2-y^2}-\frac{4y}{y-x}$

Let $y-x=-(x-y)$

‡**45.** $\frac{3a^2}{a^2-1}+\frac{2a+1}{2a-2}-\frac{2a-1}{2a+2}$

46. $\frac{2x+3}{x-6}-\frac{x^2-11x+18}{x^2-36}-\frac{x-6}{x+6}$

‡**47.** $\frac{36}{25x^2-9}-\frac{4}{5x-3}+\frac{3}{5x+3}$

48. $\frac{x+ma}{bm-x}+\frac{x-ma}{bm+x}-\frac{m^2ab}{b^2m^2-x^2}$

49. $\frac{5x-6y}{6x^2+6xy}+\frac{x+18y}{10xy-10y^2}-\frac{38x^2-2xy+15y^2}{15x^3-15xy^2}$

‡**50.** $\frac{x^2-y^2}{(x+y)^2}+\frac{x-y}{x+y}-\frac{x^2+y^2}{x^2-y^2}$

51. $\frac{3}{2x-3}-\frac{4x+12}{4x^2-9}-\frac{6}{4x^2+12x+9}$

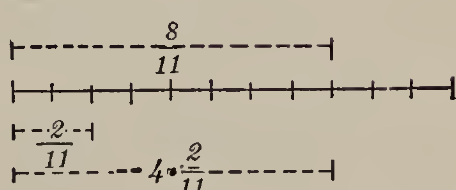
‡**52.** $\frac{2x}{x+2y}-\frac{3y}{4x+8y}-\frac{2x^2+3xy-2y^2}{x^2+4xy+4y^2}$

53. $\frac{1}{3x+2}+\frac{3}{5x-1}+\frac{2}{(3x+2)(5x-1)}$

Multiplication of Fractions

324. The number $4\cdot\frac{2}{11}$ means

$$\frac{2}{11}+\frac{2}{11}+\frac{2}{11}+\frac{2}{11}=\frac{8}{11}$$



Thus, $4\cdot\frac{2}{11}=\frac{4\cdot 2}{11}$ (See Fig. 265.)

FIG. 265

EXERCISES

1. Give the meaning of $c \cdot \frac{a}{b}$

2. Express in words the equation $c \cdot \frac{a}{b} = \frac{a \cdot c}{b}$

3. Multiply $\frac{3}{4}$ by 8; by 12; by 5; by 25; by a; by xy

4. Multiply $\frac{1}{4}$ by $\frac{1}{2}$; $\frac{1}{3}$ by $\frac{1}{5}$; $\frac{2}{3}$ by $\frac{1}{5}$; $\frac{1}{a}$ by $\frac{1}{3}$; $\frac{1}{a}$ by $\frac{2}{3}$; $\frac{1}{a}$ by $\frac{1}{b}$

($\frac{1}{2}$ by $\frac{1}{4}$, $\frac{1}{2}$ of $\frac{1}{4}$, $\frac{1}{2} \times \frac{1}{4}$, and $\frac{1}{2} \cdot \frac{1}{4}$ are all equivalent.)

5. Multiply $\frac{3}{4}$ by $\frac{5}{7}$; by $\frac{4}{5}$; by $\frac{3}{5}$; by $\frac{4}{3}$; by $\frac{a}{5}$; by $\frac{a}{3}$; by $\frac{a}{b}$

6. State the rule for multiplying two fractions and compare it with the following:

Fractions are multiplied by multiplying their numerators for the numerator of the product, and multiplying their denominators for the denominator of the product.

Since the product of fractions should generally be reduced to the simplest form, factors that are common to numerator and denominator should be *divided out* before multiplying.

7. Multiply $\frac{12}{35}$ by $\frac{15}{16}$

$$\frac{12}{35} \cdot \frac{15}{16} = \frac{12 \cdot 15}{35 \cdot 16} = \frac{3 \cdot 3}{7 \cdot 4}, \text{ etc.}$$

8. Multiply $\frac{7x}{9y}$ by $15y^2$

$$\frac{7x}{9y} \cdot 15y^2 = \frac{7x \cdot 15y^2}{9y} = \frac{7x \cdot 5y}{3}, \text{ etc.}$$

9. Multiply $\frac{7x}{2x^2-2}$ by $(2x+2)$

$$\frac{7x}{2x^2-2}(2x+2) = \frac{7x(2x+2)}{2x^2-2} = \frac{7x \cdot 2(x+1)}{2(x+1)(x-1)} = \frac{7x}{x-1}$$

10. Multiply $\frac{56x^2}{55y^2}$ by $\frac{10y}{21x}$

$$\frac{56x^2}{55y^2}\cdot\frac{10y}{21x}=\frac{56x^2\cdot 10y}{55y^2\cdot 21x}, \text{ etc.}$$

11. Multiply $\frac{3x+3y}{2x-2y}$ by $\frac{2x^2-2y^2}{3x^2+3y^2}$

$$\frac{3x+3y}{2x-2y}\cdot\frac{2x^2-2y^2}{3x^2+3y^2}=\frac{(3x+3y)(2x^2-2y^2)}{(2x-2y)(3x^2+3y^2)}$$

$$=\frac{3(x+y)2(x+y)(x-y)}{2(x-y)3(x^2+y^2)}, \text{ etc.}$$

325. Exercises 7 to 11, § 324, show that fractions may be multiplied *by writing the indicated products of the numerators over the indicated products of the denominators and then reducing the fraction obtained.*

EXERCISES

The following products are to be given in simplest form. Special effort should be made to cover this list of exercises in the minimum amount of time.

Multiply as indicated:

1. $\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{1}{8}$
2. $\frac{68}{102}\cdot\frac{95}{133}$
3. $\frac{1}{a}\cdot ac$
4. $\frac{1}{a^2}\cdot a^3$
5. $\frac{a}{b}\cdot\frac{1}{x}$
6. $\frac{a}{c}\cdot\frac{c}{d}$
7. $\frac{7xyz}{3bc}\cdot 9abc$
8. $\frac{ab}{xy}\cdot\frac{yz}{bc}$
9. $\frac{2ab}{3xy}\cdot\frac{5ax}{6by}$
10. $\frac{15ab}{16xy}\cdot\frac{24xyz}{25bc}$
11. $\frac{3ab}{4xy}\cdot\frac{5bc}{6yz}\cdot\frac{7xz}{8ac}$
12. $\frac{2a^2x}{3b^2y}\cdot\frac{6by^2}{7ax^2}\cdot\frac{5b}{4a}$

13. $\frac{a+b}{a-b} \cdot \frac{a^2-b^2}{a^2+b^2}$

14. $\frac{27x}{8y+8x} \cdot \frac{x+y}{3}$

15. $\frac{a}{a+b} \cdot \frac{b}{a-b}$

16. $\frac{6(x-y)}{5xy^2} \cdot \frac{15x^2y^3}{8(x-y)}$

17. $\frac{a^2-ab}{x^2-xy} \cdot \frac{x^2+xy}{a^2+ab}$

18. $\frac{x^2-xy}{3a+3b} \cdot \frac{(a+b)^2}{(x-y)^2}$

19. $\frac{18x^2+12xy+2y^2}{3x^3-27xy^2} \cdot \frac{5x^2y-15xy^2}{36x^2y-4y^3}$

20. $\frac{9a^2bx-9a^2by}{8cx^2u-8cx^2v} \cdot \frac{4xy^2v-4xy^2u}{15ab^2y-15ab^2x}$

21. $\frac{6ma+6mb}{35na-35nb} \cdot \frac{7as-7bs}{9ar+9br}$

22. $\frac{27pqm-27pqn}{35abx-35aby} \cdot \frac{7bpx-7bpy}{9anq-9amq}$

23. $\frac{x^2-4}{x-4} \cdot \frac{x+4}{x^2+4x+4}$

Division of Fractions

326. To divide a number by a fraction means *to find the number which multiplied by the divisor gives the dividend.*

Thus, $6 \div \frac{4}{9}$ means to find what number multiplied by $\frac{4}{9}$ will give 6. Since $\left(6 \cdot \frac{9}{4}\right) \cdot \frac{4}{9}$ gives 6, it follows that $6 \cdot \frac{9}{4}$ is the required number. Therefore $6 \div \frac{4}{9} = 6 \cdot \frac{9}{4}$.

EXERCISES

1. Using the same reasoning divide the following numbers by $\frac{5}{7}$: 3; 11; a.

2. Similarly show that $\frac{7}{8} \div \frac{5}{3} = \frac{7}{8} \cdot \frac{3}{5}$

3. Show that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$

4. Translate the equation of exercise 3 into words.

327. Reciprocals. Two numbers whose product is 1 are **reciprocals** of each other.

1. Give the reciprocals of 4, 3, $\frac{1}{3}$, $\frac{2}{5}$.

2. Compare your statement of exercise 4 with the following:

A number is divided by a fraction by **multiplying** *the dividend by the inverted divisor; that is, by* **multiplying** *the dividend by the reciprocal of the divisor.*

EXERCISES

1. Divide $25x^2$ by $\frac{15x}{y}$

$$25x^2 \div \frac{15x}{y} = 25x^2 \cdot \frac{y}{15x} = \frac{25x^2y}{15x}, \text{ etc.}$$

2. Divide $\frac{62x^2}{35p^2}$ by $\frac{93xy}{55ap}$

$$\frac{62x^2}{35p^2} \div \frac{93xy}{55ap} = \frac{62x^2}{35p^2} \cdot \frac{55ap}{93xy} = \frac{62x^2 \cdot 55ap}{35p^2 \cdot 93xy}, \text{ etc.}$$

3. Give in the simplest form: $\dfrac{\frac{ab}{4}}{\frac{b^2}{2a}}$

$$\frac{\frac{ab}{4}}{\frac{b^2}{2a}} = \frac{ab}{4} \times \frac{2a}{b^2}, \text{ etc.}$$

Find the results in the following indicated divisions and reduce them to the simplest forms:

4. $\frac{3}{5} \div \frac{2}{7}$

5. $\frac{4}{7} \div \frac{6}{11}$

6. $\frac{7}{9} \div \frac{6}{7}$

7. $\frac{3}{8} \div \frac{7}{5}$

8. $\frac{a}{b} \div \frac{c}{x}$

9. $\frac{c}{x} \div \frac{d}{f}$

10. $x \div \frac{y}{x}$

11. $ab \div \frac{xy}{z^2}$

12. $12x^3 \div \frac{16x^2}{7y}$

13. $25a^6 \div \frac{10a^2}{9x^2}$

14. $18x^2y^2z^4 \div \frac{15x^3y^3z}{ab}$

15. $(-ab) \div \frac{3d}{5ab}$

16. $\frac{16a^3b^2}{27x^5y^3} \div \frac{8a^3b}{9x^2y^4}$

17. $\frac{20x^2y^3}{21a^4c^5} \div \frac{4axy}{3a^2b^2c^2}$

18. $\frac{40a^7b^5c^6}{22m^3x^4z^5} \div \frac{35a^6b^6c^6}{88m^6xz^7}$

19. $\frac{(a-1)^2}{a+1} \div \frac{a^2-1}{a}$

20. $\frac{x^2-1}{4mn} \div \frac{x+1}{2n}$

21. $\frac{x+4}{x-4} \div \frac{x^2+16}{x^2-16}$

22. $(32x^3y^2z^2 - 40x^2y^3z^2) \div \frac{8x^3y^3z^3}{a^3b^2}$

23. $\frac{24(x-1)^2}{70(a^2-b^2)} \div \frac{30(x-1)}{28(a-b)^2(a+b)}$

24. $\frac{14x^2-7x}{12x^3+24x^2} \div \frac{2x-1}{x^2+2x}$

25. $\frac{3(x^2-4y^2)}{4(a^2-b^2)} \div \frac{x-2y}{a+b}$

26. $\frac{(a+b)^2}{a-b} \div (a^2-b^2)$

27. $\frac{3x^2-3}{x+y} \div \frac{1+x}{x^2-y^2}$

Complex Fractions

328. Complex fractions. A fraction in which either numerator or denominator, or both terms, contain fractions is a **complex fraction,** e.g., $\frac{6}{\frac{2}{3}}$, $\frac{3\frac{1}{3}}{2}$, and $\frac{\frac{3}{4}}{\frac{5}{9}}$.

The last fraction may be reduced in two ways:

(1) $\frac{\frac{3}{4}}{\frac{5}{9}} = \frac{3}{4} \times \frac{9}{5}$, etc., or (2) $\frac{\frac{3}{4}}{\frac{5}{9}} = \frac{\frac{3}{4} \cdot 4 \cdot 9}{\frac{5}{9} \cdot 4 \cdot 9} = \frac{3 \cdot 9}{5 \cdot 4}$, etc.

In the second method, numerator and denominator of the complex fraction have been multiplied by the same number, viz., the *least common multiple* (*l.c.m.*) of 4 and 9. Sometimes the use of this method is more advantageous than the first method.

EXERCISES

Reduce the following complex fractions:

1. $\dfrac{\dfrac{x-1}{x}}{x-\dfrac{1}{x}}$

$$\frac{\dfrac{x-1}{x}}{x-\dfrac{1}{x}}=\frac{\dfrac{x-1}{x}\cdot x}{\left(x-\dfrac{1}{x}\right)x}=\frac{x-1}{x^2-1}, \text{ etc.}$$

2. $\dfrac{\dfrac{2}{3}+\dfrac{3}{4}}{\dfrac{5}{6}}$

4. $\dfrac{\dfrac{4x}{5}+x}{\dfrac{4}{5}}$

6. $\dfrac{\dfrac{3x}{4}+\dfrac{7y}{15}}{\dfrac{5z}{6}}$

3. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{y}-\dfrac{1}{x}}$

5. $\dfrac{x}{\dfrac{x}{y}+1}\div\dfrac{y}{\dfrac{x}{y}-1}$

7. $\dfrac{\dfrac{x}{y}}{z}\cdot\dfrac{\dfrac{z}{y}}{x}\cdot\dfrac{y}{\dfrac{x}{z}}$

Perform the indicated operations:

8. $\left(\dfrac{m}{n}+\dfrac{p}{q}\right)\left(\dfrac{n}{m}-\dfrac{p}{q}\right)$

9. $\left(\dfrac{8x^3}{y^3}-\dfrac{y^3}{27x^3}\right)\div\left(\dfrac{2x}{y}-\dfrac{y}{3x}\right)$

10. $\left(\dfrac{2x+5y}{3x+y}-\dfrac{5x+2y}{x+3y}\right)\div\left(\dfrac{2x-3y}{2x}+\dfrac{7x-3y}{2x+6y}\right)$

Fractional Equations

329. Solve the following equations:

1. $\frac{x}{3}+\frac{x}{4}=\frac{x}{6}+4$

Multiplying each term by the least common multiple of the denominators, i.e., by 12,

$$\frac{12x}{3}+\frac{12x}{4}=\frac{12x}{6}+12\cdot 4$$

Reducing, we have $4x+3x=2x+48$, which is easily solved.

2. $\frac{15x}{2}-\frac{x}{4}=2\frac{1}{3}$

3. $5x-\frac{3-2x}{2}=2x+2\frac{1}{2}$

4. $.05(20x-3.2)=.8(4x+.12)-11.256$

Clear of fractions by multiplying *every term* by 100.

5. $1.4x-1.61-\frac{.21x+.012}{.8}=1.3x$

6. $\frac{1-2x}{.25}-\frac{2x-.5}{12.5}+\frac{2x-\frac{1}{3}}{5}=\frac{6.35-.5x}{3}$

7. $5r-13=\frac{2r-5}{4}+\frac{r+4}{4}$

8. $\frac{5r^2-3r+12}{7}=10\frac{(3r+1)(r-10)}{42}$

‡**9.** $\frac{\frac{r}{3}+\frac{1}{2}}{5}-\frac{r-\frac{r}{3}}{2}=-\frac{3}{2}$

‡**10.** $15r-\frac{6r+1}{2}-\frac{r-1}{3}=-6$

11. $2r-\frac{6r^2-2r+1}{6}+\frac{2r^2-3r}{2}=-1$

12. $\frac{10x}{2x-2}-\frac{10x}{3x-3}=4$

$l.c.m.=2 \cdot 3(x-1)$

13. $\frac{x-4}{x-2}=\frac{x-1}{x+3}$

Since exercises 13–20 are proportions, the theorem that the product of the means is equal to the product of the extremes may be used to clear them of fractions.

14. $\frac{3x-1}{4x+2}=\frac{3x+1}{4x+5}$

‡15. $\frac{x+3}{x-1}=\frac{x-1}{x-3}$

‡16. $\frac{x+2}{x+4}=\frac{x+8}{x+4}$

17. $\frac{x+1.1}{x-1.4}=\frac{x-1.7}{x+.3}$

18. $\frac{10x-2}{10x+6}=\frac{4x+5}{4x+\frac{1}{2}}$

19. $\frac{x+5}{x-3}=\frac{5x-19}{x-3}$

20. $\frac{4-x}{1-x}=\frac{15-x}{3-x}$

‡21. $\frac{5x-3}{5x+3}=\frac{x+1}{x+3}$

330. Summary of the laws of fractions.

State the laws that the following expressions formulate:

1. $\frac{m}{d}+\frac{n}{d}=\frac{m+n}{d}$

2. $\frac{m}{d}-\frac{n}{d}=\frac{m-n}{d}$

3. $\frac{m}{d}\pm\frac{n}{d}=\frac{m\pm n}{d}$

4. $\frac{a}{b}\pm\frac{c}{d}=\frac{a \cdot d\pm c \cdot b}{b \cdot d}$

5. $\frac{x}{y} \cdot \frac{y}{x}=1$

6. $\frac{a}{b}=\frac{1}{\frac{b}{a}}$

7. $n \cdot \frac{x}{y}=\frac{n \cdot x}{y}$

8. $\frac{a}{b} \cdot \frac{x}{y}=\frac{a \cdot x}{b \cdot y}$

9. $\frac{a}{b}\div n=\frac{a\div n}{b}$

10. $\frac{a}{b}\div n=\frac{a}{b \cdot n}$

11. $n\div\frac{a}{b}=n \cdot \frac{b}{a}$

12. $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}$

Problems Leading to Fractional Equations

331. Motion problems. Solve the following problems:

1. The report of a cannon shot was heard 3.4 seconds after the flash. If sound travels 1,080 ft. per second, how far away was the cannon?

The time it takes light to travel 1,080 ft. is too small to be considered in the problem.

2. In one year light travels a distance 63,000 times as great as the distance of the earth from the sun. Assuming the distance of the earth from the Pole-star to be 2,898,000 times as great as the distance of the earth from the sun, how long does it take the light of the Pole-star to reach the earth?

3. Two trains go from P to Q on different routes, one of which is 15 mi. longer than the other. The train on the shorter route takes 6 hours, and the train on the longer, running 10 mi. less per hour, takes $8\frac{1}{2}$ hours. Find the length of each route.

For the train on the short route: For the train on the longer route:

$$\begin{cases} d=x \\ t=6 \\ \therefore\ r=\dfrac{x}{6} \end{cases} \qquad \begin{cases} d=x+15 \\ t=8\frac{1}{2} \\ \therefore\ r=\dfrac{x+15}{8\frac{1}{2}} \end{cases}$$

$$\therefore\quad \frac{x}{6}-10=\frac{x+15}{8\frac{1}{2}}$$

4. A robber attempted to escape in an automobile going at the rate of 28 mi. an hour. Fifteen minutes later he was followed by the police in an automobile going at the rate of 32 mi. an hour. How soon did they overtake the robber?

5. The distance from A to B is 100 mi. A train leaving A at a certain rate, meets with an accident 20 mi. from B, reducing the speed one-half and causing it to reach B 1 hour late. What was the rate per hour before the accident?

To solve the problem find a relation between the regular time, the time before the accident, and the time after the accident.

6. A man walks beside a railway at the rate of 4 mi. per hour. A train 208 yd. long, running 30 mi. per hour, overtakes him. How long will it take the train to pass the man?

7. Two boys are running along a circular path whose length is 100 feet. When they run in opposite directions, they meet every eight seconds, and when they run in the same direction they are together every 25 seconds. What are their rates?

332. Percentage and interest problems.

Solve the following problems:

1. A property owner uses 8 per cent of the money received for rent to pay the taxes. His taxes having been raised to 11 per cent, what per cent must he raise the rent in order to keep his income the same as it was before?

Denoting by A the amount received for rent, show that his income is $\frac{92}{100}A$ under the old tax rate and $\frac{89}{100}(A+\frac{x}{100}A)$ under the advanced rate.

Thus, $\frac{89}{100}(A+\frac{x}{100}A)=\frac{92}{100}A.$

Divide each term by A and solve the equation for x.

2. A contractor needs 40,500 bricks for a building. His experience has shown that usually 3.5 per cent are spoiled. How many bricks must he order?

3. A man paid $6,200 for his house. His tax is $77, his coal bill is $72, and he spends $50 a year for repairs. If money is worth 5 per cent, how much is his monthly rental?

4. A man invested $3,000, part at 5 per cent and the remainder at 6 per cent, obtaining an income of $157 per year. How much has he invested at each rate?

333. Loss of weight problems.

If a body weighing 2 lb. in the air is suspended by a cord and weighed when immersed in water, it will weigh less than 2 pounds. It can be shown that as the weight of the water the body displaces the loss of weight is the same.

1. A mass of gold weighs 97 oz. in air and 92 oz. in water, and a mass of silver weighs 21 oz. in air and 19 oz. in water. How many ounces of gold and of silver are there in a mass of gold and silver that weighs 320 oz. in air and 298 oz. in water?

Solution: (1) Let x be the number of ounces of gold in the mass.
(2) Then $320-x$ is the number of ounces of silver.
(3) Since 97 ounces of gold lose 5 ounces, 1 ounce loses $\frac{5}{97}$ of an ounce.
(4) Since 21 ounces of silver lose 3 ounces, 1 ounce loses $\frac{3}{21}$ of an ounce.
(5) Therefore the loss of x ounces of gold is $\frac{5x}{97}$ ounces, and the loss of $320-x$ ounces of silver is $\frac{3(320-x)}{21}$.
(6) Then $\frac{5x}{97}+\frac{3(320-x)}{21}=22$ is the loss of the whole mass.
(7) The root of this equation is the required number.

2. A pound of lead loses $\frac{5}{57}$ of a pound, and a pound of iron loses $\frac{2}{15}$ of a pound when weighed in water. How many pounds of lead and of iron are there in a mass of lead and iron that weighs 159 lb. in air and 143 lb. in water?

3. If 38 oz. of gold lose 2 oz. when weighed in water and if 30 oz. of silver lose 3 oz. when weighed in water, what is the amount of each in a mass of gold and silver that weighs 106 oz. in air and 99 oz. in water?

‡**4.** If $19\frac{1}{4}$ lb. of gold and $10\frac{1}{2}$ lb. of silver each lose one pound when weighed in water, how much gold and silver is contained in a mass of gold and silver that weighs 20 lb. in air and $18\frac{3}{4}$ lb. in water?

Trigonometric Relations

334. The exercises below give practice in the operations with fractions.

Prove the following trigonometric identities:

1. $1+\tan^2 A \equiv \dfrac{1}{\cos^2 A}$

Analysis: Assume

$$1+\tan^2 A = \frac{1}{\cos^2 A}$$

Then $$1+\frac{a^2}{b^2}=\frac{1}{\frac{b^2}{c^2}} \quad \text{(See Fig. 266.)}$$

$$\therefore \quad \frac{b^2+a^2}{b^2}=\frac{c^2}{b^2} \qquad \text{Why?}$$

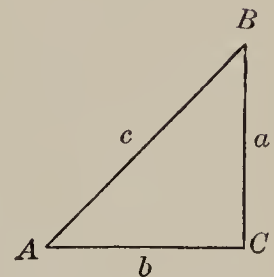


FIG. 266

Substituting for b^2+a^2 its equal c^2,

$$\frac{c^2}{b^2}=\frac{c^2}{b^2}, \text{ which is an identity.}$$

Starting from the statement $\frac{c^2}{b^2}\equiv\frac{c^2}{b^2}$, by reversing the steps of the analysis, we may now prove that $1+\tan^2 A \equiv \dfrac{1}{\cos^2 A}$.

In exercises 2–18 reversing the steps involves no particular difficulties. That part of the proof may therefore be omitted.

2. $\cos A \equiv \dfrac{\sin A}{\tan A}$

3. $\tan A \cdot \dfrac{\cos A}{\sin A} \equiv 1$

4. $\dfrac{1}{\cos A} \cdot \dfrac{1}{\tan A} \equiv \dfrac{1}{\sin A}$

5. $\cos A \cdot \tan A \cdot \dfrac{1}{\sin A} \equiv 1$

6. $\sin A \cdot \dfrac{1}{\tan A} \equiv \cos A$

‡**7.** $\dfrac{1}{\tan^2 A} \equiv \dfrac{1}{\sin^2 A} - 1$

8. $\dfrac{\sin A + \cos A}{1+\tan A} = \cos A$

‡**9.** $\dfrac{1}{\sin A} - \sin A \equiv \cos A \cdot \dfrac{1}{\tan A}$

10. $\dfrac{1}{\cos A} \cdot \dfrac{1}{\sin A} \equiv \tan A + \dfrac{1}{\tan A}$

11. $\sqrt{1-\sin^2 A} \equiv \sin A \cdot \dfrac{1}{\tan A}$

Assume $\sqrt{1-\sin^2 A} = \sin A \cdot \dfrac{1}{\tan A}$

Then $$\sqrt{1-\frac{a^2}{c^2}} = \frac{\cancel{a}}{c} \cdot \frac{b}{\cancel{a}}$$

$$\therefore \quad \sqrt{\frac{c^2-a^2}{c^2}} = \frac{b}{c}$$

$$\therefore \quad \sqrt{\frac{b^2}{c^2}} = \frac{b}{c}$$

or $$\frac{b}{c} = \frac{b}{c}$$

12. $\tan A \cdot \cos A \equiv \sqrt{1-\cos^2 A}$

‡**13.** $1+\dfrac{1}{\tan^2 A} \equiv \dfrac{1}{\sin^2 A}$

14. $\left(1+\dfrac{1}{\tan^2 A}\right)(1-\cos^2 A) \equiv 1$

15. $(1+\tan^2 A)(1-\sin^2 A) \equiv 1$

‡**16.** $\dfrac{1}{\cos A} + \tan A \equiv \dfrac{\cos A}{1-\tan A \cos A}$

17. $\dfrac{1}{\cos A} - \sin A \cdot \tan A \equiv \cos A$

‡**18.** $$\left[\frac{1}{\cos A} + \frac{1}{\sin A}\right]\left[1-\frac{1}{\tan A}\right] \equiv \left[\frac{1}{\cos A} - \frac{1}{\sin A}\right]\left[1+\frac{1}{\tan A}\right]$$

Summary

335. The chapter has reviewed and extended the laws of the operations with fractions, i.e.:

1. Addition and subtraction of fractions having the same denominator.

2. Addition and subtraction of fractions having different denominators.

3. Multiplication of fractions.

4. Division of fractions.

5. Reduction of complex fractions.

336. Fractional equations are solved by multiplying each term by the least common multiple of the denominator, and then reducing each term to the simplest form.

337. A number of trigonometric identities were proved.

CHAPTER XIV

INEQUALITIES

338. Review and extension of the axioms and theorems of inequality previously established.

1. *A line-segment, or an angle, is greater than any part of itself* (§ 6).

This axiom is to be applied only when the magnitudes and their parts are all *positive*. For, let the segment AC, Fig. 267, be considered *positive*. Then CB is *negative* and $AC+CB=AB$. For this reason AC and CB may be called *parts* of AB. One of these parts, AC, is greater in magnitude than AB.

FIG. 267

2. *The sums obtained by adding unequals to equals are unequal in the* **same** *order as the unequal addends* (§ 10).

For example,	$8>3$
and	$4=4$
Hence,	$12>7$

3. *The sums obtained by adding unequals to unequals in the same order are unequal in the* **same** *order* (§ 11).

For example,	$9>2$
and	$4>3$
Hence,	$13>5$

4. *If three magnitudes are so related that the first is greater than the second and the second greater than the third, the first is greater than the third.*

For, if $a>b$ and $b>c$, then $a+b>b+c$.
Subtracting b from both sides, $a>c$.
In obtaining the last inequality the following axiom is used:

5. *If equals are subtracted from unequals, the remainders are unequal in the* **same** *order as the unequal minuends.*

For example, $10>4$
and $3=3$
Hence, $7>1$

6. *The differences obtained by subtracting unequals from equals are unequal in the order* **opposite** *to that of the subtrahend* (§ 12).

For example, $12=12$
and $8>2$
Hence, $4<10$

7. *The products obtained by multiplying unequals by positive equals are unequal in the* **same** *order as the multiplicands.*

For example, $10<15$
$2=2$
$\therefore\ 20<30$

8. *The products obtained by multiplying unequals by negative equals are unequal in the order* **opposite** *to that of the multiplicands.*

For example, $12<15$
$-3=-3$
$-36>-45$

9. *The quotients obtained by dividing unequals by positive equals are unequal in the* **same** *order as the dividends.*

For example, $20<30$
$2=2$
$10<15$

10. *The quotients obtained by dividing unequals by negative equals are unequal in the order* **opposite** *to that of the dividends.*

For example,

$$\begin{array}{r} 50 > 40 \\ -\ 2 = -\ 2 \\ \hline -25 < -20 \end{array}$$

11. *The shortest distance between two points is the straight line-segment joining the points* (§ 3).

The following theorems express inequalities:

12. *The sum of two sides of a triangle is greater than the third side, and their arithmetical difference is less than the third side.*

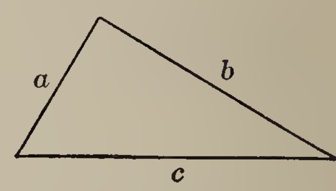


FIG. 268

The first part of this theorem follows directly from 11.

The second part follows from 5. For, let $a+b>c$, Fig. 268.

Then, $c<a+b$.

Subtracting a from both sides $c-a<b$.

Similarly, show that $b-a<c$; that $c-b<a$.

13. *The shortest distance from a point to a line is the perpendicular from the point to the line* (§ 35). Prove.

14. *If two sides of a triangle are unequal, the angles opposite them are unequal, the greater angle lying opposite the greater side* (§ 33). Prove.

15. *If two angles of a triangle are unequal, the sides opposite them are unequal, the greater side lying opposite the greater angle* (§ 34). Prove.

16. *Any point not on the perpendicular bisector of a line-segment is unequally distant from the endpoints* (§ 71). Prove.

17. *The perpendicular bisector of a line-segment is the locus of all points equidistant from the endpoints of the segment* (§ 71). Prove.

18. *Any point not on the bisector of an angle is not equidistant from the sides of the angle* (§ 72). Prove.

19. *The bisector of an angle is the locus of all points within the angle equidistant from the sides* (§ 72). Prove.

Solution of problems by means of inequalities

339. Many problems lead to relations expressed as inequalities. These *inequalities* may then be solved by using the **axioms of inequality** in the same way as *equations* are solved by using the **axioms of equality.** The following exercises will show the solution of problems by means of inequalities:

EXERCISES

1. Express relations which hold between the sides of the triangle in Fig. 269.

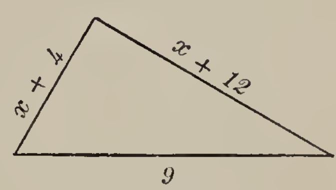


FIG. 269

2. For what values of x do the relations in exercise 1 hold?

$$x+4+9>x+12. \qquad \text{Why?}$$
$$\therefore\ 13>12. \qquad \text{Why?}$$

$\therefore$ $x=$ *any value,* i.e., any value of x will satisfy the inequality.

$$9+x+12>x+4. \qquad \text{Why?}$$
$$\therefore\ 21>4. \qquad \text{Why?}$$

$\therefore$ $x=$ *any value.*

$$x+12+x+4>9$$
$$2x+16>9$$
$$2x>-7$$
$$x>-3\tfrac{1}{2}$$

$\therefore$ any value of x *greater than* $-3\frac{1}{2}$ will satisfy all three inequalities. Why?

3. For what values of x may the following expressions represent the lengths of the sides a, b, and c, of a triangle?

a	$x-5$	$2x+3$	$x+5$	7	$2x$
b	$x+7$	$2x+2$	$8-x$	$x-3$	5
c	16	21	1	9	$4x-7$

4. Two sides of a triangle are 9 and 24 inches. Between what limits must the third side be?

Let x denote the third side.
Then $x+9>24$. Why?
$x+24>9$. Why?
$9+24>x$. Why?
Find the values of x satisfying all three inequalities.

5. There are \$50 in the treasury of a club. The club wants to buy furniture costing between \$80 and \$90. How much should be raised?

Let x be the number of dollars to be raised, etc.

‡**6.** A twentieth-century limited train wants to make the distance between New York and Chicago (1,000 miles approximately) in less than 20 hours. During the first five hours it goes at the rate of 45 miles per hour. During the next 7 hours it goes at the rate of 57 miles per hour. How fast should it go thereafter to cover the distance within the desired time?

‡**7.** A's record average speed on a 2-mile run is 6 miles per hour, and B's is $5\frac{3}{4}$ miles. How many feet can A afford to give B as a handicap?

8. Prove that *the diameter of a circle is longer than any other chord of that circle.*

Show that $AB=CO+OD>CD$, Fig. 270.

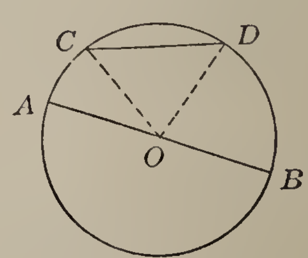


FIG. 270

9. Prove the following:

(*a*) The distance between the centers of two circles which lie *entirely outside* of each other is greater than the sum of the radii, Fig. 271.

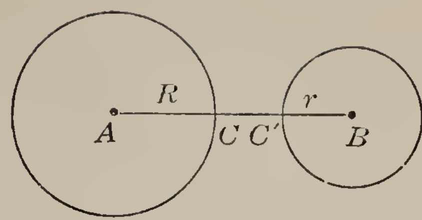


Fig. 271

(*b*) The distance between the centers of two circles touching each other *externally* is equal to the sum of the radii, Fig. 272.

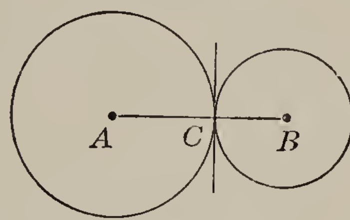


Fig. 272

(*c*) The distance between the centers of two *intersecting circles* is less than the sum of the radii, but greater than the difference, Fig. 273.

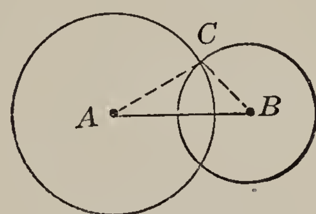


Fig. 273

(*d*) The distance between the centers of two circles touching each other *internally* is equal to the difference of the radii, Fig. 274.

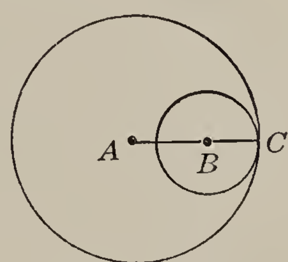


Fig. 274

(*e*) The distance between the centers of two circles, one of which lies *entirely within* the other, is less than the difference of the radii, Fig. 275.

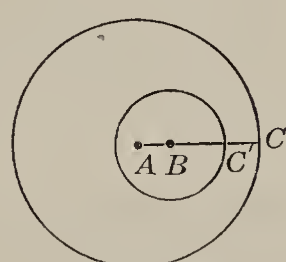


Fig. 275

10. Prove that *an exterior angle of a triangle is greater than either of the remote interior angles.*

Use § 26.

11. In Fig. 276 prove that x is greater than y.

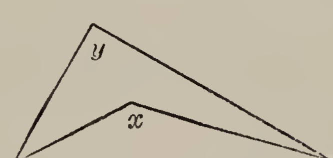


Fig. 276

12. Prove that the sum of the diagonals of a quadrilateral is less than the perimeter, but greater than the semi-perimeter.

13. The lengths of the diagonals, Fig. 277, are denoted by $5x+4$ and $4x-31$. By means of the relations in exercise 12, determine the integral values of x.

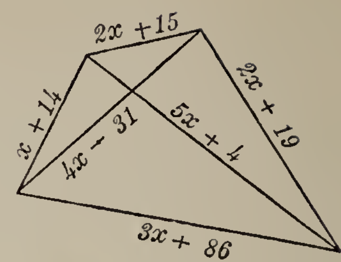


FIG. 277

14. The line joining a vertex of a triangle to the midpoint of the opposite side is a *median* of the triangle.

Prove that the median to one side of a triangle is less than one-half of the sum of the other two sides.

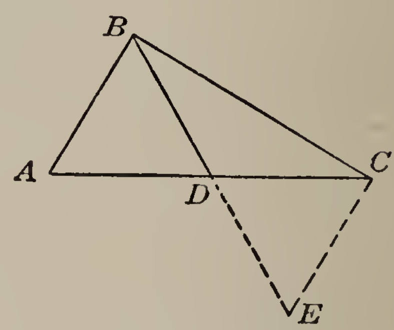


FIG. 278

In Fig. 278 extend BD making $DE=BD$ and draw EC.

Then $BE<BC+CE$.

Prove $CE=BA$.

15. Two towns are located at A and B respectively, Fig. 279. Determine a point P on the edge of a river, XY, so that the distances from P to A and B may be piped with the least amount of pipe.

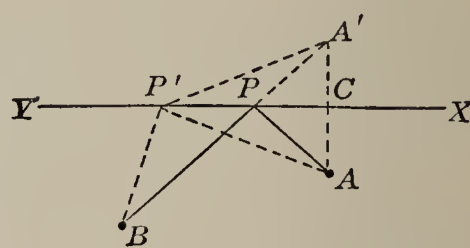


FIG. 279

Draw $AA'\perp XY$ and make $CA'=CA$.

Draw BA' meeting XY at P.

P is the required point.

Show that $BP'A>BPA$, P' being any other point on the edge of the river.

340. Theorem: *If two oblique line-segments drawn to a line from a point on the perpendicular to the line have unequal projections, the oblique line-segments are unequal.*

Let $EA\perp BC$, and $AF>AD$, Fig. 280.

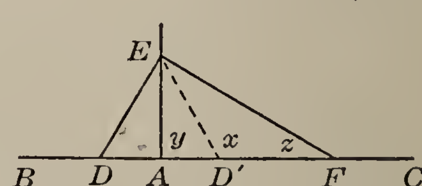


FIG. 280

Prove that $EF>ED$.

Proof: Lay off $AD' = AD$ and draw ED'.

Then $x > y$

Since $y = 90°$, $\therefore\ x > 90°$

$\therefore\ z < 90°$

$\therefore\ x > z$

$\therefore\ EF > ED'$

and $EF > ED$

341. Theorem: *Two unequal oblique line-segments drawn to a line from a point on a perpendicular to the line have unequal projections.*

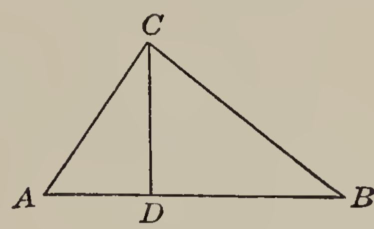


FIG. 281

Given $CB > CA$, $CD \perp AB$, Fig. 281.

To prove that $DB > DA$.

Proof (indirect method):

1. Assume $DB = DA$, then $CB = CA$. Why?
This contradicts the hypothesis.
$\therefore$ The assumption is wrong and $DB \neq DA$.*

2. Assume $DB < DA$.
Then show that $CB < CA$.
This is impossible. Why? and DB is not less than DA.

3. Since DB is not equal to DA and not less than DA, it follows that $DB > DA$.

In some of the following theorems the points and lines do not all lie in the same plane. Before studying their proofs select points and lines in the classroom to illustrate the figure given in the textbook. If the practice is followed until it becomes a habit it will add greatly to clearness of thought.

*The symbol, $\neq$, means "is not equal to."

342. Theorem: *Prove that the perpendicular is the shortest line from a point to a plane.*

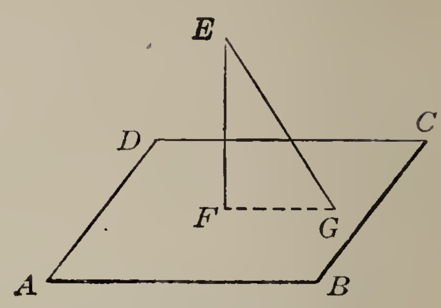


Fig. 282

Let $ABCD$, Fig. 282, represent a plane and E be any point not in the plane.

Let EF be perpendicular to $ABCD$ and G be any other point than F in $ABCD$.

Draw EG.

To prove that $EF < EG$.

Proof: In triangle EFG we have $EF \perp FG$.

For, a line perpendicular to a plane is perpendicular to any line in the plane passing through the foot of the perpendicular.

$$\therefore \quad EF < EG \text{ (§ 35).}$$

343. Distance from a point to a plane. The length of the perpendicular from a point to a plane is the **distance** from the point to the plane.

344. Theorem: *Oblique lines drawn from a point to a plane, meeting the plane at points equidistant from the foot of the perpendicular, are equal.*

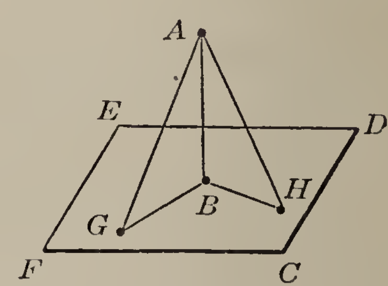


Fig. 283

Given $AB \perp CDEF$ and A any point on AB, Fig. 283.

$BG = BH$.*

To prove $AG = AH$

Proof: Show that

$\triangle ABG \cong \triangle ABH$.

* BG and BH are the *projections* of AG and AH upon the plane $CDEF$ (see §§ 353–355).

345. Theorem: *Oblique lines drawn from a point to a plane meeting the plane at points unequally distant from the foot of the perpendicular are unequal, the more remote being the greater.*

Given plane $CDEF$, $AB \perp CDEF$ and $BH > BG$, Fig. 284.

To prove $AH > AG$.

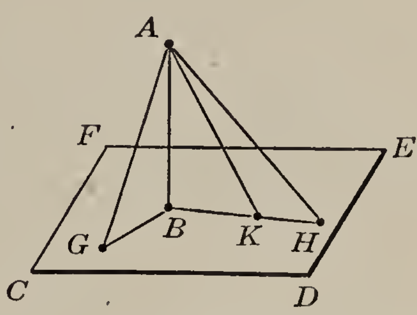


FIG. 284

Proof: Lay off BG on BH, making $BK = BG$.

Then $AK = AG$. Why?

$AH > AK$. § 340.

$AH > AG$. Why?

346. Theorem: *Equal oblique lines drawn from a point to a plane meet the plane at points equidistant from the foot of the perpendicular.* Prove.

347. Theorem: *Of two unequal oblique lines drawn from a point to a plane the greater meets the plane at the greater distance from the foot of the perpendicular.*

Let $AH > AG$, Fig. 284.

Lay off $BK = BG$. Then $AK = AG$ (§ 346).

$\therefore$ $AH > AK$, by substitution.

$\therefore$ $BH > BK$ (§ 341).

$\therefore$ $BH > BG$. Why?

EXERCISE

Given a point A on a perpendicular to a plane. Find the locus of points in that plane having a given distance from A.

348. Theorem: *If from a point inside a triangle, line-segments are drawn to the endpoints of one side, the sum of these line-segments is less than the sum of the other two sides.*

Given $\triangle ABC$, Fig. 285, and a point P inside the triangle.

To prove that

$$AP+PC<AB+BC.$$

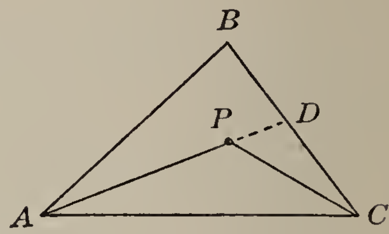


FIG. 285

Proof: Prolong AP until it intersects BC at some point, as D.

We now have:

$$AP+PD<AB+BD. \qquad \text{Why?}$$

$$PC<PD+DC. \qquad \text{Why?}$$

Adding, $\quad AP+PD+PC<AB+BD+PD+DC.$

Subtracting PD from both sides

$$AP+PC<AB+BD+DC.$$

$$\therefore \quad AP+PC<AB+BC. \qquad \text{Why?}$$

EXERCISES

1. Prove that the sum of the three line-segments joining a point inside of a triangle with the vertices, is less than the perimeter of the triangle but greater than its semi-perimeter, Fig. 286.

Use § 338, exercise 12, and § 348.

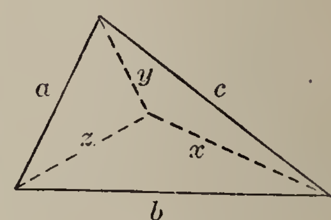


FIG. 286

2. Determine between what limits x must lie in Figs. 287 and 288.

Apply exercise 1.

What values could x have if it were required to be an integer?

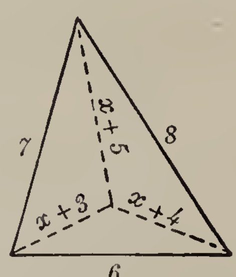


FIG. 287

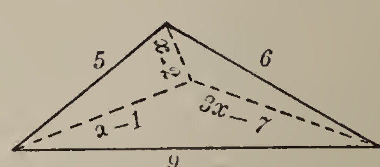


FIG. 288

3. *Construct a triangle ABC, the sides a and b and the angle A, opposite one of them, being given,* Fig. 289.

Construction: On a line of indefinite length, as AB, construct an angle equal to angle A. On one side of this angle, as AC, lay off $AD=b$.

With D as center and radius a draw a circle. This circle will either intersect AB in two points, or it will touch AB, or it will not meet AB at all.

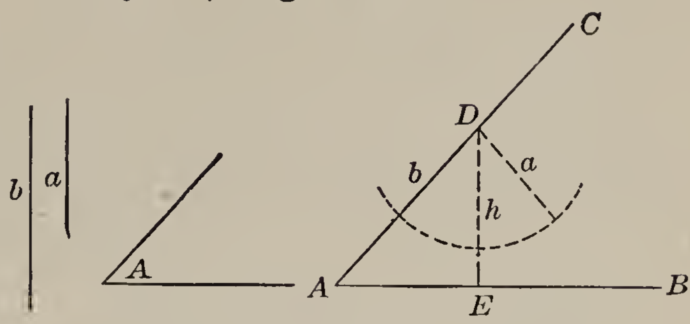


FIG. 289

Discussion: We will consider the case where $\angle A$ is acute.

1. If $a<h$, the length of the perpendicular from D to AB, the circle will not meet AB, and there is *no* triangle satisfying the given conditions, i.e., *no* solution of the problem exists, Fig. 289.

2. If $a=h$ the circle will touch AB and there is *one* solution of the problem, i.e., $\triangle ADE$, Fig. 290.

3. If $a>h$, and $<b$, the circle will intersect AB in *two* points F and F'. There are *two* solutions, i.e., $\triangle ADF$ and $\triangle ADF'$, Fig. 291.

4. If a is equal to b the circle will meet AB in A and in another point, F. There is *one* solution, i.e., $\triangle ADF$, Fig. 292.

5. If $a>b$, the circle will meet AB in two points F and F', but only $\triangle ADF$ satisfies the conditions of the problem, Fig. 293.

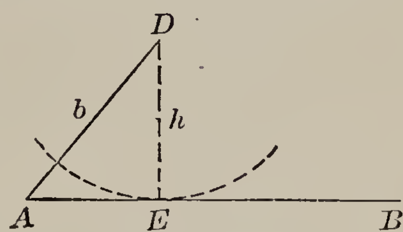


FIG. 290

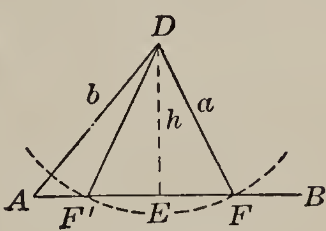


FIG. 291

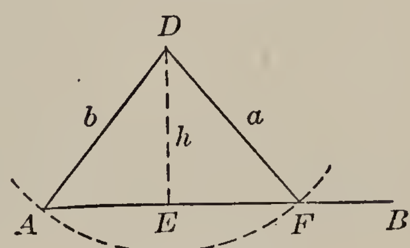


FIG. 292

4. Express trigonometrically the length of the perpendicular, h, in terms of b and A, i.e., show that $h=b \sin A$.

Find $\sin A$ from the right triangle ADE (see §248).

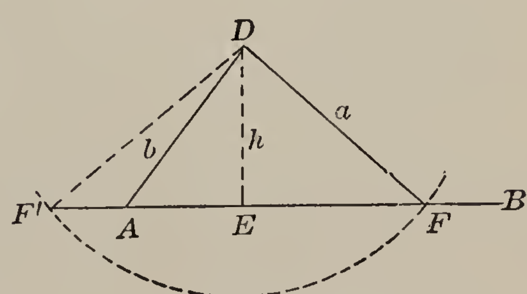


FIG. 293

349. Theorem: *In the same circle or in equal circles, unequal chords are unequally distant from the center of the circle, the shorter chord lying at the greater distance; and, conversely, chords unequally distant from the center are unequal, the chord at the greater distance being the shorter chord.*

Given $\odot P = \odot Q$, Fig. 294.
Chord $AB >$ chord DE, $PP' \perp AB$, $QQ' \perp DE$.
To prove $PP' < QQ'$.

Proof: Place $\odot Q$ on $\odot P$, so that Q falls on P, D on B, and chord DE in the position BC; then Q' will take a position as at Q''.

Draw $P'Q''$.

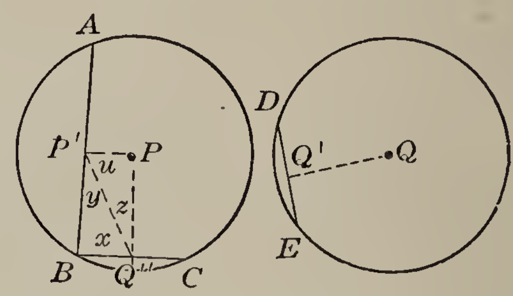


FIG. 294

	$AB > DE$.	Why?
$\therefore$	$AB > BC$.	
	$PP' \perp AB$.	Why?
	$P'B = \frac{1}{2}AB$.	Why?
	$QQ' \perp DE$.	Why?
	$PQ'' \perp BC$.	
	$BQ'' = \frac{1}{2}BC$.	Why?
Then	$P'B > BQ''$.	Why?
$\therefore$	$x > y$.	Why?
Since	$x + z = y + u$.	Why?
$\therefore$	$z < u$.	Why?
$\therefore$	$PP' < PQ''$.	Why?
$\therefore$	$PP' < QQ'$.	Why?

Conversely, **given** $\odot P = \odot Q$, Fig. 294, $PP' \perp AB$; $QQ' \perp DE$; $PP' < QQ'$.

To prove that $AB > DE$.

Proof: Proceed with the steps of the foregoing demonstration in the opposite order.

EXERCISES

1. Triangles are to be constructed with the following parts:

1. $b = 145$	$a = 178$	$A = 41°$
2. $a = 6$	$b = 3.5$	$A = 63°$
3. $a = 140$	$b = 170$	$A = 40°$
4. $b = 28$	$a = 23$	$A = 65°$

Without constructing the triangle, tell the number of solutions in each case by comparing the lengths of a, b, and h, as found by the formulas in exercise 4, § 348.

‡**2.** Construct the triangles in exercise 1 and see if the constructions verify the results obtained from the formula.

3. Discuss exercise 3, § 348, for angle A a right angle; for angle A an obtuse angle.

4. Prove that, in the same circle, a side of a regular inscribed decagon is less than a side of a regular inscribed pentagon, but that the side of the decagon is greater than half the side of the regular pentagon.

5. Show that the greater the number of sides of a regular inscribed polygon, the shorter is the length of one of its sides.

6. Prove that the distance from the center of a circle to a side of a regular inscribed polygon is greater, the greater the number of sides of the polygon.

350. Theorem: *If two sides of one triangle are equal to two sides of another triangle but the angle included between the two sides in the first is greater than the angle included by the corresponding sides in the second; then the third side in the first triangle is greater than the third side in the second.*

Given $\triangle ABC$ and DEF, Fig. 295.

$$AB = DE; \quad BC = EF; \quad \angle B > \angle E.$$

To prove that $\quad AC > DF.$

Proof: Place $\triangle DEF$ on $\triangle ABC$ so that DE falls on AB, D on A, E on B, and EF on the same side of AB as BC. Then EF must fall within $\angle ABC$. Why?

For the position of F there are three possibilities.

I. F falls *below* AC, as at F', Fig. 295.

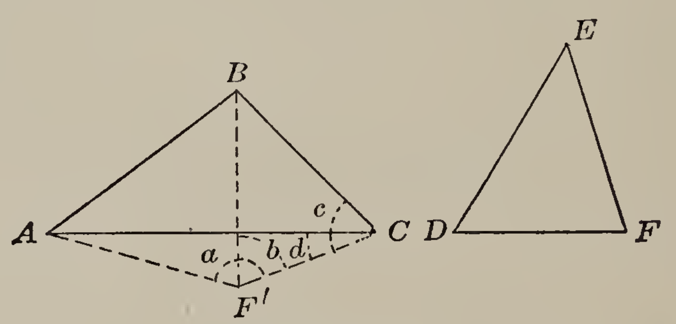


Fig. 295

Then	$a > b.$	Why?
	$b = c.$	Why?
$\therefore$	$a > c.$	Why?
	$c > d.$	Why?
$\therefore$	$a > d.$	Why?
$\therefore$	$AC > AF'$ and $AC > DF.$	Why?

II. F falls *on* AC, as at F'', Fig. 296.

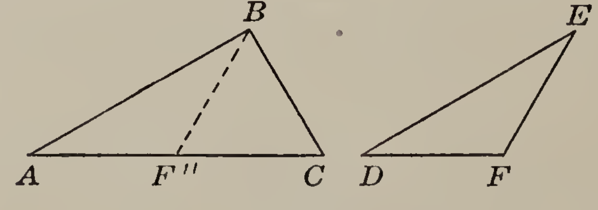


Fig. 296

Then $AC > AF''$. Why?

$AC > DF$. Why?

III. F falls *above* AC, as at F''', Fig. 297.

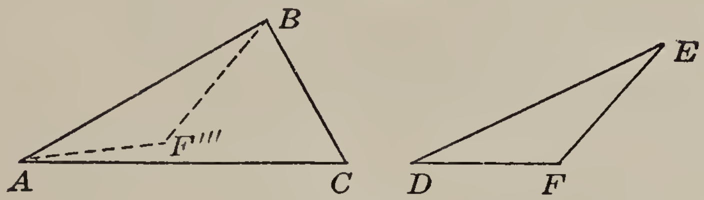


Fig. 297

$$\begin{array}{lll} \text{Then} & AF''' + F'''B < AC + CB. & \text{Why?} \\ & F'''B = \quad CB. & \text{Why?} \\ \hline \therefore & AF''' \quad < AC. & \text{Why?} \\ \text{and} & DF < AC. & \text{Why?} \end{array}$$

351. Theorem: *If two sides of one triangle are equal to two sides of another triangle, the third side of the first triangle being greater than the third side of the second; then the angle opposite the third side of the first triangle is greater than the angle opposite the third side of the second triangle.*

Given $\triangle PQR$ and XYZ, Fig. 298.

$PQ = XY$; $QR = YZ$; $PR > XZ$.

To prove that $\angle Q > \angle Y$.

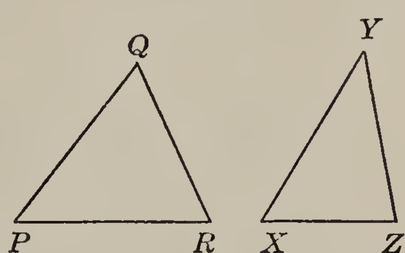


Fig. 298

Analysis: If $Q = Y$ what is known about the triangles, about PR and XZ?

Hence, can $Q = Y$ if $PR > XZ$, as here given?

What do we know about PR and XZ if $Q < Y$? Why?

Then, is $Q < Y$, if $PR > XZ$, as here given?

How, then, must angles Q and Y compare, if $PR > XZ$?

Give full proof, using the indirect method.

352. Theorem: *In the same circle or in equal circles, the arcs subtended by unequal chords are unequal in the same order as the chords; and, conversely, chords subtending unequal arcs are unequal in the same order as the arcs.*

Given $\odot A = \odot B$, Fig. 299. $\overline{CD} > \overline{EF}$.
To prove arc $CD >$ arc EF.

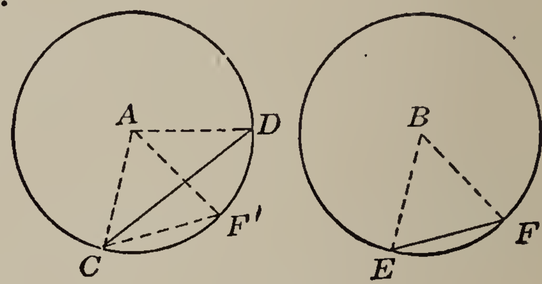


FIG. 299

Proof: Draw radii AC, AD, BE, and BF.

Show that $\angle CAD > \angle EBF$ (§ 351).

Place $\odot B$ on $\odot A$, so that EB falls on CA, E on C, B on A, and F on the same side of C as D.

Then BF must come between AD and AC, as in position AF'. Why?

Hence $\overset{\frown}{EF}$ comes in the position $\overset{\frown}{CF'}$, and F' falls on the circle between C and D.

Then, arc $CF' <$ arc CD. Why?
also, arc $CF' =$ arc EF. Why?
$\therefore$ arc $EF <$ arc CD. Why?

Conversely, **given** $\odot A = \odot B$, Fig. 299, $\overset{\frown}{CD} > \overset{\frown}{EF}$.

To prove chord $CD >$ chord EF.

Proof: Draw radii AC, AD, BF, and BE, and place $\odot B$ on $\odot A$ so that EB coincides with CA.

Since $\overset{\frown}{CD} > \overset{\frown}{EF}$, the point F will fall between C and D, as at F', and the line BF will come on the same side of AD as AC, as in position AF'.

Then, we have: $\angle CAF' < \angle CAD$. Why?
also, $\angle CAF' = \angle EBF$. Why?
and, $\angle CAD > \angle EBF$. Why?

Show that $\overline{CD} > \overline{EF}$ (§ 350).

EXERCISES

1. The length of the chords AB and BC, Fig. 300, being $6x-14$ and $4x+20$, respectively, and the lines PP' and PP'' being 16 and 10, determine x and the chords.

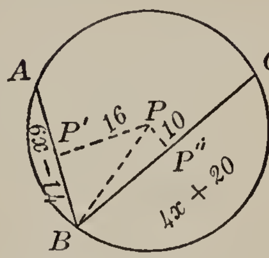


FIG. 300

We have $P'B=3x-7.$ Why?

$P''B=2x+10.$ Why?

Then, $(3x-7)^2+16^2=\overline{PB}^2.$ Why?

and $(2x+10)^2+10^2=\overline{PB}^2.$ Why?

$\therefore\ (3x-7)^2+16^2=(2x+10)^2+10^2$

or $9x^2-42x+49+256=4x^2+40x+100+100$

$5x^2-82x+105=0$

$$\therefore\ x=\frac{82\pm\sqrt{82^2-4\cdot5\cdot105}}{10}$$

$$x=\frac{82\pm68}{10}=15, \text{ or } [1\tfrac{2}{5}]$$

Then $AB=76.$

$CB=80.$

How is the truth of the theorem in § 349 illustrated by these answers?

2. The length of the lines AB and BC, PP' and PP'' (Fig. 300) being denoted by l_1, l_2, d_1, and d_2, respectively, determine the unknown number in each of the following cases. In every case test by § 349.

	l_1	l_2	d_1	d_2
1.	$2a-7$	$4a-14$	2	1
2.	6	12	$u+11$	$3u+4$
‡3.	$x+3$	$x+5$	6	4
‡4.	$4t+14$	$10t-2$	6	3

Lines and Planes in Space

353. Projection of a solid upon a plane. Imagine a model of a geometric solid, such as a cube made of wire, with only the edges and corners represented. Suppose this skeleton cube placed between a small light and the blackboard (Fig. 301). A shadow of the cube will appear on the board, giving a picture containing all the important lines and points of the solid. A drawing of this shadow will give a very good idea of the *form* of the cube. The shadow is the **projection** of the solid.

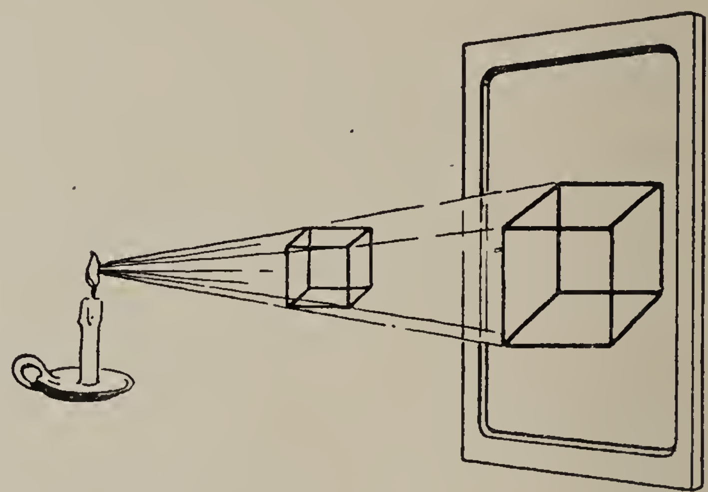


FIG. 301

By removing the light (*center of projection*) far enough, the projecting rays become nearly parallel, as in the case when the sunlight is the center of projection. The projecting rays may be perpendicular or oblique to the plane of the blackboard.

We shall consider only projections obtained by projecting rays that are *parallel* to each other and perpendicular to the plane containing the projections.

354. Projection of a point upon a plane. The foot of the perpendicular drawn from a given point to a given plane is the **projection of the point** on the plane.

Choose some point in the classroom as the tip of a gas jet, the corner of a desk, etc., and tell what its projections are on the floor, on the side wall, the end wall, and the ceiling.

355. Theorem: *The projection upon a plane, of a straight line not perpendicular to the plane, is a straight line.*

For, all projecting rays, AA', BB', CC', Fig. 302, being parallel, lie in a plane passing through AD.

Hence, the projections of all points of AD lie in the line of intersection of planes AD' and MN.

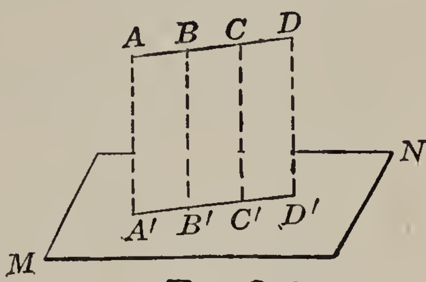


FIG. 302

356. Theorem: *The projection upon a plane, of a straight line perpendicular to the plane, is a point.* Why?

357. Theorem: *The acute angle formed by a given line and its projection upon a plane is smaller than the angle which it makes with any other line in the plane passing through the point of intersection of the given line and the plane.*

Given line AB meeting plane P at B, and BA', the projection of AB upon P. Let BC be any other line in plane P passing through B, Fig. 303.

To prove that $\angle A'BA < \angle CBA$.

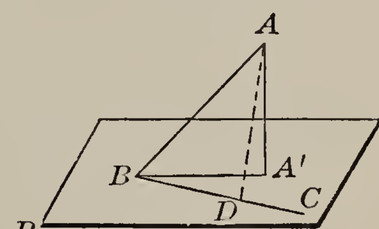


FIG. 303

Proof: On BC lay off $BD = BA'$.

Then, $AB = AB$

$BA' = BD$

and, $AA' < AD$ (§ 342)

$\therefore \angle A'BA < \angle DBA$ (§ 351).

EXERCISES

1. Find the length of the projection of AB, Fig. 303, in terms of AB and $\angle ABA'$.

2. Find the length of the projection upon a plane of a line 10 ft. long and making an angle of 60° with the plane.

Use the formula derived in exercise 1.

3. How does the length of AB, exercise 2, compare with the length of the projection?

Summary

358. The chapter has taught the meaning of the following terms:

distance from a point to a plane
median of a triangle
projection of a solid upon a plane
projection of a point upon a plane
projection of a segment upon a plane

359. The axioms and theorems on inequalities studied in the preceding chapters were reviewed and extended.

360. The use of inequalities in the solution of problems was shown.

361. The following theorems were proved:

1. *The diameter of a circle is larger than any other chord of the circle.*

2. *An exterior angle of a triangle is greater than either of the remote interior angles.*

3. *If two oblique line-segments drawn to a line from a point on a perpendicular to the line have unequal projections, the oblique line-segments are unequal.*

4. *Two unequal oblique line-segments drawn to a line from a point on a perpendicular to the line have unequal projections.*

5. *If from a point inside a triangle, line-segments are drawn to the endpoints of one side the sum of these line-segments is less than the sum of the other two sides.*

6. *In the same or in equal circles unequal chords are unequally distant from the center, the shorter chord lying at the greater distance; and the converse of this theorem.*

7. *If two sides of one triangle are equal to two sides of another triangle, but the angle included between the two sides of the first is greater than the angle included between the corresponding sides in the second, then the third side in the first is greater than the third side in the second; and the converse of this theorem.*

8. *In the same or equal circles, the arcs subtended by unequal chords are unequal in the same order as the chords, and the converse of this theorem.*

362. The points and lines in the following theorems do not all lie in the same plane:

1. *The perpendicular is the shortest distance from a point to a plane.*

2. *Oblique lines drawn from a point to a plane, meeting the plane at points equidistant from the foot of the perpendicular, are equal.*

3. *Oblique lines drawn from a point to a plane meeting the plane at points unequally distant from the foot of the perpendicular are unequal, the more remote being the greater.*

4. *Equal oblique lines drawn from a point to a plane meet the plane at points equidistant from the foot of the perpendicular.*

5. *Of two unequal oblique lines drawn from a point to a plane the greater meets the plane at the greater distance from the foot of the perpendicular.*

6. *The projection upon a plane of a straight line, not perpendicular to the plane, is a straight line.*

7. *The projection of a straight line perpendicular to the plane, upon a plane, is a point.*

8. *The acute angle formed by a given line and its projection upon a plane is smaller than the angle which it makes with any line in the plane passing through the point of intersection of the given line and the plane.*

The following construction was taught:

363. *To construct a triangle ABC, the sides a and b and the angle A, opposite one of them, being given.*

CHAPTER XV*

LINES AND PLANES IN SPACE. DIEDRAL ANGLES. THE SPHERE

364. Theorem: *If a line is perpendicular to each of two intersecting lines it is perpendicular to the plane determined by these lines.*†

Given line AB, Fig. 304, intersecting plane P at C.

$AC \perp CD$, $AC \perp CE$.

To prove $AC \perp P$.

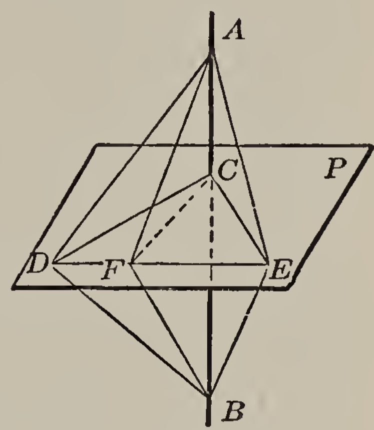


FIG. 304

Proof: Let CF be any line in P passing through C.

Draw a straight line DE intersecting CD, CF, and CE.

Draw AD, AF, and AE.

Lay off $CB = CA$ and draw BD, BF, and BE.

Show that in plane ADB, DC is the perpendicular bisector of AB.

Hence, $DA = DB$. For, any point on the perpendicular bisector of a line-segment is equidistant from the endpoints.

Similarly show that $EA = EB$.

Show that $\triangle DEA \cong \triangle DEB$.

Show that $\triangle ADF \cong \triangle BDF$ (s.a.s.).

$\therefore \quad AF = BF$.

* This chapter may be omitted if it seems desirable to shorten the course.

† Originated by Euclid, simplified by Cauchy.

Since $FA = FB$ and $CA = CB$, it follows that FC is perpendicular to AB. For, if two points on a given line are equidistant from the end-points of a segment, the given line is a perpendicular bisector of the segment.

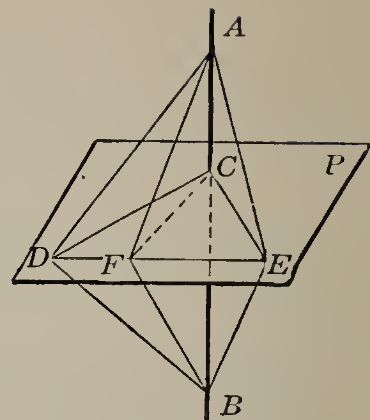


Fig. 304

Since it has been shown that AB is perpendicular to CF, which represents any line in P passing through C, it follows that AB is perpendicular to P.

365. Problem: *Through a given point in a given line pass a plane perpendicular to the given line.*

Construct two lines perpendicular to the given line at the given point. Pass a plane through those lines. This is the required plane. Prove.

366. Theorem: *All the perpendiculars to a given line at a given point lie in a plane perpendicular to the given line at that point.*

Given AB, Fig. 305, $AC \perp AB$, $AD \perp AB$, $AE \perp AB$.

To prove that lines AC, AD, AE lie in the same plane.

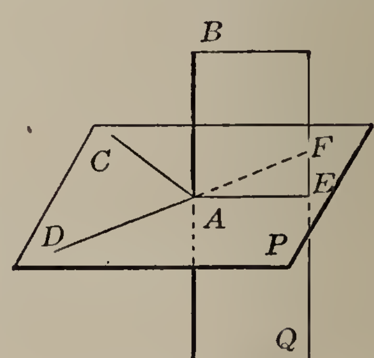


Fig. 305

Proof (indirect method):

Let P be the plane determined by AC and AD.

Then $P \perp AB$. Why?

Let AE represent any of the lines perpendicular to AB at A.

Assume that AE is not in plane P.

Then plane Q, determined by AE and AB will intersect plane P in a straight line, as AF.

Courtesy of Walter Sargent

S. MARIA DEL FIORE—FLORENCE, ITALY

The picture above illustrates the use of geometrical forms of architecture.

Then in plane Q, $AF \perp AB$. Why?
and $AE \perp AB$. Why?

The last two statements contradict the theorem that in a plane (as plane PQ) only one perpendicular can be drawn to a given line at a given point.

Therefore the assumption is wrong and AE lies in plane P.

367. Theorem: *At a given point in a given line only one plane can be constructed perpendicular to the line.*

Show that this follows from § 366.

368. Theorem: *From a given point outside of a given line one, and only one, plane can be constructed perpendicular to the line.*

Given line AB, Fig. 306, and point C not on AB.

To construct a plane through C perpendicular to AB.

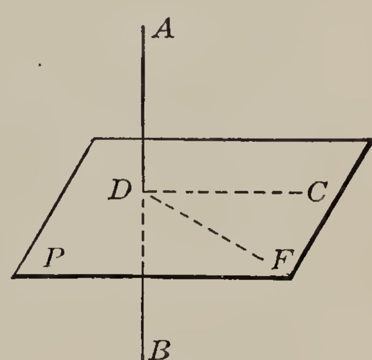


FIG. 306

Construction: Draw $CD \perp AB$.
Draw $DE \perp AB$.

Construct the plane, P, determined by CD and DE.

This is the required plane. Why?

Moreover, P is the *only* plane perpendicular to AB from C.

For, if plane Q, Fig. 307, be also perpendicular to AB, intersecting AB in D', then CD' and CD would both be perpendicular to AB. This is impossible. Why?

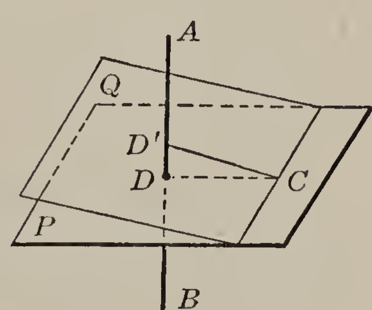


FIG. 307

369. Problem: *At a given point in a given plane construct a perpendicular to the plane.*

Given point A, Fig. 308, in plane P.

Required to construct at A a line perpendicular to plane P.

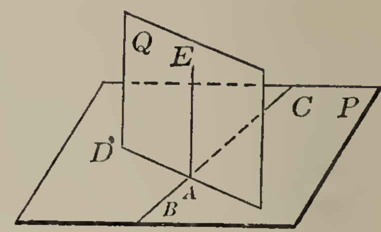


Fig. 308

Construction: Draw BC in plane P passing through A.

Construct plane $Q \perp BC$ at A, intersecting plane P in AD.

In plane Q construct $AE \perp AD$.

AE is the required perpendicular.

To prove this, show that $AE \perp AD$ and $AE \perp AB$.

370. Theorem: *Only one line can be constructed perpendicular to a given plane at a given point.*

Given $AB \perp P$, Fig. 309.

To prove that AB is the only line perpendicular to P at A.

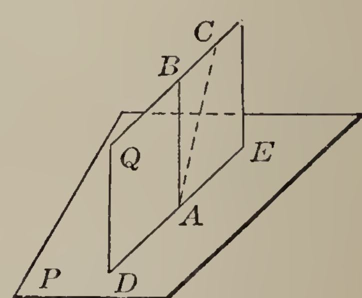


Fig. 309

Proof: Assume that AB is not the only perpendicular to P at A.

Then let AC be another perpendicular to P at A.

Pass plane Q through AB and AC cutting P in DE.

Show that AB and AC are both in Q and perpendicular to DE.

This is impossible, and the assumption that AB is not the only perpendicular to P at A is wrong.

371. Problem: *From a point outside of a plane construct a line perpendicular to the plane.*

Given plane P, Fig. 310, and point A, not in P.

To construct a perpendicular from A to P.

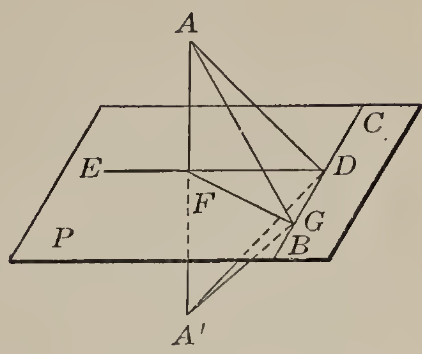


FIG. 310

Construction: In P draw a line, as BC.

Draw $AD \perp BC$.

In P draw $DE \perp BC$.

Draw $AF \perp DE$.

AF is the required line.

Proof: Draw FG any line through F in plane P meeting BC in G.

Extend AF making $FA' = FA$.

Draw $A'G$, $A'D$, and AG.

Show that $BC \perp$ plane ADF.

Show that $AD = A'D$.

Show that $\triangle ADG \cong \triangle A'DG$.

$\therefore$ $AG = GA'$. Why?

$\therefore$ FG is a perpendicular bisector of AA'. Why?

Show that $AF \perp P$.

372. Theorem: *From a given point outside of a given plane only one line can be constructed perpendicular to the plane.*

State the hypothesis and conclusion.

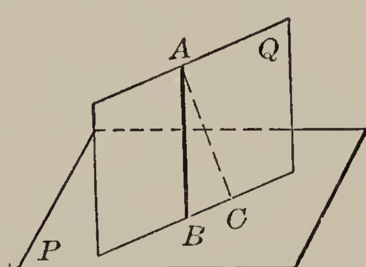


FIG. 311

Proof (indirect method):

Assume that AB, Fig. 311, is not the only perpendicular from A to P. Let AC be another perpendicular from A to P.

Draw plane Q, determined by AB and AC, intersecting P in BC.

In plane Q both AB and AC are perpendicular to BC.

This is impossible.

Hence, the assumption is wrong, and AB is the *only* perpendicular from A to P.

373. Theorem: *Lines perpendicular to the same plane are parallel.*

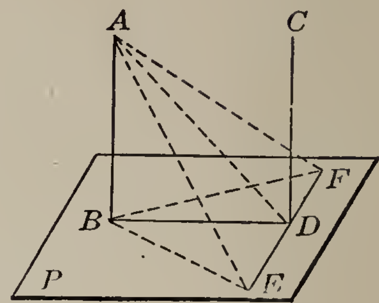


FIG. 312

Given lines AB and CD perpendicular to plane P.

To prove $AB \parallel CD$.

Proof: Draw BD.

In P draw $EF \perp BD$, and lay off $DE = DF$.

$BE = BF$ (any point on the perpendicular bisector to a line-segment is equidistant from the endpoints).

$$\therefore \quad AE = AF. \quad (\S\ 344.)$$

Show that AD is the perpendicular bisector of EF.

Thus, EF is perpendicular to DA, DB, and DC.

Therefore DB, DA, and DC lie in the same plane. Why?

$\therefore$ AB and CD lie in that plane. For, if two points of a line lie in a plane the line lies wholly in that plane.

Since AB and CD are also both perpendicular to BD, it follows that $AB \parallel CD$.

374. Theorem: *If one of two parallel lines is perpendicular to a plane, the other is perpendicular to the same plane.*

Let AB, Fig. 313, be parallel to CD.

Let AB be perpendicular to P.

If CD is not perpendicular to plane P, Fig. 313, we may draw $DC' \perp P$.

Then $DC' \parallel BA$ and $DC \parallel BA$. Why?

This is impossible. Why?

Complete the proof.

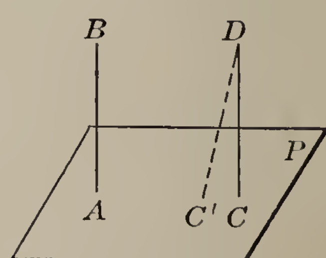


FIG. 313

375. Theorem: *Two lines parallel to the same line are parallel to each other.*

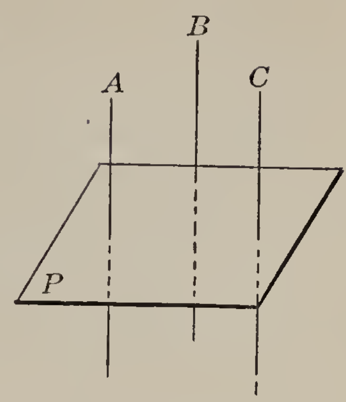


FIG. 314

Let $A \parallel B$ and $C \parallel B$, Fig. 314.
To prove $A \parallel C$.

Proof: Draw plane $P \perp B$.
Then, $A \perp P$ and $C \perp P$. Why?
$\therefore\ A \parallel C$. Why?

376. Theorem: *If two lines are parallel, a plane containing one of them and not the other, is parallel to the other.*

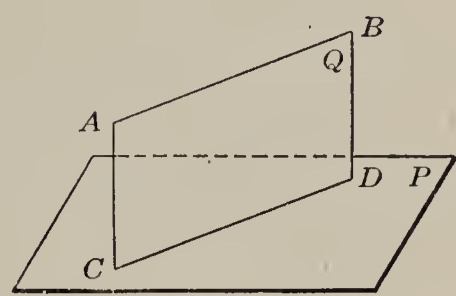


FIG. 315

Given $AB \parallel CD$, Fig. 315, and plane P containing CD, but not AB.
To prove $AB \parallel P$.

Proof: Suppose AB not parallel to P.

Then AB must meet P at some point E, if far enough extended.

Show that point E is in planes P and Q.

Then E must be on their intersection, CD.

Hence, AB and CD meet.

This contradicts the hypothesis that $AB \parallel CD$ and the assumption that AB is not parallel to P is wrong.

377. Theorem: *If one of two parallel planes is perpendicular to a line, the other is also.*

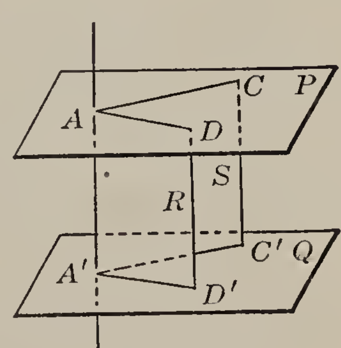


FIG. 316

Given plane $P \parallel Q$, Fig. 316.
Plane $P \perp AA'$.
To prove plane $Q \perp AA'$.

Proof: Through AA' pass planes R and S, meeting P in AC and AD, and meeting Q in $A'C'$ and $A'D'$.

Then, $AC \parallel A'C'$

and $AD \parallel A'D'$. Why?

AA' is perpendicular to AC and AD. Why?

$\therefore$ AA' is perpendicular to $A'C'$ and $A'D'$. Why?

$\therefore$ $AA' \perp Q$. Why?

378. Theorem: *If two intersecting lines are parallel to a given plane, their plane is parallel to the given plane.*

Given lines AB and AC.

AB and AC are parallel to plane P.

To prove $Q \parallel P$.

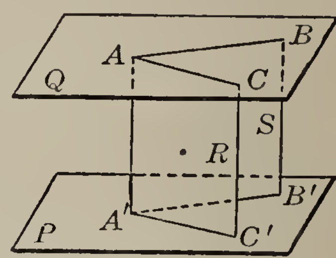


FIG. 317

Proof: Draw $AA' \perp P$.

Draw plane R, passing through AA' and AC, and plane S passing through AA' and AB.

Then, $AA' \perp A'B'$ and $A'C'$. Why?

$AC \parallel A'C'$, for, if AC meets $A'C'$, it will meet P.

Likewise, $AB \parallel A'B'$.

$\therefore$ $AA' \perp AB$ and AC. Why?

$\therefore$ $AA' \perp Q$. Why?

Show that $Q \parallel P$. Use indirect method. Apply § 368.

379. Theorem: *If two angles not in the same plane have their sides parallel and running in the same direction, the angles are equal and their planes are parallel.*

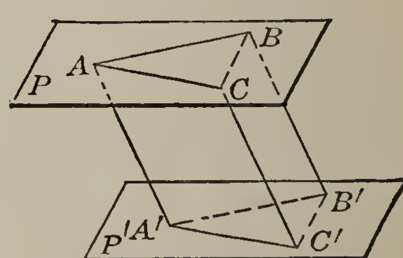


FIG. 318

Given angles A, A', Fig. 318, such that $AB \parallel A'B'$, $AC \parallel A'C'$.

To prove $\angle A = \angle A'$, $P \parallel P'$.

Proof: Draw AA'. Lay off $AB = A'B'$, $AC = A'C'$.

Draw BC and $B'C'$.

Draw CC' and BB'.

Since AB is equal and parallel to $A'B'$, $ABB'A'$ is a parallelogram and AA' is equal and parallel to BB'.

Likewise, AA' is equal and parallel to CC'.

$\therefore$ CC' is equal and parallel to BB'. Why?

$\therefore$ $\triangle ABC \cong \triangle A'B'C'$. Why?

$\therefore$ $\angle A = \angle A'$.

P is parallel to $A'C'$ and $A'B'$ (§ 376).

$\therefore$ $P \parallel P'$ (§ 378).

Diedral Angles

380. Theorem: *All plane angles of a diedral angle are equal.*

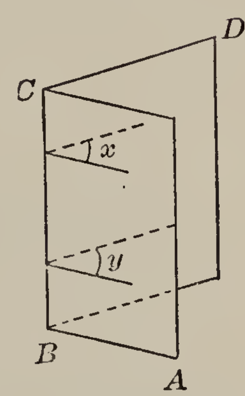


FIG. 319

Show that the sides of the plane angles x and y, Fig. 319, are parallel.

Then apply § 379.

381. Theorem: *Two diedral angles are equal if their plane angles are equal.* Conversely, *if two diedral angles are equal their plane angles are equal.*

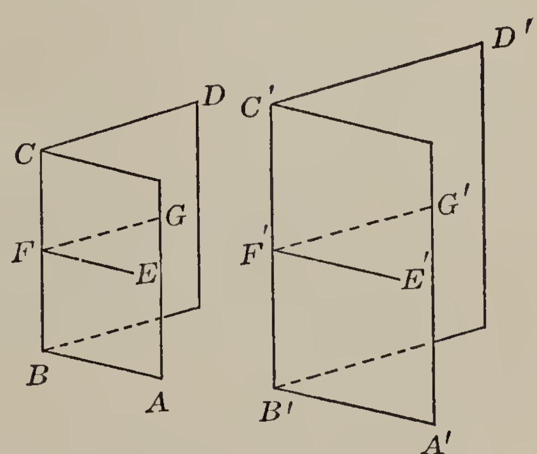


FIG. 320

Given diedral angles BC and $B'C'$ and their plane angles $EFG = E'F'G'$.

To prove $BC = B'C'$.

Proof: Place diedral angle BC on diedral angle $B'C'$, making $\angle EFG$ coincide with $E'F'G'$. This may be done because $\angle s$ EFG and $E'F'G'$ are equal.

Then CF must coincide with $C'F'$. Why?

$\therefore$ Face A must fall on face A' and face D on face D'. Why?

Hence, the diedral angles coincide and are equal.

The student may prove the converse theorem.

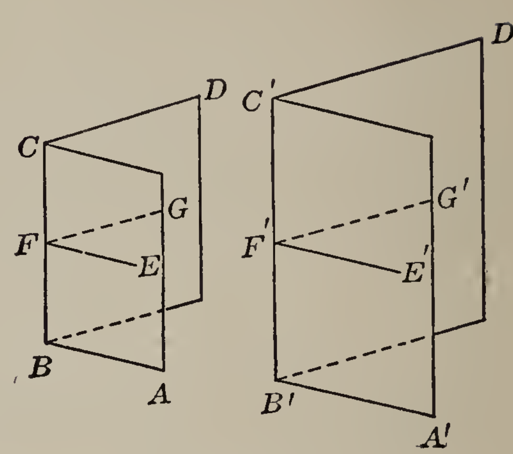


FIG. 320

A number of theorems on diedral angles are analogous to theorems on angles and may be proved in the same way. Some of these theorems are stated in the following exercises:

EXERCISES

Prove the following:

1. All right diedral angles are equal.

2. The sum of two adjacent diedral angles formed by two intersecting planes is 180°.

3. Vertical diedral angles are equal.

4. Diedral angles which are complements or supplements of the same or of equal diedral angles are equal.

5. If two parallel planes are cut by a transversal plane—

The alternate interior diedral angles are equal.

The corresponding diedral angles are equal.

The interior diedral angles on the same side are supplementary.

6. State and prove the converse of exercise 5.

7. The bisecting planes of a pair of vertical diedrals, are perpendicular.

382. Theorem: *If a line is perpendicular to a plane, every plane passing through this line is perpendicular to the plane.*

Given $AB \perp$ plane P, Fig. 321, and plane Q any plane passing through AB.

To prove that $Q \perp P$.

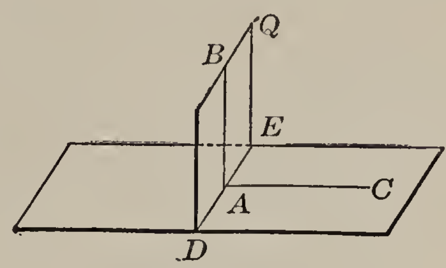


Fig. 321

Proof: In plane P draw $AC \perp DE$, the intersection of P and Q.

$BA \perp DE$. Why?

$\therefore$ $\angle BAC$ is the plane angle of $B\text{-}ED\text{-}C$. Why?

$\because$ $BA \perp AC$, $\angle BAC$ is a right angle.

$\therefore$ $Q \perp P$. Why?

EXERCISE

Show that through a line perpendicular to a given plane any number of planes may be drawn perpendicular to the given plane.

383. Theorem: *If two planes are perpendicular to each other, a line drawn in one of them perpendicular to the intersection is perpendicular to the other.*

Given $P \perp Q$, $AB \perp CD$.

To prove that $AB \perp Q$.

Proof: In plane Q draw $BE \perp CD$.

Then $\angle ABE$ is the plane angle of $A\text{-}DC\text{-}E$.

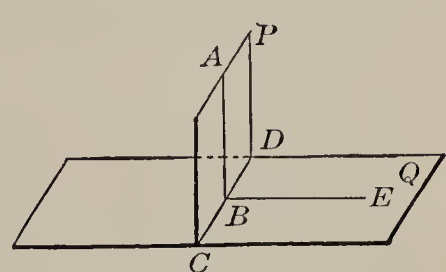


Fig. 322

$\therefore$ $\angle ABE$ is a right angle. Why?

$\therefore$ $AB \perp BE$.

$\therefore$ $AB \perp Q$. Why?

EXERCISES

Prove the following:

1. *If two planes are perpendicular to each other, a line perpendicular to one of them at a point of the intersection must lie in the other.*

Let AB, Fig. 323, be perpendicular to Q and let P be perpendicular to Q.

Suppose AB does not lie in P. Then CB may be drawn perpendicular to DE in plane P.

$CB \perp Q$ (§ 383).

But $AB \perp Q$ at the same point B.

This is impossible, etc.

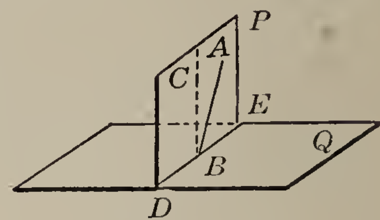


FIG. 323

2. *If from a point in one of two perpendicular planes a line is drawn perpendicular to the other it must lie in the first plane.*

Use the indirect method of proof.

384. Theorem: *If a plane is perpendicular to two planes it is perpendicular to the line of intersection.*

Given plane P, Fig. 324, perpendicular to Q and to R.

To prove $P \perp$ the line of intersection AB.

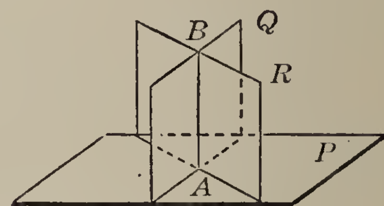


FIG. 324

Proof: At A, the point common to P, Q, and R, draw a line perpendicular to P.

This line must lie in plane Q. Why?

For the same reason it must lie in plane R.

It is therefore the intersection of Q and R.

Hence, the intersection of Q and R is a line perpendicular to plane P.

How could this theorem be applied to test whether the line of hinges of a door is perpendicular to the floor of a room, using only a carpenter's square?

385. Theorem: *Through a line not perpendicular to a given plane, one plane and only one may be passed perpendicular to the given plane.*

Given AB not $\perp$ to P, Fig. 325.

To prove that through AB one plane may be drawn perpendicular to P and only one.

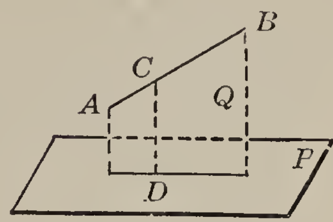


FIG. 325

Construction: From any point C on AB draw $CD \perp P$.

Draw the plane Q determined by AB and CD.

This is the required plane.

Prove that $Q \perp P$.

Q is the only plane through AB perpendicular to P. For if another plane could be passed through AB perpendicular to P, it would follow that $P \perp AB$, the intersection of the two planes. This contradicts the hypothesis.

EXERCISES

Prove the following:

1. A plane perpendicular to the edge of a diedral angle is perpendicular to the faces.

2. Through a point within a diedral angle a plane may be passed perpendicular to each face.

3. If three lines are perpendicular to each other at the same point, each line is perpendicular to the plane determined by the other two.

The Sphere

386. Sphere. Center. Radius. Diameter. A **sphere** is a solid bounded by a surface, all points of which are equidistant from a point within called the **center,** Fig. 326.

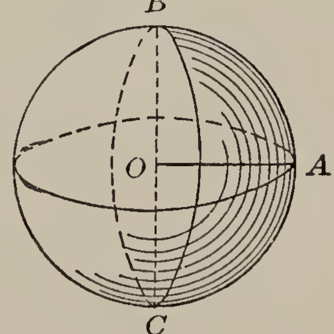


FIG. 326

A line-segment from the center to the surface of the sphere is a **radius,** as OA.

A **diameter** is a segment passing through the center and terminated by the surface, as BC.

A sphere may be produced by revolving a semicircle about the diameter.

387. Preliminary theorems:

1. *All radii of the same sphere are equal.*
2. *All diameters of the same sphere are equal.*
3. *The radii of equal spheres are equal.*
4. *Spheres having equal radii are equal.*

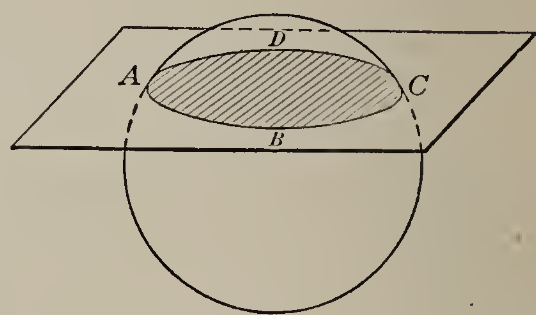


FIG. 327

388. Section of a sphere. The intersection of a plane with the surface of a sphere is a **section** of the sphere, as the curve $ABCD$, Fig. 327.

389. Theorem: *The section of a sphere made by a plane is a circle.*

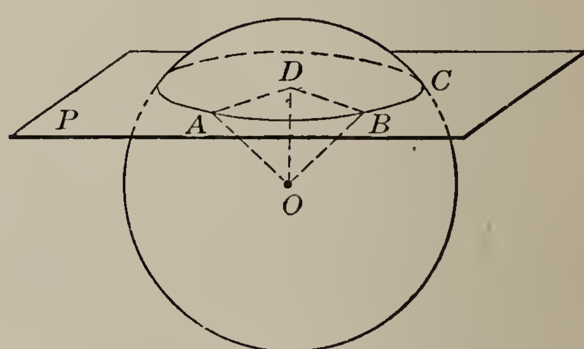


FIG. 328

Given a sphere O cut by a plane P, making the section ABC.

To prove that ABC is a circle.

Proof: Let A and B be any two points on the section ABC.

Draw the radii OA and OB.

Draw $OD \perp$ plane P.

Draw AD and DB.

Then $DA = DB$ (see § 346).

$\therefore$ ABC is a circle, since all points on ABC are equidistant from D.

390. Great circle. Small circle. Poles. Axis. A section made by a plane passing through the center of a sphere is a **great circle,** as ABC, Fig. 329.

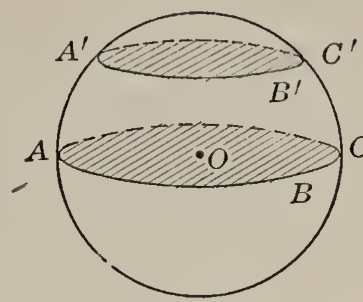


Fig. 329

A section whose plane does not pass through the center is a **small circle,** as $A'B'C'$, Fig. 329.

The diameter perpendicular to the plane of a circle of a sphere is the **axis** of the circle and the *extremities* of the diameter are the **poles** of the circle.

EXERCISES

1. Find the area of a plane section of a sphere of radius 10, which passes 6 units from the center. (Board).

Show the truth of the following theorems:

2. *The axis of a circle passes through the center.*

3. *The diameter of a sphere passing through the center of a circle is perpendicular to the plane of the circle.*

4. *All great circles of a sphere are equal.*

5. *Two great circles bisect each other.*

6. *Through two points on the surface of a sphere, not the end-points of a diameter, only one great circle can be drawn.*

How many points determine a plane?

What third point must be selected to determine a circle on the sphere?

When do two given points and the center of the sphere not determine a plane?

7. *Every great circle bisects the sphere.*

For, the two portions into which the great circle divides the surface of a sphere can be made to coincide, as all points on the surface of the sphere are equidistant from the center.

391. Spherical distance between two points. The length of the minor arc of a great circle passing through two points is the **spherical distance** between them.

Thus $\overset{\frown}{ADB}$, Fig. 330, is the spherical distance between A and B.

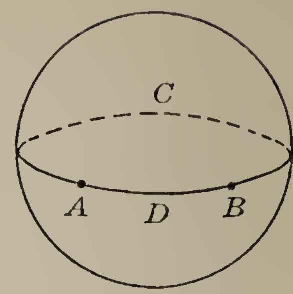


FIG. 330

392. Theorem: *All points on a circle of a sphere are equidistant from its poles.*

Given two points A and B, Fig. 331, on the circle AB of the sphere O; P and P' the poles of circle AB.

To prove that $\overset{\frown}{PA} = \overset{\frown}{PB}$.

Proof: Let the axis PP' intersect the plane of circle AB in C.

Then C is the center of circle AB. Why?

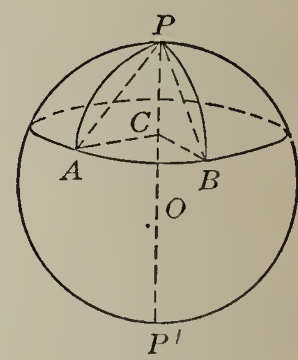


FIG. 331

$\therefore \quad CA = CB.$ Why?

$\therefore \quad \overline{PA} = \overline{PB}.$ Why?

$\therefore \quad \overset{\frown}{PA} = \overset{\frown}{PB}.$ Why?

393. Polar distance. The spherical distance from the nearer of the poles of a *small circle* to any point on the circle is the **polar distance** of the circle.

The polar distance of a *great circle* is the spherical distance to *either* pole.

394. Quadrant. One-fourth of the length of a great circle is a **quadrant.**

395. Theorem: *The polar distance of a great circle is a quadrant.* Prove.

396. Theorem: *If a point on the surface of a sphere is at the distance of a quadrant from each of two given points on the surface, it is a pole of the great circle passing through the given points.*

Given points A, B, and C on a sphere, Fig. 332; AB = a quadrant; AC = a quadrant; BCD a great circle arc.

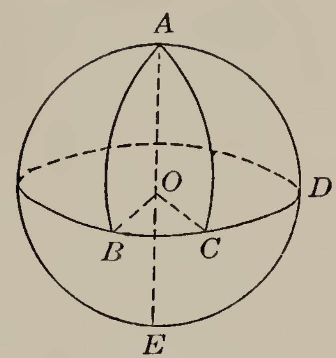


FIG. 332

To prove that A is a pole of $\overset{\frown}{BCD}$.

Analysis: If A is a pole of arc BC, what can be said of diameter AOE?

How can we show that $AOE \perp$ plane of $\odot O$?

How large are angles AOB and AOC? Give proof.

397. Theorem: *The intersection of the surfaces of two spheres is a circle whose plane is perpendicular to the line of centers of the spheres and whose center is in that line.*

Let the two intersecting spheres be generated by rotating circles, A and B, Fig. 333, about the center-line AB as an axis.

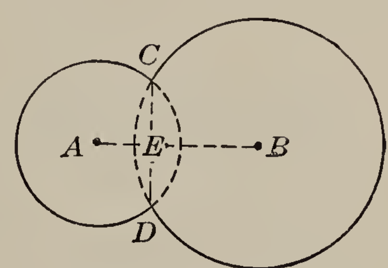


FIG. 333

To prove that the spherical surfaces intersect in a circle whose center is in AB and whose plane is perpendicular to AB.

Proof: Let CD be the common chord of circles A and B.

Then AB is the perpendicular bisector of CD. Why?

As the plane of circles A and B revolves about AB, C describes the line common to the two spheres thus generated.

Line CE always lies in the plane perpendicular to AB at E. Why?

$\therefore$ The path of C is a circle in that plane. Why?

EXERCISE

Two spheres, whose radii are 12 inches and 5 inches respectively, have their centers 13 inches apart. Find the area of the circle in which these two spheres intersect. (Harvard.)

398. Tangent line. Tangent plane. If the surface of a sphere and a line (plane) have only one point in common, the line (plane) is said to be **tangent** to the sphere.

399. Theorem: *A plane tangent to a sphere is perpendicular to the radius at the point of contact.*

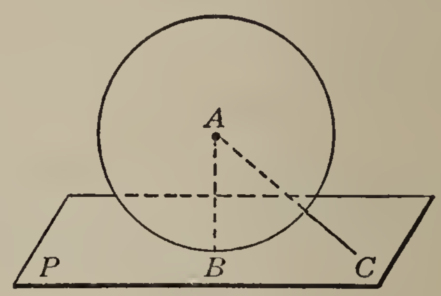


FIG. 334

Given sphere A, Fig. 334, and plane P tangent to A.

To prove that $P \perp AB$.

Proof: Let C be *any* point in P, *not* B.

Then C is outside of the sphere. Why?

$\therefore$ $AC >$ radius. Why?

$\therefore$ $AC > AB$.

Hence, AB is the shortest distance from A to plane P. Why?

$\therefore$ $P \perp AB$. Why?

400. Theorem: *A plane perpendicular to a radius of a sphere at the outer extremity is tangent to the sphere.*

To prove this, reverse the order of steps in the proof of the preceding theorem.

Summary

401. The chapter has taught the meaning of the following terms:

sphere	great circle	spherical distance between two points
center	small circle	
radius	poles	quadrant
diameter	axis of a circle	tangent line
section of a sphere	polar distance	tangent plane

402. The following theorems were proved:

1. *If a line is perpendicular to each of two intersecting lines it is perpendicular to the plane determined by these lines.*

2. *All the perpendiculars to a given line at a given point lie in a plane perpendicular to the given line at the point.*

3. *Only one plane can be constructed perpendicular to a given line at a given point.*

4. *Only one plane can be constructed perpendicular to a given line from a point outside of the line.*

5. *Only one line can be constructed perpendicular to a given plane at a given point.*

6. *From a point outside of a given plane only one line can be constructed perpendicular to the plane.*

7. *Lines perpendicular to a plane are parallel.*

8. *If one of two parallel lines is perpendicular to a plane, the other is perpendicular to the same plane.*

9. *Two lines parallel to the same line are parallel to each other.*

10. *If two lines are parallel, a plane containing one of them and not the other, is parallel to the other.*

11. *If one of two parallel planes is perpendicular to a line the other is also.*

12. *If two intersecting lines are parallel to a given plane, their plane is parallel to the given plane.*

13. *If two angles not in the same plane have their sides parallel and running in the same direction, the angles are equal and their planes are parallel.*

14. *All plane angles of a diedral angle are equal.*

15. *If two diedral angles are equal their plane angles are equal.*

16. *Two diedral angles are equal if the plane angles are equal.*

17. *If a line is perpendicular to a plane every plane through this line is perpendicular to the plane.*

18. *If two planes are perpendicular to each other a line drawn in one of them perpendicular to the intersection is perpendicular to the other.*

19. *If two planes are perpendicular to each other a line perpendicular to one of them at a point of the intersection must lie in the other.*

20. *If from a point in one of two perpendicular planes a line is drawn perpendicular to the other, it must lie in the first plane.*

21. *If a plane is perpendicular to two planes it is perpendicular to their intersection.*

22. *Through a line not perpendicular to a given plane, one plane and only one may be passed perpendicular to the given plane.*

23. *The section of a sphere made by a plane is a circle.*

24. *The axis of a circle passes through the center.*

25. *The diameter of a sphere passing through the center of a circle is perpendicular to the plane of the circle.*

26. *All great circles of a sphere are equal.*

27. *Every great circle bisects the sphere.*

28. *Through two points on the surface of a sphere, not the endpoints of a diameter, only one great circle can be drawn.*

29. *All points on a circle of a sphere are equidistant from its poles.*

30. *The polar distance of a great circle is a quadrant.*

31. *If a point on the surface of a sphere is at the distance of a quadrant from each of two given points on the surface, it is a pole of the great circle passing through the given points.*

32. *The intersection of two spherical surfaces is a circle whose plane is perpendicular to the line of centers of the spheres, and whose center is in that line.*

33. *A plane tangent to a sphere is perpendicular to the radius at the point of contact.*

34. *A plane perpendicular to a radius of a sphere at the outer extremity is tangent to the sphere.*

35. *To determine the diameter of a material sphere.*

403. The following constructions were taught:

1. Through a given point in a given line pass a plane perpendicular to a given line.

2. From a given point outside of a given line construct a plane perpendicular to the given line.

3. At a given point in a given plane construct a perpendicular to the plane.

4. From a point outside of a plane construct a line perpendicular to the plane.

5. To pass a plane perpendicular to a given plane, that shall contain a line not perpendicular to the given plane.

CHAPTER XVI

LOCI. CONCURRENT LINES

Loci

404. Locus. When a point moves it traces a path whose shape is determined by the conditions under which the point moves. Thus, a stone falling from rest moves along a *straight line,* a particle projected obliquely into space moves along a curve, which is practically a *parabola,* Fig. 335.

Fig. 335

In the study of geometry we have learned that the location of all points in a plane at a given distance from a fixed point is a *circle;* that the place of all points of a plane at equal distances from two fixed points is a *straight line,* the perpendicular bisector of the segment joining the given points.

The place of all points satisfying some specified condition and *not containing other points* is called the **locus** of the points. *Locus** is a Latin word, meaning "place."

405. Determination of a locus. To determine the locus of a point mark a number of positions of the point. From these points it will be possible to obtain a notion of the locus.

Thus, marking several positions of the pedal of a bicycle on a wall beside a walk suggests the locus of the pedal.

* The plural of *locus* is *loci.*

EXERCISES

1. A circle C, Fig. 336, is rolled without sliding along the edge of a ruler AB. Find the locus of a point P on the circle.

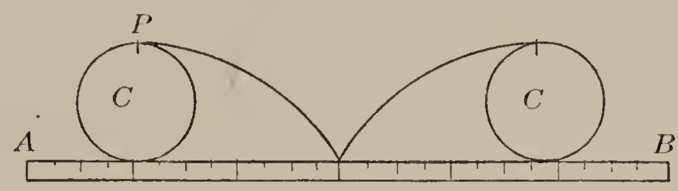


Fig. 336

Cut a circle from cardboard and roll it carefully along the ruler. By pricking through with a pin, mark a number of positions of P. Draw a smooth curve through the points thus obtained. The locus of P is called a *cycloid.*

2. Draw two perpendicular lines, Fig. 337. On a piece of tracing paper draw a segment AB and mark a point P on AB. Move AB so that B slides along OY and A along OX and mark a number of positions of P. Draw the locus of P. The locus will be a quarter of an *ellipse.*

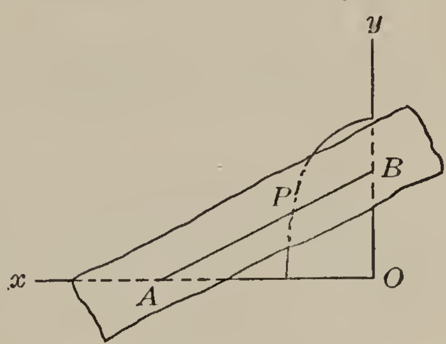


Fig. 337

3. What is the locus of points in a plane having a given distance from a given line?

Mark several points at the given distance from the given line. Their position will suggest the locus.

4. What is the locus of points in a plane at equal distances from two given parallel lines?

5. What is the locus of points in space having a given distance from a given point?

6. What is the locus of points in space equally distant from two given points?

7. What is the locus of points in space equally distant from two parallel lines?

8. What is the locus of points in space having a given distance from a given line?

9. What is the locus of points in space at equal distances from three given points? (See § 411.)

406. Proof for a locus. The locus of points satisfying given conditions must contain *all* points satisfying these conditions and *no other* points, i.e.:

I. *Every point on the locus must satisfy the given conditions.*

II. (*a*) *Every point satisfying the conditions must lie on the locus, or*

(*b*) *Any point not on the locus must not satisfy the conditions.*

407. Theorem: *The locus of points in a plane equidistant from two given points is the perpendicular bisector of the segment joining these points.*

Proof: I. Show that every point on the perpendicular bisector is equidistant from the two points.

II. Let $PA = PB$, Fig. 338. Let PC be a line drawn from P to the midpoint, C, of AB.

Show that $x = y$.

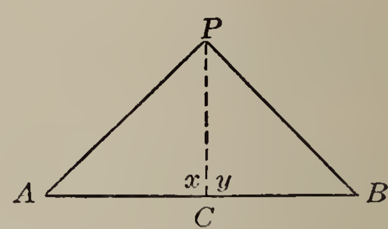


FIG. 338

408. Theorem: *The locus of points in a plane which are within an angle and equidistant from its sides is the bisector of the angle.*

Proof: I. Show that every point on the bisector is equidistant from the sides.

II. If $PB \perp AB$, Fig. 339, $PC \perp AC$ and $BP = PC$, show that $x = y$.

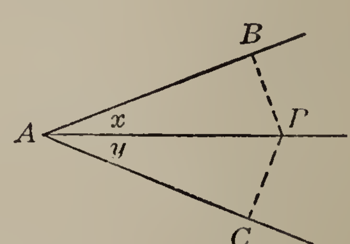


FIG. 339

409. Theorem: *The locus of points in a plane at a given distance from a given point is the circle whose center is the given point and whose radius is equal to the given distance.*

Proof: I. Every point on the circle, Fig. 340, has the given distance from the given point. Why?

II. Show that a point P, not on the circle, is not at the given distance from the given point C.

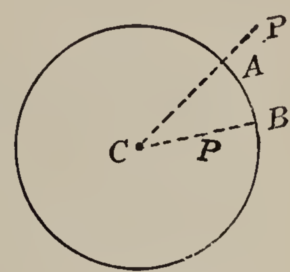


FIG. 340

410. Theorem: *The locus of points in a plane at a given distance from a given line consists of a pair of lines parallel to the given line and the given distance from it.*

Show that conditions I and II are satisfied in Fig. 341.

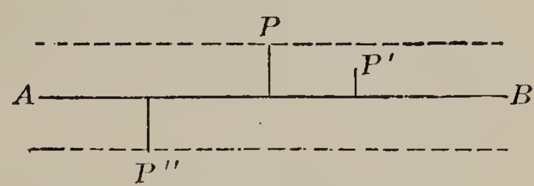


FIG. 341

EXERCISES

1. Show that the locus of the centers of all circles in a plane tangent to a given line at a given point is the perpendicular to the given line at that point.

2. Show that the locus of the centers of all circles in the same plane of given radius and tangent to a given line consists of two lines parallel to the given line and at the given distance from it.

3. Show that the locus of the vertex of an angle of given size, x, whose sides pass through two fixed points A and B consists of two arcs having AB as chord and x as inscribed angle. (See § 301 for construction of this locus.) Show that for a point D, Fig. 342, outside of the circle arc, $y < x$ and for a point E within the circle arc, $z > x$.

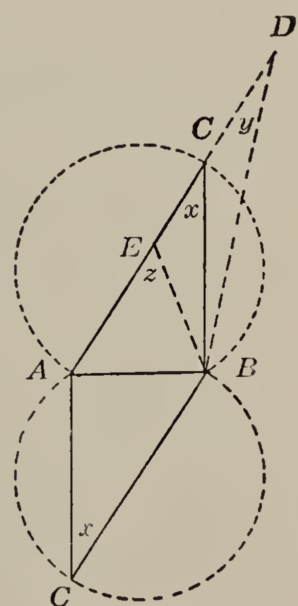


FIG. 342

4. Construct an isosceles triangle having given the base and the angle opposite the base.

5. Find the locus of the midpoints of parallel chords of a circle.

6. Find the locus of the midpoints of chords of a circle equidistant from the center.

7. Find the locus of the midpoints of all chords passing through a given point on the circle, Fig. 343.

8. Find the locus of the centers of all circles passing through two given points.

9. Find the locus of the centers of all circles tangent to a given circle at a given point.

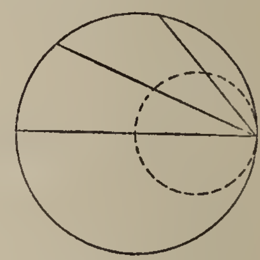


FIG. 343

10. Find the locus of the midpoints of all segments drawn from one vertex of a triangle and terminated by the opposite side.

11. Construct a circle with a given radius which shall be tangent to each of two intersecting lines.

411.* Theorem: *The locus of points in space equidistant from all points on a circle is the line perpendicular to the plane of the circle at the center.*

Proof: I. Show that any point P on the perpendicular at C, Fig. 344, is equidistant from all points of the circle. (Use § 344.)

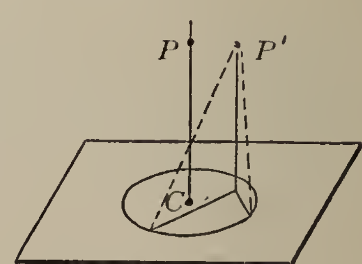


FIG. 344

II. Show that any point P' not on the perpendicular at C is not equidistant from all points of the circle. (Use § 345.)

* §§ 411–413 may be omitted, if chapter XV has been omitted.

412. Theorem: *The locus of points in space equidistant from two given points is the plane bisecting the segment joining these points, and perpendicular to it.*

Proof: I. Show that any point in plane P, Fig. 345, is equidistant from A and B.

II. Let D be any point not in plane P, and let $DA = DB$.

Show that DC is perpendicular to AB. Hence, DC must lie in plane P.

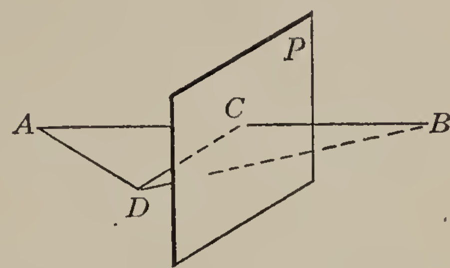


Fig. 345

413. Theorem: *The locus of a point within a diedral angle and equidistant from the faces is the plane bisecting the angle.*

Given the diedral angle $A\text{–}BC\text{–}D$, Fig. 346. Plane P bisects the diedral angle.

To prove that P is the locus of points equidistant from the faces Q and R.

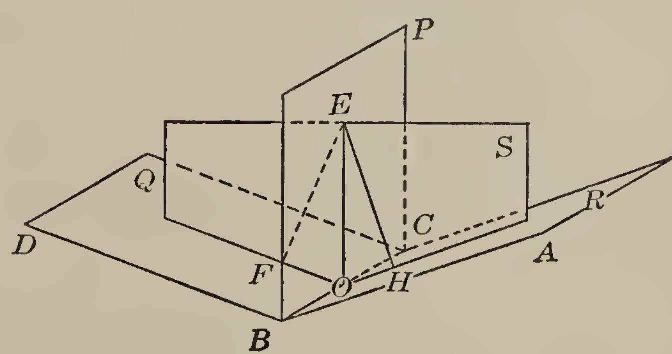


Fig. 346

Proof: I. Prove that any point, as E, in plane P, is equidistant from Q and R, as follows:

Draw $EF \perp Q$ and $EH \perp R$.

Pass plane S through EF and EH.

Then $S \perp Q$, and $S \perp R$ (§ 382).

$\therefore\ S \perp BOC$ (§ 384).

$\therefore\ BO$ is perpendicular to FO, EO, and HO. Why?

$\therefore\ \measuredangle FOE$ and HOE are *plane angles* of the diedral angles formed by P and Q, and by P and R. Why?

$\therefore \quad \angle FOE = \angle HOE.$ Why?

Prove $\triangle FOE \cong \triangle HOE.$

Then $EF = EH.$

II. Prove that every point equidistant from Q and R lies in the bisecting plane P. as follows:

Prove as in Case I that $\measuredangle FOE$ and HOE are plane angles of diedral angles PQ and PR.

Since it is given that $EF = EH$, we may prove

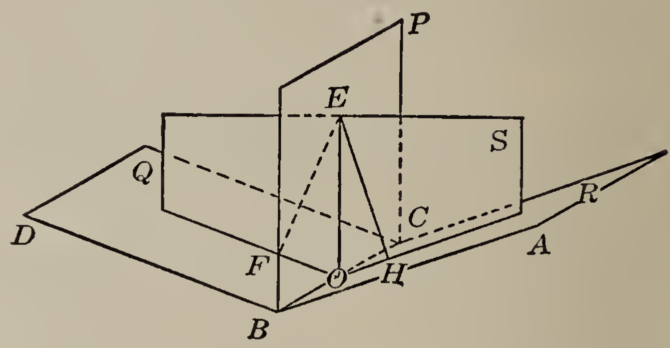


Fig. 346

$\triangle FOE \cong \triangle HOE$ (hypotenuse and one side).

$\therefore \quad \angle FOE = \angle HOE.$

$\therefore$ Diedral angle PQ = diedral angle PR.

$\therefore$ Plane P bisects Q–BC–R.

Hence, P is the required locus.

Concurrent Lines

414. Median. The **median** of a triangle is a segment drawn from a vertex to the *midpoint* of the opposite side.

415. Center of gravity of a triangle. From cardboard cut a triangle. Draw the three medians of the triangle. If the construction is made carefully, the three medians will meet in a point. If the triangle is supported by placing a pin under the point of intersection, the triangle will be found to balance. For this reason the point of intersection of the three medians of a triangle is called the **center of gravity** of the triangle.

416. Concurrent lines. If three or more lines pass through the same point, they are called **concurrent lines.**

417. Theorem: *The medians of a triangle are concurrent in a point which lies two-thirds the distance from the vertex to the midpoint of the opposite side.*

Given $\triangle ABC$, Fig. 347, with the medians AE, BF, and CD.

To prove that AE, BF, and CD are concurrent and that,

$$AO = \tfrac{2}{3}AE$$
$$BO = \tfrac{2}{3}BF$$
$$CO = \tfrac{2}{3}CD.$$

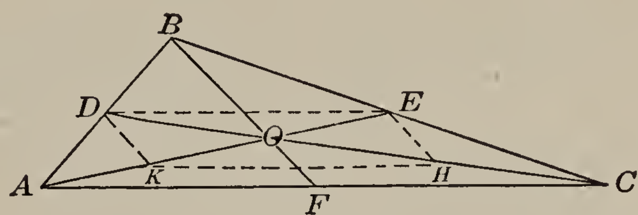


FIG. 347

Proof: AE must intersect CD at some point, as O. For, if AE does not intersect CD, it follows that

$$AE \parallel CD$$

and that $\angle EAC + \angle DCA = 180°$. Show that this is impossible.

Draw KH joining K, the midpoint of AO to H, the midpoint of OC.

Draw DE, DK, and EH.

Then, $DE \parallel AC$ and $DE = \frac{1}{2}AC$ (§ 168, exercise 2, and § 159, exercise 2).

Similarly, $KH \parallel AC$ and $KH = \frac{1}{2}AC$.

$\therefore$ $KHED$ is a *parallelogram* (§ 125).

$\therefore$ $EO = OK = KA$.

and, $DO = OH = HC$.

$\therefore$ $AO = \frac{2}{3}AE$ and $CO = \frac{2}{3}CD$.

Similarly, we may show that CD and BF meet in a point which is two-thirds the distance from B to F and from C to D, i.e., at O.

418. Trisection point. The two points dividing a segment into three equal parts are **trisection points.** Thus, the point of intersection of the medians of a triangle is a trisection point of each median.

419. Theorem: *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle.*

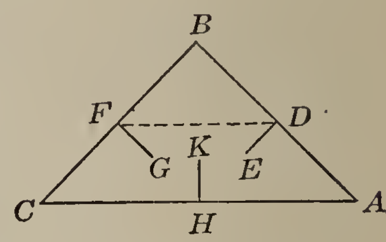


FIG. 348

Given $\triangle ABC$, Fig. 348, and DE, FG, and HK the perpendicular bisectors of AB, BC, and CA, respectively.

To prove that DE, FG, and HK are concurrent in a point equidistant from A, B, and C.

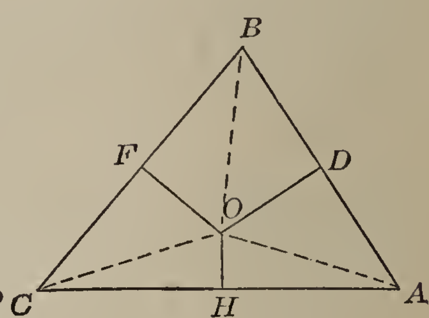


FIG. 349

Proof: Draw DF.

$$\angle EDB = 90^\circ$$
$$\angle GFB = 90^\circ$$

$\therefore\ \angle EDB + \angle GFB = 180^\circ$

$\therefore\ \angle EDF + \angle GFD < 180^\circ$. Why?

$\therefore\ DE$ and FG must intersect at some point, as O, Fig. 349.

For, if DE does not intersect FG, then

$$DE \parallel FG \text{ and}$$
$$\angle EDF + GFD = 180^\circ.$$

$OC = OB$. Why?

$OB = OA$. Why?

$\therefore\ OC = OA$. Why?

$\therefore\ HK$ must pass through O. For, the perpendicular bisector of a segment is the locus of all points equidistant from the endpoints.

EXERCISES

1. Show that the point O, Fig. 349, is the center of the circumscribed circle of triangle ABC.

2. Draw the circle circumscribed about a triangle.

3. Draw a circle passing through three points not in the same straight line.

420. Circumcenter. The point of intersection of the perpendicular bisectors of the sides of a triangle is the **circumcenter** of the triangle.

421. Theorem: *The bisectors of the angles of a triangle are concurrent in a point which is equidistant from the sides of the triangle.*

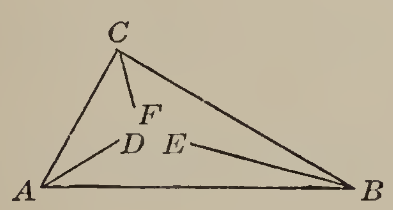


Fig. 350

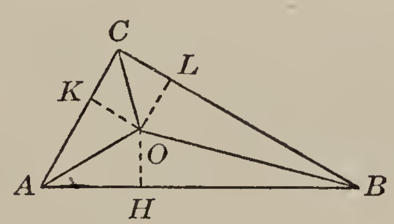


Fig. 351

Given $\triangle ABC$, Fig. 350, with AD, BE, and CF, the bisectors of $\measuredangle A$, B, and C, respectively.

To prove that AD, BE, and CF are concurrent in a point equidistant from AB, BC, and CA.

Proof: Show that AD and BE intersect, as at O, Fig. 351.

Draw $OH \perp AB$, $OK \perp AC$, $OL \perp BC$.

Then, $OH = OK$. Why?

$OH = OL$. Why?

$\therefore$ $OK = OL$. Why?

$\therefore$ CF must pass through O. Why?

EXERCISES

1. Show that the point O, Fig. 351, is the center of the circle inscribed in triangle ABC.

2. Inscribe a circle in a triangle.

422. Theorem: *The three altitudes of a triangle are concurrent.*

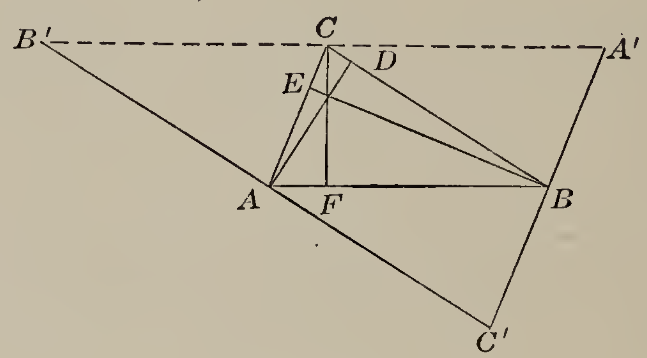


FIG. 352

Given $\triangle ABC$, Fig. 352, with $AD \perp BC$, $BE \perp AC$, and $CF \perp AB$.

To prove that AD, BE, and CF are concurrent.

Proof: Draw $B'C' \perp AD$, $C'A' \perp BE$, and $A'B' \perp CF$, forming $\triangle A'B'C'$.

Then, $AB \parallel A'B'$,
$BC \parallel B'C'$,
and $CA \parallel C'A'$. Why?

Show that $B'C = AB = CA'$.

Hence, CF is the perpendicular bisector of $A'B'$.

Similarly, show that AD is the perpendicular bisector of $A'C'$ and that BE is the perpendicular bisector of $C'A'$.

$\therefore$ AD, BE, and CF are concurrent. Why?

423. Orthocenter. The point of intersection of the three altitudes of a triangle is called the **orthocenter** of the triangle.

424. Incenter. The point of intersection of the bisectors of the interior angles of a triangle is called the **incenter** of the triangle.

EXERCISE

Show that the bisectors of one interior angle, as A, Fig. 353, and of the exterior angles at B and C are concurrent.

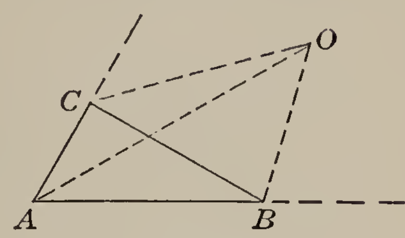


FIG. 353

425. Excenter. The point of intersection of the bisectors of two exterior angles of a triangle and the third interior angle is called an **excenter** of the triangle.

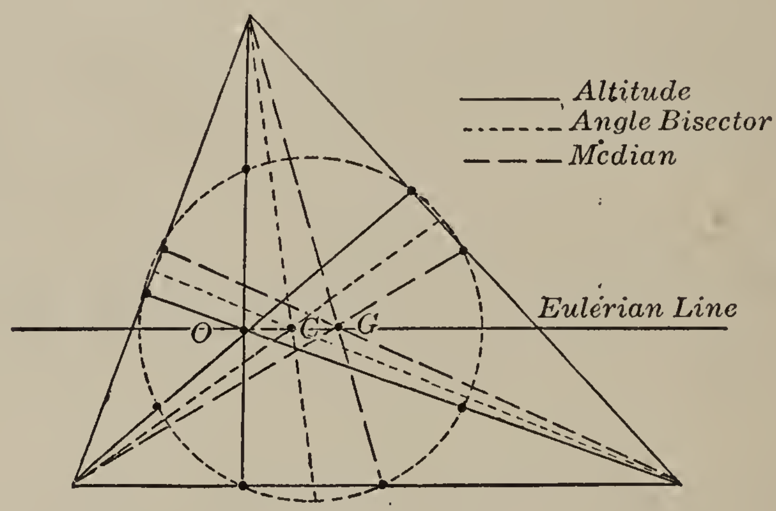
1. How many excenters are there?

2. Draw a triangle. Construct four circles tangent to the three sides.

3. Prove that the bisectors of the angles of a quadrilateral circumscribed about a circle meet at a point.

426. Historical note. The ancients even before Euclid's time were acquainted with the theorems of the *medians*, of the *altitudes*, of the *angle-bisectors* and of the *perpendicular bisectors* of the sides of a triangle, but they placed no great importance upon them. They used the *incenter*, the *circumcenter*, the *orthocenter*, and the *center of gravity* in constructions but they did not theorize about them. Greek mathematics so completely dominated the science until after mediaeval times that theorems not given by Euclid were regarded as of little moment. At the beginning of the eighteenth century the neglected theme began to be studied. In 1723 the problem was raised, to construct a triangle having given the position of its center of gravity, G, of the incenter, I, and of the orthocenter, O. Nothing worth mentioning came from this problem.

In 1765 Euler (1707–83) attacked and solved the problem of calculating the distance of the points O, G, and I from one another and from C, the circumcenter, in terms of the sides a, b, and c. He found that OCG, (see figure), is a straight line and



that $GC = \frac{1}{2}\,GO$. The straight line OCG was later named in his honor, **the Eulerian line.** In 1821 Poncelet showed that the midpoints of the sides, the feet of the altitudes, and the midpoints of the upper segments of the altitudes of a triangle all lie on the same circle.

In 1822 Feuerbach (1800–1834) also discovered this circle. He showed that its center M' bisects the segment CO, and that its radius equals half the radius of the circumscribed circle $(=r/2)$. Germans in his honor call this circle **Feuerbach's circle** but English mathematicians prefer to call it **the nine-point circle.**

Feuerbach also showed the circle to be tangent internally to the inscribed circle and externally to the escribed circle, and that the segment OG of the Eulerian line is divided by the center M' in the ratio 2:1. Since Feuerbach's time all these points and properties have been extensively studied from varied points of view, and much mathematical knowledge has resulted.

Feuerbach's circle was first given place in an elementary book on geometry by C. F. A. Jacobi in 1834. (See Tropfke, *Geschichte der Elementar-Mathematik*, II. Bd., S. 88–90.)

Summary

427. The chapter has taught the meaning of the following terms:

locus	center of gravity of a triangle	circumcenter
cycloid	concurrent lines	incenter
ellipse	trisection point	excenter
median		orthocenter

428. The proof for a locus consists in showing—

I. That every point on the locus satisfies given conditions.

II. (*a*) That every point satisfying these conditions lies on the locus, or

(*b*) That every point not on the locus does not satisfy these conditions.

429. The following theorems were proved:

1. *The locus of points in a plane equidistant from two given points is the perpendicular bisector of the segment joining these points.*

2. *The locus of points in a plane which are within an angle and equidistant from its sides is the bisector of the angle.*

3. *The locus of points in a plane at a given distance from a given point is the circle whose center is the given point and whose radius is equal to the given distance.*

4. *The locus of points in a plane at a given distance from a given line consists of a pair of lines parallel to the given line and the given distance from it.*

5. *The locus of points in space equidistant from all points on a circle is the line perpendicular to the plane of the circle at the center.*

6. *The locus of points in space equidistant from two given points is the plane bisecting the segment joining these points and perpendicular to it.*

7. *The locus of points within a diedral angle equidistant from the faces is the plane bisecting the angle.*

8. *The medians of a triangle are concurrent.*

9. *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices of the triangle.*

10. *The bisectors of the angles of a triangle are concurrent in a point which is equidistant from the sides of the triangle.*

11. *The three altitudes of a triangle are concurrent.*

CHAPTER XVII

REGULAR POLYGONS INSCRIBED IN, AND CIRCUMSCRIBED ABOUT, THE CIRCLE. LENGTH OF THE CIRCLE

Construction of Regular Polygons

430. Regular polygon. A polygon that is both *equilateral* and *equiangular* is a **regular polygon.**

431. Regular polygons in designs. Regular polygons are involved in many forms of decorative design. We use them in the tile floor, Fig. 354; in the ornamental

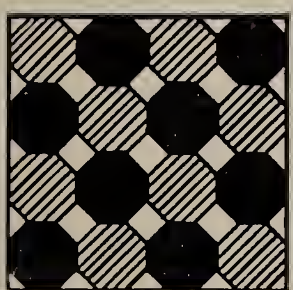


Fig. 354

Fig. 355

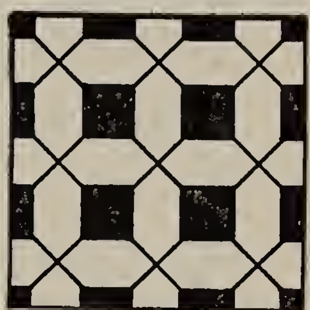


Fig. 356

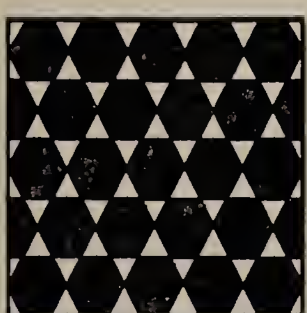


Fig. 357

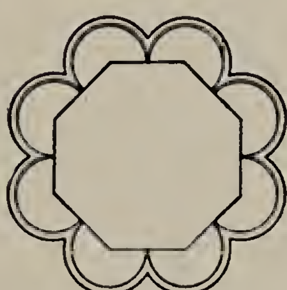


Fig. 358

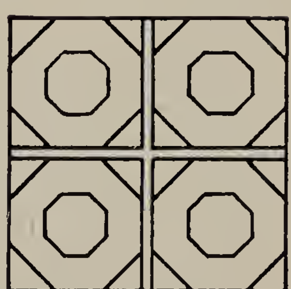


Fig. 359

window, Fig. 355; in linoleum patterns, Figs. 356–357; in paper doilies, Fig. 358; in ceiling panels, Fig. 359, floor borders, furniture designs, etc.

Point out the regular polygons in Figs. 354–359.

It is the purpose of the first part of the chapter to learn how to construct regular polygons.

EXERCISES

1. Show that an equilateral triangle is a regular polygon.

2. Draw a quadilateral that is equilateral but not equiangular. What is such a quadrilateral called?

3. Draw an equiangular quadrilateral. What is such a quadrilateral called?

4. Draw a quadrilateral that is not equiangular and not equilateral.

5. Show that a square is a regular polygon.

6. Make a sketch of a regular pentagon; hexagon; octagon (8-side).

432. Inscribed polygon. A polygon whose vertices lie on a circle is an **inscribed polygon**. The circle is said to be *circumscribed* about the polygon.

Draw an inscribed pentagon; hexagon.

433. Circumscribed polygon. A polygon whose sides are tangent to a circle is a **circumscribed polygon.** The circle is said to be *inscribed* in the polygon.

Draw a circumscribed polygon.

434. The theorems in §§ 435 and 437 will be used when we wish to prove that an inscribed or circumscribed polygon is a regular polygon. They show that the construction of regular inscribed and circumscribed polygons depends upon the problem of dividing a circle into a given number of *equal parts*.

HOUSE IN NUREMBERG, GERMANY

TOWN HALL, WERNIGERODE, GERMANY

Write an essay on the uses of mathematical forms in artistic buildings, using the pictures in this book as illustrations.

435. Theorem: *If a circle is divided into equal arcs, the chords subtending these arcs form a regular inscribed polygon.*

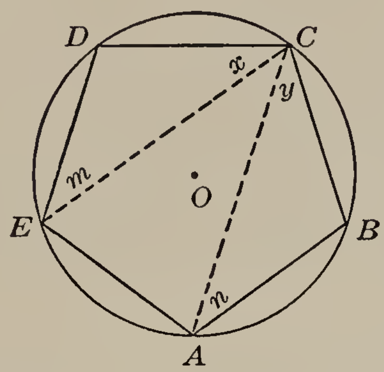


FIG. 360

Given the circle O, Fig. 360, divided into equal arcs, AB, BC, CD, etc.

The polygon $ABCD$ . . . . formed by the chords subtending these arcs.

To prove that $ABCD$ . . . . is a regular inscribed polygon.

Proof: I. Show that chords AB, BC, CD, ... are equal.

II. In triangles ABC and EDC show that $x = y$, $m = n$ (§ 298).

$\therefore \angle D = \angle B.$ Why?

Similarly, prove that the other angles of the polygon are equal.

Hence, $ABCD$ . . . . is a regular inscribed polygon. Why?

436. Theorem: *If the midpoints of the arcs subtended by the sides of a regular inscribed polygon of n sides are joined to the adjacent vertices of the polygon, a regular inscribed polygon of 2n sides is formed.* Prove.

437. Theorem: *If a circle is divided into equal arcs, the tangents drawn at the points of division form a regular circumscribed polygon.*

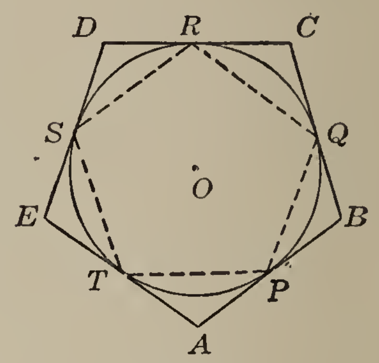


FIG. 361

Given circle O, Fig. 361; $\overset{\frown}{PQ} = \overset{\frown}{QR} = \overset{\frown}{RS}$, etc.; AB, BC, CD, etc., tangent to circle O, forming the circumscribed polygon $ABCD \ldots$.

To prove $ABCD \ldots$ a regular polygon.

Proof: Draw $PQ, QR, RS, \ldots,$ etc.

Prove $\triangle PBQ$, QCR, RDS, etc., *congruent isosceles* triangles.

$\therefore \quad \angle A = \angle B = \angle C$, etc.

Since $\quad AP = BQ,$ Why?

and $\quad PB = QC,$ Why?

$\therefore \quad AB = BC.$ Why?

Similarly, prove $\quad BC = CD = DE$, etc.

Hence, $ABCD$ is a *regular* polygon.

438. Theorem: *If tangents are drawn to a circle at the midpoints of the arcs terminated by consecutive points of contact of the sides of a regular circumscribed polygon a regular circumscribed polygon is formed having double the number of sides.* Prove. (See Fig. 362.)

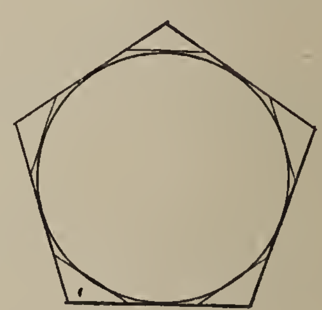


FIG. 362

EXERCISES

1. Prove that an equilateral inscribed polygon is regular.

Show that the circle is divided into equal arcs. Then apply § 435.

2. Prove that an equiangular circumscribed polygon is regular.

Show that the circle is divided into equal arcs. Then use § 437.

439. Problem: *To inscribe a square in a given circle.*

Given circle O, Fig. 363.

Required to inscribe a square in circle O.

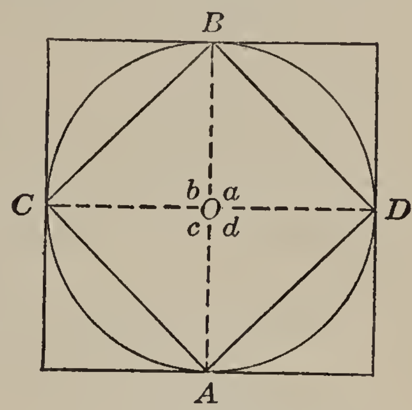


FIG. 363

Analysis: Since the square is a *regular* quadrilateral, we can inscribe a square if we can divide the circle into four equal arcs.

A circle may be divided into four equal arcs by dividing the plane around the center into four equal angles.

Since the sum of the angles around O is 360°, each of the four equal angles must be 90°.

State a way of constructing four right angles at O.

Construction: Draw the diameter AB.

Draw diameter $CD \perp AB$.

Draw AD, DB, BC, and CA.

Then $ADBC$ is the required square.

Proof: $a=b=c=d=90°$. Why?

$\therefore \quad \widehat{AD}=\widehat{DB}=\widehat{BC}=\widehat{CA}$. Why?

$\therefore \quad ADBC$ is a *regular* quadrilateral, i.e., **a square.** Why?

440. Problem: *To circumscribe a square about a given circle.*

Proceed as in the construction in § 439 and draw tangents at A, B, C, and D.

EXERCISES

1. Denoting the side of the inscribed square by a, the radius by r, prove that $a = r\sqrt{2}$.

The problem may be solved by algebra, or by trigonometry:

(*a*) Apply the theorem of Pythagoras to the sides of triangle AOD, Fig. 364.

(*b*) Find the required relation using the sine of 45°.

Notice that the equation $a = r\sqrt{2}$ expresses the fact that the side of the inscribed square *varies directly as the radius.*

Show that a is a function of r.

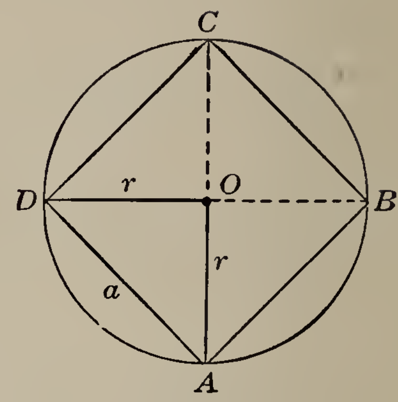


FIG. 364

2. Express the side a of the circumscribed square in terms of the radius r.

3. Express the perimeters of the inscribed and circumscribed squares in terms of the radius; in terms of the diameter.

4. Prove that the point of intersection of the diagonals of a square is the center of the inscribed and circumscribed circles.

5. Show how to construct regular polygons of 8, 16, 32, etc., sides.

6. Show that the number of sides of the polygons in exercise 5 is expressed by the formula 2^n, where n is a positive integer equal to, or greater than 2.

441. Problem: *To inscribe a regular hexagon in a given circle.*

Analysis: Into how many equal arcs must the circle be divided?

How large must the central angles be that intercept these arcs?

State a simple way of constructing an angle of 60°.

Construction: With A as center and radius AO, Fig. 365, draw an arc cutting the circle at B.

With B as center and the same radius draw the arc at C.

Similarly, draw arcs at D, E, and F.

Draw the polygon $ABCDEF$. This is the required hexagon.

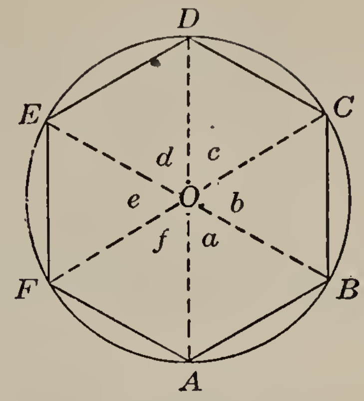


FIG. 365

Proof: Draw OA, OB, OC, etc.

Prove that $a=b=c=d=e=f=60°$.

Prove that $\widehat{AB}=\widehat{BC}\ldots\ldots\ldots=\widehat{FA}$.

Then polygon $ABCDEF$ is regular. Why?

442. Problem: *To circumscribe a regular hexagon about a given circle.*

EXERCISES

1. Express the relation between the side a of the regular inscribed hexagon and the radius r.

2. Express in terms of the radius the side of the regular circumscribed hexagon.

Draw OA and OK, Fig. 366.

Show that triangle AOK is a 60°–30° right triangle.

Hence $AO=2 \cdot AK=a$.

Find the required relation between a and r,

First by using the theorem of Pythagoras;

Secondly, by using the tangent of 30°.

Show that the side of the regular circumscribed hexagon varies directly as the radius.

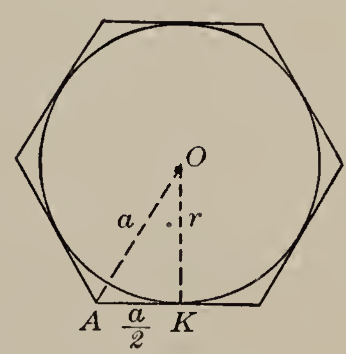


FIG. 366

Show that the side is a function of the radius.

3. Inscribe and circumscribe an equilateral triangle, a regular 12-side, 24-side, etc.

4. Show that the number of sides of the polygons in exercise 3 are given by the formula $3 \cdot 2^n$, n being a positive integer, or zero. (See exercise 7 below for value of 2^0.)

5. Show that $\frac{a^5}{a^3}=a^2$; $\frac{a^7}{a^3}=a^4$; $\frac{a^m}{a^n}=a^{m-n}$, m being greater than n, and m and n being *positive integers.*

6. Show that $\frac{a^2}{a^2}=1$; $\frac{a^3}{a^3}=1$; $\frac{a^m}{a^m}=1$.

7. Assuming that $\frac{a^m}{a^n}=a^{m-n}$ when $m=n$; show that $\frac{a^m}{a^m}=a^0$.

So far we have not defined the expression a^0. To make the results of exercises 6 and 7 agree, we shall define **a^0 to mean 1.**

8. Give the values of 2^0, 3^0, x^0, $(a+b)^0$, $(2x-y+z)^0$.

9. Express in terms of the radius r, the side of the inscribed equilateral triangle.

Show that OK, Fig. 367 is $\frac{r}{2}$ (§ see exercise 2).

Obtain the required relation first, by using the theorem of Pythagoras; secondly, by using the tangent of 60°.
Express your result in the language of variation.

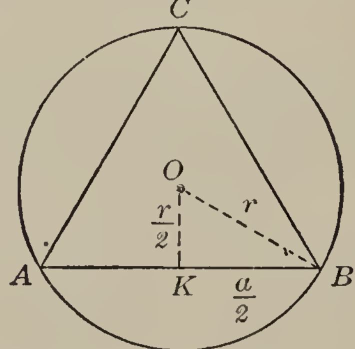


FIG. 367

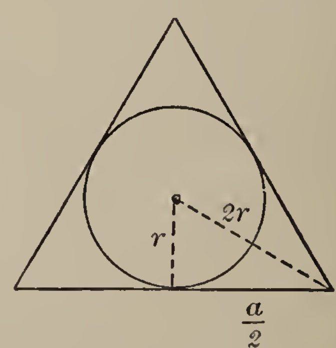


FIG. 368

10. Show that the side of the circumscribed equilateral triangle is $2r\sqrt{3}$ (Fig. 368).

(1) Use the theorem of Pythagoras.
(2) Use the tangent function.

11. Express in terms of the radius r the perimeters—
(*a*) of the regular inscribed and circumscribed hexagon,
(*b*) of the equilateral inscribed and circumscribed triangles.

Show that the perimeters vary directly as the radii.

443. Problem: *To inscribe a regular decagon in a given circle.*

Analysis: Into how many equal arcs must the circle be divided?

How large are the central angles intercepting these arcs?

Construction: The construction of an angle of 36° depends upon the problem of dividing a segment into mean and extreme ratio. (See § 315, exercise 8.)

Draw the radius AO, Fig. 369.

Divide AO into mean and extreme ratio at B, making

$$\frac{OA}{OB}=\frac{OB}{BA}.$$

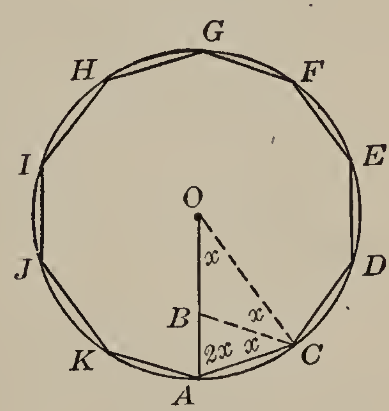


FIG. 369

With A as center and radius OB draw an arc at C.
With the same radius and center C draw an arc at D.
Similarly, draw arcs at E, F, G, H, I, J, and K.
Draw AC, CD, etc.
Polygon $ACD \ldots\ldots\ldots K$ is the required polygon.

Proof: Draw BC and OC.

$$\text{Since } \frac{OA}{OB}=\frac{OB}{BA},$$

$$\therefore \quad \frac{OA}{AC}=\frac{AC}{BA}. \qquad \text{Why?}$$

I.e., in $\triangle BCA$ and AOC two sides of one are proportional to two sides of the other.

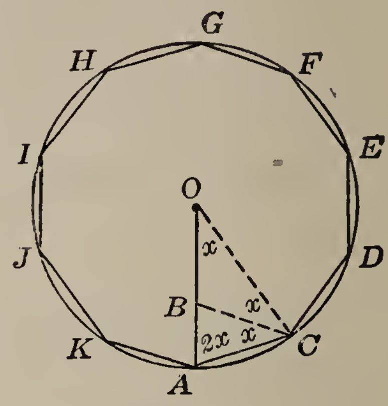


Fig. 369

Show that the included angle A is the same in both triangles.

$\therefore \quad \triangle BCA \backsim \triangle AOC.$ Why?

$\therefore \quad \dfrac{BC}{OA}=\dfrac{CA}{OC}.$ Why?

$\therefore \quad OC \cdot BC = OA \cdot CA.$ Why?

$\therefore \quad BC = CA.$ Why?

$\therefore \quad BC = OB.$ Why?

Denoting $\angle AOC$ by x, show that $OCB = x$ and that $\angle BCA = x$.

Since $\angle OCA = \angle OAC$, it follows that $\angle OAC = 2x$.

$\therefore \; 2x+2x+x=180°.$ Why?

$\therefore \quad x = 36°.$

Show that polygon $ACD \ldots\ldots\ldots K$ is a regular decagon.

EXERCISES

1. To circumscribe a regular decagon about a circle.

2. Show how to inscribe and circumscribe a regular pentagon in a given circle.

3. To inscribe and circumscribe regular polygons having 20, 40, etc., sides.

4. Show that the number of sides of the polygons in exercise 3 may be expressed by the formula $5 \cdot 2^n$, n being a positive integer or zero.

5. Express the relation between the side of the inscribed decagon and the radius of the circle.

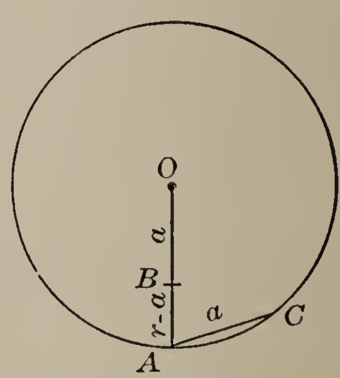


Fig. 370

Denoting $AC = OB$ by a, Fig. 370, OA by r, then $BA = r - a$.

Show that $\dfrac{r}{a} = \dfrac{a}{r-a}.$

$\therefore \quad a^2 = r^2 - ra.$

$\therefore \quad a^2 + ra - r^2 = 0.$

Solving by means of the quadratic formula,

$$a=\frac{-r\pm\sqrt{r^2+4r^2}}{2}$$

$$a=\frac{-r\pm r\sqrt{5}}{2}=\frac{r}{2}(-1\pm\sqrt{5}).$$

Show that the minus sign before the radical cannot be used *in this problem.*

$$\therefore\quad a=\frac{r}{2}\left(\sqrt{5}-1\right)=\frac{r}{2}\left(1.236\right)=.618r.$$

‡**6.** Show that the side of a regular inscribed pentagon is equal to $\frac{r}{2}\sqrt{10-2\sqrt{5}}$

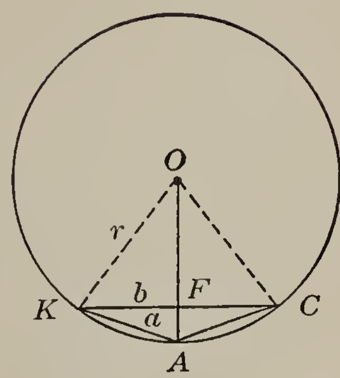


FIG. 371

Let KC, Fig. 371, be the side of the pentagon, KA and AC sides of the decagon.

Denote KF by b, OK by r, and KA by a.

Then, $\overline{OF}^2=r^2-b^2$ and $OF=\sqrt{r^2-b^2}$. Why?

$$\therefore\quad FA=r-\sqrt{r^2-b^2}.$$

Since $\overline{KF}^2=\overline{KA}^2-\overline{FA}^2$,

$b^2=a^2-(r-\sqrt{r^2-b^2})^2$. Why?

Substituting for a^2 its equal, $\frac{r^2}{4}\left(\sqrt{5}-1)\right)^2$ (exercise 5) and solving for b, we have—

$$b=\frac{r}{4}\sqrt{10-2\sqrt{5}}$$

$$\therefore\quad 2b=\frac{r}{2}\sqrt{10-2\sqrt{5}}.$$

‡**7.** Show that an approximate value of $\sqrt{10-2\sqrt{5}}$ is $2.351+$.

8. Using the sine function, find the side of the regular inscribed pentagon; decagon.

Notice the advantage of the trigonometric method over the algebraic methods used in exercises 5 and 6.

9. A man has a round table top which he wishes to change into the form of a pentagon as large as possible. The diameter of the top is $2\frac{1}{2}$ feet. What is the length of the cut required?

444. Problem: *To construct a regular 15-side in a given circle.*

Analysis: The circle must be divided into 15 equal arcs. How large are the central angles intercepting these arcs?

Notice that $24° = 60° - 36°$.

This suggests the following construction:

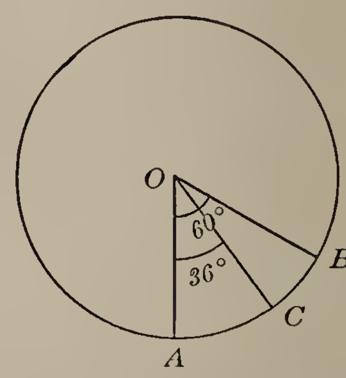


FIG. 372

Construction: At O on OA construct an angle of 60°, Fig. 372.

At O on OA construct an angle of 36°, as $\angle AOC$.

Then $\angle COB = 24°$.

$\therefore$ CB may be taken as the side of the regular inscribed 15-side.

EXERCISES

1. Show how to construct regular inscribed and circumscribed polygons having 30, 60, 120 . . . . sides.

‡**2.** Show that the number of sides of the polygons in exercise 1 is given by the formula $15 \cdot 2^n$ where n is a positive integer, or zero.†

† Gauss (1777–1855), a German mathematician, proved that by the use of an unmarked straight edge and a compass a circle can be divided into (2^k+1) equal parts, k being a number that makes 2^k+1 a prime number.

Denoting 2^k+1 by n, we have
For $k=1$, $n=3$, a prime number.
For $k=2$, $n=5$, a prime number.
For $k=3$, $n=9$, not a prime number.
For $k=4$, $n=17$, a prime number.
For $k=5$, $n=33$, not a prime number, etc.

CARL FRIEDRICH GAUSS

CARL FRIEDRICH GAUSS

CARL FRIEDRICH GAUSS was born at Brunswick, Germany, April 30, 1777, and died at Göttingen, February 23, 1855. His father was a bricklayer and did not sympathize with the son's aspirations for an education. Coupled with this was the fact that the schools of Gauss's day were very poor; but in spite of parental disapproval and very inadequate schools he became one of the greatest mathematicians of all time.

Gauss had a marvelous aptitude for calculation, and in later years used to say, perhaps only as a joke, that he could reckon before he could talk. He owed his education to the fact that one of his teachers, named Bartels, drew the attention of the reigning duke of Brunswick to the remarkable talents of the boy. The duke provided for him the means of obtaining a liberal education. As a boy Gauss studied the languages with quite as much success as mathematics.

When only nineteen, Gauss discovered a method of inscribing a regular polygon of seventeen sides in a circle. This encouraged him to pursue mathematical studies. He studied at Göttingen from 1795 to 1798. He made many of his most important discoveries while yet a student. His favorite study was higher arithmetic. In 1798 he went back to his home town of Brunswick, and for a few years earned a scanty living by private tuition.

In 1799 Gauss published a demonstration of the important theorem that *every algebraical equation has a root of the form* $a+bi$, and in 1801, a volume on higher arithmetic. His next great performance was in the field of astronomy. He invented a method for calculating the elements of a planetary orbit from three observations, by so powerful an analysis of existing data as to place him in the first rank of theoretical astronomers.

In 1807 he was appointed professor of mathematics and director of the observatory at Göttingen. He retained these offices until his death. He was devoted to his work. He never slept away from his observatory except on one occasion when he attended a scientific congress in Berlin. As a teacher he was clear and simple in exposition, and for fear his auditors might not get his train of thought perfectly he never allowed them to take notes. His writings are more difficult to follow, for he omitted the developmental details that he was so careful to supply in his lectures. His memoirs in astronomy, in geodesy, in electricity and magnetism, in electrodynamics, and in the theories of numbers and celestial mechanics are all epoch-making. Most of the whole science of mathematics has undergone a complete change of form by virtue of Gauss's work.

Gauss was the first to develop a real mathematical theory of errors. He introduced the geometrical theory of complex numbers into Germany. He was the first to use the term "complex number" in the sense it has today. He used the symbol $\equiv$ to signify *congruence*. A good description of Gauss's important work on the inscription of a regular polygon in a circle may be read in § 35 of Miller's *Historical Introduction to Mathematical Literature* (Macmillan).

The last-mentioned work, pp. 241–43, and also both Ball's and Cajori's *Histories*, give brief accounts of Gauss and his work.

‡**3.** The following is a practical method of constructing the side of a regular 10-side and 5-side.

Construction: Draw the diameter AB, Fig. 373.

Draw $OC \perp AB$.

Bisect OB at D.

With center at D and radius DC draw the arc CE.

Draw the straight line CE.

The sides of triangle EOC are equal to the sides of a regular hexagon, pentagon, and decagon, respectively.

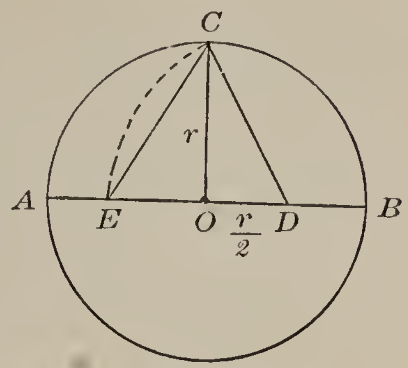


FIG. 373

Proof: I. $CO = r$ and is equal to the side of the regular inscribed hexagon.

$$\text{II. } \overline{CD}^2 = r^2 + \frac{r^2}{4}.$$

$$\therefore \quad CD = \frac{r}{2}\sqrt{5}.$$

$$EO = ED - OD = CD - OD = \frac{r}{2}\sqrt{5} - \frac{r}{2} = \frac{r}{2}\left(\sqrt{5}-1\right).$$

Hence, EO is the side of the decagon.

(See § 443, exercise 5.)

$$\text{III. } \overline{EC}^2 = r^2 + \overline{EO}^2 = r^2 + \frac{r^2}{4}\left(6 - 2\sqrt{5}\right) = \frac{4r^2 + 6r^2 - 2r^2\sqrt{5}}{4}$$

$$EC = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}, \text{ the side of the pentagon.}$$

(See § 443, exercise 6.)

4. Find the side of a decagon inscribed in a circle of radius 8; 10; 15; a.

‡**5.** The side of an inscribed pentagon is 18.8 inches. Find the radius of the circumscribed circle.

6. The side of an inscribed decagon is 14.83 inches. Find the radius of the circumscribed circle.

‡7. *If at the midpoint of the arcs subtended by the sides of a given regular inscribed polygon, tangents are drawn to the circle, they are parallel to the sides of the given polygon and form a regular circumscribed polygon.*

To prove that $AB \parallel A'B'$, Fig. 374, draw the radius OP' to the contact point of $A'B'$. Show that AB and $A'B'$ are both perpendicular to OP'.

To prove that $A'B'C'D'E'$ is regular, show that $\widehat{P'Q'} = \widehat{Q'R'} = \widehat{R'S'}$, etc.

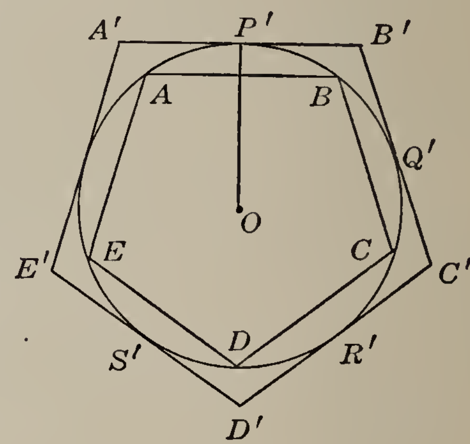


FIG. 374

8. In Fig. 374, prove that points O, B, and B' are on a straight line.

Prove that B and B' lie on the bisector of $\angle P'OQ'$.

9. Express 8 as a *theorem.*

445. Theorem: *A circle may be circumscribed about any given regular polygon.*

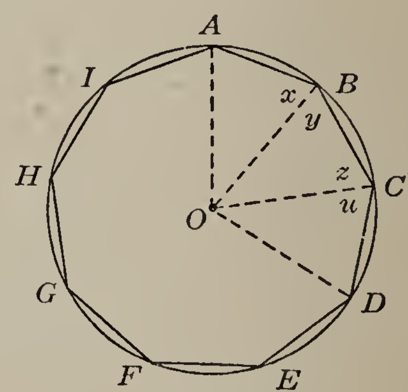


FIG. 375

Given the regular polygon $ABCD$......., Fig. 375.

To construct a circle circumscribed about $ABCD$.....

Construction: Construct a circle through A, B, and C.

This is the required circle.

Proof: It is to be proved that the circle ABC passes through D, E, etc.

$x+y=z+u.$ Why?

$y=z.$ Why?

$\therefore \quad x=u.$ Why?

Prove $\triangle AOB \cong \triangle COD.$

$\therefore \quad AO=OD$ and the circle passes through D.

Similarly, it may be shown that the circle passes through E, F, etc.

446. Theorem: *A circle may be inscribed in any given regular polygon.*

Given the regular polygon ABC......., Fig. 376.

Required to inscribe a circle within ABC........

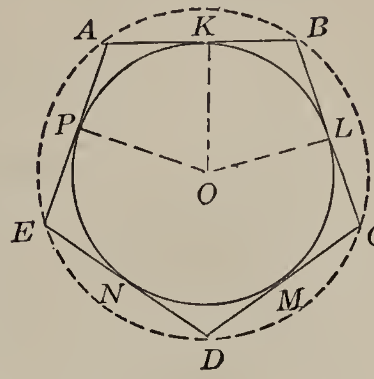


Fig. 376

Construction: Construct the center, O, of the circumscribed circle.

Draw $OK \perp AB$.

With O as center and radius OK draw circle KLM....

This is the required circle.

Proof: Draw the circumscribed circle ABC.......

Draw $OP \perp AE$.

Since chord $AB =$ chord AE, it follows that $OK = OP$. Why?

Hence, circle KLM passes through P. Why?

$\therefore$ AE is tangent to the circle. Why?

Similarly, show that ED, DC, etc., are tangents to circle KLM........

447. Theorem: *The perimeter of a regular inscribed 2n-side is greater than the perimeter of the regular n-side inscribed in the same circle.* Prove.

448. Theorem: *The perimeter of a regular circumscribed 2n-side is less than the perimeter of the regular n-side circumscribed about the same circle.* Prove.

449. Two important facts follow from the theorems in §§ 447 and 448, viz.:

1. The perimeter of the regular inscribed polygon *increases* as the number of sides *increases*.

2. The perimeter of the regular circumscribed polygon *decreases* as the number of sides *increases*.

The Length of the Circle

450. In the following discussion it will be shown that by increasing the number of sides of regular inscribed and circumscribed polygons, the perimeters approach each other more and more, and that the decimal fractions expressing these two perimeters can be made to agree to a greater and greater number of decimal places.

It is easily proved that the length of a circle is greater than the perimeter of any inscribed polygon. We will assume that the length of a circle is less than the perimeter of any circumscribed polygon.

Hence, *the length of a circle lies between the lengths of the perimeters of any pair of inscribed and circumscribed polygons.*

The determination of the length of the circle is obtained very simply by means of trigonometry:

Let AB, Fig. 377, be the side of a regular inscribed n-side.

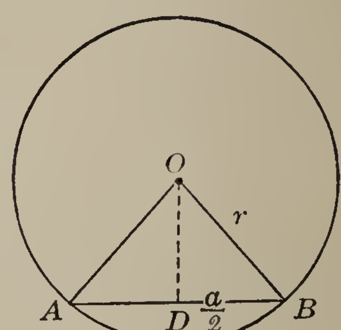


FIG. 377

Draw $OD \perp AB$.

Show that $\angle DOB = \left(\frac{360}{2n}\right)^\circ$.

Show that $\sin\left(\frac{360}{2n}\right)^\circ = \frac{\frac{a}{2}}{r}$, a denoting the side of the polygon and r the radius of the circumscribed circle.

Hence, $a = \sin\left(\frac{360}{2n}\right)^\circ d.$ Why?

$\therefore$ the perimeter, $p = n\left[\sin\left(\frac{360}{2n}\right)^\circ\right]d.$ Why? . . (A)

From Fig. 378 show that the perimeter P of the circumscribed polygon is given by

$$P = n\left[\tan\left(\frac{360}{2n}\right)^{\circ}\right]d \ldots\ldots\ldots (B)$$

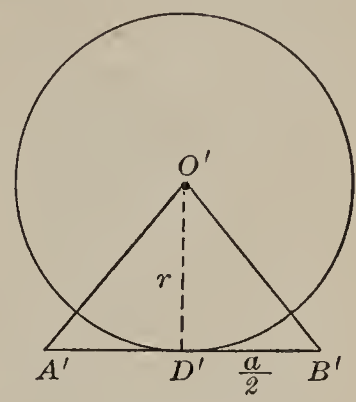


FIG. 378

By means of formulas (A) and (B) the perimeters of inscribed and circumscribed polygons may be computed, leading to the determination of approximate values of the length of the circle.

Make the computations and compare your results with the results given in the following table:

Number of sides	Perimeter of inscribed polygon	Length of circle, l	Perimeter of circumscribed polygon
3	2.5980.... · d	$2.5d < l < 5.2d$	5.1963.... · d
4	2.8284.... · d	$2.8d < l < 4.0d$	4.0000.... · d
5	2.9390.... · d	$2.9d < l < 3.7d$	3.6325.... · d
6	3.0000.... · d	$3.0d < l < 3.5d$	3.4644.... · d
7	3.0359.... · d	$3.0d < l < 3.4d$	3.3691.... · d
8	3.0614.... · d	$3.0d < l < 3.4d$	3.3137.... · d
12	3.1058.... · d	$3.1d < l < 3.3d$	3.2153.... · d
18	3.1248.... · d	$l = 3.1d$, approximately	3.1734.... · d
90	3.1410.... · d	$l = 3.141d$, approximately	3.141..... · d

The table above shows how the decimal fractions expressing the perimeters agree more and more closely as the number of sides of the polygon is increased.

The following table, which gives the decimal fractions to *six* places, shows the approach of the perimeters still better:

$P_4{}^* = 4.000000 \quad \cdot d$	$p_4{}^* = 2.828427\ldots \cdot d$
$P_6 = 3.464121\ldots \cdot d$	$p_6 = 3.000000 \quad \cdot d$
$P_8 = 3.313708\ldots \cdot d$	$p_8 = 3.061467\ldots \cdot d$
$P_{12} = 3.215390\ldots \cdot d$	$p_{12} = 3.105828\ldots \cdot d$
$P_{16} = 3.182598\ldots \cdot d$	$p_{16} = 3.121445\ldots \cdot d$
$P_{24} = 3.159659\ldots \cdot d$	$p_{24} = 3.132623\ldots \cdot d$
$P_{32} = 3.151725\ldots \cdot d$	$p_{32} = 3.136548\ldots \cdot d$
$P_{48} = 3.146086\ldots \cdot d$	$p_{48} = 3.139350\ldots \cdot d$
$P_{64} = 3.144118\ldots \cdot d$	$p_{64} = 3.140331\ldots \cdot d$
$P_{96} = 3.142714\ldots \cdot d$	$p_{96} = 3.141032\ldots \cdot d$
$P_{128} = 3.142224\ldots \cdot d$	$p_{128} = 3.141277\ldots \cdot d$
$P_{192} = 3.141873\ldots \cdot d$	$p_{192} = 3.141452\ldots \cdot d$
$P_{256} = 3.141750\ldots \cdot d$	$p_{256} = 3.141514\ldots \cdot d$
$P_{348} = 3.141662\ldots \cdot d$	$p_{348} = 3.141557\ldots \cdot d$

The last two perimeters agree to *three* decimal places. Thus, the length of the circle of diameter d which lies between these perimeters is found correct to three decimal places. It equals $3.141\ldots \cdot d$.

As the perimeters of the inscribed and circumscribed polygons with increasing numbers of sides, approach each other in length, both of them approach more and more closely the length of the circle. But however close the length of the perimeter of any polygon may come to the length of the circle, there is always another polygon the perimeter of which comes still closer to the length of the circle; and for every number given as expressing the *difference* between any perimeter and the circle we can find a polygon whose perimeter differs from the circle by *less* than that number. This is expressed by saying that the perimeters of the inscribed and circumscribed polygons *approach* the circle *as a limit.*

* The subscripts indicate the number of sides of the polygons.

As is seen by the table on p. 296, the value of this limit can be expressed more and more closely by taking polygons of a greater and greater number of sides. It cannot, however, be determined exactly.

Continuing to increase the number of sides, we find in the table above

$$P_{8192} = 3.1415928\ldots \cdot d$$

and $$p_{8192} = 3.1415926\ldots \cdot d.$$

From this it is seen that the circle, being between P_{8192} and p_{8192}, can be expressed by $C = 3.141592\ldots \cdot d$, approximately, with an error *less than 1 millionth.*

The length of the circle is therefore a *multiple* of the diameter, which, however, may not be exactly expressed in figures. The number 3.141592.... by which d is multiplied, is commonly denoted by π (the first letter of περιφέρεια, meaning *periphery* or *circumference*).

Thus, $$\boldsymbol{C = \pi d,}$$

and $$\boldsymbol{C = 2\pi r}$$

are the formulas expressing the length of the circle in terms of the diameter and radius, respectively. For our purposes it is sufficient to use $\pi = 3.14$, or $\pi = \frac{22}{7}$, which is equal to 3.14 when carried out to *two* decimal places.

451. Historical note. The determination of the value and of the nature of the number π is one of the famous problems of geometry.

Ahmes took $\pi = \left(\frac{16}{9}\right)^2$.

Archimedes (212–287 B.C.) found the value of π to be such that $3\frac{10}{71} < \pi < 3\frac{10}{70}$ by finding the values of P_{96} and p_{96}.

Ptolemy (150 A.D.) calculated $\pi = 3 + \frac{8}{60} + \frac{8}{60^2} = 3.14166.$

At the end of the sixteenth century Vieta (1579 A.D.) found the value of π to 10 decimal places, and Ludolph van Ceulen (1540–1610) to 20, 32, and 35 places. The value of π has since been carried out to more than 700 decimal places, to 30 places it is as follows:

3.141592653589793238462643383279+ (see the article "Circle" in the *Encyclopaedia Brittanica*, 11th ed.).

It was shown by Lambert (1728–1777) that the number π cannot be expressed exactly in terms of integers and hence is not a *rational number*.

Lindemann (1882) proved that π belongs to a class of numbers called *transcendental*, numbers which do not satisfy any algebraic equation with rational coefficients.

EXERCISES

1. The length of a circle is 100 inches. Find the radius.

2. Show that *the lengths of two circles are to each other as the radii or as the diameters.*

3. The distance around one of the famous large trees in California is about 100 feet. Find the diameter.

4. The radius of a fly wheel of an engine is 9 feet. If the wheel makes 40 revolutions per minute, what is the rate, in feet, per minute of a point on its outer rim?

5. The size of a man's hat is indicated by the number of inches in the diameter of a circle of length equal to the distance measured around the head where his hat rests. What size of hat does a man need, the distance around whose head is $22\frac{3}{4}$ inches?

6. Measure the distance around your own head and calculate the size of hat you need.

7. A trick circus rider performed on a tall bicycle one turn of whose driving wheel carried the bicycle 62.8 ft. forward. How tall was the wheel?

8. A circular pond is 2640.1 yd. in circumference. Find the diameter.

452. Historical note. Regular polygons have been used for decorative purposes since the beginning of mathematical history. Only such regular polygons as result from the division of the circle into 4 equal parts, i.e., squares, octagons, etc., were known in Egypt before the Eighteenth Dynasty. About this time the dodecagon appeared on presents sent to Pharaoh by his Asiatic subjects. Since the Nineteenth Dynasty chariot wheels with six spokes are shown on mural reliefs, and very rarely with four or eight spokes. The knowledge of the sextuple division of the circle was brought to Egypt from Babylon, though it is not known at what date this occurred. The Chaldaeans had a strong bias in favor of six and its multiples.

The Greeks advanced the knowledge of regular polygons. The Pythagoreans thoroughly reworked Egyptian and Babylonian knowledge and extended it by original research. They taught how to calculate the central angle for all n-gons. It is not definitely known whether the Pythagoreans could construct the regular pentagon, though they used the *pentagram* (the star pentagon) as a symbol of secrecy, and at least studied the pentagon. At all events the Greek mathematicians by the time of Eudoxus (408–355 B.C.) were masters of the division of a line into mean and extreme ratio, upon which the construction of the regular decagon depends.

That the side of a regular inscribed hexagon is equal to the radius of the circle was known in substance to the ancient Babylonians. Hippocrates (440 B.C.) mentions this property of the hexagon as a well-known theorem. The mode of calculating the sides of our most familiar regular polygons was known by the time of Hero of Alexandria (first century, B.C.).

Antipho (430 B.C.) was the first to make use of the regular inscribed polygon to approximate the area and length of the circle. Bryso, a contemporary of Antipho, improved on the latter's method, and the theory was very greatly extended by Archimedes. The latter had a method of calculating the side of a $2n$-gon from the side of an n-gon. By means of regular polygons he shut π in between the limits of $3\frac{1}{7}$ and $3\frac{10}{71}$.

After Archimedes no further advance in the theory of regular polygons was made until the thirteenth century. Jordanus Nemorarius (1237 A.D.) did not seek to *square the circle* by the aid of regular polygons, as most later writers had done, but rather to derive relations between the perimeters and areas of regular inscribed and circumscribed polygons of n and $2n$ sides.

Summary

453. The chapter has taught the meaning of the following terms:

regular polygon, circumscribed polygon, inscribed polygon

454. The following theorems may be used to prove that inscribed or circumscribed polygons are regular:

1. *If a circle is divided into equal arcs the chords subtending these arcs form a regular inscribed polygon.*

2. *If the midpoints of the arcs subtended by the sides of a regular inscribed polygon of n sides are joined to the adjacent vertices of the polygon, a regular inscribed polygon of 2n sides is formed.*

3. *If a circle is divided into equal arcs, the tangents drawn at the points of division form a regular circumscribed polygon.*

4. *If tangents are drawn to a circle at the midpoints of the arcs terminated by consecutive points of contact of the sides of a regular circumscribed polygon, a regular circumscribed polygon is formed having double the number of sides.*

455. Other theorems proved in the chapter are:

1. *If at the midpoints of the arcs subtended by the sides of a given regular inscribed polygon, tangents are drawn to*

the circle, they are parallel to the sides of the given polygon and form a regular circumscribed polygon.

2. *A circle may be circumscribed about any given regular polygon.*

3. *A circle may be inscribed in any given regular polygon.*

4. *The perimeter of a regular inscribed 2n-side is greater than the perimeter of the regular n-side inscribed in the same circle.*

5. *The perimeter of a regular circumscribed 2n-side is less than the perimeter of the regular n-side circumscribed about the same circle.*

456. The chapter has taught how to inscribe in, and to circumscribe about a circle the following regular polygons: square, hexagon, decagon, 15-side.

Other regular inscribed and circumscribed polygons may be obtained by dividing the arcs of the circle into two or more equal parts, and then joining the points of division of the circle successively by line-segments.

457. The side and perimeter of a regular inscribed or circumscribed polygon may be expressed in terms of the radius of the circle. The side and perimeter vary directly as the radius.

458. The length of a circle is expressed by the formula

$$C = \pi d, \text{ or } C = 2\pi r.$$

CHAPTER XVIII

COMPARISON OF AREAS. LITERAL EQUATIONS. AREA OF THE TRIANGLE. FACTORING

Comparison of Areas

459. Theorem: *Parallelograms having equal bases and equal altitudes are equal.**

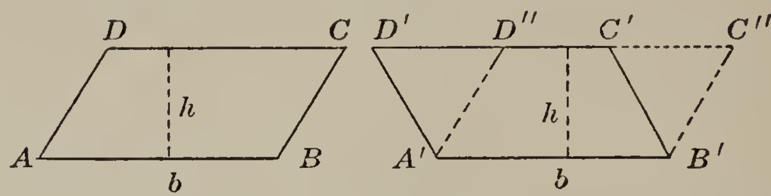


FIG. 379

Given parallelograms $ABCD$ and $A'B'C'D'$, Fig. 379, having equal altitudes h, and equal bases b.

To prove that $ABCD = A'B'C'D'$.

Proof: Imagine $ABCD$ placed upon $A'B'C'D'$, so that AB coincides with $A'B'$. Why can this be done?

Then DC must fall in the same line as $D'C'$, for the parallelograms have equal altitudes.

Prove that $\triangle D'D''A' \cong C'C''B'$ (s.a.s.).

But $\underline{D'A'B'C'' \equiv D'A'B'C''}$.

$\therefore \quad A'B'C''D'' = A'B'C'D'$

(equals subtracted from equals give equals).

$\therefore \quad ABCD = A'B'C'D'$. Why?

460. Theorem: *A parallelogram is equal to a rectangle having the same base and altitude.*

Apply the theorem in § 459.

* Equal is here used in the sense of *equal in area*, or *equivalent*.

461. Theorem: *A triangle is equal to one-half a parallelogram having the same base and altitude.*

Use the theorem that a diagonal divides a parallelogram into congruent triangles (Fig. 380).

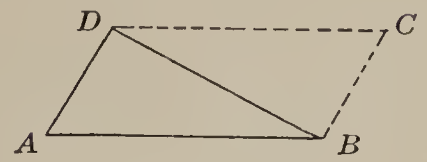


FIG. 380

462. Theorem of Pythagoras: *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the sides, including the right angle.*

Let ABC, Fig. 381, be a right triangle having a right angle at C. Let S_1, S_2, and S denote the squares on the sides a, b, and c, respectively.

To prove $S = S_1 + S_2$.

Proof: Draw $CD \perp AB$, dividing S into rectangles R_1 and R_2.

Draw AE and CF.

Show that triangle EBA and square S_1 have equal bases and altitudes.

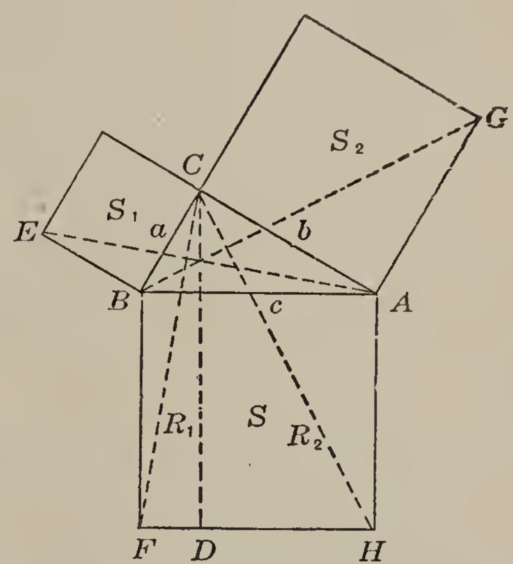


FIG. 381

Then triangle $EBA = \frac{1}{2}S_1$. Why? (1)

Similarly, prove that

triangle $FBC = \frac{1}{2}R_1$. (2)

But $\triangle ABE \cong \triangle FBC$. (3)

For $EB = BC$. Why?

$AB = BF$. Why?

And $\angle ABE = \angle FBC$. Why?

From (1), (2), (3) we have $\frac{1}{2}S_1 = \frac{1}{2}R_1$.

Therefore $S_1 = R_1$. (4)

Similarly, draw BG and CH, and prove $S_2 = R_2$.

Therefore $S = S_1 + S_2$.

463. Theorem: *The sum of the squares of two sides of a triangle is equal to twice the square of one-half of the third side increased by twice the square of the median to the third side.**

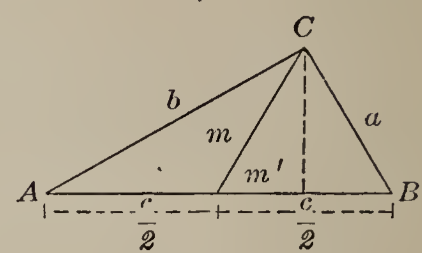


FIG. 382

Given $\triangle ABC$ having the median m to the side c, Fig. 382.

To prove that $a^2+b^2=2\left(\frac{c}{2}\right)^2+2m^2$.

Proof:
$$a^2=\left(\frac{c}{2}\right)^2+m^2-2\left(\frac{c}{2}\right)m'. \qquad \S\ 240.$$

$$b^2=\left(\frac{c}{2}\right)^2+m^2+2\left(\frac{c}{2}\right)m'. \qquad \S\ 241.$$

$$\therefore\ a^2+b^2=2\left(\frac{c}{2}\right)^2+2m^2. \qquad \text{Why?}$$

Exercises in Literal Equations in One Unknown

1. Show that the length of the median to a side of a triangle may be expressed in terms of the sides of the triangle by means of the following formula:

$$m_c=\sqrt{\frac{a^2+b^2-2\left(\frac{c}{2}\right)^2}{2}}=\tfrac{1}{2}\sqrt{2a^2+2b^2-c^2}$$

2. Find m_c when a, b, and c are respectively,

(1) 6, 10, 8 ‡(2) 5, 13, 12 ‡(3) 9, 15, 12

3. Express m_a in terms of the sides a, b, and c of the triangle ABC.

* This theorem was added to elementary geometry by Pappus who lived and taught at Alexandria at the end of the third century A.D. It enables us to find the medians of a triangle when the lengths of the sides are known.

Solve the following equations for x or y:

4. $ax-bx=a-b$

Combining the terms in x, $(a-b)x=a-b$.
Dividing both sides by $a-b$, $x=1$.

5. $x^2-c^2=3c^2-2cx+x^2$

6. $x^3-2ax+c=a+3ax+x^3$

7. $a^2+ax+cx-ac=0$

8. $-3rx+2cm=-3cx+2rm$

9. $-b^2x+a^2x=-a+b$

10. $cx+ax=a^2+c^2+2ac$

11. $3x-ax=a^2+9-6a$

12. $s^2x+r^2x-2rsx=r^2-s^2$

13. $(m+n)x+(m-n)x=2m^2$

14. $-9(y-a)+6(2y+a)=-2(y+a)$

‡15. $m(my+n)+n^2=n(my+n)+m^2$

‡16. $p(x-p)+2pq=(x+q)q$

17. $\frac{x}{a}+3=\frac{1}{2a}$

18. $\frac{x}{a}+\frac{x}{b}=\frac{1}{ab}$

19. $-\frac{x}{a}+\frac{x}{b}=\frac{-b^2+a^2}{ab}$

‡20. $\frac{x}{4b}+\frac{-x}{3b}=\frac{6a-8b}{12ab}$

‡21. $\frac{x}{a}-x=1+\frac{1}{a^2}-\frac{2}{a}$

22. $\frac{-a+b}{x}+\frac{-a-b}{x}=-a$

23. $\frac{x}{c}-x=-c+\frac{1}{c}$

24. $\frac{2x-b}{6x+s}=\frac{x-2s}{3x+b}$

‡25. $\frac{2x-a}{c}-\frac{x-a}{d}=\frac{a}{c}$

26. $\frac{cx-d}{dx}+\frac{dx-c}{cx}=2+\frac{c-d}{cdx}$

‡27. $-\frac{2x+a}{c}-\frac{-x-a}{d}=+\frac{a}{c}$

‡28. $\frac{x^2-d}{cx}+\frac{d+x}{c}=\frac{2x}{c}-\frac{d}{x}$

29. $\frac{m}{x-m}=\frac{n}{x-n}$

‡30. $\frac{x+m}{x-n}=\frac{m+n}{m-n}$

Systems of Linear Equations in Two Unknowns Having Literal Coefficients

464. Solve for x and y:

$$\begin{cases} a^2x+b^2y=a+b & (1) \\ abx-aby=b-a & (2) \end{cases}$$

$a\times(1)$ $\qquad a^3x+ab^2y=a^2+ab \qquad (3)$

$b\times(2)$ $\qquad ab^2x-ab^2y=b^2-ab \qquad (4)$

(Add. Ax.) $\qquad (a^3+ab^2)x=a^2+b^2 \qquad (5)$

$$\therefore \quad x=\frac{a^2+b^2}{a^3+ab^2}=\frac{a^2+b^2}{a(a^2+b^2)}=\frac{1}{a}$$

Find the value of y, first by eliminating the x-terms from equations (1) and (2); and then by substituting $x=\frac{1}{a}$ in equation (2). Check your results.

EXERCISES

Solve each of the following; then check:

1. $\begin{cases} x+y=1 \\ ax-by=0 \end{cases}$

2. $\begin{cases} cx+ny=1 \\ ax-by=0 \end{cases}$

3. $\begin{cases} ax+by=h \\ bx+ay=k \end{cases}$

4. $\begin{cases} cx+dy=2cd \\ bx-cy=d-c \end{cases}$

5. $\begin{cases} ax+by=2ab \\ 2bx+3ay=2b^2+3a^2 \end{cases}$

6. $\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$

*7. $\begin{cases} \frac{1}{x}+\frac{1}{y}=\frac{1}{n} \\ \frac{1}{x}-\frac{1}{y}=\frac{1}{k} \end{cases}$

8. $\begin{cases} \frac{a}{x}+\frac{b}{y}=c \\ \frac{d}{x}+\frac{c}{y}=f \end{cases}$

‡9. $\begin{cases} \frac{m}{x}+\frac{n}{y}=k \\ \frac{n}{x}+\frac{m}{y}=h \end{cases}$

* Equations in exercises 7, 8, and 9 are not linear in x and y, but in $\frac{1}{x}$ and $\frac{1}{y}$.

The Area of the Triangle

465. Since all plane figures formed by straight lines may be divided into triangles, it is important to obtain formulas for computing the area of a triangle from given parts. All other figures may then be measured by means of the triangle. We are acquainted with the following formula which gives the area of a triangle in terms of the base and altitude:

Theorem: *The area of a triangle is equal to one-half the product of the base and altitude.* (§ 56).

$$\triangle ABC = \frac{1}{2} b \cdot h$$

466. The area of a triangle may be expressed in terms of *two* sides and the included angle.

For, from Fig. 383, $\triangle ABC = \frac{1}{2} b \cdot h$.

Since $\sin A = \frac{h}{c}$,

it follows that $h = c \sin A$.

By substitution, $\triangle ABC = \frac{1}{2} bc \sin A$.

This may be expressed as a theorem as follows:

Theorem: *The area of a triangle is equal to one-half the product of two sides by the sine of the included angle.*

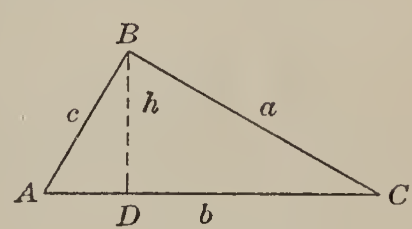


FIG. 383

EXERCISES

Show that *the area of an equilateral triangle of side a is equal to* $\frac{a^2}{4}\sqrt{3}$.

Show that *the area of a regular hexagon of side a is* $\frac{3\sqrt{3}}{2}a^2$.

467. Triangles inscribed in, or circumscribed about, a circle are frequently met.

The areas of such triangles may be expressed in terms of the sides and the radius of the circle as follows:

Let O, Fig. 384, be the center of the inscribed circle.

Draw OA, OB, and OC, dividing triangle ABC into three triangles whose sum is $\triangle ABC$.

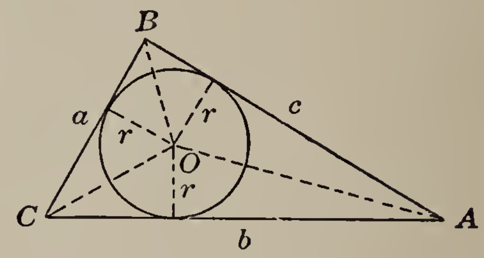


FIG. 384

Show that
$$\triangle COA = \tfrac{1}{2}r \cdot b$$
$$\triangle AOB = \tfrac{1}{2}r \cdot c$$
$$\triangle BOC = \tfrac{1}{2}r \cdot a$$

$$\therefore \quad \triangle ABC = \tfrac{1}{2}r\ (a+b+c).$$

Hence, *the area of a triangle is equal to the product of one-half the perimeter by the radius of the inscribed circle.*

It is customary to denote $\frac{1}{2}(a+b+c)$ by the symbol s.

Then, $\triangle ABC = rs.$

468. Theorem: *The area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle.*

For, let ABC, Fig. 385, be an inscribed triangle.

Draw the diameter BE.

Join EC.

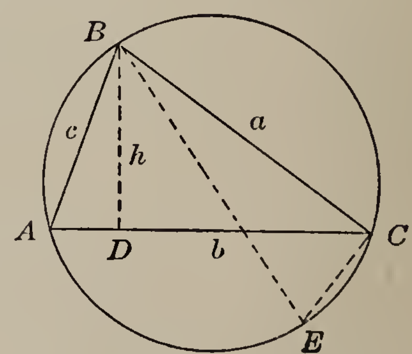


FIG. 385

$$\triangle ABC = \tfrac{1}{2}b \cdot h$$

Show $\triangle BDA \backsim \triangle BCE$

Then, $\dfrac{h}{a} = \dfrac{c}{2r}$. Why?

$$\therefore \quad h=\frac{ac}{2r}.$$

By substitution, $\triangle ABC=\frac{1}{2}\cdot b\cdot\frac{ac}{2r}$,

or, $$\triangle ABC=\frac{abc}{4r}$$

EXERCISES

1. The three sides of a triangle are 14, 8, and 12. The diameter of the circumscribed circle is 14.1. Find the area of the triangle.

2. Denoting the area of a triangle by T, then $T=\frac{abc}{4r}$. Solve the equation for r. Find, in terms of T, the radius of the circle circumscribed about a triangle whose sides are 17, 10, and 9; 8, 8, and 8; 15, 20, and 25.

3. Using the facts that the area of a triangle is $\frac{1}{2}bh$ and $\frac{abc}{2d}$, d being the diameter of the circumscribed circle, find a formula for the altitude to the side b in terms of the other sides and the diameter.

4. The sides of a triangle are 12, 10, and 8. The area is 39.7. Find the diameter of the circumscribed circle.

5. The angles of a right triangle are to each other as 1:2:3 and the altitude on the hypotenuse is 6 feet. Find the area.

‡6. Heron (1st cen. B.C.) expressed the altitude and area respectively, of an equilateral triangle as $h=a(1-\frac{1}{10}-\frac{1}{30})$, and $A=a^2(\frac{1}{3}+\frac{1}{10})$.
Calculate the errors of Heron's expressions.

469. The area of a triangle may be expressed in terms of the sides alone, thus:

Theorem: *The area of a triangle, in terms of its sides is*

$$\sqrt{s(s-a)(s-b)(s-c)}\,.$$

Given in triangle ABC, Fig. 386, the sides a, b, and c.

To prove that the area of ABC is equal to

$$\sqrt{s(s-a)(s-b)(s-c)}\cdot$$

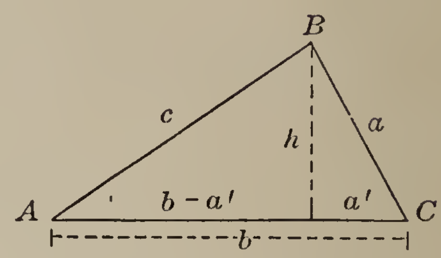


FIG. 386

Proof: Area $ABC = \frac{1}{2}b \cdot h$ (1)

This gives the area in terms of one side and the altitude h, which is not known. Let us now express h in terms of the sides and then substitute for h in equation (1).

$$h^2 = c^2 - (b-a')^2. \qquad \text{Why?} \qquad (2)$$

$$h^2 = a^2 - a'^2. \qquad \text{Why?} \qquad (3)$$

We must next eliminate a', which is not one of the three sides.

By comparison, $c^2-(b-a')^2 = a^2 - a'^2.$ (4)

Therefore, $c^2 - a^2 - b^2 + 2ba' = 0.$ (5)

Solving for a', we find $a' = \dfrac{b^2-c^2+a^2}{2b}$, (6)

Substituting in (3) the value of a' found in (6), we get

$$h^2 = a^2 - \left(\frac{b^2-c^2+a^2}{2b}\right)^2. \qquad (7)$$

Equation (7) expresses h^2 in terms of the sides a, b, and c.

We could now substitute the value of h in equation (1) and have a formula for the area of ABC in terms of a, b, and c. But in order to get a more symmetrical result, the value of h^2 in (7) will be changed *in form* before substituting in (1).

The right side of equation (7), being the difference of two squares, may be factored thus:

$$h^2=\left(a+\frac{b^2-c^2+a^2}{2b}\right)\left(a-\frac{b^2-c^2+a^2}{2b}\right).$$

Carrying out the indicated addition and subtraction within the parentheses, we have

$$h^2=\frac{2ab+b^2-c^2+a^2}{2b}\cdot\frac{2ab-b^2+c^2-a^2}{2b}.$$

$$h^2=\frac{a^2+2ab+b^2-c^2}{2b}\cdot\frac{c^2-(a^2-2ab+b^2)}{2b}. \quad \text{Why?}$$

$$h^2=\frac{(a+b)^2-c^2}{2b}\cdot\frac{c^2-(a-b)^2}{2b}. \quad \text{Why?}$$

Or $$h^2=\frac{(a+b-c)(a+b+c)}{2b}\cdot\frac{(c+a-b)(c-a+b)}{2b}. \quad (8)$$

Let $a+b+c=2s$.

Subtracting from both sides of this equation first $2c$, then $2a$ and then $2b$, we have

$$\left.\begin{aligned} a+b-c&=2s-2c=2(s-c)\\ b+c-a&=2s-2a=2(s-a)\\ c+a-b&=2s-2b=2(s-b)\end{aligned}\right\} \quad (9)$$

Substituting (9) in (8),

$$h^2=\frac{2(s-c)\cdot 2s\cdot 2(s-b)\cdot 2(s-a)}{4b^2}=$$

$$\frac{4s\cdot(s-a)(s-b)(s-c)}{b^2}.$$

Therefore,

$$h=\frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}. \quad \text{Why?} \quad (10)$$

Substituting (10) in (1),

$$ABC = \frac{1}{2}b \cdot \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)}.$$

Therefore,

$$ABC = \sqrt{s(s-a)(s-b)(s-c)}\ . \qquad (11)*$$

EXERCISES

1. The sides of a triangle are 3, 5, and 6. Find the area.

Using formula (11) of § 469,

the area $= \sqrt{7 \cdot (7-3)(7-5)(7-6)} = \sqrt{7 \cdot 4 \cdot 2 \cdot 1} = 2\sqrt{14}$, or 7.482, approximately.

2. The sides of a triangle are 34, 20, and 18. Find the area.

3. The sides of a triangle are 10, 6, and 8. Find the area.

‡**4.** The sides of a triangle are 90, 80, and 26. Find the area.

‡**5.** The sides of a triangle are 70, 58, and 16. Find the area.

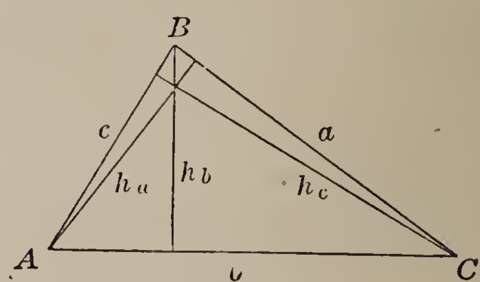


FIG. 387

470. Altitudes of a triangle. Denoting the altitudes of the triangle ABC to the sides a, b, and c by h_a, h_b, and h_c, respectively, Fig. 387, show that

$$h_b = \frac{2}{b}\sqrt{s(s-a)(s-b)(s-c)} \qquad (1)$$

(See § 469, formula [10].)

$$h_a = \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} \qquad (2)$$

$$h_c = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}\ . \qquad (3)$$

How can (2) and (3) be obtained from (1) by analogy?

* The law of formula (11) was introduced into mathematical texts by Heron of Alexandria in the first century B.C.

EXERCISES

1. In the triangle ABC, $a=10$, $b=17$, $c=21$. Find h_a.

$$h_a=\frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$$

$$s=\tfrac{1}{2}(a+b+c)=\tfrac{1}{2}(10+17+21)=24$$

$$s-a=14,\ s-b=7,\ s-c=3.$$

Substitute these values in the formula, and

$$h_a=\frac{2}{10}\sqrt{24\cdot 14\cdot 7\cdot 3}=\frac{1}{5}\sqrt{4\cdot 3\cdot 2\cdot 2\cdot 7\cdot 7\cdot 3}$$

$$=\frac{1}{5}\sqrt{4\cdot 9\cdot 4\cdot 49}=\frac{1}{5}(2\cdot 3\cdot 2\cdot 7)=\frac{84}{5}=16\frac{4}{5}.$$

2. Find the altitudes of each of the following triangles:

(1) $a=35$, $b=29$, $c=8$

(2) $a=70$, $b=65$, $c=9$

‡(3) $a=45$, $b=40$, $c=13$

3. The sides of a quadrilateral are as follows: $AB=29$, $BC=8$, $CD=28$, $DA=21$, and the diagonal $AC=30$. Find the area and the distance from D to AC.

471. Area of an equilateral triangle. *The area of an equilateral triangle is one-fourth the square of a side times the square root of 3*, or, in symbols, $A=\frac{a^2}{4}\sqrt{3}$.

The area of triangle ABC, Fig. 388, is given by the formula

$$\triangle ABC=\tfrac{1}{2}ah$$

Show that
$$h^2=a^2-\frac{a^2}{4}=\frac{3a^2}{4}$$

$$\therefore\ h=\frac{a}{2}\sqrt{3}$$

By substitution,
$$\triangle ABC=\tfrac{1}{2}a\cdot\frac{a}{2}\sqrt{3}$$

$$\therefore\ \triangle \boldsymbol{ABC}=\frac{\boldsymbol{a^2}}{\boldsymbol{4}}\sqrt{\boldsymbol{3}}$$

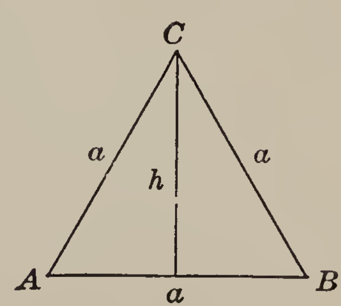


FIG. 388

EXERCISES

1. Find the areas of the following equilateral triangles, having the side equal to 12; 10; 4; 8; $c+d$; $2mn$.

2. Find the side of an equilateral triangle whose area is

$$\frac{121}{4}\sqrt{3}; \qquad 12;$$

$$25\sqrt{3}; \qquad 10\sqrt{3}.$$

In proving the formula for the area of a triangle in terms of the sides, § 469, we have factored the polynomials $2ab+b^2-c^2+a^2$ and $2ab-b^2+c^2-a^2$.

In § 472–476 we shall study further the method used in factoring these polynomials as well as some other frequently occurring polynomial forms.

Polynomials Factored by Grouping

472. *The terms of some polynomials may be grouped to show a common binomial factor.*

1. Factor $3a+3b+5a+5b$

Grouping the first two terms and the last two terms,

$$\underbrace{3a+3b}+\underbrace{5a+5b}=3(a+b)+5(a+b)=(a+b)(3+5)$$

Test by multiplication.

2. Factor $ac+bc+ad+bd$

$$\underbrace{ac+bc}+\underbrace{ad+bd}=c(a+b)+d(a+b)=(a+b)(c+d)$$

Test by multiplication.

3. Factor $14x^3-6x^2-21x+9$

$$14x^3-6x^2-21x+9=2x^2(7x-3)-3(7x-3)=(7x-3)(2x^2-3)$$

Test by substitution and by multiplication.

EXERCISES

Resolve into factors the following expressions and test results, doing as many as you can mentally:

1. $ax+bx+am+bm$
2. $ar+br+as+bs$
3. $ad+bd+ai+bt$
4. $3a+3b+ay+by$
5. $ak-bk+al-bl$
6. $ax^2-bx^2+ay^3-by^3$
7. $abc+abx+nc+nx$
8. $a^2k+a^2l+b^2k+b^2l$
9. $5au-5av+mu-mv$
10. $m^2a+ma^2+m^3a^2+m^2a^3$
11. $a^2-ad+ab-bd$
12. $x^6+5x^4+x^3+5x$
13. $6x^2-9x-10xy+15y$
14. $2m^3+m^2+6m+3$
15. $3ac+3ax-5c-5x$
16. $9-15r+27r^2-45r^3$
17. $8gh+12ah+10bg+15ab$
18. $15z-6-20zw+8w$
19. $2m^2+3km-14mn-21kn$
20. $3ax+3ab+2x^2+2bx+b+x$
21. $4x^3+4x-4x^2z-4z$
22. $1+r-r^2xy-r^3xy$
23. x^2-x^3+1-x
24. $(a+m)(c+n)-2n(a+m)$
25. $(x+y)(a+b)-(x+y)(b+c)$
26. $m(x+y)^2+(x+y)$
27. $a^2(2a+1)^2-2a-1$
28. $a-b+a^2xy-b^2xy$
29. $(c+d)(c^2+d^2)+2c^2d+2cd^2$
30. $(x+y)^2(x-y)-(x-y)^2(x+y)$

Reduce the following fractions to lowest terms:

31. $\dfrac{ax+bx+am+bm}{ar+br+as+bs}$
32. $\dfrac{3u-3v+au-av}{5bu-5bv+2ku-2kv}$
33. $\dfrac{ax^2-bx^2+ay^2-by^2}{mx^2+my^2+nx^2+ny^2}$
34. $\dfrac{x^4-2x^3+7x-14}{2x^3-4x^2+6x-12}$

473. *The terms of some polynomials may be grouped to show the difference of two squares.*

EXERCISES

Factor the following polynomials:

1. $a^2-2ab+b^2-c^2$

Grouping the first three terms, $a^2-2ab+b^2-c^2$ equals

$$(a-b)^2-c^2=(a-b+c)(a-b-c)$$

2. $x^2-6xy+9y^2-16z^2$

3. $25x^2+16y^2-4a^2+40xy$

4. $k^2-x^2-2xy-y^2$

5. $1-a^2-2ab-b^2$

6. $9m^2-a^2-4ab-4b^2$

7. $36r^2-4+20t-25t^2$

8. $x^2+2xy+y^2-a^2-2ab-b^2$

9. $a^2+2a+2bc-b^2-c^2+1$

10. $9x^2+16y^2-49a^2-4b^2+28ab+24xy$

11. $9a^2-12ab+4b^2-16x^2-8xy-y^2$

474. *The terms of some polynomials can be grouped to show a perfect square.*

EXERCISES

Factor the following polynomials:

1. $a^2+2ab+b^2+6a+6b+9$

Grouping the first three terms, the 4th and 5th terms, and keeping the last term separate, we have,

$$a^2+2ab+b^2+6a+6b+9=(a+b)^2+6(a+b)+9=(a+b+3)^2$$

2. $m^2+2mn+n^2+2m+2n+1$

3. $m^2+2mn+n^2+6am+6an+9a^2$

4. $a^2+b^2+c^2+2ab+2ac+2bc$

475. *The terms of some polynomials may be grouped to show a trinomial which can be factored by the trial method.*

EXERCISES

Factor the following:

1. $x^2+y^2+x-2xy-y-6$

Grouping the first, second, and fourth terms, the third and fifth terms, and keeping the sixth term separate,

$$x^2+y^2+x-2xy-y-6=x^2-2xy+y^2+x-y-6$$
$$=(x-y)^2+(x-y)-6=(x-y+3)(x-y+2)$$

2. $a^2+2ab+b^2+3a+3b-10$

3. $a^2-6ab+9b^2+7ac-21bc-44c^2$

4. $m^4x+m^2x-650x$

5. $4x^2+8xy+4y^2+13x+13y+3$

6. $3c^2-6cd+3d^2-2c+2d-5$

476. *Some trinomials may be factored by first changing them to complete squares.*

EXERCISES

Factor the following trinomials:

1. $x^4+x^2y^2+y^4$

By adding x^2y^2 to the trinomial $x^4+x^2y^2+y^4$, it becomes a perfect square: $x^4+2x^2y^2+y^4$. However, this changes the value of the trinomial. To keep the value unchanged x^2y^2 is subtracted from the trinomial. Thus, $x^4+x^2y^2+y^4=x^4+2x^2y^2+y^4-x^2y^2$. This may be written: $(x^2+y^2)^2-(xy)^2$. This is the difference of two squares and its factors are $(x^2+y^2+xy)(x^2+y^2-xy)$.

2. $a^4-7a^2b^2+b^4$

3. x^4+x^2+1

4. $16x^4-17x^2y^2+y^4$

5. $25x^4+31x^2y^2+16y^4$

6. $a^2x^8+a^2x^4+a^2$

7. $49a^4b^4-53a^2b^2x^2+4x^4$

8. $9x^4-10x^2y^2+y^4$

9. $4a^4-5a^2b^2+b^4$

The difference of two squares may be obtained in exercise 9 by adding and subtracting either a^2b^2, giving $4a^4-4a^2b^2+b^4-a^2b^2$, or by adding and subtracting $9a^2b^2$, giving $4a^4+4a^2b^2+b^4-9a^2b^2$.

Show that both lead to the same prime factors. Show also that exercise 8 gives two pairs of factors that lead to the same prime factors.

10. Factor the following trinomials by adding and subtracting a monomial square:

1. x^4+4

Add and subtract $4x^2$

2. $4x^4+1$

3. m^4+4

4. a^4b^4+64

5. a^4+324b^4

6. $1024x^4+y^4$

7. $81x^4+4y^4$

477. Summary of factoring. Polynomials to be factored may be classified according to the number of terms they contain.

I. If the polynomial is a **binomial** it may be of the following types:

1. The difference of two squares, as x^2-y^2. The factors are $(x+y)(x-y)$.

2. The difference of two cubes, as x^3-y^3. The factors are $(x-y)(x^2+xy+y^2)$.

3. The sum of two cubes, as x^3+y^3. The factors are $(x+y)(x^2-xy+y^2)$.

II. If a polynomial is a **trinomial** it may be of the following types:

1. The perfect square, as $x^2\pm 2xy+y^2$. The factors are $(x\pm y)^2$.

2. A trinomial which may be changed into a perfect square by adding a term, as $x^4+x^2y^2+y^4$. This is changed to $x^4+2x^2y^2+y^4-x^2y^2$ and is then factored as the difference of two squares.

3. A trinomial of the form ax^2+bx+c. Such trinomials may be factorable, having factors obtainable by the trial method.

III. **Polynomials** not of any of the types in I and II may be factored:

1. By dividing each term by a common factor, as $ax+ay$. The factors are $a(x+y)$.

2. By grouping its terms so as to change it to the form of one of the preceding types. Thus, $ax+bx+ay+by$ when grouped, takes the form $(ax+bx)+(ay+by)$. This equals $x(a+b)+y(a+b)$, which is of type III, **1.**

Similarly, the polynomial $x^2+2xy+y^2-a^2-2ab-b^2$ is changed to $x^2+2xy+y^2-(a^2+2ab+b^2)$, which is of type I, **2.**

Miscellaneous Review of Factoring

478. Factor the following polynomials:

1. $26xyz+65xy^2$

2. $7x+35x^2+14x^3$

3. $32-16a+18b^2-9b^2a$

4. $m^2+\frac{3}{4}mn-4mp-3np$

5. $121m^2n^2-64p^2q^2$

6. $81x^4y^4-z^4$

7. $32mn^4-162m$

8. $16x^2+49y^2-56xy$

9. $z^2-x^2+2xy-y^2$

10. $a^2-8ab+15b^2$

11. $m^2-4mn-77n^2$

12. $343a^3+125b^3$

13. $ab+4a-3b-12$

14. $x^2+22x+121$

15. x^3+x^2-x-1

16. $(m-n)^2-11(m-n)-12$

17. x^3-343

18. $5a+6a^2+1$

19. $56-15a+a^2$

20. $x^2y^2+30xy+104$

21. $8x^2y^2z^2-18z^4$

22. $8x^9+729$

23. a^6+25a^3+24

24. m^8-38m^4+105

25. $64-x^6$

26. $9x^2+24xy+16y^2$

27. $3b^2-14ba+8a^2$

28. $3m^2+4(2m+1)$

29. $x^2+y^2+2xy-a^2-b^2-2ab$

30. $m^2+t^2+2mt-x^2-y^2-2xy$

31. $25a^4-26a^2b^2+b^4$ (2 pairs)

32. $4x^4-13x^2y^2+9y^4$ (2 pairs)

Summary

479. The following theorems were proved in this chapter:

1. *Parallelograms having equal bases and equal altitudes are equal.*

2. *A parallelogram is equal to a rectangle having the same base and altitude.*

3. *A triangle is equal to one-half a parallelogram having the same base and altitude.*

4. *The square on the hypotenuse of a right triangle is equal to the sum of the squares on the sides including the right angle.*

5. *In a triangle the sum of the squares of two sides is equal to twice the square of one-half of the third side increased by twice the square of the median to the third side.*

6. *The area of a triangle is equal to one-half the product of the base and altitude,*

$$A=\frac{1}{2}b \cdot h.$$

7. *The area of a triangle is equal to one-half the product of two sides by the sine of the included angle,*

$$A=\frac{1}{2}ab \sin C.$$

8. *The area of a triangle is equal to one-half the perimeter times the radius of the inscribed circle,*

$$A=\frac{1}{2}p \cdot r.$$

9. *The area of a triangle is equal to the product of the three sides divided by four times the radius of the circumscribed circle,*

$$A=\frac{abc}{4r}.$$

10. *The area of a triangle is equal to*

$$A=\sqrt{s(s-a)(s-b)(s-c)}.$$

11. *The area of an equilateral triangle is one-fourth the square of a side times the square root of* 3

$$A=\frac{a^2}{4}\sqrt{3}.$$

480. The chapter has given drill in solving **literal equations** in one and two unknowns and in **factoring polynomials.**

CHAPTER XIX

AREAS OF POLYGONS. AREA OF THE CIRCLE. PROPORTIONALITY OF AREAS

Areas of Polygons

481. Area of the rectangle. The rectangle is the fundamental figure by which the areas of all other rectilinear figures are measured. In the first-year course we have seen that the area of the rectangle is given by the formula

$$S=b\cdot h$$

S denoting the area, b the base, and h the altitude. In the form of a theorem this is stated as follows:

The area of a rectangle is equal to the product of the base by the altitude.

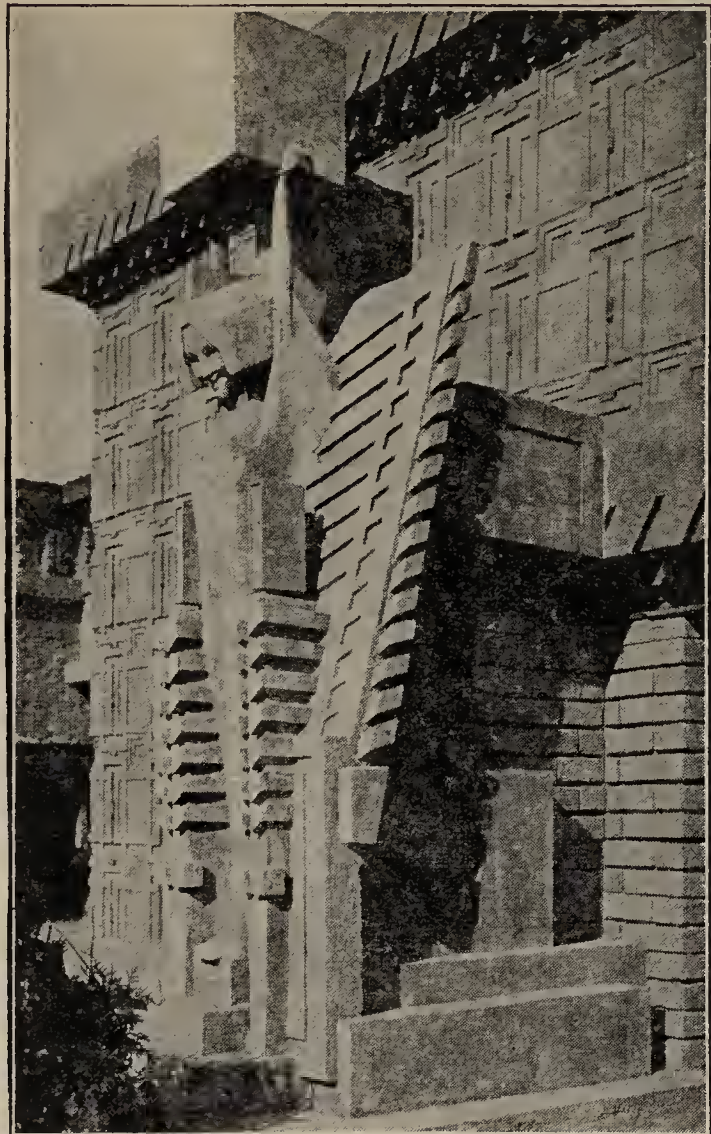
The formula, $S=b\cdot h$, which was shown to hold foı *rational* values of b and h, is also true when b and h are *irrational*. This may be shown as follows:

Let $b=\sqrt{12}=3.464101\ldots\ldots$ and $h=\sqrt{27}=5.196152\ldots\ldots$

Then the following table gives the areas of rectangles, the lengths of whose sides vary, being approximations

Rectangle	b	h	$S=b\cdot h$
I..........	3.464	5.196	17.998944
II.........	3.4641	5.1961	17.99981001
III........	3.46410	5.19615	17.999983215
IV........	3.464101	5.196152	17.999995339352

of $\sqrt{12}$ and $\sqrt{27}$ to three, four, five, and six decimal places and, therefore, *rational numbers*. Hence, the formula $S=b\cdot h$ may be applied in each case.



Courtesy of Frank Lloyd Wright

Notice the uses of mathematical forms and patterns in architectural design illustrated in these pictures

It is seen from the table that the *difference between* 18 and the several areas, I, II, III, and IV *decreases*, being less than .002, .0002, .00002, .000005, respectively. By taking b and h to a greater number of decimal places, this difference will continue to decrease, in fact it can be made less than *any assigned* number, *however small.* The area is accordingly said to *approach* 18 *as a limit.* The same result is obtained by applying the formula

$$S=b\cdot h=\sqrt{12}\cdot\sqrt{27}=\sqrt{2^2\cdot 3\cdot 3^3}=18.$$

482. Theorem: *The area of a parallelogram is equal to the product of the base and altitude.* Prove. Use §460.

483. Theorem: *The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases.* Prove (see Fig. 389).

Show that the area of a trapezoid is equal to the product of the altitude by the median (see § 161).

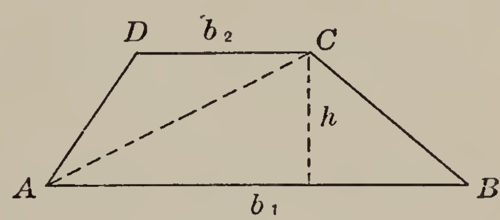


FIG. 389

484. Theorem: *The area of a regular inscribed polygon is equal to the product of one-half of the perimeter and the perpendicular from the center to the side (apothem).*

Draw AO, BO......, Fig. 390.

Denote the length of a side of the polygon by a, the perpendicular from the center to the side by h, the number of sides by n.

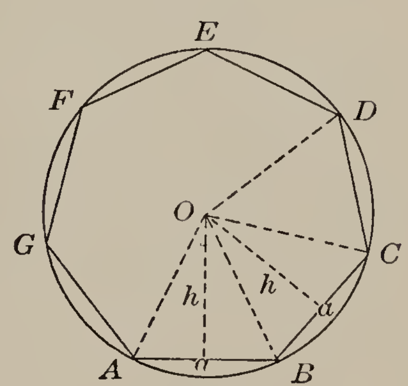


FIG. 390

Then, $\triangle AOB=\dfrac{ah}{2}$,

$\triangle BOC=\dfrac{ah}{2}$, etc. Why?

$\therefore\ ABCD\ldots=\dfrac{nah}{2}=\dfrac{p\cdot h}{2}$ Why?

485. Theorem: *The area of a regular circumscribed polygon is equal to the product of one-half the perimeter and the radius.*

Show, Fig. 391, that

$$\triangle AOB = \frac{ar}{2},$$

$$\triangle BOC = \frac{ar}{2}, \text{ etc.}$$

$$\therefore\ ABCD \ldots = \frac{nar}{2} = \frac{p \cdot r}{2}$$

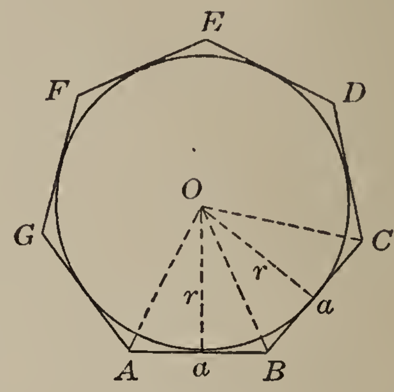


FIG. 391

EXERCISES

1. Express in terms of the radius the areas of the inscribed and circumscribed squares (see exercises 1, 2, § 440).

2. The area of a square is 16 square centimeters. Find the diameters of the inscribed and circumscribed circles.

3. Prove that the area of the equilateral inscribed triangle is $\frac{3}{4}r^2\sqrt{3}$ (see exercise 9, § 441).

Thus, the area of the equilateral inscribed triangle varies as the square of the radius. Give reason.

4. Prove that the area of the circumscribed equilateral triangle is $3r^2\sqrt{3}$.

Show that the area varies as the square of the radius. Show that the area is a function of the radius. Sketch freehand, without plotting points, the graph of this function.

5. Prove that the area of the regular inscribed hexagon is $\frac{3}{2}r^2\sqrt{3}$.

6. Prove that the area of the circumscribed regular hexagon is $2r^2\sqrt{3}$.

7. Find the area of a regular hexagon whose side is 6 inches.

8. The radius of a circle is 10. Find the area of the inscribed regular hexagon.

9. The diameter of a circle is 8. Find the area of the regular inscribed hexagon.

10. Prove that in the same circle the area of the regular inscribed hexagon is twice as large as that of the equilateral inscribed triangle.

486. Area of any polygon. The areas of polygons may be found by dividing the polygons into triangles, as

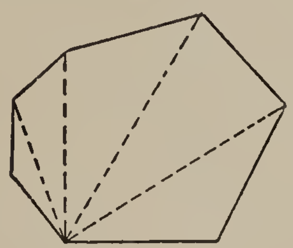


FIG. 392

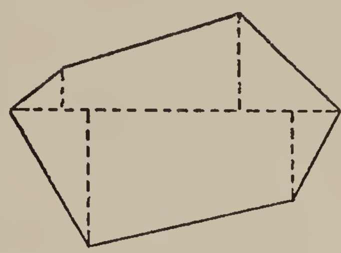


FIG. 393

in Fig. 392, or into triangles and trapezoids, as in Fig. 393.

Area of the Circle

487. If the midpoints of the arcs subtended by the sides of a given regular inscribed polygon, as triangle *ABC*, Fig. 394, are joined to the adjacent vertices of the polygon, a regular inscribed polygon, *AFBECD*, is formed having twice as many sides as the given polygon (see § 436).

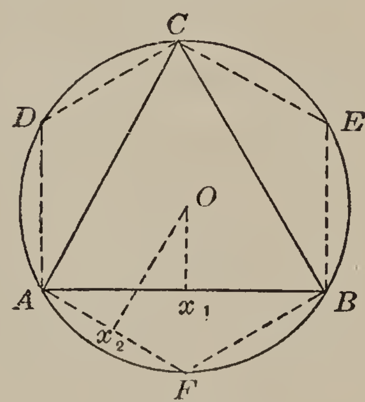


FIG. 394

The perimeter of the second polygon is greater than that of the first. Why?

If the process of doubling the number of sides is continued, the perimeter increases as the number of sides increases. It can be made to differ from the length of the circle by less than *any quantity, however small.* The perimeter is said to *approach* the circle *as a limit.*

The apothem OX *approaches* the radius *as a limit.*

The area of the polygon *approaches* the area of the circle *as a limit.*

488. If tangents are drawn at the midpoints of the arcs terminated by consecutive points of contact of the sides of a given regular circumscribed polygon, as $ABCD$, Fig. 395, a regular circumscribed polygon, as $EFGHIKLM$ is formed having twice as many sides as the given polygon (see § 438).

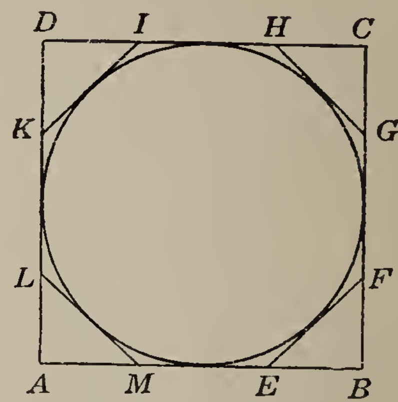


FIG. 395

The perimeter of the second polygon is less than that of the first. Why?

If the process of doubling the number of sides is continued, the perimeter decreases as the number of sides increases. It can be made to differ from the length of the circle by less than *any quantity, however small,* thus approaching the circle *as a limit.*

The area of the polygon *approaches* the area of the circle *as a limit.*

489. According to §§ 487 and 488, the area of the circle is the *common limit* approached by the areas of the inscribed and circumscribed regular polygons, as the *number* of sides *increases* indefinitely.

These areas are given by the formulas:

$\frac{ph}{2}$ and $\frac{Pr}{2}$, respectively (see §§ 484, 485).

As the number of sides of the polygons is increased indefinitely, $\frac{ph}{2}$ approaches $\frac{c \cdot r}{2}$ as a limit, for p approaches c, and h approaches r.

$\frac{Pr}{2}$ approaches $\frac{c \cdot r}{2}$ as a limit, for P approaches c.

Hence, the *common limiting value,* $\frac{cr}{2}$, expresses the area of the circle.

In words, this may be stated as follows:

Theorem: *The area of a circle is one-half the product of the length of the circle and the radius, i.e., area of circle is given by*

$$\tfrac{1}{2}cr.*$$

Since, $c=2\pi r$, it follows that the *area of a circle is given by*

$$\pi r^2.$$

Show that the area of a circle is a function of the radius and sketch the graph of this function.

490. Theorem: *The area of a sector of a circle is equal to one-half the product of the radius and the length of the arc of the sector.*

We have seen in § 297 that central angles have the same measure as the intercepted arcs and that two central angles are to each other as the intercepted arcs (§ 297, exercise 8).

Hence, $\frac{a}{b}=\frac{a'}{b'}$, Fig. 396.

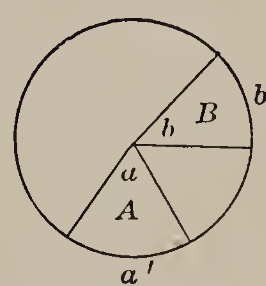


Fig. 396

Similarly, we may show that *equal* central angles include *equal* sectors and that two sectors are to each other as their central angles.

Hence, $\frac{a}{b}=\frac{A}{B}$

* A proof of the theorem is not attempted, as this is considered beyond the province of secondary-school work.

Denote by a the number of degrees in a central angle, and consider the circle as an arc whose central angle is 360°.

Then, $\dfrac{a}{360}=\dfrac{a'}{2\pi r}$, and $a'=\dfrac{\pi ra}{180}$ . . . . (A)

Similarly, $\dfrac{A}{\pi r^2}=\dfrac{a}{360}$ Why?

$\therefore\ A=\dfrac{\pi r^2 a}{360}=\dfrac{1}{2}\cdot\dfrac{\pi ra}{180}\cdot r$ Why?

or $A=\frac{1}{2}a'\cdot r$ . . . . (B)

491. Area of a segment. The *area of a segment* ACB, Fig. 397, may be found by subtracting the area of triangle, AOB, from the area of the sector, $AOBC$, the area of triangle AOB being computed by means of the formula $T=\frac{1}{2}r^2 \sin O$, § 466; or, by $T=\frac{1}{2}a\sqrt{r^2-\frac{a^2}{4}}$, § 233.

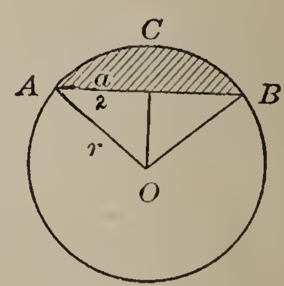


Fig. 397

Hence the area of a segment is given by the following formulas:

(1) $S=\frac{1}{2}a'r-\frac{1}{2}r^2 \sin X$, where X is the central angle subtended by the chord a.

Or by (2) $S=\frac{1}{2}a'r-\frac{1}{2}a\sqrt{r^2-\frac{a^2}{4}}$.

Where a is the length of the chord, a' the length of the arc, and r the radius of the circle.

EXERCISES

1. The area of a circle is 64. Find the diameter and length.

2. Find the diameter of a circle whose area is 1 square inch; 1 square foot; 1 square yard.

3. What is the area of the ring formed by two concentric circles, Fig. 398, whose radii are 5 inches and 6 inches, respectively; a inches and b inches, respectively?

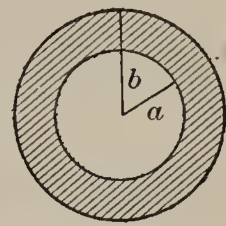


FIG. 398

4. The length of a circle is 50 inches. What is the area?

5. The area of a circle is 616 square inches. How many degrees are there in an angle at the center that intercepts an arc 11 inches long?

6. The radius of a circle is 100 feet. The length of the arc of a sector is 25 feet. Find the area of the sector.

Use formula, (B), § 490.

7. The radius of a sector is 9 inches, its area is 72 square inches. Find the length of the arc.

8. The area of a sector is a square foot, and the radius is r feet long. Find the length of the arc.

9. The radius of a circle is 8 inches. Find the area of a sector with arc 36°.

Make use of the fact that the area of the sector is $\frac{1}{10}$ of the area of the circle.

10. Find the area of the segment whose arc is 36° in a circle of radius 12 inches.

When finding the area of the triangle notice that the base of the triangle is the side of a regular 10-side, exercise 5, § 443, or use the formula $\frac{1}{2}ab \sin C$.

11. Find the area of a segment of arc 72°, in a circle of radius 20.

12. The area of a circle is 15,400 square inches. Find the area of a segment whose arc is 60°.

Proportionality of Areas

The proofs of the theorems in §§ 492–496 are very simple and are left to the student.

492. Theorem: *Two parallelograms are to each other as the products of their bases and altitudes,* i.e., $\frac{P}{P'}=\frac{bh}{b'h'}$.

493. Theorem: *Two parallelograms having equal bases are to each other as the altitudes,* i.e., $\frac{P_1}{P_2}=\frac{h_1}{h_2}$.

By alternation, $\frac{P_1}{h_1}=\frac{P_2}{h_2}$.

Thus, if the base of a parallelogram remains fixed and if the altitude varies continuously, taking successive values h_1, h_2, $h_3, \ldots\ldots$, etc., P takes the corresponding values P_1, P_2, $P_3, \ldots\ldots$, etc. However, $\frac{P}{h}$ remains constant, i.e.,

$$\frac{P_1}{h_1}=\frac{P_2}{h_2}=\frac{P_3}{h_3}=\ldots\ldots\ldots, \text{etc.}$$

Denoting this constant ratio by b, we have $P_1=bh_1$, $P_2=bh_2$, $P_3=bh_3$, etc. Show that P is a function of h. Without plotting points, sketch the graph of this function.

Hence, P is *directly proportional* to h, or P *varies directly* as h if the base b remains constant.

494. Theorem: *Two triangles are to each other as the products of the bases and altitudes.*

495. Theorem: *Areas of triangles having equal bases are to each other as the altitudes.*

496. Theorem: *Areas of triangles having equal altitudes are to each other as the bases.*

EXERCISES

1. Show that the area of a triangle having a fixed base varies directly as the altitude, i.e., show that $\frac{T}{h}$ remains constant, as h varies.

2. Show that the area of an equilateral triangle varies directly as the square of the side.

3. Show that the area of a circle varies directly as the square of the radius.

497. Theorem: *The areas of two triangles that have an angle in one equal to an angle in the other are to each other as the products of the sides including the equal angles.*

Given $\triangle ABC$ and $A'B'C'$ having $C = C'$, Fig. 399.

To prove that $\dfrac{T}{T'} = \dfrac{ab}{a'b'}$.

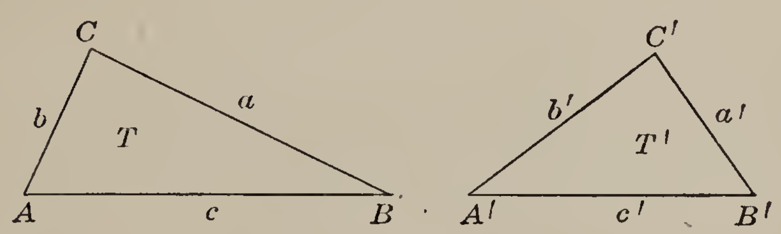


FIG. 399

Proof: $T = \frac{1}{2}ab \sin C.$ Why?

$T' = \frac{1}{2}a'b' \sin C'.$ Why?

$\therefore \dfrac{T}{T'} = \dfrac{ab}{a'b'}.$ Why?

EXERCISES

1. Two triangles have an angle in each equal. The including sides of one are 48 and 75, those of the other triangle are and 45 and 70. Find the relative areas of the triangles.

2. Two sides of a triangular building are 150 ft. and 130 feet. What part of the whole building is included by 50 ft. on the first side and 30 ft. on the second?

3. A triangular lot extends 60 ft. and 80 ft. on two sides from a corner. If a building is to front 50 ft. on the first side, how many feet on the second side should it occupy to cover $\frac{3}{4}$ of the lot?

4. Two sides, a and b, of a triangle are 9 and 15 respectively. Show where a line going through a point on a and 5 units from the common vertex of a and b must intersect the side b to bisect the surface of the triangle.

498. Theorem: *The areas of similar triangles are to each other as the squares of the homologous sides.*

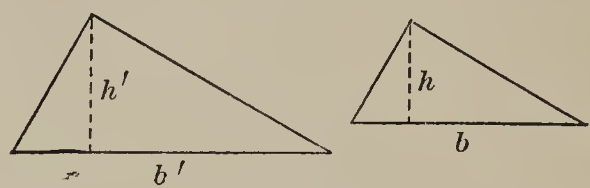


FIG. 400

Show that $\frac{T}{T'}=\frac{bh}{b'h'}=\frac{b}{b'}\cdot\frac{h}{h'}$, Fig. 400.

$$\frac{h}{h'}=\frac{b}{b'}. \qquad \text{Why?}$$

$$\therefore \quad \frac{T}{T'}=\frac{b}{b'}\cdot\frac{b}{b'}.$$

$$\therefore \quad \frac{T}{T'}=\frac{b^2}{b'^2}.$$

EXERCISES

1. The side of a triangle is 10 inches. Find the corresponding side of a similar triangle having twice the area.

2. Two similar triangles have two homologous sides 5 and 15 respectively. What is the ratio of the areas?

3. Bisect the surface of a triangle by a line drawn from a vertex to the opposite side.

499. Theorem: *The areas of similar polygons are to each other as the squares of the homologous sides.*

Given polygon $ABC\ldots\ldots \backsim$ polygon $A'B'C'\ldots\ldots$ (Fig. 401). Let P denote the area of $ABC\ldots\ldots$ and P' denote the area of $A'B'C'\ldots\ldots$

To prove $\dfrac{P}{P'}=\dfrac{d^2}{d'^2}$.

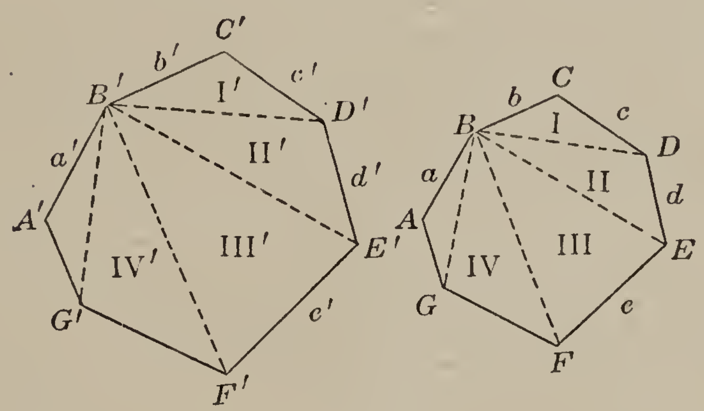


FIG. 401

Proof: Divide $ABC\ldots\ldots$ and $A'B'C'\ldots\ldots$ into triangles I, II, III, etc., and I', II', III', etc., respectively, by drawing diagonals from homologous vertices as B and B'.

Then $I \backsim I'$, $II \backsim II'$, etc. Why?

$\therefore \quad \dfrac{I}{I'}=\dfrac{c^2}{c'^2}, \dfrac{II}{II'}=\dfrac{d^2}{d'^2}\ldots\ldots$, etc. Why?

Show that $\quad \dfrac{c^2}{c'^2}=\dfrac{d^2}{d'^2}\ldots\ldots$, etc. Why?

$\therefore \quad \dfrac{I}{I'}=\dfrac{II}{II'}=\dfrac{III}{III'}\ldots\ldots$, etc. Why?

$\therefore \quad \dfrac{I+II+III+\ldots\ldots}{I'+II'+III'+\ldots\ldots}=\dfrac{II}{II'}=\dfrac{d^2}{d'^2}$ (§ 498).

$\therefore \quad \dfrac{P}{P'}=\dfrac{d^2}{d'^2}$.

EXERCISES

1. Two homologous sides of two similar triangles are 5 and 8. The area of the first is 150. Find the area of the second.

2. If one square is 9 times as large as another, what is the relative length of the homologous sides?

3. The area of a polygon is $6\frac{1}{4}$ times the area of a similar polygon. A side of the smaller is 4 feet. Find the length of the homologous side of the larger.

4. Show that if equilateral triangles are constructed on the sides of a right triangle, the triangle on the hypotenuse is equal to the sum of the triangles on the other two sides.

5. Show that if semicircles are drawn on the sides of a right triangle, the area of the semicircle on the hypotenuse is equal to the sum of the areas of the semicircles on the two sides of the right angle.

‡**6.** Semicircles are drawn on the sides of a right triangle, Fig. 402. Show that the sum of the areas of lunes I and II is equal to the area of the right triangle (theorem of Hippocrates, 430 B.C.).

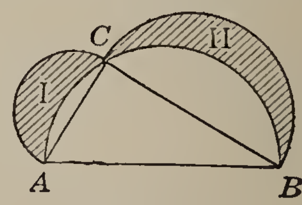


FIG. 402

7. Similar polygons, P_1, P_2, and P_3, are drawn on the sides of a right triangle as homologous sides, Fig. 403. Prove that P_3, the area of the polygon on the hypotenuse, is equal to the sum of P_1 and P_2.

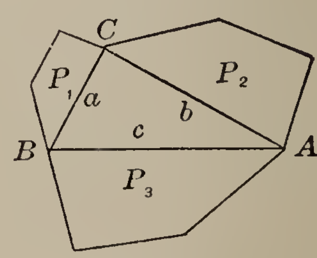


FIG. 403

Proof:

$$\frac{P_1}{P_3}=\frac{a^2}{c^2}. \qquad \text{Why?}$$

$$\frac{P_2}{P_3}=\frac{b^2}{c^2}. \qquad \text{Why?}$$

$$\therefore \frac{P_1+P_2}{P_3}=\frac{a^2+b^2}{c^2}. \qquad \text{Why?}$$

$$\therefore (P_1+P_2)c^2=P_3(a^2+b^2). \qquad \text{Why?}$$

$$\therefore P_1+P_2=P_3. \qquad \text{Why?}$$

8. The homologous sides of similar hexagons are 9 in. and 12 in., respectively. Find the homologous side of a similar hexagon equal to their sum.

500. Theorem: *The areas of two circles are to each other as the squares of the radii, or as the squares of the diameters.*

EXERCISES

1. What is the ratio of the areas of two circles whose radii are 5 in. and 10 inches?

2. The areas of two circles are in the ratio 2 to 4. What is the ratio of the diameters?

3. The radii of two circles are to each other as 3:5, and their combined area is 3850. Find the radii of the two circles.

4. The radii of two circles are to each other as 7:24, and the radius of a circle whose area is equal to their sum is 50. Find the radii of the first two circles.

Problems of Construction

501. Make the following constructions.

1. Construct a square equal to the sum of two or more given squares.

Given x, y, z, w, the sides of given squares.

Required to construct a square equal to the sum of the given squares. Fig. 404 suggests the construction.

Prove that

$$c^2 = x^2 + y^2 + z^2 + w^2.$$

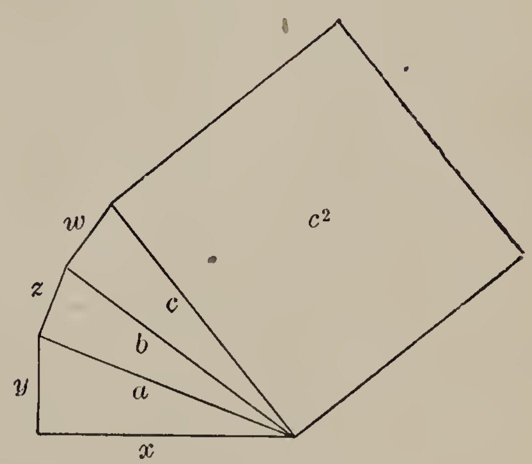


FIG. 404

2. Construct a square equal to four times a given square.

3. Construct the square root of an integral number.

Make the construction, Fig. 405, on squared paper.

Measure AC, AD, AE, AF and check by extracting the square roots of 2, 3, 4, 5.

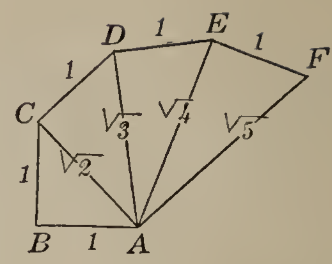


FIG. 405

4. Transform a polygon into a triangle equal to it.

Draw the diagonal AD, Fig. 406.

Through E draw $EF \parallel AD$ intersecting the extension of AB in F.

Draw DF and show that $\triangle DFA = \triangle DEA$.

Show that $FBCD$ is equal to $ABCDE$.

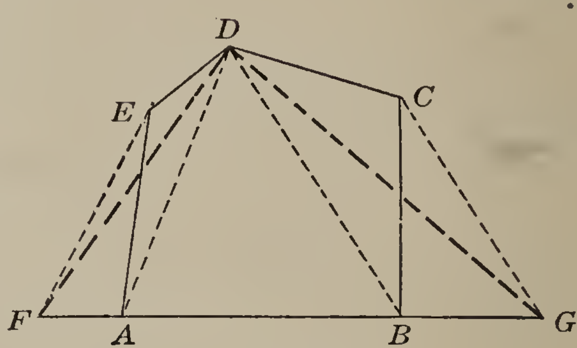


FIG. 406

This reduces the pentagon to the equivalent quadrilateral $FBCD$.

Draw the diagonal DB.
Draw $CG \parallel DB$.
Draw DG.
Show $\triangle DCB = \triangle DGB$.
Show that $FBCD = \triangle FGD$, which is the required triangle.
$\therefore \quad ABCDE = \triangle FGD$. Why?

5. Draw a square equal to a given triangle.

Analysis: Since the area of the triangle is $\frac{1}{2}bh$ and since the area of the square is a^2, we must have $a^2 = \frac{1}{2}bh$, where b and h are known, and a unknown. Hence, the problem reduces to constructing the mean proportional between $\frac{1}{2}b$ and h.

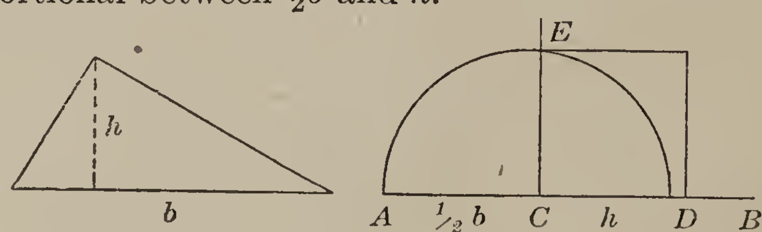


FIG. 407

Construction: On AB, Fig. 407, lay off $AC = \frac{1}{2}b$ and $CD = h$. Draw $CE \perp AD$.

Draw the semicircle on AD.
Draw a square on CE as a side. This is the required square.
Prove.

6. Explain how to draw a square equal to a given polygon.

MISCELLANEOUS PROBLEMS AND EXERCISES

‡**502.** Solve the following problems and exercises:

1. Bisect a parallelogram by a line drawn through a point on its perimeter.

2. Construct an equilateral triangle equivalent to a given triangle.

1. Transform the given triangle into an equal triangle having one angle 60°.

2. To determine the length of the side of the equilateral triangle, apply the theorem—two triangles having an angle in each equal are to each other as the products of the sides including the equal angles.

3. The base of a triangle is 18 feet. Find the length of a line parallel to the base which bisects the triangle.

4. A line parallel to the base of a triangle cuts off a triangle equal to $\frac{3}{4}$ of it. If one side of the triangle is 12, how far from the vertex does the line cut it?

5. Draw through a vertex of a triangle lines dividing it:

(1) Into two parts one of which shall be (*a*) $\frac{2}{3}$, (*b*) $\frac{1}{2}$, (*c*) $\frac{5}{9}$ of the other.

(2) Into three parts in the ratio of 2:3:4.

6. To bisect the surface of a triangle by a line through a given point P on the perimeter not at the vertex of an angle (see Fig. 408).

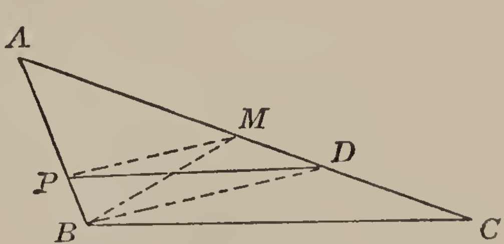


FIG. 408

Draw the median BM, also PM, $BD \parallel PM$, and PD.
Then, $\triangle PMD = \triangle PMB$. Why?
$\therefore$ $\triangle ADP = \triangle AMB = \triangle BMC = PDCB$. Why?

7. The sides of a triangle are 17, 10, and 9. The altitude of a similar triangle upon the side homologous to the side 10 in the given triangle is $14\frac{2}{5}$. Find all the sides of the second triangle.

8. The side of a square (or of any polygon, or the radius of a circle) is a. Find the side (or radius) of a similar figure k times as large.

9. The radii of two circles are 25 and 24. Find the radius of a circle equivalent to their difference.

10. The area of one of three circles is equal to the sum of the other two, and their radii are x, $x-7$, $x+1$. Find x.

11. The difference of two circles whose diameters are $x+2$ and x is equivalent to a circle whose diameter is $x-7$. Find x.

12. The area of a rectangle is 60 and diagonal is 13. Find its dimensions.

13. The perimeter of a rectangle is 46 and the area is 120. Find its dimensions.

14. The perimeter of a rectangle is 62 and the diagonal is 25. Find its area.

15. The altitude and base of a rectangle are in the ratio of 8 to 15 and the diagonal is 34 feet. Find the area.

16. The dimensions of a rectangle are in the ratio of $2ab$ to a^2-b^2, and the diagonal is $a^2c^2+b^2c^2$. Find the area.

17. Compute the altitude upon the hypotenuse of the right triangle ABC in terms of the sides of the right angle.

18. The diagonals of a rhombus are $2x-14$ and $2x$, and a side is $x+1$. Find x.

19. The homologous sides of two similar hexagons are 9 in. and 12 in. respectively. Find the homologous side of a similar hexagon (1) equal to their sum; (2) equal to their difference.

Summary

503. The following theorems have been proved in the chapter.

1. *The area of a rectangle is equal to the product of the base and the altitude.*

2. *The area of a parallelogram is equal to the product of the base and the altitude.*

3. *The area of a trapezoid is equal to one-half the product of the altitude by the sum of the bases.*

4. *The area of a regular inscribed polygon is equal to the product of one-half the perimeter and the apothem.*

5. *The area of a regular circumscribed polygon is equal to the product of one-half the perimeter and the radius.*

6. *The area of a circle is one-half the product of the length of the circle and the radius,* i.e., $A=\frac{1}{2}cr$.

7. *The area of a circle is given by the formula* $A=\pi r^2$.

8. *The area of a sector is given by the formula* $A=\frac{1}{2}a'r$.

9. *The area of a segment of a circle is given by the formulas:* $A=\frac{1}{2}a'r-\frac{1}{2}a\sqrt{r^2-\frac{a^2}{4}}$, or

$$A=\frac{1}{2}a'r-\frac{1}{2}r^2 \sin X.$$

10. *Two parallelograms are to each other as the products of the bases and altitudes.*

11. *Two parallelograms having equal bases are to each other as the altitudes.*

12. *Two triangles are to each other as the products of the bases and altitudes.*

13. *Areas of triangles having equal bases (altitudes) are to each other as the altitudes (bases).*

14. *The areas of two triangles having an angle in one equal to an angle in the other are to each other as the products of the sides including the equal angles.*

15. *The areas of similar triangles are to each other as the squares of the homologous sides.*

16. *The areas of similar polygons are to each other as the squares of the homologous sides.*

17. *The areas of two circles are to each other as the squares of the radii.*

504. The following problems of construction were taught:

1. Construct a square equal to the sum of two or more given squares.

2. Construct the square root of an integral number.

3. Transform a polygon into a triangle.

4. Draw a square equal to a given triangle.

TABLE OF SINES, COSINES, AND TANGENTS OF ANGLES FROM 1°–89°

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	.0872	.9962	.0875	50	.7660	.6428	1.1918
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	.1736	.9848	.1763	55	.8192	.5736	1.4281
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	.2588	.9659	.2679	60	.8660	.5000	1.7321
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.8988	.4384	2.0503
20	.3420	.9397	.3640	65	.9063	.4226	2.1445
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	.4226	.9063	.4663	70	.9397	.3420	2.7475
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	.5000	.8660	.5774	75	.9659	.2588	3.7321
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	.5736	.8192	.7002	80	.9848	.1736	5.6713
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	.6428	.7660	.8391	85	.9962	.0872	11.4301
41	.6561	.7547	.8693	86	.9976	.0698	14.3006
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	.7071	.7071	1.0000				

TABLE OF POWERS AND ROOTS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	55	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	65	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7.056	592,704	9.165	4.379
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	**90**	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	95	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.641

SYMBOLS

$=$	equals, equals, is equal to	Ⓢ	circles
$>$	is greater than	$\therefore$	hence, therefore.
$<$	is less than	$\because$	since
$\parallel$	parallel, is parallel to	$\equiv$	identical, is identical to
$\perp$	perpendicular, is perpendicular to	$\doteq$	approaches
$\sim$	similar, is similar to	$+$	plus
$\cong$	congruent, is congruent to	$-$	minus
$\angle$	angle	$\neq$	does not equal
$\measuredangle$	angles	rt $\angle$	right angle
▱	parallelogram	$\triangle$	triangle
▭	rectangle	$\triangle$s	triangles
$\odot$	circle	$\frown$	arc

FORMULAS

$a^2+b^2=c^2$: relation between the sides of a right triangle.

$a^2+b^2\pm 2ab'=c^2$: relations between the sides of an oblique triangle.

$c=2\pi r=\pi d$: length of a circle.

$b\cdot h$: area of a parallelogram, of rectangle.

a^2: area of a square.

$\frac{1}{2}bh$, $\frac{1}{2}ab \sin C$, $\frac{1}{2}r(a+b+c)$, $\frac{abc}{4R}$, $\sqrt{s(s-a)(s-b)(s-c)}$: area of a triangle.

$\frac{1}{4}a^2\sqrt{3}$: area of an equilateral triangle.

$hm=\frac{1}{2}h(b+b')$: area of a trapezoid.

$\frac{1}{2}p\cdot a$: area of a regular inscribed polygon.

$\frac{1}{2}p\cdot r$: area of a regular circumscribed polygon.

$\frac{1}{2}cr=\pi r^2$: area of a circle.

$$\sin A=\frac{a}{c},\ \cos A=\frac{b}{c},\ \tan A=\frac{a}{b},\ \sin^2 A+\cos^2 A=1,\ \tan A=\frac{\sin A}{\cos A}.$$

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