FIRST BOOK IN GENERAL MATHEMATICS

ATWELL

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CONTENTS

CHAPTER I . . . . Page 1

Lines and Angles

The straight line: plane: right angle: straight angle: circle: degrees, minutes and seconds: change of angular units: addition and subtraction of angles.

I. *Shop Exercise:* LEVELING DEVICES AND APPURTENANCES.

CHAPTER II . . . . Page 14

Straight Angles and Perpendiculars

Axioms: supplementary angles: vertical angles: postulates: theorems concerning perpendiculars: discussion of the straight line.

II. *Field Exercise:* LOCATION OF CORNER STAKES FOR EXCAVATION FOR THE FOUNDATION OF A BUILDING.

CHAPTER III . . . . Page 30

Parallel Lines

Perpendiculars to same line: parallels to a given line: a line perpendicular to one of two parallels: alternate interior angles: exterior-interior angles.

IIIa. *Shop Exercise:* PARALLEL RULER.

IIIb. *Field Exercise:* MARKING CORNERS BY BATTER BOARDS.

Completion of work of laying out ground for excavation, begun in Field Exercise II.

CHAPTER IV . . . . PAGE 41

TRIANGLES

Theorems for proving two triangles equal: isosceles triangle: bisector of given angle: construction of triangle when three sides are given: sum of the angles of a triangle: exterior angle of a triangle.

IVa. *Class Room Exercise:* COMBINING ANGLES OF PAPER TRIANGLES.

IVb. *Field Exercise:* MEASURING THROUGH AN OBSTRUCTION.

CHAPTER V . . . . PAGE 52

CIRCLES

Theorems involving equal arcs, equal central angles, equal chords: construction of an angle equal to a given angle: construction of a circumference through three given points not in a straight line: measurement of an inscribed angle.

V. *Field Exercise:* LAYING OUT A ROAD IN AN ARC OF A CIRCLE PASSING THROUGH THREE GIVEN POINTS.

CHAPTER VI . . . . PAGE 65

POLYGONS: USE OF LETTERS TO EXPRESS GENERAL LAWS

Classes of polygons: proof of formula for the sum of the angles of a polygon for particular polygons and for polygons in general, with introduction of algebraic symbols: the equation: construction of regular polygons.

VI. *Shop Exercise:* DESIGN OF PLATE PAD WITH ORNAMENTATION BASED ON ABOVE CONSTRUCTION.

CHAPTER VII . . . . PAGE 77

LITERAL EXPRESSIONS

Definitions: combination of terms: signs of inclosure: exponents: like terms: addition and subtraction of monomials and of polynomials: removal of signs of enclosure.

VIIa. *Field Exercise:* PROFILE OF LAND.

Determination of the various heights and depressions along a line.

VIIb. *Drawing Exercise:* A PLOT OF ABOVE.

CHAPTER VIII . . . . Page 90

The Construction and Application of Parallel Lines

Construction of a line through a given point parallel to a given straight line: theorems on parallelograms: derivation of method of dividing a given line into any number of equal parts: the vernier.

VIIIa. *Field Exercise:* staking off parallels.
Laying out a path parallel to the front of a building, or other prominent straight line.

VIIIb. *Drawing Exercise:* construction of vernier.

CHAPTER IX . . . . Page 102

Areas

Tables of measure: theorems concerning the area of the rectangle, the parallelogram, the triangle, the trapezoid: application of square root to equation.

IX. *Field Exercise:* finding area of a field by base and altitude method.
Division of the field into triangles, and measurement of their bases and altitudes.

CHAPTER X . . . . Page 111

Multiplication of Literal Expressions: Factoring

Product of a minus quantity by a minus quantity: multiplication of monomials, of polynomials: the square of the sum of two quantities, the square of the difference of two quantities, the product of the sum and the difference of two quantities (derivation of these rules algebraically, with geometrical proof): reversal of these rules for factoring.

X. *Shop Exercise:* construction of transit.
Execution of directions in the accompanying working drawings.

CHAPTER XI . . . . Page 133

The Right Triangle

Complementary angles: the right triangle: detailed geometric proof that the square of the hypotenuse of a right triangle equals the sum of the squares of the legs: quadratic surds: projection: the square of the side of any oblique triangle.

XIa. *Shop Exercise:* CONSTRUCTION OF STAIRS.

XIb. *Shop Exercise:* MODEL OF ABOVE.

CHAPTER XII . . . . PAGE 152

LITERAL FRACTIONS: HERON'S FORMULA

Reduction of literal fractions to lowest terms: least common multiple: least common denominator: combination of literal fractions: changing mixed numbers to improper fractions: fractional equations: algebraic derivations of formula for finding the area of a triangle.

XIIa. *Shop Exercise:* ERECTION OF STAIRS.
Location of stairs made in Shop Exercise XI.

XIIb. *Field Exercise:* AREA OF FIELD BY HERON'S FORMULA.
Division of a field into triangles, and measurement of their sides.

CHAPTER XIII . . . . PAGE 167

SIMILARITY

Similar triangles: proportion: measurement of inaccessible distances by proportion.

XIII. *Field Exercise:* FINDING WIDTH OF RIVER.
Application of similar triangles and proportion.

CHAPTER XIV . . . . PAGE 177

FUNCTIONS OF ANGLES

Trigonometric ratios: functions of an angle: relation between functions of an angle and its complement: table of functions to every 5°: solution of right triangles.

XIVa. *Shop Exercise:* DESIGN OF RAFTER.
Application of similar right triangles.

XIVb. *Shop Exercise:* DESIGN OF FLIGHT OF STAIRS FOR ANY GIVEN LOCATION.
Application of similar right triangles.

CHAPTER XV . . . . Page 189

Computation of Functions: Four Place Table of Sines and Cosines and Its Use

Computation of functions of 30°, 60°, and 45°: to find functions by plotting: four place table of sines and cosines: to find the sine of any given angle: to find the cosine of any given angle: to find the angle corresponding to any given sine: to find the angle corresponding to any given cosine: solution and checking of compass surveys.

XV. *Field Exercise:* finding area of field by triangulation.
Dividing a field into triangles, measuring a base line and the angles of the triangle.

CHAPTER XVI . . . . Page 207

Relations Between the Functions: Axes of Reference: Functions of Obtuse Angles

Sine divided by cosine equals the tangent: sine squared plus cosine squared equals unity: change of values of sine and cosine: axes of reference: functions of an angle in the first quadrant, and of an angle in the second quadrant: relation between the functions of an angle and the functions of its supplement: plotting change of values of sine and cosine: practical considerations in use of tables.

XVI. *Field Exercise:* measurements for topographical map of triangulation.
Location of topographical features in fields measured in Field Exercise XV.

CHAPTER XVII . . . . Page 226

Solution of Oblique Triangles

Number of cases of solvable oblique triangles: Sine Law: Cosine Law: application of preceding laws to solution of triangles: discussion of the accuracy of computations: proof of theorems to aid in checking results: a side of a triangle divided by the sine of its opposite angle is numerically equal to the diameter of the circumscribed circle: area of triangles.

XVII. *Drawing Exercise:* map of survey.
Plot of measurements made in Field Exercises XV and XVI.

CHAPTER XVIII . . . . Page 250

Simultaneous Equations: Finding Heights of Hills: Ambiguous Case

Solution of simultaneous equations: application of simultaneous equations to finding heights of hills: discussion of the ambiguous case in the solution of triangles.

XVIII. *Field Exercise:* finding the height of a hill. Application of simultaneous equations.

PREFACE

It has been urged that the investigation and progress that have characterized other branches of the school curriculum have been lacking in so far as mathematics is concerned, especially in the case of mathematics as applied to secondary schools. This is doubtless due largely to the fact that mathematics, as a pure science, is not so susceptible to theory as is a subject whose limitations are not so closely drawn, and whose subject matter is more open to speculation. In spite of this fact, however, educators have dreamed of a more ideal course in mathematics; a course which would give better and larger returns for the time spent in study, and a course which would remove from mathematics the stigma which it so often bears, of being the *bête noir* of the average high school student.

The teacher of secondary mathematics has his choice between what might be called the natural and the artificial incentives. Under the natural incentives fall the following: *a*—the uses of mathematics in the activities of life; *b*—the charm of achievement which comes with the solving of problems; *c*—the gain of mental power, of the ability to reason clearly to a definite conclusion. Among the artificial incentives, the following are the most usual and most powerful: *a*—graduation from the high school; *b*—preparation for college; *c*—the winning of some special prize or honor; *d*—the avoiding of suspicion of mental weakness.

The average teacher is content to use one or more of the artificial incentives, largely because it is easier to do so than to exert the necessary personality which is required to use successfully the natural incentives. Moreover, most secondary school text-books in mathematics seem to have been written with the purpose of requiring students to cover a certain amount of work, rather than to stimulate interest in the work being done.

The pedagogical remedy for such a state of affairs may be summed up in the words "practical use" or "relation to life." Interest can be aroused and maintained only when there is a tangible relation between the subject matter studied and the "out-of-school" life of the student. This idea is not new, but has been applied in various forms at different times. In the Colebrook School, for example, in the northern part of New Hampshire, where the school is being made over to fit the needs of the community, the course in geometry has been substituted by a course in practical mathematics which includes: *a*—the algebra of the equation; *b*—the application of geometry to practical measurements; *c*—the elementary principles of surveying.

Professor Parker, in his book on High School Methods, has the following to say regarding high-school mathematics: "The order of topics in a subject should be decided by the needs, capacities and interests of high-school students, not by the nature of the subject itself, nor the interests of a specialist in the subject. In high-school mathematics this standard will require the inclusion of many practical problems and the mixing of the easier topics of algebra and geometry, and some arithmetic, in the first year."

Mathematicians have often claimed that we should have texts covering a course in "general mathematics," rather than break up our course into algebra, geometry and trigonometry. Attempts to introduce a course which would cover these three phases of mathematics at the same time, have not proven generally successful, due largely to a lack of pedagogical and psychological principles in the working out of the content of the book.

In view of these facts the author of this work has felt that there is a place in the intermediate high school, where a course that will sum up the arithmetic of the grades, and give some insight into the coming mathematical problems of algebra and geometry, might be of use, especially if this course is linked up with the work in mechanical drawing, manual training and elementary surveying, so that the student may see the application of the mathematical theories that he is studying, and have the pleasure of actually accomplishing something in the way of a simple surveying instrument and a map of the locality. There is enough of the explorer in the blood of the average youth to lend romance to the prosaic task of land measurement, and this may be counted on to stimulate and maintain interest.

Another place which this work will fill is that of a review course in mathematics for prospective teachers. Normal students who have finished their high-school mathematics ought to have a general course covering the most important points of elementary and secondary mathematics, worked out from the standpoint of a teacher, and with emphasis on "application." Too many prospective teachers have covered their arithmetic, their algebra and their geometry, simply because it was required of them, have mastered mathematical principles, and never thought of applying these principles to problems outside the schoolroom. Such a course as that outlined in this book will compel the student to apply theory to practice.

And finally, there is the satisfaction and advantage that must be the reward of the man who knows something of mensuration: who can, in case of necessity, make his own apparatus and lay out a path, a fence line, a ditch, or the boundary of his property: who can see that his time spent in school has had a certain practical, as well as social value; and who is a better citizen, and a better neighbor, because his range of vision has been broadened, his sympathies have been extended, and instead of a theorist, he is a practical, useful exponent of the modern ideas in education.

Frederick Kurtland Fleagle,
Dean, University of Porto Rico.

March 13, 1916.

FIRST BOOK IN GENERAL MATHEMATICS

CHAPTER I

LINES AND ANGLES

1. A true **Line** has neither thickness nor width; it has length only.

NOTE: A fine line drawn on mechanical drawing paper with a hard pencil is not a true line, but is near enough to it for practical purposes.

2. The ends of a line are called its **Extremities.**

3. When one line lies completely in another line it is said to **Coincide** with it.

4. When two straight lines cross each other, they are said to **Intersect.**

5. The intersection of two straight lines is a **Point.**

6. A **Point** has no dimensions: only **Location.**

7a. A **Plane** is a flat surface, having only two dimensions—**Length** and **Breadth;** it is a surface such that a straight line joining any two points in it lies entirely in the surface.

The *surface* of the blackboard may be considered a plane.

7b. A **Plane Figure** is a figure outlined in a **Plane**—it has no thickness.

Plane figures are not actual objects—they are **Abstract Ideas.** In forming an **Abstract Idea,** the mind abstracts certain qualities of the object and considers them only.

Abstract ideas are at the basis of Arithmetic, as for example: when it is said "two plus three is five," instead of saying: "two apples plus three apples is five apples" or "two books plus three books is five books," etc. Here it is the quality of number that is abstracted from the objects added, and their other qualities such as color, or hardness, etc., are not considered; and so we are enabled to state a general truth that may be applied to all kinds of objects. So, in Reasoning about **Plane Figures,** we are abstracting the ideas of length and breadth, to derive truths which may later be applied to all kinds of objects.

8. Properties of the Straight Line:

(a) *Only one straight line can be drawn from one point to another;* hence,

(b) *Two points determine a straight line;* hence,

(c) *Two straight lines which have two points in common coincide and form but one line;* hence,

(d) *Two straight lines are equal if their extremities coincide;* also,

(e) *Two straight lines can intersect in but one point* (for if they had two points in common they would coincide and not intersect).

(f) *A straight line is the shortest distance from one point to another.*

(g) *A straight line can be produced indefinitely;* that is if the length of a line is not definitely stated, it may be extended to any length desired.

9. The amount of opening between two intersecting straight lines is called an **Angle.** The two straight lines are called the **Sides** of the angle, and the point of intersection, the **Vertex.** (The plural of vertex is **Vertices.**)

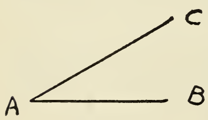


Fig. 1.

The vertex of this angle is the point A and the sides of this angle are the straight lines AB and AC. In reading the letters of the angle, the letter at the vertex is always read in the middle. Thus, this angle is read "The Angle BAC."

When no other angle has the same vertex, the angle may be designated by reading the letter at the vertex only. Thus, this angle may be read "The angle A."

Figure 1 also shows the method of marking points and straight lines—a point being designated by one letter, as "the point A," and a straight line by two letters, as "the line AB."

As the sides of an angle are straight lines, as much or as little of them as desired may be laid off from the vertex (see 8g); that is:

10. *The size of an angle is independent of the length of its sides.*

The sign $\angle$ means **"Angle."** Note that the lower line of this sign is *horizontal.* $\measuredangle$ means **"Angles."**

11. Adjacent Angles are angles which have a side in common (that is, belonging to both of them) and the same vertex.

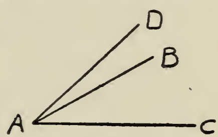


Fig. 2.

Thus, $\angle\, CAB$ and $\angle\, BAD$ are adjacent angles. This may be written "CAB and BAD are adj. $\measuredangle$."

12a. If one straight line which intersects a second straight line divides the space on one side of the second straight line in such a manner that the two adjacent angles thus formed are equal, these two adjacent angles are called **Right Angles** and the two lines are said to be **Perpendicular** to each other.

Thus CD is perpendicular to AB at the point D and the angles CDA and CDB are Right Angles. A perpendicular line may be called "a perpendicular." Thus CD is a perpendicular to AB. CDA and CDB are Right Angles.

Fig. 3.

12b. If one line is perpendicular to a second, then the second is perpendicular to the first.

Thus, in Fig. 3 it may be said either that CD is perpendicular to AB or AB is perpendicular to CD.

13. The point of intersection of the perpendicular with the other line is called the **Foot** of the perpendicular.

Thus, in Fig. 3, D is the Foot of the perpendicular CD.

The sign $\perp$ means "perpendicular."

Rt. $\angle$ means "Right Angle." Rt. $\measuredangle$ means "Right Angles."

14. A Straight Angle is an angle whose sides extend in opposite directions so as to form one straight line.

Thus ADB is a straight angle (Fig. 3).

15. *A Right Angle is half a straight angle.* (*See art.* 12 *and* 14.)

16. A **Circle** is a part of a plane bounded by a curved line, called the **Circumference** of the circle, all points of which are equally distant from a point within called the **Center.**

17. A straight line drawn from the center of the circle to the circumference is called the **Radius** of the circle. (The plural of Radius is **Radii.**)

18. *All radii of the same circle are equal.* (*See art.* 16.)

19. A straight line drawn through the center of the circle and terminated at both ends by the circumference is called a **Diameter** of the Circle.

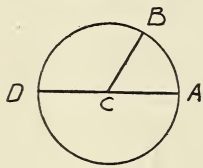


Fig. 4.

20. *A diameter is equal to two radii.* (*See art.* 17 *and* 18.)

21. An **Arc** is any part of the circumference. Thus, in Fig. 4, BA is an arc of the circle whose center is C, the straight line DA, passing through C, is a diameter, and DC, BC, and CA are Radii.

22. The angle formed between two Radii with its vertex at the center is called a **Central Angle.**

Thus (Fig. 4) BCA is a central angle. DCB is also a central angle.

23. The arc cut off by the Radii forming the sides of a central angle is called its **Intercepted Arc.**

Thus (Fig. 4) BA is the arc intercepted by the central angle BCA.

24. A circumference of a circle is divided into 360 parts, called **Degrees**. A degree is sub-divided into minutes and seconds, according to the following table:

1 Circumference	=	360 Degrees ($^\circ$)
1 Degree	=	60 Minutes ($'$)
1 Minute	=	60 Seconds ($''$)

Note that the symbol for "degree" is a small circle written above and to the right of the number of degrees, the symbol for minutes is *one* small slanting line written above and to the right of the number of minutes, and the symbol for seconds is *two* small slanting lines written above and to the right of the number of seconds. Thus, $17^\circ\ 21'\ 30''$ is read "seventeen degrees, twenty-one minutes and thirty seconds."

If Radii are drawn from the ends of each degree of arc to the center of the circle, 360 small angles will be formed with their vertices at the center of the circle. Now, since any point in a plane may be taken as the center of a circle, it follows that:

25. *The sum of the angles about a point in a plane is equal to* 360 *degrees.*

Since each degree of arc has an angle of one degree at the center of the circle it follows that:

26. *A central angle is measured by its intercepted arc.*

That is, there are just as many degrees, minutes and seconds in a central angle as there are in its intercepted arc, and so the above table serves to measure angles as well as arcs.

Since the sides of a central angle, being straight lines, may be produced indefinitely (8g), and become the Radii of any larger circle, it follows that:

27. *The size of an angular degree, or of any number of degrees, minutes and seconds, is independent of the length of the radii of the circle.*

Exercise

1. Change 27° 12′ 31″ to seconds.

Since each degree contains 60 minutes we first multiply 27 by 60. To this product we then add the 12′, obtaining the total number of minutes. We then multiply this number by 60, since there are 60″ in each minute, and to this number add the 31″ and so obtain the total number of seconds.

$$\begin{array}{r} 27^\circ \\ \underline{60} \\ 1620' \\ \underline{12} \\ 1632' \end{array} \qquad\qquad \begin{array}{r} 1632' \\ \underline{60} \\ 97920'' \\ \underline{31} \\ 97951'' \text{ Ans.} \end{array}$$

2. Change 39° 28′ 47″ to seconds.
3. " 11° 0′ 51″ " "
4. " 46° 19′ 41″ " "
5. " 67° 27′ 49″ " "
6. " 210,608″ to degrees, minutes and seconds.

$$\begin{array}{l} 60\ )\ 210{,}608''\ (\ 3510'\ 8'' \\ \quad\ \underline{180} \\ \quad\quad 306 \\ \quad\quad \underline{300} \\ \quad\quad\quad 60 \\ \quad\quad\quad \underline{60} \\ \quad\quad\quad\quad 8 \end{array}$$

$$\begin{array}{l} 60\ )\ 3510'\ (\ 58^\circ\ 30' \\ \quad\ \underline{300} \\ \quad\quad 510 \\ \quad\quad \underline{480} \\ \quad\quad\ \ 30 \end{array}$$

Since there are 60″ in each minute we divide the total number of seconds by 60 to obtain the number of minutes, and obtain 3510 for a quotient with a remainder of 8—that is, the given number of seconds is equal to 3510′ 8″. We next divide the number of minutes by 60 to obtain the number of degrees and so obtain the quotient 58 and the remainder 30. The complete answer is therefore 58° 30′ 8″.

7. Change 26,821″ to degrees, minutes and seconds.
8. " 46,429″ " " " " "
9. " 37,922″ " " " " "
10. " 67,908″ " " " " "
11. " 189,647″ " " " " "
12. Add 82° 47′ 12″, 21° 28′ 21″ and 42° 56′ 29″.

We first write the quantities in columns, keeping degrees under degrees, minutes under minutes and seconds under seconds, thus,

$$\begin{array}{r} 82^\circ\,47'\,12'' \\ 21^\circ\,28'\,21'' \\ 42^\circ\,56'\,29'' \\ \hline 147^\circ\,12'\;\;2'' \end{array}$$

Adding the seconds we obtain the number 62; from this we subtract 60″, and add 1 to the number of minutes. This leaves 2 as the number of seconds. Adding the number of minutes (not forgetting to add the 1 minute carried from the seconds column), we obtain 132 as the number of minutes. From this subtract 120 (the number of minutes in two degrees) and we have 12 as the number of minutes, with 2 to be added to the number of degrees. Adding the degrees, we obtain 147, and have as the complete answer 147° 12′ 2″.

From this example the student may understand the following:

28. Rule: *To add angles, write them one under the other, keeping the degrees, minutes and seconds in separate columns; add the seconds column, and if the number of seconds so obtained is large enough to contain* 60 *or any multiple of it, subtract the* 60 *or its multiple so contained, from the number of seconds, leaving the remainder to be the number of seconds in the answer; change the number of seconds subtracted to minutes, to be added to the minutes column: add the minutes column (including the minutes carried from the seconds column) and if the number of minutes thus obtained contains* 60 *or any multiple of* 60, *subtract the* 60 *or its multiple so contained from the number of minutes, leaving the remainder to be the number of minutes in the answer; change the number of minutes subtracted to* *degrees to carry them to the degrees column and add the degrees column. The complete answer is the sum of the quantities written under the columns of degrees, of minutes, and of seconds.*

Add:

13.
$$\begin{array}{r} 190^\circ\ 17'\ 11'' \\ 26^\circ\ 24'\ 24'' \\ 17^\circ\ \ 8'\ 16'' \\ \hline \end{array}$$

14.
$$\begin{array}{r} 18^\circ\ 39'\ 42'' \\ 31^\circ\ 32'\ 26'' \\ 42^\circ\ 14'\ 19'' \\ \hline \end{array}$$

15.
$$\begin{array}{r} 26^\circ\ 41'\ 17'' \\ 32^\circ\ 22'\ 43'' \\ 18^\circ\ 26'\ 57'' \\ 37^\circ\ 31'\ 19'' \\ \hline \end{array}$$

16.
$$\begin{array}{r} 27^\circ\ 39'\ 42'' \\ 36^\circ\ 43'\ 16'' \\ 52^\circ\ 16'\ 17'' \\ 61^\circ\ 17'\ 38'' \\ \hline \end{array}$$

17.
$$\begin{array}{r} 42^\circ\ 18'\ 36'' \\ 17^\circ\ 11'\ 47'' \\ 29^\circ\ 16'\ 29'' \\ 36^\circ\ 14'\ 42'' \\ 72^\circ\ 38'\ 51'' \\ 32^\circ\ 27'\ 49'' \\ \hline \end{array}$$

18.
$$\begin{array}{r} 61^\circ\ 18'\ 48'' \\ 32^\circ\ 17'\ 39'' \\ 41^\circ\ 18'\ 17'' \\ 27^\circ\ 25'\ 35'' \\ 33^\circ\ 18'\ 15'' \\ 62^\circ\ 31'\ 56'' \\ \hline \end{array}$$

19.
$$\begin{array}{r} 24^\circ\ 17'\ 43'' \\ 108^\circ\ 36'\ 29'' \\ 0^\circ\ 10'\ 51'' \\ 10^\circ\ 46'\ 32'' \\ 7^\circ\ \ 0'\ 41'' \\ 5^\circ\ 11'\ 17'' \\ \hline \end{array}$$

20.
$$\begin{array}{r} 11^\circ\ 39'\ 02'' \\ 24^\circ\ 36'\ 42'' \\ 28^\circ\ \ 0'\ 05'' \\ 42^\circ\ 27'\ 43'' \\ 28^\circ\ 49'\ 07'' \\ 16^\circ\ 20'\ 00'' \\ \hline \end{array}$$

21. Subtract $17^\circ\ 54'\ 32''$ from $42^\circ\ 19'\ 28''$.

$$\begin{array}{r} 42^\circ\ 19'\ 28'' \\ 17^\circ\ 54'\ 32'' \\ \hline 24^\circ\ 24'\ 56'' \end{array}$$

We first write the subtrahend beneath the minuend, keeping the degrees, minutes and seconds in separate columns. Since 28 is not large enough for 32 to be subtracted from it, we bring over $1'$ from the $19'$, leaving 18 as the number of minutes in the minuend, and add the $1'$, changed to $60''$, to the seconds in the minuend, making 88 as the number of seconds in the minuend. Subtracting 32 from 88 we have 56 as the number of seconds in the answer. In the minutes column (since 54 cannot be subtracted from

18) we borrow 1 from the number of degrees in the minuend and changing it to 60 minutes, we add it to the 18, making 78′ the number of minutes in the minuend. Subtracting we have 24′ as the number of minutes in the answer. Then subtracting 17° from 41° (the number of degrees left in the minuend), we have 24° as the remainder, making 24° 24′ 56″ as the complete answer. From this example we may understand the following:

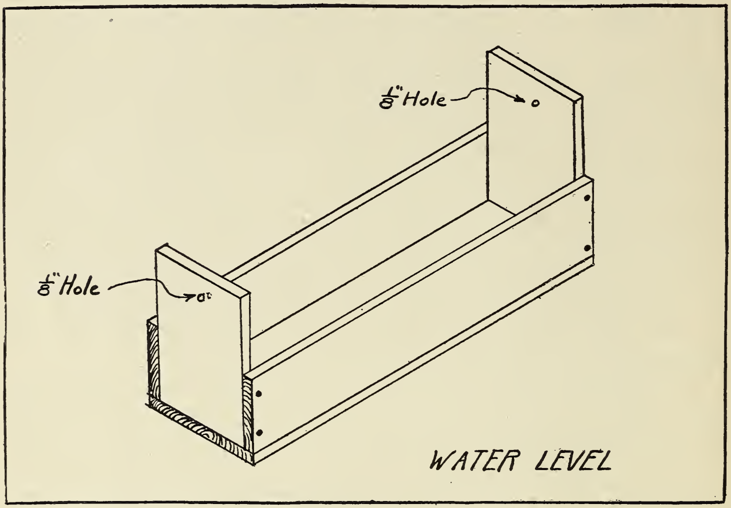
29. Rule: *To subtract one angle from another, write the subtrahend beneath the minuend, keeping degrees, minutes and seconds in separate columns. If the number of seconds in the minuend is less than that in the subtrahend, bring over one minute from the minutes column in the minuend and add it as* 60 *seconds to the seconds in the minuend. Subtract the seconds in the subtrahend from the seconds in the minuend, writing the difference as the number of seconds in the answer; repeat this process for the minutes column, writing the difference as the minutes in the answer, and finally subtract the degrees in the subtrahend from those in the minuend, writing the difference as the number of degrees in the answer.*

Do the following examples in subtraction, not forgetting that when it is necessary to bring over from the minutes column before subtracting the seconds, the number of minutes is thereby diminished by one, and that if it is necessary to borrow from the degrees column before subtracting the minutes, the number of degrees is thereby diminished by one.

22.	41° 18′ 42″ 27° 12′ 37″	**23.**	28° 17′ 6″ 12° 11′ 19″	**24.**	36° 17′ 18″ 29° 32′ 26″
25.	66° 14′ 8″ 42° 37′ 29″	**26.**	112° 0′ 16″ 89° 42′ 56″	**27.**	79° 0′ 0″ 42° 18′ 42″
28.	36° 17′ 58″ 9° 18′ 10″	**29.**	189° 0′ 12″ 72° 3′ 15″	**30.**	47° 10′ 5″ 46° 10′ 26″

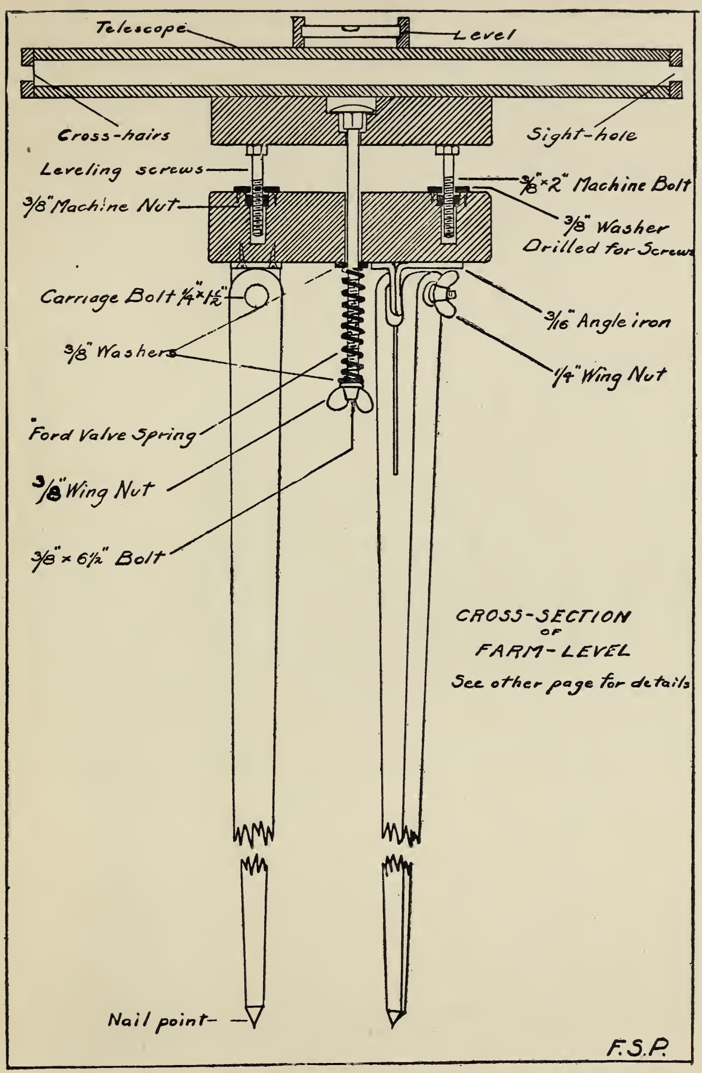
Shop Exercise

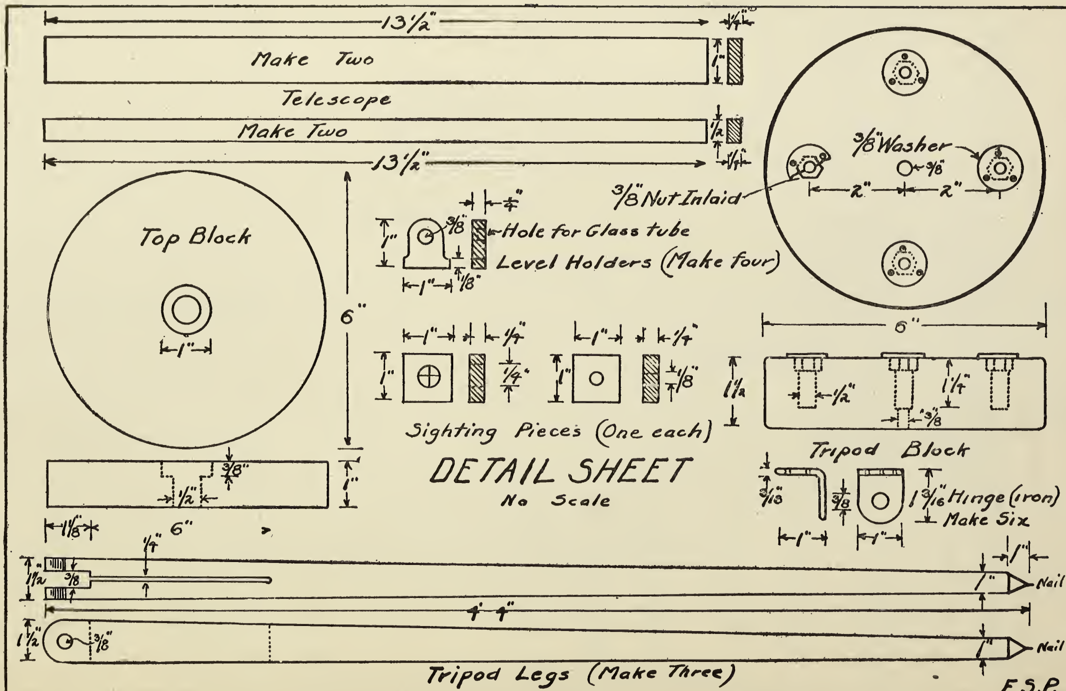
Ia. Though simple in construction, the **Water Level** shown in the accompanying illustration gives very accurate results. The pieces should be first nailed together and the implement placed on some smooth surface,—as a table top. Then, with a sharp pencil, the centers for the holes should



be marked, great care being taken that they are both the same vertical distance above the table top. The boring of these holes completes the instrument, and when it is floated in a pail nearly filled with water, the line of sight obtained by looking through the holes is a true level line.

Ib. The **Ranging Pole** is a pole about six feet long and about one and one-half inches in diameter. It should be rounded, or at least the corners should be planed off, for greater ease in handling.


Telescope
Level
Cross-hairs
Sight-hole
Leveling screws
3/8" Machine Nut
3/8"×2" Machine Bolt
3/8" Washer Drilled for Screws
Carriage Bolt 1/4"×1 1/2"
3/16" Angle iron
3/8" Washers
1/4" Wing Nut
Ford Valve Spring
3/8" Wing Nut
3/8" × 6 1/2" Bolt
CROSS-SECTION OF FARM-LEVEL
See other page for details
Nail point
F.S.P.



Make Two
Telescope
Make Two
13 1/2"
Top Block
3/8" Washer
3/8" Nut Inlaid
Hole for Glass tube
Level Holders (Make four)
Sighting Pieces (One each)
DETAIL SHEET
No Scale
Tripod Block
1 3/16" Hinge (iron)
Make Six
Nail
4' 4"
Tripod Legs (Make Three)
F.S.P.

The Ranging Pole should be brightly painted—preferably in alternate bands of red and white—the bands being about a foot wide. Two of these ranging poles are needed by a surveying section.

Ic. The **Leveling Rod** is really a long ruler. It should be made of a strip of straight, well-seasoned wood, at least six feet long. The strip should first be painted white. When it is dry, black stripes should be painted entirely across the strip at the end of each foot, and the number of feet written just above these strips. Black marks reaching halfway across the strip should be painted at the end of each six inches. Short marks may be made at the end of each inch. The bottom of the rod should be protected with a strip of tin or other metal.

Id. **Farm Level:** Work may be begun at this time on the Farm Level, as illustrated in the accompanying diagrams, which are self-explanatory. Since the completion of this instrument requires more time than it may be possible to spend in the shop before completing the next few chapters, it is suggested that the simple Water Level be made first, for use in the Field Exercises immediately following.

The "telescope," so called, in the diagram requires no lenses—it derives its name from the surprising clarity of vision obtained by looking through two holes at opposite ends of an otherwise darkened tube.

CHAPTER II

STRAIGHT ANGLES AND PERPENDICULARS

30. An **Axiom** is a self-evident truth.

For example, the properties of a straight line are axioms.

31. Axiom: *Quantities equal to the same quantity or to equal quantities are equal to each other.*

32a. Axiom: *The whole is equal to the sum of all its parts.*

32b. Axiom: *If equal subtrahends are subtracted from equal minuends, the remainders are equal.*

33a. Axiom: *The whole is greater than any of its parts.*

33b. Axiom: *The halves of equals are equal.*

34a. Axiom: *The doubles of equals are equal.*

34b. Axiom: *If equals are added to equals the sums are equal.*

If two plane angles are equal, one may be thought of as being placed on another so their vertices coincide and their sides coincide, that is:

35a. Axiom: *If two plane angles are equal they may be made to coincide throughout.*

35b. Axiom: *If two plane angles may be made to coincide throughout they are equal.*

Note: In the following proof the two angles are proved equal by placing the first angle on the second, so that their vertices coincide, and one side of the first coincides with one side of the second. (This much might be done with any two angles, whether they were equal or not.) Then it is necessary to prove the second side of the first angle coincides with the second side of the second angle, that is, that they coincide throughout.

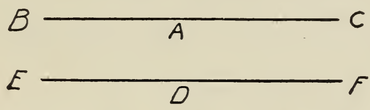
$\therefore$ is the symbol for "therefore."

$>$ is the symbol for "is greater than."

Thus "$AB > CD$" is read "AB is greater than CD."
$<$ is the symbol for "is less than."

36. A **Theorem** is a statement of which the truth is to be proved. In proving the following theorem we use only axioms and definitions, but in later theorems we may use theorems of which the truth has been already established.

37a. Theorem: *All straight angles are equal.*



Given: BAC is a straight angle and EDF is any other straight angle.

To Prove: $\angle BAC = \angle EDF$.

Proof: Place the angle BAC on the angle EDF so that the vertex A will fall on the vertex D, and side AC will fall along DF.

But BC and EF are straight lines (14).

Furthermore, part of BC (AC) coincides with part of EF (DF).

$\therefore$ BC and EF coincide (8c).

$\therefore$ $\angle\ BAC = \angle\ EDF$ (35b).

37b. Theorem: *All right angles are equal* (33b).

38. Theorem: *A straight angle contains* 180°.

Proof: If a straight line is drawn through a point in a plane it divides the sum of the angles about the point in the plane into two equal parts—for the straight angle on the one side of the point equals the straight angle on the other. But the sum of the $\measuredangle$ about a point in a plane = 360° (25).

$\therefore$ half of it = 180°. That is,
a straight angle = 180°.

Another way of stating the above theorem is:

39a. *The sum of the angles about a point in a plane on one* *side of a straight line (in the plane) passing through the point is equal to* 180°.

39b. *A right angle contains* 90° (15, 38).

40. Theorem: *If an angle contains* 180 *degrees, its sides form a straight line.*

Proof: For if it were possible to have a straight angle the sides of which did not form a straight line, on comparing this with a straight angle the sides of which did form a straight line we should have two straight angles which were not equal (35a).

But this is impossible (37a).

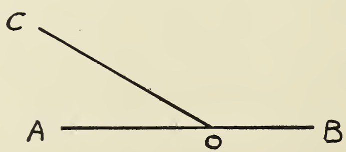
41a. The **supplement** of an angle is that angle which must be added to it to make the sum equal to 180 degrees. That is,

41b. Rule: *To find the supplement of a given angle, subtract the given angle from* 180 *degrees.*

42. *The supplements of the same angle or of equal angles are equal* (32b).

43a. Supplementary Angles are angles whose sum is equal to 180°, or a straight angle.

43b. Theorem: *If two adjacent angles have their exterior sides in a straight line, these angles are supplementary.*



Given: The exterior sides AO and OB of the adjacent angles AOC and COB are in the straight line AB.

To Prove: $\measuredangle$ AOC and COB are supplementary.

Proof: AOB is a straight line. (given)

$\therefore$ $\angle$ AOB is a st. $\angle$ (14).

But $\angle AOC + \angle COB =$ the st. $\angle AOB$ (32a).
$\therefore$ the $\measuredangle$ AOC and COB are supplementary (43a).

NOTE: Compare this theorem with art. 39a.

43c. Supplementary-Adjacent Angles are adjacent angles that are supplements of each other.

EXERCISE

Find the supplements of:

(1)	(2)	(3)	(4)	(5)
120°,	30°,	110°,	45°,	90°,

(6)
40° 50′.

180° may be written 179° 60′. The work may then be arranged as follows:

$$\begin{array}{r} 179^\circ\ 60' \\ 40^\circ\ 50' \\ \hline 139^\circ\ 10' \end{array} \text{ Ans.}$$

(7)	(8)	(9)	(10)	(11)	(12)
17° 21′,	39° 42′,	111° 27′,	82° 37′,	125° 57′,	0° 28′,

(13)
47° 28′ 13″.

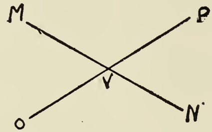
180° may be written 179° 59′ 60″. The work may then be arranged as follows:

$$\begin{array}{r} 179^\circ\ 59'\ 60'' \\ 47^\circ\ 28'\ 13'' \\ \hline 132^\circ\ 31'\ 47'' \end{array} \text{ Ans.}$$

(14)	(15)	(16)	(17)	(18)
2° 31′ 5″,	170° 19′ 53″,	158° 0′ 2″,	122° 59′ 17″,	1° 0′ 7″.

44. If the sides of an angle are prolonged so that they form another angle having the same vertex, these two angles are called **Vertical Angles.** Two intersecting straight angles form two pairs of vertical angles.

45. Theorem: *If one straight line intersects another straight line, the vertical angles thus formed are equal.*



Given: The lines MN and OP intersect at V.

To Prove: $\angle MVO = \angle PVN$.

Proof: $\angle OVM + \angle MVP = 180°$ (43b).

$\angle PVN + \angle MVP = 180°$ (43b).

That is: Both $\angle MVO$ and $\angle PVN$ are supplements of $\angle MVP$ (41a).

$\therefore \angle MVO = \angle PVN$ (42). *Q. E. D.*

By the same kind of proof show that $\angle MVP = \angle OVN$.

The preceding theorem (that is, the statement above the figure) may be readily divided into two parts: what is given and what is to be proved. This can be done with all theorems: for example, in the first theorem (37a) it is *given* that the angles are straight angles: *to prove* that they are equal.

Notice that after a theorem is stated, a figure is drawn and lettered to represent what is given in the theorem; below this figure (after the word "given") must be definitely stated what is true of the relation of the parts of the figure. Nothing can be used in the proof which is not stated as "given." For example, in the above theorem, if it were stated as given, merely, that MN and OP were intersecting straight lines—that is, if it were not stated that the lines intersected at V, then no use could be made in the proof of the fact that V is the vertex of the angles formed—for that would imply that V *was* the intersection of the lines.

It is not necessary to state that MN and OP are *straight lines*, for when a line is mentioned in mathematics it means a straight line unless otherwise stated.

What is given is sometimes called the **Hypothesis.**

The student should always make sure that the last statement in the proof is that which we are directed to obtain in the statement of which we are "to prove." For example, in the above theorem we are to *prove*

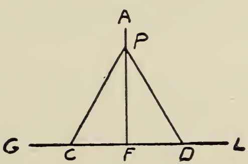
$\angle MVO = \angle PVN$; and the last statement in this proof is

$\therefore \angle MVO = \angle PVN$.

In other words, we have arrived at our answer.

To call attention to the fact that we have arrived at the answer, it is customary to mark the last statement of a completed proof with the initial letters of three Latin words meaning "which was to be proved," that is, with the letters *Q. E. D.*, the initial letters of "Quod erat demonstrandum."

46a. Theorem: *Two straight lines drawn from the same point in a perpendicular to a given line, cutting off on the given line equal lengths from the foot of the perpendicular, are equal, and make equal angles with the perpendicular.*



Given: $AF \perp$ to GL at F, P any point on AF, CP and PD two straight lines drawn from P, cutting off on GL the equal lengths CF and FD.

To Prove: $CP = PD$; $\angle CPF = \angle FPD$.

Proof: Fold the left hand half of the figure over, along the line AF, until it falls on the plane at the right of AF, and its plane coincides with that of the right hand half (7b).

$$\angle AFG = \angle AFL \text{ (37b)}$$

Then, since side AF coincides with side AF and the vertex F with the vertex F, side GF will fall along FL (35a).

C will fall on D ($CF = FD$, given).

Then, since P is common to both CP and PD, CP coincides with PD (8c).

Also, $PD = CP$ (8d).

Again, since PF coincides with PF, P with P and PC with PD, $\angle CPF = \angle FPD$ (35b). *Q. E. D.*

46b. A **Segment** is a portion of a line included between two definite points. (Thus, in the above figure, CF is a segment of GL.)

47a. To **Bisect** means to divide into two equal parts.

47b. The **Perpendicular Bisector** of a given line is that perpendicular to it which divides it into two equal parts.

48a. A **Postulate** is a construction admitted to be possible. The following are postulates.

48b. *A straight line can be drawn between any two points* (8b).

48c. *A straight line can be produced indefinitely* (8g).

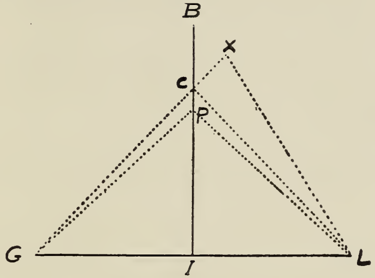
48d. *Any point in a plane may be taken as a center of a circle of any radius.*

For example, by the first postulate (48b), if A and G were given points in the preceding figure, we might draw the line AG but it would not be in accordance with the postulate to attempt to draw a line from A to G *equal to* AF—the postulate does not state that a line of a *given length* can be drawn between any two points.

Let the student beware of attempting to make a construction line fulfill two relations at once (as in the case cited above); if two are necessary it should be *assumed* that the line fulfills only one of them, and then the other must be *proved*.

49. Theorem: *All points in the perpendicular bisector of a given line are equally distant from the extremities of the line, and no point outside of the perpendicular bisector is equally distant from the extremities of the line.*

Given: GL is a straight line, BI is the perpendicular bisector of GL at I the middle point of GL; P any point in BI, X a point outside of BI.



To Prove: (1) $GP = PL$;

(2) GX is not equal to XL.

Proof: (1) Draw PG and PL.

$$GI = IL \text{ (given).}$$

$$\therefore GP = PL \text{ (46a).}$$

(2) Draw XG and XL.

Since X is not in BI, *one* of these lines must intersect BI. Let XG intersect BI at C.

Draw CL

$$XC + CL > XL \text{ (8f).}$$

But $CL = CG$ (Part 1).

Substituting for CL its equal in the above inequality:

$$XC + CG > XL$$

That is, $$GX > XL \text{ (32a).}$$

In other words, since X, *any* point outside of BI is unequally distant from G and from L, *all points* outside of BI are unequally distant from G and from L, that is, *no* point outside of BI is equally distant from G and from L.

50. A line added to a figure to assist in the proof is called a **Construction Line.** The student should be careful that every construction used is a postulate. A construction line should always be dotted, as GX in the preceding figure.

51. To Determine a line means to locate it definitely.

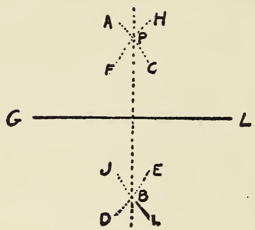
52. Theorem: *Two points equidistant from the extremities of a straight line determine the perpendicular bisector of the line.*

Proof: These two points cannot lie outside of the perpendicular bisector (49)

$\therefore$ they must lie *in* the perpendicular bisector.

A line joining these two points must coincide with the perpendicular bisector (8c). *Q. E. D.*

To construct the perpendicular bisector of a given line.



Given: GL is a straight line.

To Construct: the perpendicular bisector of GL.

Construction: With GP, any length greater than half GL, as a radius, and G as a center, construct arcs AC and DE.

(I) Any point in AC and DE will be at a distance GP from G (18).

With GP as a radius and L as center construct arcs FH and JL, intersecting AC and DE at P and B, respectively.

(II) All points in FH and JL will be at a distance GP from L. Why?

But P lies in both AC and FH.

$\therefore$ P is at a distance GP from both G and L (I, II).

That is, P is equidistant from G and L.

In like manner, B is equidistant from G and L.

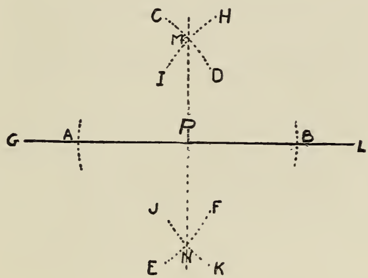
$\therefore$ PB is the perpendicular bisector of GL (52).

The student may now understand the following:

53. Rule: *To construct the perpendicular bisector of a given line; with one extremity of the given line as a center and any radius greater than half the line describe arcs of the same circle both above and below the line; with the other extremity as a center and the same radius describe arcs intersecting the first, and draw a straight line between the points of intersection. This straight line will be the perpendicular bisector of the given line.*

NOTE: This construction, and those which follow, are to be used by the student in the blackboard construction of figures. Make no figures freehand.

To erect a perpendicular to a given line at a given point in the line.



Given: P any point in the straight line GL.

To Construct: MN perpendicular to GL at P.

Construction: With any radius as AP and with P as a center describe arcs intersecting GL at A and B.

Construct MN the perpendicular bisector of AB (53).

P is equidistant from A and B (16).

$\therefore$ P lies in MN (49).

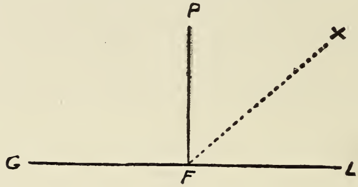
That is, MN is $\perp$ to AB at P.

That is, MN is $\perp$ to GL at P (12a, 10).

The student may now understand the following:

54. Rule: *To erect a perpendicular to a given line at a given point in the line; with the given point as a center and any radius describe arcs intersecting the given line on either side of the given point; with these points of intersection as centers and a longer radius describe arcs intersecting above and below the line; a line connecting these points of intersection will be the perpendicular desired.*

55. Theorem: *At a given point in a given straight line, there can be but one perpendicular to the line.*



Given: F any point in the straight line GL and $PF \perp$ to GL.

To Prove: PF is the only perpendicular to GL at F.

Proof: If possible let XF be a perpendicular to GL at F.

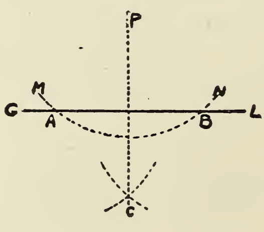
Then $\angle\ PFL$ is a rt. $\angle$ and $\angle\ XFL$ is a rt. $\angle$.

$$\therefore \angle\ PFL = \angle\ XFL \quad (37b)$$

But this is impossible (33a).

$\therefore$ PF is the only $\perp$ to GL at F. Q. E. D.

From a given external point to let fall a perpendicular upon a given line.



Given: GL is a straight line, P any external point.

To Construct: A $\perp$ to GL from P.

Construction: From P as a center, with *any* radius, draw arc MN, intersecting the given line at A and B.

P is equidistant from A and from B (18).

With A as a center and any distance greater than one half of AB as a radius describe an arc, and with B as a center and the same radius describe an arc intersecting the arc just drawn at C. Draw PC.

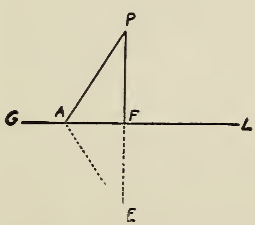
PC is the perpendicular bisector of AB (52).

That is, it is perpendicular to GL (12a, 10).

The student may now understand the following:

56. Rule: *From a given external point to let fall a perpendicular upon a given line: with the given point as a center and a radius of sufficient length describe an arc intersecting the given line in two points; with these points of intersection as centers, and a radius greater than half the distance between them, describe intersecting arcs. A straight line through the point of intersection of these arcs and through the given point will be the desired perpendicular.*

57. Theorem: *Only one perpendicular can be drawn from a given external point to a given straight line.*



Given: GL any straight line, P any external point, PF a perpendicular from P to GL, PA *any other* line drawn from P to GL.

To Prove: PA is not $\perp$ to GL.

Proof: Produce PF to E, making $FE = FP$. Draw AE.

PE is a straight line. (Construction)

$\therefore$ PAE is not a straight line (8a).

$\therefore$ $\angle$ PAE is not a st. $\angle$ (14).

Since PE is $\perp$ to GL, GL is $\perp$ to PE (12b) and by construction $PF = FE$.

$\therefore$ $\angle PAF = \angle FAE$ (46a).

That is, $\angle PAF = 1/2 \angle PAE$ (32a).

But PAE is not a st. $\angle$.

$\therefore$ half of it is not a rt. $\angle$.

That is, $\angle PAF$ is not a rt. $\angle$.

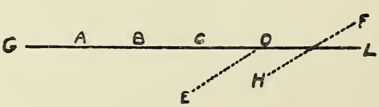
$\therefore$ PA is not $\perp$ to GL.

But PA is *any* line other than PF drawn from P to GL.

$\therefore$ PF is the only $\perp$. *Q. E. D.*

The student is now ready to consider the question "What is a straight line?" Although the student knows well enough, practically, what a straight line is, so that he would not mistake a straight line for a curved line, he will yet find it hard to compose a good mathematical definition of a straight line. A statement of an axiom about a straight line is not a *definition* of it, any more than the statement that "two points equidistant from the extremities of a straight line determine its perpendicular bisector" is a definition of a perpendicular bisector. It is therefore incorrect to state "a straight line is the shortest distance between two points" as a definition of a straight line.

To understand the following definition of a straight line let the student consider the accompanying figure.



If from anywhere in the straight line GL a segment is cut, as AB, this segment may be placed anywhere on GL, as at CD, and made to coincide with the line completely, provided both extremities, A and B, lie in the line. The segment AB would not, of course, coincide with GL if placed in the position ED or HF. Furthermore, not only will AB coincide with the line as first placed, in the position CD (with A at C and B at D), but it also will coincide with the line if *reversed*, so that A falls at D, and B at C, that is AB can be made to coincide with GL, *however* placed on GL, provided A and B lie in GL.

The student may now understand the following definition:

58. A **straight line** is a line of which any segment, *however* placed on any other part of the line, will coincide with that part if its extremities lie in that part.

The theorems in this and following chapters are called Theorems in Plane Geometry. **Plane Geometry** is the study of Plane figures.

II. Field Exercise

LAYING OUT FOUNDATION OF BUILDING

Equipment: Level, ranging pole, knotted cord,* four stakes (3 ft. long), a dozen stakes (18 in. long), mallet, hammer, nails.

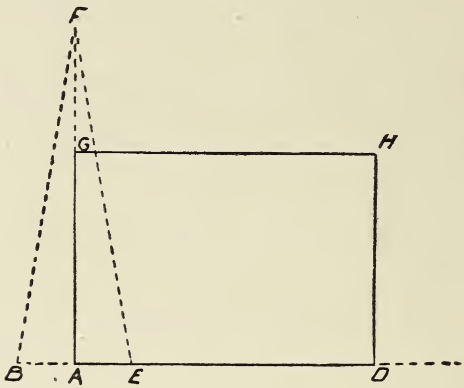
Personnel: Leader, one student to use level, one to hold ranging pole, one to drive stakes; three to hold knotted cord (one at each end and one at middle).

Procedure: Drive a long stake A to mark front corner of building. Place a stake B twenty feet to one side and here set up level so that observer may look through instrument, across corner, and along proposed front of building. The ranging pole should be held a short distance from A (on the

*The knotted cord is prepared by tying a knot at the center of a piece of cord about 100 feet long, then doubling the cord and cutting off the two ends so that they are exactly even; that is, so that the knot comes in the middle. At each end of the cord make a loop to serve as a handle. (These loops should be of the same size.)

side opposite the level) and directed by the observer until it is exactly in the line BA produced (Field Exercise I, Use of Level).

This position should be marked with a stake. Continue this process until a line has been staked out at least twenty feet longer than the desired front of the building.



Starting from A measure along this line of stakes the distance desired for the front of the building, and mark the end of this distance with the long stake D. A and D are the two front corners of the building.

To locate the two rear corners, proceed as follows: First erect a perpendicular to the line AD at A, in this manner; measure the distance BA, and locate another stake E the same distance to the other side of A—marking this stake (with pencil, or with a sheet of paper through which the top of the stake is thrust) to avoid confusing it with the other stakes. By construction, A is equally distant from B and E. One end of the knotted cord should be held at E, the other at B, and the cord drawn tightly by the knot in the middle. This knot should be marked by the stake F. FA

is perpendicular to AD (54). (If it is desired to prolong AF, this may be done by setting up the instrument at A, and sighting across F.) From A measure toward F the distance desired for the side of the building, and mark it with a stake G.

By the same method a stake H should be located for the other rear corner. To test the work the distance GH should be measured; if the length GH is the same as AD, the work is correct. To assist in measuring these distances, a nail should be driven in the center of each stake.

All the short stakes may now be taken up, and the four corner stakes, A, D, H, and G, driven in firmly.

NOTE: If the Farm Level is not completed, the Water Level may be used instead. Since the instrument is here used, not to lay off levels, but simply straight lines, it is not necessary to float it in a pail of water, but simply place it on top of a box or other support. Instead of the water level, a three foot length of water pipe (about an inch in outside diameter) may be used. The pipe should be supported between two stakes, with V-shaped notches in their tops, driven into the ground two feet or more apart.

CHAPTER III

PARALLEL LINES

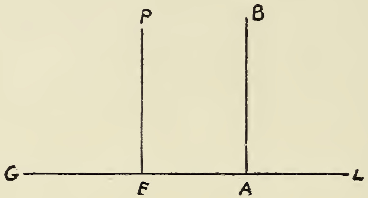
59. Parallel Lines are lines in the same plane which cannot meet, however far they may be produced.

The symbol for parallel is //.

The symbol for parallel lines is //s.

Parallel lines are often called "Parallels."

60a. Theorem: *Two or more straight lines in the same plane, perpendicular to the same straight line are parallel.*



Given: FP and $AB \perp$ to GL.

To Prove: FP and AB are //.

Proof: If FP and AB were not //, they would meet at some point. Then, from this point there would be two lines, FP and AB, both $\perp$ to GL.

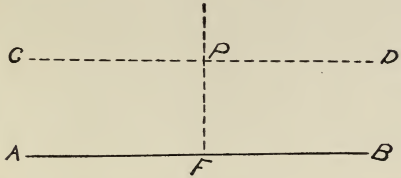
But this is impossible (57).

$\therefore$ FP and AB do not meet—that is, they are //. *Q. E. D.*

Compare the above method of proof with that used in Art. 55.

NOTE: It is upon this principle that parallel lines are drawn with the T-square—by sliding the crosspiece of the T-square up and down the edge of the drawing board.

60b. *To construct a parallel to a given line through a given external point.*



Given: AB a straight line and P a given external point.

To Construct: A line through P, // to AB.

Construction: From P let fall a $\perp$ PF upon AB (54), and produce this line upward through P.

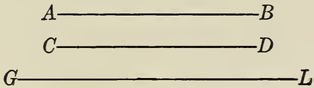
At P erect a $\perp$ (CD) to PF (54).

CD will be // to AB (60a).

In the same manner any number of perpendiculars to PF may be proved parallel to each other.

61a. Axiom: *Through a given point only one straight line can be drawn parallel to a given straight line.*

61b. *Two or more straight lines in the same plane, parallel to the same straight line are parallel to each other.*



Given: AB and CD // to GL.

To Prove: AB // to CD.

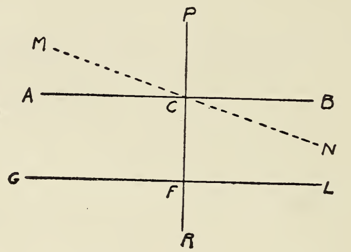
Proof: If AB and CD are not // they will meet at some point. Then from this point there would be two lines // to GL. But this is impossible (61a).

$\therefore$ AB and CD cannot meet and so are //. In like manner, the theorem could be proved to hold for any number of //s. *Q. E. D.*

Let the student prove this theorem to hold for three lines // to a given straight line.

NOTE: In the following proof it may at first seem absurd to draw the line MN and consider it as being perpendicular to PR. It should be borne in mind, however, that the line MN *symbolizes* a relation about which we are reasoning. To aid us in the proof, we are to consider a line through C perpendicular to PR and it is necessary to keep this idea distinct from the idea of a parallel to GL drawn through C.

62. Theorem: *If a straight line is perpendicular to one of two parallel lines it is perpendicular to the other also.*



Given: GL and AB are two parallel straight lines, PR a perpendicular to GL at F, cutting AB at C.

To Prove: PR is $\perp$ to AB also.

Proof: Through C draw MN $\perp$ to PR.

(Since we do not yet know whether AB is perpendicular to PR or not, we do not draw MN coinciding with AB.)

MN is // to GL (60a).

$\therefore$ MN coincides with AB (61a).

But MN is $\perp$ to PR (Construction).

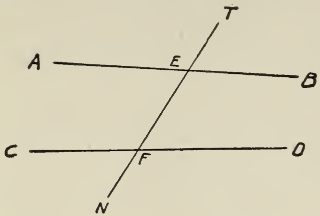
$\therefore$ AB, which coincides with MN, is also $\perp$ to PR.

That is, PR is $\perp$ to AB. *Q. E. D.*

63a. A **Transversal** is a straight line which cuts across two or more other lines.

63b. The angles formed on the inside of two given straight lines cut by a transversal on opposite sides of the transversal, one adjacent to one of the given straight lines and the other to the other given straight line, are called **Alternate Interior Angles.**

The abbreviation for alternate interior angles is "Alt. Int. $\measuredangle$s."

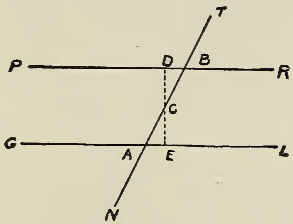


TN is the Transversal, AB and CD are the given straight lines.

Thus, $\angle$ BEF and $\angle$ EFC are Alt. Int. $\measuredangle$s: $\angle$ AEF and $\angle$ EFD are Alt. Int. $\measuredangle$s.

NOTE: "Alternate," the first part of the name, indicates that the angles are opposite sides of the transversal, while "Interior" indicates that they are inside the given straight lines.

64. Theorem: *If two parallel lines are cut by a transversal the alternate interior angles are equal.*



Given: GL and PR are two parallel lines cut by the transversal TN at points A and B.

To Prove: $\angle\ PBA = \angle\ BAL$.

Proof: Through C, the middle point of AB, draw $DE \perp$ to PR. Then DE is $\perp$ to GL also (62).

(I) That is, DB is $\perp$ to DE.

Also the fact that DE is $\perp$ to AE may be stated as AE is $\perp$ to DE (12b).

Fold the figure directly upwards about the point C, so that CE falls along CE produced, that is, along CD (8g).

Then AE, which is $\perp$ to CE, is $\perp$ to CD. (CE now coincides with CD.)

Now fold the figure ACE to the right about CE (which coincides with CD) as an axis until A falls upon the plane to the right of DC.

Since CE still coincides with CD,

(II) AE (which is $\perp$ to CE) is $\perp$ to CD.

Also, since $\angle\ ACE = \angle\ DCB$ (45), and C falls on C, AC will fall along CB (35a).

But $AC = CB$. (C is middle point of AB.)

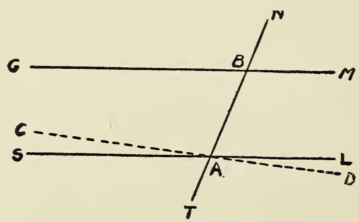
$\therefore$ A falls on B.

Now both AE and DB are $\perp$ to CD (I, II).

$\therefore$ DB and AE coincide (57), and since A coincides with B, and AC coincides with CB,

$$\angle\ PBA = \angle\ BAL \text{ (35b). } Q.\ E.\ D.$$

65. Theorem: *If the alternate interior angles, formed by a transversal intercepting two straight lines are equal, the two straight lines are parallel.*



Given: GM and SL two straight lines, cut by the transversal TN at points A and B, so that $\angle\ GBA$ equals $\angle\ BAL$.

To Prove: SL is // to GM.

Proof: Consider a line drawn through A // to GM.

Since we do not yet know whether this line coincides with SL or not, we will consider it, temporarily, as occupying the position CD.

$\angle GBA = \angle BAD$ (64).

But $\angle GBA = \angle BAL$ (Given).

$\therefore \angle BAD = \angle BAL$ (31).

Since $\measuredangle$ BAD and BAL have side BA in common, and vertex A in common, AD and AL coincide (35a).

$\therefore$ SL and CD coincide (8c).

That is SL coincides with a line // to GM and so SL is // to GM. *Q. E. D.*

Class Room Exercise

Take a piece of paper with parallel sides (such as an ordinary small blotter) and cut it across slantingly with the scissors. The angles thus formed may be made to coincide by reversing one part of the blotter and placing it over the other. What theorem does this illustrate?

66. When two given straight lines are cut by a transversal, a pair of angles on the same side of the transversal, but with one of the angles inside the given lines and the other outside the lines, are called **Exterior-Interior Angles.**

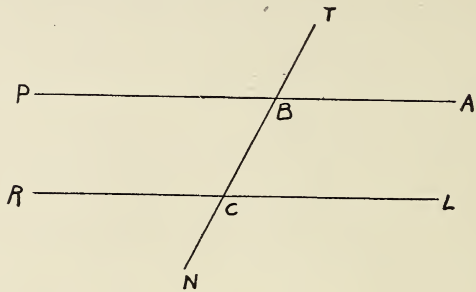
Thus in the preceding figure, NBG and BAS are exterior-interior angles.

The name "Exterior-Interior Angles" means "outside-inside angles" since one angle of the pair is outside the given lines and the other inside.

Thus when a transversal intersects two parallels, it forms but two pairs of alternate-interior angles, but forms four pairs of exterior-interior angles. Thus, in the preceding figure, the pairs of exterior-interior angles are as follows:

NBG and BAS, GBA and SAT, NBM and BAL, MBA and TAL.

67a. Theorem: *If two parallel lines are cut by a transversal, the exterior-interior angles are equal.*



Given: PA and RL are two parallel lines cut by the transversal TN at B and C.

To Prove: $\angle TBA = \angle BCL$.

Proof: $\angle TBA = \angle PBC$ (45).

$\angle PBC = \angle BCL$ (64).

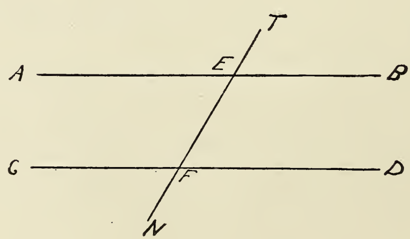
$\therefore \angle TBA = \angle BCL$ (31). *Q. E. D.*

In like manner it may be proved that $\angle TBP = \angle BCR$.

Also $\angle PBC = \angle RCN$, etc.

Name the other pair of angles in the above figure which are equal by this theorem.

67b. *If the exterior-interior angles formed by a transversal intersecting two straight lines are equal, the two straight lines are parallel.*



Given: AB and CD two straight lines cut by the transversal TN at points E and F so that $\angle\ TEB = \angle\ EFD$.

To Prove: AB is // to CD.

Proof:

$\angle\ AEF = \angle\ TEB$ (45).

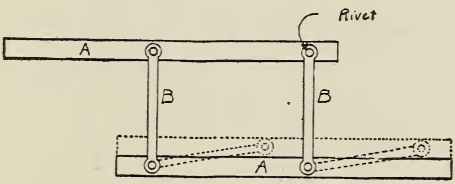
$\angle\ TEB = \angle\ EFD$ (Given).

$\therefore\ \angle\ AEF = \angle\ EFD$ (31).

$\therefore\ AB$ is // to CD (65). *Q. E. D.*

IIIa. Shop Exercise

The **Parallel Ruler** is a useful instrument for drawing lines parallel to a given line. As seen from the diagram, the distance between the bars can be varied at will.



The parts represented are:

AA, strips of wood 1/4 in. $\times$ 1 in. $\times$ 12 in.

BB, strips of brass, 4 in. long, with holes for rivets.

After constructing this instrument the student can easily design and execute a parallel ruler of larger size (using wooden strips instead of brass), for use at the blackboard.

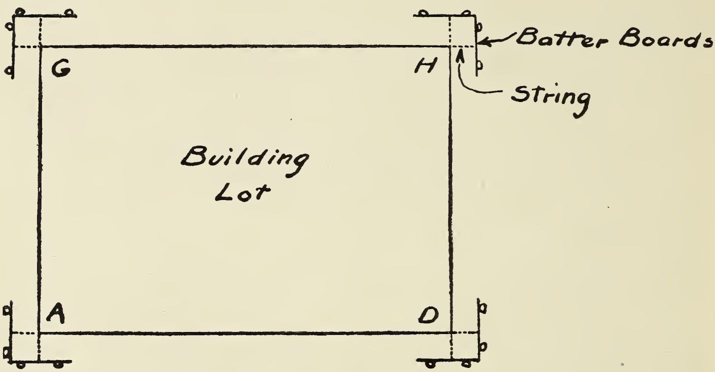
IIIb. Field Exercise

MARKING CORNERS BY BATTER BOARDS

Since it would not be possible to excavate for the foundation thus laid out (II) without disturbing the stakes marking the corners, it is customary to transfer these corner marks to "Batter Boards." As shown in the diagram (IIIb) a Batter Board is formed of two pieces of board nailed to three or four pieces of upright studding so as to form a right angle. The upper edges of these boards should be planed smooth. A Batter Board should be placed at each corner of the foundation to be marked.

Equipment: Water level (with pail of water in which to float level), box (upon which to rest pail); ball of fishline or other strong cord; 8 pieces of board, 3 feet long, planed on one edge; 12 pieces of studding, 2 inches wide, 3 inches thick, and 5 feet long; half pound sixpenny nails; hammer; plumb-bob.

Procedure: About two feet outside each corner stake, drive in the uprights, as shown in the diagram. The first



board should be fastened to two of these uprights at one end of the building, at a front corner. At first only one end of the board should be fastened to an upright (and that with a single nail) at the height desired for the masonry foundation. Locate the water level (in the pail of water, on top of the box) so that the line of sight crosses this board and traverses the front of the building. (The line of sight may be raised by pouring more water into the pail.) While the student at the water level moves it slightly from one side to the other, so that his line of sight sweeps along the top of the board, another student should move the unnailed end of the board up or down until it is level, and then nail it. Then (under the direction of the student at the water level), the board parallel to this, at the other front corner of the building, should be nailed at exactly the same level as the first board.

The water level should now be moved so that the board forming a right angle with that first located may be adjusted. (To get the line of sight in the same level as before, it is best to pour out some of the water, and then add water gradually until the line of sight reaches the same level shown by the board first nailed.) By moving the water level from one side to the other, nail this board exactly level with the board first placed, and forming a right angle with it. A board parallel to this, and at the same level, should now be located at the rear corner of the building.

The water level should now be moved to the other front corner of the building and the process repeated.

The batter boards on the two front corners are now complete—those at the rear corners still lack the end boards. These should now be put in position, each of these end boards being made level with its adjacent side boards. The work may now be tested by sighting along the length of the building with the water level; if the work has been accurately done, these two boards last located should be at the same level.

A student should now station himself at each of the front batter boards—holding one end of a length of cord so that it is parallel to the front of the building, and drawing it tight over the end boards. Another student, with the plumb-bob, should direct them in moving the cord so that it is directly above the nails in the corner stakes (II). Then nails should be driven carefully in the sides of the end boards (not the upper edge, as the hammering tends to disturb the level), and the cords fastened to them, so that they are stretched taut. In the same way, a cord should be fastened at the rear of the building and also at the two ends. As shown in the diagram, the intersections of these cords mark the corners of the foundation, while the planed edges of the board mark the level for the masonry construction. The corner posts (II) may now be removed, leaving the ground clear for excavation.

NOTE: Instead of the water level, the water pipe substitute (II) may be used, in combination with a carpenter's level—the pipe being brought to the proper alignment by driving in the stakes. In this case, for leveling the second board of the batter boards, the carpenter's level is applied directly to the upper edge of the board.

CHAPTER IV

TRIANGLES

68a. A **Triangle** is a portion of a plane bounded by three straight lines.

68b. The lines bounding a triangle are called its **sides,** and the points of intersection of the sides are called the **vertices** of the triangle. The symbol for triangle is $\triangle$.

69. Any side of a triangle may be called its **base;** the perpendicular let fall from the opposite vertex to the base is called the **altitude** of the triangle.

70. An **Isosceles Triangle** is a triangle having two equal sides.

71. Two triangles are equal if the three vertices of the one coincide with the three vertices of the other, that is, *two triangles are equal if they may be made to coincide throughout* (8c).

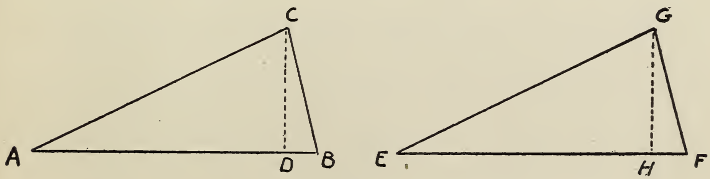
72a. The **corresponding angles** of two equal triangles are angles opposite equal sides.

72b. The **corresponding sides** of two equal triangles are sides opposite equal angles.

72c. *Corresponding parts of equal triangles are equal.*

NOTE: Corresponding parts are sometimes called **Homologous** parts.

73. *To construct a triangle equal to a given triangle.*

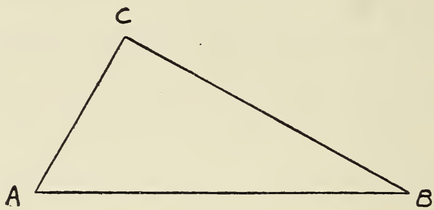


From any vertex of the given triangle (as C) let fall a perpendicular to the opposite side.

This perpendicular CD will then be the altitude of the triangle and the side AB to which it is drawn will be the base (69): to construct an equal triangle lay off a straight line equal to the base; at a point which is the same distance EH from one extremity E of this line as the foot of the altitude of the given triangle is from the extremity A of its base, erect a perpendicular (54); lay off on this perpendicular GH a distance equal to the altitude of the triangle; lines joining point G to the extremities of EF form with EF the triangle desired.

The proof of this is left to the student.

74. Theorem: *The sum of two sides of a triangle is greater than the third side.*



Given: In the triangle CAB let AB be the longest side.

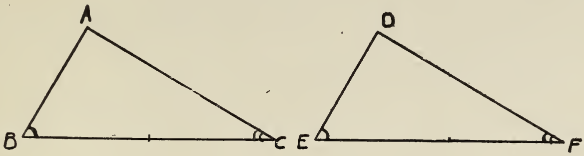
To Prove: $AC + CB > AB$.

Proof: $AC + CB > AB$ (8f). $Q. E. D.$

NOTE: Since AB is the longest line, there is no need of going through a proof to show that $AB + BC > AC$, or that $AB + AC > BC$.

Lines which are equal may be marked with the same number of cross lines, and angles which are equal by an equal number of small arcs. Thus, in the next figure, the angles B and E, being equal, are marked with one arc, and C and F with two arcs. In the succeeding figure AB and DE, being equal, are marked with one cross line, and EF and BC, being equal, are each marked with two cross lines.

75. Theorem: *Two triangles are equal if two angles and the included side of the one are equal to two angles and the included side of the other.*



Given: In $\triangle$ ABC and DEF, $BC = EF$, $\angle B = \angle E$, and $\angle C = \angle F$.

To Prove: $\triangle ABC = \triangle DEF$.

Proof: Place $\angle DEF$ on $\angle ABC$ so that EF coincides with its equal BC.

DE will fall along AB ($\angle E = \angle B$, given).

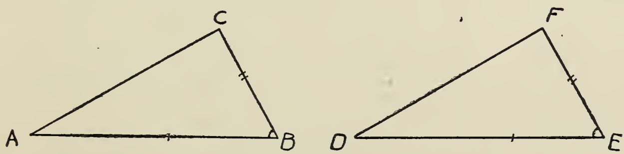
DF will fall along AC ($\angle F = \angle C$, given).

Now, AB and AC intersect at A, therefore DE and DF which coincide with them must intersect in the same point.

$\therefore$ D will fall on A (8e).

$\therefore$ $\triangle DEF = \triangle ABC$ (71). *Q. E. D.*

76. Theorem: *Two triangles are equal if two sides and the included angle of the one are equal to two sides and the included angle of the other.*



Given: In $\triangle$ ABC and DEF, $DE = AB$, $EF = BC$ and $\angle E = \angle B$.

To Prove: $\triangle DEF = \triangle ABC$.

Proof: Place $\triangle\ DEF$ on $\triangle\ ABC$ so that EF coincides with its equal, BC. Then, since E falls on B, ED will fall along AB ($\angle\ E = \angle\ B$, given).

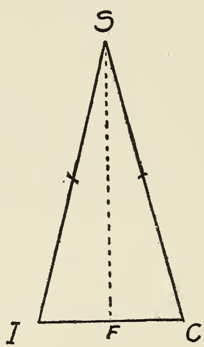
D falls on A ($ED = AB$, given).

Then, since F falls on C, DF coincides with AC and is equal to it (8c, 8d).

$\therefore\ \triangle\ DEF = \triangle\ ABC$ (71). *Q. E. D.*

77a. To say that two lines are equal by **identity,** means that they are really the same line.

77b. Theorem: *In an isosceles triangle the angles opposite the equal sides are equal.*



Given: Let ISC be an isosceles triangle, having $IS = SC$.

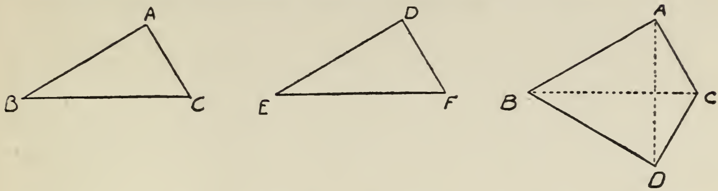
To Prove: $\angle\ I = \angle\ C$.

Proof: Let SF be drawn so as to bisect the $\angle\ S$.

In $\triangle$ SIF and SFC,

$SI = SC$	(Given)
$SF = SF$	(Identity)
$\angle\ ISF = \angle\ FSC$	(Construction)
$\therefore\ \triangle\ SIF = \triangle\ SFC$	(76)
$\therefore\ \angle\ I = \angle\ C$	(72c) *Q. E. D.*

78a. Theorem: *Two triangles are equal if the three sides of the one are equal to the three sides of the other.*



Given: In the $\triangle$s ABC and DEF, let $DE = AB$, $DF = AC$, and $EF = BC$.

To Prove: $\triangle\ DEF = \triangle\ ABC$.

Proof: Place $\triangle\ DEF$ below $\triangle\ ABC$, as shown by the dotted lines, so that EF coincides with its equal BC, D falling on the opposite side of BC from A, E coinciding with B and F with C. Draw AD.

$BA = BD$ (given).

$\therefore \triangle\ ABD$ is isosceles (70).

I. $\therefore \angle\ BAD = \angle\ BDA$ (77b).

Again, $AC = DC$ (given).

$\therefore \triangle\ ACD$ is isosceles (70).

II. $\therefore \angle\ CAD = \angle\ CDA$ (77b).

Adding equations (I and II)

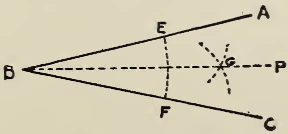
$\angle\ BAD + \angle\ CAD = \angle\ BDA + \angle\ CDA$ (34b).

$\therefore \angle\ BAC = \angle\ BDC$ (32a).

That is, $\angle BAC = \angle EDF$.

$\therefore \triangle\ ABC = \triangle\ DEF$ (76). *Q. E. D.*

To bisect a given angle.



Given: ABC any angle.

To Construct: PB bisecting ABC.

Construction: With B as a center, and any convenient radius, describe an arc intersecting AB and BC at E and F, respectively.

With E as a center and any convenient radius (greater than half the distance EF), draw an arc, and with F as a center, and the same radius, draw an arc intersecting the arc just drawn at G. BG (produced as much as desired) is the bisector of $\angle EBF$.

Proof:

$$EB = BF \text{ (18).}$$
$$EG = GF \text{ (18).}$$
$$GB = GB \text{ (Identity).}$$
$$\therefore \triangle GBE = \triangle GBF \text{ (78a).}$$
$$\therefore \angle EBG = \angle GBF \text{ (72a).}$$

The student may now understand the following:

78b. Rule: *To bisect a given angle: with the vertex of the given angle as a center, and any radius, draw an arc intersecting the sides of the given angle: with these points of intersection as centers, and any length greater than half the distance between them as a radius, draw intersecting arcs; a line drawn from this point of intersection to the vertex will be the bisector of the angle.*

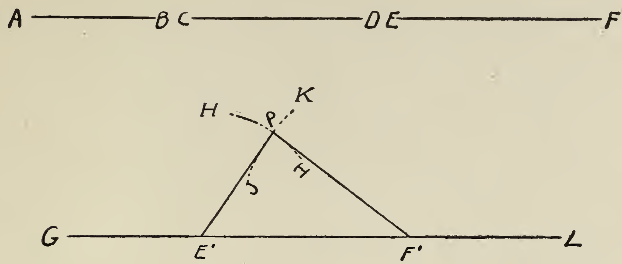
Blackboard Exercise

By aid of Art. 53 (or Art. 56) and 78b, construct an angle of 45°.

79a. To suggest that points or lines have some important relation, they may be designated by the same letters—these letters of the second point being distinguished from those of the first point, or line, by a small mark placed above and to the right of the letter, which is read **"Prime."** Thus in the following figure, the segment $E'\,F'$ of GL is so marked to indicate that it is equal to EF.

$E'\,F'$ is read, "E prime F prime."

79b. *To construct a triangle when three sides are given.*



Let the given lines be AB, CD, and EF.

Draw a line of indefinite length, as GL; on GL lay off a segment equal to one of the given lines—as $E'F'$ equal to EF.

With one of the remaining lines as a radius, as AB, and E' as a center, draw an arc, as HI.

With the remaining given side as a radius, and F' as a center, describe an arc JK, intersecting the first at P.

Then, since P is in the arc HI it is at a distance AB from E' and since it is in the arc JK it is at a distance CD from F' (18).

Draw PE' and PF'.

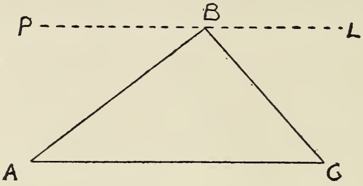
The $\triangle\ PE'F'$ is the $\triangle$ desired.

For $PE' = AB$, $PF' = CD$ and $E'F' = EF$.

The student may now understand the following:

79c. Rule: *To construct a triangle when the three sides of the triangle are given: lay off a length equal to one of the given sides—then with another of these sides as a radius and an extremity of the line just laid off as a center, draw an arc; then with the third side as a radius and the other extremity of the line first laid off as a center, draw an arc intersecting the arc first drawn; lines drawn from the point of intersection of these arcs to the extremities of the line first laid off will form with this line the triangle desired.*

80a. Theorem: *The sum of the three angles of a triangle is equal to* 180 *degrees.*



Given: ABC is a triangle.

To Prove: $\angle A + \angle B + \angle C = 180°$.

Proof: Through B draw PL // to AC.

(I) $\angle PBA + \angle ABC + \angle CBL = 180°$ (39a)

$\angle PBA = \angle BAC$ (64)

$\angle LBC = \angle BCA.$ (64)

Substituting these values in (I)

$\angle BAC + \angle ABC + \angle BCA = 180°$. *Q. E. D.*

80b. This theorem may also be stated as follows:

The sum of the three angles of a triangle is 2 *rt. angles.*

80c. *If two angles of one triangle are equal to two angles of another, then the third angle of the first triangle equals the third angle of the second.*

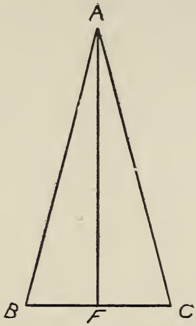
Proof: For the third angle of the first triangle is the supplement of the sum of the two given angles of that triangle (80a; 41b) and the third angle of the second triangle is the supplement of the sum of the two given angles in that triangle.

∴ The third angle of the first triangle equals the third angle of the second triangle (42). *Q. E. D.*

The above theorem (80a) was known long before the Christian era. Aristotle, who lived about 350 B. C., referred to it frequently.

Let the student prove the above theorem by drawing the construction line through C parallel to AB.

80d. Theorem: *The perpendicular from the vertex to the base of an isosceles triangle, bisects the base and the angle at the vertex.*



Given: AB and AC are the legs of the isosceles triangle ABC; AF the $\perp$ from the vertex A to the base BC.

To Prove: $BF = FC$; $\angle BAF = \angle FAC$.

Proof: In $\triangle$s BAF and FAC

$\angle B = \angle C$	(77b)
$\angle BFA = \angle CFA$	(37b)
$\therefore \angle BAF = \angle FAC$	(80c)
But $BA = AC$	(Given)
$AF = AF$	(Identity)
$\therefore \triangle BAF = \triangle FAC$	(76)
$\therefore BF = FC$	(72c) *Q. E. D.*

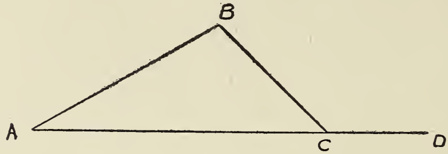
81a. If a side of a triangle is produced, the angle thus formed outside of the triangle, between the side produced and a side of the triangle, is called the **Exterior Angle** of the triangle.

Exercise

Let the student prove the following:

The bisectors of the equal angles of an isosceles triangle, together with the base, form an isosceles triangle.

81b. Theorem: *An exterior angle of a triangle is equal to the sum ol the two opposite interior angles.*



Given: $\angle BCD$ is the ext. $\angle$ of $\triangle$ ABC.

To Prove: $\angle A + \angle B = \angle BCD$.

Proof: The sum of $\angle A + \angle B$ is the supplement of angle BCA (41b).

[For $\angle A + \angle B + \angle BCA = 180°$ (80a)].

Now, $\angle BCD$ is supplement of $\angle BCA$ (43b).

$\therefore$ $\angle BCD =$ the sum of $\angle A + \angle B$ (42). *Q. E. D.*

Since an exterior angle is equal to the sum of the two opposite interior angles it follows that:

81c. *An exterior angle of a triangle is greater than either of the two opposite interior angles.*

IVa. Class Room Exercise

With a pair of scissors cut out a triangle of paper; tear off the corners and lay them on the desk so that the straight edges just touch each other. The outer edges will form a straight line (80a).

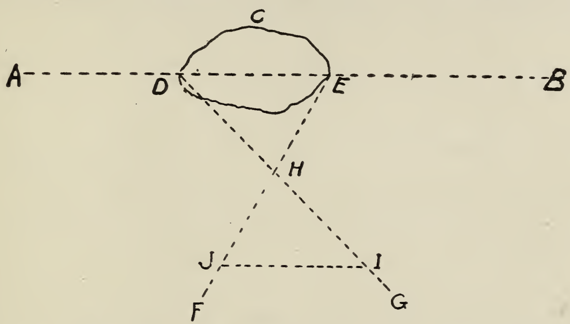
Repeat the exercise with triangles of different shapes.

IVb. Field Exercise

MEASURING THROUGH AN OBSTRUCTION

Equipment: Leveling instrument, ranging pole, stakes, cord.

Procedure: Suppose it is desired to measure the dis-



tance along the edge of a field from A to B and this line is obstructed by a thick growth of trees (C).

Locate a stake D at some point in the line AB, near the obstruction, and another stake E on the opposite side of the obstruction, and in the line AB. Set up the leveling instrument at some convenient point as F, and stake off the straight line FE (Field Exercise II). Set up the instrument again at some other point G and stake off the line DG, intersecting the first line at a point marked with the stake H. (To aid in locating this point exactly, hold the cord along the stakes in one line, and the tape along the stakes in the other.)

Measure DH and lay off the same distance from H toward the point G, marking the end of this distance with the stake I. Measure EH and lay off the equal length HJ.

Measure JI and this will give the length DE.

For $DH = HI$ and $EH = HJ$ (by construction).

$\angle DHE = \angle JHI$ (45).

$\therefore \triangle DHE = \triangle JHI$ (76).

$\therefore JI = DE$ (72c).

CHAPTER V

CIRCLES

Review Arts. 16 to 27 inclusive.

⊙ is the symbol for a circle. Ⓢ is the symbol for circles.

82a. *Two circles are equal if they have equal radii.*

For they will coincide if their centers are made to coincide. Hence:

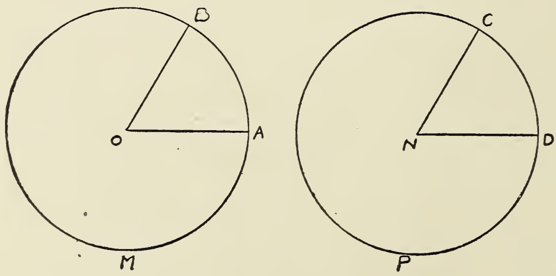
82b. *Two equal circles have equal radii.*

82c. Concentric Circles are circles having the same center.

83. The **converse** of a theorem is obtained by interchanging what is given with what is to be proved.

The two following theorems are converses of each other:

84a. Theorem: *In the same circle or in equal circles, equal central angles intercept equal arcs.*



Given: In the equal Ⓢ BAM and CDP the centers of which are O and N, the central ⦣ BOA and CND are equal.

To Prove: Arc BA = arc CD.

Proof: Place the circle CDP on circle BAM so that the $\angle CND$ shall coincide with its equal $\angle BOA$, that is, so that N falls on O, ND along OA and CN along BO.

Then D falls on A and C on B (82b).

$\therefore$ Arc CD coincides with arc BA (16). *Q. E. D.*

Note: By Art. 16, no part of CD may fall *inside* of BA or *outside* of BA.

84b. *In the same circle or in equal circles, equal arcs are intercepted by equal central angles.*

Given: (in figure of 84a) Arc $AB =$ arc CD.

To Prove: $\angle\ AOB = \angle\ DNC$.

Proof: Place $\odot\ DCP$ on $\odot\ ABM$ so that DN shall fall on its equal AO and arc DC on its equal AB (16).

Then: since C falls on B, and N on O, NC will coincide with OB (8c).

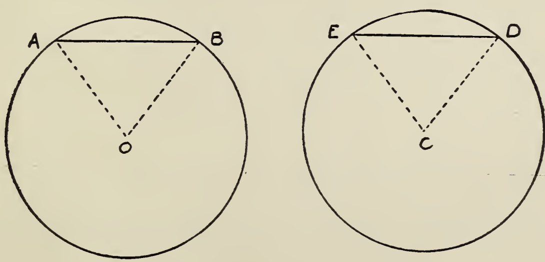
That is, the $\angle\ DNC$ will coincide with the $\angle\ AOB$.

$\therefore \angle\ AOB = \angle\ DNC$ (35b). *Q. E. D.*

85a. A **chord** of a circle is a straight line having its extremities in the circumference.

85b. The arc intercepted by the straight line is said to be **subtended** by it.

86a. Theorem: *In the same circle or in equal circles, equal arcs are subtended by equal chords.*



Given: In the equal Ⓢ the centers of which are O and C, let arcs AB and ED be equal.

To Prove: Chord AB = chord DE.

Proof: Draw radii OA, OB, CD, and CE.

$$OA = CE \qquad (82b)$$
$$OB = CD \qquad (82b)$$
$$\angle AOB = \angle ECD \qquad (84a)$$
$$\therefore \triangle AOB = \triangle DCE \qquad (76)$$
$$\therefore \text{Chord } AB = \text{chord } DE \qquad (72c).$$

86b. Theorem: *Conversely, in the same circles or in equal circles, equal chords subtend equal arcs.*

Given: In the equal Ⓢ (Fig. of 86a) the centers of which are O and C, the chords AB and DE are equal.

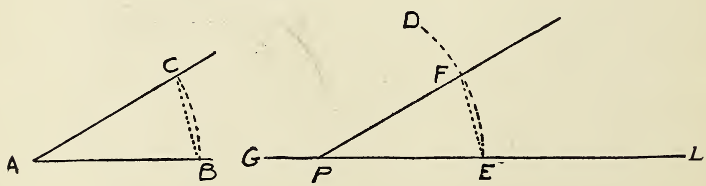
To Prove: Arc AB = arc DE.

Proof: Draw radii OA, OB, CD, CE.

In ⚠ OAB and DCE,

$$OA = CE \qquad (82b)$$
$$OB = CD \qquad (82b)$$
$$\text{Chord } AB = \text{chord } DE \qquad (\text{given})$$
$$\therefore \triangle AOB = \triangle CDE \qquad (78a)$$
$$\therefore \angle AOB = \angle DCE \qquad (72c)$$
$$\therefore \text{Arc } AB = \text{arc } ED \qquad (84a). \quad Q.E.D.$$

At a given point in a given straight line to construct an angle equal to a given angle.



Given: CAB any given angle; P a point in the straight line GL.

To Construct: An $\angle$ equal to $\angle CAB$, having the vertex P and the side PL.

Construction: With A as a center and AB as a radius draw arc CB, and with P as a center and same radius draw arc DE.

Arc CB and arc DE are arcs of equal circles (82a).

Measure chord CB and lay off same distance FE from E on arc DE.

Arc FE = arc CB (86b).

$\therefore \angle FPE = \angle CAB$ (84b).

The student may now understand the following:

87. Rule: *At a given point in a given straight line to construct an angle equal to a given angle; with the vertex of the given angle as a center and any radius draw an arc intersecting both sides of the angle; with the given point as a center and the same radius draw an arc intersecting the given line; set the compasses to measure the chord of the arc intercepted between the sides of the given angle and lay off an equal chord on the arc just drawn, beginning at its intersection with the given line. A line drawn from the other end of this arc to the given point will make the angle desired with the given line.*

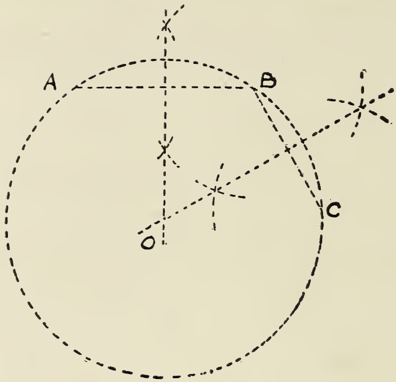
PROBLEM: By the above Rule (87) show how to construct a triangle when two angles and the side included by them are given.

HINT: Lay off a line equal to the given side of the triangle and at its extremities construct angles equal to the given angles.

PROBLEM: By the above Rule (87) show how to construct a triangle when two sides and the included angle are given.

HINT: Lay off an indefinite straight line. With this as a side construct an angle equal to the given angle. Then lay off lengths on these sides equal to the given sides.

88a. Theorem: *Through three points not in a straight line one circumference and only one can be drawn.*



Given: A, B, and C are three points not in a straight line.

To Prove: One circumference, and only one, can be drawn through A, B, and C.

Proof: Draw AB and BC. Construct the perp. bis. of AB and perp. bis. of BC. Let O be the point of intersection of these bisectors.

Since O is in the perp. bis. of AB, the distance from O to A = distance from O to B (49).

Since O is in the perp. bis. of BC, the distance from O to C = distance from O to B.

But it has just been shown that distance from O to B = distance from O to A.

$\therefore$ Distance from O to A = distance from O to B = distance from O to C (31).

$\therefore$ If O is taken as the center and a circumference drawn with OA as a radius, the circumference will pass through A, B, and C.

Now the center of the circle must be the point of intersection of the perp. bis. (49).

But these perp. bis. can have only one point of intersection (8e).

$\therefore$ O is the only point which can be used as a center of a circle.

Then, any circumference passing through the three points must have a radius of AO.

That is: any circumference drawn through A, B, and C must coincide with the circumference first drawn (16).

That is: one circumference and only one can be drawn passing through the three given points. *Q. E. D.*

If the three points were in the same straight line, then the perpendicular bisector of the line joining the first and second points would be parallel to the perpendicular bisector of the line joining the second and third points (60a). Then since the perpendicular bisector of the lines joining the points would never meet, there would be no point which might be taken as the center of the circle passing through the points—in other words, it is not possible to pass a circumference through three points in the same line. This fact may be stated as follows:

88b. *A straight line cannot meet the circumference of a circle in more than two points.*

88c. *To circumscribe a circle about a triangle* (*that is, to pass a circumference through the three vertices*). The method of construction and proof is left to the student (88a).

Exercise

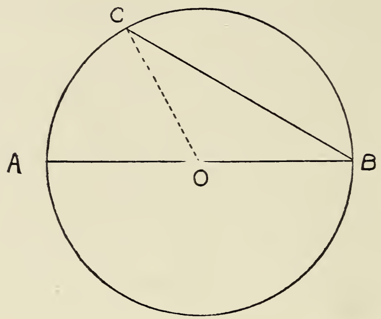
Let the student prove the following:

1. The perpendicular distances from the center of a circle to equal chords are equal.

2. A diameter perpendicular to a chord, bisects the chord and the arc subtended by it.

89a. Theorem: *An inscribed angle is measured by half the arc intercepted between its sides.*

Case 1. When one of the sides of the inscribed angle is a diameter.



Given: Let the side AB of the inscribed angle CBA pass through O, the center of the circle.

To Prove: $\angle B$ is measured by ½ arc AC.

Proof: Draw OC.

OC is the radius of the circle (17)

$\therefore \triangle OBC$ is an isosceles $\triangle$ (18, 70)

$\angle B = \angle C$ (77b)

(I) $\angle AOC = \angle B + \angle C$ (81b)

Substituting for $\angle C$ its equal $\angle B$ in equation (I)

$\angle AOC = \angle B + \angle B$

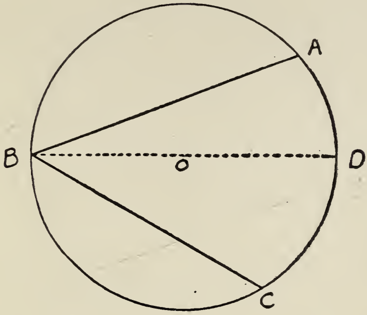
That is, $\angle AOC = 2 \angle B$

$1/2 \angle AOC = \angle B$ (33b)

Now $\angle AOC$ is measured by arc AC (26)

$\therefore \angle B$ is measured by ½ arc AC.

Case 2. When the center of the circle is within the inscribed angle.



Given: O, the center of the circle, is within the inscribed angle ABC.

To Prove: $\angle ABC$ is measured by 1/2 arc AC.

Proof: Draw BO, producing it to meet the circumference at some point D.

Since DB is a diameter of the circle,

(I) $\angle ABD$ is measured by 1/2 arc AD (Case 1).

(II) $\angle DBC$ is measured by 1/2 arc DC (Case 1).

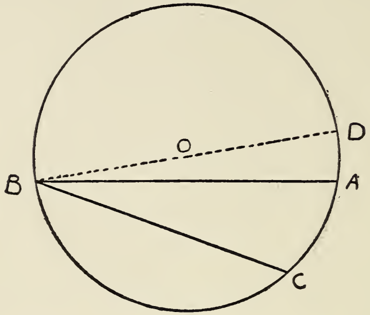
Adding:

$\angle ABD + \angle DBC$ is measured by 1/2 arc AD + 1/2 arc DC.

That is, $\angle ABD + \angle DBC$ is measured by 1/2 (arc AD + arc DC).

That is, $\angle ABC$ is measured by 1/2 arc AC (32a).

Case 3. When the center of the circle is outside of the inscribed angle.



Given: Let O, the center of the circle, be outside of the inscribed angle ABC.

To Prove: $\angle ABC$ is meas. by 1/2 arc AC.

Proof: Draw BO and produce it to meet the circumference at D.

Since BD is a diameter,

(I) $\angle DBC$ is meas. by 1/2 arc DC (Case 1).

(II) $\angle DBA$ is meas. by 1/2 arc DA (Case 1).

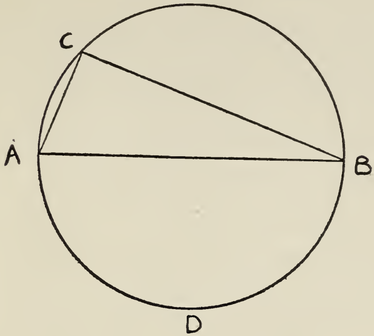
Subtracting, $\angle DBC - \angle DBA$ is meas. by 1/2 arc DC $-1/2$ arc DA. That is, $\angle DBC - \angle DBA$ is meas. by 1/2 (arc $DC -$ arc DA).

That is, $\angle ABC$ is meas. by 1/2 arc AC. *Q. E. D.*

89b. A **semi-circle** is half a circle A **semi-circumference** is half a circumference.

The following theorem may be illustrated by sliding the arms of a carpenter's square along two nails driven in a board, while holding a pencil at its vertex. Machinists test the inside of a hollow casting, which should be semi-circular in cross-section, by sliding a steel square around it; if the casting is true, the vertex of the square will always touch the casting, while the arms touch the edges.

89c. *An angle inscribed in a semi-circle is a right angle.*



Given: ACB an $\angle$ inscribed in the semi-circle ACB of the $\odot$ $ACBD$.

To Prove: $\angle\ ACB = 90°$ or a rt. $\angle$.

Proof: Since ACB is a semi-circumference, arc ADB, intercepted by AC and CB, is a semi-circumference (32a).

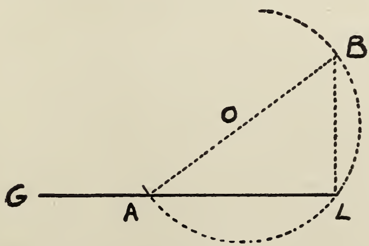
That is, arc ADB is 180° (24)

$\therefore \angle\ ACB$ is measured by 1/2 (180°) (89a)

That is, $\angle\ ACB = 90°$ or a rt. $\angle$. *Q. E. D.*

The preceding theorem is attributed to Thales of Miletus (one of the Seven Sages of Greece) who lived about 600 B. C.

At the extremity of a given line to erect a perpendicular to the line.



Given: GL is a straight line.

To Construct: A perpendicular to GL at L.

Construction: With any point as O (outside of GL, and not further to the left than the center of the line), as the center of a circle, and with OL as a radius, draw a circle intersecting GL at A. Draw AO and produce it to meet the circle at B.

A straight line drawn from B to L will be $\perp$ to GL at L.

The proof is left to the student (89c).

The student may now understand the following:

89d. Rule: *At the extremity of a given line to erect a perpendicular to the line: take any point outside of the given line (and not beyond its middle point) as a center, and with a radius equal to the distance from this point to the given extremity, draw a circumference passing through the given extremity, and intersecting the given line at some point within; from this point of intersection draw a line through the center of the circle and produce it to intersect the circumference again. A straight line drawn from this last point of intersection to the given extremity, will be the desired perpendicular.*

89e. Either part of a circle cut off by a chord is called a **segment**. Thus $BCDA$ is a segment of the circle on page 59, and the remainder, enclosed between the arc AB, and chord AB, is a segment.

Exercise

Let the student prove the following:

1. Angles inscribed in the same segment are equal.

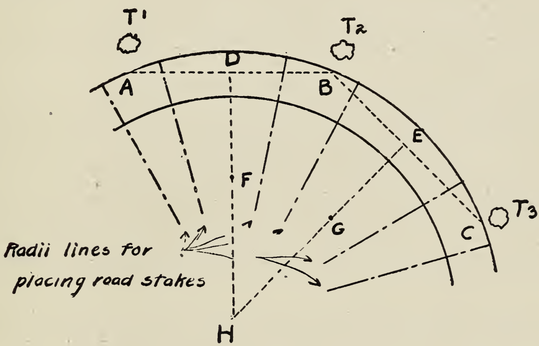
2. Draw a semi-circle on stiff cardboard. Attach a small plumb line to one end of the diameter. Hold the card upright so that the plumb line intersects the circumference at some convenient point. A line from the other end of the diameter to the point of intersection of the chord with the circumference will be level. Why?

V. Field Exercise

TO LAY OUT A ROAD IN AN ARC OF A CIRCLE PASSING THROUGH THREE POINTS

Equipment: Two dozen stakes, knotted cord, long cord, measuring tape, piece of twine, mallet.

Let T_1, T_2, T_3, be three trees (or three buildings or other objects) past which it is desired to have a road pass in an arc of a circle.



Procedure: Having decided on the requisite amount of clearance (such as two feet) between the trees and the road, lay off this distance from the trees toward the estimated position of the center of the circle, marking these distances with the stakes A, B, and C.

Lay off the lines AB and BC. With the tape find the middle points of these lines and mark them with the stakes D and E. Then locate the stake F (see Field Exercise II), which shall form with D the perpendicular bisector of AB. Likewise locate the stake G, which shall form with E the perpendicular bisector of BC.

Hold the long cord in a straight line past F and D (just touching them), so that the line DF is sufficiently extended. Sight from E across G to the point on DF produced, which is in the same straight line with E and G. Mark this point with the stake H, which is the center of the circle.

(Instead of sighting from E to G, another cord may be used to produce the line EG in the same manner as DF, and their intersection marked with the stake H.)

To test the accuracy of the construction, make a loop in the long cord about H, and stretch the cord tightly to A to mark the length. Tie a bit of twine about the cord at the point where it touches A. Then, using the cord about H as a center, see if BH and CH are the same length as AH.

In case one of the three radii thus tested is longer than the other, this excess in length may be corrected by moving the center back that amount along the radius produced.

Thus, if CH is 4 inches longer than AH and BH, move H 4 inches further from C along the line CH.

For very accurate work, to avoid the error caused by the varying stretch of the cord, the radii AH, BH and CH may be laid off with stakes, and the distances measured with the tape.

When the center of the circle has been definitely located, the extremities of other radii may be marked with stakes between A and B, and B and C.

As each radius is laid out, the distance desired for the width of the road may be measured back along the line of the radius, and marked with a stake.

This completes the work of staking out arcs of two concentric circles separated by the width desired for the road, and with the outer arc passing through the three desired points A, B, and C.

CHAPTER VI

POLYGONS: USE OF LETTERS TO EXPRESS GENERAL LAWS

90a. A **polygon** is a portion of a plane bounded by straight lines.

90b. The bounding lines are called the **sides** of the polygon, their intersections the **vertices,** and the angles between adjacent sides are the **angles** of the polygon.

By producing the sides of a polygon, we may form its exterior angles—but when the angles of a polygon are spoken of, it means the interior angles.

As we already know, a polygon of three sides is called a triangle.

90c. The distance around the outside of a polygon is called its **perimeter.**

Find the perimeter of this page in inches.

91a. A **quadrilateral** is a polygon of four sides.

91b. A **rectangle** is a quadrilateral whose opposite sides are parallel, and whose angles are right angles.

The surface of this page is a rectangle.

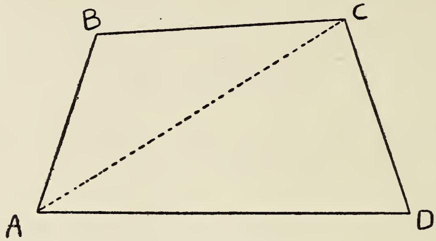
91c. A **pentagon** is a polygon of five sides.

91d. A **hexagon** is a polygon of six sides.

92. A **diagonal** of a polygon is a line joining the vertices of two angles which are not adjacent. (See AC in the next figure.)

This definition is not usually applied to triangles, though it may be—in which case the diagonals coincide with the sides of a triangle.

93. Theorem: *The sum of the interior angles of a quadrilateral equals four right angles.*



Given: $ABCD$ is a quadrilateral.

To Prove: $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s.

Proof: Draw the diagonal AC.

This divides $ABCD$ into two triangles.

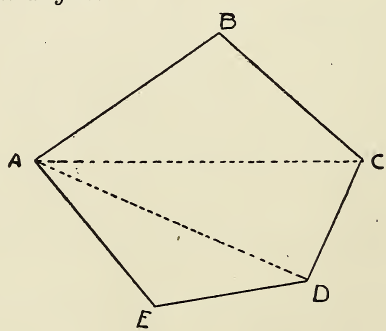
The sum of the $\angle$s of the $\triangle$s = the $\angle$s of the quadrilateral.

The sum of the $\angle$s of each $\triangle = 2$ rt. $\angle$s (80b).

$\therefore$ The sum of the $\angle$s of both triangles $= 2 \times 2$ rt. $\angle$s $= 4$ rt. $\angle$s.

That is, the sum of the $\angle$s of the quadrilateral $= 4$ rt. $\angle$s
Q. E. D.

94a. Theorem: *The sum of the interior angles of a pentagon equals six right angles.*



Given: $ABCDE$ is a pentagon.

To Prove: $\angle A + \angle B + \angle C + \angle D + \angle E = 6$ rt. ∡.

Proof: From A draw diagonals AC and AD.

This divides $ABCDE$ into three triangles. The sum of the ∡ of the ▲ = the sum of the ∡ of the pentagon.

The sum of the ∡ of each △ = 2 rt. ∡.

∴ The sum of the ∡ of all 3 ▲ = 3 × 2 rt. ∡ = 6 rt. ∡.

That is, the sum of the ∡ of the polygon = 6 rt. ∡. *Q. E. D.*

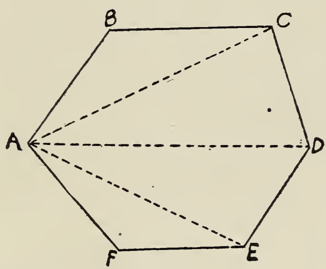
94b. "3 × 2 rt. ∡" might be written 3 (2 rt. ∡).

The symbol ( ) is called a **parenthesis.**

In the expression 3 (2 rt. ∡), 3 is the *coefficient* of the parenthesis; that is:

94c. A **coefficient** of a quantity shows how many times the quantity is taken.

95a. Theorem: *The sum of the interior angles of a hexagon equals eight right angles.*



Given: $ABCDEF$ is a hexagon.

To Prove: $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 8$ rt. ∡.

The proof is left to the student.

In like manner, it can be proved that the sum of the interior ∡ of a polygon of 7 sides = 10 rt. ∡, that the sum of the interior angles of a polygon of 8 sides = 12 rt. ∡ and so on. All the theorems concerning the sum of the interior angles of a polygon may be summed up in the following general one:

Theorem: *The sum of the interior angles of a polygon is equal to two right angles, taken as many times less two as there are sides in the polygon.*

In stating what we are to prove, we cannot now say "To prove that $\angle A + \angle B$, etc. = 4 rt. $\measuredangle$s" or "= 6 rt. $\measuredangle$s," or any other particular number of right angles, for we are to prove the theorem in *general;* we therefore say in our proof: "To prove that $\angle A + \angle B$ + etc. = $(n-2)$ 2 rt. $\measuredangle$s." By using n, instead of a particular number to represent the number of sides, we may prove a general rule,—thus including in one statement the rules regarding quadrilaterals, pentagons, hexagons, and all polygons.

95b. *This is the purpose of using letters instead of numbers in mathematics: to derive general laws.*

Note the use of parenthesis in the statement that "the sum of the angles of any polygon equals $(n-2)$ 2 rt. $\measuredangle$s." The parenthesis indicates that the quantity within it is to be considered as one quantity—that is, to work out the sum of the $\measuredangle$s of any polygon we have first to substitute the proper number for n, then subtract 2 from it and then multiply 2 rt. $\measuredangle$s by this number. Thus to find the sum of the angles of a quadrilateral by this statement, we substitute 4 for n in the parenthesis and so obtain that the sum of the angles of a quadrilateral $= (4-2)$ 2 rt. $\measuredangle$s.

Simplifying $(4-2)$, the number inside the parenthesis is 2. 2, then, is the number by which 2 rt. $\measuredangle$s is to be multiplied.

To find the sum of the interior angles of a pentagon (91c), we substitute 5 for n in the expression, "$(n-2)$ 2 rt. $\measuredangle$s," and so obtain $(5-2)$ 2 rt. $\measuredangle$s = 6 rt. $\measuredangle$s.

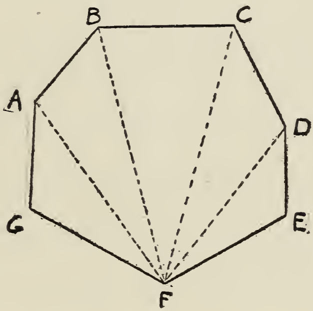
96a. *Any number of quantities enclosed in a parenthesis must be treated as but one quantity.*

96b. *If a parenthesis is preceded by a coefficient, each term within the parenthesis should be multiplied by the coefficient when the parenthesis is removed.*

The reason for this may be seen from the following example: Let $n = 3$ in the expression $2\,(n-2)$ and we have $2\,(3-2) = 2\,(1) = 2$. If we remove the parenthesis before substituting we have $2\,n - 4$. Substituting, we have $6 - 4 = 2$, the same as before.

The proofs just given with regard to the quadrilateral, pentagon and hexagon, as well as proofs which might be worked out for all other kinds of polygons, may now, by the use of the symbols just explained, be all included in the following *general proof:*

97. Theorem: *The sum of the interior angles of a polygon is equal to two right angles taken as many times less two as there are sides in the polygon.*



Given: Let $ABCDEFG$ represent a polygon of n sides.

To Prove: $\angle A + \angle B + \angle C +$ etc. $= (n-2)\ 2$ rt. $\angle$s.

Proof: From any vertex as F draw all possible diagonals.

The sum of the angles of these $\triangle$s equals the sum of the angles of the polygon (32a).

Now, in drawing lines from F to all the other vertices, a triangle would be formed for every side of the polygon except the two adjacent sides, for evidently lines from F to G and from F to E would coincide with the sides FG and FE.

That is, there will be two $\triangle$s less than there are sides, that is, there will be $(n-2)$ $\triangle$s.

Now, the sum of the $\angle s$ of each triangle = 2 rt. $\angle s$ (80b).

The sum of the angles of all the triangles, that is, the sum of the $\angle s$ of the polygon = $(n-2)$ 2 rt. $\angle s$.

In the case of a triangle $n = 3$ and the quantity inside the parenthesis reduces to 1, and so we obtain that the sum of the angles of a triangle equals (1) 2 rt. $\angle s$ = 2 rt. $\angle s$; that is, this rule holds for $\triangle s$ also.

This rule may be briefly stated as follows: The sum of the int. $\angle s$ of any polygon = $(n - 2)$ 2 rt. $\angle s$.

Such a brief statement is called a Formula.

98. A **formula** is a statement of a rule by means of letters and symbols.

Exercise

1. Find the sum of the interior angles of a polygon of seven sides.

2. Find the sum of the interior angles of a polygon of fifteen sides.

3. Find the sum of the interior angles of a polygon of eleven sides.

4. How many sides has a polygon if the sum of its angles equals fourteen right angles?

Solution: $2(n - 2)$ rt. $\angle s$ = 14 rt. $\angle s$,
or more briefly

$$2(n-2) = 14.$$
$$2n - 4 = 14.$$

Since, as stated in the above quotation, when 4 is subtracted from $2n$ we have a number equal to 14, the number equal to $2n$ must be 4 greater than 14, or 18. This may also be obtained by moving 4 from one side of the equation to the other.

Thus the equation $2n - 4 = 14$ may be written

$$2n = 14 + 4$$
$$2n = 18$$
$$\text{whence } n = 9.$$

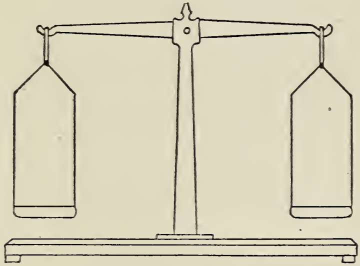
99a. Moving a quantity from one side of an equation to another is called **transposing** the quantity. When a quantity is transposed it must be changed in sign.

When (– 4) was taken from the left hand side of the above equation it was the same as if 4 had been added to that side of the equation. Suppose that a boy owes $4 at a store. Since this debt represents money that the boy must pay out it may be represented as "– 4 dollars." Now, if the boy's father pays the bill at the store, it is the same as if he had given the boy $4. That is, by removing the (– 4) dollars, he adds a (+ 4) dollars. Hence:

99b. *Removing a minus quantity is the same as adding an equal plus quantity.*

100. Either side of an equation is called a **member.** The two members of the equation $2n - 4 = 14$ are $2n - 4$ and 14 respectively.

101. *An equation may be considered as a balance.*



Thus, if equal weights are placed in the pans of the scales they balance; if we remove some weight from one side of the balance, we must remove an equal weight from the other, in order to preserve the equilibrium (that is to keep one pan from going up while the other goes down). Similarly, if we add a weight to one side of the balance, we must add a weight to the other; if we double the weight on one side, we must double the weight on the other; if we halve the weight on one side we must halve the weight on the other—in short, each member must be treated alike, that is, *an equation remains a true statement of equality if the same quantity is added to both members, or subtracted from both members, or if both members are multiplied or divided by the same quantity.*

Exercise

1. How many sides has a polygon if the sum of its interior angles equals 20 right angles?

2. What is the number of sides of a polygon if the sum of its interior angles equals 10 right angles?

3. One number is six times another and the difference of the two is 540. Find both numbers.

Solution:

Let n = smaller number.

Then $6n$ = larger number.

From the statement of the problem,

$$6n - n = 540$$

That is, $5n = 540$

$n = 108$, smaller number.

$6n = 648$, larger number.

Check: $648 - 108 = 540$

In checking a problem, we substitute the numbers found in the problem, in place of the unknowns, to see if they give a true equation.

4. One number is 9 times another, and the sum of the two numbers equals 80. Find the two numbers.

5. The sum of two numbers is 88. Three times the less equals twice the greater, plus 29. Find the numbers.

Let n = the smaller number.

Then (since the sum of the two numbers is 88),

$88 - n$ = larger number.

From the statement of the problem, 29 must be added to twice the greater, to make it equal to three times the less, that is:

$$3n = 2\,(88 - n) + 29$$
$$3n = 176 - 2n + 29$$

Transposing

$$3n + 2n = 176 + 29$$
$$5n = 205$$
$$n = 41, \text{ smaller number}$$
$$88 - n = 88 - 41 = 47, \text{ larger number.}$$

Check:
$$3\,(41) = 2\,(47) + 29$$
$$123 = 94 + 29$$
$$123 = 123$$

6. A rectangle is five times as long as it is wide. If x represents the width in feet what will represent the length? What will represent the perimeter?

7. A rectangle is 3 times as long as it is wide, and its perimeter is 264 ft. Find the length and the width.

8. The perimeter of a rectangle is 868 ft., and its length is 6 ft. more than three times the width. Find the length and the width.

HINT: Let $x =$ width, then $3\,x + 6 =$ length.

9. Find the dimensions of a rectangle whose perimeter is 500 ft., and whose length is 30 ft. less than four times the width.

10. The perimeter of a rectangle is 664 ft., and its length is 2 ft. more than twice the width. Find the length and the width.

11. The perimeter of a rectangle is 160 ft. and its length is 8 ft. more than twice its width. Find the length and the width.

12. Find the dimensions of a rectangular field whose perimeter is 700 ft. and whose length is 50 ft., less than three times the width.

13. A rectangular field is 5 times as long as it is wide, and its perimeter is 936 ft. Find the length and the width.

14. The sum of two numbers is 135. Twice the greater, minus 25, equals 5 times the less. Find the numbers.

15. The sum of two numbers is 98. Three times the greater equals 5 times the less, plus 46. Find the numbers.

16. The sum of three consecutive numbers is 39. Find the numbers.

NOTE: Since consecutive numbers are numbers which follow one another we may represent the numbers as follows:

$$\begin{aligned} \text{Let } x &= \text{one number,} \\ \text{then } x + 1 &= \text{the second number,} \\ \text{then } x + 2 &= \text{the third number.} \end{aligned}$$

17. The sum of two consecutive numbers is 41. Find the numbers.

18. The sum of four consecutive numbers is 54. Find the numbers.

19. Seven times the sum of two consecutive numbers equals 63. Find the numbers.

20. Three times the sum of three consecutive numbers equals 72. Find the numbers.

21. Find three consecutive numbers such that twice the sum of the first and second equals three times the third, plus three.

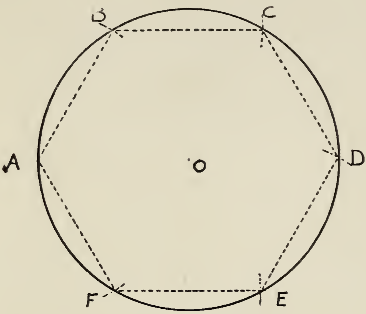
22. If three rods are fastened at their extremities so as to form a triangle, is the figure rigid? (Can the shape of the figure be readily changed?)

23. If four rods are fastened at their extremities so as to form a quadrilateral, is the figure rigid?

102. A regular polygon is a polygon which is equiangular and equilateral—that is, its sides are all equal and its angles are all equal.

The angle at a vertex of any regular polygon may be computed by finding the sum of the angles of the polygon (97) and dividing by the number of sides (since the number of sides is the same as the number of angles, and the angles are equal).

103a. *To draw a regular hexagon: Draw a circle; with the same radius lay off six chords around the circle; these will be the sides of the regular hexagon.*



The proof is left to the student.

HINT: Draw radii from the vertices (25, 77b).

By connecting alternate vertices, an equilateral triangle (one with three equal sides) is formed. (The proof is left to the student.) By connecting the three remaining vertices another equilateral triangle is formed overlapping the first. The outer edges of these triangles form a six-pointed star.

103b. By drawing a circle and in it drawing two diameters perpendicular to each other and connecting the ends of these diameters, a square is formed.

(The proof is left to the student.)

By bisecting the arcs thus formed and drawing lines to their middle points from the vertices of the square, a regular octagon (figure of eight sides) is formed.

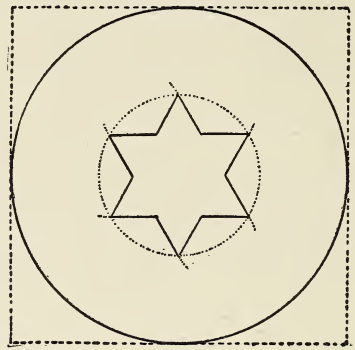
By continuing the process regular figures of 16, 32, etc., sides may be formed.

By a similar process with the hexagon, figures of 12, 24, etc., sides may be formed.

VI. Shop Exercise

PLATE PAD

Cut a piece of wood 6″ square. Draw diagonals with pencil. At intersection of diagonals place the point of the compass and describe a circle touching each of the sides. With the same center, describe another circle of less diameter (choosing a radius which will give a design in good proportion). In this circle draw a six pointed star (103a).



Follow along the outside of the star with the point of the pocket knife, pressed into the wood to a depth of about one-sixteenth of an inch. Use any standard wood-stain to color the star. (The scoring with the pen-knife prevents the stain from spreading.)

Cut around outer circle with a turning saw; smooth off with a spoke-shave and sand paper.

Other designs may be made in this way by the student.

CHAPTER VII

LITERAL EXPRESSIONS

104. A **literal expression** is one that makes use of letters to represent quantities. Thus, the expression for the sum of the interior angles of a polygon $(2n - 4)$ rt. angles is a literal expression which states a general truth—that is, one that is true no matter what the value of n is (97); or a literal expression may be used to represent an unknown quantity, as in the previous exercise, and in the following example.

Exercise

Example: 1. Three times a number, plus twice a number, plus four times the number equals 27. What is the number?

Let $x =$ the number.

Then stating the problem as an equation:

$3x + 2x + 4x = 27, \therefore x = 3$

Add

2. $5x + 7x + 11x + 2x + 6x$

3. $3y + 4y + 8y + 11y$

4. $4z + 2z + 5z + 10z$

5. $8y + 6y + 2y + 14y$

The above examples, like those in Chapter VI, are examples in Algebra, or Algebraic examples.

105. Algebra is the study of literal expressions.

106. The parts of an example in Algebra which are separated by + and − signs are called the **terms.**

Thus in: $6a + 3b - 4d$,

$6a$ is a term, $3b$ is a term, $4d$ is a term.

107. If an expression consists of only one term it is called a **monomial.** Thus, $6d$ is a monomial.

108. If an expression consists of two terms, it is called a **binomial.** Thus, $x - y$ is a binomial.

109. If an expression consists of more than two terms it is called a **polynomial.**

110. If we think of the term $6a$ as being made up of two factors, 6 and a, either term may be called a **coefficient.**

A coefficient of a term may be any one of its factors (94c). If one of the factors of a product is a number, while the rest are letters, we usually mean the number when we speak of the coefficient, though sometimes we speak of it as a **numerical coefficient.** Thus 6 is the coefficient of a in the term $6a$.

111. Any collection of terms is called an **expression.** Thus, $6a + 3b - 4d$ is an expression.

We have already learned that the sign ( ) placed around two or more terms means that they are all to be treated in the same way; to be worked with as if they were one quantity. Thus, $2\,(3a + 2d + 5c)$ means that $3a$, $2d$ and $5c$ are all to be multiplied by 2.

112. The sign [ ] is called a **bracket,** and is used in the same manner as a parenthesis.

113. The sign { } is called a **brace** and is used in the same manner as a parenthesis.

The terms to be added may also be written beneath each other. Thus,

$$\begin{array}{r} 3abc \\ 4abc \\ 2abc \\ \hline 9abc \end{array} \text{ Ans.}$$

In the following examples, add only the numerical coefficients:

6.
$$\begin{array}{r} 7x \\ 2x \\ 5x \\ \underline{8x} \end{array}$$

7.
$$\begin{array}{r} 9yz \\ 3yz \\ 4yz \\ 6yz \\ \underline{5yz} \end{array}$$

8.
$$\begin{array}{r} 4ghm \\ 9ghm \\ 6ghm \\ 3ghm \\ \underline{5ghm} \end{array}$$

9.
$$\begin{array}{r} 5uvw \\ 6uvw \\ 4uvw \\ 7uvw \\ \underline{8uvw} \end{array}$$

10.
$$\begin{array}{r} .8xy \\ 1.2xy \\ 6.3xy \\ 4.3xy \\ 5.7xy \\ \underline{.2xy} \end{array}$$

In Arithmetic we learned that if a man earned 4 dollars, and spent 2, and next day earned 3 dollars and spent 1, we express the dollars to be paid out with a minus sign in front of them, and what he received with a plus sign in front of them.

So we may write:

$$\begin{array}{l} +\ 4 \text{ dollars} \\ -\ 2 \text{ dollars} \\ +\ 3 \text{ dollars} \\ \underline{-\ 1 \text{ dollar}\ } \\ +\ 4 \text{ dollars} \end{array} \qquad\qquad \begin{array}{l} +\ 4d \\ -\ 2d \\ +\ 3d \\ \underline{-\ 1d} \\ +\ 4d \quad \text{Ans.} \end{array}$$

Since we usually omit the plus sign before a quantity if it stands alone or at the beginning of an expression, we may rewrite the example in this manner,

$$\begin{array}{r} 4d \\ -\ 2d \\ 3d \\ \underline{-\ 1d} \\ 4d \end{array} \quad \text{Ans.}$$

It is usual to omit the number 1 when it occurs as a coefficient. Thus, instead of writing "$-1d$" we may write simply "$-d$."

Do the following examples, first adding all the plus terms, then all the minus terms, and subtracting the smaller of the two sums thus obtained from the larger and giving the remainder the sign of the larger term.

11.
$$\begin{array}{r} 3x \\ -2x \\ 5x \\ \hline \end{array}$$

12.
$$\begin{array}{r} 4yz \\ 7yz \\ -2yz \\ -4yz \\ 5yz \\ \hline \end{array}$$

13.
$$\begin{array}{r} 3abc \\ 9abc \\ -7abc \\ -2abc \\ 4abc \\ \hline \end{array}$$

14.
$$\begin{array}{r} 6mn \\ -3mn \\ 4mn \\ 8mn \\ -2mn \\ \hline \end{array}$$

15.
$$\begin{array}{r} -11mn \\ 3mn \\ 4mn \\ -6mn \\ 5mn \\ \hline \end{array}$$

16.
$$\begin{array}{r} 13ab \\ -2ab \\ -7ab \\ -4ab \\ \hline \end{array}$$

17.
$$\begin{array}{r} -2x \\ -9x \\ -4x \\ -6x \\ -7x \\ \hline \end{array}$$

18.
$$\begin{array}{r} -3cde \\ 4cde \\ -5cde \\ 7cde \\ 5cde \\ -8cde \\ \hline \end{array}$$

114. If a factor occurs more than once in any term, it is not necessary to write the factor more than once, if we use a small figure called an **exponent,** written above and to the right, to show how many times the letter is taken as a factor. Thus, $7cc$ may be written $7c^2$, $5aaa$ may be written $5a^3$ and $bbbb$ may be written b^4.

We learned in Arithmetic that 2 feet + 3 square feet + 6 cubic feet cannot be combined into one term, any more than we can add bananas and oranges, so in Algebra such expressions as $2a + 3a^2 + 6a^3$ cannot be combined into one term.

115. Rule: *Terms which have different exponents cannot be added or subtracted.*

Terms which have the same letters, each letter having the same exponent in one term as it has in another, are called **like terms**. The numerical coefficient of like terms need not be the same. Thus, $10ab$ and $-7ab$ are like terms: so also are $4x^2yz^3$ and $9x^2yz^3$.

116. Rule: *Only like terms may be added or subtracted.*

Exercise

Add:

1.
$$\begin{array}{r} 5a^2bc \\ 2a^2bc \\ 7a^2bc \\ 6a^2bc \\ 8a^2bc \\ \underline{3a^2bc} \end{array}$$

2.
$$\begin{array}{r} 6cd^3 \\ -3cd^3 \\ 5cd^3 \\ 2cd^3 \\ 11cd^3 \\ \underline{4cd^3} \end{array}$$

3.
$$\begin{array}{r} 4x^2yz^3 \\ -7x^2yz^3 \\ -2x^2yz^3 \\ -3x^2yz^3 \\ -8x^2yz^3 \\ \underline{-5x^2yz^3} \end{array}$$

4.
$$\begin{array}{r} 13mnp^2 \\ -9mnp^2 \\ 4mnp^2 \\ 15mnp^2 \\ -7mnp^2 \\ \underline{8mnp^2} \end{array}$$

5.
$$\begin{array}{r} 11v^2u^3x \\ 13v^2u^3x \\ -25v^2u^3x \\ 11v^2u^3x \\ 12v^2u^3x \\ \underline{13v^2u^3x} \end{array}$$

Parentheses which contain the same quantity may be combined by combining their numerical coefficients.

Thus:

$3(m + n) + 2(m + n) + 3(m + n) = 8(m + n)$.

6. $$\begin{array}{r} 4(a+b) \\ 5(a+b) \\ 6(a+b) \\ 3(a+b) \\ \underline{9(a+b)} \end{array}$$

7. $$\begin{array}{r} 2(x+y+z) \\ 3(x+y+z) \\ -5(x+y+z) \\ 6(x+y+z) \\ 4(x+y+z) \\ \underline{-(x+y+z)} \end{array}$$

8. $$\begin{array}{r} 10(3a-2b-c) \\ 11(3a-2b-c) \\ -7(3a-2b-c) \\ 6(3a-2b-c) \\ \underline{-5(3a-2b-c)} \end{array}$$

We have learned that money to be paid out is expressed with a minus sign in front of it. Thus, if a man receives 10 dollars and then he receives a bill for 6 dollars (that is, a notice that he must pay 6 dollars) we may say that -6 has been added to his 10 dollars. Adding this -6 dollars to his 10 dollars means that the man will have 4 dollars left. In other words, to add a minus number to another number is the same as subtracting an equal plus quantity from it.

Hence we have the following:

117. Rule for Subtraction: *Change the sign of the number to be subtracted and add.*

EXERCISE

Subtract:

1. $\begin{array}{r} 7a \\ \underline{5a} \end{array}$ 2. $\begin{array}{r} 8b \\ \underline{6b} \end{array}$ 3. $\begin{array}{r} 11d \\ \underline{5d} \end{array}$ 4. $\begin{array}{r} 14m \\ \underline{9m} \end{array}$ 5. $\begin{array}{r} 1/4\,h \\ \underline{1/5\,h} \end{array}$ 6. $\begin{array}{r} .7d \\ \underline{.3d} \end{array}$

7. $\begin{array}{r} 151cd^3 \\ \underline{49cd^3} \end{array}$ 8. $\begin{array}{r} 3.1mn^2 \\ \underline{2.7mn^2} \end{array}$ 9. $\begin{array}{r} 43r^3st^4 \\ \underline{31r^3st^4} \end{array}$ 10. $\begin{array}{r} 91u^5v^3x^4 \\ \underline{51u^5v^3x^4} \end{array}$

We learned in Arithmetic that if a man owed 15 dollars, that is, had -15 dollars, and had 10 dollars, that is $+10$ dollars, the amount he was in debt after payment was 5 dollars, that is, he had -5 dollars.

So in Algebra we may subtract $15d$ from $10d$ by writing the answer as $-5d$. In like manner, any larger quantity may be subtracted from a smaller by writing the answer as minus.

Subtract:

11. $\begin{array}{r} 3s^2t \\ \underline{7s^2t} \end{array}$ **12.** $\begin{array}{r} 10x^2yz^3 \\ \underline{17x^2yz^3} \end{array}$ **13.** $\begin{array}{r} 21a^3bc^2 \\ \underline{32a^3bc^2} \end{array}$ **14.** $\begin{array}{r} 31e^2fg^3 \\ \underline{49e^2fg^3} \end{array}$

15. $\begin{array}{r} 39m^6n^5 \\ \underline{72m^6n^5} \end{array}$ **16.** $\begin{array}{r} .2ab^3 \\ \underline{7.1ab^3} \end{array}$ **17.** $\begin{array}{r} {}^{11}/_3\, eg^2h^5 \\ \underline{{}^{9}/_2\, eg^2h^5} \end{array}$ **18.** $\begin{array}{r} {}^{1}/_3\, d^3e \\ \underline{{}^{5}/_9\, d^3e} \end{array}$

If a man owed 20 dollars, and 6 dollars of the debt was paid for him, he still owes 14 dollars, that is, if from $-$ 20 dollars we subtract $-$ 6 dollars, the answer is $-$ 14 dollars. So in Algebra, if from $-20a$ we subtract $-6a$ the answer is $-14a$. That is, to subtract one minus quantity from another we follow the Rule just given for subtraction.

Subtract:

19. $\begin{array}{r} -13ab \\ \underline{-\ \ 5ab} \end{array}$ **20.** $\begin{array}{r} -28cd^2 \\ \underline{-11cd^2} \end{array}$ **21.** $\begin{array}{r} -39a^2b^5c^6 \\ \underline{-12a^2b^5c^6} \end{array}$

22. $\begin{array}{r} -4b^3d^2 \\ \underline{-3b^3d^2} \end{array}$ **23.** $\begin{array}{r} -{}^{2}/_3\, ef^2g^3 \\ -\underline{{}^{1}/_5\, ef^2g^3} \end{array}$ **24.** $\begin{array}{r} -11ab^2c^3 \\ \underline{-17ab^2c^3} \end{array}$

25. $\begin{array}{r} -4\,(2m + n + p) \\ \underline{-6\,(2m + n + p)} \end{array}$ **26.** $\begin{array}{r} -10\,(3x + y - z) \\ \underline{-10\,(3x + y - z)} \end{array}$

Exercise

Simplify the following expressions by combining similar terms:

1. $4a + 3c + 7b + 2a + 9b + 4c + 2b + a =$
2. $2x + 4y - 2 + 3x + 5y + 3z - x + 11y =$
3. $5x - 3y - z + 3x - 11y + 7x - 5y - 6z =$
4. $3mn + 2ab - 7d - 8c - 3ab - 4mn - 2ab =$
5. $3xyz + 2x - 2xyz - 11x - 4c + 10d - 13x =$
6. $3a^3 + 15bc + 4ef^2 + 11a^3 - 10bc + 7a^3 + 6bc =$

7. $3(m+n)+7x+5y+2(m+n)+4x+2y+3(m+n)-2x+3y+(m+n)$
8. $3[y+z-x]+2a+3b+4[y+z-x]-3a+9b+11[y+z-x]$

118. Rule: *To add polynomials, arrange the expressions one under the other, so that like terms are in columns, and then add these columns.*

Example:

$7a+2b+c$; $3c+2a-b$; $5c-9a+2b$; $6a+2b-c$

Arranged:

$$\begin{array}{rrr} 7a & +2b & +c \\ 2a & -\ b & +3c \\ -9a & +2b & +5c \\ 6a & +2b & -c \\ \hline 6a & +5b & +8c \end{array} \quad \text{Ans.}$$

Add:

1.
$$\begin{array}{rrr} 4a & +2c & -\ d \\ -2a & +7c & +5d \\ -8a & +6c & +2d \\ 5a & +3c & +9d \\ 2a & +8c & -6d \\ 4a & +2c & -7d \\ 3a & -5c & +5d \\ \hline \end{array}$$

2.
$$\begin{array}{rrr} 3mn & +\ x & +2y \\ 5mn & -2x & +7y \\ -6mn & +3x & -8y \\ 7mn & & +2y \\ 6mn & +2x & -4y \\ 9mn & +4x & -3y \\ 4mn & & +2y \\ \hline \end{array}$$

3.
$$\begin{array}{rrrr} 2a^2 & +7bc & +3d^4 & +\ ef \\ 4a^2 & +6bc & -2d^4 & +6ef \\ 5a^2 & +9bc & +5d^4 & +8ef \\ 7a^2 & -5bc & +4d^4 & +6ef \\ 5a^2 & -4bc & +6d^4 & +9ef \\ \hline \end{array}$$

4.
$$\begin{array}{rrr} 9(x+y) & +7z & +6 \\ 4(x+y) & +3z & +2 \\ 8(x+y) & +4z & +7 \\ 5(x+y) & +3z & +4 \\ -6(x+y) & +11z & +10 \\ 8(x+y) & +5z & +1 \\ 3(x+y) & +8z & +5 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 2mn^2 - 3x^3 - 2x + 1 \\ 2mn^2 - 2x^3 + 7x - 5 \\ 3mn^2 + 7x^3 + 4x + 6 \\ 4mn^2 + 5x^3 + 3x + 9 \\ 6mn^2 + 4x^3 + 8x + 5 \\ 11mn^2 + 8x^3 + 4x + 1 \\ \hline \end{array}$$

Perform the following examples in subtraction by changing the sign of the subtrahend (the lower number) and adding it to the minuend (the upper number).

1.
$$\begin{array}{r} 7a + 3b + 4c + d \\ 3a + b + 2c + d \\ \hline \end{array}$$

2.
$$\begin{array}{r} 4a + 2b + 3c + f \\ 5a + 3b + 6c + f \\ \hline \end{array}$$

3.
$$\begin{array}{r} -2m + 3n - 2n^2 - n^3 \\ 4m + n^2 + 2n^3 \\ \hline \end{array}$$

4.
$$\begin{array}{r} x^3 - 3x^2 + 4x + 5 \\ x^3 + 2x^2 + x + 4 \\ \hline \end{array}$$

119. Rule: *If a parenthesis, bracket or brace has a minus sign in front of it, every term within it must have its sign changed when the parenthesis, bracket or brace is removed.*

Thus, simplify:

$3a - 2b - [7a + \{2c - (7a - b - c)\} - b]$.

We first remove the bracket, and since it has a minus sign before it, we change the sign of every number within it. Thus,

$$3a - 2b - 7a - \{2c - (7a - b - c)\} + b.$$

We then remove the next sign of inclosure, the braces, and as the brace has a minus sign in front of it, we change the sign of everything within it.

$$3a - 2b - 7a - 2c + (7a - b - c) + b.$$

Since the parenthesis has a plus sign in front of it no change of sign will be made when we drop the parenthesis.

$$3a - 2b - 7a - 2c + 7a - b - c + b.$$

Combining we have, $3a - 2b - 3c$. *Answer.*

Note that in this example we removed only one bracket, parenthesis or brace at a time, and worked from the outside, in.

Exercise

Simplify:

1. $3a - [2b + 6c] + b - (2a + c) - a.$
2. $7a - [2b - \{3c + (2a + 4b - c) - 2c\}].$
3. $-[2x + x^2 - \{2x + (x^2 - x^3) + 7\} - 2].$
4. $y^4 - [y^3 + (2y^2 + 3y) - 4] - (y^4 + 2y^2).$
5. $3m + 2p + [n - \{2m + p - (m + n) + m\}].$
6. $x^3 - [4x^3 + \{2x^2 - (5x + 1) + x^3\} - 10].$
7. $-[b + a - (c - b)] + 2a + [\{2a - (3a - 5b)\} + a].$
8. $(5y - 6x) - \{4x - [5x + (8y - 5x) - 4y - 4x]\}.$

In the following examples, remove the parentheses, leaving the brackets, and simplify the results as much as possible by combining like terms within the brackets.

9. $[(x + y) + z], [(x + y) - z].$
10. $[4a + (3b - 5a)], [4a - (3b - 5a)].$
11. $[(m - 2n) + (3p - q)][(m - 2n) - (3p - q)].$
12. $[(5x - 2) + (4y - 9)][(5x - 2) - (4y - 9)].$

VIIa. Field Exercise

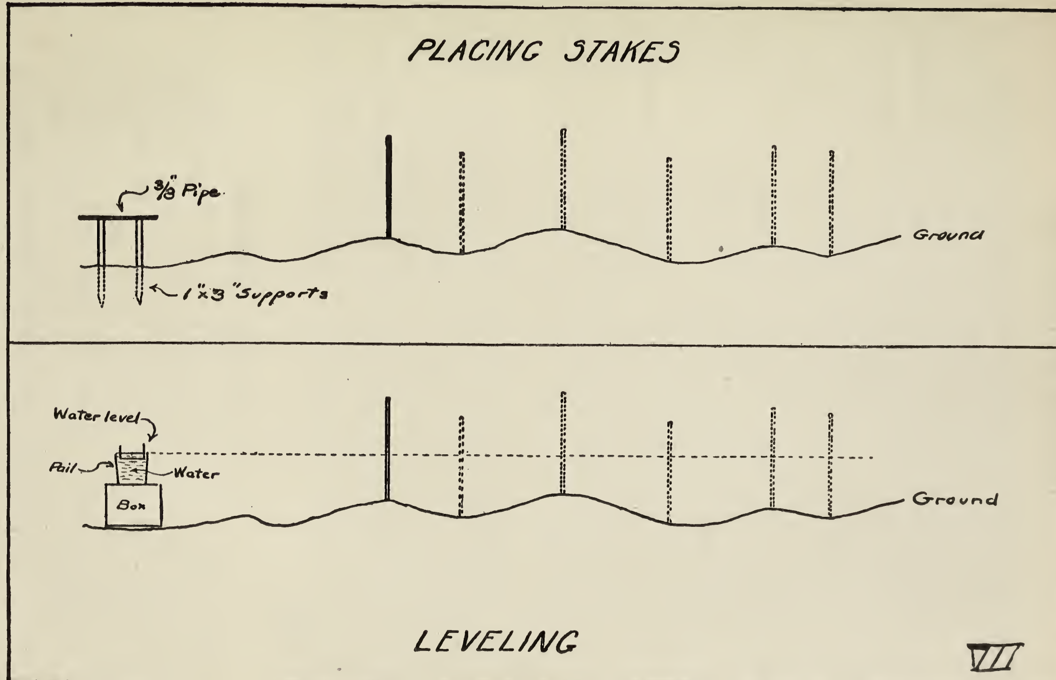
PROFILE OF LAND

The object is to determine the various heights and depressions along the line of a proposed road or ditch.

Personnel: One student to read level, one to carry ranging pole (and later leveling rod), one to drive stakes, one to keep the record of the measurements made, two to use the tape.

Equipment: Level, ranging pole, leveling rod, tape, stakes, mallet.

Procedure: It is first necessary to lay out a straight line for the ditch to follow. To do this, set up the level at one end of the proposed ditch and look toward the other end of it. The student with the ranging pole now advances along the line of the proposed ditch, seeking the first height or


PLACING STAKES
3/8" Pipe
1"x3" Supports
Ground
Water level
Pail
Water
Box
Ground
LEVELING
VII

depression, where he holds the ranging pole upright. The student at the level directs him in moving the pole until it is in his line of sight, by waving his left hand if he wishes it moved toward his left, his right hand if he wishes it moved toward his right, and extending both arms horizontally when the pole is located correctly. A stake is now driven in at this spot, and the Recorder marks it A. The ranging pole is now carried on (VII, upper diagram), its various positions being marked with stakes, until the entire line has been staked off. These stakes should be marked B, C, etc., and the distance from A to B, B to C. etc.. should be measured and recorded.

Since in this part of the work the level is used merely to lay off a straight line, the water pipe substitute may be used (as shown in the upper diagram, VII). If so, it must now be replaced by the water level, or Note at end of IIIb followed.

The leveling rod must now be held upright at each of the stakes and the heights read off. If the student holding the leveling rod holds a dark ruler across the face of the rod and moves it up or down as the observer at the level directs, it is possible to determine the level at stations so far distant that the observer could not read the figures. The readings of the leveling rod at A, B, etc., should be duly recorded.

VIIb. Drawing Exercise

Rule a line across the paper to serve as a datum line. Adopt some convenient scale—such as 1 in. = 10 ft., and lay off the horizontal distance measured to scale along the datum line, thereby locating A, B, etc. At these points draw perpendiculars below the line. Along these perpendiculars lay off to scale the readings of the level at these stations. Connect the points thus obtained, thereby showing the profile of the land.

As will be seen, this drawing reproduces the field conditions to scale—the datum line taking the place of the level line shown in the lower part of the diagram (VII).

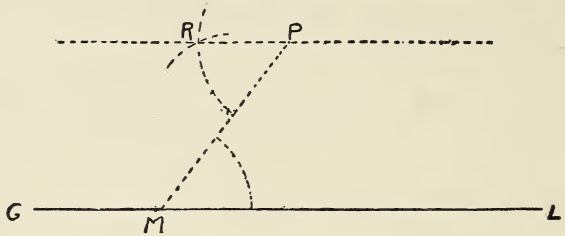
Since variations in the level are usually slight in comparison with the horizontal distances measured, it is best to use a larger scale for showing the differences in height. Thus, if in the drawing the distances between the stakes were shown on the scale of 1 in. = 10 ft., it might be advisable to use a scale of 1 in. = 1 ft. to show the differences in height—otherwise the profile of the ground as shown in the drawing might not vary much from a straight line.

Both the horizontal and the vertical scales should be marked on the drawing.

CHAPTER VIII

THE CONSTRUCTION AND APPLICATION OF PARALLEL LINES

Through a given point to construct a line parallel to a given straight line.



Given: GL a straight line, P an external point.

To Construct: A line through P, // to GL.

Construction: From P draw PM, intersecting GL at M, any point between G and L.

At P construct $\angle\ RPM = \angle\ PML$ (87).

Produce RP.

RP produced is // to GL (65).

The student may now understand the following:

120. Rule: *Through a given external point to draw a line parallel to a given straight line: through the given point draw a line making any convenient angle with the given line; with this construction line as a side and with the given point as a vertex, construct an equal angle which (with the angle just drawn) shall complete a pair of alternate interior angles; the side just drawn of this last angle, produced through the given point, will be the desired parallel.*

PROBLEM: Let the student show how to construct a line through a given point, parallel to a given straight line, by means of exterior-interior angles (67b).

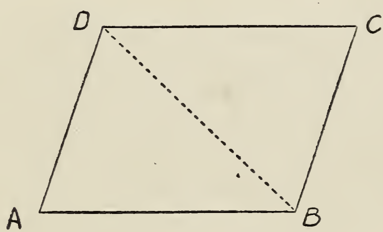
121a. A **parallelogram** is a quadrilateral which has its opposite sides parallel.

121b. If the angles of a parallelogram are right angles, the figure is called a **rectangle**. (Compare 91b.)

121c. A **square** is an equilateral rectangle.

NOTE: Equilateral means, having all sides of the same length (102).

122a. Theorem: *The opposite sides of a parallelogram are equal.*



Given: $ABCD$ is a parallelogram.

To Prove: $BC = AD$, and $AB = DC$.

Proof: Draw the diagonal BD.

In $\triangle$ ABD and BCD,

$$BD = BD \qquad \text{(Identity)}$$
$$\angle ABD = \angle BDC \qquad (64)$$
$$\angle ADB = \angle DBC \qquad (64)$$
$$\therefore \triangle ABD = \triangle DBC \qquad (75)$$
$$AD = BC \qquad (72c)$$
$$AB = DC \qquad Q.\ E.\ D.\ (72c)$$

122b. From the above proof it follows that

A diagonal divides a parallelogram into two equal triangles.

Since, if two parallel lines cross two other parallel lines, they form a parallelogram, it follows from the above theorem that:

122c. *Parallel lines comprehended between parallel lines are equal.*

122d. *The diagonals of a rectangle bisect each other.*

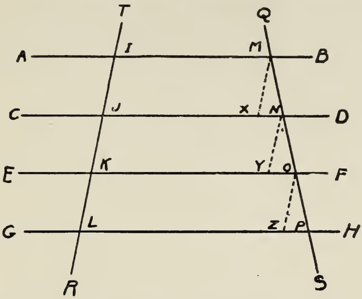
The proof is left to the student.

NOTE: This theorem is often made use of practically. For example, when it is desired to locate an electric light or other fixture in the exact center of the ceiling of a room, cords are stretched between the diagonally opposite corners of the room, and their point of intersection is marked as the center of the ceiling. When "centering" pieces for a lathe this theorem may be applied in a manner similar to that used on page 76.

NOTE: In the following demonstration, the proof that $MN = NO = OP$ depends upon showing that they are corresponding parts of the equal $\triangle$s MXN, NYO and OZP. These $\triangle$s are proved equal by showing that they have 2 $\angle$s and the included side of one = 2 $\angle$s and the included side of the other. It is here that the difficulty occurs, for, although $\angle NMX = \angle ONY = \angle POZ$ (being exterior-interior $\angle$s made by the transversal QS cutting the parallels), this method cannot be used to show that $\angle MXN = \angle NYO$, for MX and NY are not the *same* transversal, consequently the theorem with regard to exterior-interior angles (67a) does not apply. Accordingly the angles formed by TR must be used as the connecting links to show that $\angle MXN = \angle NYO = \angle OZP$.

A similar difficulty occurs in proving that $MX = NY = OZ$. Let the student not fall into the error of stating that "$MX = NY$, since parallels comprehended between parallels are equal." This theorem does not apply here; the pair of parallels which comprehend MX is not the same pair that comprehend NY, nor is it the same pair that comprehends OZ. Here, again, the transversal TR must serve to connect the parts.

123. *If three or more parallels intercept equal segments on one transversal they intercept equal segments on every transversal.*



Given: AB, CD, EF, and GH are //s cut by the transversal TR at I, J, K, and L, respectively, so that $IJ = JK = KL$; QS, any other transversal, cutting AB, CD, EF, and GH at M, N, O, and P, respectively.

To Prove: $MN = NO = OP$.

Proof: Draw MX, NY, and OZ, // to TR.

Then MX, NY, and OZ are // to each other. (61b)

$\therefore \angle ONY = \angle NMX$ (67a)

$\angle POZ = \angle ONY$ (67a)

(I) $\therefore \angle NMX = \angle ONY = \angle POZ$ (31)

Again $\angle MXN = \angle IJX$ (67a)

$\angle NYO = \angle JKY$ (67a)

$\angle OZP = \angle KLZ$ (67a)

But $\angle IJX = \angle JKY = \angle KLZ$ (67a; 31)

(II) $\therefore \angle MXN = \angle NYO = \angle OZP$ (31)

Now $MX = IJ$ (122c)

$NY = JK$ (122c)

$OZ = KL$ (122c)

But $IJ = JK = KL$ (Given).

(III) $\therefore MX = NY = OZ$ (31).

Then, by I, II, and III,

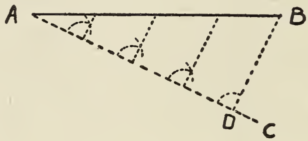
$\triangle NMX = \triangle ONY$ and $\triangle ONY = \triangle POZ$ (75).

$\therefore \triangle NMX = \triangle ONY = \triangle POZ$ (31).

$\therefore MN = NO = OP$ (72c). *Q. E. D.*

NOTE: In like manner, the theorem could be proved to hold for five parallels, or for any number of parallels.

To divide a given straight line into any required number of equal parts.



Given: AB, a straight line.

To Divide: AB into n (any required number) of equal parts.

Construction: From A lay off AC, a line of indefinite length, making any convenient angle with AB.

On AC lay off any convenient unit of length, as u, n times, beginning at A, and marking the ends of each unit.

From D, the exterior extremity of the nth division, draw DB.

At each point of division construct a // to DB (120).

AB will then be divided into n equal parts.

Proof: At A, construct a // to DB.

Then, since the transversal AD is divided into n equal parts, the transversal AB is divided into n equal parts (123).

The student may now understand the following:

124. Rule: *To divide a given straight line into any required number of equal parts: at one extremity of the line draw a line making any convenient angle with the given line; on this construction line, beginning at its point of intersection with the given line, mark off any convenient unit of length as many times as the number of equal parts into which it is desired to divide the given line: from the last point thus marked draw a straight* *line to the other extremity of the given line: through each point marked on the construction line draw parallels to the line last drawn. These parallels will divide the given line into the required number of equal parts.*

The **vernier** is a device for reading very small divisions of a line, or arc, without the use of a microscope.

The vernier may therefore be applied to a surveying instrument to read very small angular divisions (26).

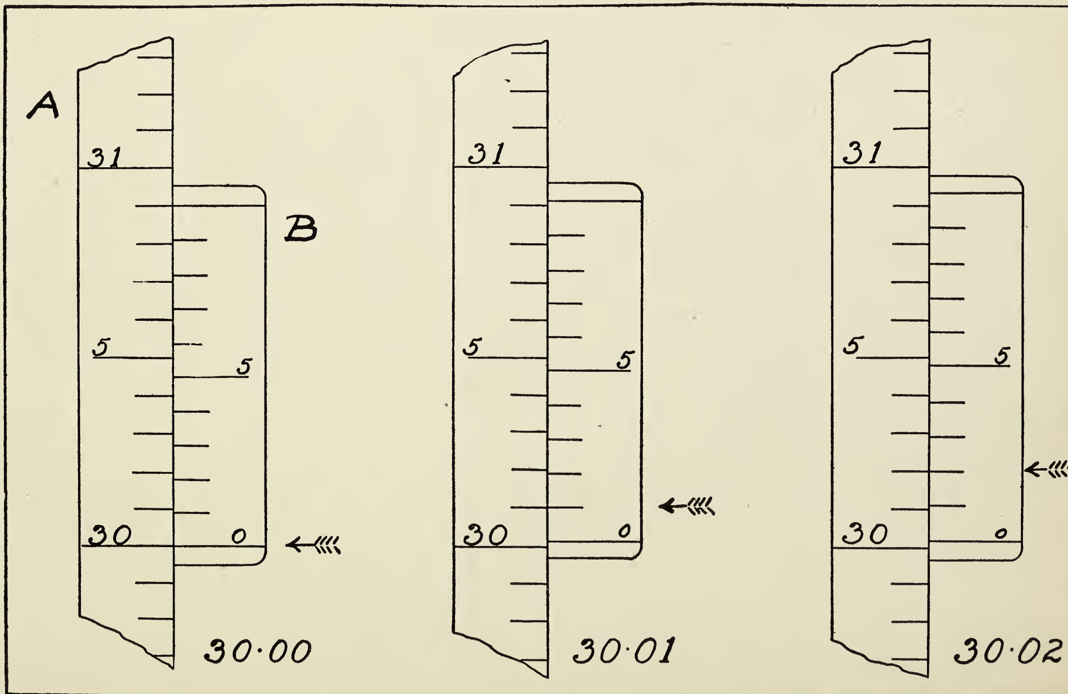
The accuracy of leveling rods (used to find difference in level) may be increased by being fitted with a vernier

The principle of the vernier may be understood from its simplest form, such as is used on the barometer. The barometer is an instrument which measures the atmospheric pressure by means of a column of mercury which is forced up into a glass tube by the pressure of the atmosphere. By the variation of the height of this column of mercury the air pressure may be determined and the weather predicted.

To measure the height of this column of mercury a scale (A) graduated in inches may be fixed near the glass tube containing the mercury. This fixed scale is graduated (that is, carefully divided) into inches and tenths of an inch, and the height of the mercury to the first decimal place (that is, to inches and tenths) may be read directly by comparison with this scale. A vernier (B) is employed to read the hundredths of an inch.

To make the vernier scale, nine of the small divisions on A are laid off (that is, a line 9/10 of an inch in length is drawn) and this line is divided into ten parts, that is, each division of the vernier is equal to $9/10 \div 10 = 9/100$ of an inch (while each division of the fixed scale is equal to $1/10 \div 10 = 10/100$ of an inch). Consequently (since $10/100 - 9/100 = 1/100$) each division of the vernier is $1/100$ of an inch smaller than the small divisions of the fixed scale.

In the figure just referred to, the zero mark on the vernier coincides with the 30-in. mark on the fixed scale. The


A
B
31
5
30
5
0
30·00
31
5
30
5
0
30·01
31
5
30
5
0
30·02

next division of the vernier does not coincide with the next mark of the fixed scale (the 1/10 mark) but is 1/100 of an inch below it; the second division on the vernier is 2/100 of an inch below the 2/10 mark on the fixed scale, the third mark on the vernier is 3/100 of an inch below the 3/10 mark on the fixed scale and so on to the tenth division of the vernier which is 10/100 below the tenth division on the fixed scale, that is, the tenth division of the vernier coincides with the 9/10 mark on the fixed scale.

Obviously, if we raise the vernier 1/100 of an inch, division 1 on the vernier will coincide with the 1/10 mark on the fixed scale. Conversely, if, in reading the barometer, we note that it is the 1/10 mark on the fixed scale which coincides with a mark on the vernier, we must add 1/100 to the height measured directly on the fixed scale and so obtain 30 + 1/100 or 30.01 for the reading of the barometer. Again, if it is the second division of the vernier which coincides with a division on the fixed scale, we must add 2/100 to the reading on the fixed scale (30) and so obtain 30 + 2/100 or 30.02 for the reading of the barometer; if it is the third division of the vernier which coincides with a division on the fixed scale, we add 3/100, and so on.

Keeping these facts in mind, the student is now prepared to take the readings of the barometer for the positions shown on the preceding page.

In the first figure, the 0 of the vernier is opposite the 30-in. mark on the fixed scale—the reading is therefore 30.00. Suppose the zero mark of the vernier were made to coincide with the 1/10 mark next following the 30-in. mark; the reading would then be 30.10.

In the second figure, the 0 mark of the vernier is between the 30 in. mark and 30.1 mark on the fixed scale. The reading from the vernier is obtained by noting that the *first* division of the vernier coincides with a division of the fixed scale. We therefore add .01 to the reading from the fixed scale, 30.00, and obtain 30.01. But if it were the 30.1 mark on the fixed scale that was just below the 0 on the vernier, and the first division of the vernier coincided with a division on the fixed scale, the vernier reading should be added to 30.10.

From these instances, the student can see that in using this vernier, the inches and tenths of an inch are read from the fixed scale, and as many hundredths added to this as the number of divisions it is necessary to count upwards along the vernier to find a vernier mark which coincides with a mark on the fixed scale. In reading the inches and tenths from the fixed scale, always read to the nearest division on the scale below the zero mark on the vernier.

The same principle may also be applied to measuring arcs and, consequently, angles (26). For this purpose the "fixed scale" may be a circular plate and the vernier marked on a circular rim which revolves around the fixed circle—or these positions may be reversed.

Suppose that an arc equal to 29° on the scale is laid off on the vernier and divided into 30 equal parts; then one division of the vernier $= 29° \div 30 = \frac{29°}{30}$; changing this value to minutes,

$$\frac{29}{30} \times 60' = 58'. \quad \text{A division of the scale, } 1°, = 60'.$$

Therefore, the difference between a division on the scale and one on the vernier $= 60' - 58' = 2'$. This vernier, therefore, enables us to read the angle to the nearest $2'$.

Problem: How should a vernier be laid off to read to $1'$? to $30''$? Draw these verniers, remembering that both fixed scale and vernier arcs must be drawn from the same center.

The method of reading all the verniers so far described may be included in the following:

125. Rule: *To use the vernier: take the reading of the fixed scale to the nearest division before the zero of the vernier, then count along the vernier to the division which coincides with a division on the fixed scale; multiply the number of the divisions so counted on the vernier by the amount that a space on the vernier differs from a space on the fixed scale, and add this product to the reading of the fixed scale.*

NOTE: The vernier is so named after its inventor, Pierre Vernier, who published a description of it in 1631.

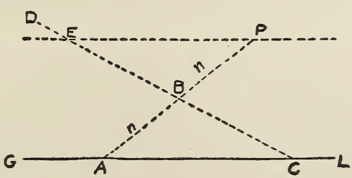
Some verniers, instead of containing one less than the number of fractional parts in one of the larger divisions of the fixed scale (as first described) contain one more: for example, instead of 9/10 as in the barometer vernier just mentioned such verniers would have a length equal to 11/10 which would be divided into ten parts.

Although such verniers have the advantage of having their graduations more distinct (since the spaces between them are greater) yet the method of reading them is more complicated.

VIIIa. Field Exercise

STAKING OFF PARALLELS

This method is of use (in laying out grounds) to make the edge of a path, or flower bed, parallel to the front of a building, or other prominent straight line. It is useful also in laying out the foundations of a building (II) so that it may be parallel to a fence or to another building.



Given: GL a straight line, P an external point.
To Construct: A line through P parallel to GL.

Construction: Put a stake at P.

Take any convenient length of tape (as $2n$) and swing it about P as a center until it intersects the given line at A.

Put a stake at A, thereby marking the extremities of the line PA, of length $2n$. Find the middle point of $2n$ on the tape stretched along PA and mark it with the stake B, thereby bisecting PA.

Put a stake at any convenient point on GL as C.

From C sight across B and place a stake at any convenient point D, in the same straight line with C and B.

Hold the tape against these stakes, measure the distance CB and lay off this same distance from B along BD, placing another stake E at the end of this distance, thereby making $BE = CB$.

Pull up stake D.

PE produced will be parallel to GL.

Proof: In ⚠ PBE and CBA

$$CB = BE \text{ (construction)}$$
$$PB = BA \quad \text{"}$$
$$\angle PBE = \angle CBA \qquad (45)$$
$$\therefore \triangle PBE = \triangle CBA \qquad (76)$$
$$\therefore \angle EPA = \angle PAC \qquad (72c)$$

Then, since PBA is a straight line (const.) PE produced is // to GL (65).

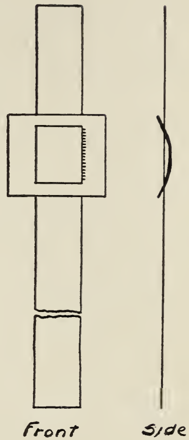
Personnel: Leader, stake-driver, two students to use tape.

Equipment: Tape, mallet, 6 stakes (if distances measured are longer than the tape, more stakes will be needed).

Procedure: Drive in a stake near a tree or other land mark, some 25 yards from the front of a building. By the above method, stake off a line parallel to the front of the building through this stake. At a distance suitable for the width of a path, stake off a line parallel to this.

VIIIb. Pasteboard Vernier.

Make a vernier of pasteboard as in the accompanying diagram. The "fixed scale" should be graduated in inches and tenths, and the vernier so graduated as to read to hundredths (125).



CHAPTER IX

AREAS

126. A **Unit of Surface** is a square whose sides are one linear unit in length.

127. The **Area** of a plain figure is the number of square feet, square inches, or other units of surface which it contains.

Table of Linear Measure

128. 12 inches (in.) = 1 foot (ft.)
3 feet = 1 yard (yd.)
5½ yards or 16½ ft. = 1 rod (rd.)
4 rods = 1 chain (ch.)
80 chains (5280 ft.) = 1 mile (mi.)

Note: Feet are often indicated by a short mark placed after a number and a little above it, inches by two short marks placed after a number and a little above it; thus, $6' \, 5''$ is read, "six feet five inches."

129. For measuring area, the units most in use in the United States are contained in the following:

Table of Square Measure

144 square inches (sq. in.) = 1 square foot (sq. ft.)
9 square feet = 1 square yard (sq. yd.)

For the measurement of larger areas:

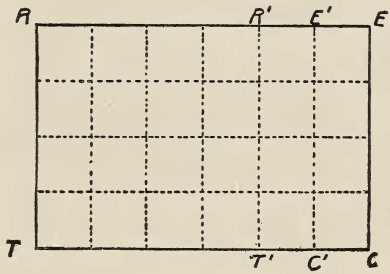
10 square chains = 1 acre (A.)
640 acres = 1 square mile (sq. mi.)

After studying the following theorem (131) let the student compute the number of yards in a square rod, the number of rods in a square chain, and the number of square feet in an acre.

130. Plain figures that have equal areas but cannot be made to coincide are called **Equivalent.**

Thus, a circle with an area of one square foot is equivalent to a square which is one square foot in area.

131. Theorem: *The area of a rectangle is equal to the product of its base by its altitude.*



Given: $RECT$ is a rectangle, m units in length and n units in width.

To Prove: Area of $RECT = m \times n$.

Proof: At the end of each unit of length in the base TC, draw $\perp$s to TC. These lines will be $\perp$ to RE also (62).

These $\perp$s will be parallel to each other (60a).

Then $E'ECC'$, $R'E'C'T'$, etc., are all rectangles (121b).

Each unit of length in TC is a side of a small rectangle, that is, there are as many small rectangles as there are units of length in TC, or m small rectangles.

At the end of each unit of length in EC, draw $\perp$s to EC.

These lines are $\perp$ to E' C' also (62).

Then $E'ECC'$ is divided into n rectangles.

These rectangles are squares (121c).

There are as many of these squares as there are units of length in EC.

That is, there are n squares in the rect. $E'ECC'$.

By prolonging the $\perp$s to $R'T'$ in the next rectangle, $R'E'C'T'$ may be divided into n squares.

Continuing the process each of the small rectangles may be divided into n squares.

There are m of these small rectangles, and each is divided into n squares,

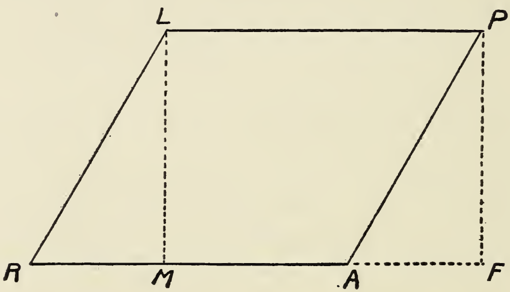
$\therefore$ the whole figure is divided into $m \times n$ squares.

But each square is a unit of surface (126).

$\therefore$ the number of units of surface in $RECT$ (that is, the area of $RECT$) $= m \times n$. *Q. E. D.*

NOTE: In the above proof, m and n are used as the length and width in order that the proof may be *general* (95b).

132. Theorem: *The area of a parallelogram is equal to the product of its base times its altitude.*



Given: ML is the altitude of the parallelogram $PARL$.

To prove: Area $PARL = ML \times PL$.

Proof: Produce RA; from P draw PF, $\perp$ to RA produced.

ML is // to PF (60a).

In $\triangle$s FPA and MLR

$\angle MRL = \angle FAP$ (67a)

$\angle LMR = \angle PFA$ (37b)

I. $\therefore \angle RLM = \angle APF$ (80c)

II. $RL = AP$ (122c)

III. $LM = PF$ (122c)

By I, II, III,

$\triangle RLM = \triangle APF$ (76).

Now, if to the figure $PAML$ we add the triangle LMR we obtain the parallelogram $PARL$, and if to the same figure $PAML$ we add the triangle AFP (the equal of RML) we obtain the rectangle $PFML$.

$\therefore$ area of the rectangle = area of the parallelogram (34b).

This might be expressed by the following equations:

$AMLP + LMR = PARL.$

$AMLP + PFA = PFML.$

$\triangle LMR = \triangle PFA$ (already proved).

$AMLP = AMLP$ (Identity).

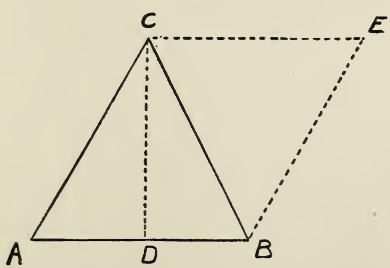
$\therefore PARL = PFML$ (34b) as stated above.

Now the area of $FMLP = PL \times ML$ (131).

Substituting for area of $FMLP$ its equal, area of $PARL$ we have:

Area of $PARL = PL \times ML$. *Q. E. D.*

133a. Theorem: *The area of a triangle is equal to half the product of its base by its altitude.*



Given: CD the altitude of $\triangle ABC$, AB the base.

To prove: Area of $\triangle ACB = 1/2\ AB \times CD$.

Proof: Draw BE // to AC, CE // to AB.

$ACEB$ is a parallelogram (121a).

$\therefore$ CB is a diagonal of the parallelogram.

$\therefore$ CB divides $ACEB$ into two equal parts.

That is, $\triangle$ ACB is 1/2 parallelogram $ACEB$ (122b).

But area of $ACEB$ = $CD \times AB$ (132).

$\therefore$ Area of $\triangle$ ACB = 1/2 $CD \times AB$. *Q. E. D.*

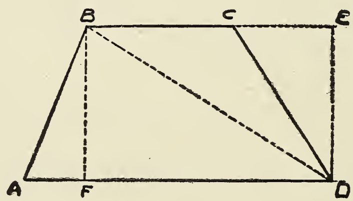
133b. This may be briefly stated as *area* $\triangle = \frac{1}{2} b \times a$. Since the order of factors is immaterial (that is, since $2 \times 3 \times 5$ is the same as $3 \times 5 \times 2$ or $2 \times 5 \times 3$) the above may be written *area* $\triangle = a \times \frac{1}{2} b$.

That is,

The area of a triangle equals the product of the altitude times one-half the base.

133c. If two sides of a quadrilateral are parallel, and two sides are not parallel, the figure is a **trapezoid.** The parallel sides of the trapezoid are called its **bases.**

133d. *The area of a trapezoid equals the product of the altitude times one-half the sum of the bases.*



Given: AD and BC are the bases of the trapezoid $ABCD$; BF its altitude.

To Prove: Area $ABCD = BF \times \frac{1}{2}(AD + BC)$.

Proof: Draw diagonal BD; produce BC and from D draw a $\perp$ meeting BC produced at E. The area $ABCD$ = area $\triangle$ ABD + area $\triangle$ BCD (32a).

I. Area $\triangle\ ABD = BF \times \frac{1}{2}\ AD$ (133a).

Area $\triangle\ BCD = ED \times \frac{1}{2}\ BC$ (133a).

But $ED = BF$ (62, 60a, 122c).

II. $\therefore$ Area $\triangle\ BCD = BF \times \frac{1}{2}\ BC$. Why?

Adding I and II,

Area $\triangle\ ABD$ + area $\triangle\ BCD = BF \times \frac{1}{2}\ AD + BF \times \frac{1}{2}\ BC$.

Or, since both $\frac{1}{2}\ AD$ and $\frac{1}{2}\ BC$ are multiplied by BF,

Area $\triangle\ ABD$ + area $\triangle\ BCD = BF \times (\frac{1}{2}\ AD + \frac{1}{2}\ BC)$.

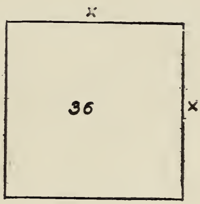
Again, since both AD and BC are multiplied by $\frac{1}{2}$,

Area $\triangle\ ABD$ + area $\triangle\ BCD = BF \times \frac{1}{2}\ (AD + BC)$.

That is, area $ABCD = BF \times \frac{1}{2}\ (AD + BC)$. *Q. E. D.*

Exercise

1. The area of a square is 36 feet. How many feet in each side?



Solution:

The side of the square may be represented by x.

Then the area equals $x \times x$ (131).

That is, $x \times x = 36$.

To find x, evidently we have to find a number, which, multiplied by itself, equals 36. This number, we know from Arithmetic, is 6. That is, $x = 6$. Ans.

In the above solution the expression, "$x \times x$" might have been written "x^2" which is read "x square" (114).

The solution of the above example might therefore have been written in this form:

I. $x^2 = 36.$

II. $x = 6.$ Ans.

The second equation is obtained from the first by extracting the square root of both sides.

From the diagram of this problem, in which the area of the square is marked 36 and the side of the square x (from which we obtained that $x = 6$) we may state the method by which we obtain (II) from (I), in the following form:

134a. Rule: *If one quantity is equal to a second, the square root of the first equals the square root of the second.*

Or,

134b. Rule: *Extracting the square root of both sides of a true equation gives a true equation.*

2. The area of a square tile is 81 in. How long is one side of the tile? What is the perimeter of the tile?

3. Two equal square tiles, placed side by side, cover an area of 50 sq. in. Find the length of the side of one of the tiles.

Solution:

$$2x^2 = 50$$
$$x^2 = 25 \text{ (101)}$$
$$x = 5$$

4. Two square tiles, placed side by side, cover an area of 128 sq. in. Find the length of the side of one of the tiles.

5. Twice the area of a certain square is 338 square in Find the length of the side of the square.

6. How many square yards of surface are there in a floor 21 feet long and 15 feet wide?

7. How many bricks will be required to build a sidewalk $6'$ wide and $72'$ long if the bricks are so placed that the face of the bricks $9''$ long and $4''$ wide is uppermost?

8. How many acres are there in a rectangular field 13 chains long and $7\frac{1}{2}$ chains wide?

HINT: Since there are 10 sq. chains in an A., to find the number of acres in a given number of square chains, the decimal point is moved one place to the left.

9. How many acres are there in a rectangular field 31 rods long and 24 rods wide?

10. The altitude of a triangle is 2.78 chains and its base is 2.75 chains. What is its area expressed as a decimal of an acre?

11. A courtyard is in the form of a trapezoid, one base of which is 24 feet and the other 32 feet. The perpendicular distance between these bases is 25 feet. Find the area of the courtyard.

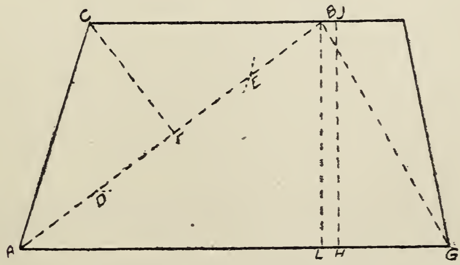
12. To find the area of a field it is divided into three triangles. The altitude of the first triangle is 5.48 ch., and its base 6.78 ch.; the altitude of the second is 6.21 ch. and its base 8.78 ch.; the altitude of the third is 7.21 ch. and its base 6.78 ch. Find the area of the field in acres.

IX. FIELD EXERCISE

FINDING AREA OF FIELD BY BASE AND ALTITUDE METHOD

EQUIPMENT: Leveling instrument; ranging pole; knotted cord; tape; dozen stakes.

PROCEDURE: The field is to be divided into triangles as



nearly equilateral as possible (as this is the best form for computation).

Start at one corner of the field as A and with the leveling instrument lay off a line intersecting the other side as at B, a perpendicular from C to AB will be the altitude of the triangle. The remainder of the field may be divided into triangles in the same way and their altitudes constructed.

In constructing the altitude of a triangle, as ACB, one of two methods may be used.

If the perpendicular distance from C to AB is less than the length of the knotted cord (or tape) the following method may be used.

Hold one end of the cord at C and describe two arcs cutting AB, at D and E, which should be marked with stakes.

(To aid in finding the exact point where the cord crosses the line AB, the tape should be stretched along the stakes.)

Measure the distance between D and E, and at the middle point drive a stake F.

CF will be perpendicular to AB at F (since C and F are each equidistant from D and E).

In case the distance of B from A and G is too long for the cord or tape to be used to construct a perpendicular from B to AG, the following method may be employed.

Estimate as closely as possible the foot of the perpendicular from B to AG and mark it with a stake as H. At H erect a perpendicular to AG as JH. This will probably not go through B, but at some distance to one side of this point.

Measure the perpendicular distance from B to this line, as JB. Lay off the distance BJ from H and mark this distance with a stake L. A perpendicular erected at L will pass through B.

The bases and altitudes of the various triangles being measured, their areas may be found and these added to find the total area of the field.

CHAPTER X

MULTIPLICATION OF LITERAL EXPRESSIONS: FACTORING

It has already been explained in Chapter VI that such an expression as $5\,(a + b)$ means that $(a + b)$ must be taken 5 times. Accordingly, if we remove the parenthesis, each term within it must be multiplied by the coefficient in front of it. Thus,

$$5\,(a + b) = 5a + 5b$$

Similarly, $$-5\,(a + b) = -5a - 5b$$

and $$x\,(y + 2) = xy + 2x$$

It has already been explained that subtracting a minus quantity is the same as adding an equal plus quantity, that is, as explained in Chapter VI, $-(-4) = +4$. Furthermore, if we subtract a minus quantity *twice* it is the same as if we had added double the plus quantity, that is, $-2\,(-4) = +8$. In the same way, $-3\,(-4) = 12$ and so on. All of these results may be included in the general statement,

$$-a\,(-b) = ab,$$

where a has any value, and b any value, that is,

135. Rule: *The product of a minus quantity by a minus quantity is a plus quantity.*

NOTE: This is stated more briefly, "*minus times minus gives plus.*"

NOTE: As the student already knows, "*minus times plus gives minus.*"

Accordingly, $-4\,(a - b) = -4a + 4b$.

Exercise

Remove Parentheses:

1. $5\,(2a + 3b + 4c)$
2. $3\,(6a^2 - 7a + 2)$
3. $-\,2\,(3x + 4y - 2z)$
4. $-\,5\,(2a - 3b - c)$
5. $-\,6\,(4b - 3c + d)$
6. $-\,4\,(8a - 7b^2 - 2c)$
7. $-\,8\,(b^2 - b + 1)$
8. $13\,(z^2 + x + 3y - 4)$

As has been already explained (114) a^2 means $a \times a$ and a^3 means $a \times a \times a$ and so on, the exponent showing how many times the letter called the base is taken. Accordingly,

$a^2 \times a = a^3$

In like manner,

$(ab) \times (ab)^2 = (ab)^3;$

since $(ab)^3 = (ab) \times (ab) \times (ab) =$

$a \times a \times a \times b \times b \times b = a^3b^3.$

That is, $(ab) \times (ab)^2 = a^3b^3.$

Similarly, $(2ab) \times (3bc) \times (2ac) = 12a^2b^2c^2.$

Applying terms previously explained, the above might be called a problem in multiplication of monomials of the same base; and the method used stated in the following:

136a. Rule: *To multiply monomials having the same base; add the exponents and write this sum as the exponent of the base, and write the product of the numerical coefficients as the coefficient in the answer.*

In multiplying monomials which have a common factor but which are not the same throughout, the above rule may be applied to the factors which are common to different terms, thus,

$$(a^2b) \times (3b^2) \times (4abc) = 12a^3b^4c$$

The method used in the above problems may be stated in the following:

136b. Rule: *To multiply monomials, write the product of all the numerical coefficients as the coefficient in the answer and after it write each literal factor with the exponent which* *is formed by adding together the exponents of this letter in all the quantities to be multiplied.*

This rule may be applied in the following examples:

Simplify:

$$6a\,(ab - 2a^2 + ac)$$

Solution:

$$6a\,(ab - 2a^2 + ac) = 6a^2b - 12a^3 + 6a^2c$$

Simplify:

$$-\,3a\,(2ab - 4ab^2 + 7a^2)$$

Solution:

$$-\,3a\,(2ab - 4ab^2 + 7a^2) = -\,6a^2b + 12a^2b^2 - 21a^3$$

Remove Parentheses in the following:

9. $2b\,(b - 2c - d)$

10. $3a\,(2a - 2b - 4c)$

11. $4ab\,(ab - 2b + 3a^2)$

12. $-\,2mn\,(n^2 - 2m + 3n)$

13. $7xy\,(2x + 3y - 5)$

14. $9x^2\,(3y - 5z + 6x)$

15. $3ab\,(2b - 3ab + b^2)$

16. $a^2\,(a + ab - 3c)$

Multiplication may also be expressed thus,

$$\begin{array}{r} 3ab + 4a + 1 \\ 2 \\ \hline \end{array}$$

This indicates that each term of the upper quantity is to be multiplied by 2. The answer, therefore, is the product, $6ab + 8a + 2$, which should be written below the line.

Multiply:

17. $\begin{array}{r} x^2 + 3x + 5 \\ x \\ \hline \end{array}$

18. $\begin{array}{r} 7x^2 + 2x + 3y - 5 \\ 2x \\ \hline \end{array}$

19. $\begin{array}{r} a^2 - b + h^2 - c^2 \\ 3b^2 \\ \hline \end{array}$

20. $\begin{array}{r} a^2 + 2ab + b^2 \\ ab \\ \hline \end{array}$

21. $\begin{array}{r} 3x^2 + 2xy + 3y^2 - 2x + 3y - 6 \\ 3y \\ \hline \end{array}$

22. $\begin{array}{r} mn + n^2 - m^2 + 3m + n \\ 2mn \\ \hline \end{array}$

The same method may be used when the multiplier is a binomial, thus,

$$\begin{array}{r} 2x + 3 \\ x + 1 \\ \hline 2x^2 + 3x \phantom{{}+3} \\ 2x + 3 \\ \hline 2x^2 + 5x + 3 \end{array}$$

We first multiply each term in the multiplicand by x and write this result as the first partial product, then we multiply the multiplicand by 1, and write this result as the second partial product, being careful to keep similar terms under each other, and if there is no similar term in the first partial product, we place the result to one side as in a column by itself. Thus, writing the second partial product, we place $2x$ under $3x$, but since there is no term free from x in the first partial product, it is necessary to place the 3 to one side. After the partial products are written down in their proper columns, we draw a line and add the columns. The same method may be used when the multiplier is a trinomial, thus,

$$\begin{array}{llllll} m^2 + & mn & + m & & & \\ 3m + & 2n & + 4 & & & \\ \hline 3m^3 & + 3m^2n & + 3m^2 & & & \\ & + 2m^2n & & + 2mn^2 & + 2mn & \\ & & + 4m^2 & & + 4mn & + 4m \\ \hline 3m^3 & + 5m^2n & + 7m^2 & + 2mn^2 & + 6mn & + 4m \end{array}$$

Rearranging, $3m^3 + 7m^2 + 5m^2n + 2mn^2 + 6mn + 4m$

The work of multiplication may often times be made easier by first arranging the terms in some definite order, generally according to descending powers of some letter. Thus, consider the following problem in multiplication:

$$\begin{array}{l} x^2 + 1 + 3x \\ 2x + 5 + 3x^2 \\ \hline \end{array}$$

If the multiplication is attempted with the terms arranged in this haphazard fashion it will be found much more confusing than when the work is arranged in this manner:

$$\begin{array}{r} x^2 + 3x + 1 \\ 3x^2 + 2x + 5 \end{array}$$

This method may be generalized for any number of terms as follows:

137. Rule: *To multiply polynomials:—both multiplicand and multiplier having been arranged in descending powers of the same letter, multiply the multiplicand by each term in the multiplier, keeping like terms in the same column, and add the partial products thus obtained.*

Exercise

Multiply:

1. $\begin{array}{r} 2x + 3y \\ \underline{2x + y} \end{array}$ 2. $\begin{array}{r} 2a + 3 \\ \underline{a + 2} \end{array}$ 3. $\begin{array}{r} 4y + 5 \\ \underline{3y + 2} \end{array}$

4. $\begin{array}{r} 4x - 3 \\ \underline{x - 5} \end{array}$ 5. $\begin{array}{l} 6 - 5x \\ \underline{x - 2} \end{array}$ 6. $\begin{array}{r} 2x - 7 \\ \underline{7 - x} \end{array}$

7. $\begin{array}{r} 3x - 1 \\ \underline{5x - 2} \end{array}$ 8. $\begin{array}{r} x^2 + x + 1 \\ \underline{x + 1} \end{array}$ 9. $\begin{array}{r} x^2 + x + 1 \\ \underline{x - 1} \end{array}$

10. $\begin{array}{r} a^2 - 3ab + b \\ \underline{a + b} \end{array}$ 11. $\begin{array}{r} a^2 + 3a - 2b \\ \underline{a - 3b} \end{array}$ 12. $\begin{array}{r} 3m^2 + 2mn + n^2 \\ \underline{m - 2n} \end{array}$

13. $\begin{array}{r} 2m^2 + 3m + 1 \\ \underline{m^2 + 2m + 1} \end{array}$ 14. $\begin{array}{l} a + b + c \\ \underline{a - 3b + 1} \end{array}$ 15. $\begin{array}{l} a^2 + 2b + 5 \\ \underline{a + b - 3} \end{array}$

138. Multiplication may be expressed by writing two or more quantities, enclosed in a parenthesis, in the same line, with no sign between them.

Thus, $(2a)$ $(3b)$ means that $2a$ is to be multiplied by $3b$, and $(2a + b + c)$ $(3a + 4b)$ $(a - b + 2c)$ means that $2a + b + c$ is to be multiplied by $3a + 4b$ and then this product is to be multiplied by $a - b + 2c$.

139. To **expand** means to carry out a process which is indicated. Thus, "Expand $(3a + b)(2a - b)$" means to multiply $3a + b$ by $2a - b$.

Expand

16. $(x^2 + xy + y^2)(x - y)$
17. $(x^2 - xy + y^2)(x + y)$
18. $(a^2 - 2a + 4)(a + 2)$
19. $(x^2 - 3x + 9)(x + 3)$

As was learned in Arithmetic, the square of a given quantity is that product obtained by multiplying a quantity by itself.

The squares of certain algebraic expressions are so important that they deserve special attention.

If we expand $(a + b)^2$, that is, if we multiply $(a + b)$ by $(a + b)$ we obtain the product $a^2 + 2ab + b^2$. This result might be stated in words instead of letters, in the following:

140. Rule: *The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first by the second, plus the square of the last.*

Expand $(2x + 3y)^2$ by the above rule.

Solution:

The square of $(2x + 3y)^2$ = the square of the first term, $(2x)(2x)$, or $4x^2$, plus twice the product of the first term by the second, that is, $2(2x)(3y)$ or $12xy$, plus the square of the last term $(3y)(3y)$ or $9y^2$. The complete answer is, then: $(2x + 3y)^2 = 4x^2 + 12xy + 9y^2$.

Expand the following by the above rule.

20. $(x + 1)^2$	21. $(y + z)^2$	22. $(m + n)^2$
23. $(2x + 1)^2$	24. $(3y + 1)^2$	25. $(a + 2b)^2$
26. $(3m + 2n)^2$	27. $(3a + 2b)^2$	28. $(3y + 4z)^2$
29. $(4c + 1)^2$	30. $(a + 5)^2$	31. $(4c + 3)^2$
32. $(6 + 7h)^2$	33. $(8 + 9m)^2$	34. $(9 + 5a)^2$

The above rule, of course, applies as well to numerical quantities as to literal quantities. By its application numbers may be squared mentally; thus,

Square 34.

Solution:

$$\begin{aligned}(34)^2 = (30+4)^2 &= 900 + 2 \times 4 \times 30 + 16 \\ &= 900 + 240 + 16 \\ &= 1140 + 16 \\ &= 1156 \text{ Ans.}\end{aligned}$$

As will be seen from the above, to apply the rule so that numbers may be squared mentally, the number is separated into parts, one of these parts being the next lower multiple of 10—as here, the number is separated into 30 and the remainder.

NOTE: In finding the middle term, it is easier (in mental work) to double the second quantity before multiplying by the first—rather than multiplying the two together and *then* multiplying by 2.

Square the following numbers mentally by the above method:

35.

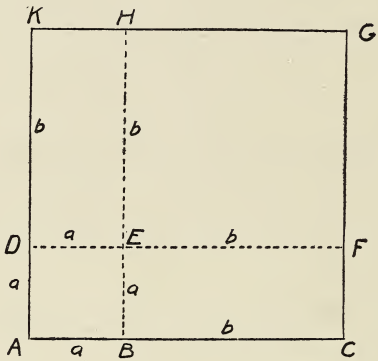
(a)	(b)	(c)	(d)	(e)	(f)
31	43	52	73	84	22
(g)	(h)	(i)	(j)	(k)	(l)
62	71	63	81	74	33

By means of the theorem which states that the area of a rectangle is equal to the product of the base by the altitude we may prove by a geometrical construction the truth of the above rule (140) and consequently of the formula of which it is the verbal expression.

To Prove: $(a + b)^2 = a^2 + 2ab + b^2$

Let AB be a line a units in length, and BC be a line b units in length. Then the line AC is $(a + b)$ units in length.

Construct a square with AC, that is, $(a + b)$, as a side, and within this square construct a square with AB, that is, a, as a side, and produce the interior sides of the smaller square to meet the sides of the larger square; thus,



(I) Since AC is $(a + b)$ units in length, the area of $ACGK$ contains $(a + b)^2$ units (131).

(II) This square $ACGK$ is made up of the square $ABED$, the square $EFGH$ and the rectangles $BCFE$ and $DEHK$.

(III) The square $ABED$ (since the side AB is a units in length) is a^2 units in area (131). The square $EFGH$ (the side of which $EF = BC = b$ units in length) contains b^2 units in area.

The rectangle $DEHK$ = the rectangle $BCFE$.

(For $DE = BE$, being sides of the same square, and $EF = EH$, being sides of the same square.)

That is rect. $DEHK$ + rect. $BCFE$ = 2 rect. $BCFE$.

But the area of $BCFE = BE \times BC = a \times b$ (131).

(IV) $\therefore$ 2 rect. $BCFE = 2ab$.

Then, by I, II, III, and IV,

$$(a + b)^2 = a^2 + 2ab + b^2. \quad Q.\ E.\ D.$$

Expand

36. $(a + b + c)^2$
37. $(3a + 2b + c)^2$
38. $(m + 2n + 3p)^2$
39. $(x + y + z)^2$
40. $(m + n + 2p)^2$
41. $(2x + 3y + 4z)^2$

Another important square is that of the difference of two quantities. This may be expressed by the formula, obtained by actual multiplication,

$$(a - b)^2 = a^2 - 2ab + b^2$$

That is, since a may have any value assigned to it, and b may have any value assigned to it, this formula is the literal expression of the following:

141. Rule: *The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the last.*

The only difference, then, between the square of the sum of two quantities and the square of the difference of two quantities, is that in the first case the sign of the middle term of the answer is plus, while in the second case the middle term is minus.

Exercise

Expand

1. $(x - y)^2$
2. $(m - n)^2$
3. $(x - 1)^2$
4. $(2a - b)^2$
5. $(3m - n)^2$
6. $(2x - 3y)^2$
7. $(a - 5)^2$
8. $(5x - 3y)^2$
9. $(3a - 7b)^2$
10. $(5m - 7n)^2$
11. $(4n - 3a)^2$
12. $(5c - 4n)^2$
13. $(6b - c)^2$
14. $(5d - 2c)^2$
15. $(9n - 1)^2$
16. $(x - 10)^2$
17. $(x - 12)^2$
18. $(2y - 5)^2$

The above rule may be used to square numbers mentally.

Square 37.

Solution:

$$\begin{aligned}(37)^2 = (40 - 3)^2 &= 1600 - 6 \times 40 + 9\\ &= 1600 - 240 + 9\\ &= 1360 + 9\\ &= 1369.\end{aligned}$$

Square the following numbers mentally:

19.	(a)	(b)	(c)	(d)	(e)
	18	29	47	58	67
	(f)	(g)	(h)	(i)	(j)
	38	46	89	78	36

When a student is called upon to square a number mentally he may use either of the formulas, $(a + b)^2 = a^2 + 2ab + b^2$, or, $(a - b)^2 = a^2 - 2ab + b^2$. The student should choose the formula which makes b as small as possible. Thus, in squaring 48 we should think of the number as $(50 - 2)$ thereby using the $(a - b)^2$ formula, rather than thinking of the number as $(40 + 8)$ and so using the $(a + b)^2$ formula. (That is, we should use the rule of art. 141, rather than the rule of 140.) The reason for the choice of a formula is obvious in the above example, since 2 is easier to use as a multiplier than 8.

Choose the more convenient of the preceding formulas. and square the following numbers mentally:

20.	(a)	(b)	(c)	(d)	(e)	(f)	(g)
	83	36	69	21	57	73	86

By means of the theorem which states that the area of a rectangle is equal to the product of the base by the altitude, we may prove by a geometrical construction the truth of the above rule (141), and consequently of the formula of which it is the verbal expression.

To Prove: $(a - b)^2 = a^2 - 2ab + b^2$

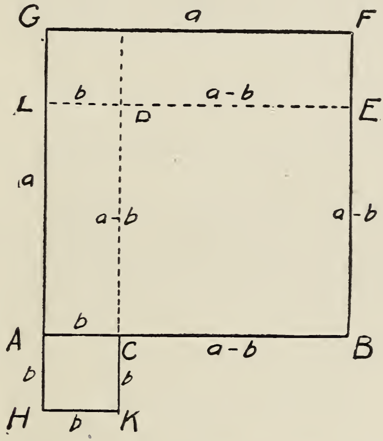
Given the line AB, a units in length, AC, b units in length, and CB, their difference, which is, therefore, $(a - b)$ units in length.

The above formula may be written:

$$(a - b)^2 = a^2 + b^2 - 2ab$$

This brings out more clearly that twice the product of a and b is subtracted from the sum of the square of a and the square of b.

The geometrical proof consists in showing that the square of which the side is $(a - b)$ is equal to the remainder left by subtracting twice the rectangle, the altitude of which is a and the base of which is b, from the area of the polygon



formed by adding the square whose side is a to the square whose side is b, that is, in showing that

$$(a - b)^2 = a^2 - 2ab + b^2$$

Upon AB (a units in length), construct a square, $ABFG$, the area of which therefore contains a^2 units.

Upon AC (b units in length), construct an exterior square, $HKCA$, the area of which therefore contains b^2 units.

(I) The area of the entire polygon (that is, $BFGHKC$) is therefore equal to $a^2 + b^2$.

(II) On the side CB $(a - b)$ units in length, construct a square, $CBED$, the area of which is therefore $(a - b)^2$ units in length.

Produce ED to L.

$EF = AC$ (since each is obtained by subtracting a side of the square $CBED$ from a side of the square $ABFG$).

$GF = LH$ [$GF = a$, and $LH = GH - GL = a + b - b = a$]

$\therefore$ Rect. $GFEL = LDKH$.

(III) That is, area $HKDEFG = 2$ area $GFEL = 2ab$.

Then, by I, II, and III, if from the area of the whole figure $(a^2 + b^2)$ we subtract the polygon $KDEFGH$, the area of which is $2ab$, we have left the square $CBED$, the area of which is $(a - b)^2$, that is

$$a^2 + b^2 - 2ab = (a - b)^2$$

Or by transposition

$$(a - b)^2 = a^2 - 2ab + b^2. \quad Q.\ E.\ D.$$

If the multiplier or multiplicand contains parentheses, it is possible to multiply them before removing the parentheses, and is sometimes more convenient to do so. Thus,

$$\begin{array}{llll}
(a + b) & + c & & \\
2\,(a + b) & + 3c & & \\
\hline
2\,(a + b)^2 & + 2c\,(a + b) & & \\
 & \, 3c\,(a + b) & + 3c^2 & \\
\hline
2\,(a + b)^2 & + 5c\,(a + b) & + 3c^2 &
\end{array}$$

We may afterwards remove the parentheses, thus,

$$2\,(a^2 + 2ab + b^2) + 5ac + 5bc + 3c^2 =$$
$$2a^2 + 4ab + 2b^2 + 5ac + 5bc + 3c^2$$

Multiply the quantities in the brackets before removing parentheses, then remove the parentheses, and combine:

21. $[(a + b) - 2d]\ [3\,(a + b) + 4d]$

22. $[5\,(a + c) - 7d]\ [4\,(a + c) - 3d]$

23. $[(m + n) - 5]\ [2\,(m + n) - 7]$

Expand the following before removing the parentheses, then remove the parentheses and combine:

24. $[(a - 2) + b]^2$

HINT: This means multiply $(a - 2) + b$ by $(a - 2) + b$ as in the above examples.

25. $[(x - y) + 5]^2$
26. $[(2x + y) - 3]^2$
27. $[(a - 3b) + 2c]^2$
28. $[(a - 3) - 2d]^2$

These problems may be expanded by the preceding formulas, using 140, if the sign between the parts is +, or 141, if −, thus:

$$\begin{aligned}[(a + b) + 2c]^2 &= (a + b)^2 + 4c(a + b) + 4c^2 \\ &= a^2 + 2ab + b^2 + 4ac + 4bc + 4c^2\end{aligned}$$

Again

$$\begin{aligned}[(a - b) - 3c]^2 &= (a - b)^2 - 6c(a - b) + 9c^2 \\ &= a^2 - 2ab + b^2 - 6ac + 6cb + 9c^2\end{aligned}$$

Choose the proper rule (either 140 or 141) and expand:

29. $[(a + b) + 3]^2$
30. $[(a + b) - 4]^2$
31. $[(b - c) - d]^2$
32. $[(a + b) - (c + d)]^2$
33. $[(m + 5) - (n - n)]^2$
34. $[(2x + y) - (a + 5b)]^2$

Another important product is that obtained by multiplying the difference of two quantities by their sum, represented by $(a - b)(a + b)$

Multiplying:

$$\begin{array}{lll} a & + b & \\ a & - b & \\ \hline a^2 & + ab & \\ & - ab & - b^2 \\ \hline a^2 & & - b^2 \end{array}$$

Since a may have any value assigned to it, and b any value assigned to it, the above result may be stated as the following:

142a. Rule: *The product of the sum and difference of two quantities is equal to the difference of their squares.*

Thus: Expand $(2x + 3y)(2x - 3y)$

Following the above rule, we square the first term $(2x)$ and obtain $4x^2$, then square the last term $(3y)$ and obtain $9y^2$, from the first square we subtract the second, obtaining the answer, $4x^2 - 9y^2$.

Exercise

Expand by the above rule:

1. $(x + y)\ (x - y)$	**2.** $(m + n)\ (m - n)$
3. $(c + d)\ (c - d)$	**4.** $(a + 3)\ (a - 3)$
5. $(3m - 2n)\ (3m + 2n)$	**6.** $(2a + 3)\ (2a - 3)$
7. $(2x + y)\ (2x - y)$	**8.** $(3y + 2z)\ (3y - 2z)$
9. $(5m + 3n)\ (5m - 3n)$	**10.** $(6 - 8z)\ (6 + 8z)$

By the above rule we may multiply certain numbers mentally; thus:

Multiply 37 by 43.

Solution:

$$(37)\ (43) = (40 - 3)\ (40 + 3) = 1600 - 9 = 1591. \quad \text{Ans.}$$

Note that in order to multiply two numbers together by the above rule, one of them must be a certain small quantity greater than a number of which the square can be readily found, while the other is the same amount less than the same number. This number need not necessarily be a multiple of 10; for example,

Multiply 13 by 11.

Solution:

$$(13)\ (11) = (12 + 1)\ (12 - 1) = 144 - 1 = 143.$$

Any number of which the square is known may be made the a in the $(a + b)\ (a - b)$ formula.

Multiply the following by the above rule:

11.

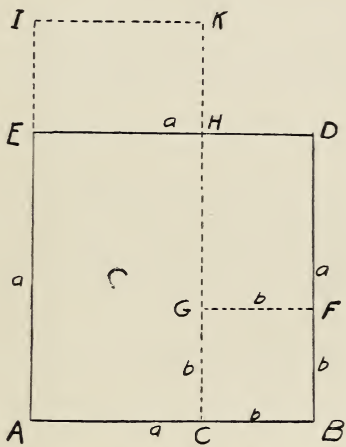
(a)	(b)	(c)
(18) (22)	(32) (28)	(33) (27)
(d)	**(e)**	**(f)**
(44) (36)	(51) (49)	(63) (57)
(g)	**(h)**	**(i)**
(82) (78)	(67) (73)	(48) (52)

By means of a geometrical construction we may prove the truth of the above rule, as expressed by the formula.

To Prove: $a^2 - b^2 = (a + b)\ (a - b)$

Given the line AB, a units in length, the line CB, b units in length.

On AB construct a square, the area of which will therefore contain a^2 units; on CB construct a square, the area of which will therefore contain b^2 units.



If from the area of the large square, a^2, we subtract the small square b^2, we have left the polygon $ACGFDE$; in other words,

(I) $a^2 - b^2 = ACGFDE$.

It now remains to be shown that

the area of $ACGFDE = (a + b)\ (a - b)$

$HG = HC - CG$, that is, $HG = (a - b)$

$EH = ED - HD$, that is, $EH = (a - b)$

$\therefore$ if EA and CH are produced to I and K, respectively, so that EI and HK will equal GF (or b), the rectangle

$EIKH$ will equal the rectangle $HDFG$ (having equal bases and equal altitudes).

Now $AIKC = EHCA + IKHE$

and $EDFGCA = EHCA + HDFG$

(II) $\therefore IKCA = EDFGCA$ (34b).

Since $EI = b$ and $EA = a$, the side $AI = (a + b)$;

$AC = (a - b)$ (Since $AB = a$, and $CB = b$)

(III) $\therefore IKCA = (a + b)\ (a - b)$

$\therefore$ by (I), (II) and (III),

$$a^2 - b^2 = (a + b)\ (a - b) \quad Q.\ E.\ D.$$

This rule may also be applied to more complicated quantities containing parentheses, or other signs of inclosure.

Expand:

$$[\,(a + b) + (c + d)\,]\ [\,(a + b) - (c + d)\,]$$

Solution:

$[\,(a + b) + (c + d)\,]\ [\,(a + b) - (c + d)\,] =$

$(a + b)^2 - (c + d)^2 =$

$a^2 + 2ab + b^2 - (c^2 - 2cd + d^2) =$

$a^2 + 2ab + b^2 - c^2 + 2cd - d^2.$

Expand:

12. $[\,(a + b) + (m + n)\,]\ [\,(a + b) - (m + n)\,].$

13. $[\,(a + b) + 2]\ [\,(a + b) - 2].$

14. $[\,(2x + y) + 5]\ [\,(2x + y) - 5].$

15. $[\,(a + b) - 2bc]\ [\,(a + b) + 2bc].$

The rule may also be applied when the quantities inside the parentheses are not of the first* degree; thus,

Expand:

$$[\,(a^2 + b^2) - 2bc]\ [\,(a^2 + b^2) + 2bc]$$

Solution:

$[\,(a^2 + b^2) - 2bc]\ [\,(a^2 + b^2) + 2bc\,] = (a^2 + b^2)^2 - 4b^2c^2$

$= a^4 + 2a^2b^2 + b^4 - 4b^2c^2$

Expand:

16. $[\,(a^2 + b^2) - 3]\ [\,(a^2 + b^2) + 3].$

17. $[\,(c + 2d) - 2ab]\ [\,(c + 2d) + 2ab]$

*Quantities in which the highest exponent is 2, are of the *second degree.*

Not only may literal expressions be multiplied, but they may also be factored.

Problem: Factor $a^2 + 3a + ab$

Solution: $a^2 + 3a + ab = a(a + 3 + b)$

Problem: Factor $6a^3b + 3ab^2 + 3ab - 12a^2b^2$

Solution: $6a^3b + 3ab^2 + 3ab - 12a^2b^2 =$
$3ab(2a^2 + b + 1 - 4ab)$.

Exercise

Factor:

1. $2a + 2b + 2c$
2. $\frac{1}{2}x + \frac{1}{2}(y + z)$
3. $xy + x^2 - 4x$
4. $m^2n - mn^2$
5. $d^2d - ab^2 + b^3 - b^2$
6. $a(a + b) - b(a + b) + 3(a + b)$

Many expressions may be resolved into their factors by reversing the laws for multiplication stated in the earlier part of the chapter.

Problem: Separate $a^2 + 2ab + b^2$ into its factors.

Solution: $a^2 + 2ab + b^2 = (a + b)(a + b)$

Separate into factors:

7. $a^2 - 2ab + b^2$
8. $x^2 + 4xy + 4y^2$
9. $a^2 + 10a + 25$
10. $m^2 - 12m + 36$
11. $4a^2 - 4ab + b^2$
12. $25x^2 - 100x + 100$
13. $x^2 + 2x(a + b) + (a + b)^2$
14. $(a^2 + b)^2 + 2(m + n)(a^2 + b) + (m + n)^2$
15. $4(a + c)^2 + 4(a + c) + 1$

Note: Problems 7 to 15 are really problems in square root.

When the expression to be factored is the difference of two squares, its factors may be obtained by reversing the rule for finding the product of the sum and difference of two quantities (142a). Thus,

Problem:

Factor $a^2 - b^2$

Solution:

$$a^2 - b^2 = (a + b)\ (a - b)$$

Factor:

16. $x^2 - y^2$
17. $a^2 - 4b^2$
18. $a^2 - (b + c)^2$
19. $x^2 - (y - z)^2$
20. $(a - b)^2 - (c + d)^2$
21. $(a^2 + b)^2 - (c + d)^2$
22. $e^2 - (f^2 + g^2 - h^2)^2$

The factors of a number may be readily checked by multiplying the factors. The product so obtained should be the original expression.

From the above problems it is evident that the rule for finding the product of the sum and difference of two quantities (142a) may be reversed as the following rule for factoring:

142b. Rule: *An expression composed of the difference of two squares may be factored into the sum of the square roots of these squares, and the difference of the square roots of these squares.*

The student should familiarize himself with the steps in the next example. It will be used later in the derivation of **Heron's Formula** for finding the area of a triangle when its three sides are known.

Problem: Factor: $4c^2b^2 - (c^2 + b^2 - a^2)^2$

Solution:

$4c^2b^2 - (c^2 + b^2 - a^2)^2 = [2cb - (c^2 + b^2 - a^2)]\ [2cb + (c^2 + b^2 - a^2)\ ].$

Removing parentheses (since the parenthesis in the first factor is preceded by a minus sign, all the terms within it are changed in sign when the parenthesis is removed; the parenthesis in the second factor being preceded by a plus sign, no change of sign is necessary):

$$[2cb - c^2 - b^2 + a^2]\ [2cb + c^2 + b^2 - a^2].$$

Since changing the order of the terms does not alter the value of an expression the quantities within the brackets may be rearranged, thus,

$$[a^2 - c^2 + 2cb - b^2]\ [b^2 + 2cb + c^2 - a^2].$$

Since quantities may be put in a parenthesis preceded by a minus sign, if they are first changed in sign (so that the two changes of signs will cancel each other when the parenthesis is again removed), the above expression may be restated as

$$[a^2 - (c^2 - 2cb + b^2)\ ]\ [\ (b^2 + 2cb + c^2) - a^2].$$

Since the quantities in the second bracket were enclosed in a parenthesis which was not preceded by a minus sign, no change of sign was necessary.

Since the quantities within the parenthesis are perfect squares, they may be expressed as follows,

$$[a^2 - (c - b)^2]\ [\ (b + c)^2 - a^2].$$

Arranged in this form it is evident that the quantity within each bracket is the difference of two squares, and may accordingly be factored (142a). (It is not necessary to retain the brackets, as the quantities within them have been factored.)

$$\{a - (c - b)\}\ \{a + (c - b)\}\ \{(b + c) - a\}\ \{(b + c) + a\}$$

The parentheses may now be removed. The expression appears, finally, as

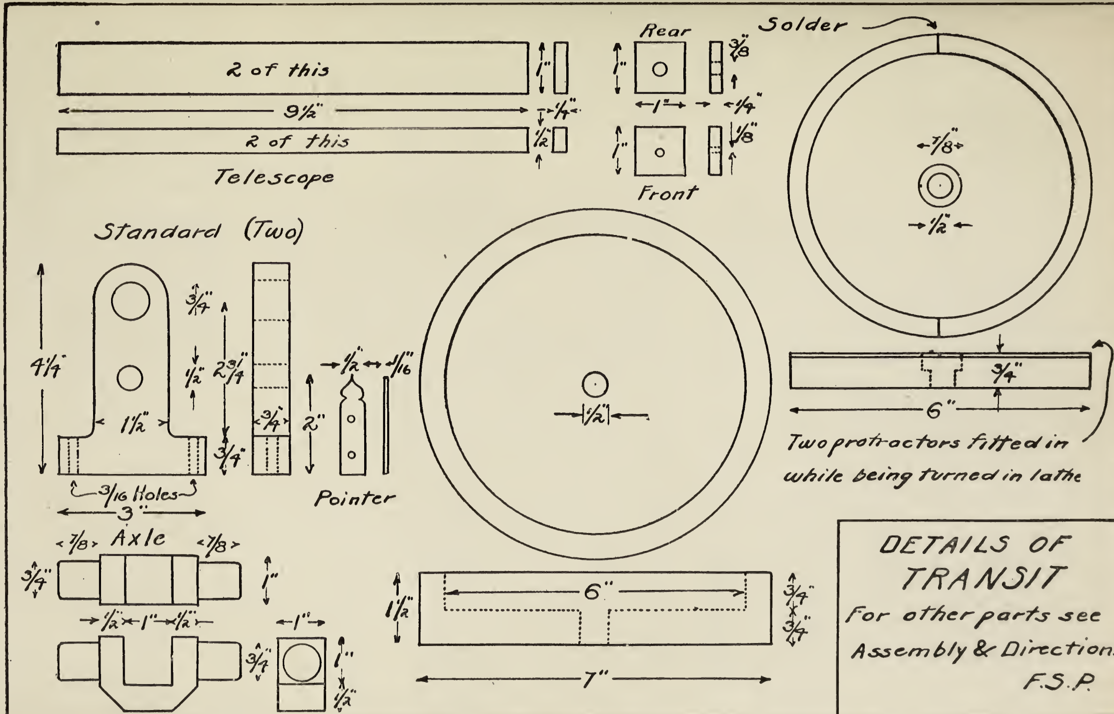
$$\{a - c + b\}\ \{a + c - b\}\ \{b + c - a\}\ \{b + c + a\}$$

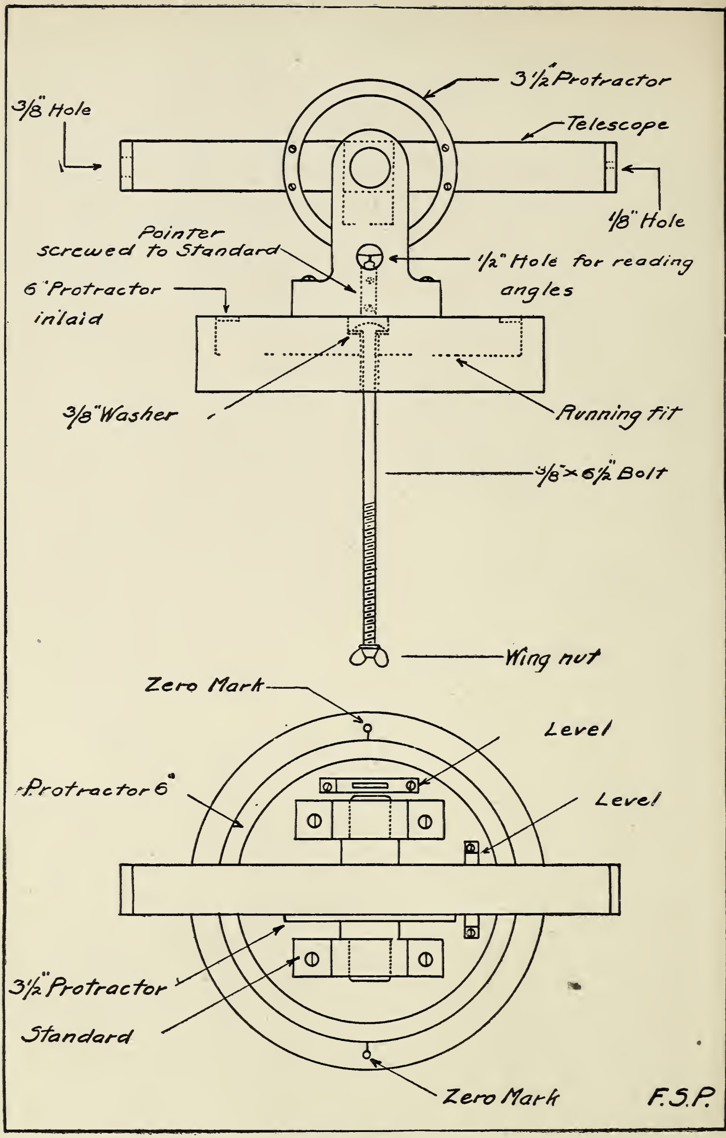
X. Shop Exercise

TRANSIT

Begin the construction of the **Transit,** as shown by the accompanying diagrams, which are self-explanatory. Instead of marking the graduations for the horizontal and vertical circles directly on the wood, it will be found more satisfactory to use paper protractors, which are listed at very moderate cost in the catalogs of many dealers in engineering supplies. One of these will be necessary for the vertical circle, and two for the horizontal.

Notice that no tripod (three-legged base) is shown in this set of drawings—since the farm level has a removable top and the base can be used with this transit.


2 of this
9½"
2 of this
Telescope
Rear
Front
Solder
Standard (Two)
3/16 Holes
Pointer
Axle
Two protractors fitted in while being turned in lathe
DETAILS OF TRANSIT
For other parts see Assembly & Directions
F.S.P.


3½" Protractor
3/8" Hole
Telescope
1/8" Hole
Pointer screwed to Standard
½" Hole for reading angles
6" Protractor inlaid
3/8" Washer
Running fit
3/8" × 6½" Bolt
Wing nut
Zero Mark
Level
Protractor 6"
Level
3½" Protractor
Standard
Zero Mark
F.S.P.

TRANSIT

CHAPTER XI

THE RIGHT TRIANGLE

143a. The **Complement** of an angle is that angle which must be added to it to make their sum equal to ninety degrees or a right angle. Hence,

143b. Rule: *To find the complement of an angle subtract it from 90 degrees.*

143c. Two angles whose sum is equal to a right angle are called **Complementary Angles.**

143d. *The complements of the same or equal angles are equal* (32b).

Exercise

Find the complements of the following angles. (Hint: Remember that if necessary 90° may be written as 89° 60′ or 89° 59′ 60″. See examples 6 and 13, Chapter II.)

1. 60°	6. 82°	11. 41° 30″
2. 45°	7. 11° 40″	12. 48° 10″ 21′
3. 47°	8. 29° 34″	13. 52° 15″ 28′
4. 59°	9. 32° 48″	14. 3° 0″ 42′
5. 64°	10. 37° 35″	15. 72° 59″ 7′

144a. If one angle of a triangle is a right angle, the triangle is called a **right triangle.**

The symbol for right triangle is rt. △.

144b. The side of right triangle which is opposite to the right angle is called the **hypotenuse.**

144c. The sides of a right triangle adjacent to the right angle are called the **legs.**

145a. An **acute angle** is an angle less than a right angle.

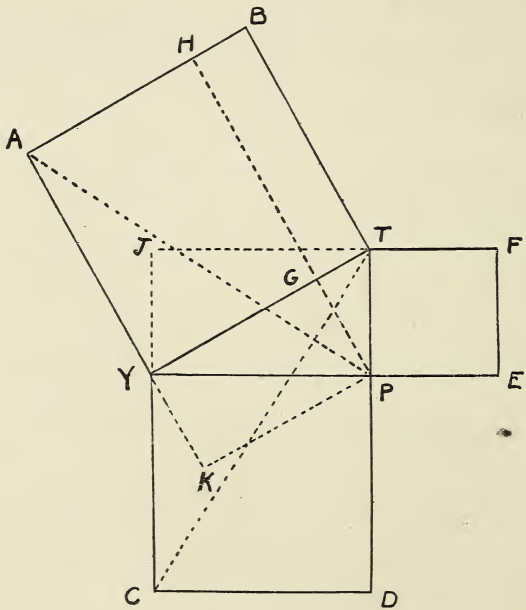
146b. An **obtuse angle** is an angle greater than a right angle.

147a. Theorem: *A triangle cannot contain more than one right angle.*

Proof: Since the sum of the *three* angles of a triangle equals *two* right angles (80a), it is impossible that two angles of a triangle should be right angles, since that would leave nothing for the third angle. *Q. E. D.*

147b. Theorem: *A triangle cannot contain more than one obtuse angle.*

148a. Theorem: *The area of the square erected on the hypotenuse of any right triangle is equal to the sum of the areas of the squares erected on the legs.*



Given: In the right triangle PYT, YT is the hypotenuse, and YP and PT are the legs. YB is the square erected on YT, and PF and YD are the squares erected on PT and YP respectively.

To Prove: Area YB = area PF + area YD.

Proof: Draw $PH \perp$ to AB, meeting YT at G.

Part I

Draw AP and CT.

To show (a) the area of $YAHG = 2$ area $\triangle AYP$;

(b) area $CYPD = 2$ area $\triangle CYT$;

(c) $\triangle AYP = \triangle CYT$,

from whence it follows that

(d) area $CYPD$ = area $AHGY$ (34a).

(a) Produce CY and draw $TJ \perp$ to CY produced.

YPT is a rt. $\angle$ (given).

Also YPD is a rt. $\angle$ (121c).

$\therefore DPT$, the sum of YPD and YPT, is a st. $\angle$ (15).

$\therefore DPT$ is a straight line (40).

That is, since CJ and DT are CY and DP produced, CJ is // to DT (8g; 121c).

Now PY is $\perp$ to CJ (121c).

And TJ is $\perp$ to CJ (const.).

$\therefore PY$ is // to TJ (60a).

$\therefore PY = TJ$ (122c).

Now area $CYPD = CY \times YP$ (131).

And area $CYT = \frac{1}{2}CY \times TJ$ (133a).

Substituting for TJ its equal, YP,

Area $CYT = \frac{1}{2}CY \times YP$.

That is (since area $CYPD = CY \times YP$), area $CYPD = 2$ area CYT.

(b) Produce AY, and draw $PK \perp$ to AY produced.

By construction, $AHGY$ is a rectangle (62, 121b).

$\therefore YG$ and KP are both $\perp$ to AY.

$\therefore YG$ and KP are // (60a).

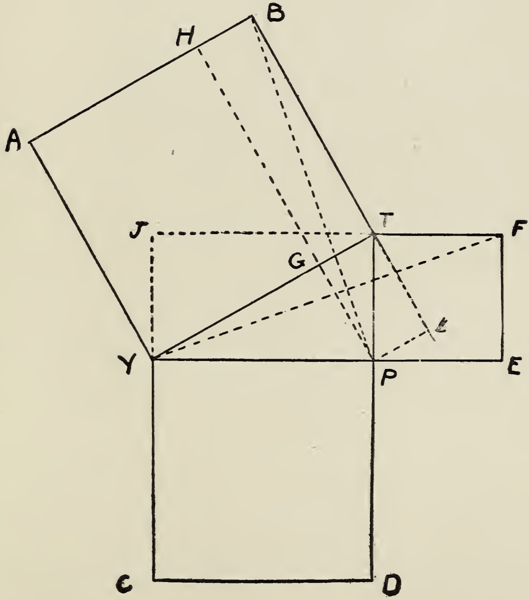
Also KA and PH are // (60a).
$\therefore$ $YG = KP$ (122c).
Area $AHGY = AY \times YG$ (131).
And area $AYP = \frac{1}{2}(AY \times KP)$ (133a).
Substituting for KP its equal YG,
Area $AYP = \frac{1}{2}(AY \times YG)$.
That is, since area $AHGY = AY \times YG$,
Area of rectangle $AHGY = 2$ area of $\triangle$ AYP.
(c) In $\triangle$s AYP and CYT,
$AY = YT$ (121c).
$CY = YP$ (121c).
Moreover, $\angle$ $CYT = 90° + \angle$ PYT.
And $\angle$ $AYP = 90° + \angle$ PYT.
$\therefore$ $\angle$ $CYT = \angle$ AYP (34b).
$\triangle$ $CYT = \triangle$ AYP (76).
Then, by (a) and (b).
(d) Area $AHGY =$ area $CYPD$ (34a).

NOTE: To avoid confusing the student by the great number of lines used in the proof a separate figure is given for Part II, but after learning the theorem the student should be able to prove all of it from one figure. Each step in the proof of Part II corresponds to a step in Part I.

Part II

Draw PB and YF.
To show (a) area $GHBT = 2$ area $\triangle$ PTB;
(b) area $PTFE = 2$ area $\triangle$ YTF;
(c) $\triangle$ $PTB = \triangle$ YTF,
from whence it follows that
(d) area $GHBT =$ area $PTFE$.
(a) $\angle$ YPT is a rt. $\angle$ (given).
$\angle$ TPE is a rt. $\angle$ (121c).
$\therefore$ Their sum $\angle$ YPE is a st. $\angle$ (15).
$\therefore$ YPE is a st. line (40).
Again, since JT is $\perp$ to CJ it is $\perp$ to DT (62).
That is, JTD is a rt. $\angle$ (12a)

and $\angle PTF$ is a rt. $\angle$ (121c).
$\therefore$ Their sum, $\angle JTF$ is a st. $\angle$ (15).
$\therefore$ JTF is a st. line (40).
YPE and JTF are // (8g, 121c)
and CJ and DT are // (Part I).
$\therefore$ $YJ = PT$ (122c).
Now area $PTFE = PT \times TF$ (131).
And area $YTF = \frac{1}{2}(YJ \times TF) = \frac{1}{2}(PT \times TF)$. (133a.)
That is, since area $PTFE = PT \times TF$, area square $PTFE = 2$ area $\triangle YTF$.



(b) Draw $PL \perp$ to BT produced.
$GHBT$ is a rectangle (62, 121b).
Then GT also is $\perp$ to BL.

$\therefore GT$ is // to PL (60a).
Furthermore PH and BL are both $\perp$ to AB.
$\therefore PH$ is // to BL (60a).
$\therefore GT = PL$ (122c).
Now the area $GHBT = GT \times TB$ (131).
And area $PTB = \frac{1}{2}(PL \times TB)$ (133a).
Substituting for PL its equal GT,
Area $PTB = \frac{1}{2}(GT \times TB)$.
That is, since area $GHBT = GT \times TB$,
Area rect. $GHBT = 2$ area $\triangle PTB$.
(c) Now in $\triangle$s PTB and YTF

$$PT = TF \qquad (121c)$$
$$YT = TB \qquad (121c)$$
$$\angle PTB = 90° + \angle PTY.$$
$$\angle YTF = 90° + \angle PTY.$$
$$\therefore \quad \angle PTB = \angle YTF \qquad (34b)$$
$$\therefore \quad \triangle PTB = \triangle YTF \qquad (76)$$

Then, by (a) and (b),
(d) Area $GHBT$ = area $PTFE$.

Part III

I. Area $YABT$ = area $AHGY$ + area $GHBT$ (32a).
But area $AHGY$ = area $CYPD$ (Part I).
And area $GHBT$ = area $PTFE$ (Part II).
Substituting these values in the above equation (I),
Area $YABT$ = area $CYPD$ + area $PTFE$. Q. E. D.

The fact that the square of the hypotenuse of a right triangle equals the sum of the squares of the two legs was known thousands of years ago in Egypt, where it was used in laying out the corners of buildings. The modern carpenter does practically the same thing when he lays out the sills of a house (the beams which rest upon the masonry foundation), by measuring along an end beam 6 feet from one corner, 8 feet from the same corner along the side beam and then laying a strip 10 feet long, so that one end coincides with the 6 foot mark, while he moves the other beam until the 8 foot mark coincides with the end of the 10 foot strip —thereby laying out the "6, 8, 10 triangle," and so making the angle between the beams a right angle.

Although this property of the right triangle has been known from remote antiquity, no general proof of it appears to have been given until the following theorem (148a) was demonstrated by Pythagoras, in whose honor it was named.

Pythagoras was a famous philosopher and mathematician who was born in the Grecian island of Samos about 582 B.C. and died in Magna Græcia about 500 B.C.

Pythagoras studied at Miletus under Thales. A generally credited tradition states that he spent several years in study in Egypt, and journeyed to Babylon and even to India. (The nature of his work in mathematics corroborates this.) Pythagoras thus learned all that was then known of mathematics. In later life he founded a famous school at Crotona in Southern Italy. He established a secret society, the members of which were known as the Pythagoreans. The Pythagoreans did much for the advancement of knowledge.

As the student knows from Arithmetic, the sign $\sqrt{}$ before a quantity means that the square root of the quantity is to be found. This sign is read "the square root of."

Thus the equation,

$$a = \sqrt{c^2 - b^2}$$

is read, "a equals the square root of c square minus b square."

The equation,

$$3 = \sqrt{9}$$

means that 3 is one of two equal factors whose product is 9.

Solve:

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
1.	$\sqrt{25}$	$\sqrt{64}$	$\sqrt{100}$	$\sqrt{144}$	$\sqrt{169}$	$\sqrt{400}$	$\sqrt{196}$

2. Carry out to two decimal places by the method learned in arithmetic:

(a)	(b)
$\sqrt{740}$	$\sqrt{1982}$

For convenience in computation the property of the right triangle proved in the Pythagorean theorem may be stated as follows:

148b. *The hypotenuse of a right triangle equals the square root of the sum of the squares of the two legs* (134a).

Letting c represent the hypotenuse, a one leg, and b the other leg of a right triangle, the Pythagorean theorem might also be expressed as

I. $c^2 = a^2 + b^2$

Whence, $c = \sqrt{a^2 + b^2}$ (134b)

From (I) by transposition, $c^2 - a^2 = b^2$,

Whence, $\sqrt{c^2 - a^2} = b$ (134b)

Or, $b = \sqrt{c^2 - a^2}$

By a similar derivation, $a = \sqrt{c^2 - b^2}$

That is,

148c. Rule: *Either leg of a right triangle equals the square root of the difference of the squares of the hypotenuse and the other leg.*

Exercise

1. One leg of a triangle is 10 feet long and the other is 18 feet long. Find the length of the hypotenuse.

2. The hypotenuse of a right triangle is 20 feet long and the base is 12 feet long. Find the altitude. (Remember that the altitude of a right triangle coincides with one leg.)

3. How long a ladder is required to reach to the top of a wall 22 feet high if the lower end of the ladder is to be placed 9 feet from the wall?

4. A flag pole is to be secured by three cables passed through holes in an iron collar 24 feet from the ground. The lower ends of these cables are passed through rings fastened in concrete bases at the vertices of an (imaginary) equilateral triangle. If each of these rings is 10 feet from the flag pole, how many feet of cable must be ordered, allowing 2 feet on each cable for fastening and slack of cable?

NOTE: The eaves of a house is the line where the roof joins the side walls.

The rafters are the sloping timbers to which the roof boards are nailed.

The timber which runs along the highest part of the framework of the roof, to which on either side are nailed the upper ends of the rafters, is called the ridge pole.

5. The height of a house from the foundation to the eaves is 18′ 6″ and the height from the ridge pole to the foundation is 26′ 10″. The width of the house is 21′. Find the length of the rafter. Adding one foot to each rafter for waste in cutting find how many linear feet must be ordered for 13 pairs of rafters.

6. It is desired to make a tent by putting canvas over a level ridge pole 10 feet long and 7 feet above the ground, the tent to be 8 feet wide at the bottom, the ends of the tent left open. How many square feet of canvas must be ordered for the tent?

As the student already knows, the symbol $\sqrt{\ }$ may be applied to quantities which are not perfect squares—as, $\sqrt{5}$.

148d. The expression for finding the square root of a quantity which is not a perfect square is called a **quadratic surd.** Because there is no number, which multiplied by itself will give 5—in other words, since 5 has no *exact* square root, it does *not* follow that $\sqrt{5}$ has no definite meaning.

Thus,

Problem: Draw a line $\sqrt{5}$ inches in length.

Solution: Draw a rt. $\triangle$, one leg of which shall be 1 inch in length, the other 2 inches. Then (by 148b),

$$\begin{aligned} \text{Hypotenuse} &= \sqrt{1^2 + 2^2} \\ \text{"} \quad &= \sqrt{1 + 4} \\ \text{"} \quad &= \sqrt{5} \end{aligned}$$

The length of the hypotenuse, then, is $\sqrt{5}$ inches, and by setting the points of the dividers at the two extremities of this hypotenuse, its length, $\sqrt{5}$, may be transferred to any diagram desired. It is evident from this that $\sqrt{5}$ has a definite meaning and is capable of exact representation.

Problem: Draw a line $\sqrt{3}$ inches long.

Solution: Draw a rt. $\angle$. On one of the sides lay off a point 1 inch from the vertex, and with this point as a center and a radius of 2 inches, describe an arc intersecting the other leg. This point of intersection determines the length of the second leg. Draw the hypotenuse. By construction, one leg of the triangle is 1 inch in length and the hypotenuse is 2 inches.

$\therefore$ the second leg $= \sqrt{4 - 1} = \sqrt{3}$

7. Draw lines having the following lengths in inches:

(a) $\sqrt{7}$	Hint: $\sqrt{4^2 - 3^2}$	(b)	$\sqrt{10}$
(c) $\sqrt{15}$		(d)	$\sqrt{17}$
(e) $\sqrt{19}$	Hint: $\sqrt{10^2 - 9^2}$	(f)	$\sqrt{21}$
(g) $\sqrt{34}$		(h)	$\sqrt{11}$

NOTE: In this chapter a quadratic surd will be called "a surd."

Many surds may be easily represented in two ways: thus,

$$\sqrt{13} = \sqrt{2^2 + 3^2} \quad \text{or} \quad \sqrt{7^2 - 6^2}$$

Problem: Draw a line $3\sqrt{5}$ inches long.

Solution: Draw an indefinite straight line. Construct a line $\sqrt{5}$ inches long. Take off this length between the points of the dividers, and lay it off 3 times along the indefinite straight line: obviously the length so determined will be $3\sqrt{5}$ inches.

8. Draw a line:

(a)	(b)	(c)
$2\sqrt{10}$ in.	$3\sqrt{17}$ in.	$4\sqrt{7}$ in.

It is often possible to simplify surds: thus,

Problem: Simplify $\sqrt{20}$.

Solution: $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.

Proof: Construct a line $\sqrt{20}$ inches long by drawing one leg of a rt. $\triangle$ 2 inches long and another leg 4 inches long. The hypotenuse will be $\sqrt{20}$ inches in length. Construct a line $\sqrt{5}$ inches long, and taking this length between the points of the dividers lay it off along the line $\sqrt{20}$ inches long. It will thus be found that the $\sqrt{5}$ will go just twice into $\sqrt{20}$. The following problems may be proved in the same way.

The square root sign (as well as the cube root sign, etc.) is often called the **radical sign.**

148e. Rule: *Surds may be reduced if it is possible to separate the quantity under the radical sign into factors, one or more of which are perfect squares. The square root of these perfect squares may be extracted and their product written outside the radical sign as the coefficient of the surd.*

Problem: Simplify $\sqrt{180a^3b}$

Solution: $\sqrt{180a^3b} = \sqrt{9 \times 4 \times 5 \times a^2 \times a \times b} =$

$$3 \times 2a\sqrt{5ab} = 6a\sqrt{5ab}$$

9. Simplify:

(a)	(b)	(c)	(d)	(e)
$\sqrt{75}$	$\sqrt{x^3}$	$\sqrt{4a^2x}$	$\sqrt{(a+b)^3}$	$\sqrt{2(a^2 - 2ab + b^2)}$

(f)	(g)	(h)	(i)
$\sqrt{27}$	$\sqrt{48}$	$\sqrt{162}$	$\sqrt{4(a+b)^2(a-b)^2c}$

Problem: Simplify $\sqrt{\dfrac{8b^2}{9}}$

Solution:

$$\sqrt{\frac{8b^2}{9}} = \sqrt{\frac{4b^2 \times 2}{9}} = \frac{2b}{3}\sqrt{2}$$

	(a)	(b)	(c)
10.	$\sqrt{\dfrac{36c^2d^2}{4}}$	$\sqrt{\dfrac{75x^2}{9}}$	$\sqrt{\dfrac{3}{4}(a+b)}$
11.	$\sqrt{\dfrac{49c^3}{9}}$	$\sqrt{\dfrac{81b^2c^3}{64}}$	$\sqrt{\dfrac{100(m+n)^2}{36}}$
12.	$\sqrt{\dfrac{8}{a^2}}$	$\sqrt{\dfrac{36b^3}{c^2}}$	$\sqrt{\dfrac{64(a+b)}{a^2}}$

In case the denominator of the fraction under the radical is not a perfect square it may be made so by multiplying both terms of the fraction by the same number: thus,

Problem: Simplify $\sqrt{\dfrac{2}{3}}$

Solution:

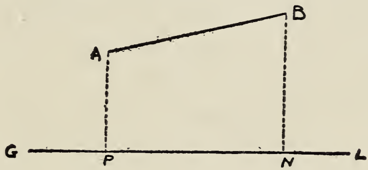
$$\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9} \times 6} = \frac{1}{3}\sqrt{6}$$

13. Simplify: (a) $\sqrt{\frac{1}{7}}$ (b) $\sqrt{\frac{2}{5}}$ (c) $\sqrt{\frac{1}{10}}$

14. (a) $\sqrt{\frac{8c}{3}}$ (b) $\sqrt{\frac{5b^2}{2}}$ (c) $\sqrt{\frac{d^2}{5}}$

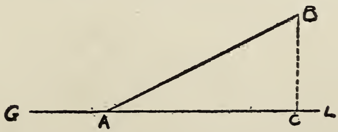
15. (a) $\sqrt{\frac{a+b}{c}}$ (b) $\sqrt{\frac{3(a-b)}{5}}$ (c) $\sqrt{\frac{7a}{(a-b)^2}}$

149. The **projection** of one line on another is that length on the second (produced if necessary) included between perpendiculars let fall upon it from the ends of the first.



Thus, the projection of AB on GL is the length of PN, the segment of GL included between perpendiculars let fall upon GL from A and from B.

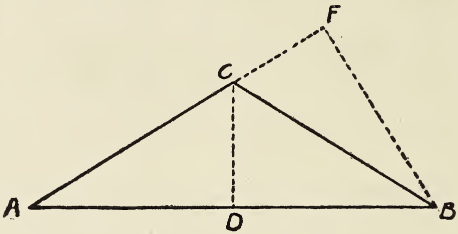
When one end of a line meets the line on which it is to be projected, the same definition applies if we consider one of the perpendiculars as of length zero.



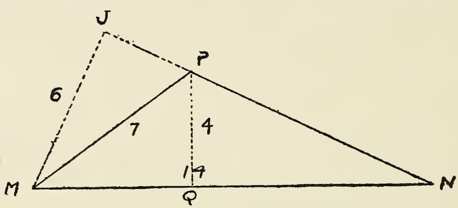
Thus, the projection of AB on GL is the length AC, the perpendicular from A to GL being of zero length.

Exercise

1. Draw a triangle with sides 5, $4\frac{1}{4}$, and 3 inches long, respectively, and find the projection of each side upon the other (produced if necessary) by drawing the perpendiculars and measuring the projection with the ruler.



2. In the triangle ABC, find the length of the projection of side AC on side AB, of side AB on side AC, given that: $AB = 16$, $BF = 8$, $AC = 9\frac{1}{2}$, $DC = 5$ (148c).



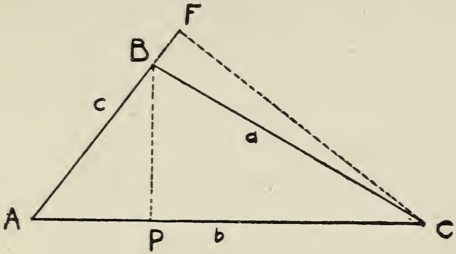
3. In the triangle MNP, find the length of the projection of the side MP on side MN, and side MP on side NP, given that $MN = 14$, $QP = 4$, $MP = 7$, $MJ = 6$.

150a. An **Oblique Triangle** is one that does not contain a right angle.

150b. Theorem: *In any oblique triangle the square of the side opposite an acute angle is equal to the sum of the squares of the two sides adjacent to the acute angle minus twice the product of either one of these two sides by the projection of the other upon that side.*

CASE I

(When the projection of the side opposite the acute angle falls upon the base.)



Given: In $\triangle ABC$, $\angle A$ is acute and a is side opposite $\angle A$; b and c are the sides opposite $\angle B$ and $\angle C$, respectively; AP the projection of c on b.

Part I

To Prove: $a^2 = b^2 + c^2 - 2\overline{AP}(b)$

NOTE: The bar is used over AP to mean "the line AP."

Proof: $\overline{PC} = b - \overline{AP}$.

Squaring both sides of this equation:

$\overline{PC}^2 = b^2 - 2\overline{AP}\ (b) + \overline{AP}^2$

Adding $\overline{BP}^2$ to both sides of this equation:

(I) $\overline{BP}^2 + \overline{PC}^2 = b^2 - 2\overline{AP}\ (b) + \overline{AP}^2 + \overline{BP}^2$

But $\overline{PB}^2 + \overline{PC}^2 = a^2$ (148a).

$\overline{AP}^2 + \overline{BP}^2 = c^2$ (148a).

Substituting these in (I)

$a^2 = b^2 - 2\overline{AP}\ (b) + c^2$

That is,

$a^2 = b^2 + c^2 - 2\overline{AP}\ (b)$.

Part II

NOTE: Since the theorem makes the statement concerning "*either one* of the two sides adjacent to the acute angle," it is necessary to prove that it holds true for both.

Given: AF is the projection of b on c in above triangle.

To Prove: $a^2 = b^2 + c^2 - 2\overline{AF}\,(c)$

Proof: $\overline{BF} = \overline{AF} - c$

Squaring both sides of this equation:

$\overline{BF}^2 = \overline{AF}^2 - 2\overline{AF}\,(c) + c^2$

Adding $\overline{FC}^2$ to both sides of this equation:

(II) $\overline{BF}^2 + \overline{FC}^2 = \overline{AF}^2 + \overline{FC}^2 - 2\overline{AF}\,(c) + c^2$

But $\overline{BF}^2 + \overline{FC}^2 = a^2$ (148a)

and $\overline{AF}^2 + \overline{FC}^2 = b^2$

Substituting in (II)

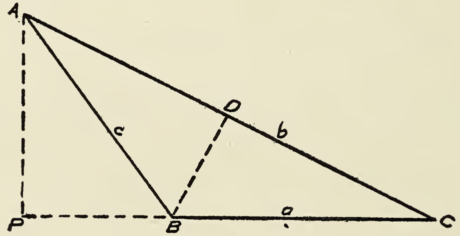
$a^2 = b^2 - 2\overline{AF}\,(c) + c^2$

That is,

$a^2 = b^2 + c^2 - 2\overline{AF}\,(c)$

CASE II

When the projection of the side opposite the acute angle falls upon the base produced.



Part I

Given: In $\triangle ABC$, $\angle C$ is acute, $\angle B$ is obtuse; side c is opposite $\angle C$, side b is opposite $\angle B$ and side a is opposite $\angle A$; $\overline{CP}$ is the projection of b on a.

To Prove:

$c^2 = a^2 + b^2 - 2a\,(\overline{PC})$

Proof:

$\overline{PB} = \overline{PC} - a$

Squaring both sides:

$\overline{PB}^2 = \overline{PC}^2 - 2a\,(\overline{PC}) + a^2$

Adding $\overline{AP}^2$ to both sides of this equation

$\overline{AP}^2 + \overline{PB}^2 = \overline{AP}^2 + \overline{PC}^2 - 2\,(a)\,\overline{PC} + a^2$

The remainder of this proof is left to the student.

BD is the perpendicular needed in Part II.

Exercise

In the following exercises, let the student draw the figures carefully to scale.

1. In $\triangle\ ABC$, find the length of a (opposite the acute $\angle\ A$), given c (or AB) $= 12$; b (or AC) $= 8'$ and AD (the projection of c on b) $= 10\ 1/2'$.

Check. The same answer should be obtained by using the projection of b on c. AE (the projection of b on c) $= 7$.

150c. Theorem: *In any oblique triangle, the square of the side opposite an obtuse angle is equal to the sum of the squares of the two sides including the acute angle, plus twice the product of either one of these including sides by the projection of the other upon that side.*

Given: In $\triangle\ ABC$, $\angle\ A$ is obtuse and a is side opposite $\angle\ A$; b and c are sides opposite $\measuredangle\ C$ and B respectively: AD the projection of c on b.

To Prove: $a^2 = b^2 + c^2 + 2b\,(AD)$

Proof: $CD = b + AD$.

Squaring both sides of this equation, etc.

The construction of the figure according to the above directions and the remainder of the proof of this theorem (which is similar to that of the preceding theorem), is left for the student.

2. In $\triangle MNQ$ find m, given that $q = 32$, $n = 42$, and MS, (the projection of n on q) $= 33$.

Check: Practically the same answer should be obtained by using MR (the projection of q on n) $= 25.14$.

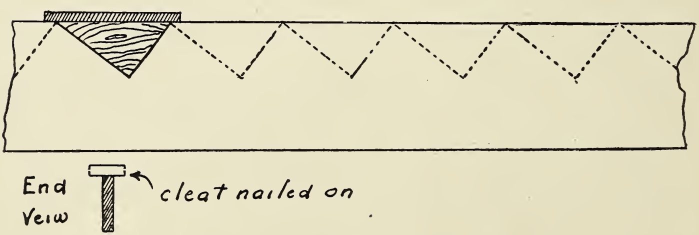
XIa. Shop Exercise

STAIRS

Definitions: The **treads** are the level boards on which the foot rests when ascending the stairs. The **risers** are the boards perpendicular to the treads. The **stringers** are the sloping supports to which the treads and risers are nailed.

Procedure: Plane one edge of a piece of white pine and lay off 15 in. along the edge. With one extremity of this line as a center and a radius of 9 in. describe an arc, and with the other extremity as a center and a radius of 12 in. describe a second arc intersecting the first. From this point of intersection draw lines to the extremity of the 15-in. line, thereby forming a right triangle. (Why?) Cut out the board along this line.

Make a strip of wood an inch wider than the thickness of the triangle and 17 in. long, and nail it along the 15-inch edge of the triangle to serve as a guide.



Lay this triangle on the board which is to be the stringer (which should be at least 9 in. wide) so that the strip is flush with the edge, and mark along the other two sides. These pencil marks, with the edge of the board, form a right triangle. Then slide the pattern along and mark another triangle, just touching the first. Continue this process until six of these triangles have been laid off.

The base of the first triangle should be produced to meet the other side of the stringer to form the bottom line. The altitude of the last triangle should be produced to meet the edge of the stringer and form the top line.

Evidently when the stringers are in place, and the lower tread-board, 1 in. thick, is nailed on, the height of its top above the floor will be 10 in. This thickness is subtracted from the height of the second riser, but when the second riser is put in place this adds the same amount as was subtracted, so the second riser is still 9 in. in height. This will be true of all the remaining risers; only the first will be 10 in. in height. To remedy this, lay off a line parallel to the bottom line and 1 in. (the thickness of the tread-board) above it. Cut on this line.

When the first riser board is nailed on, the width of the first tread is increased 1 in. When the second riser-board is nailed on, the same amount is subtracted, so the width of the first tread remains the same. This is true of all the treads up to the last. When the last riser is nailed on, however, the width of the last tread is thereby increased to 10 in. and there is no other riser to be subtracted from it. To remedy this, draw a line on the stringer parallel to the top cut, and 1 in. (the thickness of the riser) within it. Cut along this line.

Cut out the triangular pieces as marked, and the stringer is complete. The remaining stringer may be marked with this as a model. See XIIa.

XIb. Instead of making the above exercise full size, a model may be made by dividing all the above dimensions by the same number: for example, instead of laying out a triangle whose sides are 9, 12, and 15 inches, one may be laid out whose sides are 3, 4, and 5 inches long.

CHAPTER XII

LITERAL FRACTIONS: HERON'S FORMULA

151. Literal Expressions may not only take the form of Integers, but of Fractions, as well.

The terms learned in Arithmetic (Numerator, Denominator, Common Denominator, etc.) apply equally as well to literal fractions, and all the operations with fractions learned in Arithmetic may equally as well be performed with literal fractions.

Exercise

Reduce the following fractions to lowest terms:

1. $\dfrac{36x^4y^3z^2}{24x^2yz^3}$

Solution:

$$\frac{36x^4y^3z^2}{24x^2yz^3} = \frac{3\cdot 3\cdot 2\cdot 2\cdot x^2\cdot x^2\cdot y\cdot y^2\cdot z^2}{3\cdot 2\cdot 2\cdot 2\cdot x^2\cdot y\cdot z^2\cdot z} = \frac{3x^2y^2}{2z}$$

NOTE: A dot placed up midway of the letter so that it does not look like a decimal point indicates multiplication.

Factors common to both terms are cancelled as in Arithmetic. In the solution, instead of writing the x^4 as $x\cdot x\cdot x\cdot x\cdot$, it is better to write the highest power of x to be found in both terms as one factor, that is, to write the x term in the numerator as $x^2\cdot x^2$, for a glance at the denominator shows that there is an x^2 there to cancel it.

Likewise the y term of the numerator is separated into $y\cdot y^2$, since there is a y in the denominator, while the z

term is written as z^2, since there are sufficient z factors in the denominator to cancel it.

2. $\frac{12a^3b^2}{36ab}$ 3. $\frac{7x^3y^2}{14xy^2z}$ 4. $\frac{16m^2np^3}{20mn^2p}$

5. $\frac{120a^3b^2c}{75ab^3c^2}$ 6. $\frac{50x^2yz^3}{15x^3y^2z^3}$ 7. $\frac{39ax^2}{13ax^2}$

8. $\frac{88x^2y^3z^3}{66x^3yz^2}$ 9. $\frac{63a^3b^4c^4}{84a^4b^4c^2}$ 10. $\frac{54a^2b}{99a^2b^3c}$

Problem: Reduce $\frac{a^2 + 2ab}{ab}$

Solution:

$$\frac{a^2 + 2ab}{ab} = \frac{a\,(a + 2b)}{a \cdot b} = \frac{a + 2b}{b}$$

11. $\frac{b^2 + 4ab}{b^2}$ 12. $\frac{3x^2 + 6ax}{12ax}$

HINT: Do not make the mistake of considering ax as factor of both numerator and denominator in Example 12.

Problem: Reduce $\frac{a^2 - 2ab + b^2}{a^2 - b^2}$

Solution:

$$\frac{a^2 - 2ab + b^2}{a^2 - b^2} = \frac{(a - b)\,(a - b)}{(a - b)\,(a + b)} = \frac{a - b}{a + b}$$

13. $\frac{x^2 + 2xy + y^2}{x^2 - y^2}$ 14. $\frac{x^2 + 4xy + 4y^2}{x^2 - 4y^2}$

15. $\frac{a^2x^2 + 2abxy + b^2y^2}{ax + by}$

Least Common Multiple

152a. It was learned in Arithmetic that in order to add fractions it is necessary to change them to a Common Denominator and in order to find the Least Common Denominator of several fractions it is necessary to find the Lowest Common Multiple, and so was learned a method for finding the Least Common Multiple of any given number. For the same reason, it is necessary to find the Least Common Multiple of given literal expressions. The method for finding the Least Common Multiple of given literal expressions is fundamentally the same as that in Arithmetic, but is shortened somewhat by the aid of exponents, which save the work of repeating a factor. Thus, in Arithmetic, the factors of 8 would be written $2 \times 2 \times 2$, which by the aid of exponents may be written 2^3.

Problem: Find the Least Common Multiple of:

$$15a^2bc,\ 20a^3b^2c,\ 30abc^2$$

Solution:

$$\begin{aligned} 15a^2bc &= 3\cdot 5\cdot a\cdot a\cdot b\cdot c \\ 20a^3b^2c &= 2\cdot 2\cdot 5\cdot a\cdot a\cdot a\cdot b\cdot b\cdot c \\ 30abc^2 &= 3\cdot 5\cdot 2\cdot a\cdot b\cdot c\cdot c\cdot \end{aligned}$$

As in Arithmetic, take each factor the greatest number of times it occurs in any one of the three quantities to be factored.

$$3\cdot 2\cdot 2\cdot 5\cdot a\cdot a\cdot a\cdot b\cdot b\cdot c\cdot c\cdot = 60\,a^3b^2c^2,\ \text{L. C. M.}$$

By aid of exponents, we may avoid repeating factors—that is, separating a^3 into $a\cdot a\cdot a\cdot$, b^2 into $b\cdot b$, etc., by aid of the following rule:

152b. *To find the Least Common Multiple of given literal quantities find the Least Common Multiple of their numerical coefficients and then take each literal factor with the highest exponent that it is contained in any of the given literal quantities.*

Thus, in the above example, we find the Least Common Multiple of 15, 20, and 30 (which is 60) then write a with 3 as an exponent, since a^3 is the highest power of a found in the given quantities (that is, the a factor which has the largest exponent), then write b with 2 as an exponent (since b^2 is highest power of b found in the given quantities) and finally write c with 2 as an exponent (since c^2 is the highest power of c found in the given quantities) making the final answer $60a^3b^2c^2$. The process is the same when the quantities contain Binomial or Polynomial Factors.

Exercise

Find the Least Common Multiple of the following:

1. $7x^2y,\ 3x^3y^2x,\ 2xy^2z^2$
2. $4a^2bc,\ 8a^3b^2c,\ 12ab^2c^3,\ 16a^3$
3. $6m^3,\ 15n^3,\ 18p^3$
4. $24a^3h^3,\ 40ah,\ 50a^2h$
5. $32abc,\ 16a^2h,\ 8b^2c^3$
6. $3m^3,\ 9n,\ 18p^2,\ 24mn^2p$
7. $5x^3y^3z^2,\ 15zy^3z,\ 25x^2y^2z$
8. $6\,(a+b),\ 4\,(a-b)^2,\ 2\,(a^2-b^2)$

Solution: The L. C. M. of the numerical quantities is 12. We find that $(a+b)$ occurs only once in any quantity, and $(a-b)$ occurs twice as a factor: The L. C. M. is therefore $12\,(a+b)\,(a-b)^2$. Note that the exponent in this expression applies only to the factor which it follows—that is, to $(a-b)$ and not to $(a+b)$.

9. $3\,(x-y),\ 2\,(x^2-y^2),\ 5\,(x+y)$
10. $4\,(a-b),\ 2\,(a+b),\ 3\,(a^2+2ab+b^2)$
11. $5\,(x-2y),\ 10\,(x+2y),\ 15\,(x^2-4y^2)$
12. $2a,\ 3\,(a-2b),\ 2\,(a+2b)$
13. $a^2-4b^2,\ 3\,(n-b),\ 5\,(n^2-b^2),\ (n+b)$
14. $2\,(b^2+2bc+c^2),\ 3\,(b^2-2bc+c^2),\ 4\,(b^2-c^2)$

In Arithmetic it was learned that before fractions of different denomination can be added or subtracted they must be changed to the same denominator. For example:

Add: $$\frac{2}{3}+\frac{3}{4}+\frac{5}{6}+\frac{1}{2}$$

Solution:

$\frac{2}{3}$ can be changed to $\frac{8}{12}$, $\frac{3}{4}$ to $\frac{9}{12}$, etc., without change of value, since the value of a fraction is not altered if both numerator and denominator are multiplied by the same quantity —that is, $\frac{8}{12}=\frac{2\cdot 4\cdot}{3\cdot 4\cdot}=\frac{2}{3}$.

Hence, $\frac{8}{12}+\frac{9}{12}+\frac{10}{12}+\frac{6}{12}=\frac{33}{12}=\frac{11}{4}=2\frac{3}{4}$

This principle applies equally well to literal fractions.

Combine: $$\frac{2}{a}+\frac{3}{a^2}+\frac{1}{2}-\frac{1}{4a^2}$$

Solution:

By inspection we find that $4a^2$ is the L. C. D. and that a must be multiplied by $4a$ to make it equal $4a^2$. We therefore multiply both terms of $\frac{2}{a}$ by $4a$. Thus:

$$\frac{2}{a}=\frac{8a}{4a^2}$$

We find also that a^2, the denominator of $\frac{3}{a^2}$, the next fraction, should be multiplied by 4 to make it equal to $4a^2$. We therefore multiply both terms of $\frac{3}{a^2}$ by 4. Thus:

$$\frac{3}{a^2}=\frac{12}{4a^2}$$

We find, also, that 2, the denominator of the next fraction, should be multiplied by $2a^2$.

We therefore multiply both terms of $\frac{1}{2}$ by $2a^2$. Thus:

$$\frac{1}{2} = \frac{2a^2}{4a^2}$$

The last fraction $\frac{1}{4a^2}$, does not need change of form. The example now appears as follows:

$$\frac{2}{a} + \frac{3}{a^2} + \frac{1}{2} - \frac{1}{4a^2} = \frac{8a}{4a^2} + \frac{12}{4a^2} + \frac{2a^2}{4a^2} - \frac{1}{4a^2}$$

$$= \frac{8a + 12 + 2a^2 - 1}{4a^2} = \frac{2a^2 + 8a + 11}{4a^2}$$

Problem:

Combine: $$\frac{4x + 6}{8} + \frac{3x - 5}{6}$$

Solution:

$$\frac{4x + 6}{8} + \frac{3x - 5}{6}$$

$$= \frac{3(4x + 6)}{24} + \frac{4(3x - 5)}{24}$$

$$= \frac{12x + 18 + 12x - 20}{24} = \frac{24x - 2}{24}$$

$$= \frac{2(12x - 1)}{2 \cdot 12} = \frac{12x - 1}{12}$$

Exercise

Combine:

1. $\frac{2}{6b} + \frac{3}{2b^2} - \frac{2b}{3} + \frac{3}{2b}$

2. $\dfrac{1}{4} + \dfrac{h^2}{4c^2} + \dfrac{h}{c^2} - \dfrac{2h}{2c^2}$

3. $\dfrac{-3}{x^2} - \dfrac{2}{xy} + \dfrac{3}{xy^2}$

4. $\dfrac{2a}{b^2} + \dfrac{a^2}{12b^2} - \dfrac{1}{ab} + \dfrac{1}{3a^2}$

5. $\dfrac{3}{ab} - \dfrac{2b}{5a} + \dfrac{5a}{2b} - \dfrac{3ab}{10}$

6. $\dfrac{-3x^2}{4y^2} + \dfrac{x^3}{2y^3} - \dfrac{1}{x^3y^3} - \dfrac{2}{x} + \dfrac{3}{y}$

7. $\dfrac{3}{2ab} + \dfrac{3}{7a^2b} - \dfrac{b^2}{14a} + \dfrac{a^2}{b}$

8. $\dfrac{3}{m^2np} - \dfrac{4m}{np^2m} + \dfrac{m}{n^2p^2} + \dfrac{n}{p^2m} - \dfrac{2}{m^2np}$

9. $\dfrac{3}{x} - \dfrac{7}{x^2} + \dfrac{4}{2x} + \dfrac{5}{2x^2}$

10. $\dfrac{2}{xy^2} + \dfrac{3x}{2y^2} - \dfrac{3x}{2} + \dfrac{2}{xy}$

11. $\dfrac{4}{bc} - \dfrac{3b}{4c^2} - \dfrac{b}{c^2}$

12. $\dfrac{2x+9}{5} + \dfrac{6x-5}{15}$

13. $\dfrac{2x-3}{4} - \dfrac{3x+8}{6} + \dfrac{9x-4}{3}$

14. $\dfrac{3x-2}{4} + \dfrac{7x-8}{6} - \dfrac{5x+2}{15}$

15. $\dfrac{2(6x+5)}{3} + \dfrac{3(x+6)}{9} - \dfrac{4(3x-4)}{6}$

16. $\frac{2x+1}{4} - \frac{3x}{5} + \frac{4}{10} + \frac{x}{2}$

17. $\frac{3x+7}{4} + \frac{2(x-1)}{22} - \frac{8}{11}$

In Arithmetic we learned that in writing a mixed number, that is, an integer plus a fraction, it is unnecessary to write the plus sign between the whole number and the fraction—that is, $5 + \frac{2}{3}$ may be written as $5\frac{2}{3}$.

In literal expressions, however, *a whole number, which is to be combined with a fraction, must be connected with it by a plus or minus sign*—since to omit it would mean that the fraction was to be multiplied by the whole number.

In Arithmetic we learned that an integer (whole number) may be expressed as a fraction with a given denominator, by multiplying the integer by the denominator, and writing the product over the denominator.

Problem: Express 5 as a fraction whose denominator is 4.

Solution: $5 = \frac{20}{4}$

Proof: Dividing both denominator and numerator by 4 we obtain $\frac{5}{1}$ or 5.

The same method is used in dealing with literal expressions Thus:

Problem: Change a to a fraction whose denominator is b.

Solution: $a = \frac{ab}{b}$

Proof: Cancelling b in both terms of the last fraction, we have a.

Hence,

153. Rule: *Whole numbers and fractions may be combined by changing the whole numbers to fractions whose denominator is the common denominator, and then combining as when all the quantities were fractions.*

Combine:

18. $7 + \dfrac{4}{b}$

19. $3 + \dfrac{1}{b} + \dfrac{2}{a}$

20. $4 - \dfrac{2}{c} - \dfrac{3}{b} + \dfrac{2}{a}$

21. $b^2 - \dfrac{c^2 + a^2}{4c^2}$

Problem: $3 + \dfrac{4x}{x - 1}$

Solution: Since the bar (—) of a fraction serves as a parenthesis by causing the numerator to be treated as one quantity, and the denominator to be treated as one quantity, we may consider a parenthesis to be placed around $(x - 1)$ and then multiply 3 by it, putting the product over the denominator. Thus:

$$3 + \frac{4x}{(x-1)} = \frac{3\,(x-1)}{(x-1)} + \frac{4x}{(x-1)} = \frac{3x - 3 + 4x}{x - 1} = \frac{7x - 3}{x - 1}$$

22. $3 - \dfrac{x - 2}{x - 1}$

23. $5 + \dfrac{3x + 1}{x - 2}$

24. $3y + 4 + \dfrac{2}{5y + 1}$

25. $2x - 7 - \dfrac{5x}{x + 2}$

26. $2x + 1 + \dfrac{3x - 1}{x - 3}$

NOTE: The following will occur in the derivation of the formula for the area of a triangle called the **Heronian Formula:**

Problem:

Combine: $b^2 - \dfrac{(b^2 + c^2 - a^2)^2}{4c^2}$

Solution:

$$b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} = \frac{4b^2c^2}{4c^2} - \frac{(b^2 + c^2 - a^2)^2}{4c^2} =$$

$$\frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

Let the student express the numerator as the product of its prime factors. This will necessitate factoring twice. See Chap. X.

154. Rule: *Fractions occurring in equations may be eliminated by multiplying each term in the equation by the common denominator of the fractions.*

Thus:

Problem: Solve for x:

$$\frac{3x}{2} + 6 = \frac{5}{3} - \frac{x}{4}$$

Solution: The least common denominator is 12. Multiply each term in the equation by 12 (101).

$$\frac{12 \cdot 3x}{2} + 6 \cdot 12 = \frac{12 \cdot 5}{3} - \frac{12 \cdot x}{4}$$

Reducing fractions to lowest terms:

$6 \cdot 3x + 72 = 4 \cdot 5 - 3 \cdot x$

$18x + 72 = 20 - 3x$

$21x = -52$ (99a).

Dividing both sides by the coefficient of x,

$$x = -\frac{52}{21}$$

Exercise

Solve for x:

	(a)	(b)
1.	$\frac{21x}{5} = 42$	$\frac{7}{x} = 28$

2. $\frac{2x}{5} + \frac{1}{5} = \frac{3x}{10} - 3$

3. $\frac{5}{x} + \frac{19}{32} = 7$

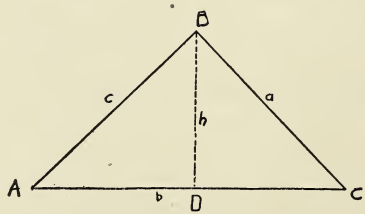
4. $\frac{x - 4}{3} + 5x = 31$ (153).

5. Solve for a: $\frac{3a - 2}{2} + 2 = \frac{a}{2} + 3$

6. Solve for a: $\frac{3a + 1}{5} + 3 = 2a - 1$

155. Heron's Formula: To compute the altitudes of a triangle in terms of its sides: (See Chap. X.)

In the $\triangle ABC$, at least one of the angles is acute. Suppose A is acute—draw BD or h, $\perp$ to AC.



(I) In the $\triangle BDA$, $h^2 = c^2 - \overline{AD}^2$ (148a)

In the $\triangle CAB$, $a^2 = c^2 + b^2 - 2b \times AD$ (150b)

Transposing a^2 and $2b \times AD$,

$$2b \times AD = c^2 + b^2 - a^2.$$

Dividing both sides of this equation by $2b$,

$$AD = \frac{c^2 + b^2 - a^2}{2b}$$

Substituting in (I),

$$h^2 = c^2 - \frac{[c^2 + b^2 - a^2]^2}{4b^2} = \frac{4c^2b^2 - [c^2 + b^2 - a^2]^2}{4b^2} =$$

$$\frac{[2cb - (c^2 + b^2 - a^2)]\,[2cb + (c^2 + b^2 - a^2)]}{4b^2}$$

Dropping parentheses,

$$h^2 = \frac{(2cb - c^2 - b^2 + a^2)\;(2cb + c^2 + b^2 - a^2)}{4b^2}$$

Changing the signs of certain terms and then enclosing them in a parenthesis preceded by a minus sign:

$$h^2 = \frac{[a^2 - (c^2 - 2cb + b^2)]\,[(c^2 + 2cb + b^2) - a^2]}{4b^2}$$

$$= \frac{[a^2 - (c - b)^2]\,[(c + b)^2 - a^2]}{4b^2}$$

$$= \frac{[a - (c - b)]\,[a + (c - b)]\,[(c + b) - a]\,[(c + b) + a]}{4b^2}$$

$$h^2 = \frac{[a - c + b]\,[a + c - b]\,[c + b - a]\,[c + b + a]}{4b^2}$$

$$\therefore h = \sqrt{\frac{[a - c + b]\,[a + c - b]\,[c + b - a]\,[c + b + a]}{4b^2}}$$

For brevity let $a + b + c = 2s$

From $a + b + c = 2s$, subtract $2c$ and we have

$$\begin{array}{r} a + b + c = 2s \\ 2c = 2c \\ \hline a + b - c = 2s - 2c \end{array}$$

Factoring:

$$a + b - c = 2(s - c)$$

From $a + b + c = 2s$, subtract $2b$ and we have

$$\begin{array}{lll} a + b + c & = 2s & \\ \quad 2b & = 2b & \\ \hline a - b + c & = 2s - 2b & = 2(s - b) \end{array}$$

From $a + b + c = 2s$, subtract $2a$ and we have

$$\begin{array}{lll} a + b + c & = 2s & \\ 2a & = 2a & \\ \hline -a + b + c & = 2s - 2a & = 2(s - a) \end{array}$$

Substituting in the above value of h

$$h = \sqrt{\frac{[2(s - c)]\,[2(s - b)]\,[2(s - a)]\,[2s]}{4b^2}}$$

$$= \sqrt{\frac{16\,(s - c)\,(s - b)\,(s - a)\,s}{4b^2}}$$

$$= \sqrt{\frac{4\,(s - c)\,(s - b)\,(s - a)\,s}{b^2}}$$

$$h = \frac{2}{b}\sqrt{(s - c)\,(s - b)\,(s - a)\,s}$$

The area of $\triangle ABC = \frac{1}{2}b \times h$ (133a).

Substituting the above value of h

$$\text{Area } ABC = \frac{1}{2} \times b \times \frac{2}{b}\sqrt{(s - c)\,(s - b)\,(s - a)\,s}$$

Cancelling:

$$\text{Area } ABC = \sqrt{(s-c)\ (s-b)\ (s-a)\ s}$$

Rearranging terms:

$$\text{Area } ABC = \sqrt{s\ (s-a)\ (s-b)\ (s-c)}$$

XIIa. Shop Exercise

ERECTION OF STAIRS

Draw a pencil line on the floor making a right angle with the wall, at that place on the floor where it is desired to locate the stringer for the stairs. At the point of intersection of this line with the wall, draw a pencil line on the wall, making a right angle with the floor. Starting from the intersection of these two lines measure off on the line on the floor a distance equal to five times the tread of the stringer and mark this point. Starting from the intersection of these two lines, measure up the line on the wall a distance equal to five times the height of the risers and mark this point.

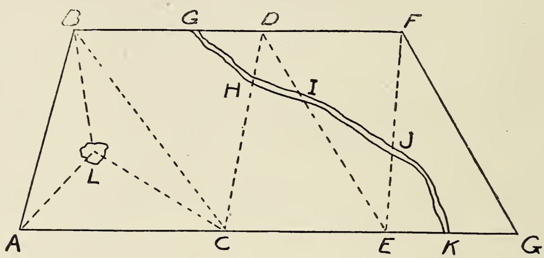
At the distance desired for the width of the stairs lay off similar lines and mark them in the same manner. The stringers may now be erected, keeping in mind the allowance of 1 in. which was made for the tread and the riser. The riser-boards and tread-boards may now be nailed in place. The tread-boards should overlap the riser-boards $\frac{1}{2}$ in. in front, and should overlap the stringers 1 in. on the sides.

XIIb. Field Exercise

AREA OF FIELD BY HERON'S FORMULA

Equipment: Leveling instrument, ranging pole, stakes, measuring tape.

Procedure: The field should be divided into triangles, which should be as nearly equilateral as practicable. Hence, having measured one end of the field, set up the instrument in one corner and lay out a line intersecting the opposite side at about the same distance from the first corner of the field as the length of the side first mentioned. For example,



having measured AB, lay off BC, measure from stake A to stake C, lay off CD, DE, EF, and measure them; also measure BC, BD, DF, CE, EM and FM. Compute areas of triangles in square chains by Heron's Formula, find their sum and change to acres.

Make a drawing of the field by adopting some convenient scale, such as 50 links = 1″, or 20 links = 1″, or whatever size will best fit the paper. Lay off the length AB to scale. Set the dividers with the length BC to scale and, with B as a center, describe an arc; set the dividers with the length AC to scale and, with A as a center, describe an arc intersecting the first arc at C. Draw in the triangle lightly. In like manner the other triangles may be drawn. The outline of the field may then be drawn in solid lines—the other lines being dotted.

If desired various topographical details occurring in the field may be located by noting their points of intersection with the various lines thus laid out. Thus, by measuring such lines as GD, DH, DI, FJ and EK, a brook may be located. A rock as at L, may be located accurately by laying off and measuring the extra lines AL, BL, and LC.

CHAPTER XIII

SIMILARITY

In general, similar figures are of the *same shape* but not of the same size. For example, a circle 1 in. in radius and one 10 in. in radius are similar. Again, a rectangle 2 in. long and 1 in. wide is similar to a rectangle 2 ft. long and 1 ft. wide.

It is evident that for two triangles to have the same shape it is necessary that the angles of one triangle be equal respectively to the angles of the other.

156. In describing similar triangles, "corresponding sides" has the same significance as in dealing with equal triangles—that is, in two similar triangles, **corresponding sides** are sides opposite equal angles.

In the example cited above, of two similar rectangles, the altitude of the smaller is contained as many times in the larger as the base of the smaller is contained in the base of the larger—that is, the *ratio* of the altitudes is the same as the *ratio* of the bases.

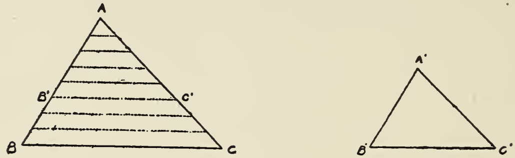
157. The **ratio** of one quantity to a second is the number of times the first contains the second.

Thus, the ratio of a yard to a foot is 3; the ratio of 10 to 5 is 2.

Obviously, a ratio is an *abstract number*. Thus, the ratio of a yard to a foot is simply 3, not 3 inches, or other units.

158. Two **similar triangles** are triangles the angles of which are equal each to each, and the corresponding sides of which are in the same ratio.

159a. Theorem: *Two mutually equiangular triangles are similar.*



Given: In the triangles ABC and $A'B'C'$, $\angle A = \angle A'$, $\angle B = \angle B'$ and $\angle C = \angle C'$.

To Prove: ABC and $A'B'C'$ are similar.

NOTE: Since the $\measuredangle$ of the $\triangle$ are equal, each to each, it is only necessary to show that the corresponding sides are in the same ratio; that is, to show that

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

Proof: Place $A'B'C'$ on triangle ABC so that $\angle A'$ shall coincide with its equal, $\angle A$, the vertex A' falling on A, the side $A'B'$ falling along AB and side $A'C'$ along AC; that is, $A'B'$ taking the position AB' and $A'C'$ taking the position AC'.

Now $\angle AB'C' = \angle ABC$ (given).

$\therefore B'C'$ is // to BC (67b).

Take some unit of length small enough to be contained without remainder a certain number of times in AB' and a certain number of times in $B'B$. Let this unit be contained m times in AB'; that is, divide AB' into m equal parts.

At the points of division construct //s to $B'C'$. Then AC' will also be divided into m equal parts (123). Let the same unit of length be contained n times in $B'B$—that is, divide $B'B$ into n equal parts. At each of these points of division construct //s to BC.

These lines are // to $B'C'$ also (61b).

CC' is divided into n equal parts (123).

That is, AB is divided into $m + n$ equal parts and AC is divided into $m + n$ equal parts.

$$\therefore \frac{AB'}{AB} = \frac{m}{m+n}$$

$$\frac{AC'}{AC} = \frac{m}{m+n}$$

$$\therefore \frac{AB'}{AB} = \frac{AC'}{AC} \quad (31)$$

Or since AB' and AC' are simply other positions of $A'B'$ and $A'C'$,

(I) $$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$

That is, the sides including the angle A are in the same ratio.

In like manner, placing the $\angle\ C'$ so that it coincides with $\angle\ C$, it may be proved that $A'B'$ is // to AB, then by taking a common unit of length as before and drawing //s it may be proved that

$$\frac{A'C'}{AC} = \frac{B'C'}{BC}$$

Then by I,

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$$

Then since $\angle\ A = \angle\ A'$, $\angle\ B = B'$, $\angle\ C = \angle\ C'$ (given) $A'B'C'$ is similar to ABC (158). *Q. E. D.*

159b. Theorem: *Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.*

For they are mutually equiangular (37b, 80c, 159a).

Problem: The base of a right triangle is 12 inches long and its altitude is 8 inches, the base of a similar rt. triangle is 15 inches long. Find the altitude.

Solution: Since the sides of similar triangles are in the same ratio,

$$\frac{12}{8} = \frac{15}{x}$$

Clearing of fractions

$$12x = 120$$
$$x = 10$$

Exercise

1. The sides of a triangle are 6, 7 and 8 units long. In a similar triangle the side corresponding to 8 is 32 units long. Find the other two sides. Hint: Form equations by aid of Art. 158.

2. What number is in the same ratio to 4 as 10 is to 5?

160. A **proportion** is a statement of equality between two equal ratios.

Thus, $\frac{3}{6} = \frac{7}{14}$ is a proportion.

This proportion may be also written in the form $3:6=7:14$. This is read "3 is to 6 as 7 is to 14."

A proportion may be stated in general terms as $a: b = c: d$. This is read "a is to b as c is to d."

161a. The **terms** of a proportion are the four quantities compared.

Thus in the above proportion the terms are a, b, c, and d.

161b. The *first* and *third* terms of a proportion are called the **antecedents.**

Thus in the above proportion a and c are antecedents. This may also be stated as "The first term of each ratio in a proportion is called its antecedent."

161c. The *second* and *fourth* terms of a proportion are called the **consequents.**

Thus, in the above proportion, b and d are consequents. This may also be stated as "The last term of each ratio of a proportion is called its consequent."

161d. The *first* and *fourth* terms of a proportion are called the **extremes.**

161e. The *second* and *third* terms of a proportion are called the **means.**

These last two definitions might be combined in this statement: "The middle terms of a proportion are called the means and the terms at the two extremities are called the extremes."

162. Theorem: *In every proportion the product of the means is equal to the product of the extremes.*

Given: $a:b = c:d$

To Prove: $ad = bc$

Proof: The given proportion may be written in this form:

$$\frac{a}{b} = \frac{c}{d} \qquad (160)$$

Multiplying both sides of this equation by bd,

$$\frac{abd}{b} = \frac{cbd}{d}$$

Simplifying, $ad = bc$. *Q. E. D.*

This theorem enables us to solve readily various problems in proportion: thus,

Problem: Find the number which is in the same ratio to 60 as 7 is to 12.

Let the number be represented by x.

Solution:

Then, $7:12 = x:60$

(or $x:60 = 7:12$)

Equating the product of the means and the extremes (162),

$12x = 420$

Whence, $x = 35$.

163. The **Fourth Proportional** to three given quantities is the fourth term of the proportion which has for its first three terms the three given quantities *taken in order*.

Thus 8 is the fourth proportional to 3, 6, and 4 in the proportion, $3:6 = 4:8$.

By aid of (162) find the **fourth proportional** to:

3. 2, 3 and 4
4. 14, 15 and 28
5. 7, 9 and 21
6. 6, 35 and 42
7. 14, 15 and 16
8. 15, 16 and 14
9. 16, 15 and 14

164a. Theorem: *If four quantities are in proportion, they are in proportion by* **alternation**; *that is, the first term is to the third as the second is to the fourth.*

Given: $a:b = c:d$

To Prove: $a:c = b:d$

Proof: From the given proportion $ad = bc$ (162).

Dividing each member of this equation by cd,

$$\frac{ad}{cd} = \frac{bc}{cd}$$

Simplifying,

$$\frac{a}{c} = \frac{b}{d}$$

Which may be written as

$$a:c = b:d \quad Q.\ E.\ D.$$

For example, this theorem might be applied to the following proportion $3:6 = 4:8$.

That this is a true proportion may first be tested by writing both ratios as fractions and then reducing these fractions to lowest terms, thus,

$$\frac{3}{6} = \frac{4}{8} \text{ that is } \frac{1}{2} = \frac{1}{2}$$

This proportion may also be tested by equating the products of the means and the extremes. Thus from $3:6 = 4:8$ it follows that $3 \times 8 = 4 \times 6$. That is $24 = 24$.

This being an identity the proportion is correct.

By alternation $3 : 6 = 4 : 8$ may be written as

$$3 : 4 = 6 : 8$$

164b. *If four quantities are in proportion, they are in proportion by* **inversion**; *that is, the second term is to the first as the fourth is to the third.*

Given: $a : b = c : d$

To Prove: $b : a = d : c$

Proof: From the given proportion $bc = ad$ (162).

Dividing each member of the equation by ac

$$\frac{bc}{ac} = \frac{ad}{ac}$$

Simplifying,

$$\frac{b}{a} = \frac{d}{c} \text{ or } b : a = d : c \quad Q.\ E.\ D.$$

165. The level of still water, as the surface of a pond on a calm day, is called the **horizontal plane,** or simply the **horizontal.**

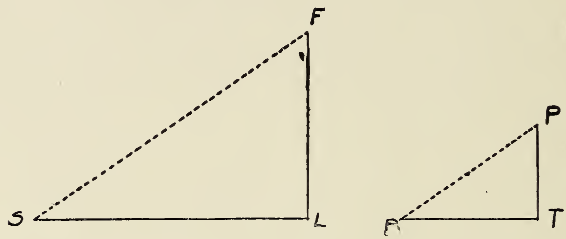
A line parallel to this line is called a **horizontal line** or simply a **horizontal.**

166. A cord which suspends a weight at one end is called a **plumb line.** The line determined by this cord is called a **vertical line.**

167. *A vertical line is perpendicular to the horizontal.*

A vertical object (as for instance a flagpole) with its shadow, forms two sides of a right triangle, the hypotenuse of which is an imaginary line from the top of the object to the furthest extremity of the shadow. It may be considered as the path of a ray of sunlight bounding the shadow.

The distance of the sun from the earth is so great in comparison with any measurement made on the earth's surface, that we may consider the rays of sunlight as *parallel lines,* without perceptible error. Thus in the following figure, which illustrates a method of finding the height of objects which cannot readily be measured directly, we may consider that the hypotenuses of the two right triangles are parallel lines.



In the above figure, FL represents a flagpole standing on a level plain, SL represents the shadow of the pole; PT represents an upright stake short enough to be measured easily, BT its shadow. As has just been explained, SF may be considered parallel to BP. Then, since BT and SL are on the same horizontal plane, $\angle PBT = \angle FSL$ since PT and FL are vertical lines,

PTB and FLS are rt. $\measuredangle$ (167)

$\therefore$ PBT and FSL are similar (159b).

$\therefore$ $PT : FL = BT : SL$

PT, BT and SL can be measured; and so the proportion can be solved for FL (162).

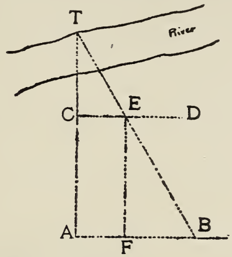
10. When a fence post 5 feet long casts a shadow 6 feet long, a flagpole casts a shadow 48 feet long. How high is the flagpole?

11. In order to find out if the trunk of a tree below the lowest branches is long enough to make a beam 32 feet long, a man measures the shadow of the trunk and finds that it is 42 feet long at the same time that a fence post 4 feet high casts a shadow $5\frac{1}{2}$ feet long. Is the trunk of the tree long enough for the purpose?

XIII. Field Exercise

FINDING WIDTH OF A RIVER

Given: In $\triangle\ TAB$, BA is $\perp$ to TA, CE is $\perp$ to TA and EF is $\perp$ to AB: T, an inaccessible point.



To Derive: The value of TC.

CE is // to AB (60a)

$\therefore\ \angle\ TEC = \angle\ TBA$ (67a)

$\therefore\ \triangle\ TEC$ is similar to $\triangle\ EBF$ (159b)

$\therefore\ TC : EF = CE : FB$

$\therefore\ TC \times FB = EF \times CE$ (162)

Dividing both sides of the equation by FB,

$$TC = \frac{EF \times CE}{FB}$$

The construction of EF may be avoided by means of the relation,

$$EF = CA \qquad (60a, 122c)$$

Substituting in the above,

$$TC = \frac{CA \times CE}{FB}$$

By measuring CA, CE, and FB and substituting in the above formula, the value of TC may be computed.

Equipment: Ranging pole, half dozen stakes, tape.

Procedure: Select a location opposite a tree, or other prominent object, on the farther bank of the river. If possible, choose a location where the ground is smooth and level. At some distance back from the river and directly opposite the tree selected, drive a stake A. Walk directly toward the tree (T) and at C drive in a stake. At C and A lay off the $\perp$s CD and AB. While one student stands at B, another walks from C toward D, carrying the ranging pole upright before him, and by direction of the observer who is sighting from B to T, he locates the pole in that point on the line CD which is in the straight line TB, marking the position with the stake E.

Measure CE, CA, AB and the distance from C to the edge of the river.

$$BA - CE = BF \quad \text{(why?)}$$

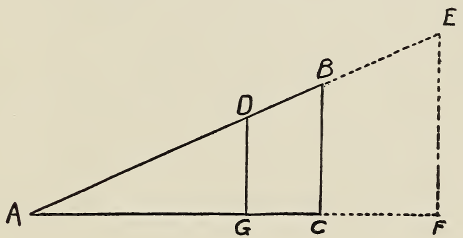
From these data compute the width of the river.

(The distance from C to the edge of the river is to be subtracted from TC.)

CHAPTER XIV

FUNCTIONS OF ANGLES

168. A very important part of mathematics, called **Trigonometry**, is based upon the fact that *the ratios between the sides of a right triangle depend entirely upon the size of the acute angle, and not upon the length of the sides.*



Thus, in the above figure, given that DG and BC are each perpendicular to AC, ADG and ABC are similar right triangles (159b).

$\therefore$ ratio $\dfrac{DG}{AG}$ = ratio $\dfrac{BC}{AC}$ (158).

In like manner it can be proved that the ratio of any other perpendicular (let fall from the hypotenuse to the base as EF) to the base, included between the foot of the $\perp$ and the vertex, as FA, is also equal to the ratio of DG to AG. That is,

$$\frac{EF}{AF} = \frac{DG}{AG}$$

We might proceed in this manner indefinitely, drawing perpendiculars from the vertex and proving that the ratio of the altitude to the base in all of the right triangles thus formed are all equal to $\frac{DG}{AG}$. In like manner it could be shown that the ratio between the altitude and the hypotenuse, or the base and the hypotenuse, are the same in all these right △ as they are in the right triangle ADG. Of course, this is merely a restatement of the properties of similar triangles, as learned in Chapter XIII, but now applied to the *angle*.

Since A represents any given acute angle, it is evident that *for any given acute angle there is only one set of values for the ratios between the sides of its right triangle.* These sets of values may be computed and arranged into a table to be used in the solution of triangles.

169a. The ratios between the sides of the right triangle of which a given angle is the acute angle, are called the **functions** of the angle.

169b. A quantity is said to be a **function** of a second quantity if its value depends upon the second quantity.

Thus, the total distance traveled by a train in a given number of hours is a function of the speed of the train—that is, it *depends* upon the number of miles traveled by the train in one hour. The total distance traveled is also a function of the number of hours during which the train is in motion.

The following ratios are called functions of an angle because they *depend entirely upon the angle.*

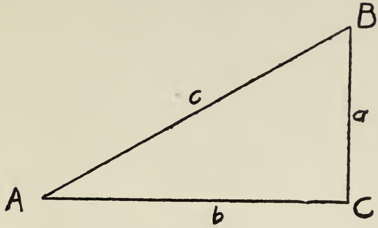
169c. The **sine** is the ratio of the opposite leg to the hypotenuse.

The **cosine** is the ratio of the adjacent leg to the hypotenuse.

The **tangent** is the ratio of the opposite leg to the adjacent leg.

The **cotangent** is the ratio of the adjacent leg to the opposite leg.

Thus, in the right triangle ACB:



$$\text{Sine } A = \frac{a}{c} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\text{Cosine } A = \frac{b}{c} = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\text{Tangent } A = \frac{a}{b} = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\text{Cotangent } A = \frac{b}{a} = \frac{\text{adjacent side}}{\text{opposite side}}$$

There are other functions of an angle, but much of the work in the solution of triangles may be done with these four.

An important relation exists between the functions of an angle and its complement. Thus, in the above figure the functions of B are as follows:

$$\text{Sine } B = \frac{b}{c}$$

$$\text{Cosine } B = \frac{a}{c}$$

$$\text{Tangent } B = \frac{b}{a}$$

$$\text{Cotangent } B = \frac{a}{b}$$

Comparing these with the functions of A, we see that the following relations exist between the functions of any angle and its complement (since A and B represent complementary angles in general).

170. Rule: *The sine of an angle is the cosine of its complement, the cosine of an angle is the sine of its complement, the tangent of an angle is the cotangent of its complement, the cotangent of an angle is the tangent of its complement.*

Notice that in the above pairs, one and only one function has the prefix *co* (meaning "with") which indicates the complementary relation. The relations between the functions of an angle and its complement may be made use of in abbreviating a table of the numerical values of the functions of angles, since it is only necessary to tabulate the values from 0° to 45°, and leave the values from 45° to 90° to be derived by aid of the above law. This has been done in the following brief table, in which the number of angular degrees for values less than 45° should be read at the left of the table, and the name of the function from the top, while if the angle is greater than 45°, the angle should be read from the right, and the name of the function from the bottom.

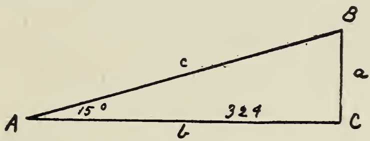
Angle	Sine	Cosine	Tang.	Cotang.	Angle
5°	.087	.996	.087	11.430	85°
10°	.174	.985	.176	5.671	80°
15°	.259	.966	.268	3.732	75°
20°	.342	.940	.364	2.748	70°
25°	.423	.906	.466	2.145	65°
30°	.500	.866	.577	1.732	60°
35°	.574	.819	.700	1.428	55°
40°	.643	.766	.839	1.192	50°
45°	.707	.707	1.000	1.000	45°
Angle	Cosine	Sine	Cotang.	Tang.	Angle

Thus, to find the cosine of 35°, we read down the column at the left until we reach 35°—then look along this line to the column which is marked "cosine" at the top, where we find .819, the function desired. Again, to find the cotangent of 55°, we read the column of degrees at the right, beginning at the bottom and reading up until we come to 55°, then looking along this line until we come to the column which is marked cotangent at the bottom, where we find .700, the value of the function desired. Notice that we arrive at the same number whether we look up the function of an angle, or the complementary function of the complement of the angle. Thus, the cosine of 35° is the sine of 55°, and the cotangent of 55° is the tangent of 35°.

Exercise

By aid of the above table solve the following:

171. Problem: In the right $\triangle ABC$ given $\angle A = 15°$, side $b = 324$, find a and c.



Solution: First drawing the right triangle and marking the known parts, we proceed to form equations which shall each involve only one unknown quantity. (The functions of known angles are considered as known quantities, since they may be obtained by referring to the table.)

$$\text{Tan } A = \frac{a}{b}$$

Substituting the value of b and also substituting the value of Tan 15° from the table,

$$.268 = \frac{a}{324}$$

Clearing of fractions

$$a = (.268)(324)$$
$$a = 86.832$$

Since the table goes only to the third decimal place this last value may be written 86.8.

In finding c we must not use an equation involving the side just found, as to do so would be to increase the error of our answers—since this table is correct only to three places of decimals, that is, the value of a found is not *exact*, and to use it for further computation with the tables would increase the error: it is better to solve for the two sides *independently*. Thus,

$$\text{Cos } A = \frac{b}{c}$$

Substituting the value of b, and also substituting the value of cos 15° from the table,

$$.966 = \frac{324}{c}$$

Clearing of fractions, $.966c = 324$

Dividing by the coefficient of c,

$$c = \frac{324}{.966} = 335.4$$

Since the table goes only to the third decimal place this last value may be written 335.

These values may be checked by the formula,

$$c^2 = a^2 + b^2 \quad (148a)$$

In squaring these quantities the results can be trusted to the first three figures only: for example, 112,225 may be written 112,000 without detracting from the accuracy of the result.

Let the student determine what percentage the 225 so discarded is of 112,000. Notice that the error of a square of a quantity is greater than the error in the quantity itself, since the error has been multiplied by the quantity.

The remaining acute angle of this triangle is found by subtracting A from $90°$.

1. Solve the right $\triangle ABC$ in which $A = 20°$, $b = 28$. (Draw a diagram as in the previous example.)

2. Solve the rt. $\triangle ABC$ in which $A = 35°$, $a = 325$.

3. Solve the rt. $\triangle ABC$ in which $A = 5°$ and $c = 284$.

4. Find the area of the right triangle in which $A = 25°$, $c = 87$. (Hint: Solve for the legs of the rt. $\angle$ and find half their product.)

NOTE: Drawings of right triangles may be made to aid in the solution of the following examples. Thus, in Ex. 5, the horizontal distance from the observer to the building is the base of the triangle, and the height of the building above the observer is the altitude of the triangle.

5. Find the height of a building, if the angle of elevation of the top is $25°$, and the distance of the observer is 62 feet, the height of the instrument being 5 feet.

(Hint: Solve the right triangle for the height of the building above the observer; then add 5 feet.)

6. How long a cable is necessary to extend from a point in an iron smokestack 48 feet above the ground and to make an angle of $40°$ with the horizontal? Add 2 feet for fastening and slack of cable.

7. How high is a tree that casts a horizontal shadow 83 feet in length when the angle of elevation of the sun is $50°$?

8. A ship is sailing due north-east at a rate of ten miles an hour. Find the rate at which she is sailing due north and also due east.

(Hint: A north-east course (line) makes an angle of $45°$ with a due east course.)

9. From the top of a lighthouse known to be 141 feet above the sea, the angle of depression of a boat is observed to be $15°$. What is the distance of the boat from the lighthouse?

(Hint: The **angle of depression** means the angle between the observer's line of sight and the horizontal. Hence, it is the complement of this angle that is included in the right triangle formed by the lighthouse, the distance along the surface of the water from the lighthouse to the boat, and the line of sight from the top of the lighthouse to the boat.)

In the solution of right triangles, the following rule may be applied to save the labor of clearing the equation of fractions:

172. Rule: *With regard to either acute angle of any right triangle:*

Sine $\times$ *hypotenuse* $=$ *opposite side;*
Cosine $\times$ *hypotenuse* $=$ *adjacent side;*
Tangent $\times$ *adjacent side* $=$ *opposite side.*

From the solution of the previous examples, the derivation of these rules is evident. To secure facility in computation, they should be committed to memory.

Solve for the remaining parts of the rt. $\triangle ABC$, given that:

10. $A = 15°,\ c = 81$

11. $B = 20°,\ a = 46$

12. $B = 65°,\ c = 105$

13. $A = 40°,\ c = 91$

14. When the angle of elevation of the sun is 40°, the shadow of a tower is 170 feet long. How high is the tower?

15. In order to locate a reef the angle of elevation of the top of a lighthouse is measured from the reef and found to be 35°. The lighthouse is known to be 142 feet high. What is the distance of the reef from the lighthouse?

(Hint: What is the complement of 35°?)

16. Two villages, A and B, are $7\frac{1}{2}$ miles apart. A man in a balloon directly above A observes the angle of depression of B to be 10°. Find the height of the balloon.

17. The boom (swinging arm) of a derrick is 26 feet long, and its lower end is fastened to the mast at a height of 3 feet from the ground. When the angle between the derrick and the mast is 50°, how far is the end of the boom from the mast and how far is the end of the boom from the ground?

18. When a lighthouse is directly east of a steamer, the captain notes the reading of the patent log (which records the distance traveled); after the steamer has gone directly north for a distance of two nautical miles, he observes the lighthouse to lie 55° East of South. How far was the steamer from the lighthouse at the time of each observation? (Use 6,080 feet as the length of the nautical mile.)

XIVa. Shop Exercise

DESIGN OF RAFTER

A common instrument for making use of the properties of similar triangles is the Carpenter's Square—often called the Steel Square. This consists of a flat piece of steel shaped like a letter L, one arm of it being usually 2 feet long and the other 16 in. long, both arms being graduated in inches commencing at the vertex.

For example, suppose that it is desired to lay out a rafter for a roof in which the top of the ridgepole is to be 8 feet above the floor of the attic, and half the width of the attic floor is 12 feet. Evidently we have here a right triangle, in which the base is 12 feet, the altitude 8 feet and the hypotenuse is the upper edge of the rafter. If now the Steel Square is laid on the timber to be marked in such a way that the 12 in. mark on the long arm coincides with the edge of the timber, and the 8 in. mark on the short arm also coincides with the edge of the timber we have a small right triangle which is similar to the large one. (For the tangent of the acute angle at the base of the small triangle, $\frac{8 \text{ in.}}{12 \text{ in.}} = \frac{2}{3}$, and the tangent of the acute angle at the base of the large triangle equals $\frac{8 \text{ ft.}}{12 \text{ ft.}} = \frac{2}{3}$, so the acute angle at the base of the small triangle equals the acute angle at the base of the large one; therefore the triangles are similar) (159b).

The "bevel" (that is, the angle at which the mark for the saw-cut is to be made with the edge of the timber) both for the top and the bottom of the rafter can be determined from the small triangle, since the corresponding angles of similar triangles are equal. The "foot" of the rafter may be marked at once by drawing the pencil along the outer edge of the long arm of the square. The problem remains to determine the length of the rafter. (Since the right triangles are similar, the corresponding sides are in the same ratio—that is, the hypotenuse of the small triangle will go into the hypotenuse of the large triangle as many times as the base of the small triangle will go into the base of the large triangle—or as the altitude of the small triangle goes into the altitude of the large triangle.) Accordingly the length of the hypotenuse of the small right triangle is to be laid off 12 times along the edge of the rafter.

Having marked the bevel along the long arm of the square, put a pencil mark on the edge of the timber where the short arm touches it (at the 8 in. mark). Move up the square along the timber so that the 12 in. mark coincides with this pencil mark and put a mark on the edge of the timber by the 8 in. graduation. Repeat this process until the square has been laid on the timber 12 times. When it is placed for the twelfth time draw a line along the short arm of the square for the top bevel.

It is evident that when the timber has been cut through with a saw along these lines, and placed in position so that the upper end is 8 feet above the horizontal, the top cut will be vertical and the lower cut will be level.

A rafter for the roof described above would be said to have a "run" of 12 feet and a "rise" of 8 feet. Lay out the pattern for cutting such rafters on a smooth board 1 in. thick, 7 in. wide and 13 feet long ($1'' \times 7'' \times 13'$) using the method above described.

From the outer point of the foot of the rafter measure back 4 in. for the width of the "plate" (the timber nailed along the edge of the attic floor to keep the rafters from slipping). At this 4 in. mark erect a perpendicular, lay off 2 inches along it and from this 2 in. mark draw a line (parallel to the bottom cut) to the outer edge of the rafter. Cut out along this line so the rafter will fit over the plate.

In a similar manner, draw a line parallel to the top bevel to allow for the ridgepole, remembering that since rafters are set up in pairs, it is necessary to allow only half the width of the ridgepole on each one.

In like manner rafters could be laid out for a roof having any given rise and run.

XIVb. Shop Exercise

DESIGN OF STAIRS

The following is the method of designing a flight of stairs for any given position—that is, to ascend a given height and extend a given amount from the wall, in other words, to have a given "rise" and "run." Suppose, for example, it was desired to lay out an extra flight of stairs to fit in the school building.

(The application of similar right triangles to this problem may be understood by considering a line drawn through the inner corners of the steps (along the stringers) erected in XIIa. This is the hypotenuse of a right triangle in which the line measured on the floor is the base and the line measured on the wall is the altitude. The small right triangles to be cut from the side pieces are similar to this.)

To solve this problem, then, we state the proportion:

Rise : Run = height of riser : width of tread.

It is desirable to have the height of the riser between 8 and 9 in. It is best, therefore, to reduce all the lengths measured to inches. The proportion then appears as follows:

Rise (in inches) : Run (in inches) $= 8\frac{1}{2} : x$.

If the value of x given by this proportion does not give a satisfactory height for the riser, try some other number instead of $8\frac{1}{2}$.

To avoid such absurdities as a fraction of a step, the width of tread must go into the run a whole number of times (and the height of a riser the same number of times into the rise).

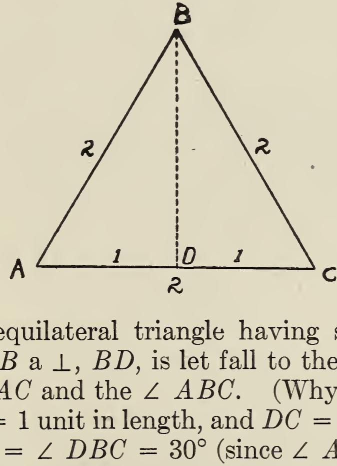
If possible to avoid it, the run of a flight of stairs should not be less than the rise, or the steps will be steep.

CHAPTER XV

COMPUTATION OF FUNCTIONS: FOUR PLACE TABLE OF SINES AND COSINES AND ITS USE

The values of the Functions of certain angles may be very readily computed.

173. *The functions of* 60° *and of* 30° *may be derived from the equilateral triangle* (77b, 80a). Thus:



ABC is an equilateral triangle having sides 2 units in length. From B a $\perp$, BD, is let fall to the base AC.

This bisects AC and the $\angle\ ABC$. (Why?)

That is $AD = 1$ unit in length, and $DC = 1$ unit in length.

The $\angle\ ABD = \angle\ DBC = 30°$ (since $\angle\ ABC$ is bisected).

Since $\triangle$ ABD and BDC are rt. $\triangle$, the length of BD may be computed from either of them.

$$\begin{aligned} BD &= \sqrt{AB^2 - AD^2} \\ &= \sqrt{4 - 1} \\ &= \sqrt{3} \end{aligned}$$

The functions of 60° may now be stated, and at the same time those of 30° may be expressed by aid of art. 170.

$$\text{Sin } 60° = \frac{\sqrt{3}}{2} = \text{Cos } 30°$$

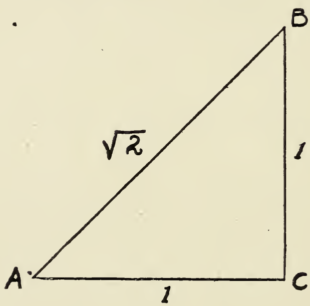
$$\text{Cos } 60° = \frac{1}{2} = \text{Sin } 30°$$

$$\text{Tan } 60° = \frac{\sqrt{3}}{1} = \sqrt{3} = \text{Cot } 30°$$

$$\text{Cot } 60° = \frac{1}{\sqrt{3}} = \text{Tan } 30°$$

The value of the sine of 60° may be found correct to four places of decimals by extracting the square root of 3 to five figures and then dividing this result by 2. In like manner the other functions of 60° and the functions of 30° may be worked out.

174. *The values of the functions of* 45° *may be computed from the isosceles right triangle.* Thus:



In the rt. $\triangle$ ABC, leg AC = leg BC.

$\therefore \angle BAC = \angle ABC$ (77b).

Then, since $\angle C$ is a rt. $\angle$, $\angle BAC = 45°$, and $\angle ABC = 45°$ (80a).

Let AC be 1 unit in length, and BC be 1 unit in length.

The length of AB may now be obtained as follows:

$$\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$$

$$AB = \sqrt{\overline{AC}^2 + \overline{BC}^2}$$

$$= \sqrt{1 + 1} = \sqrt{2}$$

The functions of 45° may now be expressed as follows:

$$\text{Sin } 45° = \frac{1}{\sqrt{2}}$$

$$\text{Cos } 45° = \frac{1}{\sqrt{2}}$$

$$\text{Tan } 45° = \frac{1}{1} = 1$$

$$\text{Cot } 45° = \frac{1}{1} = 1$$

Notice that sin 45° = cos 45° and tan 45° = cot 45°. Since 45° is its own complement, this agrees with art. 170.

The functions of different angles may be obtained by the aid of the protractor—laying off the angle on the paper carefully with a ruler and hard pencil—then drawing a perpendicular to one side of the angle from some point on the other side. The sides of the right triangle thus formed may now be measured with the ruler.

It is usually better not to take the inch as the unit of measure, but some smaller division, such as the eighth of an inch or the sixteenth. If the same unit is taken for all the sides, it is not necessary to write the lengths of the sides in the fractional form, but simply to write the number of units in each side. Thus, suppose the sides of a triangle are $\frac{21}{16}$, $\frac{28}{16}$, and $\frac{35}{16}$ long—the ratios will be the same if the sides are written as 21, 28, and 35.

Problem:

Find to four places of decimals the value of the functions of the $\angle A$, given that $a=17$, and $b=16$, in the right $\triangle ABC$.

Solution:

$$c = \sqrt{a^2 + b^2}$$
$$= \sqrt{(17)^2 + (16)^2}$$
$$= \sqrt{289 + 256} = \sqrt{545} = 23.345$$

$$\therefore \text{Sin } A = \frac{17}{23.345} = .7282$$

$$\text{Cos } A = \frac{16}{23.345} = .6854$$

$$\text{Tan} = \frac{17}{16} = 1.0625$$

$$\text{Cot } A = \frac{16}{17} = .9412$$

Exercise

In the right triangle ABC find the functions of the $\angle A$, given that,

1. $a = 7,\ b = 18$
2. $a = 34,\ c = 41$
3. $b = 28,\ c = 39$
4. $a = 17,\ c = 37$

Draw the above right triangles carefully to scale and measure the angle with a protractor.

Lay off the following angles with the protractor, using 1/16 of an inch as a unit of measure; measure off any convenient length and complete the right triangle; measure the remaining sides and compute the values of the functions:

5.	6.	7.	8.
32°	41°	76°	54°

By making the right triangles sufficiently large, the functions of any angle may be computed to the degree of accuracy desired. In making a table, however, it is best to compute the functions algebraically (as was done in obtaining the functions of 30°, 45° and 60°) from a formula, of which there are a great number.

The following table of sines and cosines has been computed from formulas and is accurate to the fourth decimal place.

175a. Natural Sines and Cosines.

∠°	Sine	Cosine		∠°	Sine	Cosine	
1°	.0175	.9998	89°	24°	.4067	.9135	66°
2°	.0349	.9994	88°	25°	.4226	.9063	65°
3°	.0523	.9986	87°	26°	.4384	.8988	64°
4°	.0698	.9976	86°	27°	.4540	.8910	63°
5°	.0872	.9962	85°	28°	.4695	.8829	62°
6°	.1045	.9945	84°	29°	.4848	.8746	61°
7°	.1219	.9925	83°	30°	.5000	.8660	60°
8°	.1392	.9903	82°	31°	.5150	.8572	59°
9°	.1564	.9877	81°	32°	.5299	.8480	58°
10°	.1736	.9848	80°	33°	.5446	.8387	57°
11°	.1908	.9816	79°	34°	.5592	.8290	56°
12°	.2079	.9781	78°	35°	.5736	.8192	55°
13°	.2250	.9744	77°	36°	.5878	.8090	54°
14°	.2419	.9703	76°	37°	.6018	.7986	53°
15°	.2588	.9659	75°	38°	.6157	.7880	52°
16°	.2756	.9613	74°	39°	.6293	.7771	51°
17°	.2924	.9563	73°	40°	.6428	.7660	50°
18°	.3090	.9511	72°	41°	.6561	.7547	49°
19°	.3256	.9455	71°	42°	.6691	.7431	48°
20°	.3420	.9397	70°	43°	.6820	.7314	47°
21°	.3584	.9336	69°	44°	.6947	.7193	46°
22°	.3746	.9272	68°	45°	.7071	.7071	45°
23°	.3907	.9205	67°				
	Cosine	Sine	∠°		Cosine	Sine	∠°

In this table, the number of degrees in an angle less than 45° is read from the left hand column, and the name of the function read at the top; the number of degrees in an angle greater than 45° is read from the right hand column, and the name of the function read at the bottom.

Problem: Find the sine of 37°.

Solution: Since this angle is less than 45°, we read down the column of figures at the left until we find 37°; the function opposite it (.6018) is the function desired, that is, we read the function from the column adjoining the column of degrees just read, as this column of functions is labelled "Sine" at the top.

Problem: Find the sine of 78°.

Solution: Since this angle is greater than 45° we read the angle from the column at the right, reading upward until we reach 78°. The function next it, on the same line (.9781), is the function desired—since this function is labelled "Sine" at the bottom.

Problem: Find the cosine of 27°.

Solution: Since the angle is less than 45° we read down the column of degrees at the left until we encounter 27°; the function on the same horizontal line with 27°, but in the second column from the number of degrees just read, (.8910) is the function desired, since this column is labelled "Cosine" at the top.

Problem: Find the cosine of 64°.

Solution: Since this angle is greater than 45° we read the angle from the column at the right, reading upward until we reach 64°. The function on the same horizontal line with 64°, but in the second column from the number of degrees just read, (.4384) is the function desired, since this column is labelled "Cosine" at the bottom.

From the above examples the following rule appears with reference to the preceding table:

175b. *The sine of a given number of degrees is the function on the same horizontal line, next adjoining the number of degrees: the cosine of a given number of degrees is the next adjoining function but one, on the same horizontal line.*

Problem: Find the sine of 58° 25′.

Solution:

$$\begin{array}{rl} \text{Sin } 59^\circ = & .8572 \\ \text{Sin } 58^\circ = & \underline{.8480} \\ \text{Tabular Difference} & .0092 \end{array}$$

Since .0092 corresponds to an increase of 1° in the angle, 25′ will correspond to the same fraction of .0092 as 25′ is of 1°. Thus, $25' = \left(\frac{25}{60}\right)^\circ = \left(\frac{5}{12}\right)^\circ$

$$\frac{5}{12} \times .0092 = \frac{5}{3} \times .0023 = \frac{.0115}{3} = .0038$$

$$\begin{array}{ll} \text{Sin } 58^\circ & = .8480 \\ \text{Correction} & = \underline{.0038} \\ \text{Sin } 58^\circ\ 25^\circ & = .8518 \end{array}$$

Since the sine is an increasing function (that is, the sine increases as the angle increases), this correction is added.

After finding a function corresponding to a given number of degrees and minutes, the student should check his work by noticing that it falls between the proper two functions in the table. Thus,

Check: This value lies between the sine of 58° and of 59° as given by the table.

175c. The change in a function corresponding to 1° is called the **Tabular Difference.** (See above problem.)

176a. *To find the sine corresponding to a given number of degrees and minutes: drop the minutes and find from the table the sine corresponding to the given number of degrees; subtract this from the next greater sine in the table to obtain the tabular difference; express the given number of minutes as a fraction of a degree, multiply the tabular difference by this fraction, and add this product to the sine first obtained from the table.*

Exercise

Find the sines corresponding to the following angles:

1.	2.	3.	4.	5.
13° 27′	49° 48′	21° 35′	56° 45′	37° 40′
6.	**7.**	**8.**	**9.**	**10.**
38° 25′	72° 42′	78° 15′	81° 24′	82° 18′

The work of finding the cosine corresponding to a given number of degrees and minutes is like that of finding the sine except that the correction is *subtracted* instead of *added*, since the cosine is a *decreasing* function—that is, it decreases as the angle *increases*.

Problem: Find the cosine of 74° 54′.

Solution:

$$\begin{aligned} \text{Cos } 74° &= .2756 \\ \text{Cos } 75° &= .2588 \\ \text{Tabular Difference } & .0168 \end{aligned}$$

$$54' = \left(\frac{54}{60}\right)^{\circ} = \left(\frac{9}{10}\right)^{\circ} = \frac{9}{10} \times .0168 = .01512$$

$$\begin{aligned} \text{Cos } 74° &= .2756 \\ \text{Corr.} &= .0151 \\ \hline \text{Cos } 74° \; 54' &= .2605 \end{aligned}$$

Check: This value lies between the cosine of 74° and the cosine of 75° as given by the table.

176b. Rule: *To find the cosine corresponding to a given number of degrees and minutes: drop the minutes and find from the table the cosine corresponding to the given number of degrees; from this subtract the next less cosine in the table to obtain the tabular difference; express the given number of minutes as a fraction of a degree, multiply the tabular difference by this fraction, and subtract this product from the cosine first obtained from the table.*

Find the cosines corresponding to the following angles:

11.	**12.**	**13.**	**14.**
28° 35′	37° 42′	49° 12′	51° 20′
15.	**16.**	**17.**	**18.**
63° 36′	10° 45′	41° 10′	44° 24′

Problem: Find the angle corresponding to the sine .6738.

Solution: We first locate the given sine between the next less and the next greater sine in the table; thus,

$$\begin{array}{ll} \text{Sin } 43^\circ & = .6820 \\ \text{Sin } 42^\circ & = .6691 \\ \hline \text{Tab. Dif.} & = .0129 \end{array} \qquad \begin{array}{ll} \text{Given Sin} & = .6738 \\ \text{Sin } 42^\circ & = .6691 \\ \hline \text{Given Dif.} & = .0047 \end{array}$$

Since .0129 corresponds to a difference of 1°, the Given Difference will correspond to that fraction of 1° which .0047 is of .0129. It is best to change this ratio $\frac{.0047}{.0129}$ to a decimal; thus,

$$\begin{array}{r|l|l} .0129 & .0047 & .36 \\ & \underline{387} & \\ & 830 & \end{array}$$

The correction to be added to 42° is therefore .36 of a degree. Changing this to minutes

$$\begin{array}{r} .36 \\ \underline{60} \\ 21.60' \end{array}$$

or to the nearest whole minute, 22′. The angle is therefore 42° 22′.

177a. Rule: *To find the angle corresponding to a given sine: locate the given sine between the next greater and the next less sine in the table; find the decimal part which the given difference is of the tabular difference, multiply this by* 60 *and add this product as minutes to the angle corresponding to the smaller sine.*

Exercise

Find the angle whose sine is:

1.	2.	3.	4.	5.
.4291	.6843	.3471	.5893	.9431
6.	**7.**	**8.**	**9.**	
.8260	.7452	.2100	.1564	

Since the cosine decreases as the angle increases, in finding the angle corresponding to a given cosine, the correction must be subtracted. Thus,

Problem: Find the angle whose cosine is .5906.

Solution: We first locate the given cosine between the next less and the next greater cosine in the table:

$$\begin{array}{rl} \text{Cos } 53° = & .6018 \\ \text{Cos } 54° = & .5878 \\ \hline \text{Tab. Dif.} = & .0140 \end{array} \qquad \begin{array}{rl} \text{Given Cos} = & .5906 \\ \text{Cos } 54° = & .5878 \\ \hline \text{Given Dif.} = & .0028 \end{array}$$

Since .0140 corresponds to 1°, .0028 will correspond to that fraction of a degree which .0028 is of .0140. It is best to change this ratio, $\frac{.0028}{.0140}$, to a decimal; thus,

$$.0140\,\big)\,.0028\,\big(\,.2$$
$$\underline{280}$$

Changing this to minutes,

$$\begin{array}{r} 60 \\ .2 \\ \hline 12.0' \end{array}$$

$54° - 12' = 53°\ 48'$ Ans.

177b. Rule: *To find the angle corresponding to a given cosine: locate the given cosine between the next greater and the next less cosine: find the decimal part which the given difference is of the tabular difference, multiply this by* 60 *and subtract this product as minutes from the angle corresponding to the smaller cosine.*

Find the angle whose cosine is:

10.	11.	12.	13.	14.
.6942	.3681	.5724	.8702	.2073
15.	**16.**	**17.**	**18.**	**19.**
.7005	.4506	.3972	.8291	.1246

In becoming accustomed to the use of these rules it is of assistance to reflect: "How shall the correction be dealt with so that the angle shall fall between the two taken from the table?" Since in the case of the Sine (the increasing function) the correction is added, it must be added to the smaller of the angles taken from the table so that the angle shall lie between those two angles: whereas, in the case of the Cosine (the decreasing function) it must be subtracted from the larger so that the answer may fall between those two angles—since to *add* to the *larger* or *subtract* from the *smaller* would make the answer fall *outside*.

Although the established custom of giving problems involving seconds gives the student the idea that a second of degree or arc is an important unit of measure, it can oftentimes be neglected in practical problems without sensible error.

Taking the circumference of the whole earth as 25,000 miles, let the student determine the number of miles in 1°, the number of miles in 1′, and then the number of feet in 1″. Since the length of arc corresponding to 1″ is so small (comparatively) when the radius is 4000 miles, the student will understand why the second is neglected in these problems.

178a. North, East, South and West are called the **Cardinal Points** of the Compass.

178b. The direction of a line as shown by the compass is called its **bearing.**

The bearing of a line which does not coincide with any of the cardinal points, may be indicated by stating the number of degrees it is East of North or West of North; also by the number of degrees it is East of South or West of South: thus, a line which makes an angle of 7° on the West side of the North and South line—going in a direction approximately South would be marked as S. 7° W. Of course, if the observer reversed his direction of movement and proceeded along the same line in the opposite direction, he would be going N. 7° E.

Gunter's Chain, used in compass surveying, is divided into a hundred links. Hence, if a side of a field was 3 chains and 41 links long it might be written as 3.41 ch. As has previously been explained (Chap. IX), to change square chains to acres (sq. ch. to A.) divide by 10 (see 129).

Problem: Starting from a point A a surveyor runs N. 26° 35′ E. 9.80 chains to B, thence N. 56° 15′ E. 8.10 chains to C, thence S. 5° 30′ W. 24.10 chains to D, thence N. 39° 30′ W. 13.90 chains to A. Calculate the area of the field $ABCD$.

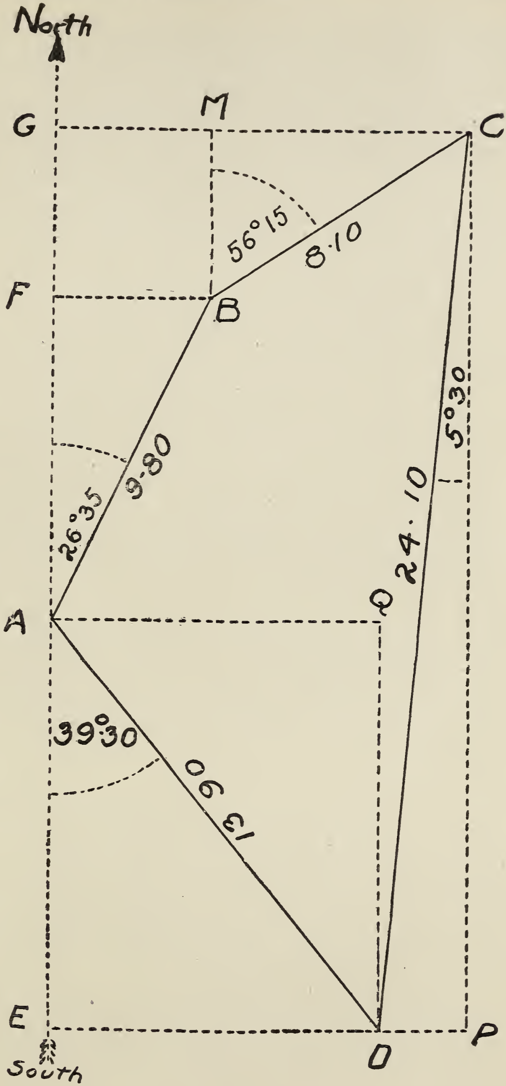
Solution: Draw a diagram of the field to scale, using the protractor to lay out the angles (remember that the top of the page is north, the right hand east, etc.) At the left side of the drawing draw a North and South line. On this line locate the most westerly point, A. Lay off AB. To assist in laying off BC, a N. and S. line BM may be drawn through B, and from this an angle of 56° 15′ laid off to the right. When BC and CD have been laid off, the drawing may be checked by connecting C and D and then measuring the length of the line and its bearing, to see if they are of the value stated. Note that the $\angle\ EAD$ is the same as the bearing of DA—that is, the same as the angle which AD would make with an N. and S. line drawn through D (64).

From the diagram it appears that,

(I) Area $ABCD$ = area trapezoid $GCDE - (\triangle\ ADE + \triangle\ ABF +$ trapezoid $FBCG)$.

The following method of solution not only gives the answer but checks the accuracy of the survey. Certain new terms must first be defined:

When the surveyor goes from A to B, the final location of B is the same as if he had moved from A to F and then from F to B. The distance moved directly north (as AF) is called the **Northing** and the distance moved directly east (as FB) is called the **Easting.** Likewise BM is a **Northing** and MC is an **Easting.** Again when the surveyor moves from C to D, the final location of D is the same as



if he had moved from C to P and thence from P to D. The distance moved directly south (as CP) is called the **Southing,** and the distance moved directly west (as PD) is called the **Westing.** Note that **Northing** is the opposite of **Southing** and **Easting** is the opposite of **Westing.**

Obviously, if the surveyor goes all around the field and returns to the *same place from which he started,* for the distance which he went north he must *return* an equal distance toward the south, and for the distance which he went east, he must *return* an equal distance west.

178c. Rule: *If a survey completely around any area is correct, the sum of the northings equals the sum of the southings, and the sum of the eastings equals the sum of the westings.*

In solving for areas of (I), the work may be arranged in the following manner to test the accuracy of the survey.

NOTE: To understand why the line EA is computed in the following solution, and the angle EAD used instead of $\angle QDA$ (the bearing of the line), draw a N. and S. line through D, and an E. and W. line through A. Let Q be their point of intersection. Evidently $QD = AE$, $AQ = ED$ (60a, 122c), $\angle QDA = \angle EAD$ (64).

	N.	S.
$AF = AB \cos 26° 35' = (9.80)(.8942) =$	8.76	
$BM = BC \cos 56° 15' = (8.10)(.5555) =$	4.50	
$EA = AD \cos 39° 30' = (13.90)(.7715) =$	10.72	
$CP = CD \cos 5° 30' = (24.10)(.9953) =$		23.99
	23.98	23.99
	E.	W.
$FB = AB \sin 26° 35' = (9.80)(.4475) =$	4.39	
$MC = BC \sin 56° 15' = (8.10)(.8314) =$	6.73	
$DP = DC \sin 5° 30' = (24.10)(.0959) =$		2.31
$ED = AD \sin 39° 30' = (13.90)(.6361) =$		8.84
	11.12	11.15

As will be seen from the above, the Northings do not exactly balance the Southings, nor the Eastings the Westings—but the error is so small that the survey may be classed as accurate. It is unusual, in the survey of an area of this size, to have any smaller error than the above. If there is a large discrepancy, however, the survey should be repeated. The error in the area is reduced by the above method of solution; since one area is subtracted from another.

Let the student compute the maximum error in the above computation, in percentage. (Hint: What per cent is 3 links of 11.12 chains?)

178d. Northings and Southings are called **Latitudes,** and Eastings and Westings are called **Departures.**

By aid of the latitudes and departures, the areas in equation (I) may be computed.

Area $\triangle ADE = \frac{1}{2}(EA \times ED) = \frac{1}{2}(10.72 \times 8.84)$	=	47.38 sq. ch.
Area $\triangle ABF = \frac{1}{2}(FA \times FB) = \frac{1}{2}(8.76 \times 4.39)$	=	19.23 " "
Area Trapezoid $FBCG = \frac{1}{2}(GC + FB) \times BM = \frac{1}{2}(15.51 \times 4.50)$	=	34.90 " "
		101.51 sq. ch.
Area $GCDE = \frac{1}{2}(GC + ED) \times CP = \frac{1}{2}(19.95 \times 23.99)$	=	239.42 sq. ch
Minus areas	=	101.51 " "
Area $ABCD$	=	137.91 sq. ch.
		or 13.79 Acres.

In the above computation GC is found by adding FB and MC. If desired, instead of dividing each different product in the above by 2, as in the above solution, which takes advantage of cancellation, this division may be left until the last, as shown by the following:

$$\tfrac{1}{2}[(GC + ED) \times CP] - \tfrac{1}{2}[EA \times ED] - \tfrac{1}{2}[FA \times FB] - \tfrac{1}{2}[(GC + FB) \times BM] =$$

$$\tfrac{1}{2}\left\{[(GC + ED) \times CP] - [EA \times ED] - [FA \times FB] - [(GC + FB) \times BM]\right\}$$

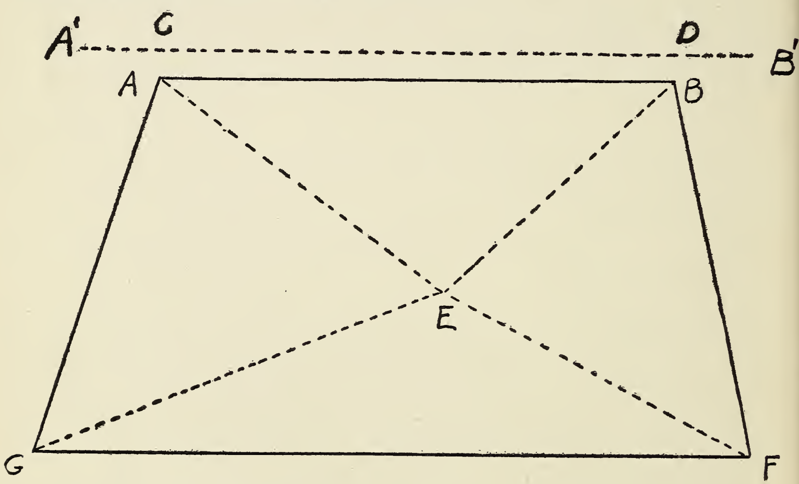
Exercise

1. A deed of a wood-lot gives the following boundaries (the letters A, B, C and D are used instead of the description of the corners as given in the deed): "Starting from a point A, N. 31° 30′ W. 10.40 chains to B, thence N. 62° E. 9.20 chains to C, thence S. 36° E. 7.60 chains to D, thence S. 45° 30′ W. 10 chains to A." Find the area of the wood-lot.

2. A surveyor, starting at A, measures S. 50° 30′ E. 6 chains to B, thence S. 58° W. 4.20 chains to C, thence N. 28° 10′ W. 5 chains to D. What is the bearing of A from D. and how long is AD?

(XV) Finding Area of Field by Triangulation:

Equipment: Transit (horizontal angle instrument), two ranging poles, measuring tape, stakes.



Procedure: At the edge of the field most suitable, measure a base line AB. In case it is not possible to measure this base along the edge of AB directly (due to a ditch, or growth of bushes, or similar obstruction) this line may be laid off to one side of AB and parallel to it (as the line $A'B'$); and the length of AB projected upon it, as CD.

As the computation of all the triangles will depend upon the base line AB, this line must be measured with considerable care. The transit should be placed at one end of the line $A'B'$, and by aid of the ranging pole a straight line should be laid off as in Field Exercise II. Then perpendiculars should be constructed from A to $A'B'$, and from B to $A'B'$, and the distance CD, between the feet of these perpendiculars, should be carefully measured.

The distance AB being known, the transit should be set up at one end as A, and a ranging pole set up at the other end B. The other ranging pole should be set up at some suitable place, E, near the center of the field, which should, if possible, be visible from all four corners of the field.

Measure $\angle\ BAE$. Move the transit to B and measure the $\angle\ ABE$. (Since one side and two angles of the triangle are known, the triangle can later be solved by aid of Chapter XVII.) In like manner set up the instrument at B and measure $\angle\ EBF$. Then move the instrument to E and measure BEF. Since the side BE will be known when the $\triangle\ AEB$ is solved, it is possible to solve $\triangle\ BEF$, since a side and two adjacent angles are known.

The $\triangle\ EFG$ will be solvable if $\measuredangle GEF$ and EFG are known since the side EF can be found by solving $\triangle\ FEB$.

Likewise the $\triangle\ AEG$ will be solvable if $\measuredangle\ AGE$ and AEG are known since the side GE can be obtained by solving $\triangle\ GEF$.

As a check on the accuracy of the work, all the angles in the diagram should be measured, when any error in observation may be detected by noting if the sum of the angles of each triangle is equal to 180°, and also if the sum of the angles about the point E is equal to 360°.

In making any correction in the observations, try to "distribute the error" proportionally. For example suppose the three angles measured in the triangle AEG are $34° 5'$, $75° 21'$ and $70° 36'$. Since the total of these amounts to $180° 2'$, it is necessary to subtract $2'$ from this. From the largest $\angle$, $75° 21'$, subtract $1'$, from the second largest $\angle$, $70° 36'$, subtract $1'$, leaving the smallest angle uncorrected. (A correction of $1'$ in an $\angle$ of $75° 21'$ is much less, proportionally, than a correction of $1'$ in an angle of $34° 5'$.)

The corrected readings of the three angles, then, are $34° 5'$, $75° 20'$ and $70° 35'$, giving a total of $180°$.

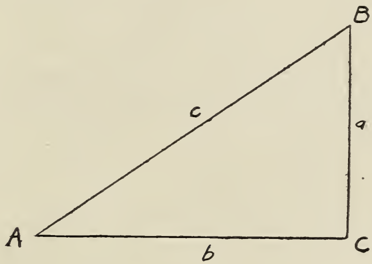
The solution of these triangles is to be left until after studying the Sine Law (Chap. XVII).

(For a large field, in which there are many obstructions to the view, it will be necessary to establish other centers of angles besides E.)

CHAPTER XVI

RELATIONS BETWEEN THE FUNCTIONS: AXES OF REFERENCE: FUNCTIONS OF OBTUSE ANGLES

179a. Theorem: *The sine of any angle divided by its cosine equals the tangent of that angle.*



Given: A, any acute angle.

To Prove:

$$\frac{\text{Sin } A}{\text{Cos } A} = \text{Tan } A$$

Proof: Complete the rt. $\triangle ABC$

$$\text{Tan } A = \frac{a}{b} \qquad (169c)$$

Since $\frac{a}{b}$ is a fraction, both of its terms may be divided by the same quantity without changing its value. Hence,

(I) $$\text{Tan } A = \frac{a}{b} = \frac{\frac{a}{c}}{\frac{b}{c}}$$

But $\frac{a}{c} = \text{Sin } A$ and $\frac{b}{c} = \text{Cos } A$ (169c).

Substituting these values in (I),

$$\text{Tan } A = \frac{\text{Sin } A}{\text{Cos } A} \quad Q.\ E.\ D.$$

By clearing this equation of fractions,

179b. Tan A Cos A = Sin A

NOTE: The proof for the case when $\angle A$ is obtuse may be worked out after studying the theorems on the functions of obtuse angles, given in the latter part of this chapter.

EXERCISE

1. Check the tangent of the following angles, as given in the table of Chapter XIV, by dividing the sine by the cosine:

(a)	(b)	(c)	(d)	(e)
20°	65°	50°	70°	15°

2. Check the sine of the following angles by multiplying the tangent by the cosine:

(a)	(b)	(c)	(d)	(e)
10°	35°	55°	70°	50°

3. By aid of the table in Chapter XV, derive to four places of decimals the tangent of the following angles:

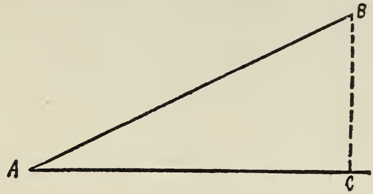
(a)	(b)	(c)	(d)
31°	47°	38° 15′	11° 24′

4. Prove that $\text{Cot } A = \frac{\text{Cos } A}{\text{Sin } A}$

Hint: Proof is similar to that of 179a.

180a. Unity means the numeral **one**. This term is used to avoid confusion when **one** is used abstractly (that is, without qualifying any object) as in the following theorem.

180b. Theorem: *The square of the sine of any angle, plus the square of the cosine of that angle, equals unity.*



Given: $\angle A$ any acute $\angle$.
To Prove: $(\text{Sin } A)^2 + (\text{Cos } A)^2 = 1$
Proof: Complete the right triangle, BAC.
$a^2 + b^2 = c^2$ (Pythagorean Theorem)
Dividing each term of this equation by c^2,

(I) $\dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} = \dfrac{c^2}{c^2}$

But

$$\frac{a^2}{c^2} = \left(\frac{a}{c}\right)^2$$

And

$$\frac{b^2}{c^2} = \left(\frac{b}{c}\right)^2$$

Substituting in I,
(II)

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

But $\dfrac{a}{c} = \text{Sin } A$, and $\dfrac{b}{c} = \text{Cos } A$

Substituting in II,
$(\text{Sin } A)^2 + (\text{Cos } A)^2 = 1.$

180c. $(\text{Sin } A)^2$ may be written $\text{Sin}^2 A$:
$(\text{Cos } A)^2$ may be written $\text{Cos}^2 A$.

The above theorem may then be conveniently expressed by this formula:

180d. $\mathrm{Sin}^2 A + \mathrm{Cos}^2 A = 1.$

The above formula may be solved for the cosine by transposing Sin^2; thus:

Writing the equation in the general form,

$\mathrm{Sin}^2 + \mathrm{Cos}^2 = 1.$

Transposing, $\mathrm{Sin}^2 = 1 - \mathrm{Cos}^2.$

Extracting the square root of both sides,

180e. $\mathrm{Sin} = \sqrt{1 - \mathrm{Cos}^2}$

In like manner, by solving for the cosine,

180f. $\mathrm{Cos} = \sqrt{1 - \mathrm{Sin}^2}.$

180g. From the two preceding formulas it is evident that the value of the sine is less than unity, since the sine is equal to the square root of unity minus a quantity. For the same reason, the value of the cosine is less than unity. This is evident, also, from the consideration of the fact that the sine is one leg of a right triangle divided by the hypotenuse, and the cosine is the other leg of a right triangle divided by the hypotenuse, for since the hypotenuse of a right triangle is longer than either leg, the numerator of each of these fractions is less than the denominator.

A further consideration of these formulas shows that *as the sine increases the cosine diminishes.* And as the sine approaches unity in value, the cosine approaches zero (nothing) in value.

These results may be summed up in the following:

180h. Rule: *The value of the sine of an angle lies between zero and unity.*

180i. Rule: *The value of the cosine of an angle lies between unity and zero.*

The above formulas may be used to derive functions or to test the accuracy of functions which have been already derived. Thus,

Given: Sin 37° = .6018; find Cos 37°.

Solution:

$$\begin{aligned}\text{Cos } 37° &= \sqrt{1 - (\text{Sin } 37°)^2}\\ &= \sqrt{1 - (.6018)^2}\\ &= \sqrt{1 - .36216324}\\ &= \sqrt{.63783676}\\ &= .7986\end{aligned}$$

Exercise

1. Given: Sin 48° = .7431, find Cos 48° by above formula and compare with value given in the table.

2. Given: Cos 39° = .7771, find Sin 39° by above formula and compare with the value given in the table.

3. Given that Cos 28° = .8829 and Sin 28° = .4695, test their accuracy by squaring both functions and adding. Their sum should equal 1.

4. If the sine of an angle, obtained as the answer to some problem, turns out to be in the form 1.6789, why should the work be repeated? (180h).

5. Find Sin 19° 30′ and Cos 19° 30′. Check as in example 3.

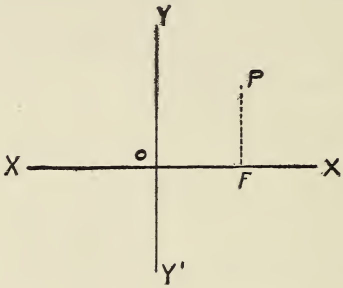
So far the functions considered have been those of angles less than 90°, but it is possible to extend the ideas of functions to include angles greater than a right angle. For dealing with such angles the following system has been found convenient:

6. Let the student prove that the product of the tangent of an angle by the cotangent is unity.

7. What is the product of the tangent and cotangent of 40° as determined from the table on page 180?

8. From the same table determine whether or not the value of the tangent and cotangent are restricted in the same manner as the sine and cosine (180h, 180i).

181a. A plane may be considered as being divided by two intersecting straight lines called **axes**, intersecting each other at right angles. Thus,



181b. The axis parallel to the top of the page is called the axis of X, or X-axis; the axis parallel to the side of the page is called the axis of Y, or Y-axis.

181c. The point of intersection of these two axes is called the **origin.**

As any point in a plane may be considered as the origin, it will be possible to derive the functions of any angle by referring them to an origin at the vertex of the angle—that is, through the vertex of any angle in a plane we may draw two straight lines intersecting at right angles.

181d. These axes divide a plane into four parts called **quadrants.**

XOY is called the First Quadrant, YOX' the Second Quadrant, $X'OY'$ the Third Quadrant, $Y'OX$ the Fourth Quadrant.

The X-axis and Y-axis together are called the **axes of reference.**

181e. The perpendicular distance from any point in the plane to the X-axis is called the **ordinate** of the point; the distance measured along the X-axis from the foot of this ordinate to the origin is called the **abscissa.**

Thus, in the above figure, OF is the abscissa of the point P and PF the ordinate.

Any point in the plane may be located by stating the length of its abscissa and ordinate.

181f. *The abscissa is always written first.*

Thus, the point (5, 3) means the point whose abscissa is 5 and whose ordinate is 3.

If the student attempted to lay out this point without further instruction, he would find four points in the plane as the point (5, 3), thus: measuring 5 units to the right of the origin, and then 3 above the X-axis, he would locate a point in the first quadrant; producing this ordinate for a length of 3 below the X-axis he would locate another point in the fourth quadrant; then measuring 5 to the left of the origin and 3 above the X-axis he would locate another point in the second quadrant; producing this ordinate below the line 3 units he would locate another point in the third quadrant. This ambiguity is removed by observing the following:

181g. Rule: *Abscissas measured to the right of the Y-axis are positive, to the left negative; ordinates measured above the X-axis are positive, below negative.*

Marked according to this method, the point in the first quadrant would be the point (5, 3), that in the second, the point $(-5, 3)$, that in the third the point $(-5, -3)$ and that in the fourth the point $(5, -3)$.

181h. The abscissa and ordinate together are called the **co-ordinates** of a point.

Evidently a point is definitely located when its co-ordinates are stated—as there is only one point in a plane which has a given abscissa and ordinate.

NOTE: A similar system is used for locating places on the earth's surface, in which the longitude gives the location of a place east or west of the meridian of Greenwich, and the latitude gives its position north or south of the equator. The meridian of Greenwich corresponds to the Y-axis, the equator to the X-axis, and their intersection to the origin.

Exercise

Lay off an X and a Y axis (at right angles to each other) and using a half inch as a unit of measure, locate the following points:

(a) $(6, 1)$ (b) $(-3, 7)$ (c) $(5, -1)$ (d) $(-5, -2)$
(e) $(-6, -3)$ (f) $(-2, 1)$ (g) $(-3, -8)$ (h) $(7, -3)$

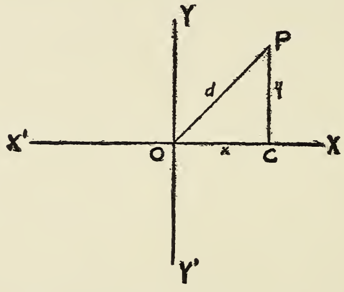
These rules may now be applied to stating the functions of angles. To avoid ambiguity, it is customary to observe the following:

182a. Rule: *An angle begins with the OX line and moves in the contrary direction to the hands of a watch.*

182b. The OX line is called the **initial line.** The other boundary line of the angle is called the **terminal line.**

182c. The **distance** is the hypotenuse of the right triangle of which the abscissa is the base and the ordinate is the altitude. The **distance** is always positive.

NOTE: The distance is always part of the terminal line.



Thus, in the accompanying figure, OX is the initial line, OC is the abscissa, PC is the ordinate, and OP is the distance. From this figure we may derive the following:

182d. *The sine is the ratio of the ordinate to the distance.*
The cosine is the ratio of the abscissa to the distance.
The tangent is the ratio of the ordinate to the abscissa.
The cotangent is the ratio of the abscissa to the ordinate.

Denoting the **abscissa** by x, the **ordinate** by y, and the **distance** by d, we have the following:

$$\text{Sin} \angle POX = \frac{y}{d}$$

$$\text{Cos} \angle POX = \frac{x}{d}$$

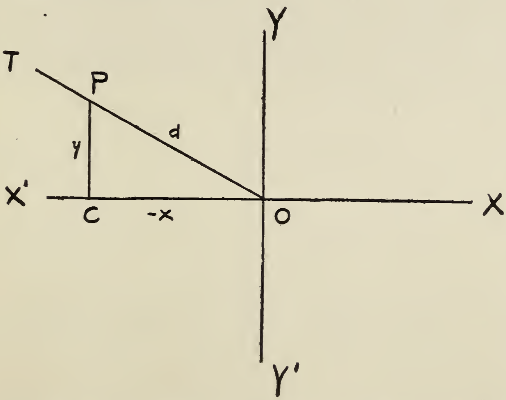
$$\text{Tan} \angle POX = \frac{y}{x}$$

$$\text{Cot} \angle POX = \frac{x}{y}$$

From an inspection of these ratios we may derive the following:

182e. Rule: *All the functions of an angle in the first quadrant are positive.*

In a similar manner, we may derive the functions of an angle in the Second Quadrant, that is, of an angle greater than 90°, but less than 180°.



Thus, in the accompanying diagram, the angle POX has been generated by the revolution of the terminal line, about

O, from the position OX to the position OT. The angle XOT is evidently greater than the right angle XOY.

To compute the functions of $\angle\ XOT$, from any point on TO, as P, a perpendicular is let fall to the X-axis, thereby forming the right triangle PCO. The functions of the angle POC may now be read off as follows:

$$\text{Sin } \angle\ POC = \frac{y}{d}$$

$$\text{Cos } \angle\ POC = \frac{-x}{d} = -\frac{x}{d}$$

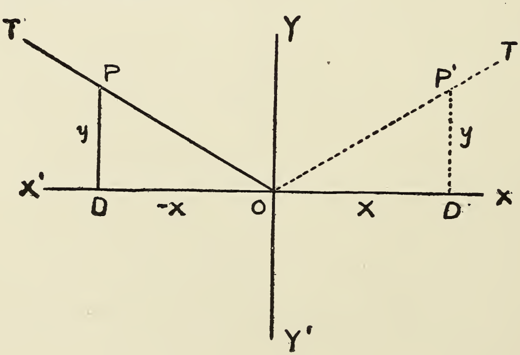
$$\text{Tan } \angle\ POC = \frac{y}{-x} = -\frac{y}{x}$$

$$\text{Cot } \angle\ POC = \frac{-x}{y} = -\frac{x}{y}$$

From an inspection of these ratios we may derive the following:

182f. Rule: *The sine of an angle in the second quadrant is positive; its cosine, tangent, and cotangent are negative.*

This rule enables us to derive a relation between the functions of an angle and those of its supplement. Thus, in the accompanying diagram:



Given: $\angle\ TOX$ an angle in the Second Quadrant, P any point on its terminal line, and PD and OD its ordinate and abscissa.

To Derive: Functions of $\angle\ TOX$ in terms of its supplement.

Derivation: On OX, with O as a vertex, construct the angle $T'OX$ (in the First Quadrant) equal to TOX'. Lay off OP' equal to OP, and from P' let fall a perpendicular $P'D'$.

The rt. $\triangle\ PDO$ = the rt. $\triangle\ P'OD'$ (Why?)

$$\therefore \quad P'D' = PD$$
$$OD' = OD$$
$$OP' = OP$$

The line OP, and its equal OP' may be represented by d, PD, and its equal $P'D'$ by y, and OD by $-x$ and OD' by x.

It is evident from the above diagram that TOX' is the supplement of TOX.

The functions of $T'OX$ (the equal of TOX') will therefore be the functions of the supplement of TOX. $\angle\ T'OX$ will be referred to as "Supplement $\angle\ TOX$." From the above diagram, we may, by inspection, obtain the following ratios:

$$\text{Sin } \angle\ TOX = \frac{y}{d} \qquad \text{Sin (Supplement } \angle\ TOX) = \frac{y}{d}$$

$$\text{Cos } \angle\ TOX = \frac{-x}{d} = -\frac{x}{d} \qquad \text{Cos (Supplement } \angle\ TOX) = \frac{x}{d}$$

$$\text{Tan } \angle\ TOX = \frac{y}{-x} = -\frac{y}{x} \qquad \text{Tan (Supplement } \angle\ TOX) = \frac{y}{x}$$

$$\text{Cot } \angle\ TOX = \frac{-x}{y} = -\frac{x}{y} \qquad \text{Cot (Supplement } \angle\ TOX) = \frac{x}{y}$$

By comparison of these two tables of functions we may obtain the following:

182g. Rule: *The sine of an angle is equal to the sine of its supplement; the cosine of an angle is equal to minus the cosine of its supplement; the tangent of an angle is equal to minus the tangent of its supplement; the cotangent of an angle is equal to minus the cotangent of its supplement.*

This rule enables us to find the functions of an angle greater than 90°, although the table reads only to 90°. For example, to find the sine of 131° 20′ we look up the sine of 48° 40′. Again, to find the cosine of 127° 15′ we look up the cosine of 52° 45′ and write it with a minus sign in front of it.

To find the angle corresponding to the cosine −.3781 we look up the angle corresponding to .3781 and then find the supplement of this angle, since the minus sign indicates that the angle is greater than 90°.

On the other hand, when we are required to look up the angle corresponding to a given sign, it is impossible to determine from the function whether we should use the angle which we take from the table (an acute angle) or its supplement, an obtuse angle.

In solving a triangle, if only one of the angles is given, this peculiarity of the sign requires the application of special rules. (Chap. XVIII.)

Exercise

1. Find the sine of the following angles:

(a) 108° 10′	(b) 124° 25′	(c) 162° 50′
(d) 118° 45′	(e) 121° 35′	(f) 134° 40′

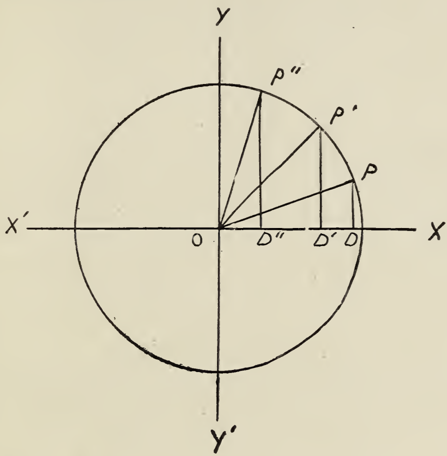
2. Find the cosine of the following angles:

(a) 126° 15′	(b) 149° 40′	(c) 152° 30′
(d) 109° 24′	(e) 115° 48′	(f) 161° 35′

3. Let the student prove that the theorems of 179a and 179b hold true for obtuse angles.

4. Let the student prove that the formulas of 180d, 180e and 180f hold true for obtuse angles.

For comparing the rate at which the functions change as the angle changes, it is convenient to take the terminal line as of fixed length—that is, as it revolves about the origin to consider its extremity as describing a circle, as in the accompanying diagram:



Let the terminal line, as it revolves counter-clockwise about the origin, from OX take the successive positions, PO, $P'O$, $P''O$, and let PD, $P'D'$, and $P''D''$ be the perpendiculars let fall from the points P, P', and P'', which are points on the circumference of a circle.

Therefore:

$$\text{Sin} \angle POX = \frac{PD}{OP}$$

$$\text{Sin} \angle P'OX = \frac{P'D'}{OP'}$$

$$\text{Sin} \angle P''OX = \frac{P''D''}{OP''}$$

Since $OP = OP' = OP''$, they may be replaced by r (radius); these ratios may then be written as follows:

$$\text{Sin } \angle\ POX = \frac{PD}{r}$$

$$\text{Sin } \angle\ P'OX = \frac{P'D'}{r}$$

$$\text{Sin } \angle\ P''OX = \frac{P''D''}{r}$$

By inspection it can be seen that as the angle increases the value of the numerators of these fractions increases, while the denominators remain constant in value—that is, the value of the fraction increases. Likewise:

$$\text{Cos } \angle\ POX = \frac{OD}{r}$$

$$\text{Cos } \angle\ P'OX = \frac{OD'}{r}$$

$$\text{Cos } \angle\ P''OX = \frac{OD''}{r}$$

By inspection it can be seen that as the angle increases the value of the numerator of these fractions decreases, while the denominator remains constant—that is, the value of the fraction decreases.

Note: If desired, this might be rendered more obvious by drawing the angles so that the lines could be exactly scaled. Thus, the successive values might be written:

$$\text{Sin } \angle\ POX = \frac{3}{10} \qquad \text{Sin } \angle\ P'OX = \frac{7}{10}$$

$$\text{Sin } \angle\ P''OX = \frac{9}{10}$$

This makes it evident that as the angle increases the sine increases.

Likewise (by drawing another figure),

$$\text{Cos} \angle POX = \frac{8}{10}$$

$$\text{Cos} \angle P'OX = \frac{5}{10}$$

$$\text{Cos} \angle P''OX = \frac{2}{10}$$

These results may be summed up in the following:

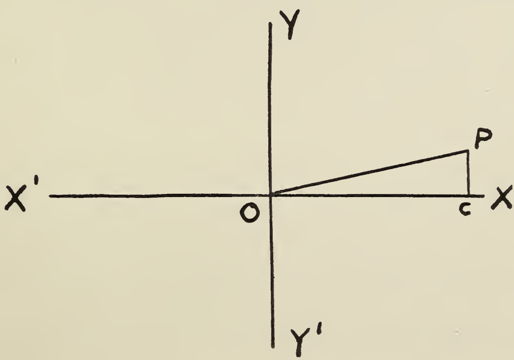
183a. Rule: *As the angle increases from* 0° *to* 90°, *the sine increases while the cosine decreases.*

Let the student compare this rule with arts. 177a and 177b.

Exercise

1. Check the accuracy of the above rule by tracing the successive values of the sine and cosine from 1° to 89° in the table.

2. Construct a diagram and prove that this rule is reversed for angles in the second quadrant.



This method of comparing the values of the functions of angles by considering the terminal line as of definite length, brings out another important fact.

When the terminal line swings from OX to the position OP, describing a small angle, the movement of its outer extremity approximates a perpendicular to the X-axis—that is, its ordinate increases very rapidly, while its abscissa decreases very slightly. In other words, for very small angles, the sine changes in value very rapidly, while the cosine changes very slowly. For angles near 90° this is reversed.

Let the student verify this statement by comparing the change for 1° in these functions for angles less than 5° with the changes for 1° with some angles near 45°.

Evidently, if any error has been made in observing the angle, if it falls in that place in the table where the function used is changing rapidly, the error of observation will cause a great error in the function to be used.

On the other hand, if, in obtaining the answer to a problem, it is necessary to look up the angle corresponding to a slowly changing function, any slight error in the work will cause a large error in the angle; in other words, if the angle falls in that part of the table where the function used is changing so slowly that any slight change in it means a great change in the angle, a slight error in the function may cause a large error in the angle. These observations lead to the following:

183b. Rule: *If possible, avoid using the tables of functions for angles between* 0° *and* 5° *or* 85° *and* 90°.

NOTE: The "errors" referred to are not actual mistakes, but errors of observation beyond the limit of the instrument to detect, while it should be remembered that the tables are correct only to a certain decimal place. In the former connection it is well to remember that with an expensive surveying instrument it is not easy to measure an angle exactly to the nearest quarter of a minute. The ordinary school arithmetics, by giving examples in degrees, minutes and seconds, are perhaps responsible for the impression that a second of arc is a quantity easily measured.

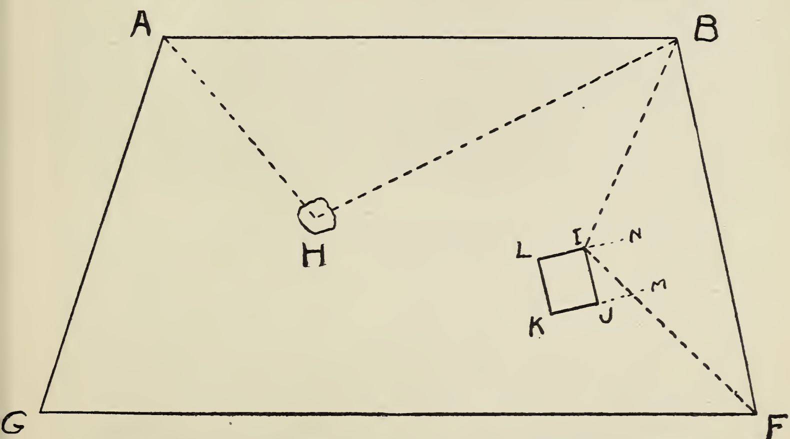
Since, in laying out a triangle in surveying, the size of one angle affects the size of the others, and it is consequently not possible to measure all the angles so that they will fall near the middle of the table, the following rule has been approved by practice:

183c. Rule: *The best form of triangle for computation is one that approximates the equilateral.*

This means practically that it is better not to lay any angle of the triangle larger than 75° or less than 35°, in order to obtain the best results in the computation of the angles or the sides. In laying out a triangle by its sides the same thing should be kept in mind—that is, the three sides of a triangle should be approximately of the same length.

(XVI) Measurements for Topographical Map of Triangulation

Equipment: Transit, ranging pole, measuring tape.



Procedure: In taking measurements for the preparation of a map of the same field measured in Field Exercise XV the various prominent features may be located by taking two observations of the object with the transit. Thus, to locate a tree (H) the angles BAH and ABH should be measured, and since AB is known the length AH and BH may be solved by the method of the following chapter.

Any other side of the field will do as well as AB for locating topographical features, since the lengths of these sides are solvable (Field Exercise XV).

This method serves very well for locating objects of no great length, horizontally, such as trees, etc., but it will not do to locate buildings.

To locate the building $IJKL$, the position of one corner of the building may be determined by the method just explained (by measuring the angles IBF and BFI). Then a stake (M) may be located in the line KJ produced, and another stake (N) located in LI produced so that $JM = IN$. (The line MN is parallel to the side IJ of the building, and if produced to intersect AB will make the same angle with AB as IJ would if produced.)

Set up the transit at M and locate the point (O) where MN produced intersects AB. Set up the instrument at O and measure BOM. Since the corner N can be located on the map, the angle BOM will enable us to have the representation of the building make the proper angles with the side of the field.

A North and South line may be determined in the following manner. (The ranging pole will serve for this determination, but it would be better if some much taller object in the field could be used.) Three hours before noon mark the shadow of the extremity of the tall object with a stake. Three hours after noon mark the extremity of the shadow with a stake. Bisect the angle between the tall object and the two stakes, and mark the point determined in the bisector with the stake, P. Produce the line from P to the tall object so that it intersects one side of the field. The angle between this line (which is the North and South line) and the side of the field will be the "bearing" of the field (remember that as we face the north, east is on the right hand and west on the left).

This determination should be repeated, with reference to the same side of the field, and the average used.

CHAPTER XVII

SOLUTION OF OBLIQUE TRIANGLES

184. In the solution of oblique triangles, four cases may be distinguished among the given parts:

(I) **One side and any two angles.**

(II) **Two sides and their included angle.**

(III) **Three sides.**

(IV) **Two sides and the angle opposite one of them.**

Notice that in each case three parts are given. One of the three given parts must be a side: if the three given parts were the angles, only the shape of the triangle could be determined—not its size (Chap. XIII). Notice that in the first three cases the given parts are the same as in the theorems for proving triangles equal (Chap. IV)—in other words a triangle is completely determined when these parts are known.

The first two cases cover nearly all the practical cases occurring in surveying with the transit; the third occurs in surveying with the chain (or tape) alone. The fourth case is called the **ambiguous case** and should be avoided in practice.

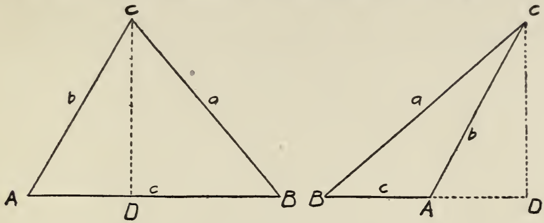
Case I. *Given one side and any two angles.*

NOTE: It is immaterial which two angles are given, since the third angle may always be found by subtracting the sum of the two given angles from 180° (80a).

The problems occurring under Case I may be solved by means of the Sine Law, which is derived as follows:

185a. Theorem: *In any triangle the sides are proportional to the sines of their opposite angle.*

Part I. When the altitude falls upon the base.



Given: ABC an acute-angled $\triangle$, in which a, b, and c are the sides opposite the $\measuredangle$ A, B, and C, respectively.

To Prove:

$$\frac{a}{b} = \frac{\text{Sin } A}{\text{Sin } B} \qquad \frac{c}{a} = \frac{\text{Sin } C}{\text{Sin } A} \qquad \frac{b}{c} = \frac{\text{Sin } B}{\text{Sin } C}$$

Proof: From C draw the altitude CD, or h.

Then, in rt. $\triangle$ CDA, $\quad \text{Sin } A = \frac{h}{b}$

In rt. $\triangle$ CDB, $\quad \text{Sin } B = \frac{h}{a}$

Dividing the first equation by the second,

$$\frac{\text{Sin } A}{\text{Sin } B} = \frac{h}{b} \div \frac{h}{a} = \frac{a}{b}$$

In like manner. by drawing the altitude from B, it may be proved that

$$\frac{\text{Sin } C}{\text{Sin } A} = \frac{c}{a}$$

and by drawing the altitude from A it may be proved that

$$\frac{\text{Sin } B}{\text{Sin } C} = \frac{b}{c}$$

Let the student complete this part of the proof in detail.

Part II. When the altitude falls upon the base produced.

Given: $\angle A$ is the obtuse $\angle$ of the $\triangle ABC$, in which a, b, and c are the sides opposite $\angle s\ A$, B, and C, respectively.

To Prove:

$$\frac{a}{b} = \frac{\text{Sin } A}{\text{Sin } B}$$

$$\frac{c}{a} = \frac{\text{Sin } C}{\text{Sin } A}$$

$$\frac{b}{c} = \frac{\text{Sin } B}{\text{Sin } C}$$

Proof: From C draw the altitude CD (or h) meeting AB produced at D.

Then in $\triangle BDC$,

$$\text{Sin } B = \frac{h}{a}$$

$\angle CAD$ is the supplement of $\angle A$,

$\therefore$ Sin A = Sin CAD (182g).

Now in $\triangle CAD$,

$$\text{Sin } CAD = \frac{h}{b}$$

Substituting for Sin CAD its equal, Sin A

$$\text{Sin } A = \frac{h}{b}$$

Dividing this last equation by the first,

$$\frac{\text{Sin } A}{\text{Sin } B} = \frac{a}{b}$$

As in Part I it may now be proved that

$$\frac{\text{Sin } C}{\text{Sin } A} = \frac{c}{a} \text{ and } \frac{\text{Sin } B}{\text{Sin } C} = \frac{b}{c}$$

185b. The above equations may be written as proportions, thus,

$$a:b = \text{Sin } A : \text{Sin } B$$
$$c:a = \text{Sin } C : \text{Sin } A$$
$$b:c = \text{Sin } B : \text{Sin } C$$

185c. By alternation,

$$\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} \qquad (164\text{a})$$

$$\frac{c}{\text{Sin } C} = \frac{a}{\text{Sin } A}, \qquad \text{whence}$$

$$\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C} \qquad (31)$$

Problem: Solve $\triangle\ ABC$, given that $\angle\ A = 62°\ 30'$, $\angle\ B = 64°\ 45'$ and side $a = 84.5$

Solution: The first step is to form a proportion involving the three given parts and one unknown part: thus,

I $\qquad \text{Sin } A : \text{Sin } B = a : b \qquad$ (185b)

Substituting the values of the given parts

II $\qquad \text{Sin } 62°\ 30' : \text{Sin } 64°\ 45' = 84.5 : b$

$$30' = \frac{30°}{60} = \frac{1°}{2} \qquad\qquad 45' = \frac{45°}{60} = \frac{3°}{4}$$

$$\begin{array}{rl} \text{Sin } 63° = & .8910 \\ \text{Sin } 62° = & .8829 \\ \hline T.D = & .0081 \\ \frac{1}{2}\ T\ D = & 0041 \end{array} \qquad\qquad \begin{array}{rr} \text{Sin } 65° = & .9063 \\ \text{Sin } 64° = & .8988 \\ \hline T.D = & .0075 \\ & 3 \\ \hline 4 \,|\, & .0225 \\ \hline & .0056 \end{array}$$

$$\begin{array}{lr} \text{Sin } 62^\circ = & .8829 \\ & .0041 \\ \hline \text{Sin } 62^\circ\ 30' = & .8870 \end{array} \qquad \begin{array}{lr} \text{Sin } 64^\circ = & .8988 \\ & .0056 \\ \hline \text{Sin } 64^\circ\ 45' = & .9044 \end{array}$$

Substituting these values in (II),

$$.8870 : .9044 = 84.5 : b$$

$$.8870\, b = .9044 \times 84.5$$

$$b = \frac{.9044 \times 84.5}{.8870} = 86.16$$

To find $\angle C$,

$$\begin{array}{lr} \angle A = & 62^\circ\ 30' \\ \angle B = & 64^\circ\ 45' \\ \hline & 127^\circ\ 15' \end{array} \qquad \begin{array}{lr} & 179^\circ\ 60' \\ & 127^\circ\ 15' \\ \hline \angle C = & 52^\circ\ 45' \end{array}$$

(III) $\text{Sin } A : \text{Sin } C = a : c.$

(IV) $\text{Sin } 62^\circ\ 30' : \text{Sin } 52^\circ\ 45' = 84.5 : c$

$$45' = \frac{3^\circ}{4}$$

$$\begin{array}{lr} \text{Sin } 53^\circ = & .7986 \\ \text{Sin } 52^\circ = & .7880 \\ \hline & .0106 \\ & 3 \\ \hline 4\ \big| & .0318 \\ \hline & .0080 \end{array} \qquad \begin{array}{lr} \text{Sin } 52^\circ = & .7880 \\ & .0080 \\ \hline \text{Sin } 52^\circ\ 45' = & .7960 \end{array}$$

Sin 62° 30′ has already been found.
Substituting these values in (IV),

$$.8870 : .7960 = 84.5 : c$$

$$.8870c = .7960 \times 84.5$$

$$c = \frac{.7960 \times 84.5}{.8870} = 75.83$$

Notice that in solving for c, the $\angle A$ and side a are used, as in solving for b. This is the best method—to use the given parts in solving for both of the unknown sides. If b had been used in solving for c, an error (due to the inaccuracy of the decimals) occurring in b would be increased in carrying it through the computation for c.

Notice the advantage by this method of computation of having the angles of nearly the same size—that is, of having a triangle that is nearly equilateral. In the above computation,

$$c = \frac{.7960 \times 84.5}{.8870}$$

Since the sine of $\angle A$ (.8870) is not very greatly different in value from the sine of $\angle C$ (.7960), in the above fraction 84.5 is approximately multiplied and divided by the same quantity—that is, any error of computation or measurement in the terms of the above fraction is not increased as it would be if the sine in the denominator were much smaller than the sine in the numerator.

Exercise

Solve the triangle ABC, given that:

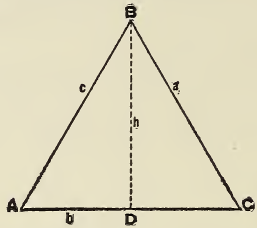
1.	$A = 61° 20'$,	$B = 62°$,	$b = 101$
2.	$A = 59° 10'$,	$C = 64° 50'$,	$c = 95$
3.	$B = 56° 40'$,	$C = 63° 25'$,	$a = 201$
4.	$A = 41° 48'$,	$B = 72° 35'$,	$b = 97$
5.	$C = 47° 36'$,	$B = 65° 35'$,	$c = 117$
6.	$C = 43° 24'$,	$B = 71° 6'$,	$a = 1.46$
7.	$A = 58° 30'$,	$C = 59° 45'$,	$b = 2.56$

Case II. *Given two sides and their included angle.*

Problems occurring under Case II may be solved by aid of the following theorem, which is known as the **Cosine Law.**

186a. Theorem: *In any oblique triangle the square of any side is equal to the sum of the squares of the other two sides minus twice their product times the cosine of their included angle.*

Part I. When the altitude falls upon the base.



Given: ABC an acute-angled triangle, in which a, b, and c are the sides opposite the $\angle$s A, B, and C, respectively.

To Prove:

$$a^2 = b^2 + c^2 - 2bc \text{ Cos } A$$
$$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$$
$$c^2 = a^2 + b^2 - 2ab \text{ Cos } C$$

Proof: From B let fall the perpendicular h, intersecting AC at D.

Then AD is the projection of c on AC (149).

I. $\therefore\ a^2 = b^2 + c^2 - 2b\,(AD)$ (150b).

But in rt. $\triangle\ ABD$,

$$\frac{AD}{c} = \text{Cos } A$$

Multiplying both sides of this equation by c,

$$AD = c \text{ Cos } A$$

Substituting this value for AD in (I),

$$a^2 = b^2 + c^2 - 2bc \text{ Cos } A.$$

In like manner, by drawing the altitude from C, it may be proved that

$$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$$

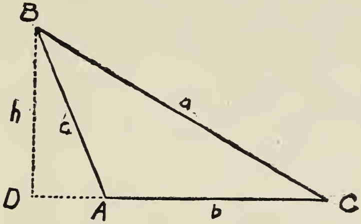
Also by drawing the altitude from A it may be proved that

$$c^2 = b^2 + a^2 - 2ab \text{ Cos } C$$

Let the student complete this part of the proof in detail.

Part II. *When the altitude falls upon the base produced.*

Given: $\angle\ A$ is the obtuse $\angle$ of the $\triangle\ ABC$, in which a, b, and c are the sides opposite $\measuredangle\ A$, B, and C, respectively.



To Prove:

$$a^2 = b^2 + c^2 - 2bc \text{ Cos } A$$
$$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$$
$$c^2 = a^2 + b^2 - 2ab \text{ Cos } C$$

Proof: From B draw the altitude h, meeting CA produced at D.

Then DA is the projection of c on b (149).

II. $\therefore$ $$a^2 = b^2 + c^2 + 2b\ (AD) \qquad (150c)$$

Since DC is a straight line, $\angle\ BAD$ is the supplement of $\angle\ BAC$ (43b).

$\therefore$ Cos $\angle\ DAB = -$ Cos $\angle\ BAC$ (182g).

Now in $\triangle\ BAD$,

$$\text{Cos } \angle\ DAB = \frac{AD}{c}$$

Whence, multiplying both sides of the equation by c, $DA = c$ Cos $\angle\ DAB$.

Substituting ($-$Cos BAC) for its equal Cos $\angle\ DAB$ in this equation, $DA = c\ (-$ Cos $\angle\ BAC)$.

Substituting this value for DA in (II),

$$a^2 = b^2 + c^2 + 2bc\ (- \text{ Cos } \angle\ BAC)$$

Whence,

$$a^2 = b^2 + c^2 - 2bc\ (\text{Cos } \angle\ BAC)$$

Since in the given triangle (exclusive of construction lines) there is only one angle at A, $\angle BAC$ may be written $\angle A$ and so the above equation becomes

$$a^2 = b^2 + c^2 - 2bc \text{ Cos } A$$

It may now be proved as before that

$$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$$
$$c^2 = a^2 + b^2 - 2ab \text{ Cos } C \qquad Q.\ E.\ D.$$

If $\angle A$ is a right angle, cos A is 0, and so the factor $2bc$ Cos A becomes 0 (since 0 times any quantity is 0); that is, if $\angle A$ is 90° the equation

$$a^2 = b^2 + c^2 - 2bc \text{ Cos } A$$

reduces to

$$a^2 = b^2 + c^2$$

Since here a represents the side opposite the right angle, the formula therefore is a statement of the Pythagorean Theorem.

In the same manner it may be proved that

$$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$$
$$c^2 = b^2 + a^2 - 2ab \text{ Cos } C$$

reduces to the Pythagorean Theorem.

Problem: Solve for the remaining parts of the triangle ABC, given that $a = 48$, $b = 52$, $C = 58° \, 10'$.

Solution: $c^2 = a^2 + b^2 - 2ab \text{ Cos } C$ (186a).

Extracting the square root of both sides of the equation,

$$c = \sqrt{a^2 + b^2 - 2ab \text{ Cos } C}$$

Substituting the values of the given parts,

$$\begin{aligned} c &= \sqrt{(48)^2 + (52)^2 - 2\,(48)\,(52)\,(.5275)} \\ &= \sqrt{2304 + 2704 - (4992)\,(.5275)} \\ &= \sqrt{5008 - 2633.28} \\ &= \sqrt{2374.72} \\ &= 48.73 \end{aligned}$$

$\angle A$ may now be found by aid of the **Sine Law,** thus

$$\text{Sin } A : a = \text{Sin } C : c$$

Substituting the value of sine of $\angle C$ (58° 10′) from the table and also substituting the values of a and of c,

$$\text{Sin } A : 48 = .8496 : 48.73$$

$$(48.73)\ (\text{Sin } A) = 48\ (.8496)$$

$$\text{Sin } A = \frac{48\ (.8496)}{48.73} = .8368$$

$$\angle A = 56^\circ\ 48'$$

$\angle B$ might now be found by adding $\angle A$ and $\angle C$ and then subtracting their sum from 180, but it is better to find $\angle B$ independently, and then use the theorem that the sum of the three angles is 180 as a check on the work, thus:

$$\text{Sin } B : b = \text{Sin } C : c$$

$$\text{Sin } B : 52 = .8496 : 48.73$$

$$(48.73)\ (\text{Sin } B) = (.8496)\ (52)$$

$$\text{Sin } B = \frac{(.8496)\ (52)}{48.73} = .9066$$

$$\angle B = 65^\circ\ 2'$$

Check

$$\begin{array}{r} \angle A = 56^\circ\ 48' \\ \angle B = 65^\circ\ \ 2' \\ \underline{\angle C = 58^\circ\ 10'} \\ 180^\circ \end{array}$$

186b. For use in solving such problems as the above, the formulas of 186a may be modified as follows:

$$a = \sqrt{b^2 + c^2 - 2bc \cos A}$$

$$b = \sqrt{a^2 + c^2 - 2ac \cos B}$$

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}$$

Exercise

Solve the $\triangle\ ABC$, given that:

1. $a = 432,\quad b = 321,\quad C = 57° 30'$.
2. $c = 371,\quad b = 325,\quad A = 62° 24'$.
3. $b = 231,\quad a = 263,\quad C = 59° 30'$.
4. $c = 432,\quad a = 394,\quad A = 49° 35'$.
5. $b = 243,\quad c = 262,\quad A = 70° 15'$.

Case III. *Given the three sides.*

By means of the Cosine Law the angles of a triangle may be computed when the three sides are known. By 186a,

$$a^2 = b^2 + c^2 - 2bc \text{ Cos } A$$

Transposing,

$$2bc \text{ Cos } A = b^2 + c^2 - a^2$$

Dividing by the coefficient of Cos A,

$$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2bc}$$

In like manner the value of Cos B may be found from the equation,

$$b^2 = a^2 + c^2 - 2ac \text{ Cos } B$$

and the value of Cos C from the equation,

$$c^2 = a^2 + b^2 - 2ab \text{ Cos } C$$

Hence,

186c.

$$\text{Cos } A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\text{Cos } B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\text{Cos } C = \frac{a^2 + b^2 - c^2}{2ab}$$

These formulas may all be included in the following:

Rule: *In any triangle the cosine of any angle is equal to the sum of the squares of the two sides including the angle, minus the square of the opposite side, all divided by twice the product of the two sides including the angle.*

Problem: Solve for the angles of the $\triangle ABC$, given that $a = 232$, $b = 325$, and $c = 251$.

Solution: First square all three sides:

$$a^2 = 53{,}824, \quad b^2 = 105{,}625, \quad c^2 = 63{,}001$$

Next substitute the given values in the formulas (186c).

Do not substitute in the first formula only, and then work that out, then substitute in the second, and work that out, etc., but substitute in all three. This is much less confusing. Thus:

$$\text{Cos } A = \frac{105{,}625 + 63{,}001 - 53{,}824}{2 \cdot 325 \cdot 251}$$

$$A =$$

$$\text{Cos } B = \frac{53{,}824 + 63{,}001 - 105{,}625}{2 \cdot 232 \cdot 251}$$

$$B =$$

$$\text{Cos } C = \frac{53{,}824 + 105{,}625 - 63{,}001}{2 \cdot 232 \cdot 325}$$

$$C = \underline{\qquad\qquad}$$

Check:

This arrangement of the work (called a "skeleton") has the advantage of separating the more exacting work of substituting in the formula from the more mechanical work of computation. The values of the cosines may now be computed. After *all three* of the cosines have been computed, the table of sines and cosines may be opened and the values of all three cosines written down at the same time.

Remember that the rule for rapid and accurate computation is "**never open the table of functions until everything possible has been done without it.**"

The work now appears as follows:

$$\text{Cos } A = \frac{105{,}625 + 63{,}001 - 53{,}824}{2 \cdot 325 \cdot 251} = .7037$$

$$A = 45° \ 17'$$

$$\text{Cos } B = \frac{53{,}824 + 63{,}001 - 105{,}625}{2 \cdot 232 \cdot 251} = .0962$$

$$\angle B = 84° \ 29'$$

$$\text{Cos } C = \frac{53{,}824 + 105{,}625 - 63{,}001}{2 \cdot 232 \cdot 325} = .6396$$

$$\angle C = 50° \ 15'$$

$$\text{Check, } 180° \ 1'$$

The three angles should always be computed separately (as in the above arrangement) and added to test the accuracy of the work. To compute two of the angles and subtract their sum from 180 would "lump" the error of computation on one angle.

Exercise

Find the angles of the $\triangle ABC$, given that

1. $a = 27$, $b = 32$, $c = 29$.
2. $a = 52$, $b = 56$, $c = 48$.
3. $a = 107$, $b = 112$, $c = 121$.
4. $a = 2.34$, $b = 2.56$, $c = 2.24$.
5. $a = 982$, $b = 1{,}003$, $c = 1{,}175$.
6. $a = 5{,}218$, $b = 4{,}927$, $c = 5{,}127$.

186d. It must be borne in mind that in practical work, all measurements are subject to errors of observation. For example, in measuring one side of a triangle errors may arise due to the stretching of the tape, the irregularity of the ground, and other causes. If, in measuring one side of a triangle sufficient care was taken (by leveling the tape, etc., and taking the mean of repeated observations) so that the result was correct to the nearest tenth of a foot, it would be absurd to combine this in computation with a side which was accurate only to the nearest foot—as the computed side would be correct only to the nearest foot.

The number of figures in the length of a side is considered as indicating the accuracy with which the measurements have been made. Thus, if one side of a rectangle is 8 feet long and another 79 feet long, the presumption is that each side is correct only to the nearest foot, but as one foot is a much larger percentage of 8 feet than of 79 feet, the longer side would be considered as being more accurately measured, and to combine these two sides in computation would disregard the principle laid down in the preceding paragraph. To obviate this, the 8 foot side should be measured to the nearest tenth of a foot. Note that if .3 of a foot were disregarded in the short side of this rectangle the area would be in error by nearly 24 sq. ft. Hence,

Rule: *Let all lines, both measured and calculated, show the same number of figures.*

Note: This rule may be disregarded if the sides are nearly equal in value; for example, a side 98 feet long may be combined in computation with one 103 feet long.

Obviously, the computed results can be no more accurate than the measurements—it is useless to carry out the computation of a side to more places of decimals than a measured side. The same principles apply to the angles.

The degree of precision in the computations, consistent with the accuracy of the data, is as follows:

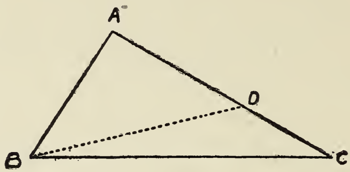
187. Two significant figures in the sides, angles correct to the nearest half degree.

Three significant figures in the sides, angles correct to the nearest 5′.

Four significant figures in the sides, angles correct to the nearest minute.

The following theorems will also be of use in checking the results:

188a. *If the sides of a triangle are unequal, the largest angle is opposite the longest side, and the smallest angle is opposite the shortest side.*



Given: In $\triangle ABC$, BC is the longest side and AB is the shortest side.

To Prove: $\angle A$ is the largest $\angle$ and $\angle C$ is the smallest $\angle$.

($\angle B$ means $\angle ABC$, and $\angle A$ means $\angle BAC$.)

Proof: Lay off $AD = AB$.

Draw BD.

($\angle ADB$ is the exterior $\angle$ of the $\triangle DBC$, and also one of the equal angles of the isosceles $\triangle ABD$.)

$\angle ADB > \angle C$ (81c)

$\angle ABD = \angle ADB$ (77b).

Substituting in the above inequality,

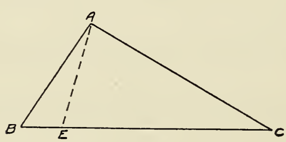
(I) $\angle ABD > \angle C$.

(II) $\angle B > \angle ABD$ (33a).

Much more so, then,

(III) $\angle B > \angle C$.

The reasoning by which (I) and (II) are combined to obtain (III) is much the same as saying, "$5 > 3$, $7 > 5$, much more so, then, is $7 > 3$."



Lay off $EC = AC$.

Draw AE.

($\angle AEC$ is the ext. $\angle$ of the $\triangle AEB$, and also one of the equal angles of the $\triangle CAE$.)

$\angle AEC > \angle B$ (81c).

But $\angle AEC = \angle EAC$ (77b).

Substituting for $\angle AEC$ its value in the above inequality,

(IV) $\angle EAC > \angle B$.

(V) But $\angle A > \angle EAC$.

Much more so, then, is

(VI) $\angle A > \angle$ B.

By (VI) and (III), since $\angle A > \angle B$ and $\angle B > \angle C$.

(VII) $\angle A > \angle C$.

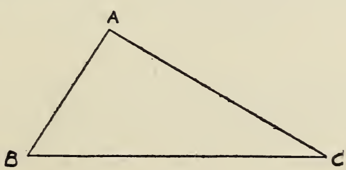
By (VI) and (VII),

$\angle A$ is the largest $\angle$ of the $\triangle$.

(Since it is greater than either of the others.) The smallest angle must be one of the remaining two, and since $\angle B > \angle C$, $\angle C$ is the smallest. *Q. E. D.*

If the results of any problem are inconsistent with this theorem, the work is wrong. Let the student apply this test to the answers to the problem of 186c.

188b. Theorem: *If the angles of a triangle are unequal, the longest side is opposite the largest angle, and the shortest side is opposite the smallest angle.*



Given: In $\triangle ABC$, $\angle A$ is the largest $\angle$ and $\angle C$ is the smallest $\angle$.

To Prove: BC is the longest side and AB is the shortest side.

Proof: The relation between BC and AC must be one of these three:

(1) $BC = AC$.

(2) $BC < AC$.

(3) $BC > AC$.

(1) BC cannot be equal to AC, for if it were $\triangle ABC$ would be isosceles and $\angle A$ would equal $\angle B$ (77b).

But this contradicts what is given, that $\angle A$ is the largest $\angle$.

$\therefore$ (1) cannot be true.

(2) BC cannot be less than AC, for if it were $\angle B$ would be larger than $\angle A$ (188a).

But this contradicts what is given, that $\angle A$ is the largest $\angle$.

$\therefore$ (2) cannot be true.

$\therefore$ (3) the only remaining relation must be true, that is,

(I) $BC > AC$.

The relation between AC and AB must be one of these three:

(4) $AC = AB$.

(5) $AC < AB$.

(6) $AC > AB$.

(4) AC cannot be equal to AB, for if it were, then $\triangle ABC$ would be isosceles, and $\angle B$ would equal $\angle C$.

But this contradicts what is given, that $\angle C$ is the smallest $\angle$.

$\therefore$ (4) cannot be true.

(5) AC cannot be less than AB, for if it were $\angle C$ would not be the smallest $\angle$.

But this contradicts what is given, that $\angle C$ is the smallest $\angle$.

$\therefore$ (5) cannot be true, that is,

(II) $AC > AB$.

By (I) and (II),

$BC > AC$, and $AC > AB$.

Hence,

(III) $BC > AB$.

By (I) and (III),

BC is the longest side of the $\triangle$.

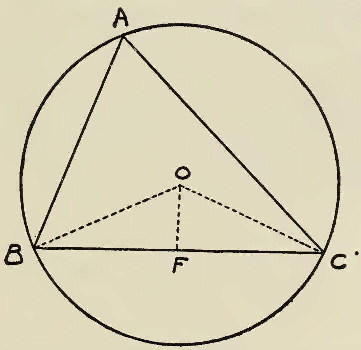
(Since it is longer than either of the others.)

The shortest side must be one of the two remaining sides, and since $AC > AB$, AB is the shortest side. *Q. E. D.*

If the results of any problem are inconsistent with this theorem the work is wrong.

Let the student apply this to the answers to the problem under the Sine Law. (Page 230.)

189. Theorem: *The ratio of any side of a triangle to the sine of the opposite angle is numerically equal to the diameter of the circumscribed circle.*



Given: O is the center of the circle circumscribed about $\triangle ABC$, and R its radius; a, b, and c the sides opposite the $\measuredangle$s A, B, and C, respectively.

To Prove: $2R = \dfrac{a}{\text{Sin } A} = \dfrac{b}{\text{Sin } B} = \dfrac{c}{\text{Sin } C}$

Proof: Draw OB and OC; draw $OF \perp$ to BC.

$\angle BOC$ is an isos. $\triangle$ (18).

OF bisects $\angle BOC$ (80d).

That is,

(I) $\angle BOF = 1/2 \angle BOC.$

Again, $\angle A$ is measured by $1/2$ arc BC (89a).

$\angle BOC$ is measured by all the arc BC (26).

(II) $\therefore \angle A = 1/2 \angle BOC.$

By (I) and (II),

(III) $\angle BOF = \angle$ A.

$BF = 1/2\, a$ (80d).

That is,

(IV) $a = 2BF.$

In the rt. $\triangle BOF$,

$BF = R \text{ Sin } \angle BOF$ (172)

$= R \text{ Sin } A$ (by III)

By (IV)

$a = 2R \text{ Sin } A.$

Dividing both sides of the equation by Sin A,

$$\frac{a}{\text{Sin } A} = 2R$$

By the Sine Law,

$$\frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C}$$

Whence,

$$2R = \frac{a}{\text{Sin } A} = \frac{b}{\text{Sin } B} = \frac{c}{\text{Sin } C} \quad \textit{Q. E. D.}$$

Exercise

1. Let the student work out another proof for the Sine Law, as follows: As in the above proof, from $\triangle BOC$ prove that $\frac{a}{\text{Sin } A} = 2R.$

In like manner, by drawing OA, and a $\perp$ from O to b, it may be proved that $\frac{b}{\text{Sin } B} = 2R.$

Also, by drawing a $\perp$ from O to c prove that $\dfrac{c}{\text{Sin } C} = 2R$.

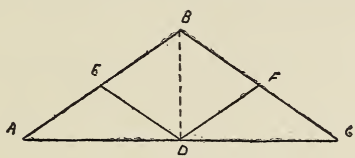
Equate these three values of $2R$.

2. Draw a triangle and circumscribe a circle about it. Measure the sides of the triangle and the diameter of the circle: change all fractions of an inch to decimals. Measure the angles with a protractor, look up their sines, and divide each side by the sine of the opposite angle. Compare the quotients thus obtained with the diameter of the circle.

3. The diagonal of a parallelogram is 8 centimeters long, the adjacent sides at one of its extremities make angles of 43° 17′ and 75° 42′ with it. Find the lengths of the sides (64, 122b).

4. An observer knows that he is 2.7 miles from one fort and 3.6 miles from another; from where he stands the distance between the forts subtends an angle of 24° 29′. Find the distance between the forts.

5. A roof in which the rafters have a rise of 9 ft. and a run of 13 ft. is to be supported by a "scissor-beam" truss constructed according to the following diagram:



The middle of each rafter is to be supported by a strut reaching to the middle point (D) of a beam stretching from A to C. Find the distance from the middle points of the rafters (E and F) to D.

Hint: First solve rt. $\triangle\ ABD$ and obtain length AB and $\angle\ ABD$. Solve $\triangle\ EBD$ in which sides $EB \left(= \dfrac{AB}{2}\right)$ and BD, and $\angle\ EBD$ are now known.

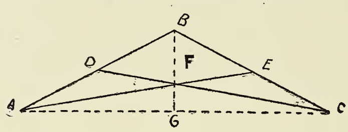
6. Starting from a point A, a surveyor runs a line due north 40 chains to B, he then runs north 64° west 40 chains to C. How far is C from A? What is the bearing of C from A?

7. At the base of a hill the angle of elevation of the summit is 9° 50′; 3000 feet farther up the hill (along a slope which is inclined at an angle of 7° 30′ to the horizontal) the angle of elevation is found to be 15° 45′. Find the height of the hill.

8. An isosceles triangle is inscribed in a circle, the radius of which is 5 in.; the base of the triangle is 5.5 in. Find the vertical angle (the angle opposite the base) and length of the other sides.

Hint: Draw a diagram; draw the radii to the three vertices.

9. A roof in which the rafters have a run of 19 feet and a rise of 10 feet is to be supported by a scissor-beam truss constructed according to the following diagram:



The middle point of each rafter (D and E) is to be supported by a strut reaching to the opposite plates (A and C), respectively. Find the length of DC and EA.

Hint: Find $\angle B$ by solving the rt. $\triangle AGB$ for $\angle ABG$ and doubling it (80d).

Find the bevel at which the upper ends of the struts are to be cut, that is, $\angle ADC$, also the lower ends, $\angle DCA$.

NOTE: It is not necessary to make any allowance for the plate cut, as that may be laid out afterwards, in the same manner as for the rafter (Shop Exercise XIVa).

Find also the distance BF.

Hint: In $\triangle BDF$, $\angle BDF$ (supplement of $\angle ADC$) and $\angle DBF$ and DB are now known.

10. Two sides of a parallelogram are 3 in. and 4 in., respectively, and the angle between them is 41°. Find the length of the two diagonals.

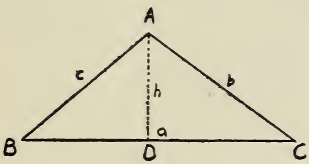
A formula for finding the area of a triangle when its three sides are given has already been derived (155). The actual labor of working out the area, however, may usually be much diminished by the following:

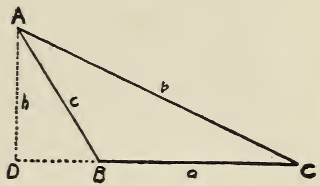
190a. Theorem: *The area of any triangle equals half the product of any two sides times the sine of their included angle.*

Given: a, b, and c the sides opposite the ∡ A, B, and C, respectively, of the △ ABC; h the altitude from vertex A to base a.

To Prove:

$$\text{Area } \triangle\, ABC = \frac{ac \text{ Sin } B}{2} = \frac{ab \text{ Sin } C}{2} = \frac{bc \text{ Sin } A}{2}$$





Proof: Area △ $ABC = \dfrac{a \times h}{2}$ (133a).

But $h = c \text{ Sin } B$ (172).

Substituting in the above, Area △ $ABC = \dfrac{ac \text{ Sin } B}{2}$

In like manner, by drawing altitudes from the other vertices, let the student prove that

$$\text{Area } \triangle\, ABC = \frac{ab \text{ Sin } C}{2}$$

$$\text{Area } \triangle\, ABC = \frac{bc \text{ Sin } A}{2}$$

Note that the above theorem applies equally well to both figures (either the acute-angled or obtuse-angled triangle).

This fact is of service in fixing the above theorem in the memory, since it indicates that the function used is the sine—as, if it were the cosine, there would be two theorems, since the cosine of an obtuse angle is negative (182g).

The solutions of triangles (such as in the preceding chapter) may be checked by finding the area with regard to two different angles; for example, by formula $\frac{ab \text{ Sin } C}{2}$, and then by $\frac{bc \text{ Sin } A}{2}$. These results should check, within the error allowable through use of approximate decimals.

Exercise

1. Find the area of the triangles in ex. 6 and ex. 8 in the above exercise.

2. Check the results of the problems under the Sine Law by finding the area of each triangle, by means of the angle and its two including sides, at two different vertices.

Example: The problem worked out under the Sine Law was as follows:

Given: $A = 62° 30'$, $B = 64° 45'$, $a = 84.5$.

Computed: $C = 52° 45'$, $b = 86.16$, $c = 75.83$.

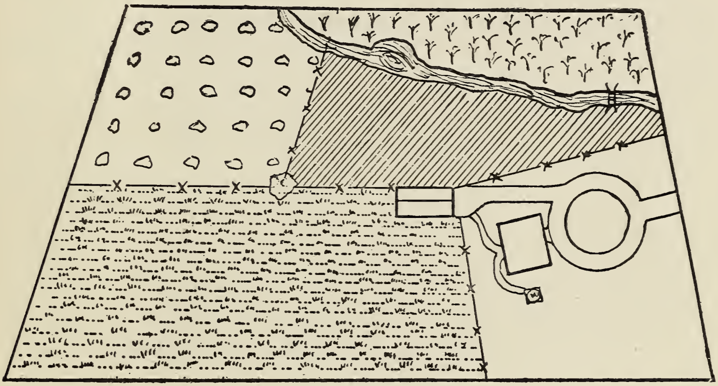
Check:
$$\frac{bc \text{ Sin } A}{2} = \frac{86.16 \times 75.83 \times .8870}{2} = 2897.61$$
$$\frac{ac \text{ Sin } B}{2} = \frac{84.50 \times 75.83 \times .9044}{2} = 2897.53$$

XVII. Drawing Exercise

MAP OF SURVEY

Perform all necessary computations to complete the work of Field Exercises XV and XVI.

Keeping in mind that the top of a map should be approximately North, make a careful drawing to scale, locating the various topographical features by the aid of the measurements made in Field Exercise XVI, or computed from them. In the upper left hand corner, within the marginal line enclosing the map, draw a line running exactly North and South; mark its upper end with an arrow-head and above it an "N," the lower end with "S." By drawing light pencil lines parallel to this, through the proper points in the drawing, the bearings of the fences, etc., measured in the previous Field Exercise may be laid off with the protractor.



The accompanying diagram shows various conventional topographical signs; upper right hand corner, cornfield, and below it ploughed land; upper left hand corner, orchard: lower left hand corner grass-land. The brook, fences, buildings and grounds can easily be identified in the drawing.

The area included by the map (found by summing the areas of the triangles measured in Field Exercise XV) should be marked on the map in acres. To complete the map, its title (in large letters) and the scale used (for example "1′ = 1 chain"), the name of the observer, and the date (in smaller letters) should be arranged neatly in the lower right hand corner within the marginal line enclosing the map.

CHAPTER XVIII

SIMULTANEOUS EQUATIONS: FINDING HEIGHTS OF HILLS: AMBIGUOUS CASE

191a. Problems involving two unknowns may be solved by two equations containing both x and y, which must be solved together, and hence are called **simultaneous equations.** Thus,

Problem: A rectangle is 5 feet longer than it is wide and its perimeter is 50 feet. Find its length and width.

Solution: Let x = length
y = width

Then

(I) $x - y = 5$

(II) $2x + 2y = 50$

Multiplying (I) by 2, and adding (II) to it

$$\begin{array}{r} 2x - 2y = 10 \\ 2x + 2y = 50 \\ \hline 4x \qquad = 60 \end{array}$$

$$x = 15$$

Substituting this value in (I)

$$15 - y = 5$$

Transposing

$$-y = -10$$

Changing signs throughout (dividing by -1)

$$y = 10$$

Check: Substituting the values of x and y in (II)

$$30 + 20 = 50$$

Equation I is the algebraic statement that the length of the rectangle is 5 feet more than its width. Equation II is obtained by collecting terms in the equation which states that the perimeter is 50 feet: $x + y + x + y = 50$.

Solve the following systems of simultaneous equations:

1. $2x + 4y = 16$
$7x + y = 17$

2. $4x - y = 11$
$2x + 6y = 64$

In 3, after multiplying by the proper factors, we may either add the y terms or subtract the x terms.

3. $3x - y = 15$
$2x + 3y = 32$

4. $5x + y = 43$
$2x + 7y = 37$

5. $9x + y = 19$
$20x - y = 10$

191b. Any method of getting rid of an unknown quantity is called a process of **elimination.**

The following method of elimination is called **substitution.**

I. $x - y = 5$

II. $2x + 2y = 50$

From (I): $x = y + 5$

Substituting in (II)

$$2(y + 5) + 2y = 50$$
$$2y + 10 + 2y = 50$$
$$4y = 40$$
$$y = 10$$

Substituting this value in

$$x = y + 5$$
$$x = 10 + 5$$
$$x = 15$$

Solve the following sets of simultaneous equations by substitution:

6. $7x + 2y = 39$
$3x - 2y = 11$

7. $3x + 2y = 26$
$9x + 5y = 71$

8. $5x - y = -1$
$3x + 2y = 28$

9. $.4x + .3y = 4.1$
$x + .4y = 7.8$

10. $.41x + .02y = 1.24$
$.92x + .05y = 2.26$

11. $.108x + .021y = .232$
$.107x - .063y = .166$

12. $x + 6y = 23.4$
$5x - 3y = 21.3$

13. $7x + 3y = 4.651$
$4x + y = 1.722$

191c. From the above examples it is evident that **elimination by substitution** consists in finding the value of one of the unknowns from one equation and substituting this value for the unknown in the other equation.

191d. Another form of this process is to find the value of one unknown in terms of the other from one equation; then find the value of the same unknown from the other equation, and equate these two values.

Thus, the above problem may be solved in this manner:

(I) $x - y = 5$
(II) $2x + 2y = 50$
From (I) by transposition
(III) $x = y + 5$
From (II) by transposition
$2x = 50 - 2y$ whence
(IV) $x = 25 - y$

Equating the values of x in (III) and (IV)

$$y + 5 = 25 - y$$

Transposing

$$2y = 20$$
$$y = 10$$

Substituting in (I)

$$x - 10 = 5$$
$$x = 15$$

Solve the following sets of simultaneous equations by equating the values of one unknown in terms of the other:

14. $6x - 2y = 26$
$4x - y = 22$

15. $.3x + 1.2y = .3534$
$.4x + .5y = .436$

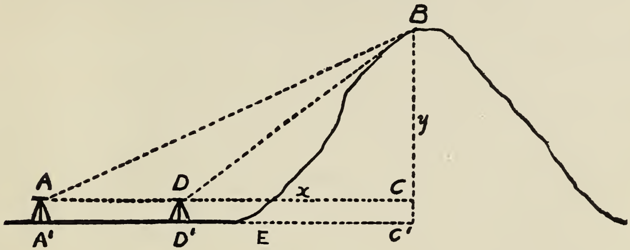
16. $.11x + .03y = 1.20$
$.02x + .11y = .95$

17. $.5x - .3y = .0273$
$1.1x - .2y = .1842$

18. $.3x - 1.1y = -3.4$
$.6x + y = 9.2$

19. $.112x + .031y = .398$
$.108x + .057y = .438$

192. By aid of simultaneous equations it is possible to find the height of objects when they could not be found by the method already learned (Chap. XIV). By that method it was necessary to measure the base of a right triangle and the angle of elevation. Sometimes it is not possible to measure this base line, as, for example, in finding the height of a hill.



In attempting to find BC', the height of the hill, it is obviously impossible to measure the part EC' of the base line, which is within the hill. In this case, BC' may be found by the following method:

Suppose the angle of elevation A to have been measured and to have been found to be 19°, the angle of elevation at D (in a straight line toward the hill) to be 24°, and the distance AD to be 300 feet.

Let DC be represented by x, and BC by y.

From rt. $\triangle\ BCA$,

$$\text{Tan } A = \frac{y}{AC} = \frac{y}{AD + x}$$

Clearing of fractions

(I) $(AD + x)\ (\text{Tan } A) = y$

From rt. $\triangle\ BCD$,

$$\text{Tan } \angle\ BDC = \frac{y}{x}$$

Clearing of fractions

(II) $x \tan \angle\ BDC = y$

Equating these two values of y

$(AD + x)\ (\tan A) = x \tan \angle\ BDC$

Substituting the numerical values for AD, tan 19° and tan 24°,

$$(300 + x)\ (.3443) = x(.4452)$$
$$103.29 + .3443x = .4452x$$
$$103.29 = .1009x$$
$$x = 1023.6$$

Substituting in (II)

$$y = (1023.6)\ (.4452) = 456 \text{ ft. Ans.}$$

Note that A and D are in the same straight line with the hill and on the same level. For smaller elevations (or whenever greater accuracy is desired) a correction should be made for the height of the instrument—that is, A and D are not really on the same level as the base of the hill, but the height of the instrument above it, and this height should be added to the computed height of the hill ($AA' = DD' = CC'$).

Exercise

1. From the first of two stations on a level road leading directly toward a hill, the angle of elevation of the top of the hill is observed to be 12°. From the second station, 200 feet nearer the hill, the angle of elevation is observed to be 18° 30′. Find the height of the hill.

2. From a station on a level plain a military engineer observes the angle of elevation of the top of a hostile fort to be 4° 15′. From a second station 320 feet directly toward the fort, the angle of elevation is 6° 20′. Find the height of the fort.

3. From the bank of a river, the angle of elevation of the top of a tree on the other bank is observed to be 32° 16′. From a station 100 feet back of the first, and in the same straight line with the tree and the first station, the angle of elevation to the top of the tree is 15° 45′. Find the width of the river.

Case IV in the solution of triangles (see art. 184) should be avoided in practice whenever possible. It is called the **Ambiguous Case,** as often-times there are two triangles which satisfy the conditions of the problem, and in case of a practical application it is necessary to determine which to use from other considerations. This ambiguity arises from the fact that the sine of an angle is equal to the sine of its supplement (Chap. XVI). (In the problems previously worked out under the Sine Law, this caused no uncertainty, as in those problems two angles were given and the third was *determined* as the supplement of the sum of the two given angles.)

193a. The classes of examples occurring under the Ambiguous Case may be divided as follows:

I. Impossible.

II. Two Solutions.

III. One Solution.

Consider the given parts as a, b and A of the $\triangle ABC$.

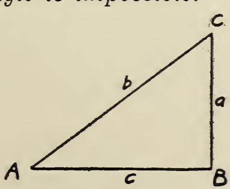
If $a < b$, and $\angle A$ is an obtuse angle, $\angle B$ must be obtuse (188a). But this is impossible (147b).

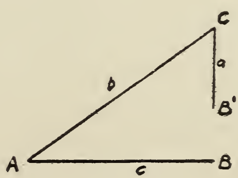
In like manner, the problem is impossible if $a = b$.

In like manner, if A is a rt. $\angle$ and $a < b$ or $a = b$ the triangle is impossible.

Let the student demonstrate this graphically by attempting to draw triangles under the above conditions, when it will become evident that such triangles will not "close." Hence,

193b. Rule: *If the given angle is a right angle or greater, the side opposite must be greater than the other given side, or the triangle is impossible.*





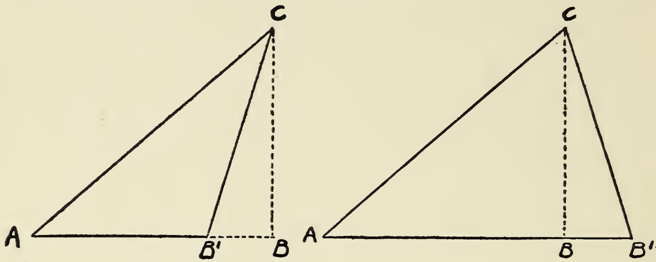
Consider first that the given parts a, b and A are parts of a right triangle.

Then $a = b$ Sin A (172).

If, however, the value of a is less than b Sin A, evidently a is not long enough to reach from C to B—that is, the triangle would not close. Hence,

193c. Rule: *If the given angle is acute, and the side opposite is less than the product of the sine of the given angle and the other given side, the triangle is impossible.*

To investigate the remaining cases, suppose that a were gradually increased. It might then cut the line AB to the left or to the right of the $\perp$ CB, as in the following figures.



In the first figure a falls within BC, in the second it falls without. Both these triangles fulfill the given conditions—having the given sides a and b, and the given angle A.

Consider now that the side a in the first figure is gradually lengthened; B', its point of intersection with AB, comes nearer and nearer to A. When $a = b$, the two lines coincide, and there is no triangle. If a continues to lengthen, B' will fall outside of A, and the angle at A will no longer be the given angle, but its supplement—and the left hand figure no longer fulfills the given conditions. In this figure, then, the value of a must lie between b Sin A and b.

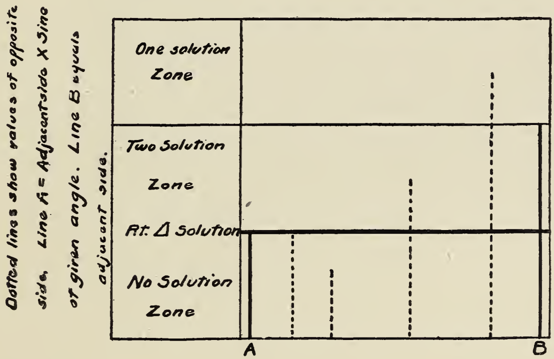
In the right hand figure, however, a may evidently be increased indefinitely (the point B' moving continually to the right) without violating the conditions of the given triangle.

Since in both figures the triangles begin with $a = b \text{ Sin } A$ (when $a < b$ the triangles do not close) and in the left hand figure the triangles cease when $a = b$, while in the right hand figure they continue indefinitely, it follows:

193d. Rule: *If the given angle is acute, and the side opposite the given angle is greater than the product of the other given side times the sine of the given angle, but less than the other given side, there are two solutions.*

193e. Rule: *If the given angle is acute, and the side opposite the given angle is greater than the other given side, there is one solution.*

The following figure is a graphic representation of these rules.



XVIII. Field Exercise

Find the height of a hill by the method outlined in this chapter.