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Correlated Mathematics for Junior Colleges

BY

ERNST R. BRESLICH

Head of the Department of Mathematics in the University High School, The University of Chicago

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EDITOR'S PREFACE

This book follows the three-year high-school course of mathematics, "along fusion lines," that was begun by Mr. Breslich five years ago. From start to finish the four books have been worked out in classrooms and in faculty conferences.

Through the economies effected by treating mathematical topics in combination and once for all, instead of once in each of the several mathematical branches, and by keeping whatever is once acquired in continual function, this four-year course is able to cover in the usual time allotment to high-school mathematics, in addition to the customary high-school courses, plane analytics and college algebra as given in American colleges, together with considerable work in differential calculus. On completing this course the student is mathematically distinctly ahead of the beginning collegiate Sophomore.

The persons responsible for this series believe that every rightly planned course, if properly administered, should at its close leave the student feeling that there is much more to be known about the subject than he has studied. They hold that a sound test of good teaching is whether at the end of the teaching the student really feels that there are immensely important things in the subject lying ahead, and that he would very much like, even if he cannot, to go on with what he has been learning. They do not believe that it is educationally wholesome to treat a course in the spirit that it is the last course the student is to have, or that, being the last required course, it is all that he really needs. The best course in any subject, and especially in any mathematical subject, is the course that best prepares the pupil both in attainment and in attitude to go on to the next. In this sense, and in this sense only, each text of the Breslich series is organized in the spirit of the preparatory-school text. When, as in these texts, mathematical material from all branches is so organized that the things studied at each level of maturity are the materials best adapted to this stage that are available in any of the branches, the organization that best prepares for the next advance is the best organization, whether the student actually makes the advance or not. It is only when the preparatory ideal becomes the *dominant* ideal that mischief is likely to be wrought.

The gratifying results of the systematic employment of this series of texts in the University High School, as shown by standard tests, by the tests of college entrance, by the ability of recent graduates to do subsequent collegiate work well, by the large percentage of University High School pupils who are continuing to elect mathematical courses, and by the number of texts of unified mathematics of both high-school and collegiate grade that have appeared since the publication of Breslich's first-year book, point to the belief that the route to essential improvement in high-school mathematics lies along the lines of the Breslich books. This belief is also strengthened by the favor this series has found among the best high schools, public and private, and with many of the foremost educators of our country. The editor feels that it is only simple justice to say that in this series of texts Mr. Breslich has rendered a significant service to mathematical education in this country.

G. W. Meyers

Chicago, Illinois
June, 1919

AUTHOR'S PREFACE

This book is designed primarily to follow the third book of the author's series of textbooks of secondary mathematics. It therefore presupposes the essentials of high-school algebra, of plane and solid geometry, and of trigonometry. It aims to combine the work which is commonly covered in separate courses in college algebra and in analytical geometry. The course is therefore suitable for the first year of the junior college and may be taught either in the fourth year of the high school or in the first year of our colleges.

As in the first three books of the series, the fundamental principle of the course is to associate closely mathematical topics which are naturally related to each other. This combination makes it possible better to motivate each topic, to show the student more clearly the meaning of the subject by means of geometrical representation, and to develop in a natural way the important concept of functional correspondence. Such an arrangement of the material gives unity to a course in which the subject is usually presented as a number of isolated topics. It has the advantage of arousing and holding the interest of the student, with the result that he gains greater power in less time.

The idea of the derivative is presented early and is used in discussing the slope of the tangent to a curve, and maximum and minimum values of a function.

As in the other books of the series, many historical notes have been distributed throughout the books.

For review purposes each chapter contains frequent summaries and a list of the formulas developed. There are also important tables and a list of mathematical formulas at the end of the book.

All the material has been tried out carefully by the author in the classroom.

The portraits appearing in the book have been taken from the *Philosophical Portrait Series*, published by the Open Court Publishing Company, Chicago.

The author is greatly indebted to Professor Charles H. Judd, Director of the School of Education, whose encouragement and interest have made the preparation of this course possible.

E. R. Breslich

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CHAPTER I

LOCATION OF A POINT. NUMBER SYSTEM

Real Numbers. Location of a Point on a Straight Line

1. Graphical representation of integral numbers. Selecting as unit length any convenient line-segment, l, we may lay off on OX, Fig. 1, the segments OA, AB, BC, etc., each equal to l. Thus, $OA=l$, $OB=l+l=2l$, $OC=l+l+l=3l$, etc.

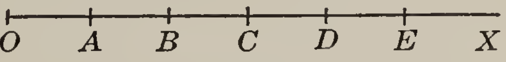

FIG. 1

The integers 1, 2, 3, etc., are the *measures* of OA, OB, OC, etc. The segments OA, OB, OC, etc., are said to *represent graphically* the integral numbers 1, 2, 3, etc.

Conversely, to every integral number corresponds a definite point on OX.

2. Origin. Abscissa. A point P, Fig. 2, on a straight line, AB, may be located by stating the distance x of P from a fixed reference point, O, on AB.

FIG. 2

The point O is the **origin,** the segment $OP=x$ is the **abscissa** of P.

3. Positive and negative numbers. The segments OA', OB', OC', etc., Fig. 3, to the left of O, differ from the segments to the right in *direction.* To distinguish between these two opposite directions, segments taken in the direction OX are called **positive.** The numbers represented

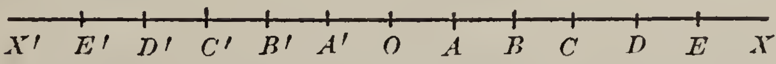

FIG. 3

by OA, OB, etc., are called **positive numbers,** and a plus (+) sign is prefixed to them. Hence, OA represents +1, OB represents +2, etc. The segments OA', OB', OC', etc., taken in the direction OX', are **negative** and represent the **negative numbers** −1, −2, −3, etc.

In this way the **scale of positive and negative integers** is obtained, Fig. 4.

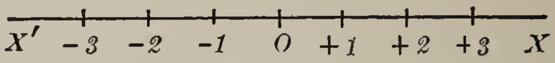

FIG. 4

4. Zero. The point O, Fig. 4, may be thought of as representing a segment having *no* length, or as representing the number *zero.**

EXERCISES

1. Locate on a straight line the segments representing the following numbers: −5, +6, −3, 0, +8.

2. Give the meaning of the following thermometer readings: −2, 0, +8, −7, −4.

5. Graphical representation of fractions. To represent graphically the *fraction* $\frac{m}{n}$, m and n being integers, we may proceed as follows:

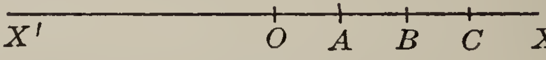

FIG. 5

Divide OA, Fig. 5, into n equal parts and denote each by $\frac{1}{n}$. This locates between O and A a number of points whose abscissas are $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots\ldots\ldots\ldots \frac{m}{n}, \ldots\ldots\ldots\ldots$

---

* Our notation of numbers, known as the Arabic figures, was borrowed by the Arabs from the Hindoos. The *principle of position* employed in this method of writing numbers required a symbol for zero. Indeed, the use of the 0 made the Arabic notation possible in which a symbol was needed to indicate a digit in the units place which would add nothing to the total.

Hence, to every fraction $\frac{m}{n}$, m and n being integers, there corresponds a point on $X'X$ such that its distance from O represents that fraction.

In the fraction $\frac{m}{n}$, m is the *numerator* and n the *denominator* of the fraction.

6. Rational numbers. Sections 1 to 5 show that all integers and fractions may be represented graphically. Integers and fractions, both positive and negative, form the domain of **rational** numbers.

7. Irrational numbers. It is easily shown that some segments of OX, Fig. 5, do not represent rational numbers. For example, the length of the diagonal of a square whose side is equal to 1 is easily shown to be equal to $\sqrt{2}$. This number is neither an integer nor a fraction, nor can it be expressed *exactly* in terms of integers and fractions. However, it may be represented geometrically by a segment of OX which has the same length as the diagonal of the square having the side equal to 1.

Numbers represented by segments of OX which cannot be expressed exactly by rational numbers are **irrational numbers.** With this agreement each segment of OX corresponds to a number, and every rational or irrational number may be represented by a segment of OX.*

There is said to be a *one-to-one correspondence* between the numbers and the points on OX.

8. Real numbers. Men of remote civilization were acquainted with the idea of *integers* and *fractions.* The

---

* Pythagoras is credited with the discovery of the existence of incommensurable ratios, such as the ratio of the diagonal of a square to the side. He and his followers made the distinction between rational and irrational numbers. Definitions for irrational numbers were given by Dedekind (1831–1916) and Cantor (born 1845).

existence of *irrational* numbers was discovered by the Greeks. They were able to show that the ratio of some segments to a segment of given length cannot be expressed *exactly* as a fraction. They called these segments *incommensurable.*

The whole set of integers, fractions, and irrational numbers forms the domain of **real numbers.**

The choice of such names as *rational* and *irrational* may seem arbitrary to the student. However, the introduction of the ideas for which these names stand was a mathematical necessity. For example, *positive* and *negative* numbers were needed because without them it was impossible to solve all equations of the form $x+a=b$. To illustrate, the equation $x+5=0$ cannot be solved in the domain of positive numbers.

Furthermore, when solving quadratic equations like $x^2+a=b$, we need the *irrational* numbers. For example, the solutions of the equation $x^2+1=4$ are $+\sqrt{3}$ and $-\sqrt{3}$.

In solving the equation $x^2+a=b$ a further difficulty arises when b is less than a, and we must interpret the meaning of $\pm\sqrt{b-a}$ for the case when $b-a$ is negative. Whatever meaning may be assigned to $-\sqrt{-n}$, it must be such that the laws of algebra formerly established for operating with numbers hold also for this new number. The name *imaginary* was assigned to the symbol $\sqrt{-a^2}$, a being any positive or negative integer, fraction, or irrational number. These numbers will be studied in §§ 12 and 13.

EXERCISES

Represent graphically the following numbers: -5, $\sqrt{3}$, π, $\frac{2}{3}$, $2\sqrt{2}$, 0.

Location of a Point in a Plane

9. The Cartesian system of co-ordinates. By taking as reference lines the two intersecting straight lines $X'X$

and $Y'Y$, Fig. 6, the position of a point P in the plane of these two lines may be determined as follows:

Draw PM parallel to YY'. Let y denote the length of MP and x the length of OM. Then the pair of numbers (x, y) locates definitely the point P. The two reference lines are usually called the **axes of co-ordinates,** or **axes,** and the point of intersection the **origin.** OX is the *x-axis* and OY the *y-axis.*

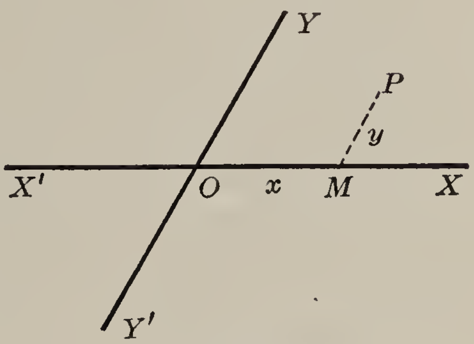

FIG. 6

$OM = x$ is the **abscissa** of P, and $MP = y$ is the **ordinate** of P. The numbers x and y are the **co-ordinates** of P. Hence we speak of the *point* (x, y) as the point whose co-ordinates are x and y.

The discovery of this system of co-ordinates is due to the philosopher René Descartes (1637), after whom it has been called the **Cartesian system of co-ordinates.** It is also called the **rectilinear system.**

Usually the axes of co-ordinates are *perpendicular* to each other, Fig. 7. In that case x and y are the two perpendicular distances of P from the axes, and the system is called a **system of rectangular co-ordinates.** The first writer who used **abscissa** consistently as a scientific term was Wolff in 1710. Leibnitz in 1694 was the first to use the word **ordinate** as a consistent technical term, and in 1692 he introduced the word **co-ordinates.**

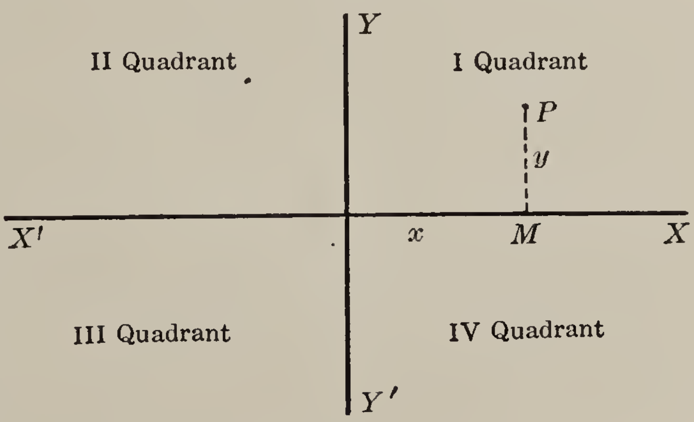

FIG. 7

The axes divide the plane into four parts called the *first*, *second*, *third*, and *fourth* **quadrants,** respectively.

Segments to the *right* of the y-axis are *positive*, those to the *left* are *negative.* Segments *above* the x-axis are *positive*, and those *below* are *negative.*

According to the system of co-ordinates just described, *to every possible pair of real numbers x and y corresponds one and only one point of the plane, and to every point of the plane corresponds a pair of two definite real numbers.*

To the student this method of locating points is not new. We use it in locating a building in a city when we say it is a certain number of blocks north and then a number of blocks west, in geography when we determine a place by means of its latitude and longitude, in navigation when we find the position of a ship.

EXERCISES

1. Plot the points whose co-ordinates are

$$(-3, 1); \quad (4, -2); \quad (0, 2); \quad (-6, 0)$$

2. Plot the triangles whose vertices are determined by the following co-ordinates:

$(1, -4)$, $(3, 2)$, and $(-5, -6)$; $(-2, -3)$, $(1, 3)$, and $(0, 0)$

10. Polar co-ordinates. A point P, Fig. 8, in a plane may be located by its direction and distance from a fixed reference point O. Point O is the **pole**; the angle XOP made by OP and the *initial line* OX is the **vectorial angle** of P. The distance OP is the **radius vector** of P. The initial line OX is the **polar axis.** Polar co-ordinates were first used by James Bernoulli in 1691.

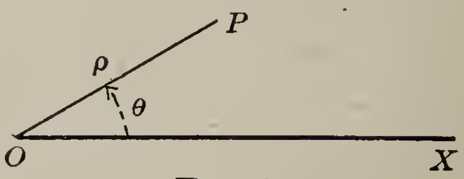

FIG. 8

The radius vector, ρ,* and the vectorial angle, θ,† are the **polar co-ordinates** of P.

---

* A Greek letter, called *Rho.*

† A Greek letter, called *Theta.*

A point P in a plane may be located definitely by letting ρ and θ have positive and negative values. Angle θ is *positive* or *negative* according as the radius vector turns *counterclockwise* or *clockwise*. The radius vector OP, Fig. 9, is regarded as *positive*, but distances measured in the direction *opposite* to OP are *negative*. Thus OP' is negative.

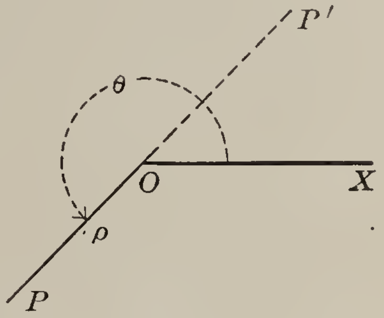

Fig. 9

EXERCISES

Locate the following points:

1. $(\rho, \theta) = \left(2, -\frac{\pi}{3}\right)$, $(-4, 60°)$, $(-2, -45°)$

2. $(3, \text{arc tan } 1)$; $(-5, \text{arc cos } \frac{1}{2})$; $\left(4, \text{arc sin } \frac{\sqrt{3}}{2}\right)$

11. Relation between Cartesian and polar co-ordinates. The following equations express polar co-ordinates in terms of Cartesian co-ordinates. Show how they are derived from Fig. 10.

$$\left\{\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \theta &= \text{arc tan } \frac{y}{x}. \\ \theta &= \text{arc sin } \frac{y}{\sqrt{x^2+y^2}} \\ \theta &= \text{arc cos } \frac{x}{\sqrt{x^2+y^2}} \end{aligned}\right.$$

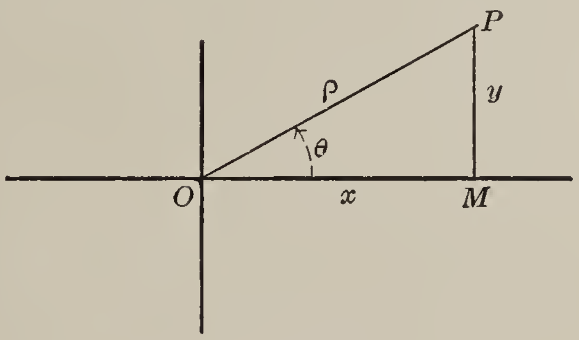

Fig. 10

The equations

$$x = \rho \cos \theta, \quad y = \rho \sin \theta, \quad x^2 + y^2 = \rho^2$$

express Cartesian co-ordinates in terms of polar co-ordinates. Prove.

EXERCISES

1. By means of the equations in § 11 find the Cartesian co-ordinates of the points: $(2,\ -30°)$; $\left(-3,\ \frac{2\pi}{3}\right)$; $(2,\ 60°)$; $\left(-2,\ \frac{\pi}{4}\right)$; $\left(16,\ \frac{\pi}{6}\right)$.

2. Find the polar co-ordinates of the points: $(2,\ -2)$; $(1,\ \sqrt{3})$; $(-2,\ -2\sqrt{3})$.

3. Change the equation $x^2-y^2=a^2$ to polar co-ordinates.

4. Change the equation $\rho=2a\ \cos\theta$ to rectangular co-ordinates.

5. Transform the equation $\rho\ \sin 2\theta=2a^2$ into rectangular co-ordinates.

6. Change the equation $x^2+y^2=2rx$ to polar co-ordinates.

Complex Numbers

12. Imaginary numbers. We have seen, § 8, that the solution of some quadratic equations leads to the square root of negative numbers. For example, no real value of x satisfies the equation $x^2+3=0$. At first mathematicians found it impossible to give a meaning to the symbol $\sqrt{-3}$ and called such numbers *imaginary* numbers, i.e., numbers having no meaning. Gradually they recognized the desirability of interpreting these new symbols so that they might be subject to the same laws of the fundamental operations of addition, subtraction, multiplication, and division as the real numbers.

This interpretation, however, is not as simple as that of negative and of irrational numbers. A *real* number may be represented by a line-segment laid off on a line $X'X$ from a fixed point O, in a given direction. Conversely, to every segment laid off on $X'X$ from O corresponds a *real* number. Hence *imaginary numbers* cannot be represented by segments of $X'X$. The following suggests the representation of imaginary numbers:

Let the segment OA, Fig. 11, represent the real number a. Turning OA about O through two right angles, the segment OA' is obtained, which is known to represent the number $-a$. Hence the multiplication of a number by the factor -1 may be interpreted geometrically as a rotation through 180° in the counterclockwise direction. We *define* $\sqrt{-1}$ as a number which multiplied by itself equals -1. Hence, $\sqrt{-1} \cdot \sqrt{-1} = -1$. This suggests the idea of interpreting multiplication of a number by $\sqrt{-1} \cdot \sqrt{-1}$ as a rotation through 180° in counterclockwise direction, and multiplication by $\sqrt{-1}$ as a rotation through 90°.

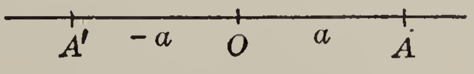

FIG. 11

13. Imaginary unit. The symbol $\sqrt{-1}$ is usually represented by the letter i.* It is called the **imaginary unit.** Since imaginary numbers, as $\sqrt{-4}$, $\sqrt{-9}$, $\sqrt{-a}$, may be changed to the forms $2\sqrt{-1} = 2i$, $3\sqrt{-1} = 3i$, $\sqrt{a}\sqrt{-1} = \sqrt{a}\, i$, they may be represented geometrically by segments of a line YY' drawn perpendicularly to the x-axis at O, Fig. 12.

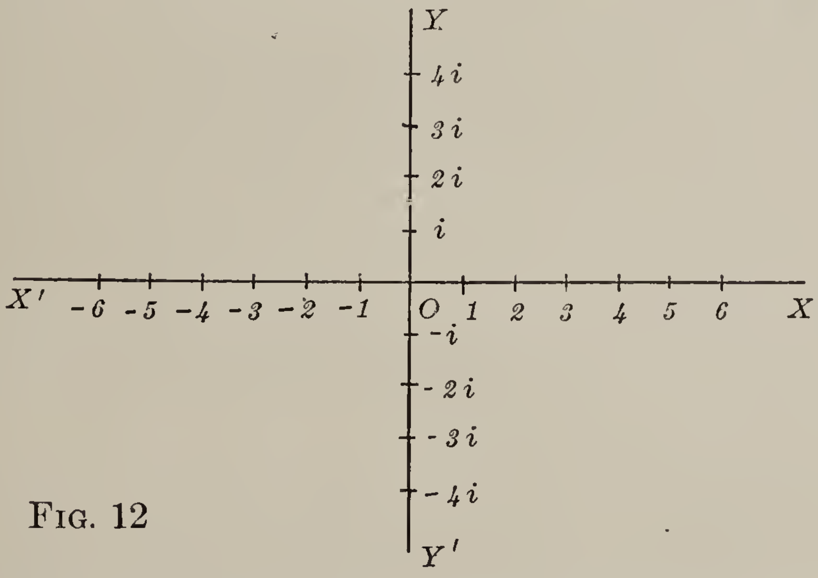

FIG. 12

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* Due to Gauss (1777–1855).

EXERCISES

Write the following numbers in the form *ai* and represent them graphically:

$$\sqrt{-9},\ \sqrt{-36},\ \sqrt{-x^2},\ \sqrt{-2}$$

14. Complex numbers. We have seen that real and imaginary numbers are represented by segments drawn from the origin on the x-axis and on the y-axis, respectively. Conversely, every segment of the x-axis, drawn from O, represents a real number, and every segment of the y-axis represents an imaginary number. This raises the question whether or not segments drawn from O, *not along the axes*, can be interpreted as representing numbers, e. g., the segment OP, Fig. 13.

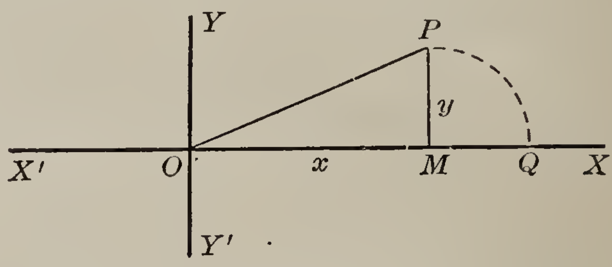

FIG. 13

The position of P depends upon two numbers, x and y, the co-ordinates of P. We may pass from O to P by passing first from O to M and then from M to P. We may also reach P by laying off $OM=x$, $MQ=y$, and then rotating MQ through 90° in the *counterclockwise* direction. This may be indicated by the symbol $x+iy$, the plus sign meaning that first x is laid off to M and then, in the same direction, y from M to Q, the factor i denoting the rotation of MQ through 90°.

In general, to any directed segment (vector) OP corresponds a symbol of the form $x+iy$, x and y being the co-ordinates of P.

The symbol $x+iy$ is called a **complex number.***

The complex number $x+iy$ reduces to a *real* number when $y=0$ and to an *imaginary* number when $x=0$.

---

* The term complex is due to Cauchy (1821).

When $x=y=0$, the number $x+iy$ reduces to 0 and is represented geometrically by the origin. Hence real and imaginary numbers are *special cases* of complex numbers.

Historical note. The history of the evolution of the imaginary and the complex number and of the system of representing them graphically is eventful and interesting. These things did not spring suddenly from a single genius nor from a single age. Like most mathematical concepts the system was the cumulative result of the attempts of many men through many centuries to realize the true meaning of the varied results of the solution of equations.

On their first appearance both negative and imaginary numbers were called *impossible* (Diophantus, Cardan, Bombelli).

In the first century the Greek Heron, in a problem on pyramids, obtained a root $\sqrt{18-144}$, and proceeded to find $\sqrt{63}$ rather than $\sqrt{-63}$. It cannot be decided whether he recognized the impossibility of finding $\sqrt{-63}$.

Diophantus (250 A.D.) avoided both irrational and imaginary numbers by putting limitations on the coefficients. Obtaining $\sqrt{\left(\frac{a}{2}\right)^2-b}$ as a solution of a certain quadratic equation, he said that $\left(\frac{a}{2}\right)^2$ must exceed b by a square number.

The Hindu Bhaskara (born 1114 A.D.) says: "There is no square root of a negative number, for such a number cannot be a square."

The geometrized algebra of the Arabs did not encounter imaginaries, so that the Arabs did not consider imaginaries.

The real history of the imaginary number begins in the sixteenth century of our era. In a book called the *Summa*, written by Pacioli in 1494, the author assumes that in $\sqrt{\frac{b^2}{4}-c}$, $\frac{b^2}{4}$ must always be equal to or greater than c. Chuquet (1484) also recognized the impossibility of finding $\sqrt{-a}$.

In 1539 Cardan regarded imaginaries as impossible, but in 1545 he attempted for the first time in history to calculate with them. He showed that $(5+\sqrt{-15})(5-\sqrt{-15})=40$. Such numbers he said had only a "formal significance."

Bombelli in 1572 used numbers like $+\sqrt{-a}$ and $-\sqrt{-a}$. He asserted that the imaginary in the "so-called irreducible case" of the solution of the cubic was only *apparent*, and he gave a rule for calculating with $\sqrt{-a}$ and with the sums of such numbers.

Vieta (1540–1603), the "father of modern algebra," never even referred to imaginary numbers.

Girard (1590?–1632) had to include them as numbers to make true his theorem that an equation of nth degree has n roots, but he saw in them only the value of making general statements possible.

Descartes (1637) had no clear conception of them.

Leibnitz (1646–1716) used the forms

$$\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}$$

and x^4+a^4

$$=(x+a\sqrt{-\sqrt{-1}})\cdot(x-a\sqrt{-\sqrt{-1}})\cdot(x+a\sqrt{\sqrt{-1}})\cdot(x-a\sqrt{\sqrt{-1}}).$$

John Bernoulli (1667–1748) established a relation between the arctangent and the logarithm of an imaginary number.

Newton discovered a rule for determining the number of imaginary roots of a given equation. This rule was further perfected by Maclaurin in 1727 and 1729 and by Campbell in 1728.

Wessel in 1797, Argand in 1806, and Gauss in 1831 all gave the graphical representation of complex numbers that is taught in this chapter. The wonder is that so large a body of theory could have been worked out without the clarifying aid of this system. See Tropfke, *Geschichte der Elementar-Mathematik*, Band I, S. 168 ff.

15. Polar form of a complex number. Since

$$x=r\cos\theta, \text{ Fig. 14,}$$

and $$y=r\sin\theta, \text{ it follows that}$$

$$x+iy=r\cos\theta+ir\sin\theta$$

JOHN BERNOULLI

JOHN BERNOULLI

JOHN BERNOULLI (1667–1748) was born and educated at Bâle, and in the university he was a pupil of his older brother James. He was professor of mathematics at Groningen from 1695 until the death of James, whom he succeeded at Bâle in 1705. He occupied the professorship at Bâle until his death in 1748.

His chief discoveries were the exponential calculus, the analytical treatment of trigonometry, the determination of orthogonal trajectories, the principle of virtual work, the solution of the brachistochrone, and the conditions of a geodesic. He was probably the first to use g for gravity and to arrive at the equation $v^2=2gh$. Prior to his time the same principle had been stated in the form of a proportion, thus: $v_1^2:v_2^2=h_1:h_2$. He introduced ϕx as a function-symbol to displace his own earlier proposal of X or ξ. The final adoption of f, F, ϕ, and ψ to represent functions was due to Euler and Lagrange.

Of most significance to mathematical science were his services as a teacher. However inappreciative he may have been of the performances of others, he was always accessible, appreciative, and fair-minded to his pupils. Notwithstanding the blemish in his character alluded to in the biographical sketch of James (facing p. 270), he seemed capable of imparting his own zeal for mathematics to his pupils. He was the most successful mathematical teacher of his age and one of the greatest since Euclid. His influence was probably the most potent single force that wrought on the Continent of Europe the general adoption of the differential calculus of Leibnitz rather than the fluxional calculus of Newton.

The illustrious Bernoulli family, which in the course of a century furnished eight distinguished members to mathematical science, was of Dutch descent, living originally at Antwerp. Driven from its country by Spanish persecutions, the family sought an asylum first at Frankfort in 1583, and finally at Bâle, in Switzerland.

[See Ball, Cajori, or *Encyclopaedia Britannica*.]

or,

$$x+iy=r(\cos\theta+i\sin\theta)$$

The distance OP or r, Fig. 14, is called the **modulus** and θ the **amplitude,** or *argument,* of the complex number.

The form $r(\cos\theta+i\sin\theta)$ is the **polar form** of $x+iy$.

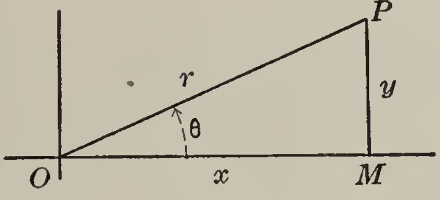

FIG. 14

EXERCISES

Draw the vectors representing the following numbers:

1. $2+i$	**4.** $-5+2i$
2. $2-i$	**5.** $-3-i\sqrt{5}$
3. $-3i$	**6.** $-2+6i$

Write the following numbers in the polar form. Draw a figure for each:

7. $1+i\sqrt{3}$	**10.** $1+i$
8. $-1-i\sqrt{3}$	**11.** $-1-i$
9. $\sqrt{3}-i$	**12.** $-1+i$

Write the following numbers in the form $x+iy$:

13. $2(\cos 45°+i\sin 45°)$	**16.** $5(\cos 90°+i\sin 90°)$
14. $4(\cos 135°+i\sin 135°)$	**17.** $2(\cos 270°+i\sin 270°)$
15. $3(\cos 30°+i\sin 30°)$	**18.** $\cos 150°+i\sin 150°$

16. Equality of complex numbers. Two complex numbers x_1+iy_1 and x_2+iy_2 are equal if, and only if, they are represented by the same vector. Hence, if two complex numbers $a+ib$ and $c+id$ are equal, we must have $a=c$ and $b=d$. Conversely, *two complex numbers* $a+ib$ *and* $c+id$ *are equal if, and only if,* $a=c$ *and* $b=d$.

Hence, *if* $a+ib=0$, *it follows that* $a=0$, $b=0$.

17. Conjugate complex numbers. In the complex number $a+ib$, a is called the *real* part and ib the *imaginary* part. Complex numbers, such as $a+ib$ and $a-ib$, whose real parts are the same and whose imaginary parts are equal numerically but differ in sign, are **conjugate complex numbers.**

EXERCISES

Find the value of x and y satisfying the following equations:

1. $x+y+(x-y)i=4+6i$
2. $x+y+i(x-y)=24.5+i(8.5)$
3. $5x+3y+i(4x-7y)=26+2i$

The Operations with Complex Numbers

18. Addition of complex numbers. The symbol $x+yi$ has been interpreted geometrically to mean that a segment $OM=x$, Fig. 15, is laid off in the direction of the x-axis, that this is followed by laying off $MN=y$, and that MN is then rotated about M through an angle of 90° in the counterclockwise direction, taking the position MP. The segment OP represents $x+yi$.

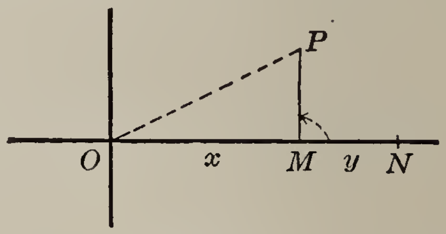

FIG. 15

By the *sum* of two complex numbers x_1+y_1i and x_2+y_2i we shall mean the following: First, the vector OP is constructed, Fig. 16, representing x_1+y_1i. Then, beginning from point P instead of from O, the vector PR is constructed *equal* to the vector OQ, which represents x_2+y_2i. The vector OR is the **sum** of x_1+y_1i and x_2+y_2i.

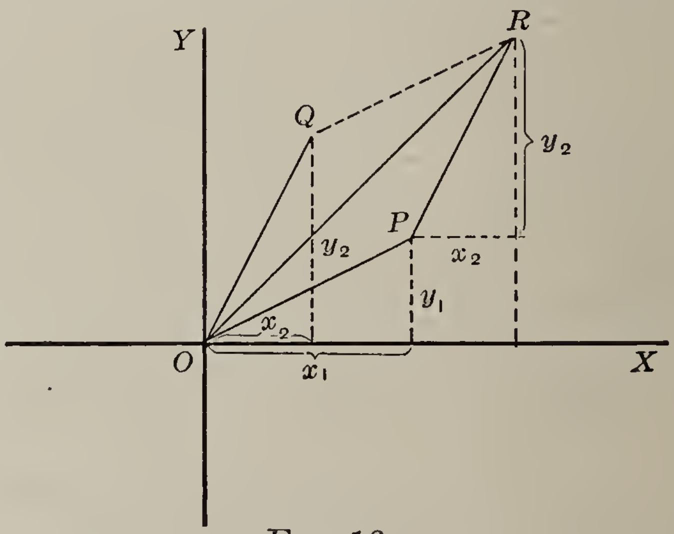

FIG. 16

Show that the sum of x_1+y_1i and x_2+y_2i may be obtained by locating the point R, whose co-ordinates are x_1+x_2 and y_1+y_2. The vector from O to R then represents the required sum.

Since OQ is equal and parallel to PR, it follows that $OPRQ$ is a parallelogram. This suggests a third construction of the sum of two complex numbers: *Draw OP and OQ representing the given complex numbers. Draw the parallelogram $OPRQ$ having OP and OQ as two adjacent sides. The diagonal which passes through O represents the sum of the given complex numbers.* Since the co-ordinates of R are x_1+x_2 and y_1+y_2, OR represents the number $(x_1+x_2)+(y_1+y_2)i$.

Hence we may *define the number* $(x_1+x_2)+(y_1+y_2)i$ as the *sum* of x_1+y_1i and x_2+y_2i, i.e.,

$$(x_1+y_1i)+(x_2+y_2i)=(x_1+x_2)+(y_1+y_2)i$$

EXERCISES

1. A steamer is moving in the direction OP, Fig. 17, and the wind is blowing it in the direction OQ. How far and in what direction will it be from the starting-point in one hour?

If the steamer were still, in one hour it would be at Q. If then the wind would stop, in another hour the steamer would cover the distance from Q to R. Hence, with the wind blowing in the direction of OQ, and the steamer starting from O and moving in the direction OP, it would actually reach the point R in one hour.

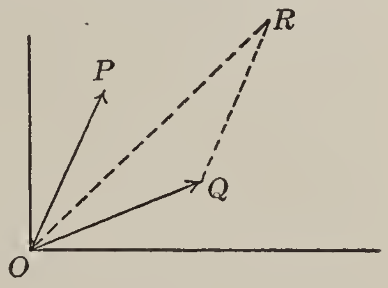

FIG. 17

Since the vector OR represents the distance of the steamer from the starting-point after one hour, it denotes its velocity and may be considered as the *sum* of the two velocities represented by OP and OQ. It is customary to call the vectors OP and OQ the *components* of OR and to call OR the *resultant* of OP and OQ. In general, the resultant of two forces represented by OP and OQ is the force represented by the diagonal OR of the parallelogram $OQRP$.

Add the following complex numbers both graphically and algebraically:

2. $(5+3i)+(-1+7i)$

Graphical addition: Let OP, Fig. 18, represent $5+3i$, and let OQ represent $-1+7i$. Draw the parallelogram $OPRQ$. The diagonal OR represents the sum.

Algebraic addition:

$$\begin{array}{r} 5+3i \\ -1+7i \\ \hline 4+10i \end{array}$$

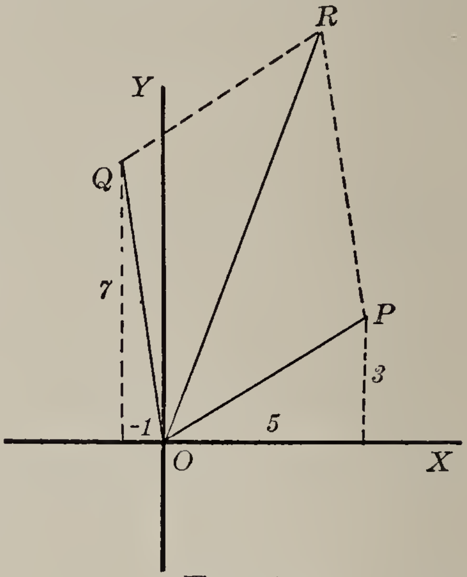

FIG. 18

3. $(4+i)+(1+5i)$

4. $(-3-2i)+(5+3i)$

5. $(-6+8i)+(-3-6i)$

6. $(8-i)+(8+i)$

7. Represent graphically a force of 15 lb. acting southeast and one of 22 lb. acting southwest at the same point. Find the magnitude and direction of the resultant.

19. Subtraction of complex numbers. We have seen that the sum of the numbers x_1+y_1i and x_2+y_2i is represented graphically by a diagonal of a parallelogram. Therefore a side of the parallelogram denotes the *difference* between the numbers represented by the adjacent side and the diagonal. Therefore, to find graphically the difference between x_1+y_1i and x_2+y_2i, let OP, Fig. 19, represent x_1+y_1i, and let OQ represent x_2+y_2i.

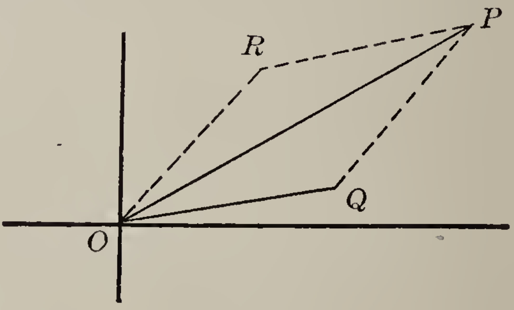

FIG. 19

Draw QP.

Draw $OR \parallel QP$, and $RP \parallel OQ$.

Then OR represents the required difference.

Show that OR represents the number

$$(x_1-x_2)+(y_1-y_2)i$$

Hence we define *the* **difference** *of two complex numbers* x_1+y_1i and x_2+y_2i *as the complex number*

$$(x_1-x_2)+(y_1-y_2)i$$

i.e., $$\boldsymbol{(x_1+y_1i)-(x_2+y_2i)=(x_1-x_2)+(y_1-y_2)i}$$

EXERCISES

Subtract the following as indicated and illustrate by a drawing:

1. $(4-3i)-(2+i)$

2. $(2+i)-(1+4i)$

3. $(7+8i)-(5+6i)$

4. $(2-3i)-(-1+i)$

20. Multiplication of complex numbers. Assuming that the ordinary laws of algebra are valid for complex numbers, we have

$$\begin{aligned}(x_1+y_1i)(x_2+y_2i)&=x_1x_2+x_2y_1i+x_1y_2i+y_1y_2i^2\\&=x_1x_2+y_1y_2i^2+(x_1y_2+x_2y_1)i\end{aligned}$$

Since $i^2=-1$, we have

$$\boldsymbol{(x_1+y_1i)(x_2+y_2i)=(x_1x_2-y_1y_2)+(x_1y_2+x_2y_1)i}$$

The *graphical* multiplication of two complex numbers is easily shown by means of the *polar* form, as follows:

Let $x_1+y_1i=r_1(\cos\theta_1+i\sin\theta_1)$

and $x_2+y_2i=r_2(\cos\theta_2+i\sin\theta_2)$

Then $(x_1+y_1i)(x_2+y_2i)$

$$\begin{aligned}&=r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\\&=r_1r_2[\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\\&\quad+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)]\end{aligned}$$

$$\therefore\quad \boldsymbol{(x_1+y_1i)(x_2+y_2i)=r_1r_2[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)]}$$

This is a *complex number whose modulus is the product of the moduli of the given numbers, and whose amplitude is the sum of their amplitudes.*

EXERCISES

Multiply as indicated and illustrate by means of a drawing:

1. $(2+2i)(1+\sqrt{3}i)$

$$(2+2i)(1+\sqrt{3}i)=2+(2+2\sqrt{3})i-2\sqrt{3}i^2$$
$$=(2+2\sqrt{3})+(2+2\sqrt{3})i$$

In the polar form:

$$2+2i=2\sqrt{2}(\cos 45°+i \sin 45°)$$
$$1+\sqrt{3}i=2 \ (\cos 30°+i \sin 30°)$$

∴ the modulus of the product is $r_1r_2=4\sqrt{2}$ and the amplitude is 75°.

∴ to construct the product, draw OR, making an angle of 75° with OX, Fig. 20, and equal in length to $4\sqrt{2}$.

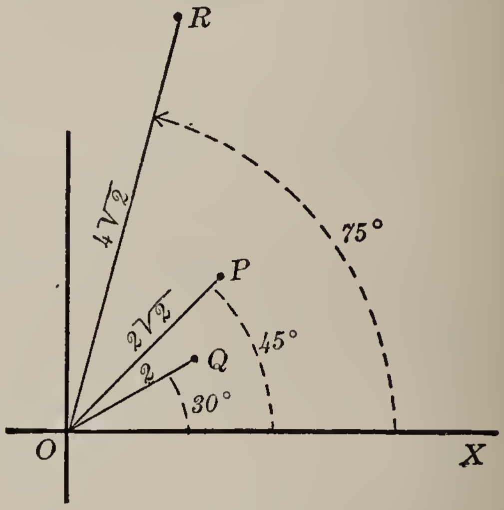

FIG. 20

2. $(2+3i)(3+5i)$

3. $(-2-2i)(2+2i)$

4. $(3+3\sqrt{3}i)(2\sqrt{3}+2i)$

5. $(\sqrt{2}+\sqrt{5}i)(\sqrt{5}+\sqrt{2}i)$

6. $(3-i)(4+3i)$

7. $(3-2i)(4+3i)$

21. The *n*th power of a complex number. If a complex number is multiplied by itself we have

$$[r \ (\cos \theta+ i \sin \theta)]^2=r^2 \ (\cos 2\theta+i \sin 2\theta), \ \S\ 20$$

Multiplying both sides by r $(\cos \theta+i \sin \theta)$, we have

$$[r(\cos \theta+i \sin \theta)]^3=r^3(\cos 3\theta+i \sin 3\theta)$$

In general, it may be proved by *mathematical induction*, § 118, that for any positive integral value of n we have

$$[r(\cos \theta+i \sin \theta)]^n=r^n(\cos n\theta+i \sin n\theta)$$

This equation is known as **De Moivre's formula.** It can be proved that the theorem holds for any *rational* value of n.

22. Division of complex numbers. The denominator of the quotient $\frac{x_1+y_1i}{x_2+y_2i}$ may be rationalized by multiplying both numerator and denominator by the conjugate of x_2+y_2i.

This gives:

$$\frac{(x_1+y_1i)(x_2-y_2i)}{(x_2+y_2i)(x_2-y_2i)}=\frac{(x_1x_2+y_1y_2)+(x_2y_1-x_1y_2)i}{x_2^2+y_2^2}$$

$$\therefore\quad \frac{x_1+y_1i}{x_2+y_2i}=\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}+\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}i$$

This shows that the result is a complex number.

The following division in the *polar form* suggests a *geometric construction* for the quotient of two complex numbers.

By rationalizing the denominator,

$$\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}=\frac{r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2-i\sin\theta_2)}{r_2^2(\cos^2\theta_2+\sin^2\theta_2)}$$

$$=\frac{r_1}{r_2}\,\frac{\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2+i(\sin\theta_1\cos\theta_2-\cos\theta_1\sin\theta_2)}{1}$$

$$\therefore\quad \frac{x_1+y_1i}{x_2+y_2i}=\frac{r_1}{r_2}\Big[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\Big]$$

Hence *the quotient of two complex numbers is a complex number whose modulus is the quotient of the moduli of the given numbers and whose amplitude is the difference of their amplitudes.* Thus construct OP_3, Fig. 21, equal to $\frac{r_1}{r_2}$ and $\angle XOP_3=\theta_1-\theta_2$. Then OP_3 represents the required quotient.

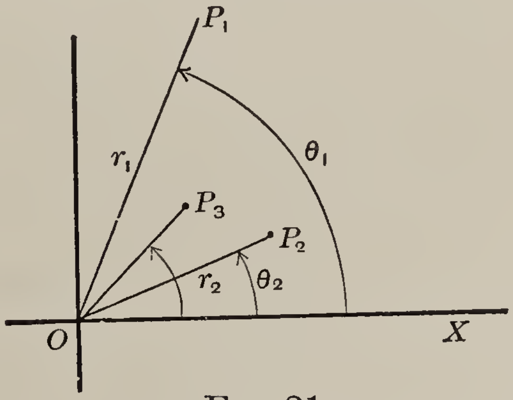

FIG. 21

EXERCISES

Divide as indicated, both algebraically and graphically:

1. $(-2+2\sqrt{3}i)\div(1+\sqrt{3}i)$

Rationalizing the denominator,

$$\frac{-2+2\sqrt{3}i}{1+\sqrt{3}i}=\frac{(-2+2\sqrt{3}i)(1-\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)}=\frac{-2+i(2\sqrt{3}+2\sqrt{3})+6}{1+3}$$

$$=\frac{4+4\sqrt{3}i}{4}=1+\sqrt{3}i$$

Writing both numbers in the polar form,

$$-2+2\sqrt{3}i=4\ (\cos 120^\circ+i\sin 120^\circ)$$

$$1+\sqrt{3}i=2\ (\cos 60^\circ+i\sin 60^\circ)$$

$$\therefore\quad \frac{-2+2\sqrt{3}i}{1+\sqrt{3}i}=2\ (\cos 60^\circ+i\sin 60^\circ)$$

2. $(2+2i)\div(1+\frac{1}{3}\sqrt{3}i)$

3. $(1+i)\div(\frac{1}{4}-\frac{1}{4}i)$

4. $(3-5i)\div(2-3i)$

5. $(2-2\sqrt{3}i)\div(1+i)$

6. $\dfrac{5-3i}{5+3i}$

7. $\dfrac{2}{-1+\sqrt{3}i}$

8. $\dfrac{3}{2i}$

9. $\dfrac{2+5i}{4i}$

10. $\dfrac{2+3i}{3-2i}$

11. $\dfrac{(2+3i)^2}{1+2i}$

12. Reduce $\dfrac{11-3i}{1-3i}$ and $\dfrac{9-2i}{-4-i}$ to the simplest form. Construct the sum.

13. Evaluate $\dfrac{2x^2-(2-3i)x+i}{x^2+x+1}$ for $x=3+2i$ and reduce the result to the form $a+bi$.

14. Simplify $\dfrac{4i\sqrt{10}}{2-3i\sqrt{5}}$

15. Reduce $\dfrac{8+i}{1+2i}$ and $\sqrt{2+12i}$ to the form $a+bi$. Then find the sum graphically.

Summary

23. The chapter has taught the meaning of the following terms:

origin
zero
abscissa, ordinate
co-ordinates
Cartesian system of co-ordinates
polar co-ordinates
positive and negative numbers
integral and fractional numbers
rational and irrational numbers
real and complex numbers
conjugate complex numbers

24. Points in a plane may be located by means of Cartesian co-ordinates and by polar co-ordinates.

25. Real and complex numbers may be represented graphically by points or line-segments.

26. Cartesian co-ordinates may be expressed in terms of polar co-ordinates and conversely.

27. Complex numbers may be added, subtracted, multiplied, and divided algebraically and graphically.

CHAPTER II

THE STRAIGHT LINE. LINEAR FUNCTION

Slope of a Line

28. Slope of a straight line. The positive angle α, Fig. 22, formed by a line AB and the x-axis, is the *inclination* of the line. The inclination may be determined by means of its trigonometrical ratios. In geometry the *tangent ratio* is used to measure the inclination of a line. *The tangent of the inclination is called the* **slope** *of the line.*

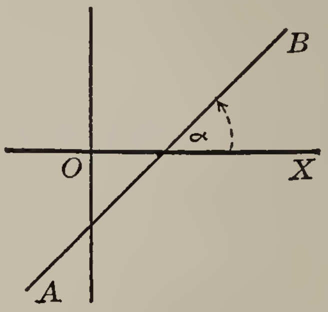

FIG. 22

Let P_1 and P_2, Fig. 23, be *any two points* of a straight line, and let the co-ordinates be (x_1, y_1) and (x_2, y_2), respectively.

Then the slope of P_1P_2 is equal to

$$\frac{QP_2}{P_1Q}=\frac{y_2-y_1}{x_2-x_1}$$

Denoting the slope by m, we have the formula

$$m=\frac{y_2-y_1}{x_2-x_1}$$

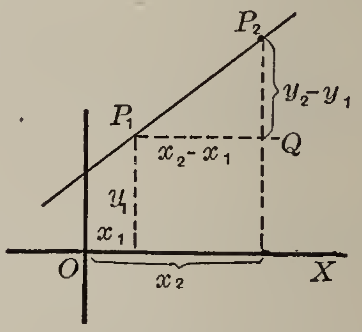

FIG. 23

The differences x_2-x_1 and y_2-y_1 are briefly denoted by Δx_1 (Delta x_1) and Δy_1 (Delta y_1). In this notation the equation $m=\frac{y_2-y_1}{x_2-x_1}$ is written $m=\frac{\Delta y_1}{\Delta x_1}$. Since the equation holds for *every* point on the straight line, the subscripts may be omitted, and we have $m=\frac{\Delta y}{\Delta x}$. The quotient $\frac{\Delta y}{\Delta x}$ is called *differential quotient.*

EXERCISES

1. Show from a figure that the slope is positive or negative according as the line rises or falls from left to right. Thus the sign of m determines whether or not the line rises or falls.

2. Show that the slope of a line parallel to the x-axis is zero.

3. Show that a line parallel to the y-axis has no slope.

4. Find the slope and inclination of each of the lines determined by the following points:

$(-2, 4)$ and $(3, 6)$; $(3, 8)$ and $(-6, -6)$; $(0, -4)$ and $(-3, 1)$.

Equation of a Straight Line

29. Notation for points. The co-ordinates of a *fixed* point are distinguished from the co-ordinates of a *variable* point by means of subscripts. Thus fixed points are denoted by (x_1, y_1), (x_2, y_2), etc., but (x, y) are the co-ordinates of *any* point in the plane.

Likewise, if a point moves on a straight line, or on a curve, its co-ordinates are x and y, without the subscripts.

If the moving point is to satisfy given conditions, these conditions may be expressed in the form of equations involving x and y, the co-ordinates of the point. Since there are several ways of determining a straight line, there are correspondingly various relations between x and y. These relations will be worked out in §§ 30 to 44.

Historical note. This type of mathematics is called analytical geometry. It used to be known as *co-ordinate geometry*. It has two distinguishing characteristics: one, the employment of co-ordinates; the other, the correlation of algebra and geometry. Before analytical geometry was invented algebra and geometry had long been studied as separate mathematical branches.

The use of co-ordinates is very old. The ancient Egyptian architect drew on the wall where he was to chisel a relief two sets of parallels at right angles. When he transferred his design, which had been traced over with a pair of parallel systems of lines at right angles, he was using co-ordinates. The Greek astronomer Hipparchus (about 120 B.C.) was employing the principle of co-ordinates when he located points on the earth's surface by longitude and latitude, using the meridian of Rhodes as *axis of ordinates.* When Heron (about 80 B.C.) mapped his land surveys on cross-lined paper and located streets, public squares, buildings, and monuments with reference to the lines and intersections, and calculated areas of tracts as sums of the rectangles formed on his rectangular paper, he was using co-ordinates.

Apollonius (250–200 B.C.) in his study of conic sections, Oresmus (1323–82 A.D.) in his study of point-sets, Kepler (1571–1630) in his studies in mensuration, all employed the principle of co-ordinates.

Furthermore, algebra and geometry had been studied in combination by individuals long before analytical geometry was invented. All the algebra the Greeks ever studied before Diophantus (*ca.* 250 A.D.) was of a geometrized type. Two books of Euclid's *Elements* were really on what we would call algebra. The principles of Arabian algebra were proved by geometrical methods. Leonardo of Pisa in 1202 urged upon all people the importance of geometry as a means of proving algebraic rules and principles. Pacioli (1494) did the same thing, as did also Regiomontanus (1436–76 A.D.), Stifel (1486–1567), Tartaglia (1500–1557), Cardan (1501–76), and Vieta (1540–1603).

It was Descartes, however, in 1637 who first published the system of representing points by number pairs, lines by equations based on co-ordinates, and the expression and investigation of properties of lines, curved or straight, by the algebraic transformations of equations, as is begun in this chapter. Geometry for more than one thousand years had been helping algebra to grow to maturity. After Descartes algebra began to repay its age-long debt. Since Descartes, algebra and geometry have grown along together.

30. The point-slope form. Let m be the slope of the line P_1P_2, Fig. 24. Let (x_1, y_1) be the co-ordinates of the given point P_1 on line P_1P_2, and let (x, y) be the co-ordinates of *any* other point P on P_1P_2.

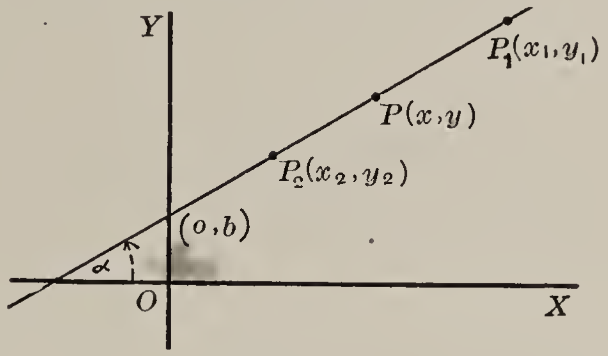

FIG. 24

Then (x, y) satisfy the equation

$$m=\frac{y-y_1}{x-x_1} \qquad \text{Why?}$$

Solving for $y-y_1$, we have

$$\boldsymbol{y-y_1=m(x-x_1)}$$

an equation of the first degree in two variables, x and y. This equation is satisfied by the co-ordinates (x, y) of *every point* on P_1P_2, and it is said to be *the equation of the straight line whose slope is m and which passes through the point P_1.*

EXERCISES

1. The slope of a line passing through the point $(4, -3)$ is 5. Find the equation of the line.

2. Find the equation of a line of slope $\frac{3}{5}$, passing through the point $(-4, -3)$.

3. Find the equation of a straight line of slope $-\frac{2}{3}$, passing through the point $(5, 7)$.

31. The slope-intercept form. Let $(0, b)$, Fig. 25, be the co-ordinates of the point of intersection of P_1P_2 and the y-axis. Substituting $(x_1, y_1)=(0, b)$ in the equation $y-y_1=m(x-x_1)$, we have

$$y-b=m(x-0),$$

or

$$\boldsymbol{y=mx+b}$$

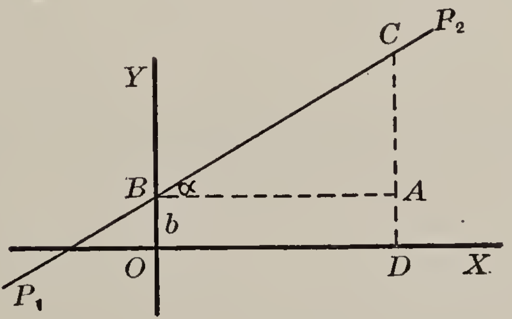

FIG. 25

The number b is the **y-intercept** of P_1P_2 and the equation $y=mx+b$ is said to be *the equation of the straight line whose slope is m and which has the y-intercept b.*

Using Fig. 25, prove that $y=mx+b$.

32. The two-point form. Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of two fixed points and let

$$y=mx+b \tag{1}$$

be the equation of the line passing through these two points. Then

$$y_1=mx_1+b \tag{2}$$

and

$$y_2=mx_2+b \tag{3}$$

Subtracting (2) from (1), and (2) from (3), we have

$$y-y_1=m(x-x_1) \tag{4}$$

and

$$y_2-y_1=m(x_2-x_1) \tag{5}$$

Dividing (4) by (5), we have

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$$

or

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

EXERCISES

1. On squared paper draw a straight line. Select two points on this line and determine its equation.

2. Write the equations of the lines passing through the points:

(5, 6) and (1, −2); (0, 2) and (6, −5); (−1, 3) and (−4, −3)

33. The determinant form. The equation of a straight line given in § 32 may also be written in the determinant form

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

This may be easily seen by expanding the determinant, § 67.

EXERCISE

Show that the condition that three points (x_1, y_1), (x_2, y_2), and (x_3, y_3) lie on a straight line is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

34. The intercept form. Let a be the x-intercept and b the y-intercept of the line PQ, Fig. 26.

Then $$\frac{y-b}{x-0} = \frac{0-b}{a-0}, \text{ § 32.}$$

$$\therefore \frac{y-b}{x} = -\frac{b}{a}$$

$$\therefore y = -\frac{b}{a}x + b$$

$$\therefore \frac{x}{a} + \frac{y}{b} = 1$$

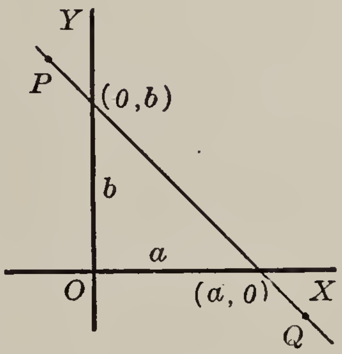

Fig. 26

This is *the intercept form of the equation of a straight line.*

EXERCISES

1. On squared paper draw a straight line cutting both axes. Find the intercepts and determine the equation of the line.

2. Find the intercepts of the lines passing through the points:

$(5, -2)$ and $(1, -3)$; $(5, -8)$ and $(-4, 3)$

35. Lines parallel to the x-axis. Since $m=0$, when the line is parallel to the x-axis, the equation $y=mx+b$ takes the form

$$y=b$$

36. Lines parallel to the y-axis. Let PQ, Fig. 27, be parallel to the y-axis. Then for all points on PQ, $x=k$, whatever be the value of y. For all points not on PQ, $x \neq k$. Hence the equation of PQ is

$$x=k$$

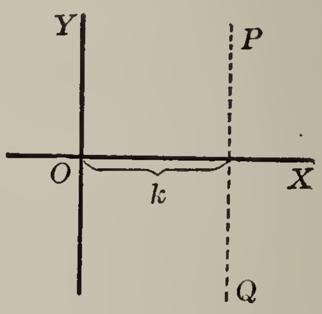

FIG. 27

37. Lines passing through the origin. Substituting $x=0$ and $y=0$ in the equation $y=mx+b$, we have $b=0$. Hence *the equation of a straight line passing through the origin is*

$$y=mx$$

38. Theorem: *Every equation of the first degree in one or two variables represents a straight line.*

The *general form* of an equation of the first degree in two variables is

$$Ax+By+C=0$$

If $B=0$, the equation $Ax+By+C=0$ may be solved for x.

This gives

$$x=-\frac{C}{A}=k$$

which is the equation of a straight line parallel to the y-axis, § 36.

If $B\neq 0$, the equation $Ax+By+C=0$ may be solved for y and changed to the form

$$y=mx+b$$

This defines a straight line cutting the y-axis at the point $(0, b)$, § 31.

If $m=0$, we have $y=b$, § 35, the equation of a line parallel to the x-axis.

If $b=0$, we have $y=mx$, § 37, the equation of a line passing through the origin.

Hence, for all values of A, B, and C the equation $Ax+By+C=0$ can be put into one of the forms $x=k$, $y=b$, $y=mx$, or $y=mx+b$, and therefore it represents a straight line.

39. Linear equation. An equation of the first degree is called a **linear equation.**

40. Linear function. A function of the first degree, as $mx+b$, is a **linear function.** In the equation $y=mx+b$, y is an *explicit* function of x; in the equation $Ax+By+C=0$, y is an *implicit* function of x.

The value of x for which $mx+b$ is equal to zero is called the **zero** of the function $mx+b$.

41. Graphing linear equations. A pair of values of x and y satisfying the equation $Ax+By+C=0$ is a *solution* of the equation. The point determined by this pair of numbers lies on the straight line defined by the equation.

Since two points determine a straight line, the graph of the equation may be obtained by plotting two solutions and drawing the straight line determined by the two points. This line is the required graph.

EXERCISES

Graph the locus of each of the following equations:

1. $x=3$

2. $y=4x$

3. $y=-5$

4. $y=-2x+3$

Let $x=0$ and find the corresponding value of y.
Then let $y=0$ and find the corresponding value of x.

5. $2x+2y=8$

6. $y=3x+5$

7. $2x-5y=0$

8. $\frac{x}{3}+\frac{y}{4}=1$

9. $y=-7$

10. $-4x+3y=12$

Write the slope-intercept form and the intercept form of each of the following equations:

11. $2y+5x-6=x+3$

12. $\frac{x}{4}+\frac{y}{3}=2$

13. $\frac{x-1}{2}=\frac{y-3}{7}$

14. $x-2y=6$

Determine C so that the graphs of the following equations pass through the point $(2, -3)$:

15. $4x+5y+C=0$

16. $7x+y+C=0$

17. $-5x+3y+C=0$

18. $10y+C=0$

19. A straight line passing through the point (5, 3) has equal intercepts. Find its equation.

20. Centigrade and Fahrenheit readings taken at the same time satisfy the equation

$$C=\frac{5}{9}F-\frac{160}{9}$$

Draw the straight line representing this equation. From the graph find C when $F=30°$, and find F when $C=100°$.

21. When a body is thrown downward (in a vacuum) with an initial velocity v_0, its velocity in t seconds is given by the equation $v=v_0+gt$. Draw the graph of this equation for $g=32$ and $v_0=3$.

22. A man invests $1,200 at 3 per cent simple interest. Find the amount a in n years.

Show that a is a linear function of n.

Notice that the coefficient of n is the *rate of increase* of the amount. What is the meaning of the coefficient of n in the graph?

23. Two variables, x and y, are so related as to have a constant ratio m. Express the functional relation between x and y in the form of an equation and show that either variable is a linear function of the other.

24. The weight, w, of an object varies directly as the volume, v. Show that w is a linear function of v. The slope of the straight line representing this function graphically denotes the density of the body.

25. The distance, s, passed over by a body moving uniformly varies directly as the time, t. Show that s is a linear function of t. What is the meaning of the slope of the line representing this function?

26. Show that one of the acute angles of a right triangle is a linear function of the other. Represent this function graphically.

42. The polar equation of a straight line. Let AB, Fig. 28, be any straight line. Let OX be the initial line, and O the pole, § 10.

Denote the length of the perpendicular from O to AB by p and the angle COX by ω.

Let P (ρ, θ) be any point on AB.

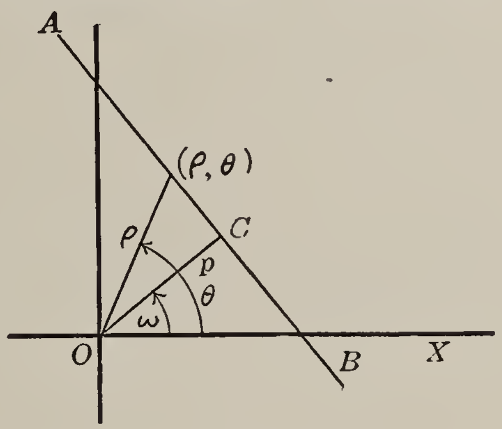

Fig. 28

Then
$$\cos POC = \frac{p}{\rho}$$

Since
$$\angle POC = \theta - \omega,$$

$$\therefore \quad \rho \cos (\theta - \omega) = p$$

This is the equation of line AB in polar co-ordinates.

EXERCISES

Draw the graphs of the following equations:

1. $\rho \cos \theta = 5$

2. $\rho \cos (\theta - 30°) = 6$

3. $\rho \cos (\theta - 120°) = 4$

4. $\rho \sin (\theta + \frac{2}{3}) = 4$

5. Find the polar equation of a straight line passing through the pole.

6. Find the polar equation of the initial line.

43. The equation of a straight line in the normal form. The equation

$$\rho \cos (\theta - \omega) = p, \text{ § 42,}$$

may be written

$$\rho \cos \theta \cos \omega + \rho \sin \theta \sin \omega = p \qquad \text{Why?}$$

$$\text{Show that} \begin{cases} \rho \cos \theta = x \\ \rho \sin \theta = y \end{cases}$$

$$\therefore \quad \boldsymbol{x \cos \omega + y \sin \omega = p}$$

This is the normal form of a linear equation.

44. Reduction of $Ax+By+C=0$ to the normal form. Since the equations

$$Ax + By + C = 0 \tag{1}$$

and

$$x \cos \omega + y \sin \omega - p = 0 \tag{2}$$

represent the same line, they must differ only by a constant factor. Thus, equation (1) may be written

$$kAx + kBy + kC = 0$$

$$\therefore \quad \cos \omega = kA$$

$$\sin \omega = kB$$

$$-p = kC$$

$$\therefore \quad \cos^2 \omega = k^2 A^2$$

$$\sin^2 \omega = k^2 B^2$$

Adding,

$$\cos^2 \omega + \sin^2 \omega = k^2 (A^2 + B^2)$$

$$\therefore \quad k=\frac{1}{\pm\sqrt{A^2+B^2}}$$

$$\therefore \quad \cos\omega=\frac{A}{\pm\sqrt{A^2+B^2}}, \ \sin\omega=\frac{B}{\pm\sqrt{A^2+B^2}},$$

$$-p=\frac{C}{\pm\sqrt{A^2+B^2}}$$

Since p is always a positive number, the last equation determines the sign of the radical. Hence the sign of the radical is opposite to that of C. Why?

EXERCISES

Reduce each of the following equations to the normal form and construct the lines from the values of ω and p. Check your results by means of the intercepts.

1. $4x+3y-25=0$

Dividing every term by $\sqrt{4^2+3^2}$, we have

$$\frac{4x}{5}+\frac{3y}{5}-5=0$$

$$\therefore \quad \cos\omega=\frac{4}{5}, \ \sin\omega=\frac{3}{5}, \ p=5$$

Since $\cos\omega$ and $\sin\omega$ are positive, the perpendicular lies in the first quadrant. Determine the value of ω and construct the line, Fig. 29.

2. $-x+3y-4=0$

3. $x+8=-2y$

4. $3x+4y+6=0$

5. $x-y=0$

6. $3x-5y=4$

7. $3x-4y=0$

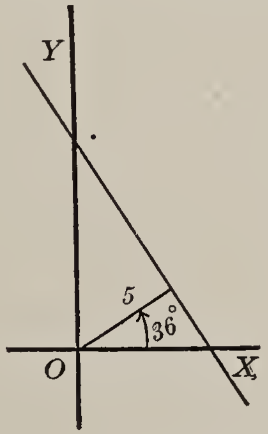

FIG. 29

Draw the lines determined by the following conditions and write their equations:

8. $p=3,\ \omega=30°$ **10.** $p=4,\ \omega=74°$

9. $p=5,\ \omega=150°$ **11.** $p=5,\ \omega=135°$

12. Find the distance between the parallel lines

$$7x-8y-40=0 \text{ and } 7x-8y-15=0.$$

13. Find the radius of the circle whose center is at the origin and which touches the line $5x+12y-25=0$.

45. Distance of a point from a line. Let AB, Fig. 30, be a given line, let $P(x_1, y_1)$ be a given point, and let d be the distance from P to AB.

Through P draw $A'B' \parallel AB$.

Let OD be perpendicular to $A'B'$ and equal to p', and let $OC=p$.

Then the equations of AB and $A'B'$, reduced to the normal form, may be written

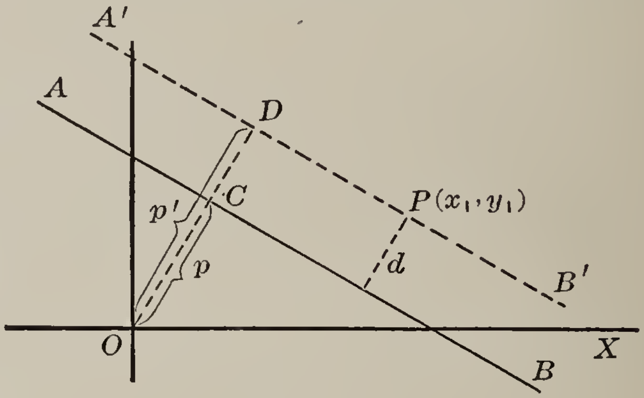

FIG. 30

$$x \cos \omega+y \sin \omega=p$$

and
$$x \cos \omega+y \sin \omega=p',$$

respectively.

Since the equation of $A'B'$ is satisfied by (x_1, y_1), it follows that

$$x_1 \cos \omega+y_1 \sin \omega=p'$$

$$\therefore \quad d=p'-p=x_1 \cos \omega+y_1 \sin \omega-p$$

$$\therefore \quad \boldsymbol{d=x_1 \cos \omega+y_1 \sin \omega-p}$$

Hence, *to find the distance from a point to a line, reduce the equation of the line to the normal form* $x \cos \omega+y \sin \omega-p=0$

and substitute the co-ordinates of the point in place of the variables x and y.

If d is *negative* the given point and the origin lie on the *same* side of the given line; if d is *positive* they lie on *opposite* sides of the given line.

EXERCISES

1. Show that

$$d=\frac{Ax_1+By_1+C}{\pm\sqrt{A^2+B^2}},$$

using the $-$ sign when C is positive and the $+$ sign when C is negative.

2. Find the distance of the point (1, 2) from the line of $3x-10=4y$.

3. Find the distance of the point $(-3, 4)$ from the line of $5x+12y=25$.

4. Find the distance of the point (1, 1) from the line of $3x+4y+6=0$.

5. Find the distance of the point (2, 3) from the line $3x+4y=-12$.

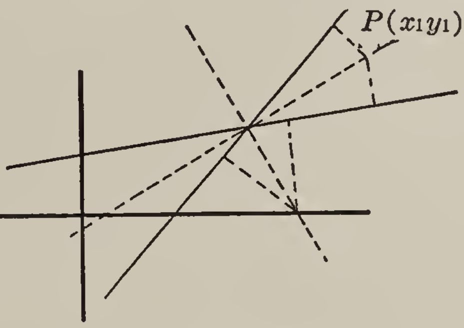

FIG. 31

46. Bisector of an angle. Let $P(x_1y_1)$, Fig. 31, represent any point on the bisector of that pair of vertical angles which contains the origin.

Since P is equally distant from the given lines it follows that the equation of the bisector is

$$x\cos\omega+y\sin\omega-p=x\cos\omega'+y\sin\omega'-p'$$

Similarly the equation of the bisector of the other pair of vertical angles is

$$x\cos\omega+y\sin\omega-p=-(x\cos\omega'+y\sin\omega'-p')$$

EXERCISE

Find the equations of the bisectors of the angles formed by the lines

$$3x-4y=12 \text{ and } 12x+5y=30$$

47. Distance between two points. Show that P_2Q, Fig. 32, is equal to y_2-y_1, and that P_1Q is equal to x_2-x_1

$$\therefore \quad d^2=(x_2-x_1)^2+(y_2-y_1)^2$$

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

or

$$\boldsymbol{d=\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

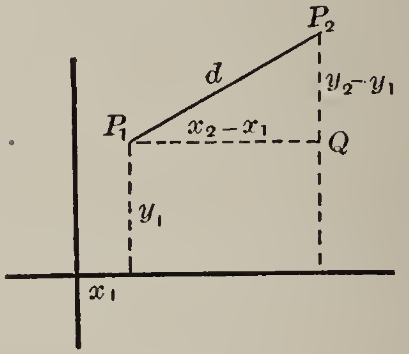

FIG. 32

48. Locus of moving point always equidistant from two given points. Suppose a point $P(x, y)$ moves, always being equidistant from two given points, $P_1(1, 2)$ and $P_2(4, 1)$. Find and plot the equation of the locus of P.

Since $$PP_1=PP_2,$$

we have

$$\sqrt{(x-1)^2+(y-2)^2}=\sqrt{(x-4)^2+(y-1)^2}, \quad \S\, 47$$

$$\therefore \quad (x-1)^2+(y-2)^2=(x-4)^2+(y-1)^2$$

This reduces to

$$6x-2y-13=0$$

Plot this equation.

EXERCISES

Find the equation of the locus of a point equidistant from the following pairs of points. Plot the locus.

$(-3, 4)$ and $(3, 2)$; $(0, 0)$ and $(4, 6)$; $(3, 2)$ and $(-5, 1)$.

49. Division of a segment in a given ratio. Let P_3, Fig. 33, divide the segment P_1P_2 into the segments m and n. The co-ordinates of P_3 may be determined as follows:

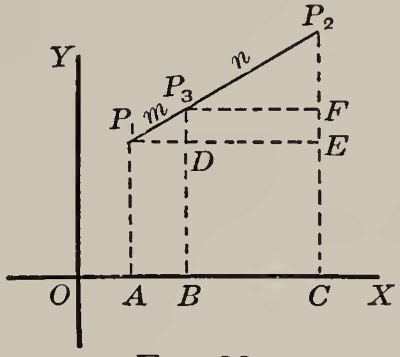

FIG. 33

$$\frac{P_1P_3}{P_3P_2}=\frac{P_1D}{DE}=\frac{P_1D}{P_3F}$$

$$\therefore \quad \frac{m}{n}=\frac{x_3-x_1}{x_2-x_3}$$

$$\therefore \quad mx_2-mx_3=nx_3-nx_1$$

$$\therefore \quad (m+n)x_3=mx_2+nx_1$$

$$\therefore \quad x_3=\frac{mx_2+nx_1}{m+n}$$

Similarly

$$y_3=\frac{my_2+ny_1}{m+n}$$

50. Midpoint of a segment. In the formulas derived in § 49, let $m=n$. Then show that

$$\begin{cases} x_3=\frac{1}{2}(x_1+x_2) \\ y_3=\frac{1}{2}(y_1+y_2) \end{cases}$$

EXERCISES

1. Derive the distance formulas, § 47, letting P_1 be in the first quadrant, P_2 in the second.

2. Prove the distance formulas for various positions of P_1 and P_2, thus showing that the formulas hold in general.

3. Find the distance between (3, −4) and (8, −2).

4. Prove that the midpoint of the hypotenuse of a right triangle is the center of the circumscribed circle.

Place the triangle, making the sides of the right angle fall along the co-ordinate axes.

5. Find the lengths of the sides of a triangle, the co-ordinates of whose vertices are (2, 2), (4, 3), (−3, 5).

6. Find the lengths of the medians of a triangle, the co-ordinates of whose vertices are (7, −3), (−4, 7), (5, −2).

7. Using the fact that the medians of a triangle are concurrent in a trisection point, find the co-ordinates of the common point.

8. Show that the points (2, 4), (−2, 4), (−1, 1), and (1, 7) are the vertices of a parallelogram.

9. Prove that the diagonals of a rectangle are equal.

Make two adjacent sides coincide with the axes.

10. Prove that the line joining the midpoints of two sides of a triangle is equal to one-half of the third side.

11. Show that a circle whose center is the point (2, −1) passes through the points (6, 2), (−1, 3), (−2, −4).

12. Find the equation of the locus of points equidistant from the points (−3, −3) and (0, 4).

13. What are the co-ordinates of the point dividing the segment (−1, 4), (5, 3) in the ratio 2:3?

14. Find the trisection points of the segment joining the points (0, 3) and (6, −3).

Summary

51. The chapter has taught the meaning of the following terms: slope, intercept.

52. The following forms of the equation of a straight line have been developed:

1. The point-slope form:

$$y-y_1=m(x-x_1)$$

2. The slope-intercept form:

$$y=mx+b$$

3. The two-point form:

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$$

4. The determinant form:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

5. The intercept form:

$$\frac{x}{a}+\frac{y}{b}=1$$

6. The polar form:

$$\rho \cos(\theta-\omega)=p$$

7. The normal form:

$$x \cos \omega+y \sin \omega=p$$

8. The equation of a line parallel to the x-axis:

$$y=b$$

9. The equation of a line parallel to the y-axis:

$$x=k$$

10. The equation of a line passing through the origin:

$$y=mx$$

11. The general form:

$$Ax+By+C=0$$

53. It has been shown:

1. *That every equation of the first degree in one or two variables represents a straight line.*

2. How to graph a linear equation.

3. That the equations of the bisectors of the angles formed by two intersecting lines are

$$x \cos \omega+y \sin \omega-p=\pm(x \cos \omega'+y \sin \omega'-p')$$

54. The following formulas have been proved:

1. The distance from a point to a line:

$$d=x_1 \cos \omega+y_1 \sin \omega-p$$

2. The distance between two points:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

3. The co-ordinates of the point dividing a segment in a given ratio:

$$(x, y)=\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

4. The co-ordinates of the midpoint of a segment:

$$(x, y)=\left(\frac{1}{2}(x_1+x_2), \frac{1}{2}(y_1+y_2)\right).$$

CHAPTER III

SEVERAL STRAIGHT LINES. SIMULTANEOUS LINEAR EQUATIONS IN TWO VARIABLES. AREAS

Several Straight Lines

55. Parallel straight lines. *If two lines are parallel, their slopes are equal,* i.e.,

$$m_1 = m_2 \qquad (1)$$

Moreover, let $A_1x + B_1y + C_1 = 0$
and $A_2x + B_2y + C_2 = 0$
be the equations of two parallel straight lines. It follows that

$$y = -\frac{A_1}{B_1}x - \frac{C_1}{B_1} \qquad \text{Why?}$$

and

$$y = -\frac{A_2}{B_2}x - \frac{C_2}{B_2} \qquad \text{Why?}$$

Since $m_1 = -\frac{A_1}{B_1}$ and $m_2 = -\frac{A_2}{B_2}$

$$\therefore \quad -\frac{A_1}{B_1} = -\frac{A_2}{B_2}$$

$$\therefore \quad \frac{A_1}{A_2} = \frac{B_1}{B_2} \qquad (2)$$

Hence, *if two straight lines are parallel, the coefficients of the variables in the equations of these lines are proportional.*

EXERCISE

State and prove the converse of the second theorem in § 55.

56. Incompatible equations. The equations of a pair of parallel straight lines are called **incompatible,** or **inconsistent.** Such equations have no *common* solution, § 61.

Hence the equations

$$A_1x+B_1y+C_1=0$$

and

$$A_2x+B_2y+C_2=0$$

are *incompatible,* have no common solution, and their lines are *parallel,* if

$$\frac{A_1}{A_2}=\frac{B_1}{B_2},$$

or if

$$m_1=m_2$$

EXERCISES

1. Show that the following pairs of lines are parallel:

$$\begin{cases}3x-4y=4\\3x-4y=9\end{cases}\qquad\begin{cases}3x+\ 5y=2\\6x+10y=2\end{cases}$$

2. Find the equation of a straight line parallel to $3x-5y=-6$ and passing through the point $(-2, 3)$.

57. Perpendicular straight lines. If two straight lines, Fig. 34, be perpendicular to each other, then

$$\alpha_1=\alpha_2+90°$$

$$\therefore\quad \tan\alpha_1=-\cot\alpha_2=-\frac{1}{\tan\alpha_2}$$

or

$$m_1=-\frac{1}{m_2},\ \S\,55$$

$$\therefore\quad -\frac{A_1}{B_1}=-\frac{1}{-\dfrac{A_2}{B_2}}$$

or

$$A_1A_2=-B_1B_2$$

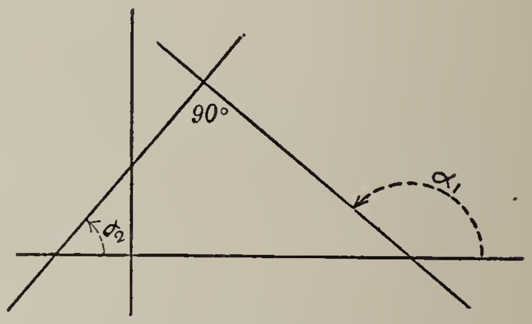

FIG. 34

Thus, if $A_1x+B_1y+C_1=0$ and $Ax_2+B_2y+C_2=0$ are equations of *perpendicular* straight lines, then

$$m_1=-\frac{1}{m_2},$$

or

$$A_1A_2=-B_1B_2$$

EXERCISES

1. Show that the lines of the equations $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$ are perpendicular to each other if $A_1A_2=-B_1B_2$.

2. Show that the lines of the equations $2x-y+3=0$ and $x+2y-7=0$ are perpendicular to each other.

3. Write an equation of a line perpendicular to the line $2x-y=8$.

4. Write the equation of the straight line perpendicular to $7x+9y=-1$ and passing through the point (5, 3).

58. Angle between two lines. Let the lines AB and CD, Fig. 35, intersect at P. Let β be the smallest angle through which CD must be turned in counterclockwise direction to coincide with AB. Let $\alpha_1>\alpha_2$.

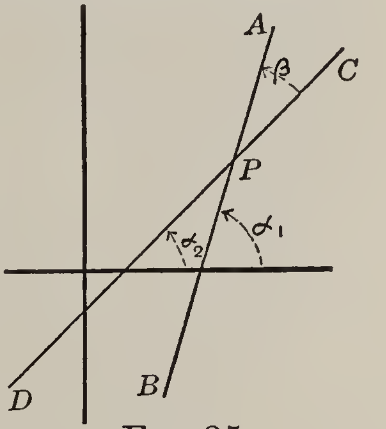

FIG. 35

Then $\beta=\alpha_1-\alpha_2$. Why?

$$\therefore\quad \tan\beta=\tan(\alpha_1-\alpha_2)$$

$$=\frac{\tan\alpha_1-\tan\alpha_2}{1+\tan\alpha_1\tan\alpha_2}$$

Since $\tan\alpha_1=m_1$ and $\tan\alpha_2=m_2$, it follows that

$$\tan\beta=\frac{m_1-m_2}{1+m_1m_2}$$

EXERCISE

Find the angle between the lines $x+2y=3$ and $3x-y=-4$.

Simultaneous Linear Equations

59. Intersection of two straight lines. Since the co-ordinates of any point on a line satisfy the equation of that line, it follows that the co-ordinates of the point of intersection of two lines satisfy each of the two equations. Hence by solving the equations simultaneously we may determine the co-ordinates of the point of intersection.

60. Solution of a system of linear equations by determinants. In a given system of linear equations in two unknowns the terms containing the unknowns may be brought to one side, and the terms not containing the unknowns to the other side, of the equation. After all similar terms have been combined, the system is of the following form:

$$\begin{cases} ax+by=c \\ a_1x+b_1y=c_1 \end{cases}$$

To eliminate y, the first equation is multiplied by b_1 and the second by b. This gives the equations

$$\begin{cases} ab_1x+bb_1y=cb_1 \\ a_1bx+b_1by=c_1b \end{cases}$$

Subtracting the second equation from the first and dividing by the coefficient of x,

$$x=\frac{cb_1-c_1b}{ab_1-a_1b}$$

Similarly,

$$y=\frac{ac_1-a_1c}{ab_1-a_1b}$$

It will be shown below how these results may be used as formulas to find the solution of the system *directly from the given equations.*

PIERRE SIMON LAPLACE

PIERRE SIMON LAPLACE

PIERRE SIMON LAPLACE (1749–1827) was born in Normandy and was educated through the aid of wealthy friends. He held himself so aloof from his relatives and benefactors that very little is known of his early life. As a young man he wrote a paper on the principles of mechanics that induced D'Alembert to recommend him for a place in a military school. Here he pursued original researches in astronomy for the seventeen years from 1771 to 1787. During this period he directed numerous memoirs of great power to the French Academy. In 1787 he set himself the task to "offer a complete solution of the great mechanical problem presented by the solar system and bring theory so closely to coincide with observation that empirical equations should no longer find a place in astronomical tables." The results of his work under this ideal are embodied in the *Exposition du système du monde* and in the *Méchanique céleste.*

The *Exposition* was published in 1796 and contains as a sort of appendix the enunciation of the famous nebular hypothesis. The *Méchanique céleste* was published in five large volumes, the first two in 1799, the third in 1802, the fourth in 1805, and the fifth in 1825. This great work was translated into English, copiously annotated, and published in Boston from 1829 to 1839 by Nathaniel Bowditch. Napoleon once said to him: "M. Laplace, they tell me you have written this large book on the system of the universe, and have not even mentioned its Creator." Laplace's curt reply was: "I had no need for that hypothesis." Nevertheless Laplace was as staunch a religionist as a scientist.

In 1812 he published his analytical theory of *probabilities*, which still remains one of the chief authorities on this subject.

Laplace was vain and selfish, ungrateful to political friends and the benefactors of his youth, pliant, if not servile, in politics, and unscrupulous about passing off the work of others as his own. Napoleon removed him from a political office for administrative inefficiency. In religion, philosophy, and science he manifested strength of character, and in later life he was both generous and appreciative of the work of his pupils. He once withheld one of his own papers from publication in order that one of his pupils might have entire credit for the investigation.

[See Ball or Cajori, *History of Mathematics*, or *Encyclopaedia Britannica.*]

Each of the expressions $cb_1 - c_1b$, $ab_1 - a_1b$, and $ac_1 - a_1c$ is of the form of the difference of two products. Such expressions are called *determinants.*[*]

The determinant $cb_1 - c_1b$ may be represented by the following symbol:

$$\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}$$

which means that from the product cb_1 we are to subtract the product bc_1.

Similarly $ab_1 - a_1b$ and $ac_1 - a_1c$ may be written

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} \text{ and } \begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}$$

Hence the solution of the system

$$\begin{cases} a\,x + b\,y = c \\ a_1x + b_1y = c_1 \end{cases}$$

---

[*] Leibnitz in a letter to L'Hôpital, of April 28, 1693, was the first to publish the essential features of the method of solution of equations by determinants, though his procedure was somewhat different from the modern form. He also drew attention to the importance of the theory of permutations and combinations in determining the factors and signs of the products. Beyond these announcements about the method, Leibnitz did nothing further with it, nor did any of his contemporaries. Aside from a "Note" in a mathematical journal of 1700, nothing further was heard of the method until Gabriel Cramer in an appendix of his book of 1750 on *The Analysis of Curves* solved a system of n equations in n unknowns by the method, showed how to use the theory of combinations and permutations with it, and convinced men of its power.

Bézout (1730–83) and Vandermonde (1735–96) both worked on the theory of determinants, and Laplace made important applications of it. Lagrange (1736–1813) applied the doctrine to the problems of analytical geometry, and Gauss, in 1801, made important investigations and improvements in the new theory. The modern name *determinants* is due to Cauchy (1789–1857). Jacobi (1804–51) completed the theory of determinants. The classic texts on the subject are Brioschi's of 1854, Baltzer's of 1857, Scott's of 1880, and Muir's of 1882 (Tropfke, I, 143–46).

takes the form

$$x=\frac{\begin{vmatrix} c & b \\ c_1 & b_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}}, \qquad y=\frac{\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}}{\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}}.$$

Notice that the two *denominators* are *the same,* the numbers in the first column being the coefficients of x and the numbers in the second column the coefficients of y *in the given equations.* This makes it easy to remember the *denominators.* The *numerator* of the fraction which gives the value of x is obtained from the *denominator* by replacing the numbers in the first column (the coefficients of x) by the constants c and c_1 respectively. The *numerator* of the fraction which gives the value of y is obtained from the *denominator* by replacing the numbers in the second column (the coefficients of y) by c and c_1.

EXERCISES

Solve the following systems:

1. $\begin{cases} 4x+6y=9 \\ 2x+9y=7 \end{cases}$

$$x=\frac{\begin{vmatrix} 9 & 6 \\ 7 & 9 \end{vmatrix}}{\begin{vmatrix} 4 & 6 \\ 2 & 9 \end{vmatrix}}=\frac{9\cdot 9-6\cdot 7}{4\cdot 9-6\cdot 2}=\frac{81-42}{36-12}=\frac{39}{24}=\frac{13}{8}$$

$$y=\frac{\begin{vmatrix} 4 & 9 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & 6 \\ 2 & 9 \end{vmatrix}}=\frac{4\cdot 7-9\cdot 2}{24}=\frac{10}{24}=\frac{5}{12}$$

Hence, $(x, y)=\left(\frac{13}{8}, \frac{5}{12}\right)$

2. $\begin{cases} 2x+3y=6 \\ 3x-5y=4 \end{cases}$

3. $\begin{cases} 5x+y=9 \\ 3x+y=5 \end{cases}$

4. $\begin{cases} 4x+2y=1 \\ 3x-2y=\frac{5}{2} \end{cases}$

5. $\begin{cases} 2x=53+y \\ 19x-17y=0 \end{cases}$

6. $\begin{cases} ax-by=c \\ dx-ey=f \end{cases}$

7. $\begin{cases} ax-by=0 \\ x-\ \ y=c \end{cases}$

8. $\begin{cases} kx+ly+n=0 \\ 3x-4y=4n \end{cases}$

9. $\begin{cases} \frac{x}{a}+\frac{y}{b}=1 \\ \frac{x}{c}+\frac{y}{d}=1 \end{cases}$

10. $\begin{cases} \frac{3x}{4}-\frac{5y}{6}=1 \\ \frac{5x}{6}-\frac{3y}{4}=2 \end{cases}$

11. $\begin{cases} 3x+4my=8mn \\ \frac{x}{7m}-7y+3n=0 \end{cases}$

12. $\begin{cases} ax+\frac{b}{y}=2ab \\ bx+\frac{a}{y}=a^2+b^2 \end{cases}$

13. $\begin{cases} (a+b)x+(a-b)y=4ab \\ (a-b)x-(a+b)y=2a^2-2b^2 \end{cases}$

14. $\begin{cases} a(x+y)+b(x-y)=a \\ (a+b)x-(a-b)y=b \end{cases}$

15. $\begin{cases} (a+b)x-(a-b)y=2ac \\ (a+c)x-(a-c)y=2ab \end{cases}$

61. Inconsistent and equivalent equations. We have seen that two linear equations in two unknowns are represented graphically by two straight lines, and that the co-ordinates of the point of intersection form the solution of the system. However, two lines do not always intersect: they may be parallel or they may coincide.

1. If we graph the system $\begin{cases} 2x+\ \ y=\ \ 5 \\ 6x+3y=18 \end{cases}$ we obtain two *parallel* lines, Fig. 36. In this case the equations have *no*

common solution. They are said to be **inconsistent,** or **incompatible.**

Moreover, if we solve the same system by determinants, we have

$$x=\frac{\begin{vmatrix}5 & 1\\ 18 & 3\end{vmatrix}}{\begin{vmatrix}2 & 1\\ 6 & 3\end{vmatrix}}=\frac{-3}{0},\quad y=\frac{\begin{vmatrix}2 & 5\\ 6 & 18\end{vmatrix}}{0}=\frac{6}{0}$$

Since it is impossible to divide the numbers -3 and 6 by zero the resulting forms also show that the equations have no common solution.

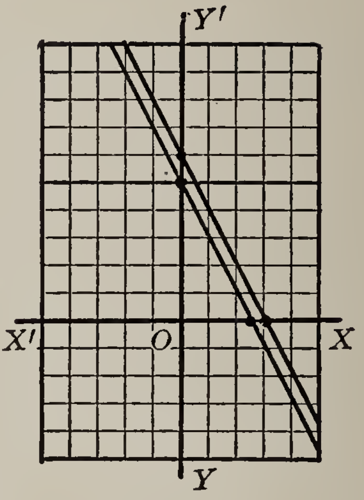

FIG. 36

2. If we graph the system $\begin{cases} x-\ \ y=\ 2\\ 5x-5y=10\end{cases}$ we find that both are represented by the *same* line, Fig. 37. Hence any solution of either equation is a solution of the other. Such equations are said to be **equivalent** or **dependent.**

The difficulty that arises here is not that there is no solution, but that there are too many.

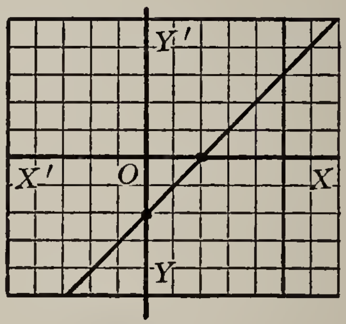

FIG. 37

Solving the same system by determinants, we have

$$x=\frac{\begin{vmatrix}2 & -1\\ 10 & -5\end{vmatrix}}{\begin{vmatrix}1 & -1\\ 5 & -5\end{vmatrix}}=\frac{-10+10}{-\ 5+\ 5}=\frac{0}{0},\quad y=\frac{\begin{vmatrix}1 & 2\\ 5 & 10\end{vmatrix}}{0}=\frac{0}{0}$$

Since a number multiplied by zero always gives zero, the expression $\frac{0}{0}$ may represent *any* number. Hence the solution is *indeterminate.* It is easily seen that one equation may be derived from the other by simple multiplication by a constant.

The two preceding examples show the following:

1. *The two linear equations*

$$\begin{cases} A_1x+B_1y+C_1=0 \\ A_2x+B_2y+C_2=0 \end{cases}$$

are simultaneous, have one, and only one, solution, and the lines intersect if the determinant

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0$$

Hence this fact may be used to determine whether a system of equations has one, and only one, solution, and whether the two straight lines intersect.

2. *The equations have no common solution, are incompatible, and the lines are parallel, if*

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$$

or if

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=k$$

3. *The equations are dependent and the straight lines are identical if*

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = 0$$

or if

$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=k$$

EXERCISES

Show which of the following systems are equivalent and which inconsistent.

1. $\begin{cases} 3x+\frac{y}{4}=6 \\ 4x+\frac{y}{3}=8 \end{cases}$

2. $\begin{cases} 3x-2y=14 \\ 9x-6y=36 \end{cases}$

3. $\begin{cases} x+\frac{2}{7}y=2 \\ \frac{x}{2}+\frac{1}{7}y=1 \end{cases}$

4. $\begin{cases} 3x+2y-7-x=12-3y \\ 2x+5y=20 \end{cases}$

5. $\begin{cases} 7x-8=4y-2x \\ 18x-8y=16 \end{cases}$

6. $\begin{cases} 3x+4y=12 \\ 6x+8y=14 \end{cases}$

7. $\begin{cases} x-y+1=0 \\ 4x+y=16 \end{cases}$

62. Pencil of lines. The totality of lines, Fig. 38, passing through one and the same point is called a **pencil of lines.**

Let and

$$\begin{cases} A_1x+B_1y+C_1=0 \\ A_2x+B_2y+C_2=0 \end{cases} \quad (1)$$

be two distinct lines of a pencil.

Let k denote an arbitrary constant.

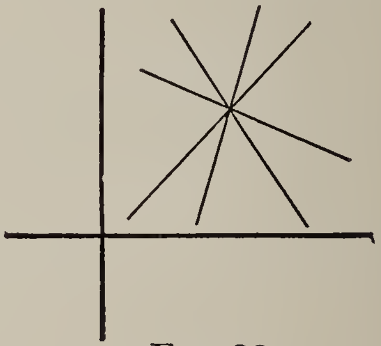

FIG. 38

Show that the equation

$$A_1x+B_1y+C_1+k(A_2x+B_2y+C_2)=0$$

represents a straight line for any value assigned to k.

Moreover, this equation is satisfied by the co-ordinates of the point of intersection of the two given equations. Why?

Therefore the equation

$$\boldsymbol{A_1x+B_1y+C_1+k(A_2x+B_2y+C_2)=0} \quad (2)$$

is the equation of a straight line passing through the point of intersection of the two lines $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$.

Since k is an arbitrary constant, equation (2) determines a system of lines passing through the point of intersection of equations (1).

The arbitrary constant k is the **parameter*** of the system.

EXERCISES

Using equation (2), § 62, solve the following problems:

1. Find the line passing through the point of intersection of the lines $4x-3y+3=0$ and $3x+5y-34=0$ and the origin.

2. Find the equation of the line determined by the point $(-1, 4)$ and the point of intersection of the lines $3x-2y-1=0$ and $2x+3y-15=0$.

3. Find the line of the pencil $2x+y-13=0$, $5x-2y+11=0$ having an inclination of 60°.

63. Pencil of parallel lines.

The equation

$$Ax+By+k=0$$

defines a straight line for every value of the arbitrary constant k. Since all those lines can be shown to be parallel, they are said to form a **pencil of parallel lines.**

64. Equation of a pair of lines.

Let

$$\begin{cases} A_1x+B_1y+C_1=0 \\ A_2x+B_2y+C_2=0 \end{cases} \qquad (1)$$

and

be the equations of two given straight lines. Show that the equation

$$(A_1x+B_1y+C_1)(A_2x+B_2y+C_2)=0 \qquad (2)$$

is satisfied by the co-ordinates of all points on the two given lines. Moreover, no other point satisfies this equation. Hence equation (2) is said to represent equations (1).

---

* The word **parameter** in the sense of an arbitrary constant in an equation was introduced into mathematics by Leibnitz.

EXERCISES

Represent graphically the following equations:

1. $x^2+6xy+9y^2+5x+15y+6=0$ Find the factors of the left member of the equation.

2. $x^2+y=x+y^2$

3. $x^2+3xy+2y^2=0$

65. Homogeneous equations. If in the equation $Ax+By+C=0$ the constant C is zero, the equation is *homogeneous.*

In general, an equation is **homogeneous** with respect to the variables if the sum of the exponents of the variables is the same in all terms.

According to § 60, the solution of the *homogeneous system*

$$\begin{cases} a_1x+b_1y=0 \\ a_2x+b_2y=0 \end{cases} \qquad (1)$$

is

$$x=\frac{\begin{vmatrix} 0 & b_1 \\ 0 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y=\frac{\begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Hence, if the denominator

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

the system of equations (1) has the solution (0, 0), which is also the *only* solution of the system. This means graphically that the two lines are *distinct,* both *passing through the origin.*

If, however, the denominator

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

then, according to § 61,

$$x=\frac{0}{0}, \quad y=\frac{0}{0}$$

which means that the system (1) has an *infinite number of solutions*, and, graphically, that the two lines coincide because they have all points in common.

If $x \neq 0$ and $y \neq 0$, it follows from

$$x=\frac{\begin{vmatrix} 0 & b_1 \\ 0 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x=0$$

Similarly,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y=0$$

Since x and y are not both zero, it follows that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0$$

This shows that *the determinant of two homogeneous linear equations in two variables vanishes if the equations are satisfied by values of x and y, excluding the case when x and y are both equal to zero.*

EXERCISE

Determine k so that the equations

$$\begin{cases} x+\ y=kx \\ 4x-2y=ky \end{cases}$$

have solutions other than (0, 0)

Simultaneous Linear Equations in Three Variables

66. Solution by elimination. Systems of equations in *three* or *more* unknowns are solved by the methods used in solving equations in *two* unknown numbers. In general, the aim should be to obtain first *two* equations in *two* unknowns by *eliminating* the third unknown, and then to solve these two equations.

EXERCISES

Solve the following systems:

1. $$\begin{cases} 4x-y+z=1 \\ x+2y+7z=7 \\ 3x-y-5z=5 \end{cases}$$

Subtracting the third equation from the first,

$$x+6z=-4$$

Multiplying the third equation by 2 and adding the resulting equation to the second equation,

$$7x-3z=17$$

Solving the system

$$\begin{cases} x+6z=-4 \\ 7x-3z=17 \end{cases}$$

we have $$(x, z)=(2, -1).$$

By substituting these values in the first equation, we find

$$y=6.$$

$\therefore$ $(x, y, z)=(2, 6, -1)$ *is the solution of the system.*

2. $$\begin{cases} 2a-b+c=1 \\ a-7b-8c=1 \\ 7a+14b+2c=7 \end{cases}$$

3. $$\begin{cases} x+2y-4z=11 \\ 2x=3y \\ y-4z=0 \end{cases}$$

4. $$\begin{cases} x+2y-z=2 \\ 3x-2y+2z=0 \\ 5x-4y+3z=1 \end{cases}$$

5. $$\begin{cases} 5x-7y-z=16 \\ 3x-2y+2z=10 \\ 2x+y+3z=6 \end{cases}$$

67. Determinant of the third order. The symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is called a **determinant of the third order.** It represents the following sum:

$$a_1b_2c_3+a_2b_3c_1+a_3b_1c_2-c_1b_2a_3-c_2b_3a_1-c_3a_2b_1$$

The nine numbers a_1, a_2, a_3, b_1, b_2, b_3, etc., are called the *elements*. The horizontal lines in the square form are the *rows* and the vertical lines the *columns* of the determinant. Each term in the expansion is a product of three elements, no two of which lie in the same row or in the same column.

A determinant of the third order may be expanded as follows:

Draw the diagonal through the first element, a_1, Fig. 39, and the parallels to it through a_2 and a_3 respectively. This gives the terms $a_1b_2c_3$, $a_2b_3c_1$, and $a_3c_2b_1$.

Then draw the diagonal through c_1 and the parallels through c_2 and c_3. The signs of the last three products are then changed. This gives the terms $-c_1b_2a_3$, $-c_2b_3a_1$, and $-c_3a_2b_1$.

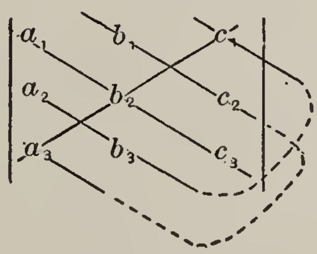

FIG. 39

EXERCISES

Evaluate the following determinants:

1. $$\begin{aligned} \begin{vmatrix} 5 & 2 & -6 \\ 1 & 4 & 7 \\ 2 & 3 & 1 \end{vmatrix} &= 5\cdot4\cdot1+1\cdot3\cdot(-6)+2\cdot7\cdot2 \\ &\quad -(-6)\cdot4\cdot2-7\cdot3\cdot5-1\cdot1\cdot2 \\ &= 20-18+28+48-105-2 \\ &= -29 \end{aligned}$$

2. $\begin{vmatrix} 1 & 3 & 8 \\ -1 & 2 & 0 \\ 1 & -4 & 5 \end{vmatrix}$ **4.** $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 8 \\ 1 & -1 & 0 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 8 & 3 & 0 \end{vmatrix}$ **5.** $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 4 & 3 & 5 \end{vmatrix}$

68. Solution by determinants. By solving the equations

$$\begin{cases} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{cases}$$

a formula may be obtained for the solution of *any* system of three *linear* equations in *three* unknowns.

Eliminating y between the first two equations, we have

$$(a_1b_2-a_2b_1)x+(b_2c_1-b_1c_2)z=d_1b_2-d_2b_1 \qquad (1)$$

Eliminating y between the first and third equations, we have

$$(a_3b_1-a_1b_3)x+(c_3b_1-b_3c_1)z=d_3b_1-d_1b_3 \qquad (2)$$

Solving equations (1) and (2), we have

$$x=\frac{d_1b_2c_3+d_2b_3c_1+d_3c_2b_1-c_1b_2d_3-c_2b_3d_1-c_3d_2b_1}{a_1b_2c_3+a_2b_3c_1+a_3c_2b_1-c_1b_2a_3-c_2b_3a_1-c_3a_2b_1}$$

According to § 67 this may be written

$$x=\frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

Notice that the *denominator* is a determinant whose elements are the coefficients of x, y, and z in the given system and that the *numerator* is derived from the denominator by replacing the coefficients of x by the constants.

Similarly,

$$y=\frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}$$

EXERCISES

Solve by determinants:

1. $\begin{cases} 2x+3y+4z=16 \\ 5x-8y+2z=\ 1 \\ 3x-y\ \ -2z=\ 5 \end{cases}$

$$x=\frac{\begin{vmatrix} 16 & 3 & 4 \\ 1 & -8 & 2 \\ 5 & -1 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 5 & -8 & 2 \\ 3 & -1 & -2 \end{vmatrix}},\quad y=\frac{\begin{vmatrix} 2 & 16 & 4 \\ 5 & 1 & 2 \\ 3 & 5 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 5 & -8 & 2 \\ 3 & -1 & -2 \end{vmatrix}},\quad z=\frac{\begin{vmatrix} 2 & 3 & 16 \\ 5 & -8 & 1 \\ 3 & -1 & 5 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 4 \\ 5 & -8 & 2 \\ 3 & -1 & -2 \end{vmatrix}}$$

$$\therefore\ (x, y, z)=(3, 2, 1)$$

2. $\begin{cases} 5x+2y-4z=-3 \\ 4x+5y+2z=20 \\ 3x-3y+5z=12 \end{cases}$

4. $\begin{cases} a+3b+9c=23 \\ a+2b+4c=15 \\ a+b+c=9 \end{cases}$

3. $\begin{cases} 3x-y+2z=9 \\ x-2y+3z=2 \\ 2x-3y+z=1 \end{cases}$

5. $\begin{cases} a+b+c=2 \\ a+3b=4 \\ b-2c=6 \end{cases}$

69. In the following problems and exercises the equations may be solved by any method.

PROBLEMS AND EXERCISES

Solve the following:

1. $$\begin{cases} \dfrac{7x+8}{5}-\dfrac{7y-1}{4}=-2 \\ \dfrac{2x-4}{2}+\dfrac{y-1}{3}=-\dfrac{1}{3} \end{cases}$$

2. $$\begin{cases} 2y-\dfrac{4x-2y}{23-y}=2y-19 \\ 3x+\dfrac{3x-9}{y-18}=3x-17 \end{cases}$$

3. $$\begin{cases} \dfrac{2+x}{3}+\dfrac{2-y}{2}=\dfrac{3(y-4x)}{4} \\ \dfrac{x-3}{2}-5=\dfrac{y+5}{3}-3(y-x) \end{cases}$$

4. $$\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}=12 \\ \dfrac{2}{x}-\dfrac{3}{y}=14 \end{cases}$$

Do not clear of fractions.

Regard $\frac{1}{x}$ and $\frac{1}{y}$ as the unknowns.

5. $$\begin{cases} \dfrac{1}{x-y}+\dfrac{1}{x+y}=15 \\ \dfrac{4}{x-y}-\dfrac{5}{x+y}=17 \end{cases}$$

6. $$\begin{cases} \dfrac{2}{x}+\dfrac{3}{y}=4 \\ \dfrac{1}{x}+\dfrac{7}{y}=6 \end{cases}$$

7. $$\begin{cases} 2x+3y+5=0 \\ 6y+5z=7 \\ 3x+10z=1 \end{cases}$$

8. $$\begin{cases} x+2y+z=-17 \\ 2x+y-z=-1 \\ 3x-y+2z=2 \end{cases}$$

9. $$\begin{cases} \dfrac{3}{4x-y}-\dfrac{5}{2x-y}=2 \\ \dfrac{3}{y-2x}+\dfrac{4}{y-4x}=\dfrac{23}{5} \end{cases}$$

10. A mixture of alcohol and water contains 10 gallons. A certain amount of water is added, and the alcohol is then 30 per cent of the total. Had double the amount of water been added the alcohol would then have been 20 per cent of the whole. How much water was actually added and how much alcohol was there? (Board.)*

11. The value of 146 francs is as great as that of 117 shillings. A dollar and 4 francs together are worth 32 cents more than 6

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* (Board) means: taken from an entrance examination given by the College Entrance Examination Board.

shillings. Find the value in cents of a franc and a shilling. (Board.)

12. A photographer has two bottles of diluted developer. In one bottle 10 per cent of the contents is developer and the rest water; in the other the mixture is half and half. How much must he draw from each bottle to make 8 oz. of a mixture in which 25 per cent is developer? (Board.)

13. A principal of \$2,500 put at simple interest and for a certain time amounts to \$2,800. If the rate of interest had been 1 per cent higher and the time two years longer, the amount would have been \$3,200. Required the time and rate. (Board.)

14. A certain number of bolts can be bought for a dollar. If 10 more could be bought for a dollar the price would be half a cent less per dozen. What is the price per dozen? (Board.)

15. A man travels 50 mi. in an automobile in $3\frac{1}{4}$ hours. If he runs at the rate of 20 mi. an hour in the country and at the rate of 8 mi. an hour when within city limits, how many miles of his journey is in the country? (Yale.)

16. A company contracted to make 252 automobiles. Two factories, working together, can make this number in 12 days. Working alone, one factory requires 7 days longer than the other to do this amount. Find the time in which each factory alone can fulfil the contract. (Sheffield.)

17. A and B together can do a piece of work in 12 days. After A has worked alone for 5 days, B finishes the work in 26 days. In what time can each alone do the work? (Sheffield.)

18. Two yachts race over a 48-mile course. Owing to difference in measurement, B is given a start of half a mile in the first trial and is beaten by 6 minutes. In the second trial, the rate of the wind being the same as before, B's start is increased to a mile and a half, and still A wins by 2 minutes. Find the rate in feet per minute of each boat. (Chicago.)

19. A sum of \$1,050 is divided into two parts and invested; the simple interest on the one part at 4 per cent for 6 yr. is the same as the simple interest on the other at 5 per cent for 12 yr.; find how the money is divided. (Princeton.)

20. A man has two sons, one six years older than the other. After two years the father's age will be twice the combined ages of his sons, and six years ago his age was four times their combined ages. How old is each? (Princeton.)

21. In buying coal A gets 1 ton more for $18 than B does; he pays $9 less for 6 tons than B pays. Find the price per ton that each pays. (Princeton.)

22. In paying two bills aggregating $175, a merchant availed himself of discount for cash, 10 per cent on one and 5 per cent on the other, and then paid them both with $166. What was the amount of each bill? (Chicago.)

23. Two locomotives, A and B, are on tracks which cross each other at right angles. When B is at the point of crossing, A has 675 ft. yet to go before reaching this point. In 5 sec. the two locomotives are at an equal distance from the crossing, and in 40 sec. more they are again at an equal distance from it. What is the rate of each in feet per second? Illustrate by a diagram.

24. A dealer has two kinds of coffee, worth 30 and 40 cents per pound respectively. How many pounds of each must be taken to make a mixture of 70 lb. worth 36 cents per pound? (Yale.)

25. A man bought a certain number of eggs. If he had bought 88 more for the same money they would have cost him less by a cent apiece; if he had bought 56 fewer they would have cost more by a cent apiece. How many eggs did he buy and at what price each? (Yale.)

26. An investment at simple interest for 6 yr. amounts to $4,960. If the rate had been 1 per cent greater the amount would have been $5,000 in 5 years. Find the rate and the sum invested. (Chicago.)

27. The sum of the three digits of a number is 16. The sum of the first and third digits is equal to the second; and if the digits in the units and in the tens places be interchanged the resulting number will be 27 less than the original number. What is the original number?

28. A chauffeur engages to accomplish a journey of 105 mi. in a specified time. After traveling 63 mi. uniformly at a rate which will just enable him to keep his agreement, his car is delayed 24 minutes. He then drives $3\frac{1}{2}$ mi. faster per hour than before and arrives exactly on time. What was his original rate? (Board.)

Areas

70. Area of a triangle having one vertex at the origin. In Fig. 40

$$\triangle OP_1P_2 = \triangle OMP_2 + \text{trapezoid } MNP_1P_2 - \triangle ONP_1$$

$$\text{Hence } \triangle OP_1P_2 = \frac{x_2y_2}{2} + \frac{(x_1-x_2)(y_1+y_2)}{2} - \frac{x_1y_1}{2}$$

$$= \tfrac{1}{2}(x_2y_2 + x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2 - x_1y_1)$$

$$= \tfrac{1}{2}(x_1y_2 - x_2y_1)$$

$$\therefore \quad \triangle OP_1P_2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

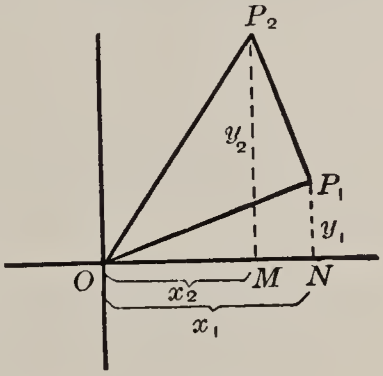

FIG. 40

The sign of this determinant will be + or − according as the origin is to the left or to the right of the directed segment P_1P_2.

EXERCISES

Find the area of each of the following triangles having the vertices as follows:

1. (0, 0); (−1, −3); (5, 3)

$$A = \frac{1}{2}\begin{vmatrix} -1 & -3 \\ 5 & 3 \end{vmatrix} = \frac{-3+15}{2} = 6$$

2. (0, 0); (2, 5); (4, 3)

71. Area of any triangle. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, Fig. 41, be the vertices of a triangle.

Draw OP_1, OP_2, and OP_3.

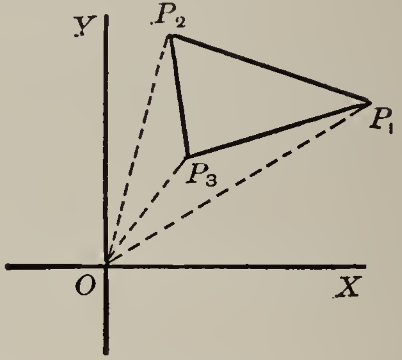

FIG. 41

Then $\triangle\ OP_1P_2 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

$\triangle\ OP_1P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}$

$\triangle\ OP_3P_2 = \frac{1}{2}\begin{vmatrix} x_3 & y_3 \\ x_2 & y_2 \end{vmatrix}$

$$\therefore \quad \triangle\ P_1P_2P_3 = \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} - \begin{vmatrix} x_3 & y_3 \\ x_2 & y_2 \end{vmatrix}\right]$$

$$= \frac{1}{2}\left[\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}\right]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

$\therefore$ **Area of** $\triangle\ P_1P_2P_3 = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

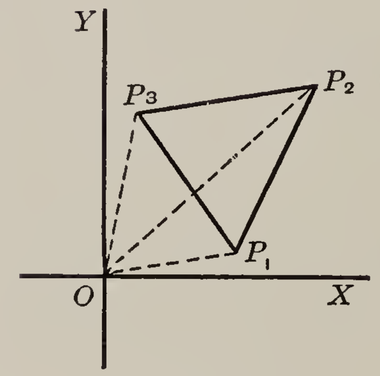

FIG. 42

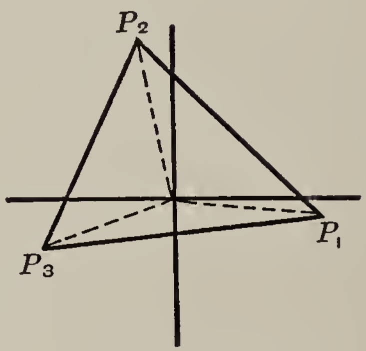

FIG. 43

EXERCISE

Prove the formula given in § 71 for the triangles in Figs. 42 and 43.

Summary

72. The chapter has taught the meaning of the following terms:

- incompatible equations
- equivalent equations
- pencil of lines, parameter
- homogeneous equations

73. The following conditions are satisfied by the equations $A_1x+B_1y+C_1=0$ and $A_2x+B_2y+C_2=0$.

If they are equations of *parallel* lines, they are *incompatible* and

$$m_1=m_2$$

and
$$\frac{A_1}{A_2}=\frac{B_1}{B_2}=k, \text{ or } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}=0$$

If they are equations of *perpendicular* lines, then

$$m_1=-\frac{1}{m_2}$$

and
$$A_1A_2=-B_1B_2$$

If they are equations of *identical* lines, they are *dependent* and

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}=\begin{vmatrix} C_1 & B_1 \\ C_2 & B_2 \end{vmatrix}=\begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}=0, \text{ or } \frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}=k$$

74. The following formulas have been proved:

1. The tangent of the angle between two lines:

$$\tan \beta=\frac{m_1-m_2}{1+m_1m_2}$$

2. A pencil of lines:

$$A_1x+B_1y+C_1+k(A_2x+B_2y+C_2)=0.$$

3. A pencil of parallel lines:

$$Ax+By+k=0$$

4. The equation of a pair of lines:

$$(A_1x+B_1y+C_1)(A_2x+B_2y+C_2)=0$$

5. The area of a triangle:

$$A=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

CHAPTER IV

DETERMINANTS

Meaning of Determinants

75. Determinants. The symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

denotes the determinant $a_1b_2-b_1a_2$, and is called a *determinant of the second order.*

The symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is a *determinant of the third order.* It is *defined* to mean the following:

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2c_3-c_2b_3)-b_1(a_2c_3-c_2a_3)+c_1(a_2b_3-b_2a_3)$$

$$= a_1b_2c_3+b_1c_2a_3+c_1a_2b_3-a_1c_2b_3-b_1a_2c_3-c_1b_2a_3$$

Show that the same expansion of a determinant of the third order may be obtained by the method given in § 67.

76. Element. Row. Column. Diagonal. The numbers a_1, a_2, a_3, b_1, b_2, b_3, etc., are the **elements** of the determinant. The horizontal lines are the **rows,** the vertical lines the **columns.** The *diagonal* $a_1b_2c_3$ is the **principal diagonal.**

77. Minor. If the row and column in which any particular element a_k stands be deleted, the determinant formed by the remaining elements is the **minor** of a_k.

Show that in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

the minor of b_2 is the determinant

$$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$$

What are the minors of c_1, a_3, c_3 in the given determinant?

The minor of an element may be denoted briefly by a capital letter having the same subscript as the element. Accordingly we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1$$

The signs prefixed to the terms in this expansion of the determinant are alternately + and −. Thus the signs of the minors of a_1, a_2, and a_3 are +, −, and +, respectively.

In general, *the sign of the minor of any element is + or − according as the sum of the number of the row and the number of the column of that element is an even or an odd number.*

78. Expansion by minors. In § 75 the determinant was expanded by minors of the *first row.* The following shows that the determinant may be expanded by elements of *any* row or column without changing its value.

For

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -a_2A_2 + b_2B_2 - c_2C_2$$

$$= -a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$$

$$= -a_2b_1c_3 + a_2c_1b_3 + b_2a_1c_3 - b_2c_1a_3 - c_2a_1b_3 + c_2b_1a_3$$

which is identically the same as the expansion given in § 75.

Expand by minors the following determinant and show that the value is zero:

$$\begin{vmatrix} 0 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - 0 \cdot \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + 0 \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0$$

This illustrates the fact that *the value of a determinant is zero when all the elements of any row, or column, are equal to zero.*

EXERCISES

Expand the determinant

$$\begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 6 \\ 0 & 2 & 0 \end{vmatrix}$$

as indicated below, and in each case find the value:

1. By minors of the first row.

2. By minors of the third column.

3. By minors of the third row.

79. Determinants of order higher than the third. An examination of the expansion of a determinant of the *third* order, § 75, shows that it is the sum of all possible products of three elements, taking as factors one, and only one, element from each row and column. The number of terms in the expansion is equal to 6, or $3 \cdot 2 \cdot 1 = 3!$ (read *factorial* 3).

Similarly, we may define a determinant of the *fourth* order as an array of $4^2=16$ numbers in four rows and four columns, as

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

and as equal to

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}$$

Since each of these terms when expanded contains $3!$ terms, the total number of terms in the expansion of a determinant of the fourth order is $4 \cdot 3! = 4 \cdot 3 \cdot 2 \cdot 1 = 4!$. Notice that each of the terms is the product of 4 elements, one, and only one, being taken from each row and column.

It is seen that a determinant of the fourth order may be expressed in terms of determinants of the third order, which again may be expressed in terms of determinants of the second order.

In the same manner it is possible to expand determinants of the fifth, the sixth, ... , or any order.

The value of a determinant may be found by expanding it by minors. However, the principles given in §§ 81 to 84 will simplify greatly the process of evaluating a determinant.

Properties of Determinants

80. The following principles are to be verified for determinants of the second and of the third order only. They hold, however, for determinants of orders higher than the third.

81. Interchange of rows and columns. If the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is revolved through an angle of 180° about the principal diagonal $a_1b_2c_3$ as an axis, we obtain the determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Notice that the first, second, and third *rows* of this determinant are respectively the first, second, and third *columns* of the given determinant.

Expand the last determinant and show that the result is the same as that obtained in § 75.

This illustrates the principle that *the value of a determinant remains the same when the rows are changed to columns and the columns to rows, the relative order of the rows and columns being preserved.*

82. Interchange of two rows, or of two columns. The determinant

$$\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is obtained from the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by interchanging the first two rows. By expanding each determinant and then comparing, the results show that the two determinants are equal numerically but differ in sign. This shows that *if two adjacent rows, or two adjacent columns, of a determinant are interchanged, the sign of the determinant is changed, but the absolute value remains the same.*

Let a determinant have two rows, or two columns, identically the same, as

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

and let the algebraic value of this determinant be denoted by D. Since the value of the determinant obtained by interchanging the two identical rows is $-D$, it follows that

$$D=-D$$

or $$2D=0$$

Hence $$D=0$$

Thus *the value of a determinant is zero, if two rows, or two columns, are identical.*

83. Multiplying a determinant by a number. Let each element of a row, or column, of a given determinant be multiplied by m, as

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Expanding this determinant by minors, we have

$$ma_1A_1-mb_1B_1+mc_1C_1$$
$$=m(a_1A_1-b_1B_1+c_1C_1)$$

This is the same as the product of the given determinant and m.

Hence, *if all the elements of a row, or column, of a determinant are multiplied by a number, the determinant is multiplied by that number.*

Using this fact, show that *a determinant vanishes if the elements of a row, or of a column, are multiples of the corresponding elements of another row or column.*

84. A determinant expressed as the sum of two determinants. A determinant may be expressed as the sum of two determinants if each element of a row, or column, is expressed as the sum of two terms, i.e.,

$$\begin{vmatrix} a_1+k_1 & b_1 & c_1 \\ a_2+k_2 & b_2 & c_2 \\ a_3+k_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$$

This equation is easily verified by expanding each determinant by minors of the elements of the first columns.

Similarly, show that

$$\begin{vmatrix} a_1+mb_1 & b_1 & c_1 \\ a_2+mb_2 & b_2 & c_2 \\ a_3+mb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + m \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

This illustrates the following principle:

The value of a determinant is not changed if each element of a row, or column, is multiplied by a number and then added to the corresponding element of any other row, or column.

85. Evaluation of determinants. The principles stated in §§ 81 to 84 may now be used to advantage in finding the value of a determinant.

Thus the value of the determinant

$$\begin{vmatrix} 13 & 3 & 2 & 16 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 1 & 15 & 14 & 4 \end{vmatrix}$$

may be found as follows.

Subtract column 3 from column 2:

$$\begin{vmatrix} 13 & 1 & 2 & 16 \\ 12 & -1 & 7 & 9 \\ 8 & -1 & 11 & 5 \\ 1 & 1 & 14 & 4 \end{vmatrix}$$

Subtract row 4 from row 1, and add it to row 2 and then to row 3:

$$\begin{vmatrix} 12 & 0 & -12 & 12 \\ 13 & 0 & 21 & 13 \\ 9 & 0 & 25 & 9 \\ 1 & 1 & 14 & 4 \end{vmatrix}$$

Take out the factor 12 and then expand by minors of the elements of column 2:

$$12\begin{vmatrix} 1 & 0 & -1 & 1 \\ 13 & 0 & 21 & 13 \\ 9 & 0 & 25 & 9 \\ 1 & 1 & 14 & 4 \end{vmatrix} = 12\begin{vmatrix} 1 & -1 & 1 \\ 13 & 21 & 13 \\ 9 & 25 & 9 \end{vmatrix}$$

Since the first and last columns are identical, the determinant vanishes.

EXERCISES

Evaluate the following determinants:

1. $\begin{vmatrix} -4 & -5 & 3 \\ 16 & 1 & 7 \\ 4 & 4 & -2 \end{vmatrix}$

Taking out +4 as a factor we have

$$+4\begin{vmatrix} -1 & -5 & 3 \\ 4 & 1 & 7 \\ +1 & 4 & -2 \end{vmatrix}$$

Adding 4 times row 1 to row 2, and then row 1 to row 3, we have

$$4\begin{vmatrix} -1 & -5 & 3 \\ 0 & -19 & 19 \\ 0 & -1 & 1 \end{vmatrix} = -4\begin{vmatrix} -19 & 19 \\ -1 & 1 \end{vmatrix} = (-4)(0) = 0$$

2. $$\begin{vmatrix} 2 & 2 & 4 & 2 \\ 2 & 5 & 3 & 6 \\ -3 & -7 & -11 & -9 \\ 0 & 6 & 9 & 9 \end{vmatrix}$$

3. $$\begin{vmatrix} 6 & 4 & 1 & 5 \\ -2 & 1 & -5 & 7 \\ 2 & -3 & 2 & 2 \\ 3 & 2 & 1 & -9 \end{vmatrix}$$

4. $$\begin{vmatrix} 2 & 1 & 7 & 9 \\ 3 & 4 & 11 & 11 \\ 6 & 4 & 10 & 2 \\ 8 & -4 & 13 & 10 \end{vmatrix}$$

5. $$\begin{vmatrix} 2 & -2 & 4 & 6 \\ 3 & -3 & 6 & 10 \\ 2 & 1 & 1 & -2 \\ 3 & 0 & 3 & 9 \end{vmatrix}$$

6. $$\begin{vmatrix} 4 & 6 & 3 & 2 \\ 3 & 7 & 6 & 0 \\ -8 & 9 & 2 & 1 \\ 3 & 7 & 5 & 2 \end{vmatrix}$$

7. $$\begin{vmatrix} 2 & 6 & 3 & -2 \\ 2 & 1 & 3 & 7 \\ 0 & 4 & 0 & -3 \\ 14 & 4 & 6 & 5 \end{vmatrix}$$

8. $$\begin{vmatrix} 2 & 1 & 3 & 7 \\ 14 & 4 & 6 & 5 \\ 2 & 5 & 3 & 4 \\ 4 & 7 & 6 & 5 \end{vmatrix}$$

9. $$\begin{vmatrix} 3 & 5 & -5 & 6 \\ 2 & 2 & -3 & 4 \\ 3 & 4 & -1 & -2 \\ 4 & 3 & -2 & 1 \end{vmatrix}$$

10. $$\begin{vmatrix} 3 & 1 & 0 & 5 \\ -1 & 0 & 2 & 3 \\ 4 & 1 & 4 & 4 \\ 0 & -1 & -2 & -1 \end{vmatrix}$$

11. $$\begin{vmatrix} 5 & 0 & 2 & 1 \\ 1 & -6 & -3 & 4 \\ 3 & 8 & 4 & -3 \\ 4 & 1 & 2 & 5 \end{vmatrix}$$

86. Factoring determinants. Exercise 1 below shows the methods of finding the factors of a determinant. Find the factors of the following determinants:

EXERCISES

1. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

First method: Let $a=b$. Then the first two rows become identical. Hence the determinant is equal to zero. Why? Therefore $a-b$ is a factor of the determinant. Why? Show similarly that $b-c$ and $c-a$ are factors of the determinant.

Show that the product of these three factors is of the same degree as the expanded determinant.

Therefore this product and the determinant can differ only by a constant factor k.

To determine k, compare the term in the principal diagonal, bc^2, with the corresponding term, kbc^2, in the product $k(a-b)(b-c)(c-a)$.

It follows that $k=1$

$$\therefore \quad \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \equiv (a-b)(b-c)(c-a)$$

Second method: Subtract row 3 from row 1 and then row 3 from row 2. Take out the factors $a-c$ and $b-c$ and expand the resulting determinant. This gives the following:

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 0 & a-c & (a-c)(a+c) \\ 0 & b-c & (b-c)(b+c) \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-c)(b-c) \begin{vmatrix} 0 & 1 & a+c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix} = (a-c)(b-c)(b+c-a-c)$$

$$= (a-c)(b-c)(b-a) = (a-b)(b-c)(c-a)$$

2. Show that

$$\begin{vmatrix} a & b & c \\ b+c & a+c & a+b \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(a-c)(a+b+c)$$

3. $$\begin{vmatrix} 1 & 1 & 1 \\ a & a^2 & a^3 \\ b & b^2 & b^3 \end{vmatrix}$$

4. $$\begin{vmatrix} x & a & a & a \\ p & x & b & b \\ q & r & x & c \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

87. Solution of systems of linear equations by means of determinants. Linear equations in three or more variables may be solved by determinants, as shown in exercise 1, below.

EXERCISES

Solve the following systems of equations:

1. $$\begin{cases} 3a+b+d=20 \\ 3c+6a+d=40 \\ 3c+a+4b=30 \\ 5c+8b+3d=50 \end{cases}$$

$$a=\frac{\begin{vmatrix} 20 & 1 & 0 & 1 \\ 40 & 0 & 3 & 1 \\ 30 & 4 & 3 & 0 \\ 50 & 8 & 5 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & 1 & 0 & 1 \\ 6 & 0 & 3 & 1 \\ 1 & 4 & 3 & 0 \\ 0 & 8 & 5 & 3 \end{vmatrix}}=5$$

Similarly find b, c, and d.

2. $$\begin{cases} 7a+14b+2c=7 \\ 2a-b+c=1 \\ a-7b-8c=1 \end{cases}$$

3. $$\begin{cases} 8x-3y-7z=85 \\ x+6y-4z=12 \\ 2x-5y+z=33 \end{cases}$$

4. $$\begin{cases} x+y+z-14=0 \\ 6x-3y-7z=0 \\ 4x-9y+7z=0 \end{cases}$$

5. $$\begin{cases} 2a-c-3=0 \\ b+4c-2=0 \\ a-3b-1=0 \end{cases}$$

6. $$\begin{cases} 2x+3y-4z=1 \\ 3x+4z-5y=2 \\ 4y+5z-6x=3 \end{cases}$$

7. $$\begin{cases} a+2b-4c-11=0 \\ 2a-3b=0 \\ b-4c=0 \end{cases}$$

8. $$\begin{cases} 3a+2c-b+2=0 \\ 2a+b-c+1=0 \\ a+2b+c+17=0 \end{cases}$$

9. $$\begin{cases} 3x+2y-z+u=2 \\ x+3y+z-u=5 \\ x+y=4 \\ z+u=6 \end{cases}$$

10. $$\begin{cases} 2x+4y-10=0 \\ 2y+3z-13=0 \\ 3z+4u-25=0 \\ 4u+5x-21=0 \end{cases}$$

11. $$\begin{cases} 2p-4q+3r+4s=-3 \\ 3p-2q+6r+5s=-1 \\ 5p+8q+9r+3s=9 \\ p-10q-3r-7s=2 \end{cases}$$

12. Determine c so that the following system of equations may have solutions other than $(x, y, z)=(0, 0, 0)$:

$$\begin{cases} x+\ y+\ z=cx \\ 2x-\ y+2z=\frac{cy}{2} \\ 2x+5y+2z=cz \end{cases}$$

See § 65.

13. Solve for x:

$$\begin{vmatrix} a & a & x \\ b & x & b \\ c & c & c \end{vmatrix}=0$$

14. Solve for x:

$$\begin{vmatrix} 1-x & 2 & 3 \\ 2 & 3-x & 5 \\ 3 & 5 & 8-x \end{vmatrix}=0$$

Summary

88. The chapter has taught the meaning of the following terms:

determinant
element
row, column
diagonal of a determinant
minor
order of a determinant

89. The following principles have been studied in this chapter:

1. *The sign of the minor of an element is* + *or* − *according as the sum of the number of the row and the number of the column of that element is an even or an odd number.*

2. *The value of a determinant remains the same when the rows are changed to columns and the columns to rows, the relative order of the rows and columns being preserved.*

3. *If all the elements of a row or column are equal to zero, the value of the determinant is zero.*

4. *If two adjacent rows, or columns, of a determinant are interchanged, the sign of the determinant is changed, but the absolute value remains the same.*

5. *The value of a determinant is zero if two rows or two columns are identically the same.*

6. *If all the elements of a row or column are multiplied by a number, the determinant is multiplied by that number.*

7. *A determinant vanishes if the elements of a row or column are multiples of the corresponding elements of another row or column.*

8. *The value of a determinant is not changed if each element of a row or column is multiplied by a number and then added to the corresponding element of any other row or column.*

90. The principles in § 89 have been used to expand, to evaluate, and to factor determinants.

Systems of linear equations may be solved by means of determinants.

CHAPTER V

QUADRATIC FUNCTION. PARABOLA

91. Quadratic function. The function $y=mx+b$, § 31, is said to be of the *first* degree in x, because the variable x occurs in it only in the first degree. This function is also called a *linear function* of x, because its graph is a straight line, § 41.

The function $y=ax^2+bx+c$, in which the highest exponent of the variable x is of the *second* degree, is a function of the *second* degree. It is also called a **quadratic function** of x.

The coefficients a, b, and c may have any real value, excluding the value $a=0$.

92. Graph of the quadratic function $y=ax^2$. This function is obtained from ax^2+bx+c by letting $b=c=0$.

If we put $a=1$, and assign to x arbitrary values, we can find the values of y given in the table, Fig. 44.

If we *plot* the points whose co-ordinates are the corresponding values of x and y given in the table and join them in succession by a smooth curve, we have the graph of the function $y=x^2$, Fig. 44. This curve is a **parabola.**

x	y
-5	25
-4	16
-3	9
-2	4
-1	1
$-\frac{1}{2}$	$\frac{1}{4}$
0	0
$+\frac{1}{2}$	$\frac{1}{4}$
$+1$	1
$+2$	4
$+3$	9
$+4$	16
$+5$	25

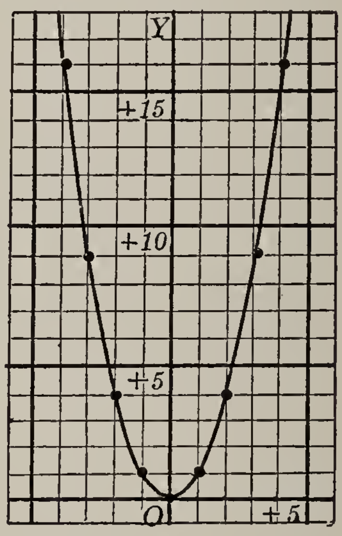

FIG. 44

EXERCISES

Plot the following parabolas with reference to the same axes:
$y=3x^2$; $y=-2x^2$; $x=2y^2$; $x=-3y^2$; $y^2=-2x$; $y^2=3x$

93. Discussion of the equation $y=ax^2$. The equations $y=3x^2$ and $y=-2x^2$ in the exercises above are obtained from the equation $y=ax^2$ by taking for a particular value $a=3$ and -2.

The graphs of these equations show that for $a>0$, and this is true in general, the parabola extends upward, and that for $a<0$ the parabola extends downward, from the origin. However, for $a=0$, $y=ax^2$ reduces to $y=0$, the equation of the x-axis.

94. Axis of symmetry of the parabola $y=ax^2$. A straight line is an **axis of symmetry** of a curve if it bisects all chords perpendicular to it. Show that for two values of x, numerically equal but opposite in sign, Fig. 45, the values of y obtained from the equation $y=ax^2$ are the same.

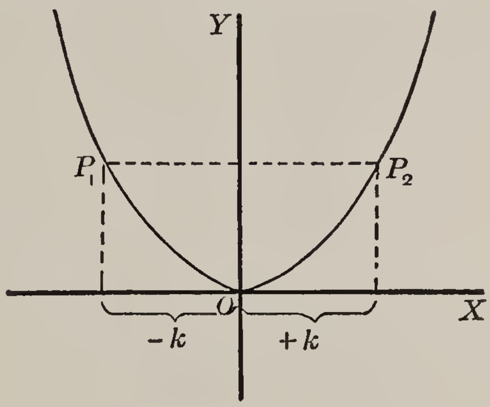

FIG. 45

Hence, if two points P_1 and P_2 on the chord of the curve have equal ordinates, the abscissas are numerically equal but opposite in sign.

Therefore the y-axis is an *axis of symmetry* of the parabola $y=ax^2$.

The point of intersection O of the axis OY of the parabola with the curve is the **vertex.**

If $x=0$, we have also $y=0$. Hence the vertex of the curve is at the origin.

As x increases, y also increases. Hence this curve is *not closed.*

95. Construction of the parabola. When a point moves so as to be always equally distant from a fixed point and a fixed straight line its locus is found to be a **parabola.**

Hence a parabola may be constructed as follows:

Let point F, Fig. 46, be a fixed point, and let AB be a fixed line.

With F as center and a radius equal to CA, draw an arc meeting CC_1 at C_1. This makes $FC_1=AC$.

Similarly construct F_1 making $FF_1=FA$; D_1 making $FD_1=DA$, etc.

Draw a smooth curve through O, C_1, F_1, D_1, etc.

This is the required locus.

The fixed point F is the **focus,** the fixed line AB the **directrix,** the point O the **vertex** of the parabola.

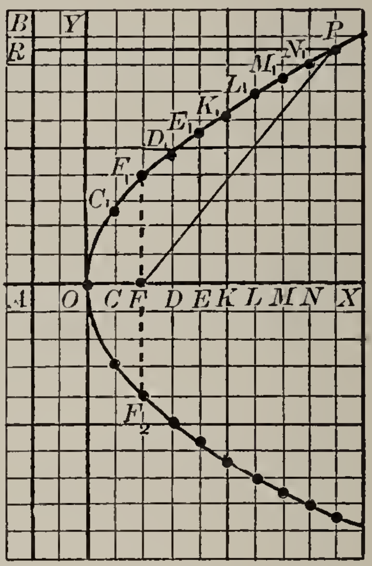

FIG. 46

An arc of a parabola may be constructed *mechanically* as follows:

Fasten one end-point of a string of fixed length, $AP+PF$, to the point F, Fig. 47, the other to the vertex A of the right triangle ABC.

Let the triangle move so that the side BC slides along the line OY.

As the triangle moves, the string is kept taut by means of a pencil point at P.

The arc VP described by P is the parabolic arc.

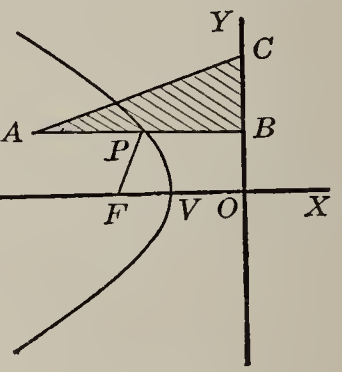

FIG. 47

96. The standard equation of the parabola. Take the vertex of the parabola as the origin.

Let AF, Fig. 46, be the x-axis, and $OY \perp AF$ the y-axis.

Let $P(x, y)$ be *any* point on the curve, and denote OF and OA by p.

Draw FP, and $PR \perp OY$.

Since $FP = RP$, and since the co-ordinates of F are $(p, 0)$, it follows that

$$\sqrt{(x-p)^2+y^2}=\sqrt{(x+p)^2}, \text{ § 47}$$

$$\therefore \quad x^2-2px+p^2+y^2=x^2+2xp+p^2$$

$$\therefore \quad \boldsymbol{y^2=4px}$$

This is a **standard form** of the equation of the parabola. The number p is the **parameter** of the parabola.

The position of the curve depends upon the value of p. If p is *positive* the curve extends to the *right* from the vertex O; if p is *negative* the curve extends to the *left*.

F_2F_1 is the **latus rectum** of the parabola.

97. The polar equation of the parabola. Let the focus F be taken as pole and the x-axis as initial line, Fig. 48.

Let ρ and θ be the polar co-ordinates of P, any point on the curve. Then the *focal radius*

$$\rho = FP = OA + p = x + p$$

Since

$$x = OA = FA + p = \rho \cos \theta + p,$$

we have

$$\rho = \rho \cos \theta + 2p.$$

Solving for ρ,

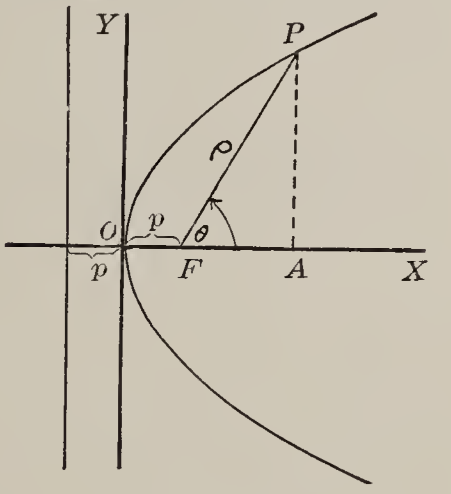

FIG. 48

$$\boldsymbol{\rho = \frac{2p}{1-\cos\theta}}$$

EXERCISES

1. Using the polar equation of the parabola, graph the curve for which $p=3$.

2. Discuss the changes of ρ as θ changes from 0 to 2π.

3. Determine p so that the curve $y^2=4px$ passes through the point (3, 5).

98. Graph of the function $y=ax^2+bx+c$. The equation $y=ax^2+bx+c$ may be re-written as follows:

$$\begin{aligned} y &= a\left(x^2+\frac{b}{a}x\right)+c \\ &= a\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)+c-\frac{b^2}{4a} \\ &= a\left(x+\frac{b}{2a}\right)^2+c-\frac{b^2}{4a} \end{aligned}$$

or
$$y-\left(c-\frac{b^2}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2$$

Denote $c-\dfrac{b^2}{4a}$ by k and $-\dfrac{b}{2a}$ by h

This gives the equation

$$\boldsymbol{y-k=a(x-h)^2}$$

The graph of the function $y=ax^2+bx+c$ differs from that of the function $y=ax^2$ only in position. For, if we

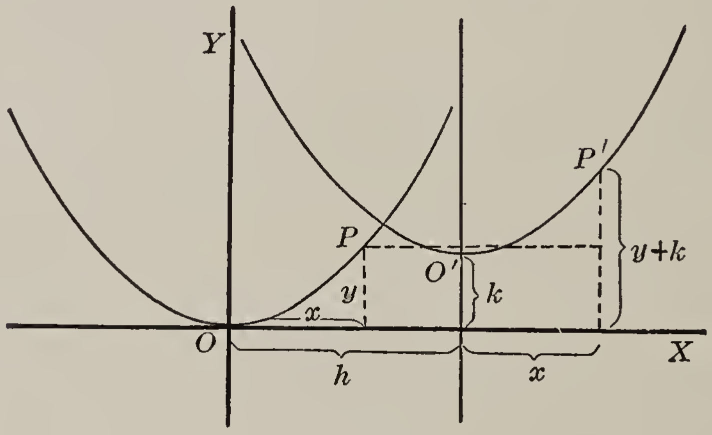

FIG. 49

place $x=x'-h$ and $y=y'-k$, Fig. 49, the equation $y=ax^2$ reduces to $y'-k=a(x'-h)^2$. This is the equation of a parabola having the same shape as $y=ax^2$, but having the vertex at the point $O'(h, k)$.

If $x=-\frac{b}{2a}+m$, or if $x=-\frac{b}{2a}-m$, the corresponding values of y, found from the equation

$$y-\left(c-\frac{b^2}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2$$

are the same. Therefore the straight line $x=-\frac{b}{2a}$ is the *axis* of the parabola.

99. Summary. The following summary will be helpful in graphing the curve $y=ax^2+bx+c$:

1. The co-ordinates of the vertex are $\left(-\frac{b}{2a}, c-\frac{b^2}{4a}\right)$.

2. The line $x=-\frac{b}{2a}$ is the axis.

3. The curve extends upward or downward according as $a>0$, or as $a<0$.

A *rough sketch* of the curve can be made by locating the vertex, as in step 1, and one other point on the curve, e.g., the point of intersection with the y-axis.

EXERCISES

Make rough sketches of the following parabolas and in each case determine the vertex and axis:

1. $y=\frac{1}{3}x^2-2x+1$

2. $y=3-x-2x^2$

3. $y=x^2-4$

4. $y=-2-3x^2$

100. Maximum and minimum of the function $y=ax^2+bx+c$. As the value of x changes from -3 to

+7, Fig. 50, the function y first decreases and then increases. At the vertex it is said to have a *turning-point.*

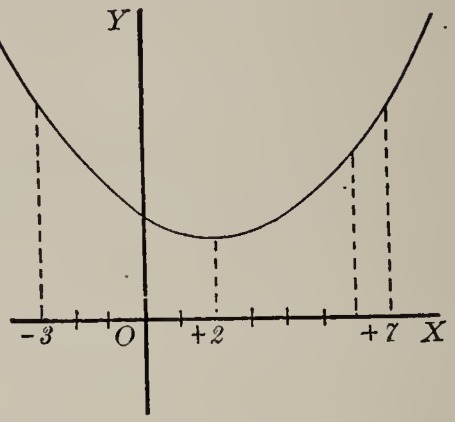

FIG. 50

Evidently, according as $a>0$, or as $a<0$, the *ordinate of the vertex* represents the *least* or *greatest* value of the function within an interval containing the turning-point. In the first case the function is said to have a **minimum** value $y=c-\frac{b^2}{4a}$ for $x=-\frac{b}{2a}$. In the last case the function has a **maximum** value.

Another method of determining the maximum or minimum of a quadratic function will be given in § 123.

EXERCISES

Determine the maximum or minimum value of each of the following functions:

1. $y=2x^2-4x+3$

2. $y=3-4x-5x^2$

3. $y=(x-1)^2+(2x-3)^2$

4. $y=-5x^2-7x+10$

5. Find the largest rectangle whose perimeter is 60.

Show that $A=30x-x^2$, and find the value of x for which A is a maximum.

6. Of all rectangles having the perimeter equal to $2a$ determine the one for which the square on the diagonal has the smallest area.

7. Show that of all the rectangles with a perimeter equal to $2s$ the one having the greatest area is a square.

101. Slope of a curve. The slope of a *straight line,* § 28, was found by means of the equation

$$m=\frac{y_2-y_1}{x_2-x_1}$$

It is constant at all points of the straight line. By the **slope of a curve** at one of its points we mean the slope of the *tangent* drawn to the curve at that point.

102. Slope of secant. Let P and P_1, Fig. 51, be two distinct points on the same curve. The straight line PP_1 is a secant of the curve. Show that the slope of PP_1 is

$$m=\frac{y_1-y}{x_1-x}$$

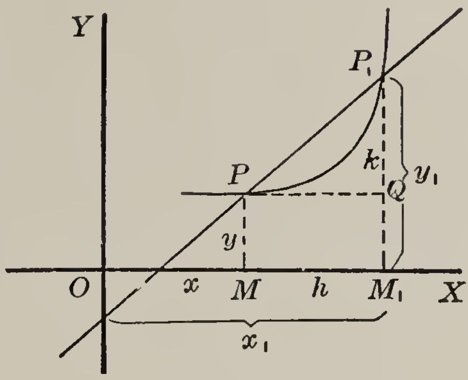

FIG. 51

Let $y=ax^2$ be the equation of the curve. Then, by substituting for y and y_1 the values ax^2 and ax_1^2, we have the slope of the secant PP_1 equal to

$$m=\frac{ax_1^2-ax^2}{x_1-x}=\frac{a(x_1^2-x^2)}{x_1-x}=a(x_1+x)$$

This result shows that the slope of the secant is a *function of* x, and therefore *not a constant.* In general, let $y=f(x)$ be the equation of the curve in Fig. 51.

Denoting MM_1 by h,

Then $OM_1=x_1$ and $M_1P_1=y_1=f(x+h)$

The slope of the secant PP_1 is $m=\dfrac{f(x+h)-f(x)}{x+h-x}$,

or
$$\boldsymbol{m=\frac{f(x+h)-f(x)}{h}}$$

Using this as a formula we have for the curve $f(x)=ax^2$

$$m=\frac{a(x+h)^2-ax^2}{h}=\frac{a[(x+h)^2-x^2]}{h}=a(2x+h)$$

Show from the figure that this is the same as the result obtained above, i.e., show that $a(x_1+x)=a(2x+h)$.

103. Tangent to a curve. Let P_1, Fig. 52, move along the curve, approaching the limiting position P. Let PT be the limiting position approached by the secant PP_1.

Then PT is called the **tangent** to the curve at the point P.

Compare this definition of a tangent to curve with the definition usually given in plane geometry.

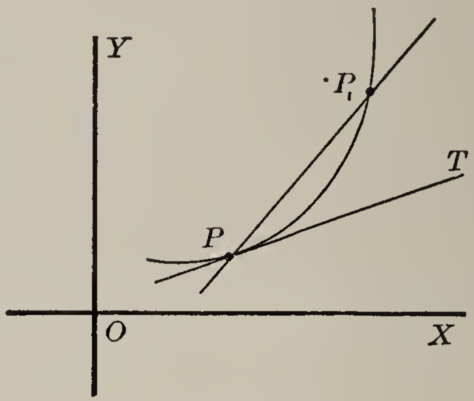

FIG. 52

104. Slope of tangent. The slope of the tangent at $\boldsymbol{P(x, y)}$ is *defined* as the *limit* which $\frac{f(x+h)-f(x)}{h}$, the slope of the secant, approaches, as h approaches zero.

For example, if the curve is the parabola $f(x)=ax^2$, the slope of the tangent at P is the limit of $\frac{a(x+h)^2-ax^2}{h}$, as h approaches zero.

Since

$$\frac{a(x+h)^2-ax^2}{h}=\frac{a(x^2+2hx+h^2-x^2)}{h}=a(2x+h)$$

it follows that the slope of the parabola at $P(x, y)$ is equal to $2ax$.

105. Rational integral function. The functions ax^2, ax^2+bx+c, ax^3+bx^2+cx+d are *rational integral functions* of x. In general, a function of the form

$$f(x)=a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\dots+a_n$$

in which a_0, a_1, a_2, $\dots$, a_n are constants and the exponents of x are positive integers, is a **rational integral function** of x of the nth degree.

EXERCISES

Which of the following functions are rational integral functions of x? Give the reason for your answer.

1. x^2
2. $3x^2+4x$
3. $5x^{-2}+2x^{-1}+7-x$
4. $x^{\frac{3}{2}}-2x^{\frac{1}{2}}+6$
5. x^4+5x^2+4
6. $x^3+2x^2-\frac{1}{x}+3$
7. $x+x^{\frac{1}{2}}-5$

106. Derivative. Let $f(x)$ be a rational integral function of x. Then the limit of the quotient

$$\frac{f(x+h)-f(x)}{h}$$

as h approaches zero is called the **derivative** of $f(x)$.

Denoting the derivative of $f(x)$ by $f'(x)$, this definition may be stated briefly thus:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

EXERCISES

1. Show that the derivative of ax^2 is $2ax$.
2. Find the derivative of ax^2+bx+c.

$$\begin{aligned} f(x+h) &= a(x+h)^2+b(x+h)+c \\ &= ax^2+2ahx+ah^2+bx+bh+c \\ &= ax^2+(2ah+b)x+ah^2+bh+c \end{aligned}$$

$$\therefore\ f(x+h)-f(x)=2ahx+ah^2+bh$$

Dividing by h, $\quad \dfrac{f(x+h)-f(x)}{h}=2ax+ah+b$

$$\therefore\ \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)=2ax+b$$

Find the derivative of each of the following functions:

3. x^2
4. x^2-4x+1
5. x^3-20x^2+2
6. x^3-2x^2+3x-7

107. Equation of a tangent. Let $P(x_1, y_1)$ be any point on the curve $y=f(x)$ and let the equation of a straight line passing through (x_1, y_1) be

$$y-y_1=m(x-x_1)$$

Since $f'(x_1)$ is the slope of the tangent to the curve $f(x)$ at the point whose abscissa is x_1, it follows that $m=f'(x_1)$. Hence the equation of the tangent to $f(x)$, at the point $P(x_1, y_1)$, is

$$\boldsymbol{y-y_1=f'(x_1)(x-x_1)}$$

EXERCISES

Find the equation of the tangent to each of the following curves:

1. $y=x^2$ at the point $(-2, 4)$ on it.

2. $y=x^3$ at the point whose abscissa is 2.

3. $y=x^2+4x$ at the point whose abscissa is 3.

108. The parabola as a path of a projectile.* Let a particle be projected with a velocity v from a point O, Fig. 53, in the direction OQ, which is inclined to the x-axis at an angle α.

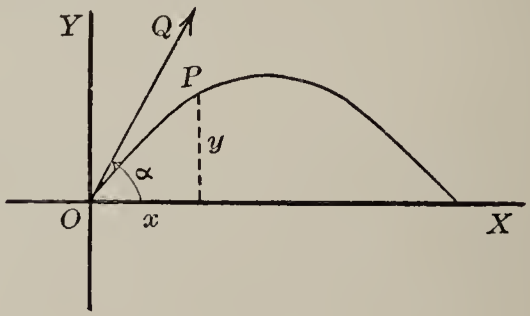

FIG. 53

Let $P(x, y)$ be the position of the particle at the time t.

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* Ancient, mediaeval, and modern mathematicians down to Tartaglia were of the opinion that the path of a projectile was a straight line, at least at the beginning and end of the flight. In 1537 Tartaglia proved the path to be curved. He also proved that a projectile reached its greatest height under an initial angle of 45°. It was reserved to the great Galileo (1564–1642 A.D.) to prove the path to be a *parabola* and to show how to calculate its height and range (Tropfke, II, 449).

It follows from a principle of mechanics that in t seconds, if the resistance of air is neglected, the particle moves horizontally a distance of $v(\cos\alpha)t$ ft. and vertically a distance of $v(\sin\alpha)t-\frac{1}{2}gt^2$ feet.

Hence

$$x=v(\cos\alpha)t$$

and

$$y=v(\sin\alpha)t-\tfrac{1}{2}gt^2$$

g being constant and equal to 32.2, approximately.

Eliminating t we have

$$y=x\tan\alpha-\frac{gx^2}{2v^2\cos^2\alpha}$$

which is the equation of the path described by the particle.

If the particle is projected horizontally, Fig. 54, angle α is zero. Therefore $\tan\alpha=0$, $\cos^2\alpha=1$, and the equation of the path of the particle reduces to

$$y=-\frac{gx^2}{2v^2}$$

from which

$$x^2=-\frac{2v^2}{g}y$$

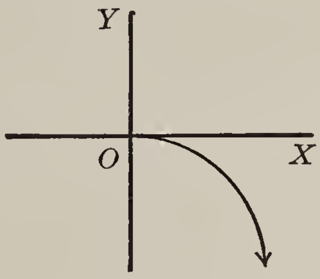

FIG. 54

Notice that this is the equation of a *parabola* whose axis is the axis of y.

EXERCISES

1. A ball is thrown horizontally with a velocity of 4 ft. per second. Find the latus rectum of the parabola described.

2. An airship traveling south at a height of 5,280 ft. and at a rate of 25 mi. per hour drops a bomb. Taking the starting-point as the origin, find the equation of the path described by the bomb. How far south of the starting-point will it strike the ground?

Summary

109. The chapter has taught the meaning of the following terms:

linear function
quadratic function
parabola
axis of symmetry
vertex of a parabola
focus, directrix
latus rectum
turning-point
maximum and minimum value of a function
slope of a curve
tangent to a curve
rational integral function
derivative

110. A parabola may be drawn by plotting the equation, by a geometric construction, and by a mechanical device. A rough sketch may be made by first locating the vertex and then one other point in the curve.

111. The following equations of the parabola were studied:

$$y^2=4px,\ y-k=a(x-h)^2$$

$$y=ax^2+bx+c$$

$$\rho=\frac{2p}{1-\cos\theta}$$

112. The **co-ordinates of the vertex** of the parabola $y=ax^2+bx+c$ are

$$(x, y)=\left(-\frac{b}{2a},\ c-\frac{b^2}{4a}\right)$$

The line $x=-\dfrac{b}{2a}$ is the **axis.**

113. The function $y=ax^2+bx+c$ has a *minimum* value $y=c-\dfrac{b^2}{4a}$ for $x=-\dfrac{b}{2a}$, if $a>0$, and a *maximum* when $a<0$.

114. The **slope of a secant** of the curve $y=f(x)$ is

$$\frac{f(x+h)-f(x)}{h}$$

The **slope of a tangent** at the point (x_1, y_1) is

$$m=\lim_{h\to 0}\frac{f(x_1+h)-f(x_1)}{h}=f'(x_1)$$

The function $f'(x)$ is the **derivative** of $f(x)$.

The **equation of the tangent** at the point (x_1, y_1) in the curve $y=f(x)$ is

$$y-y_1=f'(x_1)(x-x_1)$$

115. The path of a *projectile* is a *parabola.*

CHAPTER VI

RATIONAL INTEGRAL FUNCTIONS OF DEGREE HIGHER THAN THE SECOND

Rational Integral Function

116. Rational integral function. Linear and quadratic functions are *rational integral functions.* In general, a function of the form

$$ax^n+bx^{n-1}+cx^{n-2}+\ldots.+px+q \qquad (1)$$

is a **rational integral function of the *n*th degree,** if n is a positive integer, and if the coefficients a, b, c, $\ldots.$, q are constants.

For example, the expansion of $(x+h)^n$, for a positive integral exponent n, is a rational integral function. The expansion of $(x+h)^n$ is obtained by the binomial theorem. A proof of this theorem, whose truth was assumed in the preceding courses, is given in § 117.

117. Binomial theorem. The formula

$$(x+h)^n=x^n+nx^{n-1}h+\frac{n(n-1)}{1\cdot 2}x^{n-2}h^2+\ldots.+h^n,$$

n being a positive integer, is easily shown to hold for small values of n, by multiplying $x+h$ by itself.

It is now to be shown that it holds for *any positive* value of n.

Proof: Assume that the formula is true for $n=k$, i.e., assume that

$$(x+h)^k=x^k+kx^{k-1}h+\frac{k(k-1)}{1\cdot 2}x^{-k2}h^2+\ldots.$$
$$+\frac{k(k-1)\ldots.(k-r+2)}{1\cdot 2\cdot\ \ldots.\ \cdot(r-1)}x^{k-r+1}h^{r-1}+\ldots.+h^k$$

Multiplying both members of this equation by $x+h$,

$$\begin{aligned}(x+h)^{k+1} & \\ =x^{k+1}+kx^k h+\frac{k(k-1)}{1\cdot 2}x^{k-1}h^2+\ldots\ldots+\frac{k(k-1)\ldots\ldots(k-r+2)}{1\cdot 2\cdot\ldots\ldots\cdot(r-1)}x^{k-r+2}h^{r-1}+\ldots\ldots \\ +x^k h+\quad kx^{k-1}h^2+\ldots\ldots+\frac{k(k-1)\ldots\ldots(k-r+3)}{1\cdot 2\cdot\ldots\ldots\cdot(r-2)}x^{k-r+2}h^{r-1}+\ldots\ldots\end{aligned}$$

Combining similar terms,

$$\begin{aligned}(x+h)^{k+1}=x^{k+1}+(k+1)x^k h+\frac{(k+1)k}{1\cdot 2}\,x^{k-1}h^2+\ldots\ldots \\ +\frac{(k+1)k\ldots\ldots(k-r+3)}{1\cdot 2\cdot\ldots\ldots\cdot(r-1)}x^{k-r+2}h^{r-1}+\ldots\ldots+(k+1)xh^k+h^{k+1}\end{aligned}$$

Notice that the expansion of $(x+h)^{k+1}$ is of the *same form* as that of $(x+h)^k$, and that it may be obtained from $(x+h)^k$ by substituting $k+1$ for k. In particular notice that this is true for the rth term of $(x+h)^{k+1}$.

Hence, if the theorem is true for $n=k$, it is also true for a value of n one greater, i.e., for $n=k+1$.

But we know that the theorem holds for $n=2$. For $(x+h)^2=x^2+2hx+h^2$ follows the theorem.

Therefore it holds for $n=3$.

Again, if it holds for $n=3$, it must also hold for $n=4$.

Since we may proceed this way by taking successive values of n the theorem holds when n is *any positive integer*.

118. Mathematical induction.[*] The method of proof used in § 117 is called **mathematical induction.**

The principal steps in this method are, first, to verify the theorem for a particular case, as above for $n=2$; secondly, to show that the principle is true for $n=k+1$ if it is true for $n=k$.

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[*] Mathematical induction was invented by Maurolycus, and used in his arithmetic of 1575; but it was first brought into prominence by Pascal. See W. H. Bussey, "The Origin of Mathematical Induction," *American Mathematical Monthly*, XXIV, 199–207.

The necessity of the first step may be seen as follows: From the statement

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}+2 \qquad (1)$$

we get, by adding $\frac{1}{(k+1)(k+2)}$ to both sides, the statement

$$\frac{1}{1\cdot 2}+\ldots\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}+2$$
$$=\frac{k^2+2k+1}{(k+1)(k+2)}+2$$

$$\therefore \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots\ldots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}+2 \qquad (2)$$

The form of this result shows that if statement (1) is true for k it is also true for $k+1$. It does not follow, however, that statement (1) is true for every value of k, e.g., $k=1$. For in that case the left member has the value $\frac{1}{2}$ while the value of the right member is equal to $2\frac{2}{3}$.

The necessity of the second step may be seen from the following example.

It may easily be verified that the statement

$$1\cdot 2+2\cdot 3+3\cdot 4+4\cdot 5+\ldots\ldots+n(n+1)=3n^2-3n+2$$

is true for $n=1$, 2, and 3, but that it does not hold for the next greater value $n=4$.

EXERCISES

Write down the general term of each of the following series of numbers:

1. $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots$

2. $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots.$

3. $\frac{3}{1\cdot 2}+\frac{4}{2\cdot 3}+\frac{5}{3\cdot 4}+\ldots.$

Write down the sum of the first five terms of the series whose general terms are as follows:

4. 2^n

5. $n(n+1)(n+2)$

6. $\frac{1}{n+1}$

7. $\frac{n}{3^n}$

Prove the following by mathematical induction:

8. $1+3+5+7+\ldots.+(2n-1)=n^2$

For $n=1$, we have $1 = 1^2$
For $n=2$, we have $1+3 = 2^2$
For $n=3$, we have $1+3+5=3^2$
Hence the statement is true for $n=1$, 2, and 3.
Assume that it is true for $n=r$
Then $1+3+5+7+\ldots.+(2r-1)=r^2$
Adding the equation $2(r+1)-1=2(r+1)-1$
we have

$$1+3+5+7+\ldots.+(2r-1)+[2(r+1)-1]=r^2+2r+2-1, \text{ or } (r+1)^2$$

Hence $1+3+5+7+\ldots.+(2r-1)+[2(r+1)-1]=(r+1)^2$
The last statement is of the same form as the assumed statement and may be obtained from it by substituting $r+1$ for r.

It follows that, if the statement is true for $n=r$, it is also true for $n=r+1$.

Since we know it to be true for $n=2$, it follows that it is true for $n=3$, 4, etc.

9. $1^2+2^2+3^2+\ldots.+n^2=\frac{1}{6}n(n+1)(2n+1)$

10. $2+2^2+2^3+\ldots.+2^n=2(2^n-1)$

11. $1^3+2^3+3^3+\ldots.+n^3=\frac{n^2(n+1)^2}{4}$

12. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots.+\frac{1}{n(n+1)}=\frac{n}{n+1}$

13. $1+2^2+2^4+2^6+\dots+2^{2n-2}=\dfrac{2^{2n}-1}{3}$

14. $1\cdot 2+2\cdot 3+3\cdot 4+\dots+(n)(n+1)=\dfrac{n(n+1)(n+2)}{3}$

119. Graph of a rational integral function. The graph of a *linear* function, as $f(x)=ax+b$, has been found to be a *straight line.* If the function is not of the first degree, the graph is usually a *curved line.* Thus the graph of the function $f(x)=ax^2+bx+c$ is a *parabola.* Both of these functions are rational integral functions. It is one of the purposes of this chapter to make a further study of the graphs of rational integral functions.

120. Evaluation of functions. To graph the function $f(x)$, we must select a number of values of x and then compute the corresponding values of $f(x)$.

The value of $f(x)$ for $x=a$ may be denoted by $f(a)$. This value may be found by *substituting* in $f(x)$ for x the number a. But often it is obtained more easily by the process of *synthetic division.*[*]

The following example illustrates the process:

Given $f(x)=2x^4-5x^3+7x-8$. Find $f(3)$.

$$\begin{array}{rrrrr} 2 & -5 & 0 & 7 & -8 \\ & 6 & +3 & +9 & +48 \\ \hline 2 & +1 & +3 & +16 & +40 \end{array}$$

Explanation. Write down the coefficients: 2, -5, 0; 7, -8.

Bring down the coefficient of x^4, i.e., 2.

Multiply 2 by 3, and add the product to -5. This gives $+1$.

Multiply $+1$ by 3, and add the product to 0. This gives $+3$.

Multiply $+3$ by 3, and add the product to 7. This gives $+16$.

---

[*] *Third-Year Mathematics,* pp. 12–14.

Multiply +16 by 3, and add the product to −8. This gives +40.

$$\therefore \quad f(3)=40$$

EXERCISES

1. If $f(x)=2x^4-5x^3+7x-8$, find $f(2)$, $f(-1)$, $f(5)$, by using synthetic division.

2. Draw the graph of

$$f(x)=x^3-6x^2+7x+4, \text{ Fig. 55.}$$

Assume for x the values 0, 1, 2, 3, 4, 5, −1, −2, −3, and find the corresponding values of $f(x)$ by means of synthetic division.

Plot the pairs of corresponding numbers and join the points thus found by means of a smooth curve.

3. Draw the graph of

$$f(x)=-x^3+5x^2-2x-8.$$

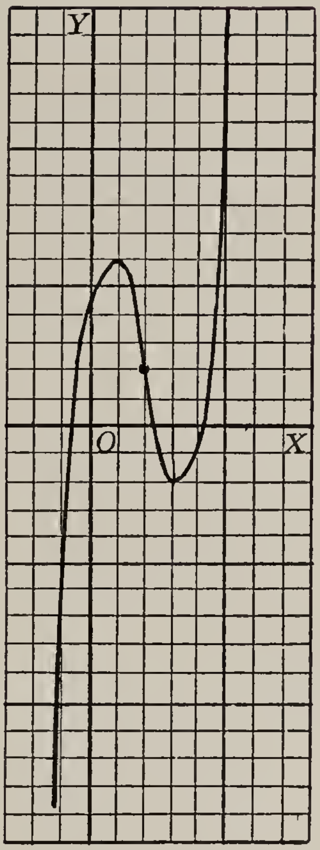

FIG. 55

121. Continuity of $f(x)$. When making the graph of $f(x)=\frac{1}{x}$, Fig. 56, we find the following pairs of corresponding values of x and $f(x)$.

x	-3	-2	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	$+1$	$+\frac{3}{2}$	$+2$
$f(x)$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	No value	$+2$	$+1$	$+\frac{2}{3}$	$+\frac{1}{2}$

Notice that the graph of this function is broken, or *discontinuous*, for $x=0$. The function $f(x)=\frac{1}{x}$ is said to be a *discontinuous* function.

In order that a function may be *continuous* in the vicinity of $x=a$, it must have a real and finite value corresponding to $x=a$.

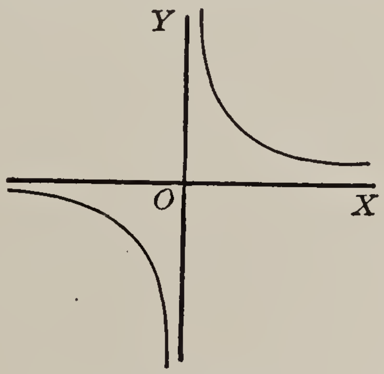

FIG. 56

The curve in Fig. 57 is also discontinuous. In this case $f(x)$ approaches two different values according as x approaches a from the left or from the right.

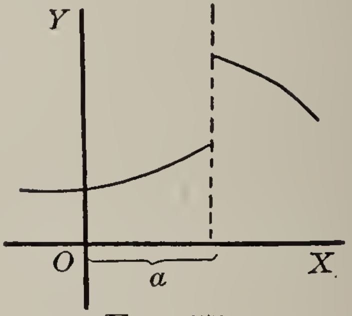

FIG. 57

Such discontinuities cannot occur if $f(x)$ is a rational integral function. For in that case to every real and finite value of x there corresponds one, and only one, real and finite value of $f(x)$.

More precisely, *$f(x)$ is said to be* **continuous** *in the vicinity $x=x_1$,* if $\lim\limits_{h \to 0} [f(x_1+h)-f(x_1)]=0$.

That this is true for a rational integral function may be seen as follows:

Substituting in (1), § 116, the value x_1+h for x, we have

$$f(x_1+h)=a(x_1+h)^n+b(x_1+h)^{n-1}+c(x_1+h)^{n-2}+\dots\dots \\ +p(x_1+h)+q$$

Expanding the terms in $f(x_1+h)$ by means of the binomial theorem

$$a(x_1+h)^n \quad =a\left[x_1^n+nx_1^{n-1}h+\frac{n(n-1)}{1\cdot 2}x_1^{n-2}h^2+\dots\dots\right]$$

$$b(x_1+h)^{n-1}=b\left[x_1^{n-1}+(n-1)x_1^{n-2}h \right. \\ \left. +\frac{(n-1)(n-2)}{1\cdot 2}x_1^{n-3}h^2+\dots\dots\right]$$

$$c(x_1+h)^{n-2}=c\left[x_1^{n-2}+(n-2)x_1^{n-3}h \right. \\ \left. +\frac{(n-2)(n-3)}{1\cdot 2}x_1^{n-4}h^2+\dots\dots\right]$$

. . . .

$$p(x_1+h) \quad =p(x_1+h)$$

$$q=q$$

Adding,

$$\begin{aligned} f(x_1+h) = {} & ax_1^n+bx_1^{n-1}+cx_1^{n-2}+\ldots\ldots+px_1+q \\ & +h[nax_1^{n-1}+(n-1)bx_1^{n-2} \\ & \qquad +(n-2)cx_1^{n-3}+\ldots\ldots+p] \\ & +h^2\left[\frac{n(n-1)}{1\cdot 2}ax_1^{n-2}+\ldots\ldots\right] \\ & +\ldots\ldots, \text{etc.} \end{aligned}$$

$$\text{But } f(x_1) = ax_1^n+bx_1^{n-1}+cx_1^{n-2}+\ldots\ldots+px_1+q$$

Subtracting,

$$\begin{aligned} f(x_1+h)-f(x_1) = {} & h[nax_1^{n-1}+(n-1)bx_1^{n-2}+(n-2)cx_1^{n-3} \\ & +\ldots\ldots+p] \\ & +h^2\left[\frac{n(n-1)}{1\cdot 2}ax_1^{n-2}+\ldots\right]+\text{ etc.} \end{aligned}$$

Notice that every term of the right member of this equation approaches zero as h approaches zero.

Therefore

$$\lim_{h\to 0}\,[f(x_1+h)-f(x_1)]=0 \text{ and } \lim_{h\to 0}\,f(x_1+h)=f(x_1).$$

Hence *a rational integral function is continuous for all finite values of x.*

It follows that as x changes from any value a to another value b, $f(x)$ changes from $f(a)$ to $f(b)$ and takes every value between $f(a)$ and $f(b)$ *at least once.*

Moreover, if $f(a)<0$ and $f(b)>0$, it follows that there is at least one value of x between a and b for which $f(x)$ is *equal* to zero.

Graphically this means that the curve, Fig. 58, is an unbroken line and crosses the x-axis between A and B *at least once.*

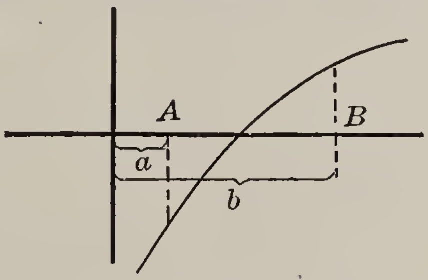

FIG. 58

EXERCISES

Tabulate $f(x)$ for assumed values of x and locate the places where the graphs of the following functions cross the x-axis:

1. $2x^2+x-6$ **2.** x^3-5x^2+2x+8 **3.** $x^4-6x^3+24x-16$

122. Derivative of a rational integral function of *x*. The derivative of $f(x)$ has been defined as

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}, \text{ § 106}$$

From § 121,

$$\begin{aligned} f(x+h)-f(x) = h[nax^{n-1}&+(n-1)bx^{n-2} \\ &+(n-2)cx^{n-3}+\ldots\ldots+p] \\ &+h^2\left[\frac{n(n-1)}{1 \cdot 2}ax^{n-2}+\ldots\ldots\right]+ \text{ etc.} \end{aligned}$$

$$\begin{aligned} \therefore \quad \frac{f(x+h)-f(x)}{h} = nax^{n-1}&+(n-1)bx^{n-2} \\ &+(n-2)cx^{n-3}+\ldots\ldots+p \\ &+h\left[\frac{n(n-1)}{1 \cdot 2}ax^{n-2}+\ldots\ldots\right] \\ &+h^2[\ldots\ldots] \\ &+\ldots\ldots, \text{ etc.} \end{aligned}$$

$$\begin{aligned} \therefore \quad \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = nax^{n-1}&+(n-1)bx^{n-2} \\ &+(n-2)cx^{n-3}+\ldots\ldots+p \end{aligned}$$

or, $\boldsymbol{f'(x)=nax^{n-1}+(n-1)bx^{n-2}+(n-2)cx^{n-3}+\ldots\ldots+p}$

This shows that, if $f(x)$ is an integral rational function of x, the derivative $f'(x)$ may be found by the following simple law:

Multiply the coefficient of each term by the exponent of x in that term, and then diminish the exponent of x by unity.

ISAAC BARROW

ISAAC BARROW

ISAAC BARROW (1630–77) was born in London and died in Cambridge, England. He attended school at Charterhouse, London, where he did poorly, then at Felstead, where he did better, and finally completed his schooling at Trinity College, where he devoted himself to literature, science, and particularly to natural philosophy. He then studied for the medical profession and later traveled in France, Italy, and Eastern Europe. In 1659 he returned to England through Germany and Holland.

In 1660 Barrow was appointed to the Greek professorship at Cambridge. In 1662 he was elected professor of geometry at Gresham College, became a fellow of the Royal Society in 1663, and in the same year was chosen to be the first occupant of the new Lucasian chair of mathematics at Cambridge. He held this chair until 1669, when he resigned it in favor of his great pupil Isaac Newton. He then turned to the study of divinity, became a clergyman, and in 1672 was made a master of Trinity College by Royal patent. In 1675 he was chosen vice-chancellor of Cambridge University. He died May 4, 1677, and was buried in Westminster Abbey.

Ball says of his personality that he was "low in stature, lean, and of a pale complexion" and slovenly in dress. Nevertheless he was noted for his strength, courage, and ready and caustic wit.

He published a complete edition of Euclid's *Elements* in Latin in 1655 and in English in 1660. In 1657 he published an edition of Euclid's *Data*. His lectures of 1664–66 were published in 1683 in a volume called *Lectiones mathematicae*. In 1669 he published *Lectiones opticae et geometricae*.

His chief services to science are his *method of tangents*, which probably suggested the fluxional calculus to Newton, and the fact that he was the principal teacher and stimulator of Isaac Newton.

Barrow's method of tangents consisted of choosing two neighboring points, P and Q, on a curve, drawing their ordinates, and a line, QR, parallel to the horizontal axis, thus forming a little curvilinear triangle, PQR, which he called the *differential triangle*. He regarded this differential triangle as containing, *in embryo*, all the properties of the curve. In fact, his procedure differed from that of the fluxional calculus only in language and notation. It was suggestion enough for a Newton.

Barrow was a great genius. If he had devoted himself continuously to any one of the numerous things which he pursued for a brief period, and if he had been granted a long life, he would have attained high eminence.

[See Ball, Cajori, or *Encyclopaedia Britannica*.]

EXERCISES

1. If $f(x)=ax^2+bx+c$, show by the law given in § 122 that $f'(x)=2ax+b$ and compare this result with that found in § 106.

Find the derivative of each of the following functions:

2. $f(x)=3x^2-7x+2$

3. $f(y)=4y^2+3y-6$

4. $f(x)=6x^4-2x^3+3x+1$

123. Maxima and minima. When a curve, Fig. 59, runs *upward toward the right*, the angle of inclination of the tangent T_1P_1 toward the x-axis is *acute*. Hence the slope is *positive*.

When the curve runs *downward toward the right* the angle of inclination of the tangent P_2T_2 is *obtuse*. Hence the slope is *negative*.

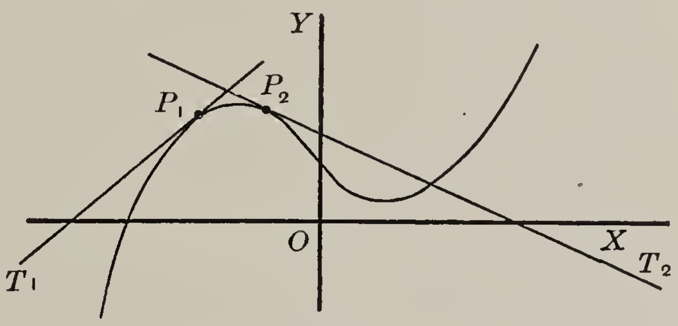

FIG. 59

If the slope of the tangent is *zero* the tangent is *parallel* to the x-axis.

A rational integral function of x is *increasing* when an increase in x causes the function to *increase*, and it is *decreasing* when an increase in x causes the function to *decrease*.

Hence the function is increasing when the graph runs upward to the right, and it is decreasing when the graph runs downward to the right.

Or the function is *increasing* when the derivative is *positive* for a given value of x, and *decreasing* when the derivative is *negative* for a given value of x.

If for a given value of x the derivative is zero, the function *may* have a maximum or a minimum.

It has a *maximum* value for $x=a$ if it changes from an increasing to a decreasing function, and it has a *minimum* value for $x=a$ if it changes from a decreasing to an increasing function.

More precisely, this may be stated by means of the derivative as follows:

Let h be a positive number taken very small. *Then* $f(a)$ *is a maximum of* $f(x)$, *when* $f'(a-h)>0, f'(a)=0$, *and* $f'(a+h)<0$,

And $f(a)$ *is a minimum of* $f(x)$, *when* $f'(a-h)<0, f'(a)=0$, *and* $f'(a+h)>0$.

That the condition $f'(a)=0$ alone is *not sufficient* for a maximum or a minimum is shown by the following example:

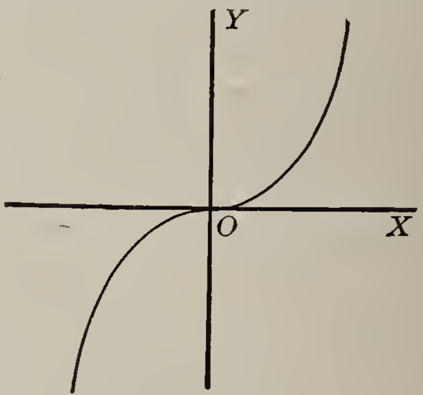

Fig. 60

Let $f(x)=x^3$, Fig. 60.

Then $f'(x)=3x^2$.

For $x=0$, $f'(x)=0$, but it is positive for all positive and negative values of x. Hence the curve is increasing for $x<0$ and for $x>0$, and it has neither a maximum nor a minimum in the vicinity of $x=0$.

From the preceding discussion we may now formulate the following rule for determining the maxima and minima of a rational integral function:

Find $f'(x)$.

Let $f'(x)=0$ *and solve this equation.*

Let a *be any one of the roots of the equation* $f'(x)=0$.

Determine the sign of $f'(a-h)$ *and of* $f'(a+h)$.

If the sign of $f'(x)$ *changes from* $+$ *to* $-$, *as* x *passes through* a, $f(a)$ *is a maximum value of* $f(x)$.

If the sign of $f'(x)$ *changes from* $-$ *to* $+$, *as* x *passes through* a, $f(a)$ *is a minimum value of* $f(x)$.

124. Calculation of $f'(x \pm h)$. According to § 121,

$$f(x+h)=ax^n+bx^{n-1}+cx^{n-2}+\dots\dots+px+q$$
$$+h[nax^{n-1}+(n-1)bx^{n-2}+(n-2)cx^{n-3}+\dots\dots+p]$$
$$+h^2\left[\frac{n(n-1)}{1\cdot 2}ax^{n-2}+\frac{(n-1)(n-2)bx^{n-3}}{1\cdot 2}+\dots\dots\right]$$
$$+\dots\dots, \text{ etc.}$$

Notice that the first line in this expansion is $f(x)$, that the function in the parenthesis in the second line is the derivative of $f(x)$, and that the function in the parenthesis in the third line is $\frac{1}{1\cdot 2}$ multiplied by the derivative of $f'(x)$.

The derivative of $f'(x)$ is called the **second derivative** of $f(x)$ and is denoted by $f''(x)$.

Similarly, the third, fourth, $\dots\dots$, kth derivatives of $f(x)$ are denoted by $f'''(x)$, $f^{(iv)}(x)$, $\dots\dots$, $f^{(k)}(x)$, respectively.

Using this notation, we have

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{1\cdot 2}f''(x)+\frac{h^3}{1\cdot 2\cdot 3}f'''(x)+\dots\dots$$

This formula is a particular case of a theorem known as *Taylor's theorem.*[*]

Since $f(x)$ is a rational integral function of x, the formula just found may be used to find the value of $f(x+h)$.

---

[*] Brook Taylor (1685–1731) was one of Newton's admirers. In his book *Methodus incrementorum* he added the then new mathematical branch, now called "finite differences," and gave a statement of this theorem together with a worthless proof. The value of the theorem was not recognized until Lagrange, fifty years later, showed its great power. One hundred years after its discovery, Cauchy gave the first rigorous proof of the theoreum.

EXERCISES

1. If $f(x)=3x^4-2x^2-3x+1$, find $f(x+h)$

$$\begin{aligned} f'(x)&=12x^3-4x-3\\ f''(x)&=36x^2-4\\ f'''(x)&=72x\\ f^{(\text{iv})}(x)&=72\\ f^{(\text{v})}(x)&=0 \end{aligned}$$

$$\begin{aligned} \therefore\ f(x+h)&=3x^4-2x^2-3x+1+h(12x^3-4x-3)+\frac{h^2}{1\cdot 2}(36x^2-4)\\ &\qquad+\frac{h^3}{1\cdot 2\cdot 3}(72x)+\frac{h^4}{1\cdot 2\cdot 3\cdot 4}(72)\\ &=3x^4-2x^2-3x+1+h(12x^3-4x-3)+h^2(18x^2-2)\\ &\qquad+h^3(12x)+h^4(3) \end{aligned}$$

2. Using the result in exercise 1, find $f(1+h)$; $f(2+h)$; $f(-1+h)$.

Find the maxima and minima of the following functions:

3. $f(x)=x^3+18x^2+105x$

4. $y=3x^3-2x+4$

125. Remainder theorem. Let $f(x)$ be a rational integral function of the nth degree,

$$\text{i.e., }\ f(x)=ax^n+bx^{n-1}+cx^{n-2}+\ldots\ldots+px+q$$
$$\text{Then }\ f(r)=ar^n+br^{n-1}+cr^{n-2}+\ldots\ldots+pr+q$$

Subtracting,

$$\begin{aligned} f(x)-f(r)&=a(x^n-r^n)+b(x^{n-1}-r^{n-1})+\ldots\ldots+p(x-r)\\ &=(x-r)[a(x^{n-1}+x^{n-2}r+x^{n-3}r^2+\ldots\ldots+r^{n-1})\\ &\qquad+b(x^{n-2}+x^{n-3}r+\ldots\ldots+r^{n-2})+\ldots\ldots+p] \end{aligned}$$

Denoting the function within the brackets by $Q(x)$,

$$f(x)-f(r)=(x-r)Q(x)$$

Hence $$f(x)=(x-r)Q(x)+f(r) \tag{1}$$

where $Q(x)$ is a rational integral function of (x) of degree $n-1$.

Comparing (1) with the relation

$$\text{dividend} \equiv \text{divisor} \times \text{quotient} + \text{remainder}$$

we conclude that the remainder obtained by dividing $f(x)$ by $x-r$ is $f(r)$.

This result may be expressed in the form of a theorem as follows:

If a rational integral function $f(x)$ *be divided by* $x-a$, *the remainder obtained is equal to the value of the function for* $x=a$.

More briefly this may be stated thus: *If* $f(x)$ *is divided by* $x-a$, *the remainder is* $f(a)$.

This theorem is known as the **remainder theorem.**

126. Factor theorem. *If* $f(x)$, *divided by* $x-a$, *gives the remainder zero, then* $x-a$ *is a* **factor** *of* $f(x)$.

For the remainder R is then equal to $f(a)$, § 125. It follows that $x-a$ is a factor of $f(x)$ if $f(a)=0$.

EXERCISES

State and prove the converse of the factor theorem.

Summary

127. The chapter has taught the meaning of the following terms:

rational integral function
mathematical induction
continuity of a function
first, second, and higher derivatives
increasing and decreasing function
maximum and minimum value of a function
the general term of a series

128. Rules have been formulated:

1. For determining the maxima and minima of a rational integral function.

2. For finding the value of $f(x+h)$ in terms of powers of h. (Taylor's Theorem.)

3. For finding the derivative of a rational integral function $f(x)$.

129. The following theorems have been proved:

1. *The binomial theorem for any positive integral exponent.*

2. *The remainder theorem.*

3. *The factor theorem.*

4. *A rational integral function is continuous for all finite values of* x.

CHAPTER VII

SOLUTION OF EQUATIONS WITH NUMERICAL COEFFICIENTS

Algebraic Solution

130. Algebraic solution. If $f(x)$ is a rational integral function of the *first* degree, it is *always* possible to solve the equation $f(x)=0$. For if $f(x)=ax+b$, the solution of $ax+b=0$ is readily obtained from the formula

$$x=-\frac{b}{a}$$

If $f(x)$ is a rational integral function of x of the *second* degree, as ax^2+bx+c, the roots of the equation

$$f(x)=ax^2+bx+c=0$$

are given by the general formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Notice that in both cases *the roots of the equations are functions of the coefficients.* The solutions obtained by means of formulas are called **algebraic solutions,** if the formulas involve only a finite number of the operations of addition, subtraction, multiplication, division, and extraction of roots.

It will be shown in chapter viii that the *cubic* equation

$$ax^3+bx^2+cx+d=0$$

and the *biquadratic* equation

$$ax^4+bx^3+cx^2+dx+c=0$$

can also be solved algebraically. However, the solution by means of special formulas, when applied to equations containing only numerical coefficients, is not as practical as the general methods to be given in §§ 143 to 152.

It can be shown by higher mathematics that the equation

$$f(x)=ax^n+bx^{n-1}+\ldots+px+q=0$$

cannot generally be solved algebraically if $n>4$.

The great mathematician Niels Henrik Abel, in 1824, was the first to prove it to be impossible to give an *algebraic solution* of a general quintic equation. Hermite (1822–1901) solved the equation of the fifth degree by means of *elliptic functions,* and Klein has given the simplest solution by *transcendental functions.*

131. Number of roots of an equation. We have seen that every *linear* equation has one root, and that every *quadratic* equation has *two* roots, real or complex. We shall now *assume* that *every rational integral equation of the nth degree whose coefficients are real or complex numbers has at least one root, real or complex.*

Let r_1 be a root of $f(x)=0$.

Then $x-r_1$ is a factor of $f(x)$, § 126.

$\therefore$ $f(x)=(x-r_1)f_1x$, where $f_1(x)$ is the quotient $\dfrac{f(x)}{x-r_1}$ and a rational integral function of x of degree $n-1$.

Hence, according to our assumption above, $f_1(x)=0$ has at least one root, r_2. It follows that

$$f_1(x)=(x-r_2)f_2(x)$$

where $f_2(x)$ is of degree $n-2$. By substitution,

$$f(x)=(x-r_1)(x-r_2)f_2(x)$$

Continuing this process we shall have

$$f(x)=(x-r_1)(x-r_2)\ldots.(x-r_n)A$$

where A is the coefficient of x^n and does not contain x.

This shows that *every rational integral function of x is the product of n linear factors.*

If we substitute for x any one of the values $r_1, r_2, \ldots., r_n$, $f(x)$ will become zero. If we substitute for x any other value, the factors $(x-r_1)$, $(x-r_2)$, ..., $(x-r_n)$ are all different from zero, and $f(x)$ cannot be equal to zero.

Hence *every rational integral equation of the nth degree whose coefficients are real or complex numbers has n, and only n, roots, real or complex.*

This theorem is called the **fundamental theorem of algebra.** The theorem was first stated in its complete form by D'Alembert (1717–83). Gauss (1777–1855) was the first to give a satisfactory proof.

132. Multiple roots. Some of the n factors of $f(x)$ may be equal to each other, as in $f(x)=5(x-1)^2(x-3)$. In that case the equation $f(x)=0$ is said to have *equal roots,* or **multiple roots.**

Graphical Representation of the Roots of $f(x)=0$

133. Real and distinct roots. Let x_1, x_2, ..., x_n be the real and distinct roots of $f(x)=0$ and let

$$x_1<x_2<\ldots.<x_n$$

For example, let $x_1=1$, $x_2=3$, and $x_3=4$, and let $f(x)=2(x-1)(x-3)(x-4)$.

When $x<1$, all the binominal factors of $f(x)$ are negative. Therefore $f(x)$ is *negative,* and the graph is *below* the x-axis, Fig. 61. Thus, when $x=0$, $f(x)=2(-1)(-3)(-4)=-24$.

When $1<x<3$, the first factor of $f(x)$ is *positive* and the other two are *negative*. Therefore $f(x)$ is *positive*, and the graph is *above* the x-axis. Thus, when $x=2$, $f(x)=2(1)(-1)(-2)=4$.

When $3<x<4$, the first two factors are *positive* and the last *negative*. Therefore $f(x)$ is *negative*, and the graph is *below* the x-axis.

When $x>4$, all factors of $f(x)$ are *positive*, and the graph is *above* the x-axis.

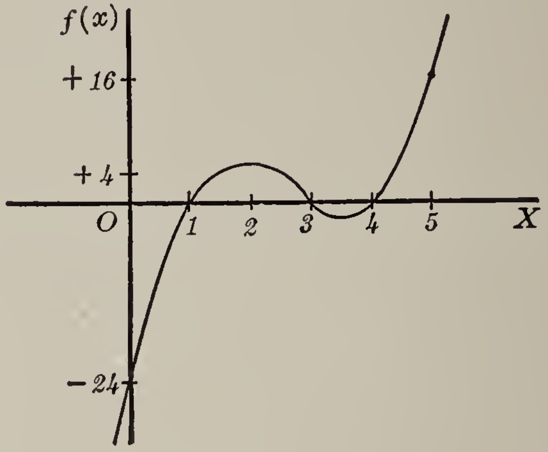

FIG. 61

Hence the graph crosses the x-axis at the points $x=1$, 3, and 4.

EXERCISES

Plot the graph of each of the following equations:

1. $x(x-2)(x-4)=0$

2. $(x+2)(x-2)(x)=0$

3. $\frac{1}{6}(x-\frac{1}{2})(x-2)(x-3)=0$

4. $3(x+\frac{2}{3})(x-1)(x-4)=0$

134. Real multiple roots. Let

$$f(x)=a_0(x-r_1)(x-r_2)^2(x-r_3)=0$$

and let $r_1<r_2<r_3$. Show that

$f(x)$ is positive when $x<r_1$;
$f(x)$ is negative when $r_1<x<r_2$;
$f(x)$ is zero when $x=r_2$;
$f(x)$ is negative when $r_2<x<r_3$.

Hence at $x=r_2$ the graph does *not cross* the x-axis but *touches* it at point C, Fig. 62.

We may think of point C as being obtained by moving the dotted line in the figure downward until A and B coincide at the point of tangency C.

By letting r_3 vary and approach r_2, $f(x)$ will take the form $a_0(x-r_1)(x-r_2)^3$ and

when $x<r_1$, $f(x)$ is positive;
when $r_1<x<r_2$, $f(x)$ is negative;
when $x=r_2$, $f(x)$ is zero;
when $x>r_2$, $f(x)$ is positive.

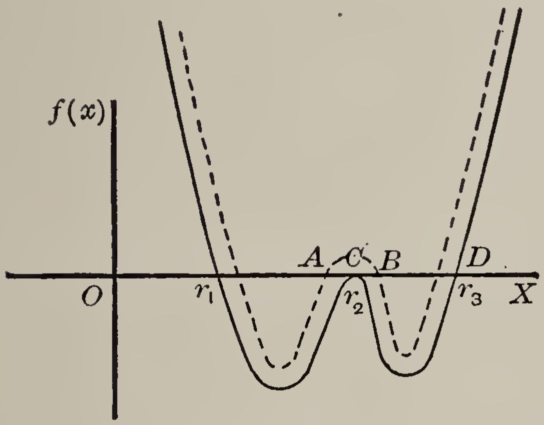

FIG. 62

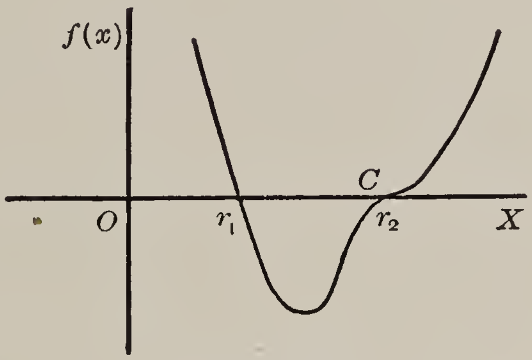

FIG. 63

Hence the curve has crossed the x-axis at point C, Fig. 63.

We may think of point D, Fig. 62, as having moved to the left until it coincides with points A and B at point C.

In general, if a factor $(x-r_n)$ occurs an *even* number of times in $f(x)$ the curve does *not* cross the x-axis; if it occurs an *odd* number of times the curve *crosses* the x-axis.

EXERCISES

Draw the graph of each of the following equations:

1. $3(x-1)(x-3)^2=0$
2. $\frac{1}{4}(x+2)x^2(x-3)=0$
3. $(x-2)(x-3)^3=0$
4. $x(x-1)^2(x-3)^3=0$

135. Complex roots. Let $a+bi$ be a root of the equation $f(x)=0$. Then $a+bi$ satisfies the equation $f(x)=0$ and

$$f(a+bi)=a(a+bi)^n+b(a+bi)^{n-1}+\dots+p(a+bi)+q=0$$

By expanding the powers of $a+bi$, and then combining first the real terms and then the imaginary terms, this equation takes the form

$$f(a+bi)=A+Bi=0$$

It follows that $A=0$, $B=0$, § 16.

Similarly, if we substitute for x the value $a-bi$, we have

$$f(a-bi)=A-Bi$$

Since $\qquad A=0$, and $B=0$

$$A-Bi=0$$

or $\qquad f(a-bi)=0$

This means that $a-bi$ is also a root of the equation $f(x)=0$.

Hence the following theorem has been proved:

If a rational integral equation with real coefficients has a complex root $a+bi$, then $a-bi$ is also a root of the equation.

Hence, if $f(x)$ contains a factor of the form $x-(a+bi)$, it must also contain the factor $x-(a-bi)$, and therefore the quadratic factor

$$[x-(a+bi)][x-(a-bi)]=(x-a)^2+b^2$$

This factor is positive for all real values of x and *cannot have* the value zero. Hence *it cannot cause $f(x)$ to become zero nor the graph to touch or cut the x-axis.*

EXERCISES

Draw the graphs of the following equations:

1. $(x+2)(x^2-3x+5)=0$

Verify the following table and compare the results with the graph in Fig. 64.

x	<-2	-2	>-2	-1	0	1	2	3
$f(x)$	Negative	0	Positive	9	10	9	12	25

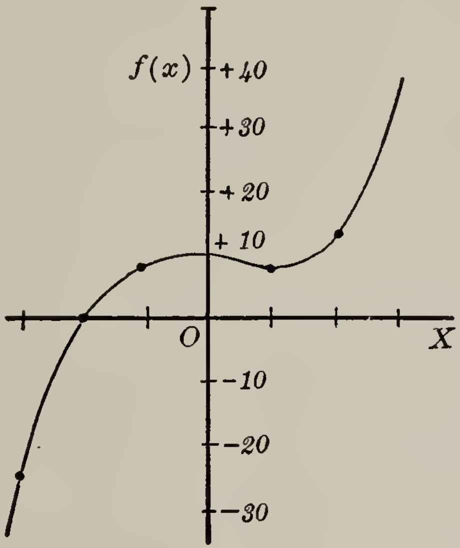

FIG. 64

2. $\frac{1}{2}(x+4)(x^2-3x+4)=0$

3. $x(x+3)(x^2-8x+20)=0$

Solve the following equations:

4. One root of $2x^3-15x^2+46x-42=0$ is $3+\sqrt{-5}$. Find the others.

Apply the theorem in § 135.

5. One root of the equation $x^4-8x^3+21x^2-26x+14=0$ is $3+\sqrt{2}$. Find the others.

6. One root of the equation $x^4-3x^2-42x-40=0$ is $\frac{-3+\sqrt{-31}}{2}$. Find the others.

Location of Real Roots

136. Location of real roots. The problem of locating the *real* roots of $f(x)=0$ is simplified by the use of the theorems stated in the following sections.

137. Descartes' rule of signs. Let $a, b, c, \ldots, p, q$ be the coefficients of x in a rational integral function arranged in descending order.

If two successive coefficients have *opposite* signs, we have what is called a **variation** of signs. Thus in

$$x^4-3x^3-2x^2+5x-4$$

the signs are

$$+\quad -\quad -\quad +\quad -$$

and there are three variations.

When all the signs are the same, $f(x)$ cannot vanish if a *positive* number is substituted for x. Hence the equation $f(x)=0$ cannot have a positive root if there are no variations of sign.

If $f(x)$ is arranged according to descending powers of x and contains all powers of x from n to 0, show that it cannot vanish for *negative* values of x in case the signs are alternately positive and negative.

Thus the number of variations of signs in $f(x)$ gives some information regarding the existence of the positive and negative roots of the equation $f(x)=0$.

The following consideration establishes a relation between the number of positive roots of $f(x)$ and the number of variations of signs.

Let $f_1(x)=2x^8+6x^7-2x^6-x^5-2x^4+3x^3-7x^2+4x+3$ represent a rational integral function of x, arranged according to descending powers of x.

The signs of $f_1(x)$ may be arranged as follows:

$$+\,+\qquad -\,-\,-\qquad +\qquad -\qquad +\,+,$$

or

$$+(+\ +)-(+\ +\ +)+(+)-(+)+(+\ +) \tag{1}$$

If $f_1(x)$ is multiplied by $x-r_1$, r_1 being a positive number, the following signs are obtained

$$\begin{array}{cccccccccc} + & + & - & - & - & + & - & + & + & \\ & - & - & + & + & + & - & + & - & - \end{array}$$

These may be grouped, giving the signs of $(x-r_1)f_1(x)$, as follows:

$$\begin{array}{ll} & +(+\ \ +)-(+\ \ +\ \ +)+(+)-(+)+(+\ \ +) \\ & +(\quad\ \ -)-(+\ \ -\ \ -)+(+)-(+)+(+\ \ -)\ - \\ \hline \text{Adding} & +(+\ \ \pm)-(+\ \ \pm\ \ \pm)+(+)-(+)+(+\ \ \pm)- \end{array} \quad (2)$$

We may now compare the signs in $f_1(x)$ as given in (1) with the signs in $(x-r_1)f_1(x)$ given in (2). We observe that the number of variations of signs in (2), disregarding the variations due to the ambiguous signs within the parentheses, is *one more* than the number of variations in (1). If the ambiguous signs in (2) are replaced by either sign, the number of variations either remains the same or is increased. Hence the number of variations of signs in $f_1(x)$ is *at least one less* than the number of variations in $(x-r_1)f_1(x)$.

Let $r_1, r_2, r_3, \ldots, r_k$ be the only *positive* roots of $f(x)=0$.

Since the number of variations in $\dfrac{f(x)}{x-r_1}$ is *at least one less* than the variations in $f(x)$, it follows that $f(x)$ has *at least one* variation.

Since the quotient $\dfrac{f(x)}{(x-r_1)(x-r_2)}$ has *at least two* variations less than $f(x)$, $f(x)$ must have *at least two* variations.

Similarly, since the quotient $\dfrac{f(x)}{(x-r_1)(x-r_2)\ldots.(x-r_k)}$ has *at least k* variations less than $f(x)$, $f(x)$ must have *at* *least k* variations, i.e., at least as many variations as the number of positive roots of $f(x)=0$.

This result may be stated in the form of a theorem as follows:

Theorem. *The number of positive roots of* $f(x)=0$ *cannot be greater than the number of variations of sign in* $f(x)$.

Since the *negative* roots of $f(x)$ are the *positive* roots of $f(-x)=0$, it follows that *the number of negative roots of* $f(x)=0$ *cannot be greater than the number of variations in* $f(-x)$.

The theorem given above is known as **Descartes' rule of signs.**

EXERCISES

By means of Descartes' rule state some conclusions regarding the roots of the following equations:

1. $x^3+5x-7=0$

Writing the signs, we have

$f(x)=++-\therefore$ there cannot be more than one positive root.

$f(-x)=---\therefore$ there are no negative roots.

$\therefore$ two roots are complex.

2. $x^3+3x+7=0$

3. $x^3+3x+2=0$

4. $x^3+1=0$

5. $2x^3-7x^2+3x-1=0$

6. $y^3-y=21.3$

7. $z^3+z+3=0$

8. $y^3-4y^2-3y+19=0$

9. $x^4-2x^2+1=0$

10. $x^4-5x^3+20x-16=0$

11. $x^5-3x^2+6=0$

12. $x^7+2x^4-x^2-5=0$

13. $x^7-2x^6+x^4-1=0$

138. Theorem. *The first term in the rational integral function*

$$f(x)=ax^n+bx^{n-1}+\ldots\ldots+px+q$$

can be made to exceed the sum of the remaining terms by taking a sufficiently large value of x.

For, let $f(x) = ax^n + bx^{n-1} + \ldots . + px + q$, and let m be the greatest of the coefficients $a, b, \ldots ., p, q$.

Comparing the first term ax^n with the sum of the remaining terms, we have

$$\frac{ax^n}{bx^{n-1}+cx^{n-2}+\ldots .+px+q} > \frac{ax^n}{m(x^{n-1}+x^{n-2}+\ldots .+x+1)}$$

$$= \frac{ax^n}{\frac{m(x^n-1)}{x-1}} > \frac{ax^n(x-1)}{mx^n} = \frac{a}{m}(x-1)$$

$$\therefore \frac{ax^n}{bx^{n-1}+cx^{n-2}+\ldots+px+q} > \frac{a}{m}(x-1)$$

By taking x sufficiently large, $\frac{a}{m}(x-1)$ can be made greater than 1.

$\therefore \quad \frac{ax^n}{bx^{n-1}+\ldots .+q}$ can be made greater than 1.

$\therefore \quad ax^n$ can be made greater than $bx^{n-1} + \ldots . + q$.

EXERCISES

1. Show that $2x^4 - 3x^3 + 7x^2 - 4x - 2$ becomes positive for positive and negative values of x, provided x be taken sufficiently large numerically.

2. Show that $3x^3 - 4x^2 + 7x - 2$ becomes positive when x is positive and increases numerically, and that it becomes negative when x is negative and increases numerically.

139. Theorem. *If* $f(a)$ *and* $f(b)$ *have opposite signs, an odd number of real roots lies between* a *and* b.

For, let $f(x)$ be a rational integral function. Let A and B be two points on the graph of $f(x)$, Fig. 65, on opposite sides of the x-axis.

Since the graph is a continuous curve, § 121, it must cross the x-axis an *odd* number of times in passing from A to B.

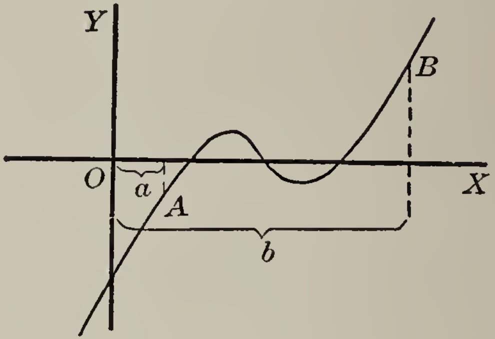

FIG. 65

The values of x, between a and b, for which $f(x)$ is equal to zero, are roots of $f(x)=0$.

Since $f(x)$ is a continuous function, it must pass through zero *at least once*, when changing from a positive to a negative value, or from a negative to a positive value.

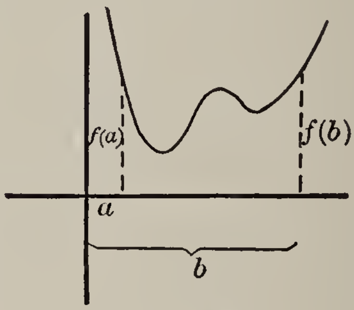

FIG. 66

140. Theorem. *If $f(a)$ and $f(b)$ have like signs, either no real root or an even number of real roots lies between a and b.*

Graphically this means that the curve may not intersect the x-axis at all, as in Fig. 66; or touch it; or cut it in two places, as in Fig. 67; or in four places, as in Fig. 68.

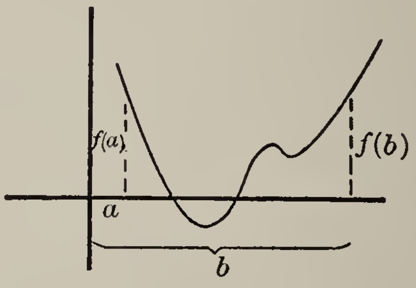

FIG. 67

141. Theorem. *Every equation of odd degree has at least one real root.*

For, if a is sufficiently large numerically, and is negative, $f(a)$ has the same sign as the first term and is therefore negative. Likewise, if a is sufficiently large, numerically, and is positive, $f(a)$ is positive.

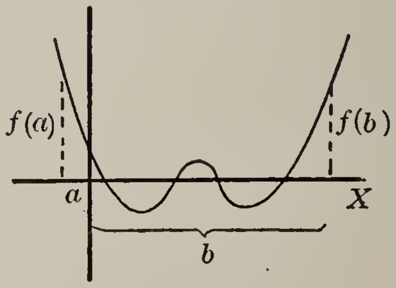

FIG. 68

Hence $f(x)$ has at least one real root, § 139.

142. Locating the real roots of $f(x)=0$. The theorems stated in §§ 137 to 141 are now to be applied in locating the real roots of an equation. Exercise 1 below illustrates the process in detail.

EXERCISES

Locate the roots of the following equations approximately:

1. $f(x)=x^3+5x-7=0$

1. The signs of $f(x)$ and $f(-x)$ are:

$f(x)=++-\therefore f(x)$ has *not more than one positive* root, § 137.

$f(-x)=---\therefore$ there are *no negative* roots, § 137.

$\therefore$ two roots are imaginary, and the third is positive. Why?

2. A rough graph of $f(x)$ may be obtained as follows:

For sufficiently large positive values of x the function $f(x)$ is positive, § 138. See Fig. 69.

For $x=0$, $f(x)=-7$.

For negative values of x, sufficiently large numerically, the function $f(x)$ is negative, § 138.

3. Let $x=1$, then $f(x)=-1$.

Let $x=2$, then $f(x)=+11$.

This shows that the positive root lies between 1 and 2.

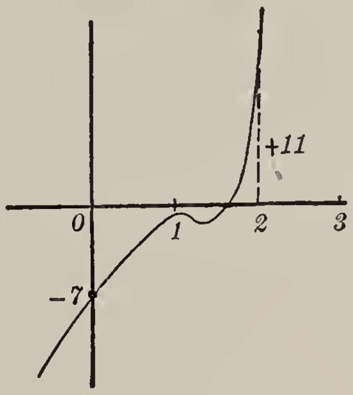

FIG. 69

2. $x^3+2x^2+8=0$

3. $x^3-3x-1=0$

4. $x^3-2x^2-3x+5=0$

5. $x^3+4x^2-2x-40=0$

6. $x^5+2x^3-5x^2+x+11=0$

7. $6x^4+29x^3-54x^2-51x-10=0$

Rational Roots

143. Integral roots. *If an equation of the form*

$$x^n+bx^{n-1}+cx^{n-2}+\dots+px+q=0$$

in which the coefficient of the highest power of x is unity, and in which all coefficients are integers, has rational roots, they are integers.

To prove this, assume that not all of the roots of the equation are integers.

Let $\frac{s}{t}$ be a fractional root where s and t have no common factor.

We may then substitute $\frac{s}{t}$ for x in the given equation. This gives

$$\frac{s^n}{t^n}+b\frac{s^{n-1}}{t^{n-1}}+c\frac{s^{n-2}}{t^{n-2}}+\ldots\ldots+p\frac{s}{t}+q=0$$

Multiplying by t^{n-1};

$$\frac{s^n}{t}+bs^{n-1}+cts^{n-2}+\ldots\ldots+pt^{n-2}s+qt^{n-1}=0$$

$$\therefore \quad \frac{s^n}{t}=-(bs^{n-1}+cts^{n-2}+\ldots\ldots+pt^{n-2}s+qt^{n-1})$$

Since the right member of this equation is an integer, $\frac{s^n}{t}$ must be integral.

This is impossible, since s does not contain t as a factor.

$\therefore$ the assumption that $\frac{s}{t}$ is a fractional root is wrong, and all the roots of the equation are integers.

144. Theorem. *If the equation* $f(x)=0$, § 143, *has an integral root* s, *then* s *is a divisor of* q.

Since $f(s)=0$, we have

$$s^n+bs^{n-1}+cs^{n-2}+\ldots\ldots+ps=-q$$

Dividing by s,

$$s^{n-1}+bs^{n-2}+cs^{n-3}+\ldots\ldots+p=-\frac{q}{s}$$

Since the left member is integral, q must be divisible by s, which was to be proved.

145. Finding the integral roots of $f(x)=0$. We may now find the *integral* roots of a given equation by trial, using only the integral factors of the constant term,

§ 144. As soon as a root is found, the equation may be *depressed* to one of a degree lower by 1, by dividing it synthetically by that root.

We may then proceed in the same way to find an integral root of the depressed equation.

EXERCISES

Solve the following equations:

1. $x^4-12x^3+48x^2-68x+15=0$

By Descartes' rule of signs the equation has *no negative* roots. Why?

The integral roots are divisors of 15, § 144.

Hence we may try the numbers 1, 3, 5, and 15.

By synthetic division,

$$\begin{array}{rrrrrl} 1 & -12 & +48 & -68 & +15 & \underline{|1} \\ & 1 & -11 & 37 & -31 & \\ \hline 1 & -11 & 37 & -31 & \boxed{-16} & \end{array} \qquad \therefore\ 1 \text{ is not a root of } f(x)=0$$

$$\begin{array}{rrrrrl} 1 & -12 & +48 & -68 & +15 & \underline{|3} \\ & 3 & -27 & 63 & -15 & \\ \hline 1 & -9 & +21 & -5 & \boxed{0} & \end{array} \qquad \therefore\ 3 \text{ is a root of } f(x)=0$$

Dividing by 5,
$$\begin{array}{rrrr} & 5 & -20 & +5 \\ \hline 1 & -4 & 1 & \boxed{0} \end{array} \qquad \therefore\ 5 \text{ is a root of } f(x)=0$$

The equation has now been depressed to the quadratic equation

$$x^2-4x+1=0$$

which may be solved by means of the quadratic formula.

2. $x^3-5x^2+2x+8=0$

3. $x^3-9x^2+23x-15=0$

4. $x^3+x^2-17x+15=0$

5. $x^3-6x^2+7x+4=0$

6. $x^4-10x^2+9=0$

7. $x^4-20x^2-21x-20=0$

8. Two roots of the equation $6x^4+7x^3-37x^2-8x+12=0$ are 2 and .5. Find the others.

146. Rational fractional roots. The problem of finding the fractional roots of $f(x)=0$ is reduced to that of finding integral roots by transforming the given equation into one whose roots are integers and multiples of the roots of the given equation. The following section will show that it is possible to multiply the roots of an equation by a given number without first finding the roots themselves.

147. To multiply the roots of an equation by *m*. Let

$$f(x)=ax^n+bx^{n-1}+cx^{n-2}+\dots+px+q=0$$

Let y be a root of the transformed equation and equal to m times the corresponding root of the given equation, i.e., $y=mx$.

Since $x=\frac{y}{m}$, we may substitute $\frac{y}{m}$ for x in $f(x)$ and thus eliminate x.

This gives

$$a\frac{y^n}{m^n}+b\frac{y^{n-1}}{m^{n-1}}+c\frac{y^{n-2}}{m^{n-2}}+\dots+p\frac{y}{m}+q=0$$

Multiplying by m^n,

$$ay^n+mby^{n-1}+m^2cy^{n-2}+\dots+m^{n-1}py+m^nq=0$$

Notice that *this equation is of the same form as the given equation and may be obtained from it by multiplying the coefficient of the second term by m, that of the third term by m², etc.*

EXERCISES

Multiply the roots of the following equations as indicated:

1. $x^3-\frac{1}{2}x^2+\frac{1}{4}x-3=0$, by 2

Multiplying the second term by 2, the third by 2^2, and the fourth by 2^3, we have the equation

$$y^3-y^2+y-24=0$$

the roots of which are 2 times the corresponding roots of the given equation.

2. $x^3+3x^2-4x+1=0$, by -2

3. $x^3-\frac{1}{5}x^2-\frac{1}{25}x+\frac{2}{5}=0$, by 5

4. $x^4-\frac{x^3}{3}+\frac{x^2}{4}+2x-\frac{1}{16}=0$, by 6

5. $x^4-2x^3+13x^2+.2x-.05=0$, by 10

Solve the following equations:

6. $3x^3+8x^2+x-2=0$

Dividing by 3,

$$x^3+\tfrac{8}{3}x^2+\frac{x}{3}-\tfrac{2}{3}=0$$

Multiplying the roots by 3,

$$y^3+8y^2+3y-18=0$$

Thus the original equation has been transformed to an equation which has no fractional root, and which may be solved as the equation in exercise 1, § 145. By dividing the roots of this equation by 3 we obtain the roots of the given equation.

7. $4x^3-16x^2-9x+36=0$

8. $8x^3-4x^2-2x+1=0$

9. $2x^3+3x^2+5x+2=0$

Irrational Roots

148. Geometrical interpretation of the process of finding irrational roots. The irrational roots of an equation may be found by certain processes of approximation, one of which is explained in detail in § 150.

The following discussion illustrates the geometrical meaning of this process.

Suppose $x_1=1.342\ldots.$ to be an irrational root of the equation $f(x)=0$.

1. The integral part of this irrational root may be found by substituting in $f(x)$ for x integral numbers as

1, 2, 3, 4, etc. For, if the values of the function $f(x)$ corresponding to two successive values of x have opposite signs, there is at least one root of the equation between them.

The graph of $f(x)$ in Fig. 70 shows that the equation $f(x)=0$ has one root between 1 and 2.

2. Expressing this root as a decimal fraction, we have

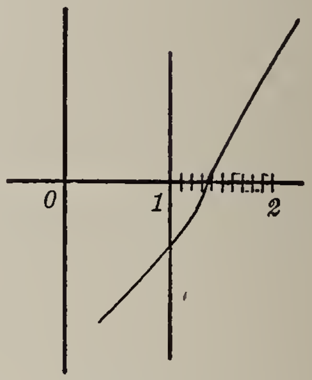

FIG. 70

$$x_1=1+\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\ldots$$

a, b, c, etc., representing the decimal figures. In Fig. 70 a is equal to 3.

3. To find a, the origin is moved to the point $x=1$.

Algebraically this is done by denoting $x-1$ by the number y. It then follows that $x=y+1$.

Substituting for x its equal $y+1$ in the equation $f(x)=0$, we have an equation in y which may be denoted by $g(y)=0$. This equation has a root

$$y_1=\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\ldots$$

corresponding to the root

$$x_1=1+\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}\ldots \text{ of } f(x)=0$$

Multiplying the roots of the equation $g(y)=0$ by 10, we obtain the equation $h(z)=0$, which has a root

$$z_1=a+\frac{b}{10}+\frac{c}{100}\ldots$$

corresponding to y_1.

Since a is the *integral* part of an irrational root of the equation $h(z)=0$, we may find it by trial in the same way as we found the integral part of the root x_1 of $f(x)=0$.

4. The process of finding b and the required remaining decimal figures is a repetition of the process explained in 3.

Notice that the process of finding an irrational root of $f(x)=0$ consists of three steps:

a) *The integral part of the root of* $f(x)=0$ *is located by trial.*

b) *The origin is translated, or the equation* $f(x)=0$ *is transformed into the equation* $g(y)=0$ *such that* $y=x-h$, *where* h *is the integral part of the required root of* $f(x)=0$.

c) *The roots of* $g(y)=0$ *are multiplied by 10.*

We are familiar with the processes of steps a) and c). We shall now learn a process by which to make the transformation in step b), § 149. This is called the process of *diminishing the roots of an equation.*

149. To transform an equation $f(x)=0$ into another whose roots are those of $f(x)=0$ each diminished by a constant k. Let

$$f(x)=ax^n+bx^{n-1}+\dots\dots+px+q=0$$

Let $x=k+y$ be the value of a root of $f(x)=0$

Then $f(k+y)=0$

This is an equation in y, and since $y=x-k$, every root of the equation $f(k+y)=0$ is equal to a corresponding root of $f(x)=0$, diminished by k.

According to § 124,

$$f(x+h)=f(x)+\frac{h}{1}f'(x)+\frac{h^2}{1\cdot 2}f''(x)+\frac{h^3}{1\cdot 2\cdot 3}f'''(x)$$
$$+\dots\dots+\frac{h^n}{1\cdot 2\cdots n}f^{(n)}(x)$$

$$\therefore\quad f(k+y)=f(k)+\frac{y}{1}f'(k)+\frac{y^2}{1\cdot 2}f''(k)+\frac{y^3}{1\cdot 2\cdot 3}f'''(k)$$
$$+\dots\dots+\frac{y^n}{1\cdot 2\cdots n}f^{(n)}(k)=0$$

This is the transformed equation.

The following example illustrates the steps in the process of obtaining the equation $f(k+y)=0$.

Let $$f(x)=Ax^4+Bx^3+Cx^2+Dx+E=0$$

Then $$f'(x)=4Ax^3+3Bx^2+2Cx+D$$

$$f''(x)=12Ax^2+6Bx+2C$$

$$f'''(x)=24Ax+6B$$

$$f^{(\text{iv})}(x)=24A$$

Hence the coefficients of y in the equation $f(k+y)=0$ are

$$f(k)=Ak^4+Bk^3+Ck^2+Dk+E$$

$$f'(k)=4Ak^3+3Bk^2+2Ck+D$$

$$\frac{f''(k)}{1\cdot 2}=6Ak^2+3Bk+C$$

$$\frac{f'''(k)}{1\cdot 2\cdot 3}=4Ak+B$$

$$\frac{f^{(\text{iv})}(k)}{1\cdot 2\cdot 3\cdot 4}=A$$

We could now substitute these values in the equation

$$f(k+y)=f(k)+yf'(k)+\ldots.$$

and thus get the desired transformed equation.

However, these coefficients may be obtained in a more simple way, as may be seen from the following example. It shows that it is necessary only to write the coefficients of x of $f(x)=0$ and then to divide synthetically by k. The successive remainders are the required coefficients of y in the equation $f(k+y)=0$.

$$
\begin{array}{lllll|l}
A & +B & +C & +D & +E & \underline{k} \\
 & +Ak & +Ak^2+Bk & +Ak^3+Bk^2+Ck & +Ak^4+Bk^3+Ck^2+Dk & \\
\hline
A & Ak+B & +Ak^2+Bk+C & +Ak^3+Bk^2+Ck+D & +\mathbf{Ak^4+Bk^3+Ck^2+Dk+E}\,[=f(k)] & \\
 & Ak & 2Ak^2+Bk & +3Ak^3+2Bk^2+Ck & & \\
\cline{1-4}
A & +2Ak+B & +3Ak^2+2Bk+C & +\mathbf{4Ak^3+3Bk^2+2Ck+D}\,[=f'(k)] & & \\
 & Ak & 3Ak^2+Bk & & & \\
\cline{1-3}
A & +3Ak+B & +\mathbf{6Ak^2+3Bk+C}\left[=\dfrac{f''(k)}{1\cdot 2}\right] & & & \\
 & Ak & & & & \\
\cline{1-2}
A & +\mathbf{4Ak+B}\left[=\dfrac{f'''(k)}{1\cdot 2\cdot 3}\right] & & & & \\
\mathbf{A}\left[=\dfrac{f^{(\text{iv})}(k)}{1\cdot 2\cdot 3\cdot 4}\right] & & & & &
\end{array}
$$

The same process may easily be verified for equations of degree other than the fourth.

To *increase* the roots by a, *diminish* them by $-a$.

EXERCISES

Form equations whose roots are the roots of the following equations diminished by the number indicated in parentheses:

1. $5x^4-4x^3+3x^2+4x-5=0$ (2)

$$
\begin{array}{rrrrr|l}
5 & -4 & +3 & +4 & -5 & \underline{2} \\
 & 10 & 12 & 30 & 68 & \\
\hline
5 & 6 & 15 & 34 & \mathbf{63} & \\
 & 10 & 32 & 94 & & \\
\cline{1-4}
5 & 16 & 47 & \mathbf{128} & & \\
 & 10 & 52 & & & \\
\cline{1-3}
5 & 26 & \mathbf{99} & & & \\
 & 10 & & & & \\
\cline{1-2}
\mathbf{5} & \mathbf{36} & & & &
\end{array}
$$

$\therefore$ $5y^4+36y^3+99y^2+128y+63=0$ is the required equation.

2. $x^2-5x+6=0$ (3)

3. $x^3-6x^2+8x+10=0$ (2)

4. $x^3-4x^2+6x+2=0$ (3)

5. $x^3-4x^2-3x-29=0$ (-6)

6. $2x^4-3x^2+2x+5=0$ (-1)

150. Horner's method of finding positive irrational roots of $f(x)=0$.* This method is explained by means of the following example:

The equation

$$f(x)=5x^3-3x^2-6x+3=0$$

has one positive root between 1 and 2. Find the value of this root correct to two decimal places.

Let $x_1=1+\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\dots.$

1. Diminish the roots of $f(x)=0$ by 1 as follows:

$$\begin{array}{rrrr|l}
5 & -3 & -6 & +3 & \underline{1} \\
 & 5 & 2 & -4 & \\ \hline
5 & 2 & -4 & \mathbf{-1} & \\
 & 5 & 7 & & \\ \cline{1-3}
5 & 7 & \mathbf{3} & & \\
 & 5 & & & \\ \cline{1-2}
\mathbf{5} & \mathbf{12} & & &
\end{array}$$

, which gives the equation

$$g(y)=5y^3+12y^2+3y-1=0,$$

which has a root

$$y_1=\frac{a}{10}+\frac{b}{100}+\frac{c}{1000}+\dots.$$

corresponding to the root x_1 of the equation $f(x)=0$.

2. Since y_1 is a decimal fraction, the terms $5y^3$ and $12y^2$ will usually be considerably smaller than the term $3y$. Hence, to get an *approximate* value for a we may neglect the first two terms and solve the resulting equation $3y-1=0$. This gives $y=.3$ approximately, and suggests that $a=3$ approximately.

---

* William George Horner (1786–1837), an English algebraist, published his method in the *Philosophical Transactions* in 1819.

To determine the *exact* value of a, find the value of $g(y)$ for $y=.3$. If $g(.3)$ is negative, find $g(.4)$; if this is negative, find $g(.5)$, etc., until $g(y)$ changes sign and becomes positive. If, however, $g(.3)$ is positive, find $g(.2)$. If this is negative, $a=2$, but if $g(.2)$ is positive, find $g(.1)$, etc., until $g(y)$ becomes positive.

3. The work of finding the value of $g(y)$, when y is a decimal fraction, may be simplified by multiplying the roots of $g(y)=0$ by 10. This gives the equation

$$h(z)=5z^3+120z^2+300z-1000=0$$

4. Dividing synthetically by 2, we have

$$\begin{array}{rrrrl} 5 & +120 & +300 & -1000 & \underline{|2} \\ & 10 & +260 & +1120 & \\ \hline 5 & +130 & +560 & +\boxed{120} & =h(2) \end{array}$$

showing that $h(z)$ is positive for $z=2$.

Dividing synthetically by 1, we have

$$\begin{array}{rrrrl} 5 & +120 & +300 & -1000 & \underline{|1} \\ & 5 & +125 & +\ 425 & \\ \hline 5 & +125 & +425 & \boxed{-575} & =h(1) \end{array}$$

showing that $h(z)$ is negative for $z=1$.

Since the function is negative for 1 and positive for 2 it follows that $a=1$.

1′. Diminish the roots of $h(z)=0$ by 1.

$$\begin{array}{rrrrl} 5 & +\ 120 & 300 & -1000 & \underline{|1} \\ & 5 & 125 & +\ 425 & \\ \hline 5 & 125 & 425 & -\ \mathbf{575} & \\ & 5 & 130 & & \\ \cline{1-3} 5 & 130 & \mathbf{555} & & \\ & 5 & & & \\ \cline{1-2} \mathbf{5} & \mathbf{135} & & & \end{array}$$

This gives the transformed equation

$$k(u)=5u^3+135u^2+555u-575=0$$

2′. To find an approximate value for b, solve the equation

$$555u=575$$

which gives $u=1$, approximately.

Since u is a decimal fraction, we may find the values of $k(.9)$, $k(.8)$, etc., until the function becomes negative.

3′. Multiplying the roots of the equation $k(u)=0$ by 10, we have

$$5t^3+1350t^2+55500t-575000=0$$

4′. For $t=9$ this fraction is positive. For $t=8$ it is negative.

Hence $b=8$, and $x_1=1.18$, approximately.

The preceding process may now be arranged in the following condensed form:

$$f(x)=5x^3-3x^2-6x+3=0$$

$$\begin{array}{rrrrl}
5 & -3 & -6 & +3 & \underline{|1} \\
 & 5 & 2 & -4 & \\
\cline{1-4}
5 & 2 & -4 & -1 & \\
 & 5 & 7 & & \\
\cline{1-3}
5 & 7 & 3 & & \\
 & 5 & & & \\
\cline{1-2}
\mathbf{5} & \mathbf{12} & & & \\
5 & 120 & 300 & -1000 & \underline{|1} \\
 & 5 & 125 & 425 & \\
\cline{1-4}
5 & 125 & 425 & -\mathbf{575} & \\
 & 5 & 130 & & \\
\cline{1-3}
5 & 130 & \mathbf{555} & & \\
 & 5 & & & \\
\cline{1-2}
\mathbf{5} & \mathbf{135} & & & \\
5 & 1350 & 55500 & -575000 & \underline{|8} \\
 & 40 & 11120 & 532960 & \\
\cline{1-4}
5 & 1390 & 66620 & -42040 &
\end{array}$$

151. Negative irrational roots. *Negative* roots of $f(x)=0$ correspond to the *positive* roots of $f(-x)=0$. Hence, to find the *negative* irrational roots of $f(x)=0$, transform the equation into $f(-x)=0$ and then find the *positive* irrational roots of this equation. These roots with their signs changed are the required roots.

152. Summary. The following is a summary of the steps involved in finding the *real* roots of $f(x)=0$, $f(x)$ being a rational integral function of x.

1. Find all the *integral* roots, §§ 143 to 145.
2. Find the *fractional* roots, §§ 146, 147.
3. Determine two consecutive integers between which an *irrational* root lies and find the approximate value of such a root by Horner's method, § 150. All other irrational roots may be found in the same way.

EXERCISES

Find the real roots of the following equations, correct to two decimal places:

1. $x^3+3x^2-2x-5=0$

2. $x^3+x^2-4x-2=0$

3. $2x^3+5x-100=0$

4. $x^3+x^2-2x-1=0$

5. $x^4-4x-2=0$

6. $16x^4+16x^2-15=0$

7. $x^4-11x^3+38x^2=51x-27$

8. $x^4-3x^3+3=0$

9. $8x^4+24x^3-x-3=0$

10. $2x^5-4x-3x^2+6=0$

11. $2y^4-9y^3+11y^2+7y-15=0$

12. $x^5+3x^4-15x^3-35x^2+54x+72=0$

Solve the following problems:

13. Find the three cube roots of unity.

14. Find the real cube root of 67.

15. The volume of a spherical segment of one base is given by the formula

$$V=\pi rh^2-\frac{\pi}{3}h^3$$

A hollow sphere of radius 10 inches is partly filled with a gallon of water. What is the depth of the water to three significant figures?

Let $\pi = 3.14$, approximately, and let one gallon $= 231$ cubic inches.

16. Find an approximate solution, correct to two decimal places, of the system $y = x^3$, $2x + y = 2$.

Relation between Roots and Coefficients

153. Identity. According to § 131 every rational integral equation of nth degree whose coefficients are real or complex numbers has n, and only n, roots; for such a function can always be transformed into the form

$$f(x) = A(x - r_1)(x - r_2) \dots (x - r_n)$$

If, however, A should be equal to zero, all coefficients of x in $f(x)$ would be zero, and *any* value of x would satisfy the equation $f(x) = 0$. Furthermore, $f(x)$ can become zero for values of x different from r_1, $r_2 \dots r_n$, only if $A = 0$.

Hence, *if the function* $f(x) = ax^n + bx^{n-1} + cx^{n-2} + \dots + px + q$ *is equal to zero for more than n values of x, it follows that*

$$a = b = c = \dots = p = q = 0$$

If $f(x) = 0$ for *all* values of x, the equation is an **identity.**

154. Theorem: *If*

$$ax^n + bx^{n-1} + cx^{n-2} + \dots + px + q \equiv a_1x^n + b_1x^{n-1} + c_1x^{n-2} + \dots + p_1x + q_1$$

then

$$a = a_1,\ b = b_1,\ c = c_1,\ \dots,\ p = p_1,\ q = q_1$$

For, changing the equation to the form $f(x) \equiv 0$, we have

$$(a-a_1)x^n+(b-b_1)x^{n-1}+(c-c_1)x^{n-2}+\ldots\ldots+(p-p_1)x$$
$$+(q-q_1) \equiv 0$$

Therefore

$$a-a_1=b-b_1=c-c_1=\ldots\ldots=p-p_1=q-q_1=0, \text{ § 153,}$$

and $a=a_1$, $b=b_1$, $c=c_1$, etc.

155. Relations between roots and coefficients. Let $r_1, r_2, \ldots\ldots, r_n$ be the roots of the equation

$$f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=0$$

Then

$$f(x) \equiv (x-r_1)(x-r_2)\ldots\ldots(x-r_n)$$

Multiplying as indicated,

$$\begin{aligned} f(x) \equiv x^n & \\ & -x^{n-1}(r_1+r_2+r_3+\ldots\ldots+r_n) \\ & +x^{n-2}(r_1r_2+r_1r_3+\ldots\ldots+r_2r_3+r_2r_4 \\ & \qquad\qquad +\ldots\ldots\ldots+r_{n-1}r_n) \\ & +x^{n-3}\ (r_1r_2r_3+r_1r_2r_4+\ldots\ldots r_{n-2}r_{n-1}r_n) \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & +(-1)^n(r_1r_2r_3\cdots\cdots r_n) \end{aligned}$$

According to § 154, we have

$$\begin{aligned} p_1 &= -(r_1+r_2+\ldots\ldots+r_n) \\ p_2 &= +(r_1r_2+\ldots\ldots+r_2r_3+\ldots\ldots+r_{n-1}r_n) \\ p_3 &= -(r_1r_2r_3+\ldots\ldots+r_{n-2}r_{n-1}r_n) \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ p_n &= (-1)^n(r_1r_2\ldots\ldots r_n) \end{aligned}$$

Hence, *if the equation $f(x)=0$ is reduced to form*

$$x^n+p_1x^{n-1}+\ldots\ldots+p_{n-1}x+p_n=0,$$

the coefficient of x^{n-1} *is equal to the sum of the roots with the sign changed; the coefficient of* x^{n-k} *is equal to the sum of all possible products taking k roots at the time, with the signs changed when k is odd; the constant term is equal to the product of the roots with the sign changed when n is odd.*

EXERCISES

Form the equations whose roots are:

1. 1, 2, 3

$$f(x) = x^3 - (1+2+3)x^2 + (1\cdot2+1\cdot3+2\cdot3)x - (1\cdot2\cdot3) = 0$$
$$= x^3 - 6x^2 + 11x - 6 = 0$$

2. $1, 3, -4$

3. $2, -1, 3, 0$

4. $\pm1, \pm2$

5. $\frac{1}{3}, \frac{1}{2}, \frac{5}{6}$

6. $1 \pm \sqrt{5}, 2 \pm \sqrt{3}$

Using the first two roots, we have the function x^2-2x-4; the last two roots give the function x^2-4x+1.

$\therefore f(x) = (x^2-2x-4)(x^2-4x+1) = 0$

7. $4 \pm \sqrt{3}, -1 \pm \sqrt{5}$

8. $2 \pm \sqrt{-3}, -3 \pm \sqrt{-2}$

9. $\dfrac{1 \pm i\sqrt{3}}{2}, \dfrac{-1 \pm i\sqrt{3}}{2}$

10. Two of the roots are $2 - \sqrt{3}$ and $\sqrt{5}$

11. The roots of an equation are $+2, -3, 5, 4, -1$. Write the second, fifth, and sixth terms.

Solve the following equations:

12. $y^3 - 6y^2 - 4y + 24 = 0$, the roots being in arithmetical progression.

Denote the roots by $a-d$, a, $a+d$

Then $-(a-d+a+a+d) = -6$

$$\therefore 3a = 6$$
$$a = 2$$
$$-(a-d)(a)(a+d) = -24$$
$$a(a^2-d^2) = -24$$
$$4 - d^2 = -12$$
$$d^2 = 16$$
$$d = \pm4$$

$\therefore$ The roots are $-2, 2, 6$

13. $x^3-\frac{11}{2}x^2+6x+\frac{9}{2}=0$, two roots being equal

14. $y^3-8y^2+5y+14=0$, the sum of two roots being equal to 9.

15. $y^3-\frac{9}{2}y^2-\frac{27}{2}y+27=0$, the roots being in geometrical progression.

16. One root of $4x^4-14x^3+16x^2-9x+2=0$ is an integer; a second is the reciprocal of the first. Find all the roots.

17. The roots of the equation $x^3-6x^2+kx+10=0$ are in arithmetical progression. Solve the equation and find the value of k.

18. In the equation $x^4+x^3+3x^2+4x+6=0$ the sum of two roots is -2 and the product of the other two is 3. Solve the equation.

Summary

156. The chapter has taught the meaning of the following terms:

algebraic solution
multiple roots
complex roots
depressed equation
transformed equation
variation of signs

157. It was shown graphically that if a factor $x-r$ occurs an *even* number of times in $f(x)$, the curve does *not* cross the x-axis; if it occurs an *odd* number of times, the curve *crosses* the x-axis.

A complex root cannot cause $f(x)$ to become zero nor the graph to touch or cut the x-axis.

158. An equation may be transformed:

1. Into another whose roots are equal to the roots of the given equation, multiplied by a constant.

2. Into another whose roots are equal to the roots of the given equation, diminished by a constant.

159. The following theorems have been taught:

1. *Every rational integral equation of the nth degree whose coefficients are real or complex numbers has at least one root, real or complex.*

2. *Every rational integral function of x is the product of n linear factors.*

3. *Every rational integral equation of the nth degree whose coefficients are real or complex numbers has n, and only n, roots, real or complex* (fundamental theorem of algebra).

4. *If a rational integral equation with real coefficients has a complex root $a+bi$, then $a-bi$ is also a root.*

5. *The number of positive roots of $f(x)=0$ cannot be greater than the number of variations of sign in $f(x)$, the number of negative roots cannot be greater than the number of variations in $f(-x)$* (Descartes' rule of signs).

6. *The first term in a rational integral function of x can be made to exceed the sum of the remaining terms by taking a sufficiently large value of x.*

7. *If $f(a)$ and $f(b)$ have opposite signs, an odd number of real roots lies between a and b.*

8. *If $f(a)$ and $f(b)$ have like signs, either no real root, or an even number of roots, lies between a and b.*

9. *Every equation of odd degree has at least one real root.*

10. *If a rational integral equation with integral coefficients, in which the coefficient of the highest power of x is unity, has rational roots, they are integers; if it has an integral root, this root is a divisor of the constant term; the roots may be expressed in terms of the coefficients.*

160. To find the roots of a rational integral equation of nth degree in one unknown we may proceed as follows:

1. By means of Descartes' rule obtain some information regarding the positive, negative, or complex roots.

2. Locate the roots approximately by making a table of corresponding values of x and $f(x)$.

3. If the equation has integral roots, depress the equation to one of lower degree.

4. Find the fractional roots by transforming the depressed equation into one whose roots are integers.

5. Approximate the irrational roots by Horner's method.

CHAPTER VIII

ALGEBRAIC SOLUTION OF THE GENERAL CUBIC AND BIQUADRATIC EQUATIONS

The Cubic Equation

161. To transform an equation into another in which some particular term is missing. The first step in the solution of a cubic equation, described in § 162, is to eliminate one of its terms. This may be done as follows:

Let $f(x)$ be any rational integral equation, e.g., let

$$f(x)=ax^3+bx^2+cx+d=0$$

Let $x=y+k$,

Then by § 3, we have

$$f(y+k)=f(y)+\frac{k}{1}f'(y)+\frac{k^2}{1\cdot 2}f''(y)+\ldots\ldots=0$$

According to § 124, the calculation of $f(y+k)$ may be arranged in the following manner:

$$\begin{array}{llll}
a & b & c & d \\
 & ak & ak^2+bk & ak^3+bk^2+ck \\
\hline
a & ak+b & ak^2+bk+c & \boldsymbol{ak^3+bk^2+ck+d} \\
 & ak & 2ak^2+bk & \\
\cline{1-3}
a & 2ak+b & \boldsymbol{3ak^2+2bk+c} & \\
 & ak & & \\
\cline{1-2}
\boldsymbol{a} & \boldsymbol{3ak+b} & &
\end{array}$$

$$\therefore\quad f(y+k)=$$
$$ay^3+(3ak+b)y^2+(3ak^2+2bk+c)y+(ak^3+bk^2+ck+d)=0$$

By equating to zero the coefficient of any particular term and solving the resulting equation for k, a value of k may be determined which will make that term vanish.

Whatever may be the degree of $f(x)$, the same method may be used to transform $f(x)$ into another function in which some particular term is missing.

162. Solution of the cubic. Let the equation

$$ax^3+bx^2+cx+d=0 \qquad (1)$$

be the *general cubic equation.*

Transform this equation by substituting $y+k$ for x. In the transformed equation let $k=-\frac{b}{3a}$.

Then, by § 161, we have

$$ay^3+\left(c-\frac{b^2}{3a}\right)y+\left(d-\frac{cb}{3a}+\frac{2b^3}{27a^2}\right)=0 \qquad (2)$$

Divide both members of this equation by a. Then denote the coefficient of y by p and the constant term by q. This gives the equation

$$y^3+py+q=0 \qquad (3)$$

where

$$p=\frac{3ac-b^2}{3a^2}, \text{ and } q=\frac{d}{a}-\frac{cb}{3a^2}+\frac{2b^3}{27a^3} \qquad (4)$$

The form of the function y^3+py+q may be changed to $y(y^2+p)+q$. This function vanishes if

$$y(y^2+p)=-q \qquad (5)$$

To find a solution of this equation, let

$$p=-3uv \text{ and } -q=u^3+v^3$$

Then equation (5) takes the form

$$y(y^2-3uv)=u^3+v^3 \qquad (6)$$

It is easily seen that $y=u+v$ satisfies equation (6), and therefore equation (3).

We must next express u and v in terms of p and q. Then the sum $u+v$ is the required solution of equation (3).

From the equation

$$-3uv=p$$

it follows that

$$v=-\frac{p}{3u}$$

Substituting this value for v in the equation

$$u^3+v^3=-q$$

we have

$$u^3-\frac{p^3}{27u^3}+q=0$$

or

$$27u^6+27qu^3-p^3=0$$

$$\therefore \quad u^3=-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$$

$$\therefore \quad u=\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

and

$$v=-\frac{p}{3u}=\frac{-p}{3\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}$$

Multiplying numerator and denominator by

$$\sqrt[3]{-\frac{q}{2}\mp\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

$$v=\frac{-p\sqrt[3]{-\frac{q}{2}\mp\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\sqrt[3]{\left[-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right]\left[-\frac{q}{2}\mp\sqrt[3]{\frac{q^2}{4}+\frac{p^3}{27}}\right]}}$$

$$=\frac{-p\sqrt[3]{-\frac{q}{2}\mp\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\sqrt[3]{-\frac{p^3}{27}}}=\frac{-p\sqrt[3]{-\frac{q}{2}\mp\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}}{3\cdot-\frac{p}{3}}$$

or

$$v=\sqrt[3]{-\frac{q}{2}\mp\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

$$\therefore\quad y=u+v=\sqrt[3]{-\frac{q}{2}\pm\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}\mp\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

Since this sum is the same whether the upper or lower signs before the radical sign be taken, we may choose

$$y=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}+\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

Let u_1 be one of the three cube roots of $-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$.

Denoting by v_1 the value of v which satisfies the equation

$$-3uv=p$$

one solution of equation (3) will be given by

$$y=u_1+v_1$$

Moreover, if u_1 is one of the three cube roots of $-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}$, the other two are $u_2=\omega u_1$ and $u_3=\omega^2 u_1$, where ω and ω^2 are the complex cube roots of unity, $-\frac{1}{2}(1+i\sqrt{3})$ and $-\frac{1}{2}(1-i\sqrt{3})$.

Hence the corresponding values of v are

$$v_2 = -\frac{p}{3u_2} = -\frac{p}{3\omega u_1} = +\frac{v_1}{\omega} = +\frac{\omega^2 v_1}{\omega^3} = \omega^2 v_1$$

and

$$v_3 = -\frac{p}{3u_3} = -\frac{p}{3\omega^2 u_1} = \frac{v_1}{\omega^2} = \frac{\omega v_1}{\omega^3} = \omega v_1$$

Therefore the three roots of the equation

$$y^3 + py + q = 0$$

are

$$y_1 = u_1 + v_1$$

$$y_2 = \omega u_1 + \omega^2 v_1$$

$$y_3 = \omega^2 u_1 + \omega v_1$$

where

$$u_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} \text{ and } v_1 = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

the cube roots being selected so that the relation

$$-3u_1 v_1 = p$$

may be satisfied.

Historical note. These formulas are known as **Cardan's formulas**. They were discovered by the Italian mathematicians Tartaglia and Ferro and were first published by Cardan (1501–76) in the treatise known as *Ars Magna* (1545).

The story of the solution of the cubic is interesting. In an algebra published about the middle of the eleventh century the Arab Omar Alkayyami gave a classification of cubics, together with methods of solution by geometrical construction. No attempt was made at an algebraic solution.

In his *Liber abaci* of 1202 Leonardo of Pisa introduced the study of algebra into Italy from Arabian sources. The Italians were long the chief cultivators of algebra. In a book called the

Summa, in 1494, Pacioli gave the Arabic classification, stated that in the existing state of science the cubic could not be solved, but directed the attention of mathematicians to the solution of the cubic as the next need in the development of the science. Mathematicians continued working on the problem. An anonymous tract from learned Italian circles of the fourteenth century is witness of these struggles. The French author Chuquet as late as 1484 conceded the solution of the cubic to be impossible, but he did not despair that it would yet be found.

Finally, the sixteenth century brought the light. Scipio del Ferro, about whom we know almost nothing except that he was a professor at Bologna from 1496 to 1526, was the first to find an algebraic solution of the form $x^3+ax=b$. Both of the later contestants for the honor of priority of discovery, Tartaglia and Cardan, agree to this. Ferro may have found his solution in an Arab work. At all events he told his discovery to numerous acquaintances, among them a young friend Fiori (or Florido), who was not a mathematician. Fiori got this knowledge according to Tartaglia in 1506 and according to Cardan in 1515.

In 1530 Colla proposed to Tartaglia a problem which depended on a cubic of the form $x^3+px^2=q$. Tartaglia's attention was thus directed to the problem of the cubic, and he wrote to Colla that he could solve *numerical* equations of the form $x^3+px^2=q$. Fiori, believing Tartaglia an impostor, proclaimed his own knowledge of the solution of the form $x^3+px=q$, and in accordance with the custom of the time challenged Tartaglia to a contest. Each contestant was to propose 30 problems to the other and to deposit a stake with a notary. The one who solved the larger number of these competitive problems in 30 days was to have the stake.

Tartaglia, suspecting that Fiori's problems would lead to the form $x^3+px=q$, bestirred himself to find a solution of this form of the cubic. He says that he found the solution-formula 8 days before the contest, which began February 20, 1535. The next day (February 13) he found the formula for $x^3=ax+b$. Tartaglia solved all his problems in less than two hours, and Fiori could not solve any of his, all of which led to the form $x^3+px^2=q$. Tartaglia was declared the victor, and afterward composed some verses to commemorate his victory.

Fiori and others besought Tartaglia to make known his method, but Tartaglia persistently refused. In a letter of February 12, 1539, Cardan, then a most significant mathematician, urged him to divulge his solution. At first Tartaglia refused. Later Cardan, after repeated entreaty, and under the most solemn oath of secrecy, secured enough fragmentary hints from Tartaglia to enable him, good mathematician that he was, to construct the solution. Cardan broke his pledge, when he included it in his great *Ars Magna* of 1545. This being the earliest publication of the method, and the facts in the case being then unknown, the solution has until recent years been known as Cardan's solution. Tartaglia was reserving the solution as the "crowning glory" of his own work, which was published the following year, 1546. See Tropfke, Band I, S. 274-77.

163. Discussion of the roots. The nature of the roots of the equation

$$y^3+py^2+q=0$$

depends upon the value of

$$\frac{q^2}{4}+\frac{p^3}{27}$$

This is called the **discriminant** of the cubic equation. Let p and q be real numbers.

If $\frac{q^2}{4}+\frac{p^3}{27}>0$, one root is real and two are conjugate complex.

If $\frac{q^2}{4}+\frac{p^3}{27}=0$, all roots are real and two are equal. For

$$u_1=v_1 \text{ and } y_1=2u_1=-2\sqrt[3]{\frac{q}{2}},\ y_2=-(\omega+\omega^2)u_1,\ y_3=-(\omega^2+\omega)u_1$$

But $\omega^2+\omega+1=0$. $\therefore\ y_2=y_3=u_1$

If $\frac{q^2}{4}+\frac{p^3}{27}<0$, it may be shown that all roots are real.

Although the solution of the cubic equation given in § 162 is of theoretical value, in solving numerical equations the method is not practical, and the methods of chapter vii are to be used.

EXERCISES

Solve the following equations by means of the formulas in § 162:

1. $x^3-3x^2+6x-4=0$

$a=1,\ b=-3,\ c=6,\ d=-4=0$

$k=1,\ p=3,\ q=0$

$\therefore\ u_1=1,\ v_1=-1$

$\therefore\ y_1=0,\ y_2=\omega-\omega^2=i\sqrt{3},\ y_3=\omega^2-\omega=-i\sqrt{3}$

$\therefore\ x_1=1,\ x_2=1+i\sqrt{3},\ x_3=1-i\sqrt{3}$

2. $x^3-7x+6=0$

3. $x^3-4x^2-3x+12=0$

4. $6x^3-7x^2-14x+15=0$

The Biquadratic Equation

164. Solution of the biquadratic. Every equation of the form

$$Ax^4+Bx^3+Cx^2+Dx+E=0 \tag{1}$$

may be transformed into one in which the term in x^3 is missing. Hence it is sufficient to be able to solve the equation

$$f(y)=y^4+py^2+qy+r=0 \tag{2}$$

which contains no term in y^3.

As in § 162, let

$$y=u+v+w \tag{3}$$

Squaring,

$$y^2 = u^2 + v^2 + w^2 + 2(uv + uw + vw),$$

or

$$y^2 - (u^2 + v^2 + w^2) = 2(uv + uw + vw)$$

Squaring again,

$$\begin{aligned} y^4 - 2(u^2+v^2+w^2)y^2 + (u^2+v^2+w^2)^2 &= 4(u^2v^2+u^2w^2+v^2w^2) \\ &\quad +8(u^2vw+uv^2w+uvw^2) \end{aligned}$$

or

$$\begin{aligned} y^4 - 2(u^2+v^2+w^2)y^2 - 8uvw(u+v+w) + (u^2+v^2+w^2)^2 \\ -4(u^2v^2+u^2w^2+v^2w^2) = 0 \end{aligned}$$

or

$$\begin{aligned} y^4 - 2(u^2+v^2+w^2)y^2 - 8uvwy + (u^2+v^2+w^2)^2 - 4(u^2v^2 \\ +u^2w^2+v^2w^2) = 0 \qquad (4) \end{aligned}$$

Equating coefficients of similar terms in equations (2) and (4),

$$y^2 + v^2 + w^2 = -\frac{p}{2} \qquad (5)$$

$$uvw = -\frac{q}{8} \qquad (6)$$

and $(u^2+v^2+w^2)^2 - 4(u^2v^2+u^2w^2+v^2w^2) = r.$
The last equation may be written

$$\frac{p^2}{4} - 4(u^2v^2+u^2w^2+v^2w^2) = r$$

Hence

$$u^2v^2 + u^2w^2 + v^2w^2 = \frac{p^2-4r}{16} \qquad (7)$$

From equations (5), (7), and (6) we obtain the equations

$$\left.\begin{aligned} -(u^2+v^2+w^2) &= \frac{p}{2} \\ +(u^2v^2+u^2w^2+v^2w^2) &= \frac{p^2-4r}{16} \\ -(u^2v^2w^2) &= \frac{-q^2}{64} \end{aligned}\right\} \qquad (8)$$

It follows from § 155 that u^2, v^2, and w^2 are the roots of the equation

$$t^3+\frac{p}{2}t^2+\frac{p^2-4r}{16}t-\frac{q^2}{64}=0^* \tag{9}$$

Denoting the roots of this equation by t_1, t_2, and t_3, respectively,

$$y_1=u+v+w=\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3}$$

the signs of the radicals involved being selected to satisfy the relation

$$uvw=-\frac{q}{8}$$

The following are the combinations satisfying this condition:

$$\begin{aligned}\sqrt{t_1}\sqrt{t_2}\sqrt{t_3}=\sqrt{t_1}(-\sqrt{t_2})(-\sqrt{t_3})=(-\sqrt{t_1})\sqrt{t_2}(-\sqrt{t_3})\\ =(-\sqrt{t_1})(-\sqrt{t_2})\sqrt{t_3}\end{aligned}$$

Hence the roots of equation (2) are

$$\left.\begin{aligned}y_1&=\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3}\\ y_2&=\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3}\\ y_3&=-\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3}\\ y_4&=-\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3}\end{aligned}\right\} \tag{10}$$

The solution of the biquadratic equation in one unknown was first given by Ferrari (1522–62). Other solutions were given by Descartes and Euler.

EXERCISES

Solve the following equations:

1. $x^4+x^2+4x=3$

2. $x^4-10x^2+20x=16$

3. $x^4-9x^2-12x+10=0$

---

* Equation (9) is known as *Euler's reducing cubic.*

Summary

165. An equation may be transformed into another in which a particular term is missing. In solving cubic and biquadratic equations this transformation is used to remove the second term.

166. *The roots of the cubic equation*

$$y^3+py+q=0$$

are

$$y_1=u_1+v_1; \quad y_2=\omega u_1+\omega^2 v_1; \quad y_3=\omega^2 u_1+\omega v_1,$$

where

$$u_1=\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

and

$$v_1=\sqrt[3]{-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$$

and

$$-3u_1v_1=p$$

167. *If in the equation* $y^3+py+q=0$, *p and q are real numbers, and*

if $\frac{q^2}{4}+\frac{p^3}{27}>0$, *one root is real and two are conjugate complex;*

if $\frac{q^2}{4}+\frac{p^3}{27}=0$, *all roots are real and two are equal;*

if $\frac{q^2}{4}+\frac{p^3}{27}<0$, *all roots are real.*

168. *The roots of the biquadratic equation*

$$y^4+py^2+qy+r=0$$

are

$$y_1=\sqrt{t_1}+\sqrt{t_2}+\sqrt{t_3};\quad y_3=-\sqrt{t_1}+\sqrt{t_2}-\sqrt{t_3}$$
$$y_2=\sqrt{t_1}-\sqrt{t_2}-\sqrt{t_3};\quad y_4=-\sqrt{t_1}-\sqrt{t_2}+\sqrt{t_3}$$

where t_1, t_2, *and* t_3 *are the roots of the equation*

$$t^3+\frac{pt^2}{2}+\frac{p^2-4r}{16}t-\frac{q^2}{64}=0$$

CHAPTER IX

LIMITS

Theorems on Limits

169. Notion of limit. Illustrations of limits are frequent in elementary mathematics. In geometry we took as the area of a circle the limiting value of the area of a regular inscribed polygon of n sides as n increases without bound. In the study of the geometrical progression

$$S_n = 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots + \frac{1}{2^{n-1}} + \dots$$

we found the limiting value of S_n as n increases without bound. We defined the tangent to a curve at a given point as the limiting position of a variable secant passing through that point, § 103.

The limit of a variable x may be defined as follows:

Let δ be any positive number assigned in advance and selected as small as we please. Then the variable x is said to approach to a constant a as a **limit,** *if its law of variation is such that the numerical difference between x and a will ultimately become and remain less than δ.*

In symbols the statement *the limit of x is equal to a* is written $\lim x = a$; *x approaches a as a limit* is written $x \rightarrow a$; the *numerical difference between x and a* is indicated by $|x - a|$.

It is important to note that according to the definition $|x - a|$ must not only *become* but also *remain* less than δ. For example, in the geometrical progression

$$S_n = \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \tfrac{1}{32} + \tfrac{1}{64} + \dots$$

we may tabulate the sums S_1, S_2, etc., as follows:

S_1	S_2	S_3	S_4	S_5	S_6
$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	$\frac{63}{64}$

As n takes the successive values 1, 2, 3, and 4, $|S_n-\frac{15}{16}|$ *becomes* less than any assigned positive number δ, being equal to zero when $n=4$. But as n increases beyond 4, $|S_n-\frac{15}{16}|$ increases and does *not remain* less than any assigned value of δ, e.g., δ less than $\frac{1}{32}$, Fig. 71.

Hence S_n does *not* approach $\frac{15}{16}$ as a limit. The variable S_n in this example is an increasing variable and always *less* than the limit.

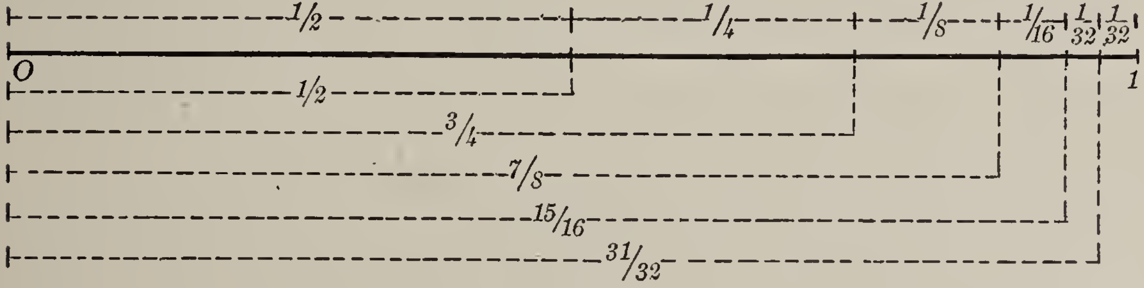

FIG. 71

However, a variable may be *greater* than its limit. For example, the area A_n of a regular circumscribed polygon having n sides decreases as n increases and is always greater than the area A of the circle as it approaches A as a limit.

Moreover, a variable may be alternately greater than and less than its limit. Such a variable is the sum of n terms of the geometrical progression 1, $-\frac{1}{2}$, $+\frac{1}{4}$, $-\frac{1}{8}$, $+\frac{1}{16}$, etc. The limit of this sum is $\frac{2}{3}$. The successive values of S_n are 1, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, etc. These sums are alternately greater than and less than the limit $\frac{2}{3}$, although $|S_n-\frac{2}{3}|$ decreases as n grows larger and larger.

It follows from the definition of a limit that *a variable can have but one limit.*

170. Infinitesimal. A variable x whose limit is zero is an **infinitesimal.** In symbols, lim $x=0$, or $x \rightarrow 0$.

Thus $x-a$ *is an infinitesimal, if* x *is a variable approaching* a *as a limit.* In symbols, $\lim\limits_{x \to a} |x-a|=0$.

Notice that an infinitesimal is a *variable,* not a *constant,* number.

171. Infinite. The sum of n terms of the series 1, 2, 4, 8, 16 ...., as n increases without bound, has no limit, as it ultimately becomes and remains *greater* than any assigned positive number, however large. In general, a *variable* which ultimately becomes and remains greater than any previously assigned positive number, however large, is said to become **infinite.** The statement *the variable* x *becomes infinite* is expressed in symbols thus: $x \rightarrow \infty$.

Show that the following two statements are immediate inferences of the preceding definitions:

If n *is a finite constant, and if* x *is an infinitesimal, then* $\frac{n}{x}$ *becomes infinite,* i.e., $\frac{n}{x} \rightarrow \infty$, as $x \rightarrow 0$.

If n *is a finite constant and if* x *becomes infinite, then* $\frac{n}{x}$ is an infinitesimal, i.e., $\lim\limits_{x \to \infty} \frac{n}{x}=0$.

172. Theorem. *If* u *and* v *be two infinitesimals, and* X *and* Y *two variables, always less than a finite positive number* k*, then* $Xu+Yv$ *is an infinitesimal.*

For, let δ be a positive number, however small, then u and v will ultimately become and remain less than δ, i.e.,

$$
\begin{array}{lll}
 & |u| < \delta \text{ and } |v| < \delta, & \S\ 170 \\
 & |X| < k \text{ and } |Y| < k, & \text{given} \\
\therefore & |Xu| < \delta k \text{ and } |Yv| < \delta k &
\end{array}
$$

Since $|Xu+Yv| \leqq |Xu|+|Yv|$

$\therefore \quad |Xu+Yv| < 2\delta k.$

Hence, if we select δ so that $2\delta k < \epsilon$, any positive number, however small, we shall have

$$|Xu+Yv| < \epsilon$$

$\therefore \quad \lim\limits_{\substack{u \to 0 \\ v \to 0}} (Xu+Yv)=0$, and $Xu+Yv$ is an infinitesimal.

Similarly the theorem may be shown to hold for any *finite* number of variables.

173. Theorem. *Let x and y be two variables approaching the limits a and b, respectively. Then the sum, difference, product, and quotient also approach limits, and*

1. $\lim\limits_{\substack{x \to a \\ y \to b}} (x+y) = \lim\limits_{x \to a} x + \lim\limits_{y \to b} y$
2. $\lim\limits_{\substack{x \to a \\ y \to b}} (x-y) = \lim\limits_{x \to a} x - \lim\limits_{y \to b} y$
3. $\lim\limits_{\substack{x \to a \\ y \to b}} (xy) = \left(\lim\limits_{x \to a} x\right)\left(\lim\limits_{y \to b} y\right)$
4. $\lim\limits_{\substack{x \to a \\ y \to b}} \left(\frac{x}{y}\right) = \frac{\lim\limits_{x \to a} x}{\lim\limits_{y \to b} y}$, if $b \neq 0$

For, $|x-a| \to 0$ and $|y-b| \to 0$, §169

$\therefore \quad X(x-a)+Y(y-b) \to 0$, § 172

i.e., $\lim\,[X(x-a)+Y(y-b)]=0$

Let $X=Y=1$

Then $\lim\,[x-a+y-b] \;=0$

or $\lim\,[x+y-(a+b)]=0$

which means that

$$\lim\,(x+y)=a+b=\lim x+\lim y$$

To show case 2, proceed similarly, letting

$$X=1, Y=-1$$

To show case 3, let $x-a=u$ and $y-b=v$, u and v being infinitesimals.

Then

$$x=a+u \text{ and } y=b+v$$

Multiplying,

$$xy=ab+bu+(a+u)v$$

$$xy-ab=bu+(a+u)v$$

But $bu+(a+u)v$ is an infinitesimal, § 172.

$\therefore$ $xy-ab$ is an infinitesimal.

Hence $\lim xy=ab=(\lim x)(\lim y)$.

Similarly this may be shown to be true for more than two variables.

To show case 4, divide $x=a+u$ by $y=b+v$.

Then $$\frac{x}{y}=\frac{a}{b+v}+\frac{u}{b+v}$$

$$=\frac{a}{b}+\left(\frac{a}{b+v}-\frac{a}{b}\right)+\frac{u}{b+v}$$

$$=\frac{a}{b}-\frac{a}{b(b+v)}v+\frac{1}{b+v}u$$

$$\therefore \quad \frac{x}{y}-\frac{a}{b}=\frac{+1}{b+v}u+\frac{-a}{b(b+v)}v$$

Since $b\neq0$, $\frac{1}{b+v}$ approaches $\frac{1}{b}$, and $\frac{-a}{b(b+v)}$ approaches $-\frac{a}{b^2}$.

Hence, by § 172,

$$\frac{1}{b+v}u+\frac{-a}{b(b+v)}v \text{ is an infinitesimal.}$$

$$\therefore\quad \frac{x}{y}-\frac{a}{b} \text{ is an infinitesimal.}$$

$$\therefore\ \lim \frac{x}{y}=\frac{a}{b}=\frac{\lim x}{\lim y}, \text{ if } \lim y \neq 0.$$

The following statements are immediate consequences of case 3:

5. $\lim\limits_{x \to a} x^n = a^n = (\lim\limits_{x \to a} x)^n$

6. $\lim\limits_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} = \sqrt[n]{\lim\limits_{x \to a} x}$

7. $\lim\limits_{x \to a} kx = ka = k \lim\limits_{x \to a} x$, k being a constant

Indeterminate Forms

174. Limiting value of a function. If $f(x)$ is a rational integral function of x, and if a is a finite number, then, § 121,

$$\lim_{x \to a} f(x) = f(a)$$

If a rational function $f(x) = \dfrac{\phi(x)}{\psi(x)}$, $\phi(x)$ and $\psi(x)$ being rational integral functions of x, and if a is a finite number, then

$$\lim_{x \to a} f(x) = \frac{\lim\limits_{x \to a} \phi(x)}{\lim\limits_{x \to a} \psi(x)} = \frac{\phi(a)}{\psi(a)} = f(a)$$

provided that $\psi(x)$ does not vanish for $x = a$.

If, however, $\psi(a) = 0$, and if the rational function $f(x)$ is in its lowest terms,

$$f(x) \to \infty$$

EXERCISES

Find the limiting value of each of the following functions:

1. $\frac{x+3}{x+1}$, as $x \to 1$

$$\lim_{x \to 1} \frac{x+3}{x+1} = \frac{1+3}{1+1} = 2$$

2. $\frac{x^2+1}{x-3}$, as $x \to 3$

Since $f(3) = \frac{9+1}{0}$ the function has no finite limit, i.e.,

$$\frac{x^2+1}{x-3} \to \infty, \text{ as } x \to 3$$

3. $\frac{x^2-4}{x+1}$, as $x \to 2$

$$\lim_{x \to 2} f(x) = \frac{0}{3} = 0$$

4. $\frac{x^2-9}{x-2}$, as $x \to 3$

5. $\frac{x^2-4x+3}{x^2+3x+2}$, as $x \to 2$

6. $\frac{x^2+5x+6}{x^2-4}$, as $x \to 2$

7. $\frac{x^3+27}{x^3-27}$, as $x \to -3$

175. Indeterminate forms. The function

$$f(x) = \frac{x^2-9}{x-3}$$

takes the form $\frac{0}{0}$ when $x=3$. This form has no arithmetical meaning. Reducing $f(x)$ to its lowest terms, we have

$$f(x) = \frac{x^2-9}{x-3} = x+3$$

which holds for all values of x except $x=3$.

If we let x approach 3 as a limit, without letting it actually reach 3, we have

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} (x+3) = 6$$

Since the fraction $f(x)=\dfrac{x^2-9}{x\ -3}$ gives the meaningless symbol $\dfrac{0}{0}$, if 3 is substituted for x, it is convenient to *assign* to $f(x)$ for $x=3$ the value $\lim\limits_{x\to 3}\dfrac{x^2-9}{x\ -3}=6$.

The function $f(x)=\dfrac{x^2+3}{2x^2+x-1}$ takes the form $\dfrac{\infty}{\infty}$, as $x\to\infty$. This symbol also has no arithmetical meaning.

Dividing numerator and denominator of $f(x)$ by x^2, we have

$$\frac{1+\dfrac{3}{x^2}}{2+\dfrac{1}{x}-\dfrac{1}{x^2}}$$

Since $\dfrac{3}{x^2}$, $\dfrac{1}{x}$, and $-\dfrac{1}{x^2}$ are infinitesimals when $x\to\infty$, we have

$$\lim_{x\to\infty}\frac{x^2+3}{2x^2+x-1}=\lim_{x\to\infty}\frac{1+\dfrac{3}{x^2}}{2+\dfrac{1}{x}-\dfrac{1}{x^2}}=\frac{1}{2}$$

The two preceding examples illustrate the method of finding the limiting value of $f(x)$ when it takes the form $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

The symbols $\dfrac{0}{0}$, $\dfrac{\infty}{\infty}$, are called **indeterminate forms.** Other indeterminate forms are $0\cdot\infty$ and $\infty-\infty$. The form $0\cdot\infty$ may be reduced to the form $\dfrac{0}{0}$ by first multiplying the two functions and then finding the limit. To find the limit of the form $\infty-\infty$ first perform the indicated subtraction of the two functions and then pass to the limit.

EXERCISES

Find the limiting value of each of the following:

1. $\dfrac{x^3-1}{x-1}$, as $x \to 1$

2. $\dfrac{n-1}{n+1}$, as $n \to \infty$

3. $\dfrac{x+1}{x-2}$, as $x \to 2$

4. $\dfrac{x-3}{x(x-3)^2}$, as $x \to 3$

5. $\dfrac{(x-1)^2}{x(x-1)^2}$, as $x \to 1$

6. $\dfrac{x+1}{x}$, as $x \to \infty$

7. $\dfrac{x^3-2x}{3x^3+5x^2}$, as $x \to \infty$

8. $\dfrac{x^2-x+3}{2x^2+x-4}$, as $x \to \infty$

9. $\dfrac{x^2+x+1}{x+1}$, as $x \to -1$

10. $\dfrac{x^2+x-2}{2x+4}$, as $x \to \infty$

11. $(x^2+2x-3) \cdot \dfrac{1}{x-1}$, as $x \to 1$

12. $\dfrac{3}{x}-\dfrac{1}{x(x+3)}$, as $x \to 0$

Summary

176. The chapter has taught the meaning of the following terms:

limit of a variable
infinitesimal, infinite
indeterminate forms
limiting value of a function

177. The following theorems have been studied:

1. *If* u *and* v *be two infinitesimals and* X *and* Y *two variables, always less than a finite number* k*, then* $Xu+Yv$ *is an infinitesimal.*

2. *If* x *and* y *be two variables approaching the limits* a *and* b*, then*

$$\lim_{\substack{x \to a \\ y \to b}} (x+y) = \lim_{x \to a} x + \lim_{y \to b} y$$

$$\lim_{\substack{x \to a \\ y \to b}} (xy) = \left(\lim_{x \to a} x\right)\left(\lim_{y \to b} y\right)$$

$$\lim_{\substack{x \to a \\ y \to b}} \left(\frac{x}{y}\right) = \frac{\lim\limits_{x \to a} x}{\lim\limits_{y \to b} y}, \text{ if } b \neq 0$$

CHAPTER X

INFINITE SERIES

178. Sum of a series. The sum of numbers arranged in a sequence according to some law, such as

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots\ldots+\frac{1}{2^{n-1}}$$

or

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\ldots+\frac{1}{n}$$

is called the *sum of a series of numbers,* or, briefly, the **sum of a series.**

In general, the sum of a series is represented by the expression

$$u_1+u_2+u_3+\ldots\ldots+u_k+\ldots\ldots+u_n$$

This may be denoted briefly by S_n. In place of the capital S the Greek letter sigma, $\sum$, is also used. Thus the statement *the sum of terms* u_k is written $\sum u_k$. Similarly we write the sum of k terms from $k=1$ to $k=n$:

$$\sum_{k=1}^{n} u_k=u_1+u_2+u_3+\ldots\ldots+u_n$$

$$\sum_{k=1}^{n}\frac{1}{2^{-1k}}=1+\tfrac{1}{2}+\tfrac{1}{4}+\ldots\ldots+\frac{1}{2^{n-1}}$$

$$\sum_{k=1}^{n}\frac{1}{k}=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\ldots+\frac{1}{n}$$

Since the value of the sum of k terms of a series depends upon the value of k, it may be thought of as a function of k.

179. Infinite series. If the number of terms of a series is allowed to increase without bound, the series is said to have *infinitely many* terms and is called an **infinite series.** The following are examples of infinite series:

$$\sum_{k=1}^{\infty}\frac{1}{2^{k-1}}=1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\ldots. \qquad (1)$$

$$\sum_{k=1}^{\infty}k=1+2+3+4+5+\ldots. \qquad (2)$$

$$\sum_{k=0}^{\infty}(-1)^k=1-1+1-1+1-1+\ldots. \qquad (3)$$

180. Convergent and divergent series. The three examples of infinite series given in § 179 will be used to illustrate the meaning of the words *convergent* and *divergent.*

1. The sum S_n of n terms of series (1) is $\dfrac{a-ar^n}{1-r}$ $=2-\dfrac{1}{2^{n-1}}$. Show that S_n approaches the definite limit 2 as n grows without bound. The number 2 is called the *sum* of the infinite series.

In general, *an infinite series is said to be* **convergent** *if the sum of the first n terms approaches a definite finite limit as n increases without bound. This limit is the* **sum** *of the series.*

2. The sum S_n of n terms of series (2) grows beyond bound as n becomes sufficiently large, i.e., it becomes *infinite.* The series is said to be *divergent.*

3. For series (3), S_n remains finite as n increases without bound, but it does *not approach a limit.* It is a *divergent* series. It is often called *oscillating* series, because S_n takes alternately the values 1, 0, 1, 0, etc. In general, *an infinite series is* **divergent** *if it is not convergent.*

Series All of Whose Terms Are Positive

181. We shall now take up the problem of determining whether a series is convergent or divergent. First we shall consider the case of series all of whose terms are *positive.* The theorems in §§ 182 and 183 will be needed in the discussion.

182. Theorem. *If a series* $u_1+u_2+u_3+\dots$ *is convergent, it follows that* $\lim\limits_{n\to\infty} u_n=0$.

$$\text{For, } u_n=(u_1+u_2+u_3+\dots+u_{n-1}+u_n)-(u_1+u_2+u_3+\dots+u_{n-1})$$

$$=S_n-S_{n-1}$$

$$\text{Let } S=\sum_{n=1}^{\infty} u_n.$$

$$\text{Then } \lim_{n\to\infty} S_n=S \text{ and } \lim_{n\to\infty} S_{n-1}=S$$

$$\text{and } \lim_{n\to\infty}(S-S_n)=0 \text{ and } \lim_{n\to\infty}(S-S_{n-1})=0$$

$$\therefore \lim_{n\to\infty}[(S-S_{n-1})-(S-S_n)]=0$$

$$\text{or } \lim_{n\to\infty}(S_n-S_{n-1})=0$$

$$\text{or } \lim_{n\to\infty} u_n=0.$$

Thus, it has been shown that this theorem holds for a convergent series. This does not mean, however, that a series is convergent if $\lim\limits_{n\to\infty} u_n=0$.

183. Theorem. *An infinite series of positive terms is convergent if, as n increases beyond bound, S_n remains less than some finite number N.*

For a series of positive terms cannot oscillate, and it increases as n increases.

Hence, if we assume the series not to be convergent, S_n must grow without bound as n increases.

But this is impossible, since S_n is always less than N.

Therefore the series is convergent.

184. Comparison test. By means of the theorems in §§ 182 and 183 it is possible to compare a given series with a series known to be either convergent or divergent, and thus to establish the convergence or divergence of the given series. The method is known as the **comparison test.**

185. Theorem. *Let*

$$u_1+u_2+u_3+\dots\dots+u_n+\dots\dots \qquad (1)$$

be a given series of positive terms, and let

$$v_1+v_2+v_3+\dots\dots+v_n+\dots\dots \qquad (2)$$

be a known convergent series of positive terms. If, from some particular term on, each term of series (1) *is equal to or less than the corresponding term of* (2), *series* (1) *is convergent.*

This may be seen as follows. For a sufficiently large value of n the following inequalities hold:

$$u_{n+1} \leqq v_{n+1},\ u_{n+2} \leqq v_{n+2}, \dots\dots$$

$$\therefore\quad u_{n+1}+u_{n+2}+\dots\dots \leqq v_{n+1}+v_{n+2}+\dots\dots$$

Let $\qquad S_n=u_1+u_2+u_3+\dots\dots+u_n$

and $\qquad S'_n=v_1+v_2+v_3+\dots\dots+v_n$

Since series (2) is convergent, S'_n approaches a definite finite limit. Hence $v_{n+1}+v_{n+2}+\dots$ may be made as small as we please by taking n large enough.

It follows that $u_{n+1}+u_{n+2}+\dots$ may be made as small as we please by taking n large enough.

Hence the sum $u_1+u_2+\dots+u_n$ will always be less than some finite number N, and series (1) is convergent.

186. Theorem. *Let*

$$u_1+u_2+u_3+\dots+u_n+\dots \tag{1}$$

be a given series of positive terms, and let

$$v_1+v_2+v_3+\dots+v_n+\dots \tag{2}$$

be a known divergent series of positive terms. If, from some particular term on, each term of series (1) *is equal to or greater than the corresponding term of* (2), *series* (1) *is divergent.*

Assuming series (1) not to be divergent, it would follow that series (2) is convergent.

Since this is not true, the assumption is incorrect, and series (1) is divergent.

Notice that the convergence or divergence of a series is not changed by adding to, or subtracting from, it a finite number of terms.

187. Theorem. *Let*

$$u_1+u_2+u_3+\dots+u_n+\dots$$

and

$$v_1+v_2+v_3+\dots+v_n+\dots$$

be two given series with positive terms. If the ratio of each term of one to the corresponding term of the other be finite, the series are either both convergent or both divergent.

For, let R be a finite number greater than the largest value which the ratio $\frac{u_n}{v_n}$ may have for all possible values of n.

$$\text{Then}\quad \frac{u_1}{v_1}<R;\quad \frac{u_2}{v_2}<R;\quad \frac{u_3}{v_3}<R;\ \text{etc.}$$

$$\therefore\quad u_1<Rv_1;\ u_2<Rv_2;\ u_3<Rv_3;\ \text{etc.}$$

$$\therefore\quad u_1+u_2+u_3+\ldots.<R(v_1+v_2+v_3+\ldots.)$$

If $v_1+v_2+v_3+\ldots.$ is *convergent*, $R(v_1+v_2+v_3+\ldots.)$ is finite, R being finite.

$\therefore\quad u_1+u_2+u_3+\ldots.$ is convergent.

Similarly, let r be a finite number less than the smallest value of the ratio $\frac{u_n}{v_n}$.

Then $u_1+u_2+u_3+\ldots.>r(v_1+v_2+v_3+\ldots.)$

Therefore if $v_1+v_2+v_3+\ldots.$ is *divergent*, r being a finite number, it follows that

$$u_1+u_2+u_3+\ldots.$$

is also divergent.

188. Test series for comparison. Two standard series, known to be convergent or divergent, are particularly useful for applying the comparison tests of §§ 184 and 186. They are as follows:

1. The geometric series

$$a+ar+ar^2+\ldots.$$

which is known to be *convergent* when $r<1$, and *divergent* when $r\geqq 1$.

2. The series

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots.\frac{1}{n^p}+\ldots.,$$

which is *convergent* when $p>1$, and *divergent* when $p\leqq 1$.

This may be proved as follows:

First, *let* $p>1$.

Show that the following statements are true:

$$\frac{1}{2^p}+\frac{1}{3^p}<\frac{1}{2^p}+\frac{1}{2^p}=\frac{2}{2^p}\qquad=\frac{1}{2^{p-1}}$$

$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}<\frac{4}{4^p}\qquad=\frac{1}{4^{p-1}}$$

$$\frac{1}{8^p}+\frac{1}{9^p}+\dots\dots+\frac{1}{15^p}<\frac{8}{8^p}=\frac{1}{8^{p-1}},\text{ etc.}$$

Adding,

$$\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots\dots<\frac{1}{2^{p-1}}+\frac{1}{4^{p-1}}+\frac{1}{8^{p-1}}+\dots\dots$$

The right member of this inequality is a geometrical progression for which $r=\frac{1}{2^{p-1}}$.

Since $p>1$, $\therefore$ $\frac{1}{2^{p-1}}<1$, and the right member is a convergent series.

$\therefore$ the left member

$$\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\dots\dots$$

is convergent when $p>1$.

Next, *let* $p=1$.

Then

$$1+\tfrac{1}{2}=1+\tfrac{1}{2}$$
$$\tfrac{1}{3}+\tfrac{1}{4}>\tfrac{1}{4}+\tfrac{1}{4}=\tfrac{1}{2}$$
$$\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8}>\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}=\tfrac{1}{2},\text{ etc.}$$

Adding,

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\dots\dots>1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\dots\dots$$

The right member of this inequality is divergent. Therefore the left member is divergent. This series is known as the *harmonic* series.

Finally, *let* $p<1$

Since every term of the series

$$\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots$$

is greater than the corresponding term of the divergent series

$$\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots$$

the first series is divergent.

EXERCISES

Test the following series for convergence or divergence:

1. $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

2. $1+\frac{1}{\sqrt{2^3}}+\frac{1}{\sqrt{3^3}}+\cdots$

3. $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots$

4. $1+\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\cdots$

Compare with $1+\frac{1}{2}+\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2\cdot 2}+\cdots$

5. $\frac{1}{2}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 2^2}+\frac{1}{5\cdot 2^3}+\cdots$

6. $\frac{2}{1}+\frac{3}{2}+\frac{4}{3}+\cdots$

7. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$

8. $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots$

9. $\frac{1}{2}+\frac{1}{1+2^2}+\frac{1}{1+3^3}+\cdots$

10. $\frac{2}{1+2\sqrt{2}}+\frac{3}{1+3\sqrt{3}}+\frac{4}{1+4\sqrt{4}}+\cdots$

189. Ratio test. This test is an application of the following theorem:

The series of positive terms

$$u_1+u_2+u_3+\ldots+u_n+u_{n+1}+\ldots$$

is **convergent** *if, as n grows without bound, the ratio of the* $(n+1)$*th term to the nth term approaches a limit r, and if* $r<1$. *The series is* **divergent,** *if* $r>1$.

If $r=1$, *the ratio test does not enable us to decide the question of convergence or divergence.*

First, *let* $r<1$.

Since

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=r$$

it is possible to find a positive integer m large enough that for all values of n greater than m the difference

$$\frac{u_{m+1}}{u_m}-r$$

will become and remain less than a positive number δ taken small enough to make $r+\delta<1$, Fig. 72.

FIG. 72

Denoting $r+\delta$ by a,

$$\frac{u_{m+1}}{u_m}<a;\ \frac{u_{m+2}}{u_{m+1}}<a;\ \frac{u_{m+3}}{u_{m+2}}<a;\ \text{etc.}$$

$$\therefore\quad u_{m+1}<au_m$$

$$u_{m+2}<au_{m+1}<a^2u_m$$

$$u_{m+3}<au_{m+2}<a^3u_m,\ \text{etc.}$$

$\therefore$ every term of the series

$$u_{m+1}+u_{m+2}+u_{m+3}+\ldots$$

is less than the corresponding term of the series

$$au_m + a^2u_m + a^3u_m + \dots.$$

which is equal to

$$u_m(a + a^2 + a^3 + \dots.)$$

However, this series is convergent, since $a < 1$.

$\therefore$ the series $u_{m+1} + u_{m+2} + u_{m+3} + \dots.$ is convergent.

$\therefore$ $u_1 + u_2 + u_3 + u_4 + \dots. + u_m + u_{m+1} + \dots.$ is convergent.

Next, *let* $r > 1$.

By taking m large enough, for all values of $n > m$ the following inequalities will hold:

$$\frac{u_{m+1}}{u_m} > 1; \quad \frac{u_{m+2}}{u_{m+1}} > 1; \quad \frac{u_{m+3}}{u_{m+2}} > 1; \text{ etc.}$$

$$\begin{aligned} \therefore \quad & u_{m+1} > u_m \\ & u_{m+2} > u_{m+1} > u_m \\ & u_{m+3} > u_{m+2} > u_m, \text{ etc.} \end{aligned}$$

Hence $u_{m+1} + u_{m+2} + u_{m+3} + \dots. > u_m + u_m + u_m$

$\therefore$ $u_1 + u_2 + u_3 + \dots. + u_m + u_{m+1} + \dots.$ is divergent for $r > 1$. *If* $r = 1$, the series may be convergent or divergent, as the following two examples illustrate:

1. For the *divergent* harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots.$$

the ratio of the $(n+1)$th term to the nth term is

$$\frac{u_{n+1}}{u_n} = \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \frac{n}{n+1}$$

Hence, $$\lim_{n \to \infty} \frac{u_{n+1}}{u_n} = \lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1 + \dfrac{1}{n}} = 1$$

2. In the series

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots\ldots+\frac{1}{n(n+1)}$$

the ratio of the $(n+1)$th term to the nth term is

$$\frac{u_{n+1}}{u_n}=\frac{\dfrac{1}{(n+1)(n+2)}}{\dfrac{1}{n(n+1)}}=\frac{n}{n+2}=\frac{1}{1+\dfrac{2}{n}}$$

$$\therefore\quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1}{1+\dfrac{2}{n}}=1$$

However, this series is *convergent,* as may be seen from the following:

$$S_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots\ldots+\frac{1}{n(n+1)}$$

$$=(1-\tfrac{1}{2})+(\tfrac{1}{2}-\tfrac{1}{3})+(\tfrac{1}{3}-\tfrac{1}{4})+\ldots\ldots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=1-\frac{1}{n+1}$$

$\therefore\quad \lim_{n\to\infty} S_n=1$, and the series is convergent.

EXERCISES

Apply the ratio test to the following series:

1. $3+\frac{3^2}{2}+\frac{3^3}{3}+\ldots\ldots+\frac{3^n}{n}+\ldots\ldots$

$$\frac{u_{n+1}}{u_n}=\frac{3^{n+1}\cdot n}{(n+1)3^n}=\frac{3}{1+\dfrac{1}{n}}$$

$$\therefore\quad \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=3$$

$\therefore$ the series is divergent.

2. $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\frac{4}{2^4}+\ldots.+\frac{n}{2^n}+\ldots.$

3. $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots.+\frac{1}{n!}+\ldots.$

4. $\frac{2}{1\cdot3}+\frac{4}{3\cdot5}+\frac{6}{5\cdot7}+\ldots.+\frac{2n}{(2n-1)(2n+1)}+\ldots.$

5. $\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\ldots.+\frac{1}{n(n+1)(n+2)}+\ldots.$

6. $\frac{3}{1\cdot2}+\frac{5}{2^2\cdot3}+\frac{7}{3^2\cdot4}+\ldots.$

7. $1+\frac{1}{2^n}+\frac{2!}{3^2}+\frac{3!}{4^2}+\ldots.$

8. $\frac{5}{2^2}+\frac{5^2}{3^2}+\frac{5^3}{4^2}+\ldots.$

9. $\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+\ldots.$

Series Having Positive and Negative Terms

190. The following theorem will be useful in examining for convergence a series having positive and negative terms.

191. Theorem. *An infinite series* $u_1+u_2+u_3+\ldots.$, *whose terms are all real but not all of the same sign, is convergent, if the positive series* $|u_1|+|u_2|+|u_3|+\ldots.$ *is convergent.*

For

$$u_1+u_2+u_3+\ldots.<|u_1|+|u_2|+|u_3|+\ldots.$$

since the corresponding terms in both members are equal numerically, and since some of the terms of the left member are negative.

Since the right member is a convergent series, its sum approaches a definite finite limit as the number of terms grows without bound.

Hence it is possible to select a number m such that for $n>m$ the sum

$$|u_{m+1}|+|u_{m+2}|+|u_{m+3}|+\ldots.$$

becomes and remains less than a positive number δ, however small.

$$\therefore \quad u_{m+1}+u_{m+2}+u_{m+3}+\dots <\delta$$

Hence the series

$$u_1+u_2+u_3+\dots$$

approaches a definite finite limit as n grows without bound and is convergent.

192. Absolutely convergent series. If a convergent series with positive and negative terms remains convergent when the signs of the negative terms are changed, it is an **absolutely convergent series.**

For example, the series

$$1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{8}+\tfrac{1}{16}-\dots$$

is *absolutely convergent.*

193. Ratio test. The ratio test for a series with positive and negative terms is an application of the following theorem:

Theorem. *Let*

$$u_1+u_2+u_3+\dots$$

be an infinite series.

1. *If* $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$, *the series is convergent.*

2. *If* $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|>1$, *the series is divergent.*

3. If $\lim\limits_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=1$, *the question of convergence or divergence must be decided by some other method.*

For, if for a series of positive terms

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|<1$$

that series is convergent.

Hence the series must be convergent if the signs of some of the terms are changed. This shows that statement 1 is true.

For any convergent series

$$\lim_{n \to \infty} u_n = 0, \ \S\ 182$$

This is impossible for a series for which

$$\lim_{n \to \infty} \left| \frac{u_{n+1}}{u_n} \right| > 1$$

Hence the series must be divergent, which shows the truth of statement 2.

194. Alternating series. A series whose terms are alternately positive and negative is an **alternating series.** For example, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$ is an alternating series.

195. Theorem. *An alternating series is convergent if each term is numerically less than the preceding term, and if the limit of the nth term is zero as n increases beyond bound.*

For, let

$$u_1 - u_2 + u_3 - u_4 + \ldots.$$

be the given alternating series.

Let n be an even number. Then S_n may be written

$$S_n = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \ldots. + (u_{n-1} - u_n)$$

This is a series of *positive* terms because $u_1 > u_2 > u_3 \ldots.$

But S_n may also be written

$$S_n = u_1 - (u_2 - u_3) - (u_4 - u_5) - \ldots. - (u_{n-2} - u_{n-1}) - u_n$$

Since all the differences in the parentheses are positive, it follows that

$$S_n < u_1$$

∴ The given series is convergent.

Hence the sum S_n of an *even* number of terms approaches a definite finite limit S.

Since

$$S_{n+1}=S_n+u_{n+1}$$

it follows that

$$\lim_{n\to\infty} S_{n+1}=\lim_{n\to\infty} S_n+\lim_{n\to\infty} u_{n+1}$$

$$\therefore \lim_{n\to\infty} S_{n+1}=S+0$$

∴ The sum S_{n+1} of an *odd* number of terms approaches the same limit as S_n.

∴ The given alternating series is convergent.

EXERCISES

Test the following series for convergence or divergence:

1. $\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-\dots$

2. $\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}-\dots$

3. $\frac{1}{2}\cdot\frac{3}{4}-\frac{1}{3}\cdot\frac{4}{5}+\frac{1}{4}\cdot\frac{5}{6}-\dots$

4. $1-\frac{3}{2^2}+\frac{5}{3^2}-\dots$

5. $1+\frac{4}{5}+\frac{9}{5^2}+\frac{16}{5^3}+\dots$

6. $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$

Series Whose Terms Are Functions of x

196. The following problem illustrates the method of testing for convergence or divergence a series whose terms are functions of x.

Consider the series

$$\frac{1}{2}+\frac{2x}{2^2}+\frac{3x^2}{2^3}+\dots+\frac{nx^{n-1}}{2^n}+\dots$$

We have

$$\frac{u_{n+1}}{u_n}=\frac{\dfrac{(n+1)x^n}{2^{n+1}}}{\dfrac{nx^{n-1}}{2^n}}=\frac{(n+1)x^n\cdot 2^n}{2^{n+1}nx^{n-1}}=\frac{(n+1)x}{2n}=\frac{\left(1+\dfrac{1}{n}\right)x}{2}$$

$$\therefore\quad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|=\tfrac{1}{2}x$$

$\therefore$ the series is *convergent* if $\left|\frac{x}{2}\right|<1$, or if $|x|<2$.

The series is *divergent* if $\left|\frac{x}{2}\right|>1$, or if $|x|>2$.

When $\left|\frac{x}{2}\right|=1$, or when $|x|=2$, we may substitute ± 2 for x in the given series and examine the resulting series for convergence or divergence.

The convergence or divergence of a series decides its usefulness. For example, the series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots.;\quad 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots.;$$

$$\text{and}\quad \frac{x}{1}-\frac{x^2}{2}+\frac{x^3}{3}-\ldots.$$

are used to define respectively the functions sin x, cos x, and log $(1+x)$ for all values of x for which the series are convergent

EXERCISES

Examine the following series for convergence or divergence:

1. $x-\frac{x^2}{3!}+\frac{x^3}{5!}-\frac{x^4}{7!}+\ldots.$

2. $\frac{1}{1+x^2}+\frac{1}{1+2x^2}+\frac{1}{1+3x^2}+\ldots.$

3. $\frac{x}{1\cdot 3}-\frac{x^3}{3\cdot 3^3}+\frac{x^5}{5\cdot 3^5}+\ldots.$

4. $\frac{1}{1\cdot 2}+\frac{x}{2\cdot 3}+\frac{x^2}{3\cdot 4}$

5. $\frac{1\cdot 3}{2}+\frac{3\cdot 5x}{2^2}+\frac{5\cdot 7x^2}{2^3}+\ldots.$

6. $\frac{x}{1}+\frac{x^2}{2}+\frac{x^3}{3}+\ldots.$

Summary

197. The chapter has taught the meaning of the following terms:

sum of a series
infinite series
convergent series
divergent series
absolutely convergent series
alternating series

198. The convergence of a series may be established by the comparison test or by the ratio test.

199. The geometric series $a+ar+ar^2+\ldots.$ and the series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots.+\frac{1}{n^p}+\ldots.$ are useful for applying the comparison test.

200. The following theorems have been studied:

1. *If a series* $u_1+u_2+u_3+\ldots.$ *is convergent, it follows that* $\lim_{n\to\infty} u_n=0$.

2. *An infinite series of positive terms is convergent if* S_n *remains less than some finite number* N *as* n *increases without bound.*

3. *An infinite series of positive terms is convergent if, from some particular term on, each term is equal to, or less than, the corresponding term of a known convergent series of positive terms.*

4. *An infinite series of positive terms is divergent if, from some particular term on, each term is equal to, or greater than, the corresponding term of a known divergent series of positive terms.*

5. *If the ratio of each term of one of two given series with positive terms to the corresponding term of the other be finite, the series are either both convergent or both divergent.*

6. *A series of positive terms is convergent if the limit of the ratio of the* $(n+1)$*th term to the nth term as n increases without bound is less than* 1, *and divergent if the limit is greater than* 1.

7. *An infinite series* $u_1+u_2+u_3+\dots$ *whose terms are all real, but not all of the same sign, is convergent if the series* $|u_1|+|u_2|+|u_3|+\dots$ *is convergent.*

8. *An infinite series* $u_1+u_2+u_3+\dots$ *is convergent or divergent according as* $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$ *is less than* 1, *or greater than* 1.

9. *An alternating series is convergent if each term is numerically less than the preceding term, and if the limit of the nth term is zero as n increases without bound.*

CHAPTER XI

PARTIAL FRACTIONS

201. Partial fractions. The identity

$$\frac{x-5}{x^2-1}\equiv\frac{3}{x+1}+\frac{-2}{x-1}$$

expresses the proper fraction $\frac{x-5}{x^2-1}$ as the sum of two *partial fractions* $\frac{3}{x+1}$ and $\frac{-2}{x-1}$. In this chapter we shall study the problem of decomposing a *proper rational* fraction into a sum of simpler fractions which cannot themselves be still further resolved. Such fractions are called **partial fractions.** Our problem is clearly the inverse of the problem of expressing the sum of several given fractions as a single fraction.

It is sufficient to study only the case of *proper* fractions, i.e., fractions whose numerator is of degree lower than that of its denominator. For any *improper* fraction can always be changed to the sum of an integral number and a proper fraction. For example,

$$\frac{3x^2-2}{x^2-5}\equiv 3+\frac{13}{x^2-5}$$

202. Case I. *Let*

$$R(x)=\frac{f(x)}{g(x)}=\frac{ax^{n-1}+b\,x^{n-2}+c\,x^{n-3}+\dots+l}{a_1x^n+b_1x^{n-1}+c_1x^{n-2}+\dots+l_1}$$

be a proper rational fraction reduced to its lowest terms, and let $g(x)$ be the product of n linear factors, all of which are distinct, i.e.,

$$g(x)=(x-r_1)(x-r_2)(x-r_3)\dots(x-r_n)$$

Let us assume that $R(x)$ may be decomposed into partial proper fractions whose denominators are $x-r_1$, $x-r_2$, ..., $x-r_n$, respectively. Since the degree of the numerator in each fraction is lower than that of the denominator, the numerators are *constants*.

Thus we have

$$R(x)=\frac{A}{x-r_1}+\frac{B}{x-r_2}+\frac{C}{x-r_3}+\ldots.+\frac{L}{x-r_n}$$

Adding the partial fractions, we have

$$R(x)=\frac{A(x-r_2)\ldots.(x-r_n)+B(x-r_1)(x-r_3)\ldots.(x-r_n)+\ldots.+L(x-r_1)\ldots.(x-r_{n-1})}{(x-r_1)(x-r_2)\ldots.(x-r_n)}$$

It is clear that the degree of the numerator of this fraction is lower than that of the denominator. Hence the sum of the partial fractions is a *proper* fraction.

The denominator of this fraction is identically the same as $g(x)$, and the numerator when simplified is of the same form as $f(x)$.

Hence, if we determine A, B, C, ..., L, so that

$$f(x)=A(x-r_2)\ldots.(x-r_n)+B(x-r_1)(x-r_3)\ldots.(x-r_n)$$
$$+\ldots.+L(x-r_1)\ldots.(x-r_{n-1})$$

then

$$\frac{A}{x-r_1},\ \frac{B}{x-r_2},\ \ldots.,\ \frac{L}{x-r_n}$$

are the required partial fractions of $R(x)$.

Exercise 1, below, illustrates the method of resolving a given fraction into partial fractions.

EXERCISES

Resolve the following into the simplest partial fractions:

1. $\frac{x^2}{(x^2-1)(x-2)}$

To determine the partial fractions, let us assume that the fraction may be decomposed into partial fractions whose numerators are constants. Then

$$\frac{x^2}{(x^2-1)(x-2)} \equiv \frac{A}{(x-1)} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$\therefore \quad x^2 \equiv A(x+1)(x-2) + B(x-1)(x-2) + C(x-1)(x+1)$$

Since this holds for all values of x, it holds for $x=-1$, 2, and 1.

When $x=-1$, we have by substitution,

$$1=0+B(-2)(-3)+0$$

$$\therefore \quad B=\tfrac{1}{6}$$

When $x=2$,

$$4=C(1)(3)$$

$$\therefore \quad C=\tfrac{4}{3}$$

When $x=1$,

$$1=A(2)(-1)$$

$$\therefore \quad A=-\tfrac{1}{2}$$

Hence

$$\frac{x^2}{(x^2-1)(x-2)} \equiv -\frac{1}{2(x-1)} + \frac{1}{6(x+1)} + \frac{4}{3(x-2)}$$

2. $\dfrac{2x-3}{(x-1)(x+2)}$

3. $\dfrac{x^2+2}{x(x-1)(x-2)}$

4. $\dfrac{3x(x+4)}{(2x+1)(x+1)(x-2)}$

5. $\dfrac{2x+3}{x^2-5x+6}$

6. $\dfrac{2x+3}{x^3+x^2-2x}$

7. $\dfrac{(1+3x)(1-2x)}{x(x^2-1)}$

203. Case II. *Let the denominator of the fraction to be decomposed into partial fractions be the product of linear factors, not all of which are distinct.*

Suppose the denominator has a factor $(x-a)^m$. Corresponding to this factor there must be at least one partial fraction with denominator $(x-a)^m$. However, there may be other partial fractions with denominators of the forms

$$(x-a)^{m-1},\ (x-a)^{m-2},\ \ldots\ldots,\ (x-a)^2,\ x-a$$

Hence we shall assume, corresponding to every factor of the form $(x-a)^m$, the sum of the partial fractions

$$\frac{A}{(x-a)^m}+\frac{B}{(x-a)^{m-1}}+\ldots.+\frac{L}{(x-a)^2}+\frac{M}{x-a}$$

Exercise 1, below, illustrates the method of determining the coefficients A, B, C, ...

EXERCISES

Resolve the following into the simplest partial fractions:

1. $\dfrac{3x^2+1}{(x+1)(x-1)^2}$

Assume that the fraction may be decomposed into partial fractions whose denominators are constants. Then

$$\frac{3x^2+1}{(x+1)(x-1)^2}\equiv\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Then $3x^2+1\equiv A(x-1)^2+B(x+1)(x-1)+C(x+1)$
Let $x=1$
Then $3+1=C(2)$
$\therefore\ C=2$
Let $x=-1$
Then $3+1=A(-2)^2$
$\therefore\ A=1$
Select any other convenient value for x, as $x=2$.
Then $12+1=A(1)^2+B(3)(1)+C(3)$
or $A+3B+3C=13$
$\therefore\ 3B=6$
and $B=2$

$$\therefore\ \frac{3x^2+1}{(x+1)(x-1)^2}=\frac{1}{x+1}+\frac{2}{x-1}+\frac{2}{(x-1)^2}$$

2. $\dfrac{5}{(x^2-1)(x+1)}$

3. $\dfrac{3}{(x-1)^3}$

4. $\dfrac{x+1}{x(x+1)^3}$

5. $\dfrac{6x^2-x+1}{x(1-x^2)}$

6. $\dfrac{x^2+1}{x^3(x-1)}$

7. $\dfrac{x^2+2x}{x^3-x^2-x+1}$

204. Case III. *Let some of the factors of the denominator be real quadratic which cannot be separated into linear factors.*

For every such factor occurring only once we shall assume a partial fraction whose numerator is of the first degree, as

$$\frac{Ax+B}{x^2+px+q}$$

For every factor of the form x^2+px+q occurring m times we shall assume the sum of the partial fractions

$$\frac{Ax+B}{(x^2+px+q)^m}+\frac{Cx+D}{(x^2+px+q)^{m-1}}+\dots\dots+\frac{Lx+M}{x^2+px+q}$$

EXERCISES

Resolve the following into the simplest partial fractions:

1. $\dfrac{8x^4+5x^3+13x^2+14x+13}{(x^2+1)^2(x+2)}$

For the factors x^2+1 and $(x^2+1)^2$ assume numerators of the form $mx+n$; for the factor $x+2$ assume a constant numerator.

Then $\dfrac{8x^4+5x^3+13x^2+14x+13}{(x^2+1)^2(x+2)}=\dfrac{Ax+B}{(x^2+1)^2}+\dfrac{Cx+D}{x^2+1}+\dfrac{E}{x+2}$

$\therefore\ 8x^4+5x^3+13x^2+14x+13$

$$\equiv(Ax+B)(x+2)+(Cx+D)(x^2+1)(x+2)+E(x^2+1)^2$$

$$\equiv\begin{array}{r|l r|l r|l r|l r}
 & & & & A & x^2 & +2A & x & +2B \\
 & & & & & & +\ B & & \\
C & x^4 & +2C & x^3 & +\ C & & +2C & & \\
 & & +\ D & & +2D & & +\ D & & +2D \\
E & & & & +2E & & & & +\ E
\end{array}$$

Equating the coefficients of similar terms,

$$\begin{aligned}
C+E&=8\\
2C+D&=5\\
A+C+2D+2E&=13\\
2A+B+2C+D&=14\\
2B+2D+E&=13
\end{aligned}$$

Solving these equations we have

$$A=2,\ B=5,\ C=3,\ D=-1,\ E=5$$

$$\therefore\quad 8x^4+5x^3+13x^2+14x+13=\frac{2x+5}{(x^2+1)^2}+\frac{3x-1}{x^2+1}+\frac{5}{x+2}$$

2. $\dfrac{4}{x^3-1}$

3. $\dfrac{1-3x}{2-x+2x^2-x^3}$

4. $\dfrac{1+x^3}{x(1+x+x^2)}$

5. $\dfrac{2}{(x+1)(x^2-x+1)}$

6. $\dfrac{1+2x-5x^2+x^3}{(x^2+5)(x^2+1)}$

7. $\dfrac{x^2-4x+5}{(x^2-2x+1)(x^2+1)}$

Summary

205. In decomposing a proper rational fraction into the simplest partial fractions we have considered three cases:

1. If the denominator is the product of distinct linear factors, the numerators of the assumed partial fractions are constants.

2. If the denominator is the product of linear factors not all of which are distinct, the numerators of the assumed partial fractions are constants and the denominators of the form $(x-a)^m$.

3. If some of the factors of the denominator are real quadratic, such as x^2+px+q, and not factorable into linear factors, the numerators of the assumed partial fractions are constants for every denominator of the form $(x-a)^m$, but for every denominator of the form $(x^2+px+q)^m$ the numerators are of the form $mx+n$.

CHAPTER XII

PERMUTATIONS AND COMBINATIONS

Permutations

206. Elements. Permutations. A given number of things may be arranged in one row in a number of ways. Suppose we have three books to be arranged on a shelf. This may be done as follows.

Denoting the books by the letters a, b, and c, we may select book a first and then either b and c, or c and b. This gives the arrangements abc and acb.

Selecting b as the first book, we have the arrangements bac and bca.

Finally, by taking c for the first place, we have the arrangements cab and cba.

Thus the three books may be arranged on the shelf in the six different ways:

$$\begin{array}{ccc} abc & bac & cab \\ acb & bca & cba \end{array}$$

The things to be arranged are called *elements.*

Arrangements of n elements, differing from each other only in the order of the elements, are called **permutations.** Hence, according to the problem above, three elements have six permutations.

To find the number of permutations of *four* elements we may proceed as follows.

Since there are four elements, each of which may occupy the first place, we may fill the first place in four different ways. In each case the three other places may then be filled in six different ways. This gives 4×6 different arrangements, or 24 permutations. They are as follows:

abcd
abdc
acbd
acdb
adbc
adcb
bacd
badc
bcad
bcda
bdac
bdca
cabd
cadb
cbad
cbda
cdab
cdba
dabc
dacb
dbac
dbca
dcab
dcba

Similarly the permutations of *five* elements may be arranged in five columns, each column containing 24 permutations. Hence there are 5×24 permutations of five elements.

207. The number of permutations of *n* elements. Let the number of permutations of n elements be denoted by P_n.

For example, let it be required to find in how many ways we may arrange n books on a shelf. Denoting the books by numbers 1, 2, 3, 4, ... , we have for two books the permutations 12 and 21, i.e., two permutations.

For three books,

123
132
213
231
312
321

we have $3 \cdot 2$, or 6 permutations.

For four books we shall have $4 \cdot 3 \cdot 2$, or 24 permutations, etc.

Arranging these results in a table,

n	P_n
1	1
2	$1 \cdot 2$
3	$1 \cdot 2 \cdot 3$
4	$1 \cdot 2 \cdot 3 \cdot 4$
5	$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$

we infer that the number of permutations of n elements is given by the formula

$$\boldsymbol{P_n = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots\ldots \cdot n}$$

The products 1, $1 \cdot 2$, $1 \cdot 2 \cdot 3$, ..., $1 \cdot 2 \cdot 3 \cdot \ldots\ldots \cdot n$, are called *factorial* 1, *factorial* 2, *factorial* 3, ..., *factorial n*, respectively, and are commonly denoted by the symbols $1!$, $2!$, $3!$, ..., $n!$, or $\underline{|1}$, $\underline{|2}$, $\underline{|3}$, ..., $\underline{|n}$.

Hence $\boldsymbol{P_n = n!}$

This formula may be proved by mathematical induction, § 118. We assume first that

$$P_n = n(n-1)(n-2) \cdot \ldots\ldots \cdot 3 \cdot 2 \cdot 1$$

To get P_{n+1} we arrange the $n+1$ elements in $n+1$ columns, § 206, each column containing P_n permutations.
$\therefore \quad P_{n+1} = (n+1)P_n = (n+1)(n)(n-1)(n-2) \ldots . 3 \cdot 2 \cdot 1$
Hence, if the formula is true for n elements it is also true for $(n+1)$ elements.

Since we know it to hold for $n=4$, it must also hold for $n=5$, and therefore for $n=6$, etc., for all positive integral values of n.

208. Fundamental principle. In the development of the formula in § 207 we made use of the following principle:

If one thing can be done in m different ways, and if, after this has been done, another can be done in n ways, then the two things together can be done in mn ways.

EXERCISES

1. Form the permutations of the letters a, b, c, d, and e.

2. Form the permutations containing a, b, c, d, and beginning with $d\,b$.

3. How many permutations can be formed with 8 elements?

4. How many permutations can be formed with the letters in the world "April"?

5. In how many ways can 7 people be seated on a bench?

209. Permutations of n things, not all of which are different. Thus far we have considered only the case where all elements are distinct. If some of the n things are alike, the number of permutations may be found as follows:

Attach subscripts to the things that are alike and form the number of permutations as if all elements were distinct. For example, we change $abbb$ to $ab_1b_2b_3$. This gives the following four groups of six permutations having a in the first, second, third, and fourth places, respectively.

$ab_1b_2b_3$	$b_1ab_2b_3$	$b_1b_2ab_3$	$b_1b_2b_3a$
$ab_1b_3b_2$	$b_1ab_3b_2$	$b_2b_1ab_3$	$b_1b_3b_2a$
$ab_2b_1b_3$	$b_2ab_1b_3$	$b_1b_3ab_2$	$b_2b_1b_3a$
$ab_2b_3b_1$	$b_2ab_3b_1$	$b_3b_1ab_2$	$b_2b_3b_1a$
$ab_3b_1b_2$	$b_3ab_1b_2$	$b_2b_3ab_1$	$b_3b_1b_2a$
$ab_3b_2b_1$	$b_3ab_2b_1$	$b_3b_2ab_1$	$b_3b_2b_1a$

If we leave off the subscripts there will be only one permutation in each group. Thus, instead of 4! permutations, we have only $\frac{1}{6}$ (4!), or $\frac{4!}{3!}$ permutations.

In general, to find the permutations of n things, p of which are alike, denote the like things by subscripts. They may then be arranged in groups, such that the permutations in each group differ only in the position of the p elements, which were given alike. Since each group contains as many permutations as can be formed with p elements, i.e., $p!$, there are $\frac{n!}{p!}$ groups, Leaving off the subscripts, all the permutations in one group are the same.

Hence there are $\frac{n!}{p!}$ permutations of n things, p of them being alike.

If among the n given things there are still q other elements that are alike, we divide the $\frac{n!}{p!}$ permutations into groups whose $q!$ permutations differ only in the position of these q elements.

Hence there are $\frac{\frac{n!}{p!}}{q!}$ or $\frac{n!}{p!q!}$ groups.

Leaving off the subscripts of the q elements, each group will have only one permutation, and the number of permutations is the same as the number of groups, i.e., $\frac{n!}{p!q!}$

By continuing this process we find that *the number of permutations of n things of which p are of one kind, q of another, etc., is given by the formula*

$$P=\frac{n!}{p!q!\ldots.}$$

EXERCISES

1. How many permutations can be formed from the letters *aabbbc?*

2. In how many different ways can the letters of the word "short" be arranged?

3. In how many different ways can the letters of the word "mathematics" be arranged?

4. In how many ways can 3 white, 1 blue, and 2 red balls be arranged?

210. Permutations of n things, taken r at a time. From four different flags, a, b, c, and d, signals of two flags each are to be made. We may first select flag a and choose with it one of the remaining flags, b, c, or d. Thus we have the three arrangements ab, ac, and ad. Similarly, by choosing first one of the other flags, three arrangements are possible in each case.

Thus we have $4 \cdot 3 = 4[4-(2-1)]$ possible cases:

ab	ba	ca	da
ac	bc	cb	db
ad	bd	cd	dc

If signals of three flags each were to be formed, we would have two possibilities for each of the cases above. Thus, corresponding to ab we have abc and abd. Hence, altogether we shall have $4 \cdot 3 \cdot 2$ cases.

Similarly we may show that the number of five things taken two at a time is $5 \cdot 4$ or $5[5-(2-1)]$.

Three at a time $5 \cdot 4 \cdot 3$ or $5 \cdot 4$ or $[5-(3-1)]]$

And four at a time $5 \cdot 4 \cdot 3 \cdot 2$ or $5 \cdot 4 \cdot 3[5-(4-1)]$.

In general, the number of n things taken r at a time is given by the formula

$$_nP_r = n(n-1)(n-2)\ldots.(n-r+1)$$

The formula is easily proved by mathematical induction, as follows:

We have seen that

$$_nP_r = {_nP_{r-1}}(n-r+1)$$

Assume

$$_nP_r = n(n-1)(n-2)\ldots.(n-r+1)$$

Then

$$_nP_{r+1} = {_nP_r}(n-r) = n(n-1)(n-2)\ldots.(n-r+1)(n-r)$$

which shows that the formula holds for $r+1$ if it holds for r.

Since we know the formula to hold for $r=1$ and 2, it follows that it holds for any value of r.

EXERCISES

1. In how many ways can 10 soldiers be arranged in a row?

2. How many permutations are there of the letters of the word "player" when 3 are taken at a time?

3. In how many ways can a baseball team be formed from 18 ball players, each of whom will take any position?

4. How many signals can be formed in the 7 flags of different colors, displaying 4 at a time?

5. How many permutations can be formed with the letters of the word "Illinois"?

6. How many different numbers containing 6 digits can be formed with the figures 1122334?

7. How many different signals can be made with 10 flags, 3 of which are white, 4 blue, 2 black, and 1 red?

8. In how many ways can 5 pupils be seated in 8 seats?

211. Permutations in a circle. When the objects are arranged in a circle, or in any closed curve, the relative order is not changed if all objects are shifted the same number of places. Hence the position of one object is immaterial. We may suppose one of the n objects in a fixed position and find the number of arrangements of the $n-1$ remaining objects. For example, if four persons, A, B, C, and D, are seated at a table, the arrangement is the same whether we start from A, B, C, or D, as long as we proceed in the same direction.

Since $n-1$ objects can be arranged in $(n-1)!$ orders the *number of circular permutations of n objects is* $(n-1)!$.

EXERCISES

1. In how many ways can 6 persons be seated at a round table?

2. Show that n keys can be arranged in a ring in $\frac{(n-1)!}{2}$ ways.

Consider that by revolving the ring about a diameter through an angle of 180° each arrangement is found to occur twice.

3. How many different bracelets can be formed with 30 beads of different colors?

Combinations

212. Combinations. If A, B, and C are appointed as a committee, the character of the committee is not changed by changing the order in which A, B, and C may be arranged. The orders ABC, ACB, BCA, BAC, etc., are different *permutations* of the three men, but they are said to be the same *combination.* In general, a group of objects in which the order of arrangement is not considered is called a **combination.**

213. Number of combinations. We have denoted the number of permutations of n things taken r at a time by ${}_nP_r$. From every combination of r things we can form $r!$ permutations. Hence the number of combinations of n things taken r at a time multiplied by $r!$ is equal to ${}_nP_r$. In symbols this may be written briefly

$$r!({}_nC_r)={}_nP_r$$

or

$$\boldsymbol{{}_nC_r=\frac{{}_nP_r}{r!}=\frac{n(n-1)\ldots.(n-r+1)}{r!}}$$

Multiplying numerator and denominator by $(n-r)!$ this takes the form

$${}_nC_r=\frac{n!}{r!(n-r)!}$$

Hence

$${}_nC_r = {}_nC_{n-r}$$

or *the number of combinations of n things taken r at a time is the same as the number of combinations of n things taken n−r at a time.*

214. The total number of combinations. In the binomial formula

$$(a+b)^n = a^n + na^nb + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \dots.$$

the coefficients are 1, ${}_nC_1$, ${}_nC_2$, etc.

If we let $a = b = 1$, we have

$$\mathbf{2^n = 1 + {}_nC_1 + {}_nC_2 + {}_nC_3 + \dots . {}_nC_n}$$

Hence *the total number of combinations of n things taking* 1, 2, 3, ..., *n at a time is* $2^n - 1$.

EXERCISES

1. In how many ways can a committee of 4 men be appointed from 15 men?

2. How many straight lines can be drawn through 10 points, no 3 of which are in a straight line?

3. In how many ways can a group of 3 lawyers and 4 merchants be selected from 7 lawyers and 6 merchants?

4. How many different crews of 8 men can be selected from a group of 16 men?

5. There are 2 roads from A to B, 3 from B to C, and 3 from C to D. By how many routes can a man travel from A to D?

6. In how many ways can 8 men be arranged in a row so that neither of 2 given men may be at either end of the row? (Board.)

7. How many triangles can be drawn with each vertex in 1 of 20 given points, no 3 of which are in the same straight line? How many such triangles can be drawn if 4 of the given points lie in a straight line? (Board.)

8. From 14 men, how many committees of 4 can be formed? Of these, how many include one particular man, A? How many include A but not B? (Board.)

9. How many planes are determined by 30 points if no 4 of the points lie in the same plane?

10. A club consists of 8 men and 6 women. How many different committees of 7 members each can be selected from the club, each committee to consist of 3 men and 4 women? (Williams.)

11. How many even numbers of 3 digits each can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8? (Williams.)

12. In how many ways can a committee of 3 be selected from 10 persons so that a particular person A shall always be (1) included and (2) excluded? (Williams.)

13. A person has 22 friends, of whom 14 are men. In how many ways can he make up dinner parties of 17 guests, it being required that 10 guests of each party be men? (Williams.)

14. In a certain school there are 60 pupils and 5 teachers. An athletic committee of 3 teachers and 2 pupils is to be chosen. How many such committees could be formed? (Harvard.)

15. Six flags of different colors can be hoisted, either singly or any number at a time, one above another. How many different signals can be made with them? (Harvard.)

Summary

215. The chapter has taught the meaning of the following new terms:

permutation, circular permutation, combination

216. The following important principle has been used: *If one thing can be done in m different ways, and if, after this has been done, another can be done in n ways, then the two things together can be done in mn ways.*

217. The following formulas have been developed in this chapter:

1. $P_n = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots\ldots \cdot n$, where P_n denotes the number of permutations of n elements.

2. $P = \frac{n!}{p!n!\ldots\ldots}$, where P denotes the number of permutations of n things, of which p are of one kind, q of another, etc.

3. ${}_nP_r = n(n-1)(n-2) \cdot \ldots\ldots \cdot (n-r+1)$, where ${}_nP_r$ denotes the number of permutations of n things taken r at a time.

4. $P_n = (n-1)!$, where P_n is the number of circular permutations of n objects.

5. ${}_nC_r = \frac{n(n-1)\ldots\ldots(n-r+1)}{r!} = \frac{n!}{r!(n-r)!}$, where ${}_nC_r$ is the number of combinations of n things taken r at a time.

CHAPTER XIII

THE CIRCLE

General Quadratic Function of Two Variables

218. General quadratic function of two variables. Most of the functions studied in the preceding chapters are functions of *one* variable. However, functions of *several* variables are not unknown to the student. For example, in the study of simultaneous equations he has met *linear* functions of the form $ax+by+c$, and such *quadratic* functions as xy and ax^2+by^2-c.

The function

$$f(x, y)=Ax^2+2Hxy+By^2+2Gx+2Fy+C$$

where the coefficients A, B, C, F, G, and H may have *any* values independent of x and y, excluding the case $A=B=H=0$, is called the **general rational integral function of the second degree.** The terms containing x^2, y^2, and xy are terms of the *second* degree.

Show that the functions ax^2+by^2+c and xy are special cases of the general function.

219. General quadratic equation. The equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

is the most **general equation of the second degree.** Any *pair of values* of x and y which satisfies this equation is a **solution** of the equation.

Before taking up the study of the *general* equation we shall consider in this chapter and in chapter xiv some of the *special* cases.

The Circle

220. Standard equation of the circle. *The circle may be defined as the locus of a point moving in a plane so that it is always equidistant from a fixed point.* To find the equation of the circle, Fig. 73, denote the co-ordinates of the center by (h, k), the co-ordinates of *any* point P on the circle by (x, y), and the distance from P to C by r. Then, according to the definition above, $CP=r$ for any position of P.

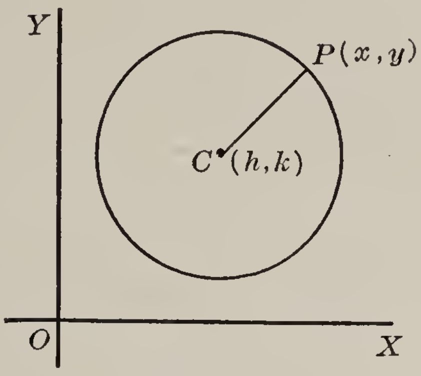

FIG. 73

Show that

$$CP=\sqrt{(x-h)^2+(y-k)^2}$$

$$\therefore \quad \overline{CP}^2=(x-h)^2+(y-k)^2$$

Hence the equation

$$(x-h)^2+(y-k)^2=r^2 \dots \qquad (1)$$

is satisfied by the co-ordinates (x, y) of *any* point P on the circle.

Show that the co-ordinates of no other point satisfy equation (1).

Equation (1) is a standard equation of a circle whose center is $C(h, k)$, and whose radius is equal to r.

If $h=k=0$, the *center of the circle is at the origin* and the equation reduces to

$$x^2+y^2=r^2$$

EXERCISES

Write the equations of the following circles and make a sketch of each:

1. Center $(+2, +4)$, radius $=6$

2. Center $(-3, 2)$, radius $=5$

3. Center (1, 0), radius $=7$

4. Center (3, 4) and passing through the origin

5. Center on the x-axis, radius $=a$

6. Center on the y-axis, radius $=b$

7. Tangent to x-axis, radius $=4$

8. Tangent to both axes, radius $=8$

221. General equation of the circle. The equation

$$(x-h)^2+(y-k)^2=r^2 \tag{1}$$

when expanded reduces to

$$x^2+y^2-2hx-2ky+(h^2+k^2-r^2)=0 \tag{2}$$

Comparison of equation (2) with the general quadratic equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0 \tag{3}$$

suggests to let $H=0$, and $A=B$. This gives the equation

$$\mathbf{Ax^2+Ay^2+2Gx+2Fy+C=0} \tag{4}$$

Equation (4) may be shown to be the equation of a circle for all real values of A, G, F, and C except $A=0$, as follows: First, divide every term by A. This gives

$$x^2+y^2+2\frac{G}{A}x+2\frac{F}{A}y+\frac{C}{A}=0 \tag{5}$$

By comparing (5) and (2) it is seen that for

$$h=-\frac{G}{A},\ k=-\frac{F}{A},\ \text{and}\ \frac{C}{A}=\frac{G^2}{A^2}+\frac{F^2}{A^2}-r^2 \tag{6}$$

equation (5) reduces to equation (2).

Hence, to reduce equation (5) to equation (2), add and subtract $\frac{G^2}{A^2}$ and $\frac{F^2}{A^2}$ to complete the squares in equation (5). This gives the equation

$$\left(x^2+2\frac{G}{A}x+\frac{G^2}{A^2}\right)+\left(y^2+2\frac{F}{A}y+\frac{F^2}{A^2}\right)+\left(\frac{C}{A}-\frac{G^2}{A^2}-\frac{F^2}{A^2}\right)=0$$

This may be written

$$\left(x+\frac{G}{A}\right)^2+\left(y+\frac{F}{A}\right)^2=\left(\sqrt{\frac{G^2}{A^2}+\frac{F^2}{A^2}-\frac{C}{A}}\right)^2 \qquad (7)$$

Equation (7) expresses the fact that the square of the distance from (x, y) to $\left(-\frac{G}{A}, -\frac{F}{A}\right)$ is constant and equal to $\frac{G^2}{A^2}+\frac{F^2}{A^2}-\frac{C}{A}$. This means that equation (7), which is only another form of equation (4), is the equation of a circle whose center is at the point $\left(-\frac{G}{A}, -\frac{F}{A}\right)$, and whose radius is equal to

$$\frac{\sqrt{G^2+F^2-CA}}{A}$$

Note the following *special cases:*

1. When $G^2+F^2=C \cdot A$, the radius is zero, the equation is satisfied only by the point $(x, y)=\left(-\frac{G}{A}, -\frac{F}{A}\right)$, and the circle is a *point circle.*

2. When $G^2+F^2<C \cdot A$, equation (7) is not satisfied by any real values of x and y. In this case we speak of the circle as an *imaginary circle.*

It follows that *an equation of the form*

$$\mathbf{Ax^2+Ay^2+2Gx+2Fy+C=0} \qquad (8)$$

represents a circle if $A \neq 0$, *and if* $G^2+F^2>A \cdot C$. *Equation* (8) *is the* **general equation of the circle.**

$$\left.\begin{array}{l}\textit{The center is the point}\\ \qquad (h, k)=\left(-\dfrac{G}{A}, -\dfrac{F}{A}\right)\\ \textit{The length of the radius is}\\ \qquad r=\sqrt{\dfrac{G^2}{A^2}+\dfrac{F^2}{A^2}-\dfrac{C}{A}}\end{array}\right\} \quad (9)$$

222. Special positions of the circle. Equation (8) of the circle is of a simple form for special positions of the circle.

If the center is at the origin,

$$-\frac{G}{A}=h=0, \quad -\frac{F}{A}=k=0, \quad \frac{C}{A}=-r^2$$

and the equation is

$$x^2+y^2=r^2$$

If the center is on the x-axis,

$$k=-\frac{F}{A}=0 \text{ and } F=0$$

If the center is on the y-axis,

$$h=-\frac{G}{A}=0 \text{ and } G=0$$

EXERCISE

Write the equation of the circle whose center is on the x-axis; on the y-axis; on the positive side of the x-axis and a distance r from the origin.

223. To find the co-ordinates of the center and radius of a given circle. The co-ordinates of the center and the radius of a given circle may be found either by formula (9), § 221, or by changing the given equation to form (1), § 220. The following example illustrates the methods:

Let the equation of a circle be

$$7x^2+7y^2-4x-y-3=0$$

1. To find the center and radius, divide by 7 and complete the squares. This gives the equation

$$x^2-\frac{4}{7}x+\frac{4}{49}+y^2-\frac{1}{7}y+\frac{1}{196}=\frac{3}{7}+\frac{4}{49}+\frac{1}{196}$$

or

$$(x-\frac{2}{7})^2+(y-\frac{1}{14})^2=\frac{101}{196}$$

Hence

$$h=\frac{2}{7},\ k=\frac{1}{14},\ r=\frac{\sqrt{101}}{14}$$

2. According to formula (9), § 221,

$$h=-\frac{G}{A}=\frac{2}{7},\ k=-\frac{F}{A}=\frac{1}{14},\ r=\sqrt{\frac{2^2}{7^2}+\frac{(\frac{1}{2})^2}{7^2}+\frac{3}{7}}=\frac{\sqrt{101}}{14}$$

EXERCISES

Find the centers and radii of the following circles. Draw the circle, if possible.

1. $x^2+y^2=-4x$

2. $x^2+y^2+6y=0$

3. $x^2+y^2-6x-8y=24$

4. $2x^2+2y^2+10x-6y-1=0$

5. $x^2+y^2+10x+110=0$

6. $4x^2+4y^2-4x+16y-19=0$

7. Prove that the co-ordinate axes are axes of symmetry of the circle $x^2+y^2=r^2$.

Show that $x=\pm\sqrt{r^2-y^2}$, $y=\pm\sqrt{r^2-x^2}$. Use § 94.

8. Prove that the circle $x^2+y^2=r^2$ is symmetric with respect to the origin.

Show that if (x_1, y_1) lies on the circle, $(-x_1, -y_1)$ is also on the circle.

9. Find the intercepts of the circle $ax^2+ay^2+bx+cy+d=0$ and compare the product of the intercepts on the x-axis with that of the intercepts on the y-axis.

224. The equation of a circle derived from given conditions. The equation

$$Ax^2+Ay^2+2Gx+2Fy+C=0$$

contains four constants. However, by dividing by A and substituting for $\frac{2G}{A}$, $\frac{2F}{A}$, and $\frac{C}{A}$ the constants b, c, and d, respectively, the equation changes to the form

$$x^2+y^2+bx+cy+d=0 \tag{1}$$

which involves only *three* arbitrary constants. Similarly the equation of a circle in the form

$$(x-h)^2+(y-k)^2=r^2 \tag{2}$$

contains three arbitrary constants, h, k, and r.

Geometrically a circle may be determined by *three* conditions, e.g., by three points not upon the same straight line. To obtain the equation the co-ordinates of the given points may be substituted in equations (1) or (2) and the resulting equations solved simultaneously. Exercises 1 and 2 below illustrate the method.

EXERCISES

1. Find the equation of the circle circumscribed about a triangle whose vertices are $A(-1, 2)$, $B(0, -3)$, $C(2, 3)$.

Substituting the co-ordinates of A, B, and C in equation (1),

$$\begin{aligned}1+4-b+2c+d&=0\\0+9+0-3c+d&=0\\4+9+2b+3c+d&=0\end{aligned}$$

Solving for b, c, and d, we have

$$b=-\tfrac{11}{4},\ c=\tfrac{1}{4}\ d=-\tfrac{33}{4}$$

Show that the equation of the circle is

$$4x^2+4y^2-11x+y-33=0$$

2. Find the equation of a circle which passes through the points (0, 4) and (6, 0), and whose radius is $\sqrt{13}$.

Substituting in equation (2) we have

$$(0-h)^2+(4-k)^2=13$$
$$(6-h)^2+(0-k)^2=13$$

Solve this system to determine the values of h and k.

Find the equations of a circle passing through the following points:

3. (0, 0), (5, 0), (0, 4)

4. (−1, 2), (0, −3), (2, 3)

5. $(a, 0)$, $(-a, 0)$, $(0, b)$

6. (2, −1), (3, −2), origin

7. Find the equation of the circle the center of which is at the point (3, 4), and which passes through the point (4, −3).

8. Prove that the locus of a point which moves so that the sum of the squares from two fixed points is constant is a circle.

9. Find the equation of a circle the center of which is (6, 8), and which touches the line $4x+3y+1=0$.

10. Find the equation of the circle circumscribed about the triangle whose sides are determined by the equations $x+y+1=0$, $x-y-1=0$, $y=4$.

11. Find the locus of a point twice as far from the origin as from the point (0, 6).

12. Find the locus of the vertex of a triangle whose distances from the other two vertices are in the ratio $\frac{m}{n}$.

13. Find the equation of the circle passing through the point (−2, 5) and tangent to both axes.

14. Find the equation of the circle passing through the points (2, −3) and (4, −1) and having its center on the line $3y+x=18$.

15. Find the co-ordinates of the point of intersection of

$$\begin{cases} x+2y+1=0 \\ x^2+y^2-12x=0 \end{cases}$$

Tangent to a Circle

225. Differentiation of implicit functions. It has been seen in § 106 that the slope of the tangent to a curve $y=f(x)$ at the point (x_1, y_1) is the value of $f'(x)$ for $x=x_1$.

To find $f'(x)$ when the function is of the form $f(x, y)$, we may first solve the equation $f(x, y)=0$ for y, and then find the derivative of y. This process is often inconvenient, as for example in the case of the quadratic function

$$f(x, y)=Ax^2+2Hxy+By^2+2Gx+2Fy+C$$

226. Slope of the tangent to $f(x, y)=0$. A simple process of finding the slope of the tangent to the curve $f(x, y)=0$ may be derived as follows:

Let $\frac{k}{h}$ be the slope of the secant P_1P_2, Fig. 74. Since both points, $P_1(x_1, y_1)$ and P_2, $x+h$, $y+k$, lie on the curve $f(x, y)=0$, their co-ordinates satisfy the equation.

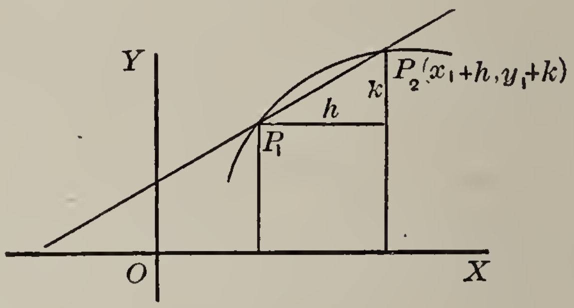

Fig. 74

Hence,

$$f(x_1+h, y_1+k)=A(x_1+h)^2+2H(x_1+h)(y_1+k)+B(y_1+k)^2$$
$$+2G(x_1+h)+2F(y_1+k)+C=0$$

and

$$f(x_1, y_1)=Ax_1^2+2Hx_1y_1+By_1^2+2Gx_1+2Fy_1+C=0$$

Subtracting the second equation from the first gives

$$A(2x_1h+h^2)+2H(x_1k+y_1h+hk)+B(2y_1k+k^2)+2Gh$$
$$+2Fk=0$$

$$\therefore \quad k(2Hx_1+2By_1+2F+Bk)+h(2Ax_1+2Hy_1+2G+Ah+2Hk)=0$$

$$\therefore \quad \frac{k}{h}=-\frac{2Ax_1+2Hy_1+2G+Ah+2Hk}{2Hx_1+2By_1+2F+Bk}$$

which is the slope of the secant P_1P_2.

The slope m of the tangent to $f(x, y)=0$ at the point (x_1, y_1) is the limit of $\frac{k}{h}$ as $k\rightarrow 0$, $h\rightarrow 0$.

Hence

$$m=-\frac{2Ax_1+2Hy_1+2G}{2Hx_1+2By_1+2F}$$

or

$$m=-\frac{Ax_1+Hy_1+G}{Hx_1+By_1+F}$$

This shows that the slope of the tangent to $f(x, y)=0$ may be obtained briefly by the following rule:

1. *Differentiate* $f(x, y)$ *with respect to* x, *regarding* y *as constant. This gives* $f_x'(x_1, y_1)=2(Ax_1+Hy_1+G)$.
2. *Differentiate* $f(x, y)$ *with respect to* y, *regarding* x *as a constant. This gives* $f_y'(x_1, y_1)=2(Hx_1+By_1+F)$.
3. *Divide* $f_x'(x_1, y_1)$ *by* $f_y'(x_1, y_1)$ *and prefix the* $-$ *sign.*

EXERCISES

Find the slope m of each of the following curves at the point $P(x, y)$:

1. $y-x^2=0$

2. $x^2+y^2=4$

3. $x^2-xy+3=0$

4. $x^3+xy+2=0$

5. $2y^2-3x+5=0$

6. $x^2y-6x+3=0$

227. Equation of the tangent to $f(x, y)=0$. Let (x_1, y_1) be the point of contact, and let m be the slope of the tangent. Then according to the point-slope equation of a straight line the equation of the tangent to $f(x, y)=0$ is

$$y-y_1=m(x-x_1)$$

228. Equation of the tangent to a circle. To find the slope of the tangent to the circle

$$f(x, y)=x^2+y^2-r^2=0$$

at the point (x_1, y_1), find $f_x'(x_1, y_1)=2x_1$, and $f_y'(x_1, y_1)=2y_1$. Hence

$$m=-\frac{x_1}{y_1}$$

The equation of the tangent is

$$y-y_1=-\frac{x_1}{y_1}(x-x_1)$$

which reduces to

$$y_1y+x_1x=y_1^2+x_1^2$$

or

$$x_1x+y_1y=r^2$$

Note the resemblance between this equation and that of the given circle and formulate a rule for remembering the equation of the tangent.

229. Normal to a curve. The **normal** to a curve is the line perpendicular to the tangent to the curve at the point of contact.

230. Slope of the normal. The **slope** of the normal is the negative reciprocal of the slope of the tangent, § 57.

Show that the slope of the normal to the circle $x^2+y^2=r^2$ is

$$m=\frac{y_1}{x_1}$$

EXERCISES

1. Find the equation of the normal to the circle at the point (x_1, y_1).

Find the equation of the tangent and normal to each of the following curves at the point (x_1, y_1):

2. $y=x^2$

3. $y^2=x^3$

4. $xy=4$

5. $x^2+y^2=25$

6. Show that the equation of the tangent to the circle

$$Ax^2+Ay^2+2Gx+2Fy+C=0$$

at the point (x_1, y_1) is

$$Ax_1x+Ay_1y+G(x+x_1)+F(y+y_1)+C=0$$

7. Find the equation of a circle the center of which is (3, 4) and which is tangent to the line $4x-3y+22=0$.

8. Find the equations of the tangent and normal to the circle $x^2+y^2-4x-6y-12=0$ at the point (6, 0).

9. Find the equations of the tangent and normal to the circle $x^2+y^2-3x-11y-40=0$ at the point (2, −3).

10. Find the equation of the circle tangent to the x-axis and passing through the points (4, 9) and (−3, 2).

11. Find the equations of the tangents to the circle $x^2+y^2=25$ at the ends of the diameter passing through the point (3, −4).

12. At the points of intersection of the circle $x^2+y^2=25$ and the straight line $x+2y=10$ tangents are drawn to the circle. Find the point of intersection of the tangents and the angle between them.

13. Determine the relation between a, b, and c such that the line $ax+by+c=0$ is tangent to the circle $x^2+y^2=r^2$.

14. Find the point of contact and the equation of the tangent to the circle $(x-2)^2+(y-3)^2=r^2$, having the slope $\frac{-x_1-2}{y_1-3}$.

15. Find the angle between the lines $3x-2y=6$ and $x^2+y^2=36$.

231. Subtangent. Subnormal. The projection QN of the normal PN, Fig. 75, on the x-axis is called the **subnormal.** The projection TQ of the tangent TP is called the **subtangent.**

Since $\tan \alpha = m = \frac{y_1}{TQ}$, it follows that the

$$\textbf{subtangent } TQ = \frac{y_1}{m}$$

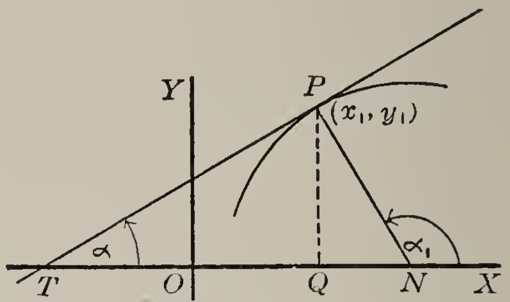

FIG. 75

Since $\tan \alpha_1 = n = \frac{y_1}{QN}$, it follows that $QN = \frac{y_1}{n}$.

$$\text{But } n = -\frac{1}{m}$$

$\therefore$ the **subnormal** $QN = -my_1$.

EXERCISES

1. Find the length of the tangent PT, Fig. 75, in terms of the ordinate of P and the slope of PT.

2. Find the length of the normal PN, Fig. 75, in terms of the ordinate of P, and the slope of the tangent at P.

3. Find the lengths of the subtangent, subnormal, tangent, and normal to the circle $x^2+y^2=r^2$ at the point (x_1, y_1).

232. The length of a tangent to a circle from a given point. Denoting TP, Fig. 76, by t, the radius TC by r, and CP by d, show that

$$t^2 = d^2 - r^2$$

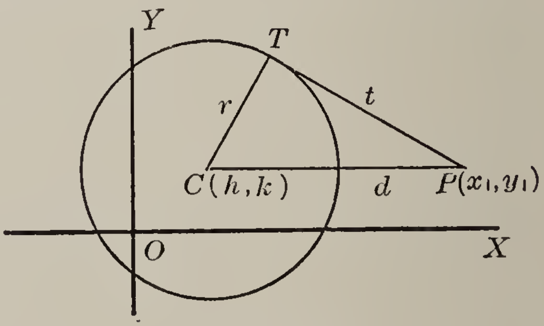

FIG. 76

But by § 47

$$d^2 = (x_1 - h)^2 + (y_1 - k)^2$$

Hence

$$\boldsymbol{t^2 = (x_1 - h)^2 + (y_1 - k)^2 - r^2} \qquad (1)$$

Equation (1), p. 206, shows that *the square of the length of the tangent from a given point* (x_1, y_1) *to the circle* $(x-h)^2+(y-k)^2=r^2$ *is found by substituting the co-ordinates of the point in the equation of the circle.*

EXERCISES

1. Show that the length of the tangent drawn from a point (x_1, y_1) to the circle $x^2+y^2+2Gx+2Fy+C=0$ is given by the formula

$$t^2=x_1^2+y_1^2+2Gx_1+2Fy_1+C$$

2. Find the length of the tangent to the circle

$$x^2+y^2-4x-6y+10=0$$

from the point (4, 5).

3. Determine a point such that the length of the tangent drawn from that point to the circles $x^2+y^2-8x+12=0$ and $x^2+y^2-1=0$ is equal to 6.

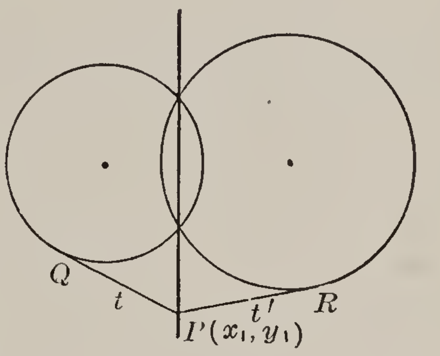

Fig. 77

233. Systems of circles. Radical axis. Let the equations of two given circles, Fig. 77, be

$$x^2+y^2+2G_1x+2F_1y+C_1=0 \quad (1)$$

and

$$x^2+y^2+2G_2x+2F_2y+C_2=0 \quad (2)$$

Then the equation

$$\boldsymbol{(x^2+y^2+2G_1x+2F_1y+C_1)+k(x^2+y^2+2G_2x+2F_2y+C_2)=0} \quad (3)$$

represents a *system of circles passing through the points common to the given circles.*

For we may show that equation (3) in general represents a circle, and that the co-ordinates of a point satisfy equation (3) if they satisfy (1) and (2).

When $k=-1$ equation (3) reduces to

$$2(G_1-G_2)x+2(F_1-F_2)y+C_1-C_2=0$$

which is the equation of a straight line passing through the points of intersection of the given circles. This straight line is called the **radical axis** of the two given circles.

The radical axis of two intersecting circles contains the common chord.

EXERCISES

1. Show that the tangents drawn to two circles from any point on the radical axis are equal.

Denote the equations of the circles by

$$x^2+y^2+2Gx+2Fy+C=0$$

and $$x^2+y^2+2G'x+2F'y+C'=0$$

Then

$$t^2=x_1^2+y_1^2+2Gx_1+2Fy_1+C$$

and $$t'^2=x_1^2+y_1^2+2G'x_1+2F'y_1+C'$$

$$\therefore\quad t^2-t'^2=2(G-G')x_1+2(F-F')y_1+C-C'$$

Since (x_1, y_1) is on the radical axis,

$$2(G-G')x_1+2(F-F')y_1+C-C'=0$$

$$\therefore\quad t^2-t'^2=0$$

and $$t=t'$$

2. Find the equation of the radical axis of the following circles:

$$x^2+y^2-4x-6y-1=0$$
$$x^2+y^2+6x+4y+7=0$$

3. Show that the radical axis of two circles is perpendicular to the line joining their centers.

4. If the radical axes of three circles, taken in pairs, are not parallel, prove that they are concurrent.

5. Find the point of intersection of the radical axes (*radical center*) of the three following circles:

$$
\begin{aligned}
x^2+y^2-6x-4y+12&=0\\
x^2+y^2-20x+91&=0\\
x^2+y^2&=64
\end{aligned}
$$

6. Find the circle passing through the intersections of the following circles:

$$
\begin{aligned}
x^2+y^2+2x-14y+25&=0\\
x^2+y^2&=25
\end{aligned}
$$

and through the point (6, 8).

7. Find the equation of the radical axis of the following circles:

$$
\begin{aligned}
x^2+y^2+6x-16&=0\\
x^2+y^2-10x+21&=0
\end{aligned}
$$

Graph the circles and the radical axis.

234. The polar equation of the circle. The polar equation of a circle, Fig. 78, may be obtained from the equation

$$(x-h)^2+(y-k)^2=r^2$$

by making the following substitutions:

$$
\begin{cases}
x=\rho\cos\theta\\
y=\rho\sin\theta\\
h=b\cos\alpha\\
k=b\sin\alpha
\end{cases}
$$

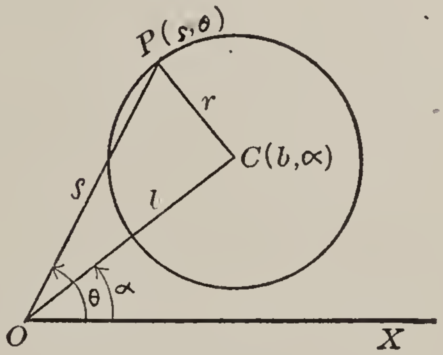

FIG. 78

It may also be obtained directly from the law of cosines

$$\rho^2+b^2-2\overline{OP}\cdot\overline{OC}\cos(\theta-\alpha)=r^2$$

This gives

$$\boldsymbol{\rho^2-2\rho b\cos(\theta-\alpha)+b^2-r^2=0} \qquad (1)$$

Equation (1), p. 209, represents a circle because it states the relation between ρ and θ for any point on the circle.

When the *pole is at the center,* $b=0$ and equation (1) reduces to

$$\rho=r$$

When the *pole is on the circle,* $b=r$ and equation (1) reduces to

$$\rho=2r \cos (\theta-\alpha)$$

This may be written

$$\rho=2r \cos \theta \cos \alpha+2r \sin \theta \sin \alpha$$

which reduces to

$$\rho=p \cos \theta+q \sin \theta$$

where p and q are the intercepts on the polar axis and on the 90°– axis respectively.

When the *pole is on the circle and the polar axis tangent to the circle,* equation (1) reduces to

$$\rho=2r \sin \theta$$

235. Parametric equations. If the co-ordinates (x, y) of a point on a curve can be expressed as functions of a third variable t, the equations

$$x=f(t) \text{ and } y=g(t) \qquad (1)$$

are **parametric equations,** and t is a **parameter.** The equation of the curve in Cartesian co-ordinates may then be obtained by eliminating t from equation (1).

For example, let

$$x=a \cos t,\ y=a \sin t \qquad (2)$$

To eliminate t, square both equations and add. This gives

$$x^2+y^2=a^2 \cos^2 t+a^2 \sin^2 t=a^2$$

or

$$x^2+y^2=a^2$$

the equation of a circle.

Hence the *parametric equations* of a circle are

$$x = a \cos t, \quad y = a \sin t$$

The parameter t represents the vectorial angle.

To graph these equations, find corresponding values of x and y for assumed values of t and plot the points (x, y).

EXERCISES

1. Let $a=5$ and tabulate corresponding values of t, x, and y which satisfy equations (2), § 234. Draw the graph.

2. Show that the equation $x=t^2$ and $y=2t$ are parametric equations of the parabola.

3. Graph the circle

$$\begin{cases} x=4 \sin t \\ y=4 \cos t \end{cases}$$

Summary

236. This chapter has taught the meaning of the following terms:

general quadratic function in two variables

normal, subnormal

subtangent

radical axis

systems of circles

237. The following standard equations of a circle have been developed in this chapter:

1. $(x-h)^2+(y-k)^2=r^2$, where (h, k) are the coordinates of the center and r is the radius.

2. $x^2+y^2=r^2$, where r is the radius and the center is at the origin.

3. $Ax^2+Ay^2+2Gx+2Fy+C=0$ if $G^2+F^2>CA$, and $A \neq 0$, the center being at the point $\left(-\frac{G}{A}, -\frac{F}{A}\right)$, and the radius being equal to $\sqrt{\frac{G^2+F^2+CA}{A}}$.

4. $\rho^2-2b\rho\cos(\theta-\alpha)+b^2-r^2=0$, where (b, α) are the polar co-ordinates of the center.

5. $\begin{cases} x=a\cos t \\ y=a\sin t \end{cases}$, where a is the radius and the parameter t represents the vectorial angle.

238. The following are special cases of the general equation of a circle:

1. When $G^2+F^2=C\cdot A$, $r=0$, and the circle is a *point circle:*

$$(x, y)=\left(-\frac{G}{A}, -\frac{F}{A}\right)$$

2. When $G^2+F^2<C\cdot A$ the circle is an *imaginary circle.*

239. The following are special cases of the polar equation of a circle:

1. $\rho=r$ when the pole is at the center.

2. $\rho=2r\cos(\theta-\alpha)$ when the pole is on the circle, which may be written in the form

$$\rho=a\cos\theta+b\sin\theta$$

where a and b are the intercepts on the polar axis and on the 90°– axis respectively.

3. $\rho=2r\sin\theta$ when the pole is on the circle and the polar axis tangent to the circle.

240. The equation of a circle may be derived from three given conditions, e.g., from three given points.

241. The following formulas have been proved:

1. The slope of the tangent to $f(x, y)=0$ at the point (x_1, y_1):

$$m=-\frac{f_x'(x_1, y_1)}{f_y'(x_1, y_1)}$$

2. The equation of the tangent to the circle $x^2+y^2=r^2$:

$$x_1x+y_1y=r^2$$

3. The equation of the tangent to the circle

$$Ax^2+Ay^2+2gx+2Fy+C=0:$$

$$Ax_1x+Ay_1y+G(x+x_1)+F(y+y_1)+C=0$$

4. The subtangent and subnormal at the point (x_1, y_1):

$$\text{subtangent}=\frac{y_1}{m}$$

$$\text{subnormal}=-my_1$$

where m is the slope of the tangent.

5. The length of a tangent from the point (x_1, y_1) to the circle:

$$t^2=(x_1-h)^2+(y_1-k)^2-r^2$$

6. A system of circles passing through the points common to two given circles:

$$(x^2+y^2+2G_1x+2F_1y+C_1)+k(x^2+y^2+2G_2x+2F_2y+C_2)=0$$

7. The equation of the radical axis of two given circles:

$$2(G_1-G_2)x+2(F_1-F_2)y+(C_1-C_2)=0$$

CHAPTER XIV

ELLIPSE. HYPERBOLA. PARABOLA

The Ellipse

242. Ellipse. Foci. *The locus of a point the sum of whose distances from two fixed points is constant is called an* **ellipse.**

The fixed points are the **foci** of the ellipse.

Let F and F_1 be the foci, Fig. 79, and let P be any point on the locus.

Then FP and F_1P are the **focal radii** of the ellipse.

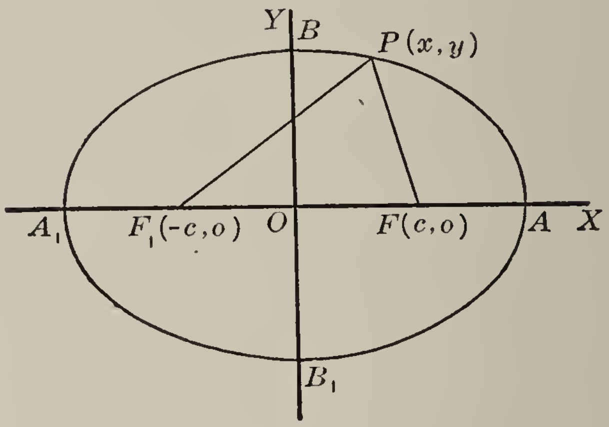

FIG. 79

243. Equation of the ellipse. Denote the constant sum $FP+PF_1$ by $2a$. Let O be the mid-point of FF_1, and let $OF=OF_1=c$. Take the origin at O and OF as the x-axis. Then the *co-ordinates of the foci* are $(c, 0)$ and $(-c, 0)$.

Show that the equation $F_1P+PF=2a$, written in terms of the co-ordinates x and y is

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

Squaring both members, we have

$$x^2+y^2+c^2+\sqrt{(x^2+y^2+c^2+2cx)(x^2+y^2+c^2-2cx)}=2a^2$$

$$\therefore \quad (x^2+y^2+c^2)^2-4c^2x^2=[2a^2-(x^2+y^2+c^2)]^2$$

$$\therefore \quad -4c^2x^2=4a^4-4a^2(x^2+y^2+c^2)$$

$$\therefore \quad (a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2)$$

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

Substituting b^2 for a^2-c^2 this equation reduces to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This is a *standard form* of the equation of the ellipse.

244. Discussion of the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

When we substitute for x the value 0, the corresponding values of y are $+b$ and $-b$.

When $y=0$, we find x to be $+a$ and $-a$.

Hence the *intercepts* on the x-axis and y-axis are a and b respectively.

Hence $OA=OA_1=a$, $OB=OB_1=b$.

Since $F_1B=BF$, and since $F_1B+BF=2a$, it follows that $F_1B=BF=a$.

This may be used to derive the relation

$$a^2=b^2+c^2$$

A_1A and B_1B are called the **major** and **minor axis,** respectively.

If a is less than b the foci of the ellipse lie on the y-axis.

The more nearly a and b are equal, the smaller is the value of c and the nearer are the foci to the origin. When $a=b$ the value of c is zero, and the equation of the ellipse reduces to that of a circle $x^2+y^2=a^2$.

Hence the circle may be considered as a special case of an ellipse, the two foci coinciding at the center.

Extent of the ellipse.—Solving the equation

$$b^2x^2+a^2y^2=a^2b^2$$

for x and for y, we have

$$x=\pm\frac{a}{b}\sqrt{b^2-y^2} \text{ and } y=\pm\frac{b}{a}\sqrt{a^2-x^2} \qquad (1)$$

Hence for real values of x the value of y must be numerically less than b (written briefly $|y|<b$); and for real values of y, x must be numerically less than $a(|x|<a)$.

This shows that the ellipse lies wholly within the rectangle $MNRS$, Fig. 80, whose center is the origin and whose sides are parallel to the axes and equal to $2a$ and $2b$ respectively.

Symmetry of the ellipse.—Equations (1) show that the ellipse is symmetrical with respect to the axes, as for every value of x for which y is real there are two values of y equal numerically and opposite in sign. Similarly, for every value of y for which x is real there are two values of x equal numerically and opposite in sign.

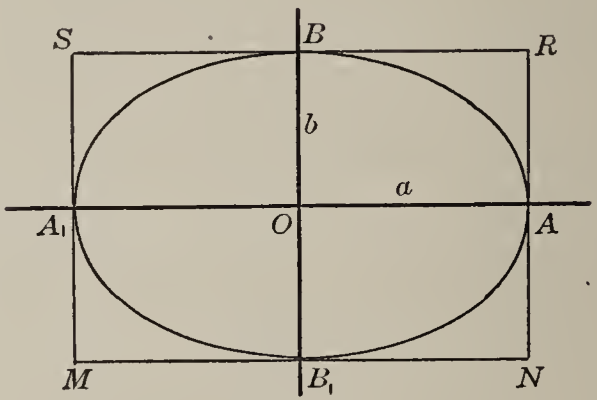

FIG. 80

245. Focal radii. Denoting the lengths of the *focal radii* F_1P and FP, Fig. 81, by r_1 and r, we have

$$r=\sqrt{(x-c)^2+y^2}$$

$$=\sqrt{(x-c)^2+\frac{b^2(a^2-x^2)}{a^2}}$$

$$=\sqrt{\frac{(x^2-2cx+c^2)a^2+a^2b^2-b^2x^2}{a^2}}$$

Substitute b^2+c^2 for a^2. Then

$$r=\sqrt{\frac{a^4-2ca^2x+c^2x^2}{a^2}}$$

$$=\left(\frac{a^2-cx}{a}\right)$$

$$\therefore \quad r=\left(a-\frac{c}{a}x\right)$$

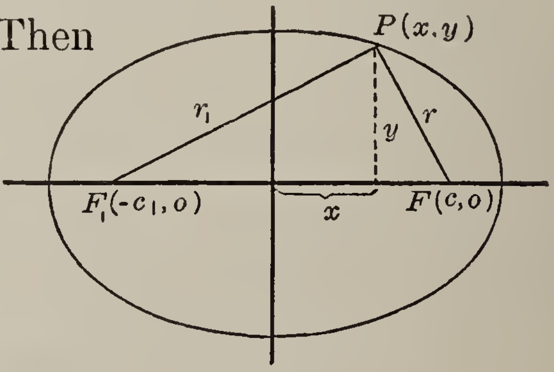

FIG. 81

Similarly,

$$r_1=\left(a+\frac{c}{a}x\right)$$

It is easily seen that these results satisfy the condition

$$r+r_1=2a$$

246. Mechanical construction of an ellipse. The relation between the focal radii shown in § 245 suggests that the arc of an ellipse may be traced mechanically as follows:

The ends of a string are fastened to two points, F_1 and F, Fig. 82.

The string is kept taut by means of a moving pencil point P. This point then describes the arc of an ellipse.

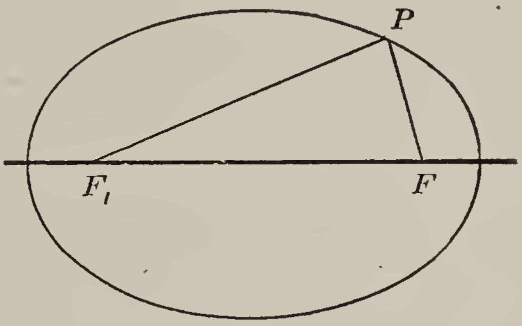

Fig. 82

247. Eccentricity. The ratio of $2c$, the distance between the two foci, to the major axis $2a$, is the **eccentricity** e of the ellipse.

Thus

$$e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$$

Since $a>c$, it follows that the *eccentricity of the ellipse is less than unity.*

EXERCISES

1. Show as in § 243 that the equation of an ellipse, with the center at the origin, whose foci are on the y-axis is $\frac{y^2}{a^2}+\frac{x^2}{b^2}=1$ or $a^2x^2+b^2y^2=a^2b^2$.

2. Find the semi-major axis, the semi-minor axis, and the eccentricity of each of the following ellipses. Sketch the ellipse in each case.

$$\frac{x^2}{16}+\frac{y^2}{9}=1;\quad x^2+2y^2=4;\quad 3x^2+2y^2=6;\quad 15x^2+14y^2=960$$

3. Find the co-ordinates of the points of intersection of $x^2+3y^2=3$ and $x-y=1$; also find the co-ordinates of the foci of the ellipse.

4. Find the equation of the ellipse the major axis of which is 6 and the foci of which are $(\pm 2, 0)$.

5. Find the equation of the ellipse the major axis of which is 10 and which passes through the point (3, 1).

6. Find the equation of the ellipse whose vertices are (9, 0), (−9, 0), (0, 6), and (0, −6).

7. The sum of the focal radii of an ellipse is 10 and the distance between the foci is 8, the origin being at the center. Find the equation.

8. Find the points of intersection of the straight line $x+3y=3$ with the ellipse $9x^2+16y^2=144$.

9. Find the locus of the mid-points of the ordinates of the ellipse whose equation is $b^2x^2+a^2y^2-a^2b^2=0$.

10. Find the locus of a point P on a straight line AB which moves so that its extremities are always on two lines perpendicular to each other.

Let $AP=m$, $PB=n$.

248. Latus rectum. The **latus rectum** of an ellipse is the chord through a focus perpendicular to the major axis, as MN, Fig. 83.

To find the length of the latus rectum, let $x=\pm c$ in the equation

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

Show that

$$2y=\frac{2b^2}{a}$$

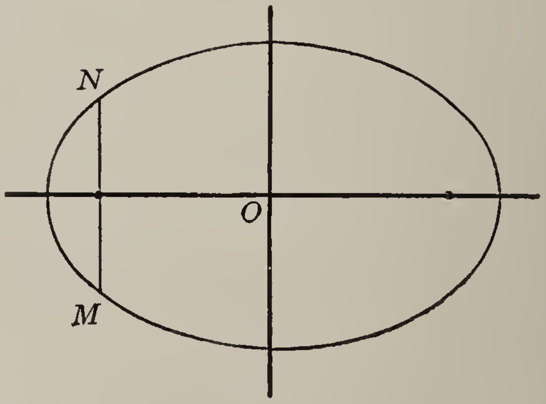

FIG. 83

EXERCISES

1. Find the length of the latus rectum of each of the following ellipses: $2x^2+3y^2=6$; $25x^2+9y^2=225$

2. Show that the semi-latus rectum is the third proportional to the semi-major and semi-minor axes.

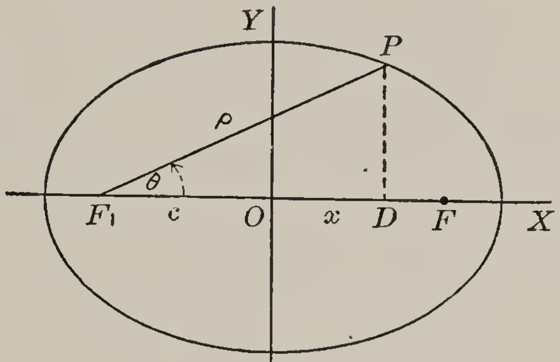

FIG. 84

249. Polar equation of the ellipse. Let F_1, Fig. 84, be the pole and F_1X the polar axis. Then $F_1P=a+ex$, §§ 245, 247. Since $OD=x=F_1D-F_1O=\rho\cos\theta-c$, it follows that

$$F_1P=\rho=a+e\rho\cos\theta-ec$$

$$\therefore \quad (1-e\cos\theta)\rho=a-ec$$

$$\therefore \quad \rho=\frac{a-ec}{1-e\cos\theta}$$

Since $$e=\frac{c}{a}$$

$$\rho=\frac{\dfrac{a^2-c^2}{a}}{1-e\cos\theta}=\frac{\dfrac{b^2}{a}}{1-e\cos\theta}$$

Let $$\frac{b^2}{a}=p$$

Then the **polar equation of the ellipse** takes the form

$$\rho=\frac{p}{1-e\cos\theta}\text{ where } e<1$$

Compare this equation with the polar equation of the parabola, § 97.

250. Discussion of the polar equation. The values of ρ corresponding to $\theta=0°$ and $180°$ give the *intercepts* on the polar axis. The intercepts on the 90°-axis are found by letting $\theta=90°$ and $270°$.

Hence the intercepts are as follows:

θ	0°	90°	180°	270°
ρ	$a+c$	$\frac{b^2}{a}$	$a-c$	$\frac{b^2}{a}$

Extent of the curve.—When $\theta=0$, $1-e\cos\theta$ is least, and the *largest* value of ρ is obtained. When $\theta=180°$, $1-e\cos\theta$ is largest, and ρ has the smallest numerical value. For all values of θ the value of ρ is finite, since $1-e\cos\theta$ cannot equal zero.

Symmetry.—Since $\cos\theta=\cos(-\theta)$, the values of ρ for θ and $(-\theta)$ are the same. Hence the curve is symmetric with respect to the polar axis.

EXERCISES

1. Discuss the equation $\rho=\frac{8}{5-2\cos\theta}$. What curve is represented by this equation?

2. Reduce the equation $\rho=\frac{8}{5-2\cos\theta}$ to the standard polar form.

3. Find a polar equation of the ellipse by changing the equation $b^2x^2+a^2y^2=a^2b^2$ to polar co-ordinates, using the center as pole.

Let $x=\rho\cos\theta$, $y=\rho\sin\theta$.

Substitute these values for x and y in the given equation and simplify.

251. Parametric equations. The equations

$$x=a\cos t,\ y=b\sin t$$

are **parametric equations** of the ellipse. For the equation $b^2x^2+a^2y^2=a^2b^2$ is satisfied by the corresponding values of x and y for all values of t.

To find a geometric meaning of t, construct a circle having the major axis as diameter, Fig. 85.

Extend the ordinate MP of the point $P(x, y)$ until it meets this circle at $P_1(x, y_1)$, and denote the vectorial angle of P_1 by t.

Then $x = a \cos t$,

$$y_1 = a \sin t.$$

Since $y_1 = \sqrt{a^2 - x^2}$, and $y = \frac{b}{a}\sqrt{a^2 - x^2}$, it follows that

$$y = \frac{b}{a} y_1 \text{ and } y_1 = \frac{a}{b} y$$

Hence $\frac{a}{b} y = a \sin t$, by substitution, and $y = b \sin t$.

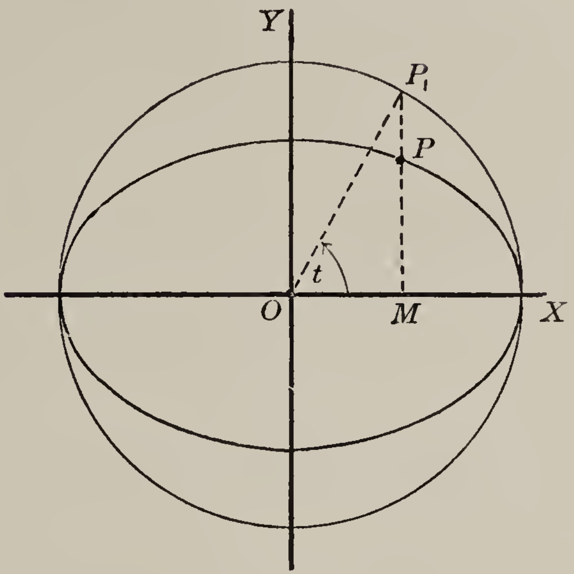

FIG. 85

The angle t is the **eccentric angle** of the point $P(x, y)$ of the ellipse.

252. Geometrical construction of an ellipse. The parametric equations of the ellipse suggest the following construction of the curve.

Let a and b be the semi-major and semi-minor axes.

Draw the concentric circles, Fig. 86, whose radii are a and b respectively.

Draw OP_1 meeting the smaller circle at P_2.

Draw $M_1P_1 \perp OX$ and

$$P_2P \| OX.$$

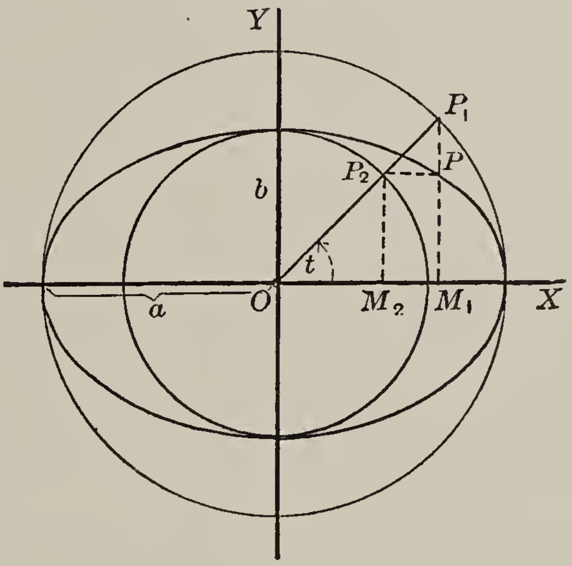

FIG. 86

The point P thus determined is a point of the required ellipse.

For let $t = \angle P_1OM_1$. Then $OM_1 = a \cos t$, and $M_1P = M_2P_2 = b \sin t$. The concentric circles are the **auxiliary circles** of the ellipse.

EXERCISES

1. Construct the curve represented by the following parametric equations:

$$x=t,\ y=\frac{a^3}{a^2+t^2},\ \text{for } a=2$$

This curve is known as the *witch.*

2. Construct the following loci:

$$x=1+t,\ y=\tfrac{1}{2}t^3;\quad x=3t,\ y=t^2(3t-1)$$

253. Equation of the tangent and normal to an ellipse. Let

$$f(x,\ y)=b^2x^2+a^2y^2-a^2b^2$$

Then the *slope of the tangent* at the point $(x_1,\ y_1)$ is

$$-\frac{f_x'(x_1,\ y_1)}{f_y'(x_1,\ y_1)}=-\frac{2b^2x_1}{2a^2y_1}=-\frac{b^2x_1}{a^2y_1},\quad \S\ 226$$

or

$$m=-\frac{b^2x_1}{a^2y_1}$$

$\therefore$ The *slope of the normal* is given by

$$n=\frac{a^2y_1}{b^2x_1}$$

Hence the **equation of the tangent** is

$$y-y_1=m(x-x_1)=-\frac{b^2x_1(x-x_1)}{a^2y_1}$$

Show that this reduces to

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

State how this equation may be obtained directly by inspection from the equation of the ellipse.

Show that the **equation of the normal** is

$$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$$

EXERCISES

1. Find the slope of the tangent and normal to the following curves:

$9x^2+16y^2=73$, at the point $(1, 2)$
$5x^2+18y^2=182$, at the point $(2, -3)$

2. Write the equations of the tangent and normal to each of the following curves:

$4x^2+25y^2=100$, at the point $(4, \frac{6}{5})$
$x^2+4y^2=16$, at the point $(0, -2)$

3. Find the equation of the tangent to the ellipse $\frac{x^2}{8}+\frac{y^2}{2}=1$ having the slope equal to .5.

Find the slope at the point $P(x_1, y_1)$ and then determine (x_1, y_1) by solving the system of equations

$$m_1=.5$$

and
$$\frac{x_1^2}{8}+\frac{y_1^2}{2}=1$$

When (x_1, y_1) are known the equation of the tangent is given in § 253.

4. Find the angle between $x^2+4y^2=5$ and $y=\frac{x^2}{8}$ at one of the points of intersection.

5. Prove that the tangents drawn at the end-points of the latus rectum to a parabola are perpendicular to each other.

6. Prove that the two tangents to the curve $b^2x^2+a^2y^2=a^2b^2$, which are perpendicular to each other, meet in a point which lies on the curve $x^2+y^2=a^2+b^2$.

254. Length of tangent, normal, subtangent, and subnormal.

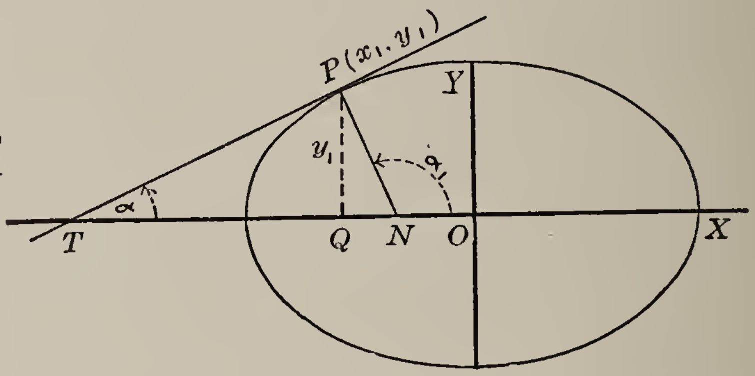

FIG. 87

Show as in § 231 that the *subtangent*, Fig. 87, is given by the equation

$$TQ = \frac{y_1}{m} = -\frac{a^2 y_1{}^2}{b^2 x_1}$$

Show that the *subnormal* is given by the equation

$$QN = my_1 = \frac{-b^2 x_1}{a^2}$$

EXERCISES

1. By means of the theorem of Pythagoras find the length of the tangent PT, and of the normal PN, Fig. 87.

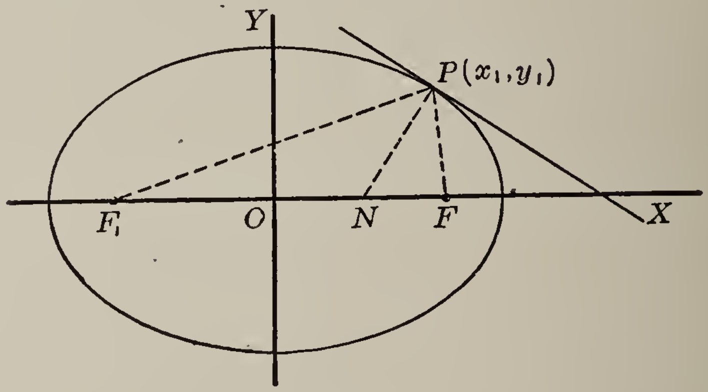

FIG. 88

2. *The angle between the focal radii drawn to a point of an ellipse is bisected by the normal at that point.* Prove.

Proof: The equation of the normal, Fig. 88, is

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

For $$y = 0$$

$$x = ON = -\frac{b^2 x_1}{a^2} + x_1 = \frac{(a^2 - b^2) x_1}{a^2} = e^2 x_1$$

$\therefore$ $$F_1N = F_1O + ON = ae + e^2 x_1$$

and $$NF = OF - ON = ae - e^2 x_1$$

$\therefore$ $$\frac{F_1N}{NF} = \frac{a + ex_1}{a - ex_1} = \frac{F_1P}{PF}$$

Hence, by plane geometry, the normal NP bisects angle F_1PF. This shows that a ray of light, or heat, from one focus of an elliptic surface is reflected to the other focus.

3. By means of exercise 2 show how to construct geometrically the tangent to a given ellipse at any point.

4. Find the equation of the tangent to $9x^2+36y^2=18$, parallel to the line $x-3y+15=0$.

5. Find the equation of the tangent to the ellipse in exercise 4 drawn from an exterior point (1, 2).

Show that the point of contact is found by solving the equation

$$2-y=-\tfrac{1}{4}\frac{x_1}{y_1}(1-x_1), \quad \S\,253$$

and

$$\frac{x_1{}^2}{8}+\frac{y_1{}^2}{2}=1$$

6. Find the equation of the tangent from the point (6, −1) to the ellipse $x^2+4y^2=9$.

7. Find the lengths of the subtangent, subnormal, tangent, and normal to the equation $x^2+4y^2=8$ at the point (2, 1).

The Hyperbola

255. Hyperbola. Foci. The **hyperbola** is *the locus of a point the difference of whose distances from two fixed points is constant.* The fixed points are the **foci** of the hyperbola. The distances from a point on the curve to the foci are the **focal radii.**

256. Equation of the hyperbola. Denote the constant difference F_1P-FP by $2a$, Fig. 89.

Let $F_1F=2c$.

Take the mid-point O of F_1F as origin and OF as x-axis.

Then show that the equation

$$F_1P-FP=2a$$

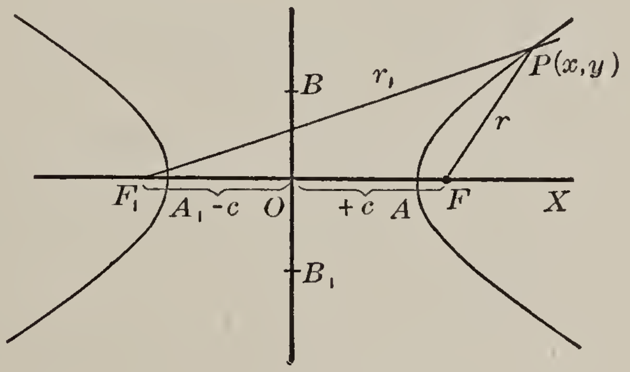

FIG. 89

may be written

$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=2a$$

As in § 243 show that this may be reduced to

$$(a^2-c^2)x^2+a^2y^2=a^2(a^2-c^2) \qquad (1)$$

Since $F_1P-FP<F_1F$, it follows that $2a<2c$, or

$$a<c, \text{ and } a^2<c^2$$

∴ Let $a^2-c^2=-b^2$

Then equation (1) reduces to

$$b^2x^2-a^2y^2=a^2b^2$$

or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

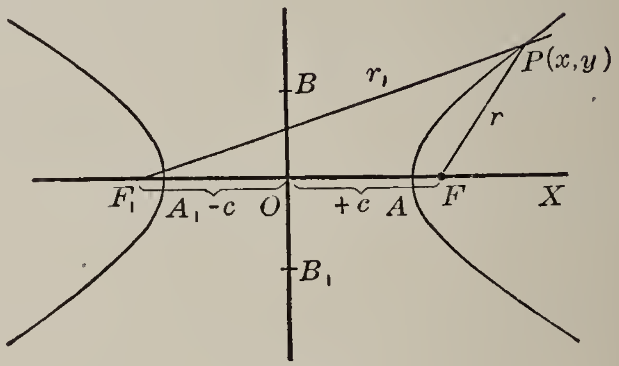

FIG. 89

which is a *standard form* of the equation of the hyperbola.

257. Discussion of the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$**.** For $y=0$ we have $x=\pm a$, which are the *intercepts* on the x-axis. Hence the hyperbola intersects the x-axis in two points A_1 and A.

The segment A_1A is the **transverse axis** of the hyperbola. A_1 and A are the *vertices* of the curve. The segment B_1OB for which $OB_1=OB=b$, is called the **conjugate axis.**

For $x=0$, y is imaginary. Hence the curve does not intersect the y-axis.

Extent of the hyperbola.—Solving the equation

$$b^2x^2-a^2y^2=a^2b^2$$

we have

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2},\ x=\pm\frac{a}{b}\sqrt{y^2+b^2}$$

Hence for numerical values of $x<a$ the values of y are complex. If x is numerically greater than a and increases, y is real and increases. This shows that the curves consist of two infinite branches, no points of which lie in the strip between the lines $x=+a$ and $x=-a$.

Symmetry of the hyperbola.—To every value of $x>a$ correspond two values of y numerically equal and opposite in sign. The x-axis therefore divides each of the branches of the curve into two congruent symmetric parts.

The equation $x=\pm\frac{a}{b}\sqrt{y^2+b^2}$ shows that the curve is symmetrical with respect to the y-axis.

258. Focal radii. As in § 245, we find

$$F_1P=r_1=\frac{cx}{a}+a$$

and

$$FP=r=\frac{cx}{a}-a$$

259. Eccentricity. Since $c>a$, it follows that for the hyperbola the *eccentricity*

$$e=\frac{c}{a}>1$$

260. Equilateral hyperbola. When $a=b$ the equation $b^2x^2-a^2y^2=a^2b^2$ reduces to

$$x^2-y^2=a^2$$

This is called an **equilateral** or **rectangular hyperbola.**

261. Latus rectum. The chord MN, Fig. 90, drawn through a focus and perpendicular to the transverse axis is the **latus rectum.**

Show that

$$MN=2\frac{b^2}{a}$$

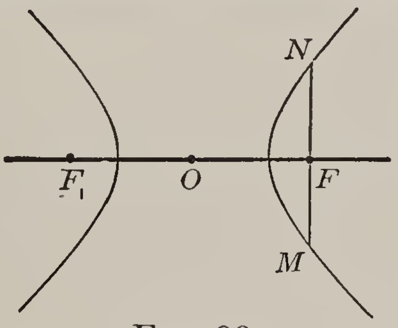

FIG. 90

262. Asymptotes. Let $y=mx$ be the equation of a straight line through the origin. By solving simultaneously the system

$$y=mx$$
$$b^2x^2-a^2y^2=a^2b^2$$

we have the co-ordinates of the points of intersection

$$(x,\ y)=\left(\frac{\pm ab}{\sqrt{b^2-a^2m^2}},\ \frac{\pm amb}{\sqrt{b^2-a^2m^2}}\right)$$

It follows that there is no point of intersection when

$$b^2<a^2m^2, \text{ or } b<am$$

Of special interest are the lines for which

$$b^2=a^2m^2$$

i.e., $m=\pm\frac{b}{a}$, Fig. 91,

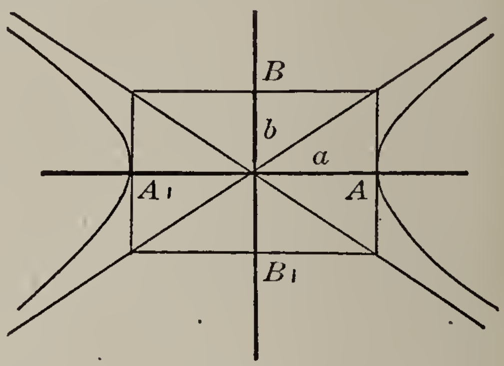

FIG. 91

For, changing the equation

$$y=\pm\frac{b}{a}\sqrt{x^2-a^2}$$

to the form

$$y=\pm\frac{b}{a}x\sqrt{1-\left(\frac{a}{x}\right)^2}$$

it is seen that as x increases without bound the branches of the hyperbola approach more and more the line

$$y=\pm\frac{b}{a}x$$

These lines are called the **asymptotes** of the hyperbola.

To *construct* them, draw the rectangle formed by the lines $x=\pm a$ and $y=\pm b$. Then draw the diagonals of this rectangle.

The asymptotes are helpful in drawing the hyperbola.

BRIDGE OF SANT ANGELO, ROME

The Romans used the arch very extensively in constructing their bridges

263. The equation of the hyperbola whose foci are on the y-axis. This equation may be obtained by interchanging the variables x and y. It is therefore

$$\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$$

The *transverse* axis is $2a$, the *vertices* are the points $(0, \pm a)$, and the *foci* are the points $(0, \pm c)$.

264. Conjugate hyperbolas. The hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \text{ and } \frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

are **conjugate hyperbolas**, Fig. 92. The transverse and conjugate axis of either are respectively the conjugate and transverse axis of the other.

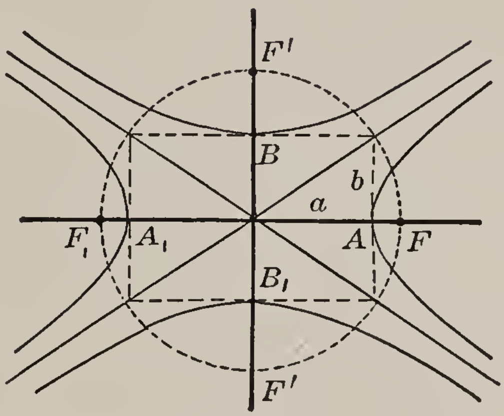

FIG. 92

Since the equation

$$c^2=a^2+b^2$$

gives the same value of c for both curves, the *foci* of both are at the same distance from the origin. Hence they lie on a circle whose radius is c, and whose center is the origin. They are the points

$$F_1(-c, 0),\ F(c, 0),\ F_1'(0, -c),\ F'(0, c)$$

The *eccentricity* of the conjugate hyperbola is

$$e=\frac{c}{b}$$

EXERCISES

1. Discuss the equation $4x^2-9y^2=36$ as to extent, symmetry, intercepts, latus rectum, and asymptotes. Make a sketch of the curve.

2. Discuss as in exercise 1 the equations $9x^2-y^2=16$; $x^2-y^2=9$.

3. Find the points on the curve $b^2x^2-a^2y^2=a^2b^2$, for which the abscissa is equal to the ordinate.

4. Find the points of intersection of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$ and the line $3y-4x=2$.

5. Write the equation of the hyperbola conjugate to

$$6x^2-3y^2=8.$$

6. Find the locus of a point which moves so that the difference of its distances from the points $(0, \pm 6)$ is equal to 10.

7. Find the equation of the hyperbola whose center is at the origin whose foci are on the x-axis, if it passes through the points $(3\sqrt{2}, 5)$ and $(-9, 5\sqrt{8})$.

8. Sketch the hyperbola $x^2-9y^2=9$ and its conjugate.

9. Find the foci of the curve $9x^2-16y^2=144$.

10. Find the equation of the equilateral hyperbola passing through the point $(-5, 2)$.

11. Find the equation of the hyperbola passing through $(1, -3)$ and $(2, 4)$.

12. Find the lengths of the focal radii at the point for which $y=1$ and $x>0$, the equation of the curve being $4x^2-9y^2=36$.

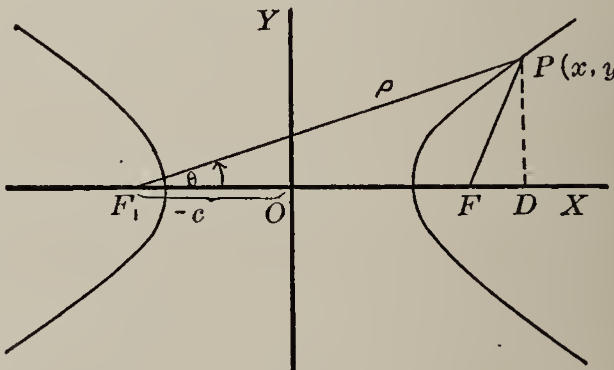

FIG. 93

265. Polar equation of the hyperbola. As in §250, the polar equation of the hyperbola, Fig. 93, is found to be

$$\rho=\frac{p}{1-e\cos\theta}$$

where

$$e>1,\ e=\frac{c}{a},\ p=\frac{-b^2}{a},\ b^2=c^2-a^2$$

266. Discussion of the polar equation. Show that the *intercepts* are as follows:

θ	0	90°	180°	270°
ρ	$a+c$	$\frac{-b^2}{a}$	$-(c-a)$	$\frac{-b^2}{a}$

Extent of the curve.—The value of ρ increases without bound as $1-e\cos\theta$ approaches zero, i.e., as $\cos\theta \rightarrow \frac{a}{c}$.

When $\cos\theta>\frac{a}{c}$, show that $1-e\cos\theta<0$ and $\rho>0$. The point P lies, therefore, on the branch of the curve to the right of O.

When $\cos\theta<\frac{a}{c}$, show that $1-e\cos\theta>0$ and $\rho<0$, and P lies on the branch to the left of O.

The following table gives the changes of ρ and θ for $\theta=0°\ldots.360°$

θ	0	....	arcos $\frac{a}{c}$	....	90°	....	180°	270°	....	arcos $\frac{a}{c}$	....	360°
ρ	$a+c$	+	∞	−	$-\frac{b^2}{a}$	−	$-(c-a)$	$-\frac{b^2}{a}$	−	∞	+	$a+c$

Symmetry.—Since $\cos(-\theta)=\cos\theta$, the curve is symmetric with respect to the polar axis.

267. Equation of the tangent and normal. As in § 253 the **slope of a tangent** to the hyperbola may be found to be

$$m=\frac{b^2x_1}{a^2y_1}$$

The **slope of the normal** is

$$m = -\frac{a^2y^2}{b^2x_1}$$

The **equation of the tangent** is

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$

which reduces to

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1$$

The **equation of the normal** is

$$y - y_1 = -\frac{a^2y_1}{b^2x_1}(x - x_1)$$

These equations may also be obtained by substituting $(-b^2)$ for b^2 in the corresponding equations for the ellipse.

268. Length of tangent, normal, subtangent, and subnormal. The lengths of the subtangent and subnormal may be derived as in § 254. They may also be obtained by substituting $(-b^2)$ for b^2 in the formulas of § 267.

Hence for the hyperbola the **subtangent** is $\frac{a^2y_1^2}{b^2x_1}$ and the **subnormal** is $\frac{b^2x_1}{a^2}$.

The tangent and normal are found by means of the theorem of Pythagoras.

EXERCISES

1. Find the equation of the tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at the point $(-6\frac{2}{3}, 4)$.

2. Find the equations of the tangents drawn from the point $(+1/2, 0)$ to the hyperbola $\frac{x^2}{2} - y^2 = 1$.

3. Prove that the angle between the focal radii of any point on the hyperbola is bisected by the tangent at the point.

See exercise 2, § 254.

4. Show that the parametric equations of the hyperbola are $x=a\sec\theta$ and $y=b\tan\theta$.

Eliminate the parameter θ.

5. Plot the equation

$$\rho=\frac{2}{1-2\cos}$$

The Parabola

269. The equation of the parabola has been discussed in chapter v. The following is a review and extension of the important formulas:

1. *Standard form* of the equation of the parabola:

$$y^2=4px$$

2. *Polar equation:*

$$\rho=\frac{2p}{1-\cos\theta}$$

3. *Tangent* to the parabola:

$$y_1y=2p(x+x_1)$$

4. *Normal* to the parabola:

$$y-y_1=-\frac{y_1}{2p}(x-x_1)$$

5. *Subtangent:*

$$2x_1$$

6. *Subnormal:*

$$2p$$

7. *Focal radius:*

$$r=x_1+p$$

EXERCISES

1. Find the co-ordinates of the focus and the length of the focal radius from the point (2, 4) if the equation of the parabola is $y^2=8x$.

2. Prove that the parametric equations of the parabola are $x=\frac{t^2}{4p}$, $y=t$.

3. Find the angle between the curves $y^2=8x$ and $.8x^2+.2y^2=1$.

4. Find the equation of the tangent to $y^2=4x$ which makes an angle of 45° with the x-axis.

5. Find the equation of the chord of the parabola $y^2=4x$ which is bisected by the point (4, 2).

6. Prove that the angle between the two tangents to a parabola is equal to one-half the angle between the focal radii drawn to the points of contact.

7. Plot and find the co-ordinates of the points of intersection of $y^2=4x+5$ and $y=2x+1$.

Historical note. The ellipse, parabola, and hyperbola have been studied since the days of Democritus (460–370 B.C.) and of Archytas of Tarentum (430–365 B.C.). The former studied the plane sections of a cone parallel to the base, and the latter studied the mutual intersections of cones and cylinders. Probably Archytas did not recognize his intersections to be plane curves.

Plato, however, urged upon scholars the importance of studying the geometry of solids as well as of planes, and Menaechmus (350 B.C.), a disciple of Plato, is credited with being the discoverer of these curves as *sections of cones.* He regarded the ellipse as a section of an acute-angled cone, by a plane passed perpendicular to an element, a parabola as a similar section of a right-angled cone, and a hyperbola as such a section of an obtuse-angled cone. Each type was a section of a special form of cone. Menaechmus solved the then celebrated problem of the duplication of the cube in two ways, one with the aid of a parabola and hyperbola, and the other with two intersecting parabolas. His work on these curves became so famous that for many years after his time the curves were called the "Menaechmian triads." Menaechmus' solutions of the cube-duplication problem stimulated other mathematicians to devise other curves to solve one or another of the three classical problems of antiquity. Thus we have the quadratrix of Hippias, the spirals of Archimedes, the conchoid of Nicomedes (*ca.* 180 B.C.), and the cissoid of Diocles (*ca.* 180 B.C.).

Aristaeus about 320 B.C. wrote a great work in five books on the ellipse, parabola, and hyperbola, calling them *space loci.*

Euclid about 300 B.C. wrote on the theory of these curves. Euclid's work is lost, but we are told that Apollonius (between 250 and 200 B.C.), whose monumental work on conic sections constitutes the crown of ancient mathematics, was greatly dependent on Euclid's work, and in an extant manuscript by Euclid there is a passage in which he says that the section of a right circular cone made by a plane not parallel to the base is a "long shield" (i.e., an *ellipse*). He thus recognized that an ellipse could be cut from a cone by a plane in another position than perpendicular to an element.

Archimedes (287–212 B.C.) wrote on the conics, his chief contributions being the determination of the areas of the parabola and the ellipse.

Apollonius (260–200 B.C.) wrote a great work in eight books on the theory of the conic sections, collecting and systematizing all existing knowledge of the curves and immensely extending it. His work introduced the modern nomenclature and the geometrical method of treatment and soon drove out of circulation all other works on the subject, Euclid's included. Apollonius treated the curves as plane loci, not necessarily to be regarded as sections of cones. It was nearly a thousand years after Apollonius before any further advance was made in the theory of the conics.

It may be said that while the methods of Apollonius were geometrical they nevertheless were equivalent to the use of co-ordinates. The method of treatment given in this text is called analytical, and it employs co-ordinates, but it also employs algebraic equations as a means of revealing the geometrical properties of the conic sections. Thus employment of algebraic equations to develop the properties of the conics was developed comparatively recently. Heron, in the first century before Christ, carried Apollonius' methods into surveying, and the Arabs in the eighth and ninth centuries after Christ applied Appolonius' methods even to the solution of cubic equations. A drawing, with accompanying explanatory text, made by an unknown European author of the tenth or eleventh century after Christ applies co-ordinates to some astronomical problems, and Bishop Nicholas Oresmus (1323–82) taught the astronomical uses of abscissas and ordinates in the College of Navarre in Paris, France, but he had no thought of the use of the algebraic equation as applied to co-ordinates. He located a series of separate points but could not handle a continuous succession of points.

The second essential of the analytical method is the application of algebraic equations to the study of geometry.

Algebra arose out of the pure calculatory processes of arithmetic. These processes were first stated as mere abbreviated rules of reckoning, and later these rules were more compactly written as formulas. The Arabs made some progress in showing the correctness of algebraic rules by the aid of geometrical reasoning. Leonardo of Pisa in 1202 introduced Arabian methods into Europe and urged the importance of proving algebraic formulas by geometry. Subsequently most significant mathematicians both advocated and used geometry to prove algebraic things. Pacioli (1494), Tartaglia (1500–57), Cardan (1501–76), Benedetti (1530–90), Vieta (1540–1603), all followed this practice, and in 1630 Ghetaldi published a book on algebraic geometry, in which, with Vieta's aid, he systematized both how geometry might be used to rationalize algebra and, by the introduction of an unknown, how algebraic equations might be employed to derive geometrical properties. Vieta, Pascal, Cavalieri, Fermat, and many others employed algebraic equations in geometrical study. This method indeed characterized the spirit of the mathematical age just before Descartes. But Descartes in 1637 sketched a method by which algebraic equations, in conjunction with co-ordinates, could be applied as a most powerful means of discovering properties of geometrical figures. Descartes' method, as perfected by his immediate successors, is the analytical method, which is the method used in this text. The particular application of the analytical method of the study of the conic sections that is used here embodies, of course, many improvements and simplifications that have been introduced in very recent times. (Mainly from Tropfke, Band II, Th. 12 u. 13.)

Summary

270. The chapter has taught the meaning of the following terms:

focal radii
latus rectum
focus
eccentricity
parametric equations
eccentric angle
auxiliary circles
subtangent
subnormal
hyperbola
major and minor axis
transverse axis
conjugate axis
equilateral hyperbola
asymptotes
conjugate hyperbolas

271. The table on pp. 238 and 239 contains a list of equations and formulas developed in this chapter.

	Circle	Parabola	Ellipse	Hyperbola
Cartesian Equations	$x^2+y^2=a^2$ $(x-h)^2+(y-k)^2=a^2$, where h and k are the co-ordinates of the center	$y^2=4px$ $(y-k)^2=4p(x-h)$, where $p=$ the distance from the vertex to the focus	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are the semi-major and -minor axes	Conjugate hyperbolas: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ Equilateral hyperbola: $x^2-y^2=a^2$
Polar Equations	$\rho=2a\cos\theta$, where a is the radius	$\rho=\frac{2p}{1-\cos\theta}$, where $p=$ the distance from the vertex to the focus	$\rho=\frac{p}{1-e\cos\theta}$, where $p=\frac{b^2}{a}$, $e<1$	$\rho=\frac{p}{1-e\cos\theta}$, where $p=-\frac{b^2}{a}$, $e>1$
Parametric Equations	$\begin{cases}x=a\cos t\\y=a\sin t\end{cases}$	$x=\frac{t^2}{4p}$, $y=t$	$\begin{cases}x=a\cos t\\y=b\sin t\end{cases}$	$\begin{cases}x=a\sec t\\y=b\tan t\end{cases}$
Asymptotes				$y=\pm\frac{b}{a}x$
Foci		$(p, 0)$	$(\pm c, 0)$	$(\pm c, 0)$

	Circle	Parabola	Ellipse	Hyperbola
Intercepts	$\pm a,\ \pm a$		$\pm a,\ \pm b$ $a^2=b^2+c^2$	$\pm a$ $a^2=c^2-b^2$
Extent of the Curve	$\lvert x\rvert \leqq a,\ \lvert y\rvert \leqq a$	$x \geqq 0$	$\lvert x\rvert \leqq a,\ \lvert y\rvert \leqq b$	$\lvert x\rvert \geqq a, y=$ any value
Axes of Symmetry	Any diameter	x-axis	x-axis, y-axis	x-axis, y-axis
Eccentricity		$e=1$	$e=\frac{c}{a}<1$	$e=\frac{c}{a}>1$
Focal Radii		x_1+p	$a\pm ex$, $r_1+r=2a$	$ex\pm a$, $r_1-r=2a$
Latus Rectum		$4p$	$\frac{2b^2}{a}$	$\frac{2b^2}{a}$
Tangent	$x_1x+y_1y=a^2$	$y_1y=2p(x+x_1)$	$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$	$\frac{x_1x}{a^2}-\frac{y_1y}{b^2}=1$
Normal	$y=\frac{y_1}{x_1}x$	$y-y_1=-\frac{y_1}{2p}(x-x_1)$	$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$	$y-y_1=-\frac{a^2y_1}{b^2x_1}(x-x_1)$
Subtangent, Subnormal		$2x_1,\ 2p$	$-\frac{a^2y_1^2}{b^2x_1},\ -\frac{b^2x_1}{a^2}$	$\frac{a^2y_1^2}{b^2x_1},\ \frac{b^2x_1}{a^2}$

CHAPTER XV

CONIC SECTIONS. TRANSFORMATION OF CO-ORDINATES

Plane Sections of a Right Circular Cone

272. Conic sections. The circle, parabola, ellipse, and hyperbola may be obtained as curves in which a right circular cone is cut by a plane. For this reason they are called **conic sections,** or **conics.***

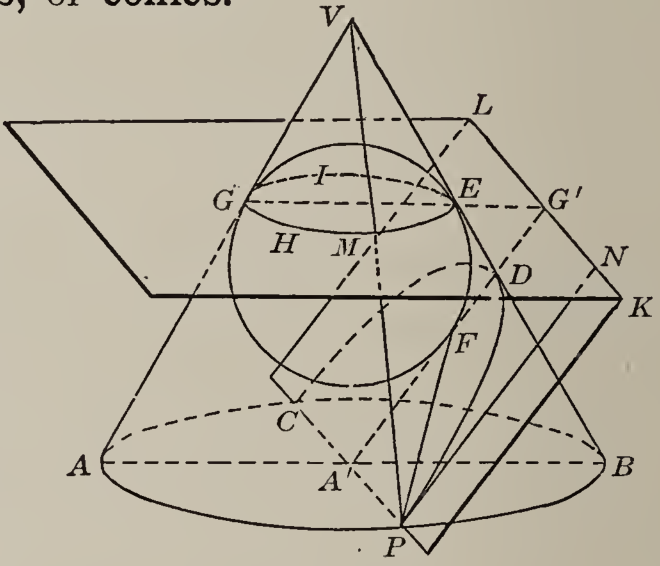

FIG. 94

273. The parabola. We shall first consider the case when the cutting plane is parallel to an element of the cone.

Let AVB, Fig. 94, be a right circular cone, and let CDP be a section made by a plane parallel to VA.

Let the plane AVB represent a section of the cone passing through the vertex and perpendicular to the plane CDP.

Inscribe a sphere in the cone tangent to the plane CDP at F. Let the sphere touch the cone in the circle

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* The conic sections were first studied by Menaechmus (375–325 B.C.), and they were long known as "Menaechmian triads." Most Greek mathematicians after Menaechmus discussed them. Euclid (about 300–275 B.C.) wrote a book on conic sections. Apollonius (about 260–200 B.C.) completed their study. Apollonius introduced the word *parabola,* together with *ellipse* and *hyperbola* into science (Ball, pp. 47, 60, 79).

$GHEI$. Plane GHE is perpendicular to the axis of the cone and therefore to plane AVB. Since plane CDP is also perpendicular to AVB, the line of intersection KL must be perpendicular to plane AVB.

Let P be any point on the section CDP.

Draw VP meeting GHE at M, $PN \| FD$.

Then $PF = PM = AG = A'G' = PN$.

Hence *any point* P *on the section* CDP *is equidistant from* F *and* KL, *or* $\dfrac{PF}{PN} = 1$ (1)

Let D be the origin and $A'G'$ the x-axis.

Then the co-ordinates of P are

$$x = DA', \; y = A'P$$

Show that

$$y^2 = (A'P)^2 = (AA') \cdot (A'B)$$

$$\therefore \quad \frac{y^2}{x} = (AA') \cdot \frac{A'B}{DA'} = \frac{GG' \cdot GE}{VG}$$

Since for any point P on the given section CDP the lines GG', GE, and VG remain constant, we have

$$\frac{y^2}{x} = \text{constant}$$

Denoting this constant by $4p$, we obtain the equation

$$\boldsymbol{y^2 = 4px}$$

which is the equation of a parabola.

274. Definition of a conic section. Equation (1), § 273, illustrates a fundamental property common to all conic sections, which may be defined as follows:

A conic is the locus of a point which moves in a plane so that the distance from a fixed point in the plane is in constant ratio to the distance from a fixed line in that same plane.

The fixed point is called the **focus,** the fixed line the **directrix,** and the constant ratio the **eccentricity.**

When the eccentricity $e=1$ the equation of the conic reduces to the equation of the **parabola.** When $e<1$ the conic is an **ellipse,** and when $e>1$ the conic is a **hyperbola.**

275. The ellipse. Let the plane CPD, Fig. 95, cut all elements of the cone. Inscribe a sphere touching the cone in the circle GME and touching plane CDP in F.

Inscribe a second sphere touching the cone in circle $G_1M_1E_1$ and plane CDP in F_1.

Draw PF and PF_1.

Show that

$$PF=PM$$

and that

$$PF_1=PM_1$$

$$\therefore \quad PF+PF_1=MM_1$$

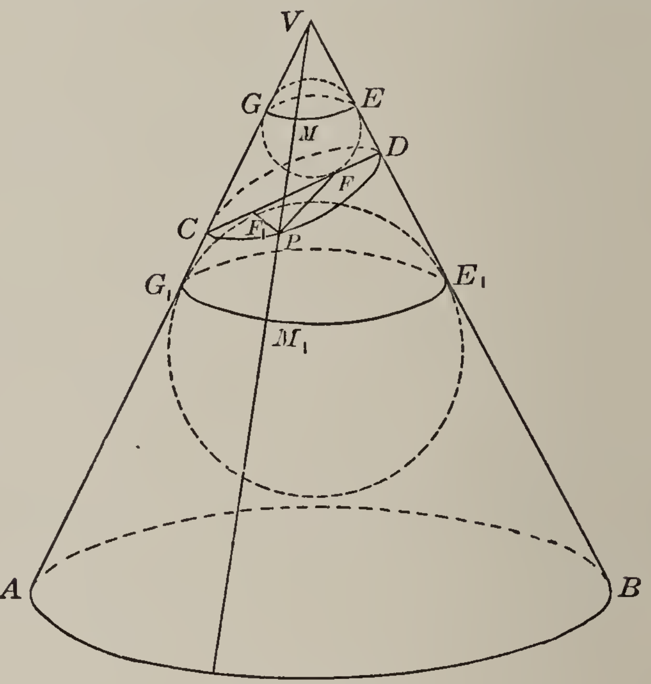

FIG. 95

Since MM_1 is constant for every point P in the section CDP, it follows that the locus of P is an **ellipse.**

The points F_1 and F are the **foci** of the ellipse. The lines of intersection of planes GME and CDP and of CDP and $G_1M_1E_1$ are parallel and equidistant from the foci. They are the two **directrices** of the ellipse.

276. The circle. When plane CDP, Fig. 95, is parallel to the base of the cone, the section is a **circle.**

The circle may then be regarded as a particular case of an ellipse.

277. The hyperbola. If the plane CDP, Fig. 96, cuts both sheets of the cone, the section is a **hyperbola.**

For

$$PF_1 = PM_1$$

and

$$PF = PM$$

$$PF_1 - PF = MM_1$$

which is constant for any point P.

F and F_1 are the **foci** of the hyperbola. The **directrices** are the lines of intersection of plane CDP with planes GME and $G_1M_1E_1$.

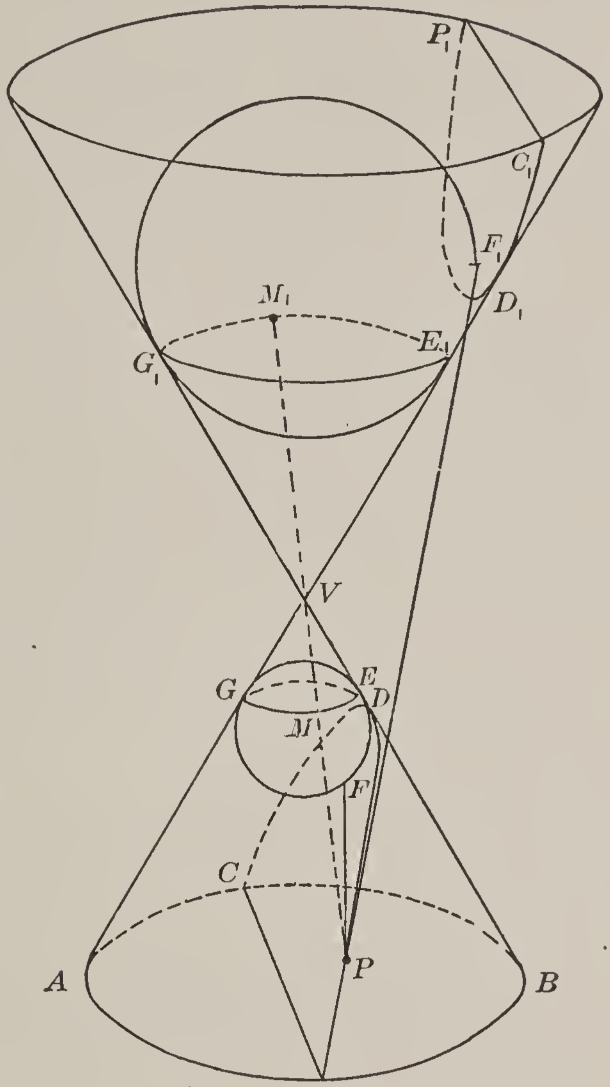

FIG. 96

278. Limiting cases. The parabola may be regarded as the limiting case of both the ellipse and the hyperbola. For if the cutting plane CDP, Fig. 95, is made to turn about D, the point C moves away to an infinite distance when the plane becomes parallel to the element VA.

Similarly the branch $C_1D_1P_1$, Fig. 96, moves off to an infinite distance as the plane CDP is made to rotate about D and becomes parallel to VA.

If plane CDP becomes a tangent plane to the cone, the section approaches a straight line as a limit.

If the cutting plane passes through V, the section will be one straight line, two intersecting straight lines, or a point, according as the plane is tangent to the cone, intersects the cone in an element, or contains the vertex as the only point of the cone.

279. Directrix. Since for any point P, Fig. 97, on the ellipse we have the equation

$$\frac{F_1P}{PN_1}=e$$

it follows that the equation holds for point A_1 on the ellipse, i.e.,

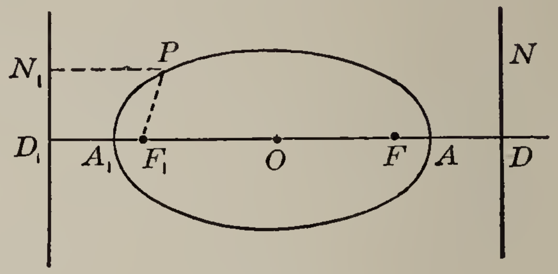

FIG. 97

$$\frac{F_1A_1}{A_1D_1}=e,$$

or

$$\frac{a-c}{OD_1-a}=e$$

$$OD_1=\frac{ae+a-c}{e}=\frac{a}{e}$$

Hence the equation of the directrix D_1N_1 is

$$x=-\frac{a}{e}$$

Similarly the equation of the directrix DN is

$$x=+\frac{a}{e}$$

For the hyperbola, Fig. 98.

$$\frac{F_1P}{PN_1}=e$$

$$\therefore \quad \frac{F_1A_1}{A_1D_1}=e$$

$$\frac{c-a}{a-OD_1}=e$$

$$OD_1=\frac{ae-c+a}{e}=\frac{a}{e}$$

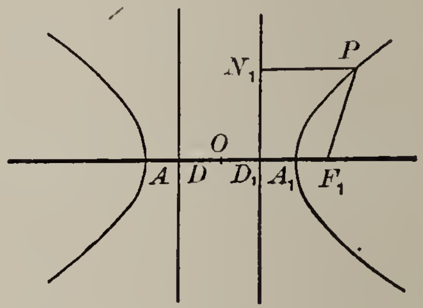

FIG. 98

Hence the equations of the directrices of the hyperbola are

$$x = \pm \frac{a}{e}$$

EXERCISES

1. Find the foci, directrices, and latus rectum of the following conics and sketch each:

1. $x^2+9y^2=9$
2. $x^2-9y^2=9$
3. $9x^2-16y^2=144$
4. $9x^2+25y^2=225$
5. $4x^2+9y^2=36$
6. $9x^2-36y^2=324$

2. Write the equation of the ellipse whose center is the origin and whose major axis is the x-axis if the directrix is $x=-3$ and the eccentricity .6; if the equation of the directrix is $x=8$ and the distance between the foci 10.

3. Find the lengths the semi-axes and the center of the hyperbola whose focus is (3, 0), whose directrix is $x=1$, and for which $e=\frac{3}{2}$.

Transformation of Co-ordinates

280. Changes of axes. It is often possible to simplify the equation of a locus by making a change of the position of the axes of co-ordinates and thus referring the equation to the new axes. This is known as a **transformation of co-ordinates.**

281. Translation of axes. When the new axes $O'X'$ and $O'Y'$, Fig. 99, are parallel respectively to the old axes OX and OY, the transformation is called a **translation of the axes.**

Let (h, k) be the co-ordinates of the new origin O' referred to the old axes.

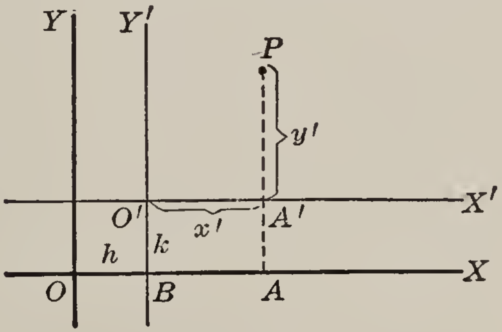

FIG. 99

Let a point P have the co-ordinates (x, y) when referred to the old axes and (x', y') when referred to the new axes. Then

$$OA = x, \quad AP = y$$
$$O'A' = x', \quad A'P = y'$$

Since $OA = OB + BA$,

$$\boldsymbol{x = h + x'} \qquad (1)$$

Since $AP = AA' + A'P$,

$$\boldsymbol{y = k + y'} \qquad (2)$$

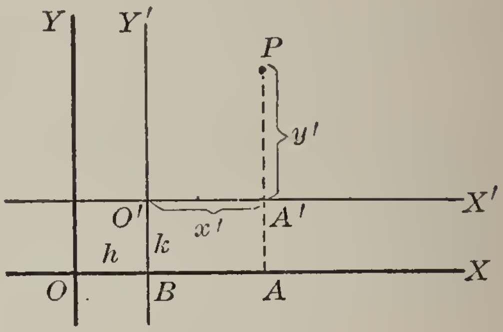

Fig. 99

Equations (1) and (2) are the **formulas of translation.** Hence, if in an equation of a locus we substitute $h + x'$ for x and $k + y'$ for y, the new equation is the equation of the locus referred to the point (h, k) as origin, the new axes being parallel to the old.

Thus no point of the locus changes position, although the equation is changed to a different form.

EXERCISES

1. Prove equations (1) and (2), § 281, when O' is located as in Fig. 100.

2. Find the co-ordinates (x', y') of the point $(x, y) = (-2, 3)$ with respect to a new pair of axes intersecting at the point $(2, -1)$ and parallel to the original axes.

Since $(h, k) = (2, -1)$, it follows that

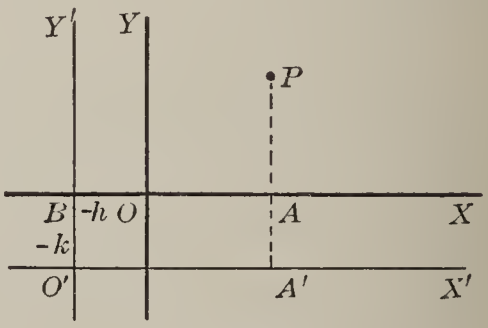

Fig. 100

$$x = 2 + x', \; y = -1 + y'$$
$$\therefore \quad x' = x - 2, \; y' = y + 1$$
$$\text{or} \quad x' = -4, \quad y' = 4$$

Transform the following equations by translating the axes, in each case moving the origin to the point indicated:

3. $2x^2+3y^2-12x+2y+10=0$; new origin $(3, -2)$

Here $x=3+x', \ y=-2+y'$

By substitution the equation is changed to

$$2(3+x')^2+3(-2+y')^2-12(3+x')+2(-2+y')+10=0$$

which reduces to $2x'^2+3y'^2-10y'-12=0$

The primes may now be dropped, since the old axes are no longer considered.

4. $y=3x-5$; new origin $(3, -2)$. Graph the given and the transformed equation and both pairs of axes.

5. $y^2-3x+3y+12=0$; new origin $(1, 5)$

6. x^2+y^2+4x+8; new origin $(2, -3)$

7. $y^2-3y-2x+4=0$; new origin $(2, -1)$

282. Equations of the conic whose principal axis is parallel to one of the co-ordinate axes. The standard equations of the conics were obtained by choosing the origin as center and one of the co-ordinate axes as principal axis. By translating the origin to the point (h, k), the standard equations are transformed to the following:

Circle $(x-h)^2+(y-k)^2=a^2$

Ellipse, axis $y=k$, $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$

axis $x=h$, $\dfrac{(y-k)^2}{a^2}+\dfrac{(x-h)^2}{b^2}=1$

Hyperbola, axis $y=k$, $\dfrac{(x-h)^2}{a^2}-\dfrac{(y-k)^2}{b^2}=1$

axis $x=h$, $\dfrac{(y-k)^2}{a^2}-\dfrac{(x-h)^2}{b^2}=1$

Parabola, vertex (h, k),

axis $y=k$, $(y-k)^2=4p(x-h)$

axis $x=h$, $(x-h)^2=4p(y-k)$

The equations in § 282 are all of the general form

$$Ax^2+By^2+2Gx+2Fy+C=0$$

They may be derived from this form by completing the squares.

EXERCISES

Simplify the following equations:

1. $x^2-y^2-4x-12y-20=0$

Completing the squares

$$(x^2-4x+4)-2(y^2-6y+9)=6$$

or
$$\frac{(x-2)^2}{6}-\frac{(y-3)^2}{3}=1$$

a conic whose center is (2, 3).

2. $9x^2+4y^2-54x+40y+145=0$

3. $9x^2-16y^2+90x+128y-175=0$

283. Removal of terms of the first degree. An important use of transforming the co-ordinates is the simplification of equations. For the origin can often be so chosen as to remove from the equation the terms of the first degree in x and y. The following example illustrates the method.

Simplify the equation

$$x^2+y^2-2x+2y+7=0 \quad (1)$$

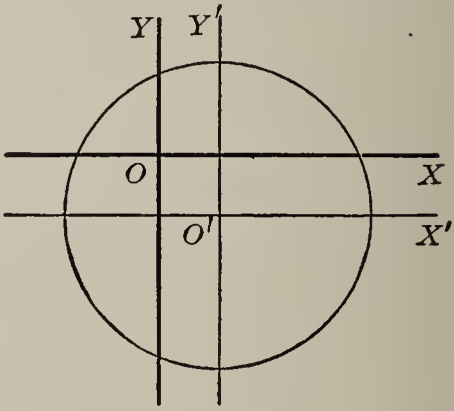

FIG. 101

Let $x=h+x'$ and $y=k+y'$, Fig. 101.

Then equation (1) becomes

$$(h+x')^2+(k+y')^2-2(h+x')+2(k+y')+7=0$$

Dropping the primes and collecting terms gives the equation

$$x^2+y^2+(2h-2)x+(2k+2)y+(h^2+k^2-2h+2k+7)=0$$

Let $$2h-2=0 \text{ and } 2k+2=0$$

or let $$h=1,\ k=-1$$

Then the equation reduces to

$$x^2+y^2=9$$

This is the equation of a *circle* whose radius is 3.

Therefore the given equation represents the same circle referred to the original axes.

EXERCISES

In the following equations remove the terms of the first degree by translating the axes:

1. $x^2+y^2-2x+2y+3=0$

2. $4x^2+9y^2-16x-18y-11=0$

3. $9x^2+4y^2-36x+16y=-16$

4. $x^2+3y^2+x-9y=-4$

5. $2x^2-4y^2+4x+4y-1=0$

6. By transforming the co-ordinate axes, simplify the equation

$$x^2+4y^2-16x+24y+84=0$$

Sketch the curve, also the old and the new axes.

284. Rotation of axes. Let the angle XOY, Fig. 102, be turned about the origin O to the new position $X'OY'$, and denote angle XOX' by θ.

Let $(x,\ y)$ be the co-ordinates of a point P referred to the axes OX and OY, and let $(x',\ y')$ be the co-ordinates of P referred to the axes OX' and OY'.

Draw $Q'R \perp QP$.

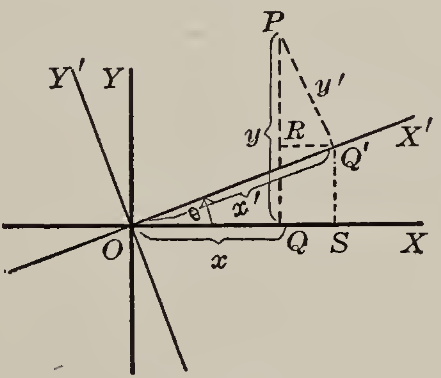

FIG. 102

Then

$$x = OQ = OS - QS = x' \cos \theta - y' \sin \theta$$

and

$$y = QP = SQ' + RP = x' \sin \theta + y' \cos \theta$$

Hence, when the axes are rotated through a given angle without moving the origin the relations between the old co-ordinates and the new co-ordinates of a point are

$$\begin{cases} x = x' \cos \theta - y' \sin \theta \\ y = x' \sin \theta + y' \cos \theta \end{cases}$$

They are called the **formulas of rotation.**

By means of these formulas the equation of a locus may be transformed into a new equation referred to new axes having the same origin but making a given angle with the original axes. This transformation is called **rotation of the axes.**

The angle θ may be so chosen that the new equation shall have no term in xy. This will be used to simplify a given equation; see § 290.

EXERCISES

Transform each of the following equations by rotating the axes through the angle indicated:

1. $xy = 8$; $\theta = 45°$

Let $$x = x' \cos \theta - y' \sin \theta = \frac{x'\sqrt{2}}{2} - \frac{y'\sqrt{2}}{2}$$

and $$y = x' \sin \theta + y' \cos \theta = \frac{x'\sqrt{2}}{2} + \frac{y'\sqrt{2}}{2}$$

Substituting in the given equation and reducing, we have

$$\frac{x'^2 - y'^2}{2} = 8$$

or $$x^2 - y^2 = 16$$

which is the equation of an equilateral hyperbola.

2. $x^2+12xy+9y^2=16;\ \theta=\frac{\pi}{4}$

3. $29x^2-24xy+36y^2-180=0;\ \theta=\text{arc tan } \frac{3}{4}$

4. $x^2+y^2=a^2;\ \theta=\frac{\pi}{6}$

Show that this equation is unaltered by rotating the axes through any angle.

5. $16y^2-24xy+9x^2-20x+110y=75;\ \theta=\text{arc sin } \frac{3}{5}$

Summary

285. If a right circular cone is cut by a plane the section may be a circle, parabola, ellipse, hyperbola, or two intersecting straight lines.

286. The following equations and formulas have been developed:

Curves	Equations	Directrices	Equations if the Principal Axis Is Parallel to One of the Co-ordinate Axes
Ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$x=\pm\frac{a}{e}$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, if the axis is $y=k$ $\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1$, if the axis is $x=h$
Hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$x=\pm\frac{a}{e}$	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$, if the axis is $y=k$ $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$, if the axis is $x=h$
Circle	$x^2+y^2=a^2$		$(x-h)^2+(x-k)^2=a^2$, if the center is at the point (h, k)
Parabola	$y^2=4px$		$(y-k)^2=4p(x-h)$, if the axis is $y=k$ $(x-h)^2=4p(y-k)$, if the axis is $x=h$

Formulas of translation: $\begin{cases} x=h+x' \\ y=k+y' \end{cases}$

Formulas of rotation: $\begin{cases} x=x'\cos\theta-y'\sin\theta \\ y=x'\sin\theta+y'\cos\theta \end{cases}$

CHAPTER XVI

THE GENERAL EQUATION OF THE SECOND DEGREE. DIAMETERS

The General Equation of the Second Degree

287. General equation of a conic. The most general equation of the second degree is of the form

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0 \tag{1}$$

where A, H, and B cannot all be zero.

The following shows that *the most general equation of a conic is of the form of equation* (1).

Let $P(x, y)$ be any point on the conic, Fig. 103.

Let DG be the directrix given by the equation

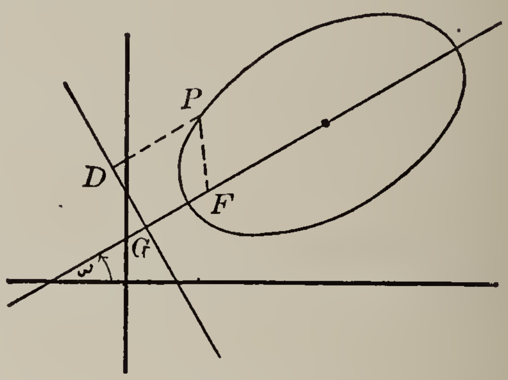

Fig. 103

$$ax+by+c=0$$

Then
$$PD=x\cos\omega+y\sin\omega-p$$
$$PF=e\cdot PD \tag{2}$$

Let (h, k) be the co-ordinates of the focus F.

Then equation (2) may be written

$$\sqrt{(x-h)^2+(y-k)^2}=e(x\cos\omega+y\sin\omega-p) \tag{3}$$

By squaring both sides and collecting terms equation (3) reduces to

$$(1-e^2\cos^2\omega)x^2-2(e^2\cos\omega\sin\omega)xy+(1-e^2\sin^2\omega)y^2 \\ +2(e^2p\cos\omega-h)x+2(e^2p\sin^2\omega-k)y+h^2+k^2 \\ -e^2p^2=0 \tag{4}$$

which is of the form of equation (1).

288. A conic passing through five given points. Equations (1) and (4), § 287, involve five arbitrary constants. They are h, k, p, ω, and e in equation (4), and five ratios between the coefficients A, H, B, G, F, and C in equation (1). *A conic section is therefore determined by five independent conditions.* Thus, *in general, a conic can be passed through five points in the same plane.*

The equation of a conic to be determined by five points may be found by first substituting the co-ordinates of the given points in the general equation

$$x^2+axy+by^2+cx+dy+e=0$$

The values of a, b, c, d, and e may then be determined by solving the system of the five linear equations thus obtained.

The following is a shorter method of finding the equation of a conic to be determined by five points.

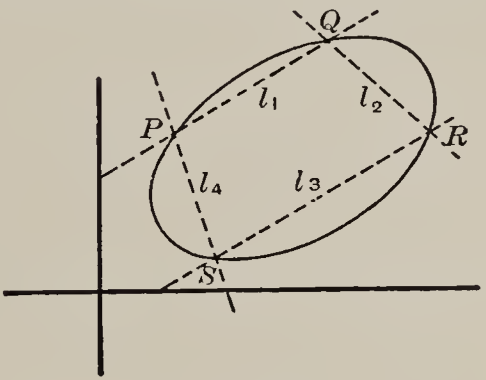

FIG. 104

Let P, Q, R, and S, Fig. 104, be four of the given points.

Form the equations of the lines PQ, QR, RS, and SP.

Suppose these equations are denoted by l_1, l_2, l_3, and l_4, respectively. Then $l_1l_3=0$ and $l_2l_4=0$ are two equations of the second degree in x and y, having the points T, Q, R, S in common.

Hence the quadratic equation

$$l_1l_3+kl_2l_4=0$$

represents a system of conics passing through the four points P, Q, R, and S. The value of k is determined by substituting in this equation the co-ordinates of the fifth given point.

It will be seen, § 295, that a *parabola* is determined by four points.

EXERCISES

Find the equation of the conic passing through the following points:

1. $(1, -2)$, $(2, -1)$, $(1, 2)$, $(-3, -1)$, $(-2, 4)$

$l_1 = -x+y+3=0$

$l_2 = 3x-4y+5=0$

$l_3 = -3x-y+5=0$

$l_4 = -x-4y-7=0$

$$\therefore \quad l_1l_2+kl_3l_4=(-x+y+3)(3x-4y+5) +k(-3x-y+5)(-x-4y-7)=0$$

Substituting $(x, y)=(-2, 4)$ we have

$$(9)(-17)+k(7)(-21)=0$$

$$k=-\tfrac{51}{49}$$

2. $(1, 1)$, $(-1, 2)$, $(-2, -1)$, $(0, -2)$, $(3, -3)$

3. $(0, 0)$, $(0, -2)$, $(2, -1)$, $(-2, 3)$, $(4, 3)$

4. $(1, -3)$, $(2, 4)$, $(-3, 1)$, $(4, 2)$, $(0, 0)$

5. $(1, 1)$, $(-1, 2)$, $(2, 3)$, $(-3, -1)$, $(0, -4)$

289. The general equation of the second degree is a conic. We have seen that the equations of the conics are special cases of the most general equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

In §§ 290–93 we shall show that *this equation always represents a conic or a limiting case of a conic.*

290. Removal of the *xy*-term. By means of the rotation formulas the general equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0 \tag{1}$$

is transformed to

$$A'x'^2+2H'x'y'+B'y'^2+2G'x'+2F'y'+C'=0$$

where

$$\begin{aligned}
A' &= A\cos^2\theta + 2H\sin\theta\cos\theta + B\sin^2\theta\\
H' &= (B-A)\sin\theta\cos\theta + H(\cos^2\theta - \sin^2\theta)\\
B' &= A\sin^2\theta - 2H\sin\theta\cos\theta + B\cos^2\theta\\
G' &= G\cos\theta + F\sin\theta\\
F' &= F\cos\theta - G\sin\theta
\end{aligned}$$

Let $H' = (B-A)\sin\theta\cos\theta + H(\cos^2\theta - \sin^2\theta) = 0.$

This equation is equivalent to

$$(B-A)\sin 2\theta + 2H\cos 2\theta = 0$$

$$\therefore\quad \tan 2\theta = \frac{2H}{A-B},\ \text{or}\ \theta = \tfrac{1}{2}\text{arc}\tan\frac{2H}{A-B}$$

As 2θ changes from 0° to 180°, or as θ takes all possible values between 0° and 90°, $\tan 2\theta$ assumes all positive and negative values. Hence we *can always find a value of* θ, *less than 90°, which satisfies the equation* $\tan 2\theta = \frac{2H}{A-B}$.

The term $x'y'$ *in the transformed equation will vanish if the axes are rotated through the angle* θ.

Hence, in place of equation (1) we may discuss the simpler transformed equation

$$A'x^2 + B'y^2 + 2G'x + 2F'y + C' = 0$$

EXERCISES

Simplify the following equations by removing the xy-term:

1. $8x^2 + 4xy + 5y^2 = 36$

Here $\tan 2\theta = \frac{2H}{A-B} = \frac{4}{3}$

Hence, $\cos 2\theta = \frac{3}{5}$

$$\therefore\quad \sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}} = \sqrt{\tfrac{1}{5}}$$

and $\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}} = \sqrt{\tfrac{4}{5}}$

The formulas of rotation are

$$\begin{cases} x=x'\sqrt{\frac{4}{5}}-y'\sqrt{\frac{1}{5}} \\ y=x'\sqrt{\frac{1}{5}}+y'\sqrt{\frac{4}{5}} \end{cases}$$

$$\therefore \quad 9x^2+4y^2=36$$

2. $x^2-2xy+y^2-2y-1=0$

3. $16y^2-24xy+9x^2-20x+110y-75=0$

4. Transform the equation

$$\rho=\frac{1}{1-2\cos\theta}$$

into Cartesian co-ordinates, find the center of the curve, and its intercepts on the axes. Sketch the curve.

5. Transform the co-ordinates so that the xy-term disappears from the equation

$$x^2+xy+y^2=4$$

Sketch the curve on the new axes and find the co-ordinates of the foci.

291. Discussion of the equation

$$Ax^2+By^2+2Gx+2Fy+C=0$$

By completing the squares this equation may be written

$$A\left(x+\frac{G}{A}\right)^2+B\left(y+\frac{F}{B}\right)^2=\frac{AF^2+BG^2-ABC}{AB}$$

By translating the axes and moving the origin to the point $\left(-\frac{G}{A}, -\frac{F}{B}\right)$, this equation reduces to

$$Ax'^2+By'^2=k$$

If A and B have like signs, this is the equation of an **ellipse,** if k has the same sign as A and B. If the sign of k is opposite to that of A and B the equation is not satisfied by real values of x and y, and the locus is *imaginary.* If $k=0$ the equation is satisfied only by the *point* $(x', y')=(0, 0)$ or $(x, y)=\left(-\frac{G}{A}, -\frac{F}{B}\right)$. This point may be considered a limiting case of the ellipse.

If A and B have opposite signs, the equation

$$Ax'^2+By'^2=k$$

is a **hyperbola** when $k\neq 0$.

If $k=0$, the equation represents *two straight lines* passing through the points $(x', y')=(0, 0)$, or $(x, y)=\left(-\frac{G}{A}, -\frac{F}{B}\right)$. They may be considered a limiting case of the hyperbola.

If either A or B is zero, let us suppose $A=0$, $B\neq 0$. Then the original equation is of the form

$$By^2+2Gx+2Fy+C=0$$

Completing the square we have

$$B\left(y+\frac{F}{B}\right)^2=-2G\left(x+\frac{C}{2G}-\frac{F^2}{2GB}\right)$$

If $G\neq 0$, the origin may be moved to the point

$$(x, y)=\left[\left(-\frac{C}{2G}+\frac{F^2}{2GB}+x'\right), \left(-\frac{F}{B}+y'\right)\right]$$

The equation then is transformed to

$$y'^2=-\frac{2G}{B}x'$$

the locus of a **parabola.**

If $G=0$, the equation is of the form

$$By^2+2Fy+C=0$$

By factoring we have

$$B(y-r_1)(y-r_2)=0$$

r_1 and r_2 being the roots of the equation $By^2+2Fy+C=0$. Hence the locus either consists of *two parallel or two coinciding lines, or is imaginary,* according as r_1 and r_2 are real or imaginary.

292. The value of $AB-H^2$ is not changed by a rotation of the axes. For

$$
\begin{array}{llll}
A'B'-H'^2= & & & \\
\quad [A\cos^2\theta & +\,2H\sin\theta\cos\theta+B\sin^2\theta] & [A\sin^2\theta-2H\sin\theta\cos\theta & \\
\quad +B\cos^2\theta] & -[(B-A)\sin\theta\cos\theta+H(\cos^2\theta-\sin^2\theta)]^2 & & \\
=A^2\sin^2\theta\cos^2\theta & +2AH\sin^3\theta\cos\theta+AB\sin^4\theta & & \\
-4H^2\sin^2\theta\cos^2\theta & -2BH\sin^3\theta\cos\theta & -2AH\sin\theta\cos^3\theta & \\
+\;B^2\sin^2\theta\cos^2\theta & +2BH\sin\theta\cos^3\theta & & +AB\cos^4\theta \\
-(B-A)^2\sin^2\theta\cos^2\theta & +2(B-A)H\sin^3\theta\cos\theta-H^2\sin^4\theta & -2(B-A)H\sin\theta\cos^3\theta & \\
 & & & -H^2\cos^4\theta \\
+2H^2\sin^2\theta\cos^2\theta & & & \\
=(AB-H^2)\sin^4\theta+2(AB-H^2)\sin^2\theta\cos^2\theta & & +(AB-H^2)\cos^4\theta & \\
=(AB-H^2)[\sin^2\theta+\cos^2\theta]^2=AB-H^2 & & &
\end{array}
$$

293. Discussion of the equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

By rotating the axes through an angle

$$\theta=\tfrac{1}{2}\text{ arc tan }\frac{2H}{A-B}$$

the equation is transformed to

$$A'x'^2+B'y'^2+2G'x'+2F'y'+C'=0$$

and
$$AB-H^2=A'B'-H'^2=A'B'$$

Hence, according to § 291, if A' and B' have like signs; or

if **$AB-H^2>0$,** the original equation represents an **ellipse,** or a *point,* or an *imaginary locus;*

if **$AB-H^2<0$,** the equation represents a **hyperbola,** or *two intersecting straight lines;*

if **$AB-H^2=0$,** the equation represents a **parabola,** *two parallel lines, two coinciding lines,* or *an imaginary locus.*

These results may be tabulated as follows:

	$A'F'^2+B'G'^2-A'B'C' \neq 0$	$A'F'^2+B'G'^2-A'B'C'=0$
$AB-H^2>0$	The locus is an ellipse real or imaginary.	The locus is a point.
$AB-H^2<0$	The locus is a hyperbola.	The locus consists of two intersecting straight lines.
$AB-H^2=0$	The locus is a parabola.	The locus consists of two parallel straight lines, two coinciding lines, or is imaginary.

The expression $A'F'^2+B'G'^2-A'B'C'$ may be expressed in terms of the coefficients of the original equations by means of the values given in § 290. The same results may be obtained by removing the terms of the first degree from the original equation and then computing the numerator of the constant term. The result is

$$\Delta = ABC+2FGH-AF^2-BG^2-CH^2$$

or

$$\Delta = \begin{vmatrix} A & H & C \\ H & B & F \\ G & F & C \end{vmatrix}$$

This is called the **discriminant** of the equation

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

294. Co-ordinates of the center of a central conic. Let h and k be the co-ordinates of the center. The origin may be moved to the point (h, k) by translation of the axes, using the formulas

$$\begin{cases} x=h+x' \\ y=k+y' \end{cases}$$

The general equation then reduces to

$$Ax'^2-2Hx'y'+By'^2+2(Ah+Hk+G)x' \\ +2(Hh+Bk+F)y'+C'=0$$

Hence h and k, the co-ordinates of the center, satisfy the equations

$$\begin{cases} Ah+Hk+G=0 \\ Hh+Bk+F=0 \end{cases} \qquad (1)$$

Solving for h and k

$$h=\frac{HF-BG}{AB-H^2}, \quad k=\frac{HG-AF}{AB-H^2}$$

Equations (1) have one common solution if

$$AB-H^2 \neq 0$$

295. Simplification of numerical equations of a conic. The following suggestions are useful in reducing an equation to the simplest form:

Determine $AB-H^2$.

If $\boldsymbol{AB-H^2 \neq 0}$, and the conic is an ellipse or a hyperbola, find the co-ordinates of the center, § 294. The origin should then be moved to the center, thus removing the first-degree terms.

Remove the xy term by turning the axes through an angle

$$\theta=\tfrac{1}{2}\text{ arc tan }\frac{2H}{A-B}$$

If $\boldsymbol{AB-H^2=0}$, and if the conic is a parabola, the terms of the first degree cannot be removed, since the equations in h and k, § 294, are inconsistent.

Hence the xy-term is removed first. The transformed equation is then of the form

$$B'y^2+2G'x+2F'y+C'=0$$

The origin is then moved to the point

$$(h, k)=\left[\left(\frac{-C'}{2G'}+\frac{F'^2}{2G'B'}+x''\right), \left(-\frac{F'}{B'}+y''\right)\right]$$

reducing the equation to the form

$$y''^2=-2\frac{G'}{B'}x''$$

EXERCISES

Simplify the following equations and determine the locus:

1. $32x^2+8xy+5y^2+16x-16y-16=0$

2. $4x^2-4xy+y^2-2y-1=0$

3. $x^2+2yx+y^2-4=0$

4. $3x^2-3xy-y^2+15x+10y-24=0$

5. $8x^2-3y^2+16x-6y+11=0$

6. $4x^2+8xy+4y^2+4x+3=0$

7. $36x^2-48xy+16y^2-6x+4y-6=0$

8. $4x^2+12xy+9y^2-30x+80y+200=0$

9. Find the vertices, foci, and asymptotes of the hyperbola $x^2-4y^2+2x+12y=9$.

10. Sketch the curve $x=2-3\cos t$, $y=3+2\sin t$, and find its Cartesian equation.

Find the equation of a parabola passing through the following four points:

11. (0, −3), (−4, −2), (2, 0), (2, 7)

Use § 288 and the equation $AB-H^2=0$

12. (6, −1), (4, −4), (9, 4), (5, −2)

13. (4, 0), (0, 4), (1, 1), (9, 1)

Diameters

296. Diameters. The locus of the middle points of a system of parallel chords of a conic is a **diameter** of the conic.

297. Diameter of a parabola. Let the equation of the parabola be $y^2=4px$.

Let QR be any chord, Fig. 105, $P_1(x_1, y_1)$ the middle point of QR, and $P(x, y)$ any point on QR.

Denote the angle made by QR and OX by θ, and the distance P_1P by r.

Then the equation of QR is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

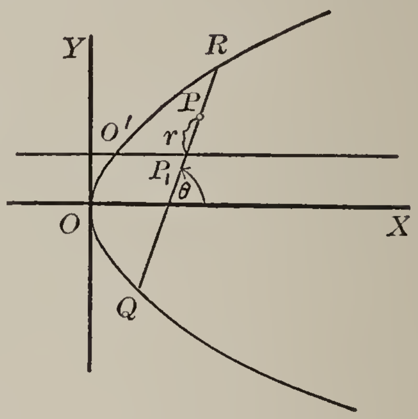

FIG. 105

It follows that

$$x=x_1+r\cos\theta,\ y=y_1+r\sin\theta \qquad (1)$$

By substituting the results in (1) in the equation $y^2=4px$ we have

$$r^2(\sin^2\theta)+2r(y_1\sin\theta-2p\cos\theta)+(y_1{}^2-4px_1)=0$$

If P falls on R, or on Q, the two values of r formed by solving this equation are numerically equal and opposite in sign.

Hence
$$y_1\sin\theta-2p\cos\theta=0$$
$$y_1=2p\cot\theta$$
or
$$y_1=\frac{2p}{m}$$

where m is the slope of the chord.

As QR moves parallel to its first position, m remains constant, but the point P_1 always satisfies the equation

$$y=\frac{2p}{m}$$

This shows that the locus of P_1, *the diameter of the parabola, is a straight line parallel to the axis.*

Let O' be the point of intersection of the parabola with the diameter.

Then the co-ordinates of O' are

$$(x', y') = \left(\frac{p}{m^2}, \frac{2p}{m}\right)$$

Show that the equation of the tangent at O' reduces to

$$y = mx + \frac{p}{m}$$

Hence *the tangent at the end-point of the diameter is parallel to the chords bisected by the diameter.*

298. Diameters of an ellipse and hyperbola. Substituting the values in equation (1), § 297, in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

we have

$$\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right)r^2 + 2\left(\frac{x_1\cos\theta}{a^2} + \frac{y_1\sin\theta}{b^2}\right)r + \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 = 0$$

Hence

$$\frac{x_1\cos\theta}{a^2} + \frac{y_1\sin\theta}{b^2} = 0$$

Dropping the subscripts, the *equation of the diameter of the ellipse is*

$$y = -\left(\frac{b^2}{a^2}\cot\theta\right)x$$

or

$$y = -\frac{b^2}{a^2m}x$$

The equation shows that the diameters of an ellipse pass through the center.

Similarly *the equation of the diameter of the hyperbola*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

is found to be

$$y=\frac{b^2}{a^2m}x$$

EXERCISES

1. Find the equation of the diameter of $4x^2-8y^2=96$, which bisects all the chords parallel to the line $6x-8y=10$.

2. Find the equation of the chord of $16x^2+8y^2=32$ which is bisected at the point $(-4, 1)$.

3. Find the equation of the diameter of the parabola $2y^2+6x=0$ bisecting the chords whose slope is 2.

299. Conjugate diameters. We have seen that the equation of a diameter of the *ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$y=-\frac{b^2}{a^2m}x$$

Let m_1 be the slope of the diameter.
Then

$$m_1=-\frac{b^2}{a^2m}$$

when

$$mm_1=-\frac{b^2}{a^2}$$

This is the relation between the slope of the diameter $y=m_1x$ and the parallel chords of slope m bisected by the diameter.

If the diameter $y=mx$, Fig. 106, bisects all the chords parallel to the diameter $y=m_1x$ the relation between their slopes is

$$mm_1=-\frac{b^2}{a^2} \quad (1)$$

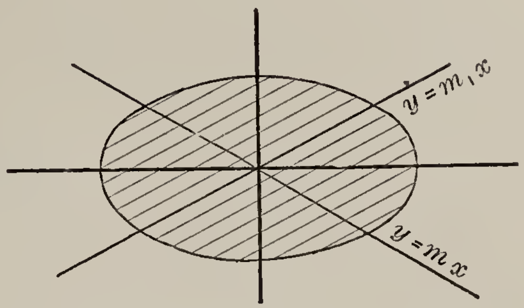

Fig. 106

If the slopes m and m_1 satisfy the equation

$$mm_1=-\frac{b^2}{a^2}$$

they are called **conjugate diameters.**

Similarly it may be shown that the conjugate diameters of the **hyperbola,** Fig. 107, satisfy the relation

$$mm_1=\frac{b^2}{a^2}$$

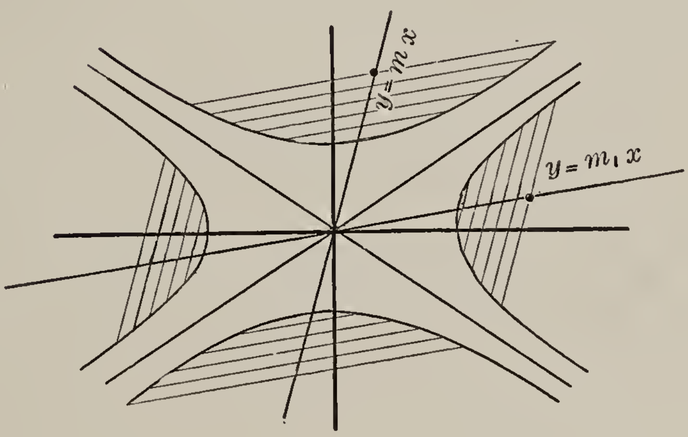

Fig. 107

EXERCISES

1. Show that the axes of an ellipse or hyperbola are the only pair of conjugate diameters perpendicular to each other.

2. Prove that if one of two conjugate diameters of an ellipse makes an acute angle with the x-axis, the other makes an obtuse angle with the x-axis.

3. Prove that if one of two conjugate diameters of a hyperbola makes an acute angle with the transverse axis, the other makes an acute angle with it.

4. If the slope of a diameter of an ellipse is $\frac{b}{a}$ show that the slope of the conjugate diameter is $-\frac{b}{a}$. Show that the two diameters are equal, and that their equations are $y=\pm\frac{b}{a}x$.

5. Show that the equal conjugate diameters of a hyperbola coincide with one of the asymptotes.

6. Show that two conjugate diameters of a circle are always perpendicular to each other.

Summary

300. The chapter has taught the meaning of the following terms:

discriminant
central conic
diameter of a conic
conjugate diameters

301. The following theorems have been proved:

1. *The general equation of the second degree is a conic.*

2. *A conic is determined by five points.*

3. *The term in xy in an equation of a conic will vanish if the axes are rotated through an angle*

$$\theta=\tfrac{1}{2}\text{ arc tan }\frac{2H}{A-B}$$

4. *The value of* $AB-H^2$ *is not changed by a rotation of the axes.*

302. The locus of the general equation of the second degree may be determined as follows:

	If $A'F'^2+B'G'^2-A'B'C'\neq 0$	If $A'F'^2+B'G'^2-A'B'C'=0$
If $AB-H^2>0$	The locus is an ellipse real or imaginary.	The locus is a point.
If $AB-H^2<0$	The locus is a hyperbola.	The locus consists of two intersecting straight lines.
If $AB-H^2=0$	The locus is a parabola.	The locus consists of two parallel straight lines, two coinciding lines, or is imaginary.

CHAPTER XVII

LOCI

Plotting the Locus of an Equation

303. Plotting the locus of an equation. A conic section has been defined as the locus of a point satisfying a given geometric property. For example, the circle is the locus of a point moving in a plane and having a fixed distance from a given point in the plane. This property may be expressed as an equation stating a relation between the co-ordinates of the variable point, such as $x^2+y^2=r^2$, or $\rho=2r\cos\theta$. The locus may then be constructed by plotting the equation. In general this involves the following steps:

1. Solve the equation for one of the co-ordinates.
2. Tabulate a series of pairs of corresponding values of the co-ordinates.
3. Plot the points corresponding to these pairs of values.
4. Join the points by means of a smooth curve. The method of plotting curves will be illustrated in §§ 304 to 308.

304. The exponential curve. The *exponential curve* is defined by the equation

$$y=a^x$$

In constructing the graph let $a=2$.

Verify the table in Fig. 108.

x	y
0	1
1	2
2	4
3	9
4	16
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
-3	$\frac{1}{9}$
-4	$\frac{1}{16}$

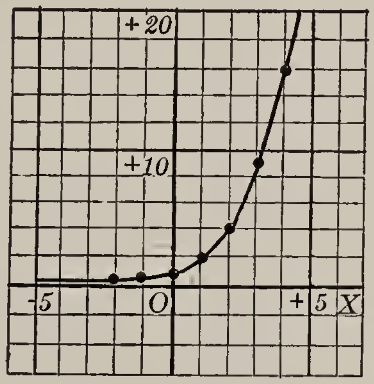

Fig. 108

Discussion: For all values of $a > 1$ the value of y is positive for any value of x. Hence the curve does not cross the x-axis.

As x increases, a^x also increases, and $\lim\limits_{x \to +\infty} a^x = +\infty$.

As x decreases, a^x decreases, and $\lim\limits_{x \to -\infty} a^x = 0$, i.e., the curve approaches the x-axis as an asymptote.

When $x = 0$, $a^x = 1$. Hence the curve crosses the y-axis at the point $(0, 1)$ for all values of $a > 1$.

305. The logarithmic curve. The *logarithmic curve* is defined by the equation $y = \log_a x$. According to the definition of a logarithm this relation is equivalent to the equation $x = a^y$, which may be obtained from the exponential equation $y = a^x$ by interchanging the variables. The graph of the curve is given in Fig. 109.

x	y
.01	-2
.1	-1
1	0
10	1
20	1.3

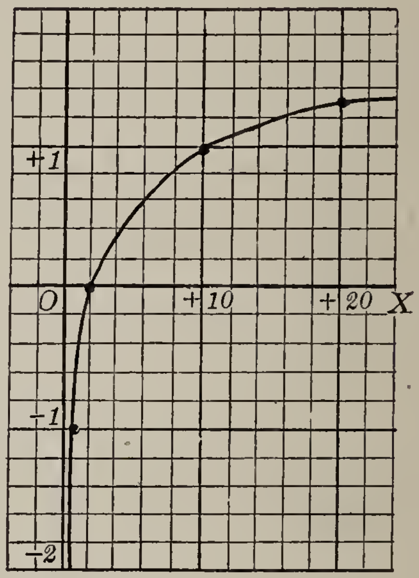

FIG. 109

306. The four-leafed rose. To graph the equation

$$\rho = a \sin 2\theta$$

let $a = 1$ and tabulate corresponding pairs of values of ρ and θ, Fig. 110.

The locus is found to be a four-leafed curve, if plotted for all four quadrants.

Show that as θ varies from 0° to 360° the point describes the leaves in the first, fourth, third, and second quadrants.

θ	ρ
0°	0.0
15°	0.5
30°	0.9
45°	1.0
60°	0.9
75°	0.5
90°	0.0

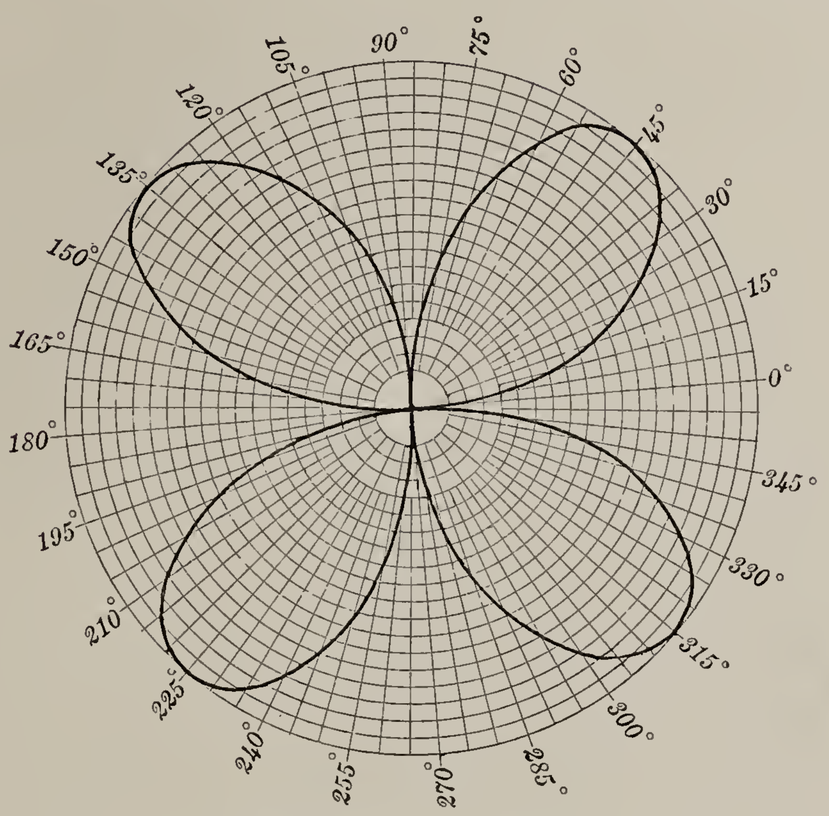

FIG. 110

EXERCISES

1. Plot the four-leafed rose $\rho = a \cos 2\theta$; the three-leafed rose $\rho =\ \sin 3\theta$; the five-leafed rose $\rho = a \sin 5\theta$.

2. Show that the equation $\rho = a \sin 2\theta$, transformed into rectangular co-ordinates, takes the form $(x^2+y^2)^2 = 4a^2x^2y^2$.

307. The lemniscate. To obtain the locus of the equation

$$\rho^2 = 2a^2 \cos 2\theta$$

solve the equation for ρ, which gives $\rho = \pm a\sqrt{2 \cos 2\theta}$.

Let $a = 1$ and let θ vary from 0° to 360°. The following table shows the corresponding approximate values of ρ:

θ	0	15°	30°	45°	increasing to 135°	135°	increasing to 180°
ρ	± 1	$\pm .9$	$\pm .7$	± 0	imaginary	0	increasing to ± 1

The curve is shown in Fig. 111. As θ varies from 0° to 45° the point describes the parts of the curve which lie in the first and third quadrant. As θ varies from 135° to 180° the parts in the second and fourth quadrant are described.

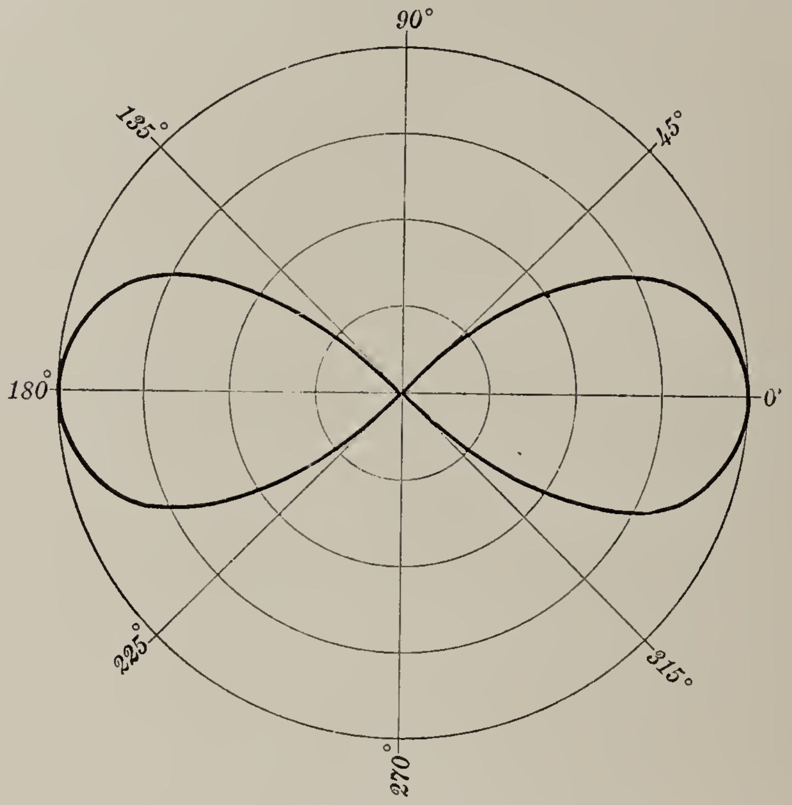

FIG. 111

EXERCISES

1. Find the intercepts and axes of symmetry of the locus of the equation $\rho^2=2a^2 \cos 2\theta$.

2. Transform the equation $\rho^2=2a^2 \cos 2\theta$ into the equation $(x^2+y^2)^2=2a^2(x^2-y^2)$.

Historical note. The lemniscate was invented and intensively studied by James Bernoulli (1654–1705), a professor of mathematics from 1687 to 1705 in the Swiss city of Basle. The curve is commonly referred to as the lemniscate of Bernoulli. It is a special case of the Cassinian oval, § 313. Fagnano (1682–1786) also devoted considerable study to this curve. The word lemniscate means "like a hanging ribbon."

JAMES BERNOULLI

JAMES BERNOULLI

JAMES BERNOULLI (1654–1705) was born and educated at Bâle. After finishing his linguistic and philosophical studies at the university at Bâle, he accidentally saw some geometrical figures one day and became seized with a passion for mathematical study. He applied himself secretly to mathematical pursuits against his father's desire to make a clergyman of him. From 1676 to 1682 he traveled to France, England, and Holland and made the acquaintance of distinguished scientists of those countries. Returning to Bâle in 1682 he opened a public seminary on experimental physics and devoted himself to physical and mathematical investigation. He mastered the then newly created calculus by his own almost unaided efforts and was always thenceforth a strong advocate of the *differential calculus*. He was the first to use the word *integral* as applied to the calculus.

He wrote and published tractates on *Teaching Mathematics to the Blind*, on *Dialling*, on *Comets*, on *Gravitation of the Aether*, etc. Some of his tracts of this period were tinged with the philosophy of Descartes, and some were said to contain matters worthy of Newton's *Principia*.

He contributed much to mathematical methods, was the first to solve Leibnitz' problem of the isochronous curve, and proposed the problem of the catenary. He contributed the curves known as the *elastica*, the *lintearia*, and the *velaria*. In his study of spirals he became so impressed with the remarkable properties of the logarithmic spiral that he requested it to be engraved on his tombstone with the words, *Eadem numero mutata resurgo*. The request was fulfilled.

In 1696 he proposed the famous isoperimetrical problem and offered a reward for its solution. With numerous other mathematicians his brother John offered a solution. James showed John's solution to be incorrect. John altered his solution, resubmitted it, and claimed the reward. James showed the amended solution to be still incorrect and called it no solution at all. James then gave a correct solution of his own. John republished another incorrect solution but did not concede its incorrectness until 1718, after James's death, and then he tried to "palm off" his brother's solution, thinly disguised, for his own incorrect one.

James Bernoulli was professor of mathematics at the university at Bâle from 1687 until his death in 1705, when he was succeeded by his brother John. He was once rector of the university, and numerous honors were bestowed upon him. His chief continuous work was his *Ars conjectandi*, a book on the calculus of probabilities. This contains what are now called "Bernoulli's theorem" and "Bernoulli's numbers." These numbers are the coefficients of $\frac{x^n}{n!}$ in the expansion $(e^x-1)^{-1}$, though Bernoulli did not so regard them.

[See Ball, Cajori, or *Encyclopaedia Britannica*.]

308. The limaçon and cardioid. The equations of the *limaçon*

$$\rho = a \pm b \cos \theta$$

and $\rho = a \pm b \sin \theta$ lead to different forms of the locus according as a is less than, greater than, or equal to, b.

θ	δ
0°	3.
30°	2.7
45°	2.4
60°	2.
90°	1.
120°	0.
135°	−.4
150°	−.7
180°	−.1

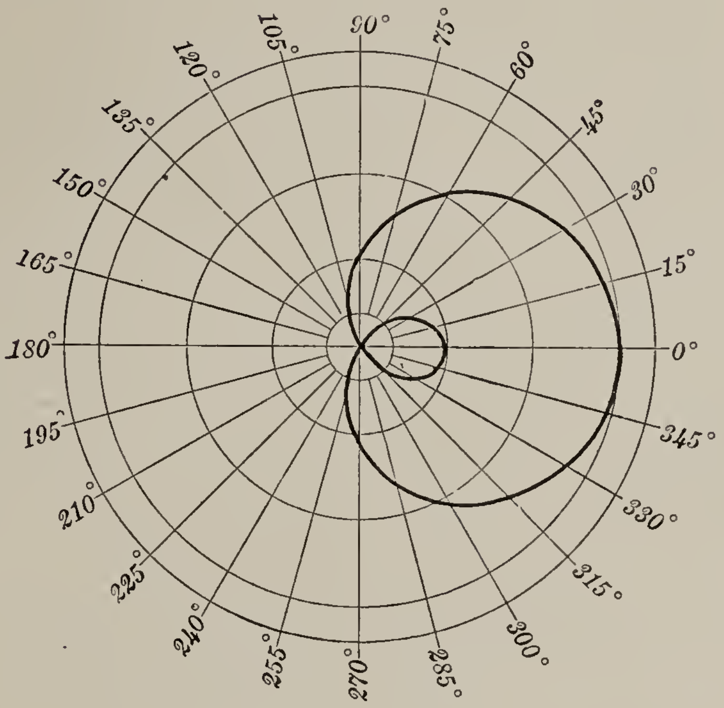

Fig. 112

Fig. 112 is the locus of the equation $\rho = 1 + 2 \cos \theta$. When $a = b$ the locus is called the *cardioid*, Fig. 113.

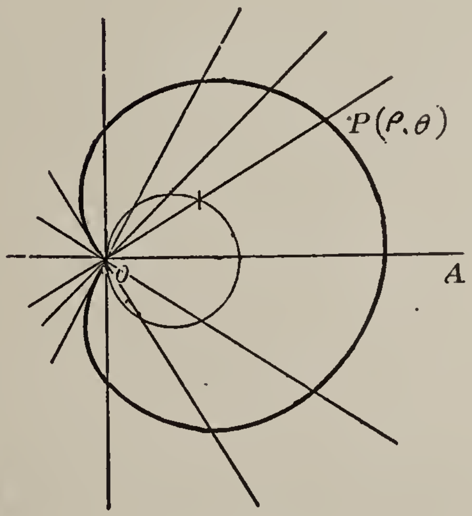

Fig. 113

EXERCISES

1. Transform the equation $\rho=a\pm b\cos\theta$ into rectangular co-ordinates.

2. Plot and discuss the locus of each of the equations,

$$\rho=1-2\cos\theta;\ \rho=1+2\sin\theta;\ \rho=1-2\sin\theta$$

3. Plot and discuss each of the following cardioids,

$$\rho=1-\cos\theta;\ \rho=1+\cos\theta;\ \rho=1-\sin\theta;\ \rho=1+\sin\theta$$

Historical note. The limaçon, commonly called the limaçon of Pascal, was invented and given the name limaçon by Blaise Pascal (1623–62). Pascal was a renowned theologian, analyst, geometrician, and philosopher. The limaçon is a special case of the Cartesian oval, which may be defined as a curve having two focal points, in which a constant multiple of one radius vector of any point differs from the other radius vector of the same point by a constant quantity. The word *limaçon* means "snail-like."

Determination of the Equation from the Geometric Property of the Locus

309. The cissoid of Diocles. Let OBA, Fig. 114, be a circle with diameter OA. Let the variable secant OB, revolving about O, meet the tangent AC at D. From O lay off $OP=BD$. It is required to determine the equation of the locus of P.

Let a be the radius of the circle and denote $\angle POA$ by θ.

Then

$$\cos\theta=\frac{2a}{OD},\ \text{and}\ OD=2a\sec\theta$$

Similarly

$$\cos\theta=\frac{OB}{2a},\ \text{and}\ OB=2a\cos\theta$$

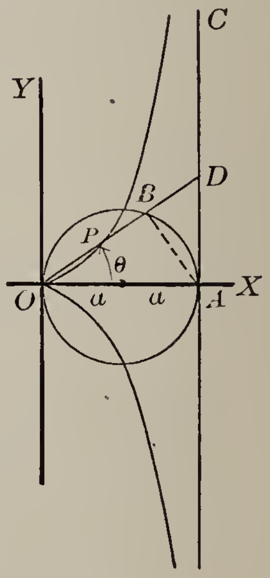

FIG. 114

Since $\rho = OP = OD - PD = OD - OB$

it follows that

$$\rho = 2a(\sec\theta - \cos\theta) = 2a\left(\frac{1-\cos^2\theta}{\cos\theta}\right)$$

$$= \frac{2a\sin^2\theta}{\cos\theta}$$

$$\therefore \quad \boldsymbol{\rho = 2a\tan\theta\sin\theta}$$

1. Transform the polar equation of the cissoid into the equation

$$y^2 = \frac{x^3}{2a - x}$$

2. Show that the line $x = 2a$ is an asymptote of the cissoid, that the curve is symmetric with respect to the x-axis, and that the values of x must lie between 0 and $2a$.

Historical note. The cissoid was devised by a Greek mathematician named Diocles, of the first Alexandrian school. Diocles lived about 180 B.C. Little else is known of him except that he solved the problem proposed by Archimedes, to draw a plane to divide a sphere into two parts whose volumes shall be in a given ratio.

The Greek word cissoid means "ivy-like," and Diocles named his curve from its resemblance to the vine of the ivy hanging from a trellis.

Diocles devised the cissoid to solve the problem of inserting two mean proportionals between two given lines, to which Hippocrates, many years before, had reduced the problem of duplicating a cube. To insert two mean proportionals, x and y, between any line, a, and its double, $2a$, we take the proportion:

$$a : x = x : y = y : 2a$$

and solve it for x, finding easily that

$$x^3 = 2a^3, \text{ whence } x = \sqrt[3]{2a}$$

The cissoid enables us to construct x. If then a denote the length of the edge of a given cube, x will be the length of edge of a cube of twice the volume.

Taking the proportion

$$a : x = x : y = y : na$$

we readily find

$$x^3 = na^3, \text{ or } x = a\sqrt[3]{n}$$

Now letting $a=1$, we have $x=\sqrt[3]{n}$, and since n is *any* number, and x can be constructed from the cissoid, this curve enables us to construct a line which equals the cube root of *any* number.

310. The witch of Agnesi. Let OAB, Fig. 115, be a circle whose radius is a, and whose diameter is OA.

Let the variable secant OBC intersect the tangent AD at C.

Draw $CE \perp OX$ and $BP \perp CE$.

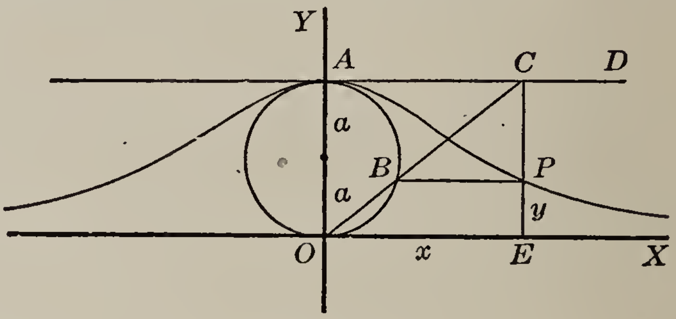

FIG. 115

It is required to find the equation of the locus of P, as OC turns about point O.

Show that

$$\frac{2a-y}{2a} = \frac{BC}{CO} = \frac{x^2}{CO^2} = \frac{x^2}{x^2+4a^2}$$

$$\therefore \quad 1 - \frac{y}{2a} = \frac{x^2}{x^2+4a^2}$$

$$\therefore \quad \frac{y}{2a} = \frac{x^2+4a^2-x^2}{x^2+4a^2}$$

$$\therefore \quad y = \frac{8a^3}{x^2+4a^2}$$

Discuss this equation as to asymptotes and symmetry.

Historical note. The witch of Agnesi was invented by an Italian lady, Maria Gaetana Agnesi (1728–99). She was made a member of the Academy of Bologna in 1748 and was appointed professor of mathematics in the University of Bologna in 1750. She was an excellent writer and an accomplished scholar. Her writings were translated into both English and French. Madame Agnesi called the curve the *versiera*, but later mathematicians attached the inventor's own name to it.

311. The strophoid. From the point $A(a, 0)$, Fig. 116, draw AB to any point B on the y-axis.

On AB lay off $BP = BP' = BO$.

It is required to find the locus of P and P' as AB turns about point A.

Denote the co-ordinates of P by ρ and θ.

Then in $\triangle OPA$ we have

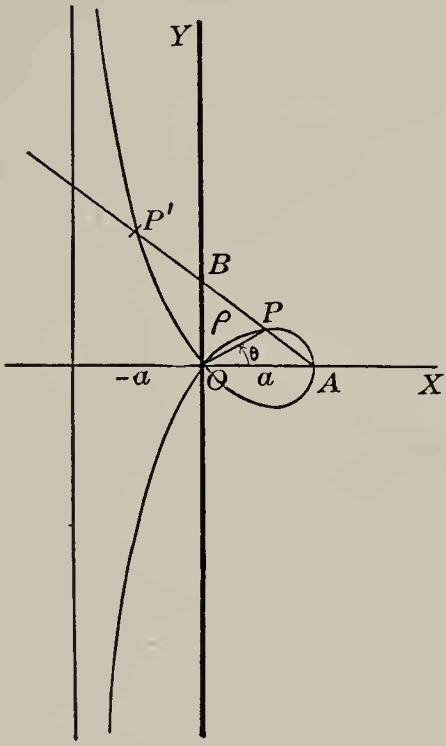

FIG. 116

$$\frac{\rho}{a} = \frac{\sin OAP}{\sin OPA}$$

$$\angle OPA = 180° - \angle BPO$$

$$\angle BPO = \angle BOP = 90 - \theta$$

$$\therefore \quad \angle OPA = 90 + \theta$$

$$\therefore \quad \angle OAP = 180 - (90 + \theta + \theta) = 90 - 2\theta$$

$$\therefore \quad \frac{\rho}{a} = \frac{\sin (90 - 2\theta)}{\sin (90 + \theta)} = \frac{\cos 2\theta}{\cos \theta}$$

$$\therefore \quad \boldsymbol{\rho = 2a \cos \theta - a \sec \theta}$$

EXERCISES

1. Transform the equation of the strophoid into the equation

$$y = \pm x \sqrt{\frac{a - x}{a + x}}.$$

2. Show that the locus of the equation in exercise 1 is symmetrical with respect to the x-axis; that x must lie between $-a$ and $+a$; that $x = -a$ is an asymptote.

312. The conchoid of Nicomedes. Let OA, Fig. 117, be perpendicular to BC. Let the line OD turn about point O, meeting the straight line BC at point D. On OD lay off $DP = DP' =$ a constant length b. Find the locus of P and P'.

Let (ρ, θ) be the co-ordinates of P.

Then

$$\rho = OD + DP = \frac{a}{\cos\theta} \pm b$$

or

$$\boldsymbol{\rho = \frac{a}{\cos\theta} \pm b}$$

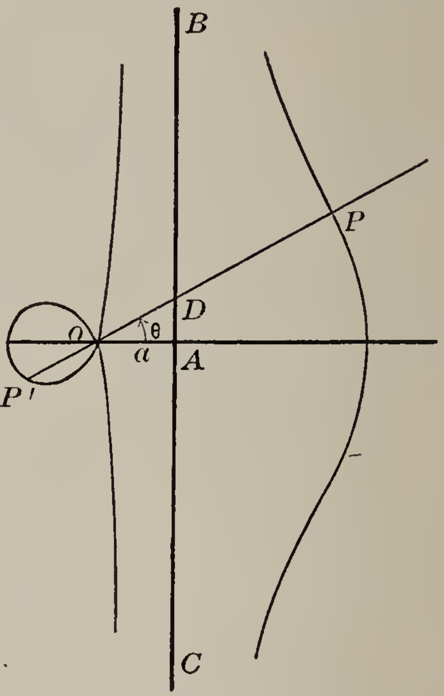

FIG. 117

EXERCISES

1. Transform the equation $\rho = \frac{a}{\cos\theta} \pm b$ into the equation $(x^2 + y^2)(x - a)^2 = b^2x^2$ and discuss the locus as to asymptotes and symmetry.

2. Let a line-segment AB move so that the end-points A and B are always on the x-axis and y-axis, respectively. From the origin O draw OP perpendicular to AB. Show that the equation of the locus of P is $\rho = \sin 2\theta$, the equation of the four-leafed rose.

Historical note. The conchoid, or mussel-shaped curve, was invented by the Greek Nicomedes, who lived in the second century B.C. and about the same time as Diocles. Like the cissoid, it was invented to solve the Delian problem of "duplicating the cube." It is also readily applied to the equally celebrated problem of the "trisection of a given angle." Both the cissoid and the conchoid have been of interest to mathematicians for more than twenty centuries.

313. The ovals of Cassini. Let a point $P(x, y)$, Fig. 118, move so that the product of its distances from two fixed points $F_1(-a, 0)$ and $F(a, 0)$ remains constant. It is required to find the locus of P.

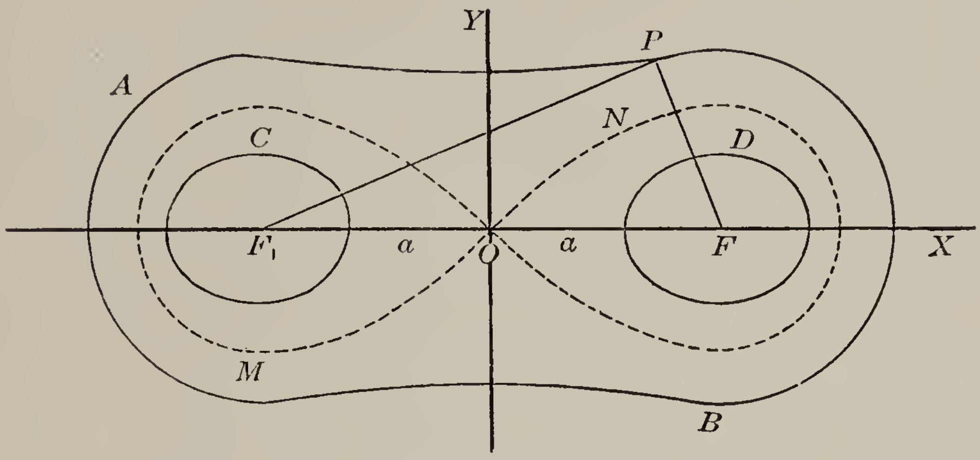

FIG. 118

Show that $F_1P = \sqrt{(x+a)^2+y^2}$

and that $FP = \sqrt{(x-a)^2+y^2}$

Let $\sqrt{(x+a)^2+y^2}\sqrt{(x-a)^2+y^2} = c^2$

Hence the required equation is

$$[(x+a)^2+y^2][(x-a)^2+y^2] = c^4$$

When $c = a$, the equation reduces to

$$(x^2+y^2)^2 - 2a^2(x^2-y^2) = 0$$

which is the equation of the *lemniscate* MN.

When $c > a$, the locus is the oval PAB.

When $c < a$, the locus consists of the ovals C and D.

314. The spiral of Archimedes. This is the locus of a point whose radius vector ρ is directly proportional to the vectorial angle θ, i.e., $\rho = a\theta$.

The table, Fig. 119, gives approximate corresponding values of ρ and θ for $a=1$.

θ	ρ
0	0
$\frac{\pi}{12}$	.26
$\frac{\pi}{6}$	.52
$\frac{\pi}{4}$	.79
$\frac{\pi}{3}$	1.05
$\frac{\pi}{2}$	1.57
$\frac{2\pi}{3}$	2.09
π	3.14
$\frac{3\pi}{2}$	4.71

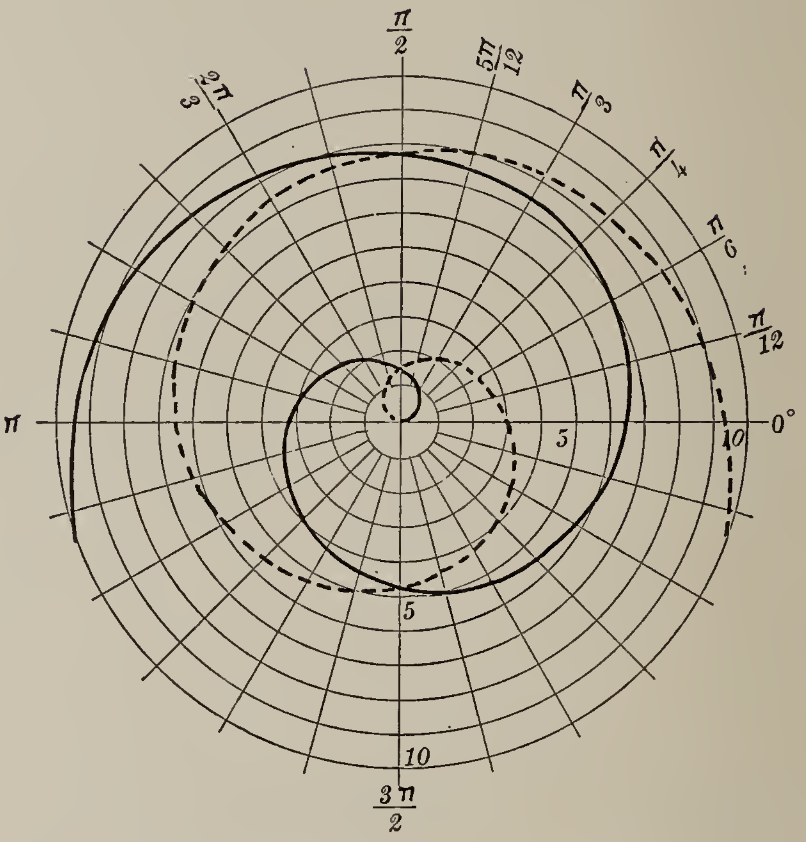

Fig. 119

As θ increases indefinitely, ρ also increases without limit, the curve winding around O.

The dotted curve is the spiral obtained when $a=-1$.

EXERCISES

1. Plot the *hyperbolic spiral* $\rho^{\theta}=a$. Find the asymptote.

2. Plot the *logarithmic spiral* $\rho=b^{a\theta}$, or the equivalent equation $a\theta=\log_b \rho$.

315. The cycloid. If a circle rolls along a straight line the curve traced by any point of the circle is called a *cycloid*.

Take as origin, Fig. 120, the point O on OX, where a point P of the circle touches the line OX.

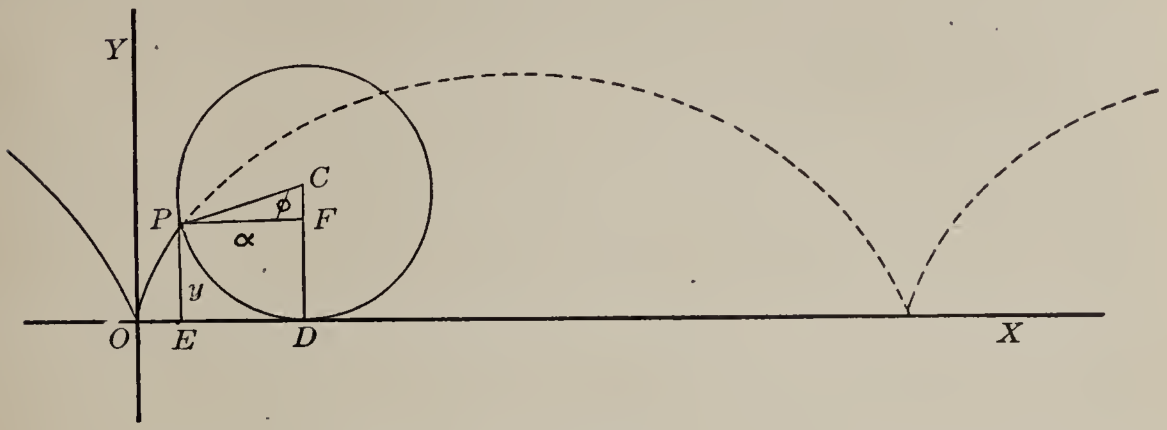

FIG. 120

Let a denote the radius of the circle, and ϕ the central angle PCD.

Then $$x = OD - ED = OD - PF$$

and $$y = DC - FC$$

Show that $$OD = PD = a\phi$$

$$PF = a \sin \phi$$

and $$FC = a \cos \phi$$

$$\therefore \quad x = a\phi - a \sin \phi$$

and $$y = a - a \cos \phi \text{, which may be written}$$

$$\begin{cases} x = a(\phi - \sin \phi) \\ y = a(1 - \cos \phi) \end{cases}$$

These are the *parameter equations* of the cycloid. Show that the curve has an unlimited number of equal arches, forming a *cusp* wherever P touches the line OX.

Historical note. A point fixed with respect to any curve which rolls on any other curve generates a *roulette.* If the rolling curve is a circle and the fixed curve is a straight line the curve generated by the point on the rolling circle is a *cycloid.* If both the fixed and rolling curves are circles, and the generating point is on the circumference of the rolling circle, the curve generated is an *epicycloid* if the rolling circle is on the outside of the fixed circle, and is a *hypocycloid* if the rolling circle is on the inside of the fixed circle.

Galileo in 1630 was the first to call attention to the cycloid, suggesting that as the shape was particularly graceful it should be used for arches of bridges. In 1634 Roberval found its area. Descartes, doubting the correctness of Roberval's solution, defied both him and Fermat to find its tangent. Fermat solved it at once. In 1658 Pascal solved many problems that had been proposed on the cycloid. Wallis, Huygens, and many subsequent mathematicians worked on it. It was the second curve that was rectified. This was done by Wren in 1673.

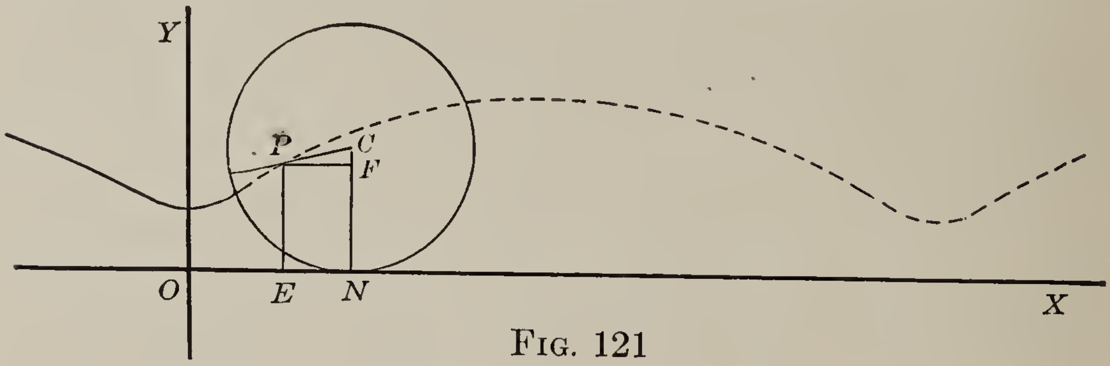

FIG. 121

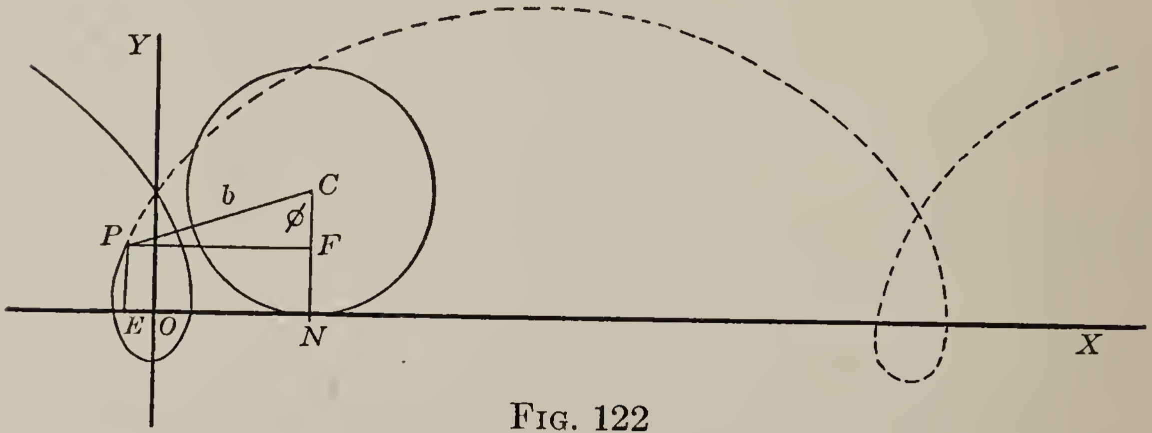

FIG. 122

316. The trochoid. Let P be any point on the radius, Fig. 121, or on the extension of the radius of circle C, Fig. 122. The locus of P is called a *trochoid*. Show that

$$\begin{cases} x = a\phi - b\sin\phi \\ y = a - b\cos\phi \end{cases}$$

317. The epicycloid. Let the circle C, Fig. 123, roll upon the outside of circle O. The locus of any point P on circle C is called an *epicycloid.*

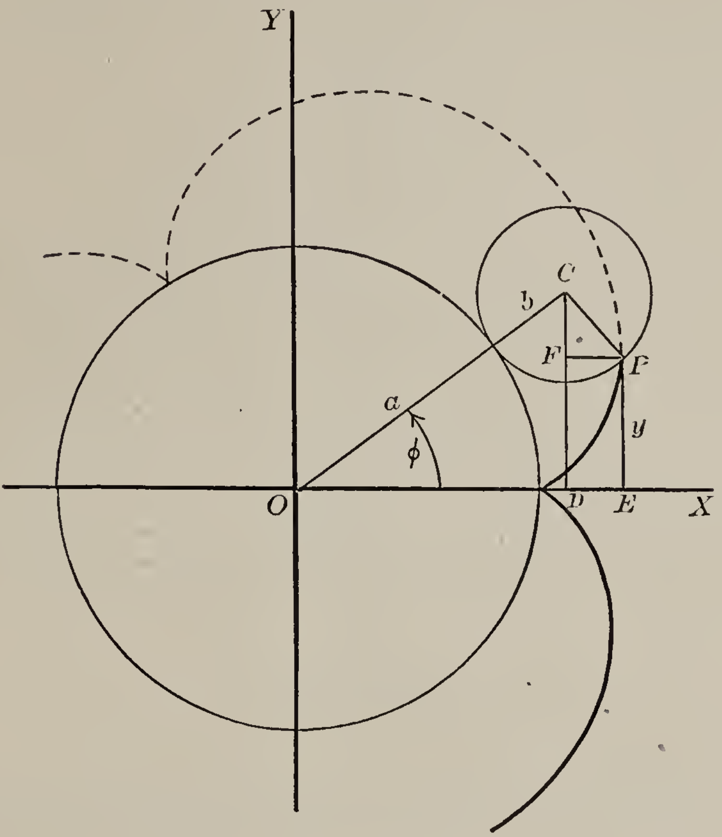

Fig. 123

Denote the radii of circles O and C by a and b, respectively, $\angle OCP$ by θ, and $\angle COE$ by ϕ.

Then $$x = OD + DE = OD + PF$$

and $$y = DC - CF$$

Show that

$$OD = (a+b) \cos \phi;$$

$$\angle CPF = 180 - (\theta + \phi),$$

$$PF = a \cos [180 - (\theta + \phi)] = -a \cos (\theta + \phi)$$

$$DC = (a+b) \sin \phi$$

$$CF = a \sin [180 - (\theta + \phi)] = a \sin (\theta + \phi)$$

$$\begin{cases} x = (a+b) \cos \phi - a \cos (\theta + \phi) \\ y = (a+b) \sin \phi - a \sin (\theta + \phi) \end{cases}$$

318. The hypocycloid. If circle C, Fig. 123, rolls on the inside of circle O, the locus of P is called *hypocycloid,* Fig. 124.

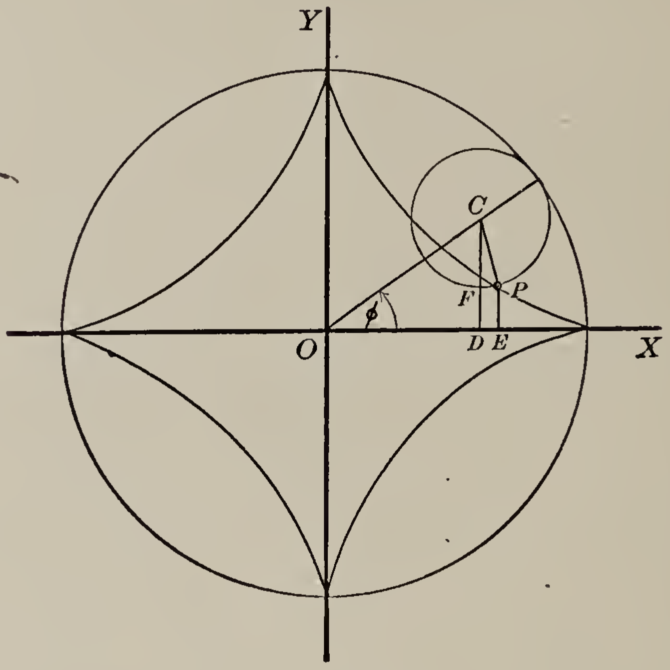

FIG. 124

Show that the parameter equations of the hypocycloid are

$$\begin{cases} x=(b-a)\cos\phi+a\cos\dfrac{b-a}{a}\phi \\ y=(b-a)\sin\phi-a\sin\dfrac{b-a}{a}\phi \end{cases}$$

When the rolling circle makes exactly four revolutions in rolling along circle O, as in Fig. 124, the locus of P is called an *astroid.*

EXERCISES

1. Show that the equations of the astroid are $x=a\cos^3\phi$, $y=a\sin^3\phi$.

Let $a=4b$ and express x and y in terms of functions of ϕ. Use the formulas $\sin 3\phi=3\sin\phi-4\sin^3\phi$, and $\cos 3\phi=4\cos^3\phi-3\cos\phi$.

2. Transform the parametric equations of the astroid into the rectangular equation $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

3. Transform the parametric equations

$$\begin{cases} x=\dfrac{3at}{1+t^3} \\ y=\dfrac{3at^2}{1+t^3} \end{cases}$$

into the equation

$$x^3+y^3=3axy$$

4. Plot the locus of exercise 3.

5. Plot the locus of each of the equations

$$y=\frac{1}{x-2};\quad y=\frac{x-2}{x-3};\quad y=\frac{(x-1)(x-3)}{(x-2)(x-4)}$$

Summary

319. The equation of the following loci have been studied in this chapter.

Exponential curve:

$$y=a^x$$

Logarithmic curve:

$$y=\log_a x$$

Four-leafed rose:

$$\rho=a \sin 2\theta$$
$$(x^2+y^2)^2=4a^2x^2y^2$$

Five-leafed rose:

$$\rho=a \sin 5\theta$$

Lemniscate:

$$\rho^2=2a^2 \cos 2\theta$$
$$(x^2+y^2)^2=2a^2(x^2-y^2)$$

Limaçon:

$$\rho=a\pm b\cos\theta$$
$$\rho=a\pm b\sin\theta$$

Cardioid:

$$\rho=a(1\pm\cos\theta)$$
$$\rho=a(1\pm\sin\theta)$$

Cissoid:

$$\rho=2a\tan\theta\sin\theta$$

$$y^2=\frac{x^3}{2a-x}$$

Witch:

$$y=\frac{8a^3}{x^2+4a^2}$$

Strophoid:

$$\rho=2a\cos\theta-a\sec\theta$$

$$y=\pm x\sqrt{\frac{a-x}{a+x}}$$

Conchoid:

$$\rho=\frac{a}{\cos\theta}\pm b$$

Ovals:

$$[(x+a)^2+y^2][(x-a)^2+y^2]=c^4$$

Spirals:

$$\rho=a\theta,\ \rho\theta=a,\ \rho=b^{a\theta}$$

Cycloid:

$$\begin{cases}x=a(\phi-\sin\phi)\\ y=a(1-\cos\phi)\end{cases}$$

Trochoid:

$$\begin{cases}x=a\phi-b\sin\phi\\ y=a-b\cos\phi\end{cases}$$

Epicycloid:

$$\begin{cases} x=(a+b)\cos\phi-a\cos(\theta+\phi) \\ y=(a+b)\sin\phi-a\sin(\theta+\phi) \end{cases}$$

Hypocycloid:

$$\begin{cases} x=(b-a)\cos\phi+a\cos\dfrac{b-a}{a}\phi \\ y=(b-a)\sin\phi-a\sin\dfrac{b-a}{a}\phi \end{cases}$$

Astroid:

$$\begin{cases} x=a\cos^3\phi \\ y=a\sin^3\phi \end{cases}$$

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

LOGARITHMS OF NUMBERS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474

LOGARITHMS OF NUMBERS—*Continued*

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	9525	8531	8537	8543	8549	8555	9561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	9698	8704	9710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
100	0000	0004	0008	0013	0017	0021	0026	0030	0035	0039

TABLE OF POWERS AND ROOTS

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.259	52	2,704	140,608	7.211	3.732
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.779
5	25	125	2.236	1.709	**55**	3,025	166,375	7.416	3.802
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.825
7	49	343	2.645	1.912	57	3,249	185,193	7.549	3.848
8	64	512	2.828	2.000	58	3,364	195,112	7.615	3.870
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.892
10	100	1,000	3.162	2.154	**60**	3,600	216,000	7.745	3.914
11	121	1,331	3.316	2.223	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.957
13	169	2,197	3.605	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.741	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.872	2.466	**65**	4,225	274,625	8.062	4.020
16	256	4,096	4.000	2.519	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.061
18	324	5,832	4.242	2.620	68	4,624	314,432	8.246	4.081
19	361	6,859	4.358	2.668	69	4,761	328,509	8.306	4.101
20	400	8,000	4.472	2.714	**70**	4,900	343,000	8.366	4.121
21	441	9,261	4.582	2.758	71	5,041	357,911	8.426	4.140
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.795	2.843	73	5,329	389,017	8.544	4.179
24	576	13,824	4.898	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	**75**	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.717	4.235
27	729	19,683	5.196	3.000	77	5,929	456,533	8.774	4.254
28	784	21,952	5.291	3.036	78	6,084	474,552	8.831	4.272
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.290
30	900	27,000	5.477	3.107	**80**	6,400	512,000	8.944	4.308
31	961	29,791	5.567	3.141	81	6,561	531,441	9.000	4.326
32	1,024	32,768	5.656	3.174	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.744	3.207	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.830	3.239	84	7,056	592,704	9.165	4.379
35	1,225	42,875	6.916	3.271	**85**	7,225	614,125	9.219	4.396
36	1,296	46,656	6.000	3.301	86	7,396	636,056	9.273	4.414
37	1,369	50,653	6.082	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.361	88	7,744	681,472	9.380	4.447
39	1,521	59,319	6.244	3.391	89	7,921	704,969	9.433	4.464
40	1,600	64,000	6.324	3.419	**90**	8,100	729,000	9.486	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.497
42	1,764	74,088	6.480	3.476	92	8,464	778,688	9.591	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.643	4.530
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.546
45	2,025	91,125	6.708	3.556	**95**	9,025	857,375	9.746	4.562
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.797	4.578
47	2,209	103,823	6.855	3.608	97	9,409	912,673	9.848	4.594
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	6.928	3.659	99	9,801	970,299	9.949	4.626
50	2,500	125,000	7.071	3.684	**100**	10,000	1,000,000	10.000	4.641

TABLE OF SINES, COSINES, AND TANGENTS OF ANGLES FROM 1°–90°

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2	.0349	.9994	.0349	47	.7314	.6820	1.0724
3	.0523	.9986	.0524	48	.7431	.6691	1.1106
4	.0698	.9976	.0699	49	.7547	.6561	1.1504
5	.0872	.9962	.0875	**50**	.7660	.6428	1.1918
6	.1045	.9945	.1051	51	.7771	.6293	1.2349
7	.1219	.9925	.1228	52	.7880	.6157	1.2799
8	.1392	.9903	.1405	53	.7986	.6018	1.3270
9	.1564	.9877	.1584	54	.8090	.5878	1.3764
10	.1736	.9848	.1763	**55**	.8192	.5736	1.4281
11	.1908	.9816	.1944	56	.8290	.5592	1.4826
12	.2079	.9781	.2126	57	.8387	.5446	1.5399
13	.2250	.9744	.2309	58	.8480	.5299	1.6003
14	.2419	.9703	.2493	59	.8572	.5150	1.6643
15	.2588	.9659	.2679	**60**	.8660	.5000	1.7321
16	.2756	.9613	.2867	61	.8746	.4848	1.8040
17	.2924	.9563	.3057	62	.8829	.4695	1.8807
18	.3090	.9511	.3249	63	.8910	.4540	1.9626
19	.3256	.9455	.3443	64	.9888	.4384	2.0503
20	.3420	.9397	.3640	**65**	.9063	.4226	2.1445
21	.3584	.9336	.3839	66	.9135	.4067	2.2460
22	.3746	.9272	.4040	67	.9205	.3907	2.3559
23	.3907	.9205	.4245	68	.9272	.3746	2.4751
24	.4067	.9135	.4452	69	.9336	.3584	2.6051
25	.4226	.9063	.4663	**70**	.9397	.3420	2.7475
26	.4384	.8988	.4877	71	.9455	.3256	2.9042
27	.4540	.8910	.5095	72	.9511	.3090	3.0777
28	.4695	.8829	.5317	73	.9563	.2924	3.2709
29	.4848	.8746	.5543	74	.9613	.2756	3.4874
30	.5000	.8660	.5774	**75**	.9659	.2588	3.7321
31	.5150	.8572	.6009	76	.9703	.2419	4.0108
32	.5299	.8480	.6249	77	.9744	.2250	4.3315
33	.5446	.8387	.6494	78	.9781	.2079	4.7046
34	.5592	.8290	.6745	79	.9816	.1908	5.1446
35	.5736	.8192	.7002	**80**	.9848	.1736	5.6713
36	.5878	.8090	.7265	81	.9877	.1564	6.3138
37	.6018	.7986	.7536	82	.9903	.1392	7.1154
38	.6157	.7880	.7813	83	.9925	.1219	8.1443
39	.6293	.7771	.8098	84	.9945	.1045	9.5144
40	.6428	.7660	.8391	**85**	.9962	.0872	11.4301
41	.6561	.7547	.8693	86	.9976	.0698	14.3006
42	.6691	.7431	.9004	87	.9986	.0523	19.0811
43	.6820	.7314	.9325	88	.9994	.0349	28.6363
44	.6947	.7193	.9657	89	.9998	.0175	57.2900
45	.7071	.7071	1.0000	**90**	1.0000	.0000	∞

FORMULAS

PLANE GEOMETRY

1. Length of circle $= 2\pi r = 3.14159d$
2. Area of circle $= \pi r^2$
3. Area of triangle $= \frac{1}{2}bh = \frac{1}{2}ab \sin C = \frac{1}{2}r(a+b+c)$

$$= \frac{abc}{4R} = \sqrt{s(s-a)(s-b)(s-c)}$$

4. Area of parallelogram $= bh$
5. Area of square $= a^2$
6. Area of equilateral triangle $= \frac{a^2}{4}\sqrt{3}$
7. Area of trapezoid $= \frac{1}{2}h(b_1+b_2) = hm$

SOLID GEOMETRY

1. Volume of prism $= ba$
2. Volume of pyramid $= \frac{1}{3}ba$
3. Volume of right circular cylinder $= \pi r^2 a$
4. Total surface of right circular cylinder $= 2\pi r(r+a)$
5. Lateral surface of right circular cylinder $= 2\pi ra$
6. Volume of right circular cone $= \frac{1}{3}\pi r^2 a$
7. Lateral surface of right circular cone $= \pi rs$
8. Total surface of right circular cone $= \pi r(r+s)$
9. Surface of sphere $= 4\pi r^2$
10. Volume of sphere $= \frac{4}{3}\pi r^3$

SERIES

1. Arithmetical progression:

$$l = a + (n-1)d; \quad s = \frac{n}{2}(a+l)$$

2. Geometrical progression:

$$l=ar^{n-1}; \quad s=\frac{a-ar^n}{1-r}; \quad \text{if } r<1 \text{ and } n\to\infty,\ s=\frac{a}{1-r}$$

3. Binomial theorem:

$$(a+b)^n=a^n+\frac{n}{1}a^{n-1}b+\frac{n(n-1)}{1\cdot 2}a^{n-2}b^2$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3+\text{etc.}$$

The kth term

$$=\frac{n(n-1)(n-2)\ .\ .\ .\ .\ (n-k+2)}{1\cdot 2\cdot 3\ .\ .\ .\ .\ k-1}a^{n-k+1}b^{k-1}$$

LOGARITHMS

1. $\log ab=\log a+\log b$
2. $\log \frac{a}{b}=\log a-\log b$
3. $\log a^n=n \log a$
4. $\log \sqrt[n]{a}=\frac{\log a}{n}$
5. $\log 1=0$
6. $\log_a N=\frac{\log_b N}{\log_b a}$
7. $\text{colog}\, N=\log \frac{1}{N}=(10-\log N)-10$

QUADRATIC EQUATION

If $ax^2+bx+c=0$, $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

If $b^2-4ac=0$, the roots are real and equal.
If $b^2-4ac>0$, the roots are real and unequal.
If $b^2-4ac<0$, the roots are complex.

TRIGONOMETRIC FORMULAS

1. $\sin\alpha=\frac{a}{c}$, $\cos\alpha=\frac{b}{c}$, $\tan\alpha=\frac{a}{b}$,

$\csc\alpha=\frac{c}{a}$, $\sec\alpha=\frac{c}{b}$, $\cot\alpha=\frac{b}{a}$

2. $\sin^2\alpha+\cos^2\alpha=1$

3. $\sec^2\alpha=1+\tan^2\alpha$

4. $\csc^2\alpha=1+\cot^2\alpha$

5. $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$

6. $\cot\alpha=\frac{\cos\alpha}{\sin\alpha}$

7. $\sec\alpha=\frac{1}{\cos\alpha}$

8. $\csc\alpha=\frac{1}{\sin\alpha}$

9. $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

10. $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

11. $\tan(\alpha\pm\beta)=\frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$

12. $\sin\alpha+\sin\beta=2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

13. $\sin\alpha-\sin\beta=2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

14. $\cos\alpha+\cos\beta=2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$

15. $\cos\alpha-\cos\beta=-2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)$

16. $\sin\alpha\sin\beta=\frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta)$

17. $\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$

18. $\sin\alpha\cos\beta=\frac{1}{2}\sin(\alpha+\beta)+\frac{1}{2}\sin(\alpha-\beta)$

19. $\sin 2\alpha=2\sin\alpha\cos\alpha$

20. $\cos 2\alpha=\cos^2\alpha-\sin^2\alpha$

$=2\cos^2\alpha-1=1-2\sin^2\alpha$

21. $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

22. $\sin\frac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

23. $\cos\frac{1}{2}\alpha = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

24. $\tan\frac{1}{2}\alpha = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

25. $\sin\left(\frac{\pi}{2}\pm\theta\right) = \cos\theta$

26. $\cos\left(\frac{\pi}{2}\pm\theta\right) = \mp\sin\theta$

27. $\tan\left(\frac{\pi}{2}\pm\theta\right) = \mp\cot\theta$

28. $\sin(\pi\pm\theta) = \mp\sin\theta$

29. $\cos(\pi\pm\theta) = -\cos\theta$

30. $\tan(\pi\pm\theta) = \pm\tan\theta$

31. $\sin(-x) = -\sin x$

32. $\cos(-x) = \cos x$

33. $\tan(-x) = -\tan x$

34. $\csc(-x) = -\csc x$

35. $\sec(-x) = \sec x$

36. $\cot(-x) = -\cot x$

37. $\sin 30° = \frac{1}{2}$

38. $\sin 45° = \frac{1}{2}\sqrt{2}$

39. $\sin 60° = \frac{1}{2}\sqrt{3}$

40. $\cos 30° = \frac{1}{2}\sqrt{3}$

41. $\cos 45° = \frac{1}{2}\sqrt{2}$

42. $\cos 60° = \frac{1}{2}$

TRIANGLES

43. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

44. $a^2 = b^2 + c^2 - 2bc\cos A$

45. $\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}$

46. $\frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}$

47. $\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}$

If $s = \frac{1}{2}(a+b+c)$:

48. $\sin\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}$.

49. $\cos\frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}$.

50. $\tan\frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

If r = radius of inscribed circle:

51. $r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

52. $\tan\frac{1}{2}A=\frac{r}{s-a}$.

53. $\tan\frac{1}{2}B=\frac{r}{s-b}$.

54. $\tan\frac{1}{2}C=\frac{r}{s-c}$.

55. $$\text{Area}=\frac{1}{2}ab\sin C=\frac{c^2}{2}\cdot\frac{\sin A\sin B}{\sin C}$$
$$=\sqrt{s(s-a)(s-b)(s-c)}$$

56. Diameter of circumscribed circle
$$=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

GREEK ALPHABET

Names	Letters		Names	Letters	
Alpha	Α	α	Nu	Ν	ν
Beta	Β	β	Xi	Ξ	ξ
Gamma	Γ	γ	Omicron	Ο	ο
Delta	Δ	δ	Pi	Π	π
Epsilon	Ε	ϵ	Rho	Ρ	ρ
Zeta	Ζ	ζ	Sigma	Σ	σ
Eta	Η	η	Tau	Τ	τ
Theta	Θ	θ	Upsilon	ϒ	υ
Iota	Ι	ι	Phi	Φ	ϕ
Kappa	Κ	κ	Chi	Χ	λ
Lambda	Λ	λ	Psi	Ψ	ψ
Mu	Μ	μ	Omega	Ω	ω

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